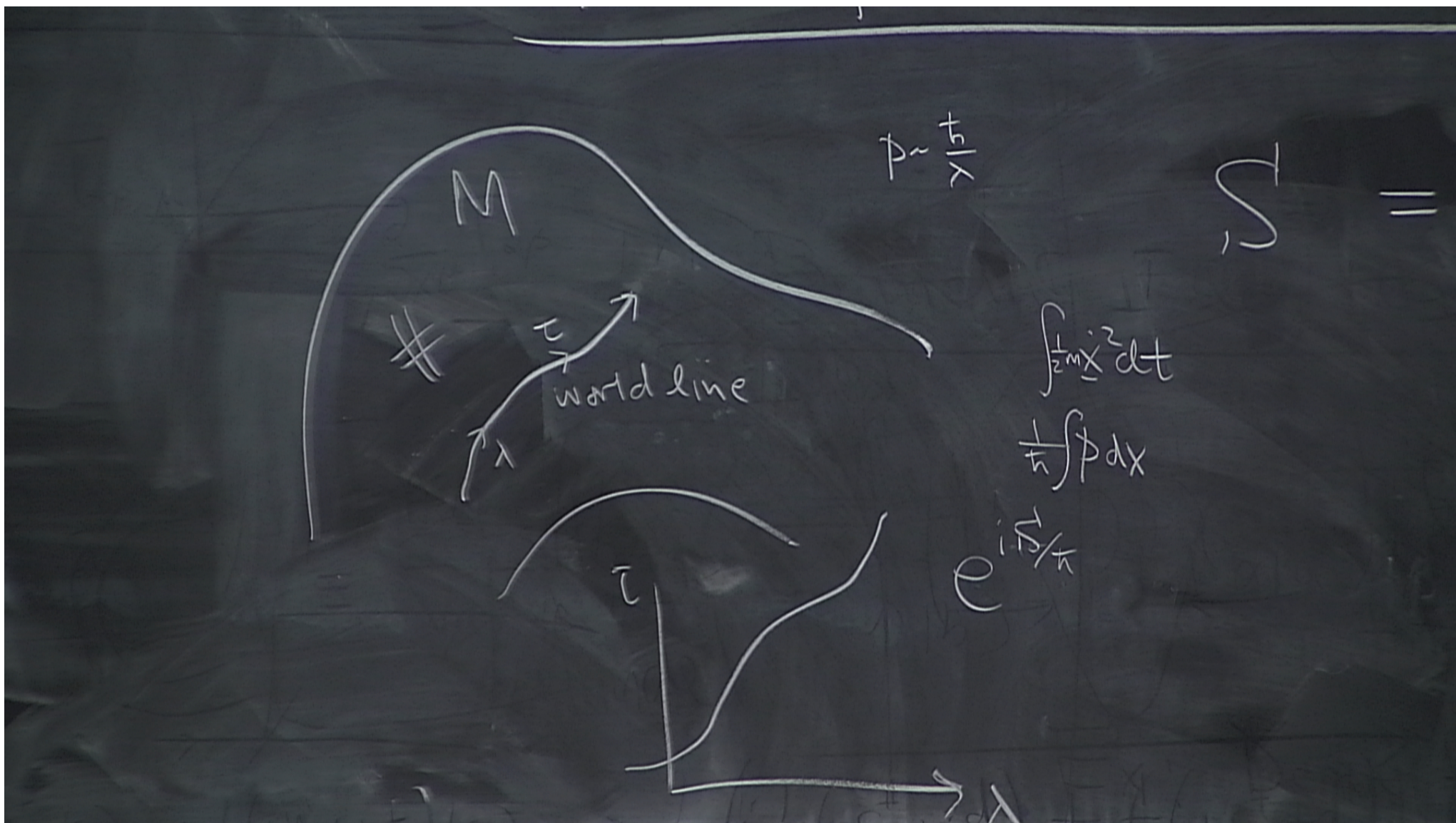


Title: Relativity - Lecture 8

Date: Sep 21, 2011 10:30 AM

URL: <http://pirsa.pi.local/11090033>

Abstract:





parameter with dim<sup>n</sup>s of momentum.

$$S = -mc \int d\tau = -mc \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \\ = -mc \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

- reparameterisation-invariant i.e. replacing



relativistic particle

parameter with dim's of momentum.

$$S = -mc \int d\tau = -mc \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

$$= -mc \int d\lambda \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

$$d\tilde{\lambda} = \frac{d\tilde{\lambda}}{d\lambda} d\lambda$$

- reparameterisation-invariant i.e. replacing  $\lambda$  by any  $\tilde{\lambda}(\lambda)$  (monotonically increasing) leaves action invariant.



$$\begin{aligned}
 \delta S' &= S'[x^M(\lambda) + \delta x^M(\lambda)] - S'[x^M(\lambda)] \\
 &= -mc \int d\lambda \frac{1}{2\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left( - \overset{\delta g_{\mu\nu}}{g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu} - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu \right)
 \end{aligned}$$

$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$





$$\begin{aligned}
 \delta S' &= S'[x^M(\lambda) + \delta x^M(\lambda)] - S'[x^M(\lambda)] \\
 &= -mc \int d\lambda \frac{1}{2} \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left( -g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu \right) \\
 &\quad \underbrace{\hspace{10em}}_{\substack{\delta g_{\mu\nu} \\ \text{integrate by parts.}}} \\
 &= \frac{d}{d\lambda} \left( \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \right) \frac{d}{d\lambda} \left( \frac{1}{2} (-g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - g_{\mu\nu} \dot{x}^\nu \delta \dot{x}^\mu) \right)
 \end{aligned}$$



$$\delta S' = S'[x^M(\lambda) + \delta x^M(\lambda)] - S'[x^M(\lambda)]$$

$$= -mc \int d\lambda \frac{1}{2} \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left( -g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu \right)$$

integrate by parts.

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\frac{d}{d\tau} = \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \frac{d}{d\lambda}$$

$$\frac{1}{2} \left( -g_{\mu\nu,\alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \dot{x}^\nu \delta \dot{x}^\mu \right)$$

integrate by parts

$$\left( -g_{\mu\nu,\alpha} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu)' \right) \delta x^\alpha$$



$$\begin{aligned}
 \delta S' &= S'[x^M(\lambda) + \delta x^M(\lambda)] - S'[x^M(\lambda)] \\
 &= -mc \int d\lambda \frac{1}{2} \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \left( -g_{\mu\nu, \alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu \right) \\
 &\quad \underbrace{\hspace{10em}}_{\text{integrate by parts.}} \\
 \frac{d}{d\tau} &= \frac{1}{\sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}} \frac{d}{d\lambda} \\
 &\quad \frac{1}{2} \left( -g_{\mu\nu, \alpha} \delta x^\alpha \dot{x}^\mu \dot{x}^\nu - 2g_{\mu\nu} \dot{x}^\nu \delta \dot{x}^\mu \right) \underbrace{\hspace{5em}}_{\text{integrate by parts}} \Rightarrow \\
 &\quad \int \left( -g_{\mu\nu, \alpha} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu)' \right) \delta x^\alpha \\
 &= 0 \quad \forall \delta x^\alpha(\lambda)
 \end{aligned}$$



$$2g_{\mu\nu} \delta \dot{x}^\mu \dot{x}^\nu$$

integrate by parts.

$$\dot{g}_{\mu\nu} = g_{\mu\nu,\alpha} \dot{x}^\alpha$$

$$\Rightarrow \boxed{(g_{\mu\nu} \dot{x}^\nu)' = \frac{1}{2} g_{\alpha\beta,\mu} \dot{x}^\alpha \dot{x}^\beta}$$

integ by parts

$$\Rightarrow \ddot{x}^\mu + \frac{1}{2} ($$



ate by parts.

$$\Rightarrow \boxed{(g_{\mu\nu} \dot{X}^\nu)' = \frac{1}{2} g_{\alpha\beta, \mu} \dot{X}^\alpha \dot{X}^\beta}$$

$$\Rightarrow \ddot{X}^\mu + \underbrace{\frac{1}{2} g^{\mu\beta}}_{\left\{ \begin{matrix} \mu \\ \lambda\nu \end{matrix} \right\}} (g_{\lambda\beta, \nu} + g_{\nu\beta, \lambda} - g_{\lambda\nu, \beta}) \dot{X}^\lambda \dot{X}^\nu =$$



$$\int d\lambda \left( -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu) \right) \delta x^\mu = 0 \quad \forall \delta x^\mu(\lambda)$$

$$\Rightarrow \ddot{x}^\mu + \underbrace{\frac{1}{2} g^{\mu\beta} (g_{\lambda\beta,\nu} + g_{\nu\beta,\lambda} - g_{\lambda\nu,\beta})}_{\sum_{\lambda\nu}^{\mu\beta}} \dot{x}^\lambda \dot{x}^\nu = 0$$

Improved formulation: (no  $\sqrt{\quad}$   $\rightarrow$  quantisation better)

$$S = \frac{m}{2} \int d\lambda \left( \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

$e_\mu^a$

$e = \text{"einbein"} \sim \sqrt{\text{metric along world line}}$

$$e'(\lambda) = \frac{d\lambda}{d\lambda'} e(\lambda)$$

$$e' d\lambda' = e d\lambda$$



$$\left\{ \begin{matrix} \mu \\ \lambda \nu \end{matrix} \right\}$$

atisation better)

$$e_\mu^a$$

$$e = \text{"einbein"} \sim \sqrt{\text{metric along world line}}$$

$$e'(\lambda') = \frac{d\lambda}{d\lambda'} e(\lambda)$$

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$$\left\{ \begin{matrix} \mu \\ \lambda \nu \end{matrix} \right\}$$

parameterisation better)

$$e_\mu^a$$

$$e = \text{"einbein"} \sim \sqrt{\text{metric along world line}}$$

$$e'(\lambda') = \frac{d\lambda}{d\lambda'} e(\lambda)$$

$$e' d\lambda' = e d\lambda = \text{reparameterisation invariant.}$$



Improved formulation: (no  $\sqrt{\quad}$   $\rightarrow$  quantisation)

$$S = \frac{m}{2} \int d\lambda \left( \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

$$\frac{\partial S}{\partial e} : \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} = -1$$



$$\int d\lambda \left( -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\mu) \right) \delta x^\nu$$

integrate by parts

$$= 0 \quad \forall \delta x^\nu(\lambda)$$

$$\Rightarrow (g_{\mu\nu} \dot{x}^\mu)^\cdot = \frac{1}{2} g_{\mu\rho, \lambda} \dot{x}^\mu \dot{x}^\lambda$$

$$\Rightarrow \ddot{x}^\lambda + \underbrace{\frac{1}{2} g^{\mu\rho}}_{\substack{\text{metric} \\ \lambda\nu}} (g_{\lambda\rho, \nu} + g_{\nu\rho, \lambda} - g_{\nu\rho, \mu}) \dot{x}^\mu \dot{x}^\nu = 0$$

Improved formulation: (no  $\sqrt{\quad}$   $\rightarrow$  quantisation better)

(set  $c=1$ )

$$S' = \frac{m}{2} \int d\lambda \left( \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

$e_\mu^a$

$e$  = "einbein"  $\sim \sqrt{\text{metric along world line}}$

$$e'(\lambda) = \frac{d\lambda}{d\lambda'} e(\lambda)$$

$$\frac{\delta S'}{\delta e} : \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} = -1 \quad \text{ie} \quad g_{\mu\nu} u^\mu u^\nu = -1$$

$e' d\lambda' = e d\lambda$  = reparametrisation invariant.



$\{\lambda, \nu\}$

quantisation better)

$e_\mu^a$

e)

$e = \text{"einbein"} \sim \sqrt{\text{metric along world line}}$

$$e'(\lambda') = \frac{d\lambda}{d\lambda'} e(\lambda)$$

$$g_{\mu\nu} u^\mu u^\nu = -1$$

$e' d\lambda' = e d\lambda = \text{reparameterisation invariant.}$

$$u^\mu = \frac{1}{e} \dot{x}^\mu$$



quantisation:

(no  $\sqrt{\quad}$   $\rightarrow$  quantisation better)

$$S = \frac{m}{2} \int d\lambda \left( \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

$\downarrow$  massive particle  
 $0 \Rightarrow$  photon  
 $+e \Rightarrow$  tachyon.  $e = e'(\lambda')$

$$\frac{\delta S}{\delta e} = -\frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} = -1 \quad \text{like} \quad g_{\mu\nu} u^\mu u^\nu = -1$$

$e' d\lambda'$   
 $u^\mu = \frac{dx^\mu}{d\lambda'}$



$$\int d\lambda \left( -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\mu) \right) \delta x^\nu$$

integrate by parts

$$= 0 \quad \forall \quad \delta x^\nu(\lambda)$$

$$\Rightarrow (g_{\mu\nu} \dot{x}^\mu)^\cdot = \frac{1}{2} g_{\mu\beta, \lambda} \dot{x}^\mu \dot{x}^\beta$$

$$\Rightarrow \ddot{x}^\lambda + \underbrace{\frac{1}{2} g^{\mu\beta}}_{\sum_{\lambda\nu}} (g_{\lambda\mu, \nu} + g_{\nu\mu, \lambda} - g_{\mu\nu, \lambda}) \dot{x}^\mu \dot{x}^\nu = 0$$

Improved formulation:

(no  $\sqrt{\quad}$   $\rightarrow$  quantisation better)

$e_\mu^a$

$= 1$ )

$$S = \frac{m}{2} \int d\lambda \left( \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e} - e \right)$$

$\downarrow$   
massive particle  
 $0 \rightarrow$  photon  
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$e = \text{"einbein"} \sim \sqrt{\text{metric along world line}}$

$$\frac{\delta S}{\delta e} : \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} = -1$$

$$\text{ie } g_{\mu\nu} u^\mu u^\nu = -1$$

$$e'(\lambda) = \frac{d\lambda}{d\lambda'} e(\lambda)$$

$e' d\lambda' = e d\lambda = \text{reparametrisation invariant.}$

$$u^\mu = \frac{1}{e} \dot{x}^\mu$$



formulation:

(no  $\sqrt{\quad}$   $\rightarrow$  quantisation better  
massive particle

$$S' = \frac{m}{2} \int d\lambda \left( \frac{(-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu)}{e} - e \right)$$

$0 \Rightarrow$  photon  
 $+e \Rightarrow$  tachyon.  
 $\rightarrow$  general

$$\frac{\delta S'}{\delta e} = \frac{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{e^2} = -1 \quad \text{Ricci-like} \quad |e| \quad g_{\mu\nu} u^\mu u^\nu = -1$$

$$\Rightarrow e = \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad \Rightarrow S' = -m \int \sqrt{\quad}$$





$$a \rightarrow e^{-i\phi} a$$

$$a^\dagger \rightarrow e^{i\phi} a^\dagger$$

Ex: vary  $S(e, x)$  w.r.t  $x$  and recover the geodesic equation, with  $dz = e dx$ .

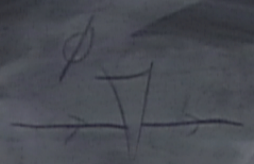




# From Newton to Einstein

Slowly moving particles:  $\left| \frac{dx}{d\tau} \right| \ll 1$ ,  $\frac{dct}{d\tau} \sim 1$

geodesic equation



$$a \rightarrow e^{-\phi} a$$

$$a' \rightarrow e^{-\phi} a'$$



## From Newton to Einstein

Slowly moving particles:

$$\left| \frac{dx}{d\tau} \right| \ll 1$$

$$\frac{dct}{d\tau} \sim 1$$

$$d\tau = \sqrt{cdt^2 - dx^2} \quad \text{at a point}$$

geodesic equation

$$\frac{d^2 x^M}{d\tau^2} + \Gamma^M$$



# From Newton to Einstein

slowly moving particles:

$$\left| \frac{dx}{d\tau} \right| \ll 1$$

$$\frac{dct}{d\tau} \sim 1$$

$$d\tau = \sqrt{cdt^2 - dx^2} \quad \text{at a point}$$

geodesic equation

$$\frac{d^2 x^M}{d\tau^2} + \Gamma_{00}^M \left( \frac{d(ct)}{d\tau} \right)^2 = 0$$

i.e. drop  $\frac{dx}{d\tau}$  terms



# From Newton to Einstein

Slowly moving particles:

$$\left| \frac{dx}{d\tau} \right| \ll 1, \quad \frac{dct}{d\tau} \sim 1$$

$$d\tau = \sqrt{c^2 dt^2 - dx^2} \quad \text{at a point}$$

geodesic equation

$$\frac{d^2 x^M}{d\tau^2} + \Gamma_{00}^M \left( \frac{d(ct)}{d\tau} \right)^2 = 0$$

i.e. drop  $\frac{dx}{d\tau}$  terms

Assume metric is stationary,  $\frac{\partial}{\partial t} g_{\mu\nu} = 0$ , and  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Gamma_{00}^M = \frac{1}{2} g^{M\alpha} (g_{\alpha,0} + g_{0,\alpha} - g_{00,\alpha}) = -\frac{1}{2} g^{M\alpha} g_{00,\alpha}$$

small correction

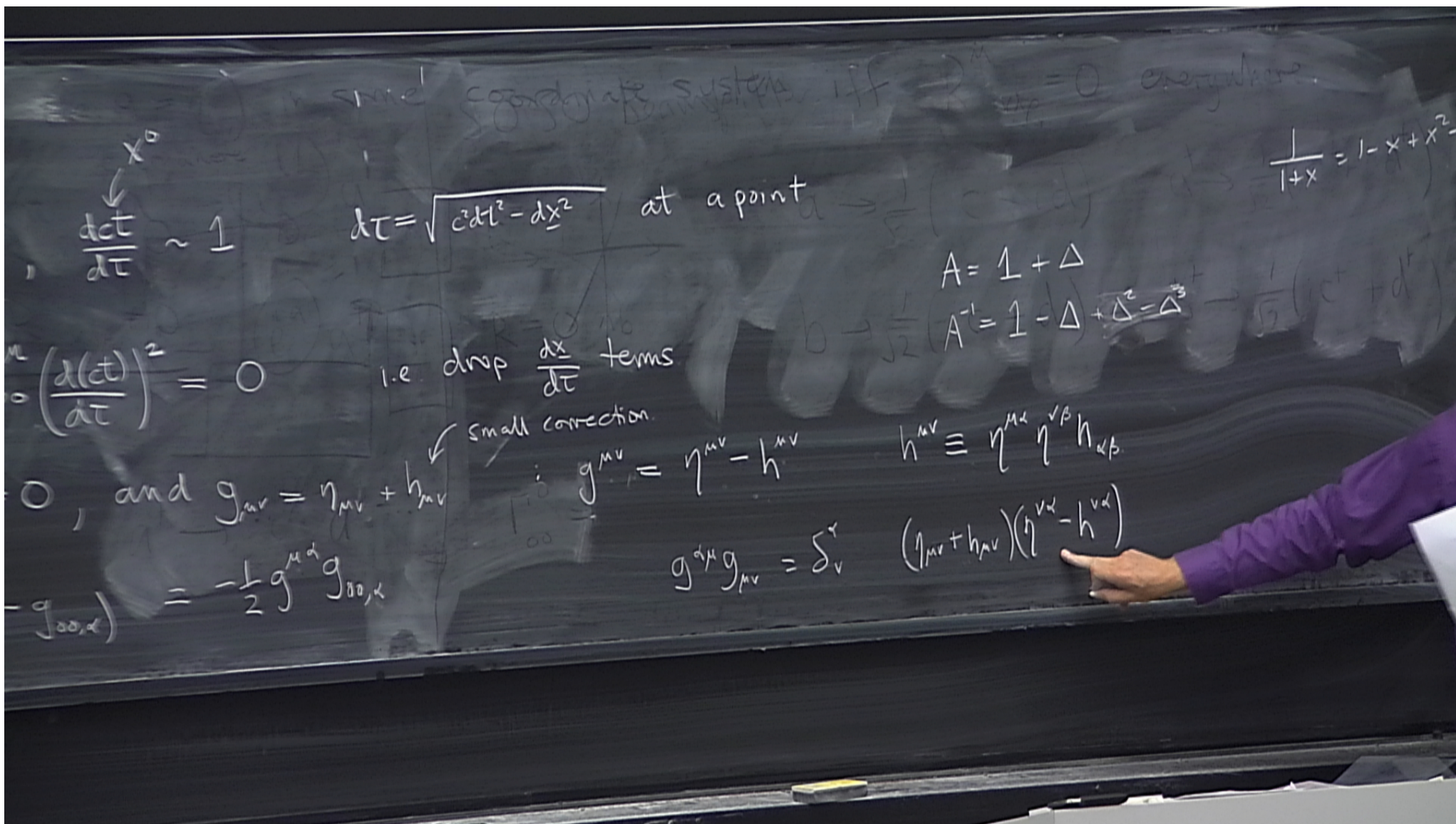


$$A = 1 + \Delta$$

$$\frac{1}{\sqrt{2}} (A^{-1} = 1 - \Delta + \Delta^2 - \Delta^3 + \dots) \rightarrow \frac{1}{\sqrt{2}} (c^+ + d^+)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$







since coordinate system iff  $\partial_{\mu} g_{\mu\nu} = 0$  everywhere.

$\frac{dct}{d\tau} \sim 1$

$d\tau = \sqrt{c^2 dt^2 - dx^2}$  at a point  $a \rightarrow \frac{1}{\sqrt{2}} (c^2 + d^2)$

$\frac{1}{1+x} = 1 - x + x^2 - \dots$

$A = 1 + \Delta$

$A^{-1} = 1 - \Delta + \Delta^2 - \Delta^3 + \dots \rightarrow \frac{1}{\sqrt{2}} (c^2 + d^2)$

$\left( \frac{d(ct)}{d\tau} \right)^2 = 0$  i.e. drop  $\frac{dx}{d\tau}$  terms

small correction.

$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$

$g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}_{\mu}$

$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - h^{\nu\alpha}) = \delta^{\alpha}_{\mu} + h_{\mu}^{\alpha} - h_{\mu}^{\alpha}$

cancel

$g_{00,\alpha} = -\frac{1}{2} g^{\mu\nu} g_{00,\alpha}$



$\frac{dx}{dt}$  terms  
 all correction

at a point  $a \rightarrow \frac{1}{\sqrt{2}}(c - id) \rightarrow \frac{1}{\sqrt{2}}(1+x) = 1 - x + x^2 - x^3 \dots$

$A = 1 + \Delta$   
 $A^{-1} = 1 - \Delta + \Delta^2 - \Delta^3 \dots \rightarrow \frac{1}{\sqrt{2}}(c^+ + d^+)$

$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$   
 $g^{\alpha\mu} g_{\mu\nu} = \delta^\alpha_\nu$

$h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$

$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - h^{\nu\alpha}) = \delta^\alpha_\mu + h_{\mu}{}^\alpha - h_{\mu}{}^\alpha - h_{\mu\nu} h^{\nu\alpha}$

cancel  
 cancel by cont.



$\frac{dx}{dt}$  terms  
 all correction

at a point  $a \rightarrow \frac{1}{\sqrt{2}}(c + cd)$

$A = 1 + \Delta$   
 $A^{-1} = 1 - \Delta + \Delta^2 - \Delta^3 \rightarrow \frac{1}{\sqrt{2}}(c^t + d^t)$

$\frac{1}{1+x} = 1 - x + x^2 - x^3$

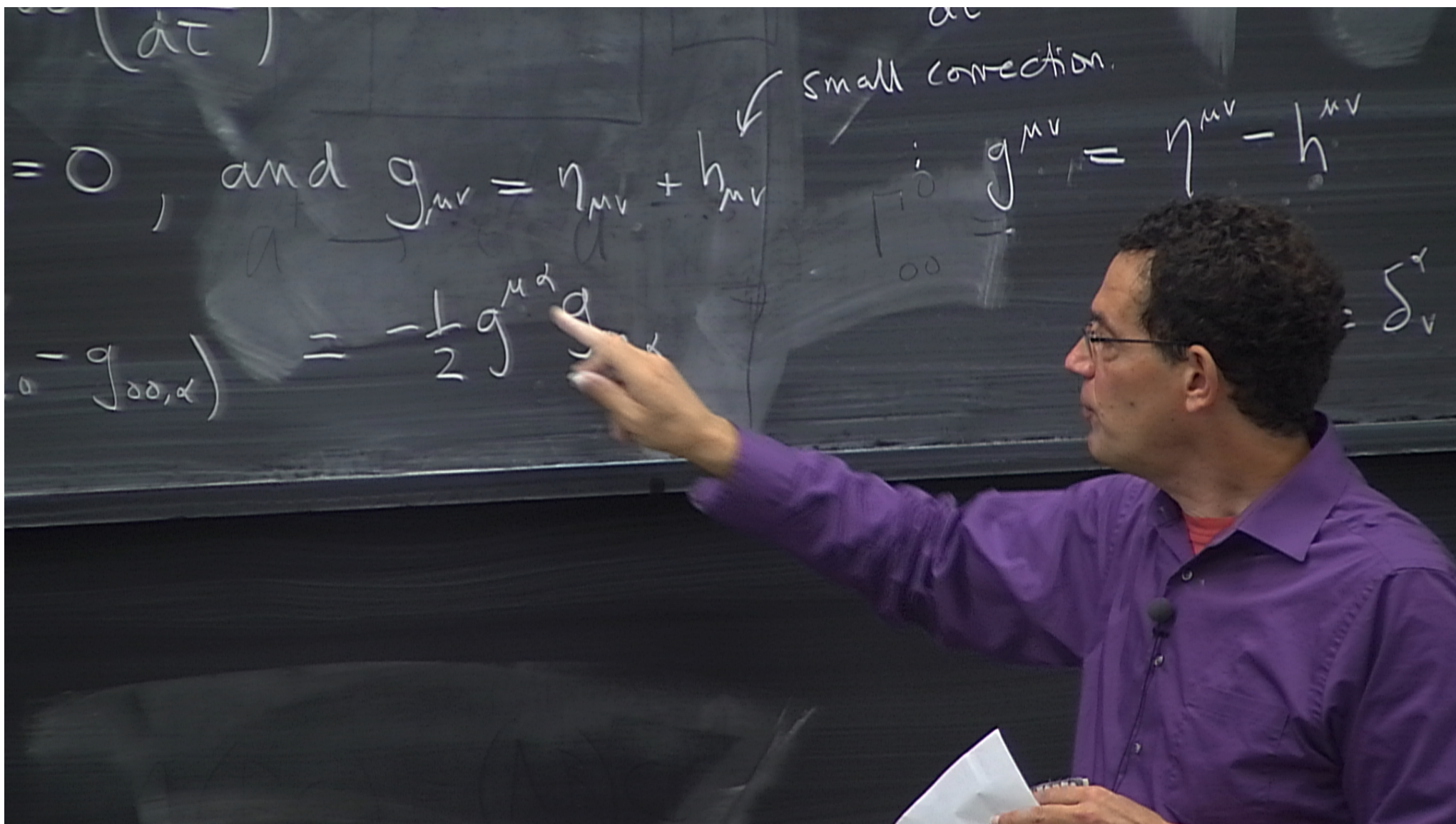
$g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu}$   
 $h^{\mu\nu} \equiv \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$

$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\nu$

$(\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\alpha} - h^{\nu\alpha}) = \delta^\alpha_\mu + h_{\mu}{}^\alpha - h_{\mu}{}^\alpha - h_{\mu\nu} h^{\nu\alpha}$   
 +chh

cancel  
 cancel by  $g_{\mu\nu}$







$(\partial \tau)$   
 $= 0$ , and  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  ↙ small correction.  
 $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$   
 $-g_{00,\alpha} = -\frac{1}{2} g^{\mu\alpha} g_{00,\alpha} \approx -\frac{1}{2} \eta^{\mu\alpha}$   
 $g^{\alpha\mu} g_{\mu\nu} = \delta^\alpha_\nu$



$(\partial_t)$   
 $= 0$ , and  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  ↙ small correction.  
 $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$   
 $-g_{00,\alpha} = -\frac{1}{2}g^{\mu\alpha}g_{00,\alpha} \approx -\frac{1}{2}\eta^{\mu\alpha}h_{00,\alpha}$  }  $g^{\alpha\mu}g_{\mu\nu} = \delta^\alpha_\nu$







Slowly moving particles:  $\left| \frac{dx}{d\tau} \right| \ll 1$ ,  $\frac{dct}{d\tau} \sim 1$   $d\tau = \sqrt{c^2 dt^2 - dx^2}$  at a point

geodesic equation  $\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \left( \frac{dx^\alpha}{d\tau} \right) \left( \frac{dx^\beta}{d\tau} \right) = 0$  <sup>①</sup> i.e. drop  $\frac{dx}{d\tau}$  terms

Assume metric is stationary,  $\frac{\partial}{\partial t} g_{\mu\nu} = 0$ , and  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$  <sup>small correction.</sup>  $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$$\Gamma_{00}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\alpha 0,0} + g_{0\alpha,0} - g_{00,\alpha}) = -\frac{1}{2} g^{\mu\alpha} g_{00,\alpha} \approx -\frac{1}{2} \eta^{\mu\alpha} h_{00,\alpha}$$

<sup>②</sup> to 1st order in  $h$ .  $g^{\mu\alpha} g_{\mu\nu} = \delta^\alpha_\nu$

①, ②  $\Rightarrow \frac{d^2 x}{d\tau^2} = \frac{1}{2} \left( \frac{dct}{d\tau} \right)^2 \nabla h_{00}$

$$\frac{1}{2} \left( \frac{N(N-1)}{2} \right) \left( \frac{N(N-1)}{2} + 1 \right) = \frac{1}{8} N(N-1)(N^2 - N + 2) = A$$

$N$  dim's.

fields

$\frac{N(N-1)}{2}$

values

$\frac{N(N-1)(N-2)(N-3)}{4!}$  constraints

$\Rightarrow \# \text{ independent components} = A - B = \frac{N^2(N-1)}{2}$



Slowly moving particles:  $\left| \frac{dx}{d\tau} \right| \ll 1$ ,  $\frac{dct}{d\tau} \sim 1$   $d\tau = \sqrt{c^2 dt^2 - dx^2}$  at a point

geodesic equation  $\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\alpha\beta}^\mu \left( \frac{dx^\alpha}{d\tau} \right) \left( \frac{dx^\beta}{d\tau} \right) = 0$   $\textcircled{1}$  i.e. drop  $\frac{dx}{d\tau}$  terms

Assume metric is stationary,  $\frac{\partial}{\partial t} g_{\mu\nu} = 0$ , and  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$   $\textcircled{2}$  small correction.  $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

$\Gamma_{00}^\mu = \frac{1}{2} g^{\mu\alpha} (g_{\alpha 0,0} + g_{0\alpha,0} - g_{00,\alpha}) = -\frac{1}{2} g^{\mu\alpha} g_{00,\alpha} \approx -\frac{1}{2} \eta^{\mu\alpha} h_{00,\alpha}$  to 1st order in  $h$ .  $g^{\mu\alpha} g_{\mu\nu} =$

$\textcircled{1}, \textcircled{2} \Rightarrow \frac{d^2 x}{d\tau^2} = \frac{1}{2} \left( \frac{d(ct)}{d\tau} \right)^2 \nabla h_{00}$

$\Rightarrow \frac{d(ct)}{d\tau^2} = 0 \Rightarrow ct = A + B\tau$



$n=i$       ①, ②       $\Rightarrow$        $\frac{d^2 x}{d\tau^2} = \frac{1}{2} \left( \frac{d(ct)}{d\tau} \right)^2 \frac{\nabla h_{00}}{\left( \frac{N(N-1)}{2} + 1 \right)} = \frac{1}{2} N(N-1) \dots$

$\Rightarrow$        $\frac{d(ct)}{d\tau} = 0 \Rightarrow ct = A + B\tau$   
 $\Rightarrow d(ct) = B d\tau$

$\Rightarrow \# \text{ indpt components} = A - B = \frac{N^2}{12}$

$N$  dim's.  
 takes  $\frac{N(N-1)}{2}$  values (antisymm)







$$\textcircled{1}, \textcircled{2} \Rightarrow \frac{d^2 x}{d\tau^2} = \frac{1}{2} \left( \frac{d(ct)}{d\tau} \right)^2 \nabla h_{00} \quad \textcircled{3}$$

$$\frac{d^2(ct)}{d\tau^2} = 0 \Rightarrow ct = A + B\tau$$

$$\Rightarrow d(ct) = B d\tau$$

$\textcircled{3} \Rightarrow$

$$\frac{d^2 x}{dt^2} = \frac{1}{2} c^2 \nabla h_{00}$$

c.f Newton

$$m \frac{d^2 x}{dt^2} = m(-\nabla \phi) \Rightarrow$$

$$h_{00} = -\frac{2\phi}{c^2}$$

values  
(antisym)

$\Rightarrow$  # indep components =



$$= 0 \quad \forall \quad \sigma_{\mu\nu}(\lambda)$$

$$\left\{ \begin{matrix} \mu \\ \lambda \nu \end{matrix} \right\}$$

Surface of earth  $\phi = -\frac{GM}{r} \quad : \quad \frac{\phi}{c^2} \sim 10^{-9}$



Surface of earth  $\phi = -\frac{GM_E}{r_E} : \frac{\phi}{c^2} \sim 10^{-9}$

Sun  $\phi = -\frac{GM_S}{r_S} \frac{\phi}{c^2} \sim 10^{-6}$

Neutron star  $\sim 10^{-1} !$



$$\begin{aligned}
 &= -mc \int d\lambda \frac{1}{2\sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}} \left( -g_{\mu\nu, \alpha} \delta X^\alpha \dot{X}^\mu \dot{X}^\nu - 2g_{\mu\nu} \delta \dot{X}^\mu \dot{X}^\nu \right) \quad \dot{g}_{\mu\nu} = g_{\mu\nu, \alpha} \dot{X}^\alpha \\
 &\quad \text{integrate by parts} \\
 &\quad \left( d\lambda \frac{1}{2} \left( -g_{\mu\nu} \delta X^\alpha \dot{X}^\mu \dot{X}^\nu - 2g_{\mu\nu} \dot{X}^\nu \delta \dot{X}^\mu \right) \right) \\
 &\quad \int d\lambda \left( -g_{\mu\nu, \alpha} \dot{X}^\alpha \dot{X}^\mu \dot{X}^\nu + 2(g_{\mu\nu} \dot{X}^\nu) \right) \delta \dot{X}^\mu \quad \text{integrate by parts} \\
 &= 0 \quad \forall \delta \dot{X}^\mu(\lambda) \\
 &\Rightarrow \boxed{(g_{\mu\nu} \dot{X}^\nu)^\cdot = \frac{1}{2} g_{\alpha\beta, \mu} \dot{X}^\alpha \dot{X}^\beta} \\
 &\Rightarrow \ddot{X}^\mu + \frac{1}{2} g^{\mu\beta} (g_{\lambda\beta, \nu} + g_{\nu\beta, \lambda} - g_{\lambda\nu, \beta}) \dot{X}^\lambda \dot{X}^\nu = 0 \quad \text{S.M. 2}
 \end{aligned}$$

What defines  $g_{\mu\nu}$ ? (and thus  $\phi$ ).

Newton:  $\nabla^2 \phi = 4\pi G \rho_m$  (Poisson)

$\Rightarrow \phi = -G \int \frac{d^3 \mathbf{r}' \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$



$\nabla_{\mu\nu}$ ? (and thus  $\phi$ ).

$$\nabla^2 \phi = 4\pi G \rho_m \quad (\text{Poisson})$$

$$\Rightarrow \phi = -G \int \frac{d^3 \underline{r}' \rho(\underline{r}')}{|\underline{r} - \underline{r}'|}$$

$$\nabla^2 \left( \frac{1}{4\pi |\underline{r} - \underline{r}'|} \right) = -\delta^3(\underline{r} - \underline{r}')$$



$T_{\mu\nu}$  ?

(and thus  $\phi$ ).

$$= \sum -\frac{GdM}{|\mathbf{r}-\mathbf{r}'|}$$

$$\nabla^2 \phi = 4\pi G \rho_m \quad (\text{Poisson})$$

$$\Rightarrow \phi(\mathbf{r}) = -G \int \frac{d^3 \mathbf{r}' \rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\nabla^2 \left( \frac{1}{4\pi |\mathbf{r}-\mathbf{r}'|} \right) = -\delta^3(\mathbf{r}-\mathbf{r}')$$



$\phi$

$$= \sum \frac{-G dM}{|\underline{r} - \underline{r}'|}$$

mass density

$$\Rightarrow \phi(\underline{r}) = -G \int \frac{d^3 \underline{r}' \rho_m(\underline{r}')}{|\underline{r} - \underline{r}'|}$$

$$\nabla^2 \left( \frac{1}{4\pi |\underline{r} - \underline{r}'|} \right) = -\delta^3(\underline{r} - \underline{r}')$$



$$\int d\lambda \left( -g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + 2(g_{\mu\nu} \dot{x}^\nu) \right) \delta x^\mu = 0 \quad \forall \delta x^\mu(\lambda)$$

$$\Rightarrow \ddot{x}^\mu + \underbrace{\frac{1}{2} g^{\mu\alpha} (g_{\alpha\beta,\nu} + g_{\nu\beta,\alpha} - g_{\mu\alpha,\beta})}_{\text{Christoffel symbol}} \dot{x}^\alpha \dot{x}^\beta = 0$$

What determines  $g_{\mu\nu}$ ? (and thus  $\phi$ ).

Newton:  $\nabla^2 \phi = 4\pi G \rho_m$  (Poisson)

$\Rightarrow$  expect  $\nabla^2 g_{00} = -\frac{8\pi G}{c^2} \rho_m$

$$= \sum \frac{-G dM}{|\mathbf{r} - \mathbf{r}'|} \quad \text{mass density}$$

$$\Rightarrow \phi(\mathbf{r}) = -G \int \frac{d^3 \mathbf{r}' \rho_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$\nabla^2 \left( \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \right) = -\delta^3(\mathbf{r} - \mathbf{r}')$$



$$= 0 \quad \forall \quad dx^\mu(\lambda)$$

W) + determines  $g_{\mu\nu}$ ? (and thus  $\phi$ ).

Newton:  $\nabla^2 \phi = 4\pi G \rho_m$  (Poisson)  $\Rightarrow \phi(r) = -$

$\Rightarrow$  expect  $\nabla^2 g_{00} \sim -\frac{8\pi G}{c^2} \rho_m \sim -\frac{8\pi G \rho}{c^2}$

---

$\rho = \text{energy dens}$

stress-energy tensor



What determines  $g_{\mu\nu}$ ? (and thus  $\phi$ ).

Newton:  $\nabla^2 \phi = 4\pi G \rho_m$  (Poisson)  $\Rightarrow \phi(r) = -$

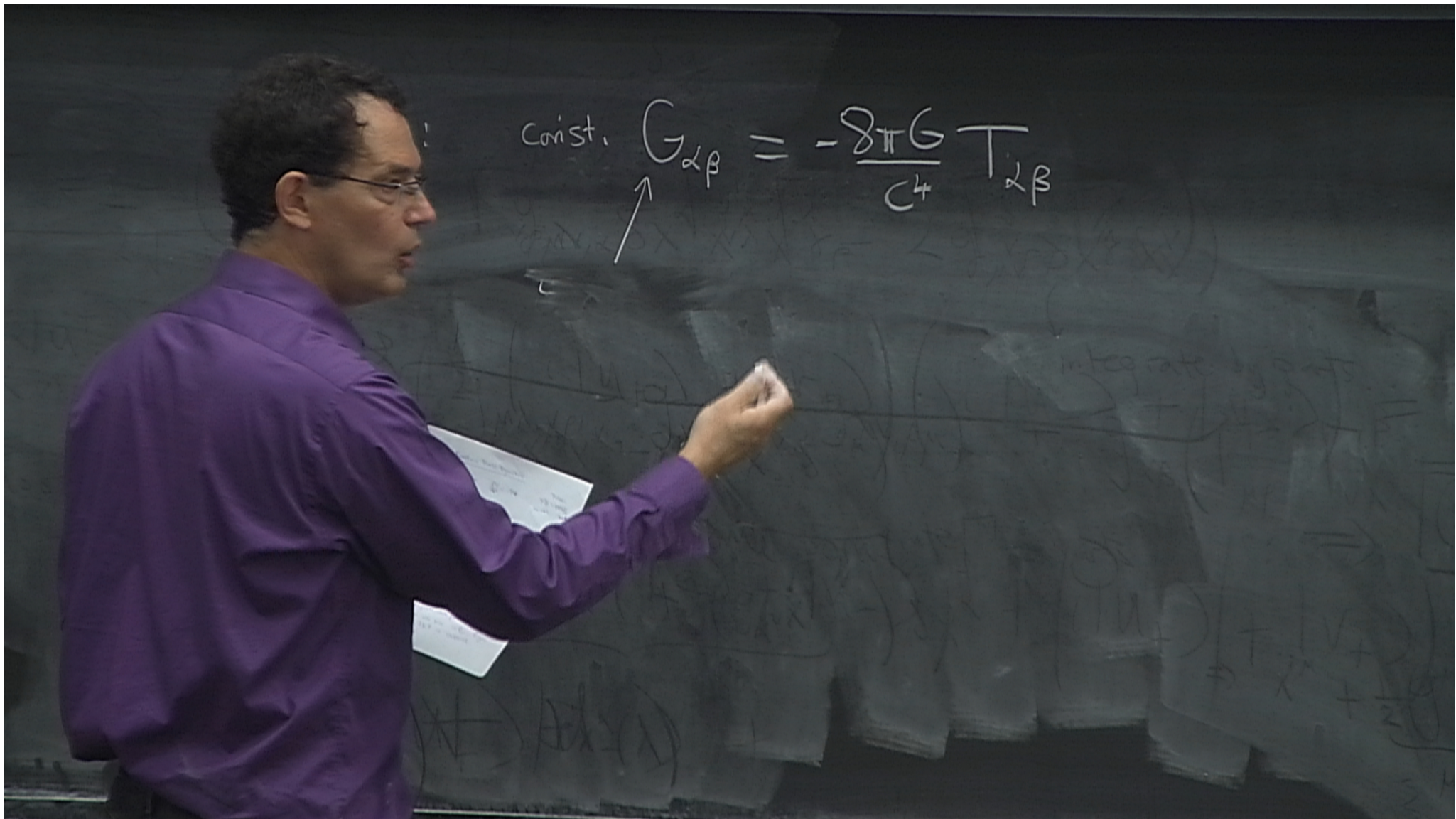
$\Rightarrow$  expect  $\nabla^2 g_{00} \sim -\frac{8\pi G}{c^2} \rho_m \sim -\frac{8\pi G \rho}{c^4}$ ,  $\nabla^2 \left( \frac{1}{4\pi} \right)$

but  $\rho = T_{00}$   
 $\Uparrow$   
stress-energy  
tensor

$$\Rightarrow \boxed{\nabla^2 g_{00} = -\frac{8\pi G}{c^4} T_{00}}$$

$\rho =$  energy dens







What determines  $g_{\mu\nu}$ ? (and thus  $\phi$ ).

Newton:  $\nabla^2 \phi = 4\pi G \rho_m$  <sup>(Poisson)</sup>  $\Rightarrow \phi(\mathbf{r}) = -G \int \frac{d^3\mathbf{r}' \rho_m(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$   $= \sum \frac{-G dM}{|\mathbf{r} - \mathbf{r}'|}$  <sub>mass density</sub>  $E = mc^2$

$\Rightarrow$  expect  $\nabla^2 g_{00} \sim -\frac{8\pi G}{c^2} \rho_m \sim -\frac{8\pi G \rho}{c^2}$ ,  $\nabla^2 \left( \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \right) = -\delta^3(\mathbf{r} - \mathbf{r}')$

but  $\rho = T_{00}$   
 $\Uparrow$   
 stress-energy tensor

$\Rightarrow \nabla^2 g_{00} = -\frac{8\pi G}{c^2} T_{00}$   
 $\downarrow$   $\downarrow$   
 Riemann  $T_{\alpha\beta}$

must turn this into a tensor equation.

Guess:  $\boxed{\text{const. } G_{\alpha\beta} = -\frac{8\pi G}{c^2} T_{\alpha\beta}}$

$R_{\alpha\beta}$ ,  $g_{\alpha\beta} R$  : for consistency, since  $\nabla_\alpha T^{\alpha\beta} = 0$ , must have  $\nabla_\alpha (G^{\alpha\beta}) = 0$

recall  $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$ ,  $\nabla_\alpha G^{\alpha\beta} = 0$ .



Calculate  $G_{\mu\nu}$  for  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ,  $h_{\mu\nu} \ll 1$   
 $h^2 \Rightarrow$  neglect.

$$R_{\rho\sigma\mu\nu} = \frac{1}{2} (h_{\rho\sigma,\mu\nu} - h_{\rho\mu,\sigma\nu} - h_{\rho\nu,\sigma\mu} + h_{\mu\nu,\sigma\rho}) + \text{terms} (N^2 - N + 2)$$

values  
(antisym)

$$\frac{d^2 x}{dt^2} = \frac{1}{2} c^2 \nabla h_{00}$$

$\Rightarrow$  # indep components =

$$A - B = \frac{N}{2} (N-1)$$



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$$R_{0000} = 0 ; R_{0i0j} = -\frac{1}{2} h_{00,ij}$$

Values  
(antisym)

$$\frac{d^2 x}{dt^2} = -\frac{1}{2} c^2 \nabla^2 h_{00}$$

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$$R_{\rho\sigma\mu\nu} = \frac{1}{2} (h_{\rho\sigma,\mu\nu} - h_{\rho\mu,\sigma\nu} - h_{\rho\nu,\sigma\mu} + h_{\rho\nu,\mu\sigma}) + \text{terms } (N-N+2)$$

$$R_{0000} = 0 ; R_{0i0j} = -\frac{1}{2} h_{00,ij}$$

$$\Rightarrow R_{00} = -\frac{1}{2} \nabla^2 h_{00} = -\frac{1}{2} \nabla^2 g_{00}$$

values  
(antisym)



$$G_{\alpha\beta} = K T_{\alpha\beta}$$

$$= R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

#

○

1

6

20

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$R_{1212}$

$\pi \ln 5$

$\Rightarrow$

$$\Lambda_{00} = -\frac{2\phi}{c^2}$$

1

2

3

4



$$G_{\alpha\beta} = K T_{\alpha\beta}$$

$$= R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$K T'_{\lambda} = R - 2R = -R$$

$$g^{\alpha\beta} g_{\alpha\beta} = \delta^{\alpha}_{\alpha} = 4$$



$$G_{\alpha\beta} = K T_{\alpha\beta}$$

$$= R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$K T^\lambda_\lambda = R - 2R = -R$$

$$R_{\alpha\beta} = K \left( T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^\lambda_\lambda \right)$$

$$g^{\alpha\beta} g_{\alpha\beta} = \delta^\alpha_\alpha = 4$$



$\Rightarrow$  Einstein  
equation.

$$G_{\alpha\beta} = +\frac{8\pi G}{c^4} T_{\alpha\beta}$$



