

Title: Relativity - Lecture 7

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Abstract:

$$\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\mu} = 0$$

$SO(3,1)$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$$

$$W_\mu = W_\mu^{ab} \sum_{ab}$$

$$D_\mu = \partial_\mu + W_\mu \quad [D_\mu, D_\nu] \propto R_{\mu\nu}^{ab} = R_{\mu\nu\rho\lambda} e^{a\rho} e^{b\lambda}$$

$$\frac{D^2 \delta x^\mu}{ds^2} = R^\mu_{\nu\lambda\rho} \delta x^\rho \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds}$$

$$\begin{aligned} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\alpha &= T_{\mu\nu}^\beta \nabla_\beta V_\alpha - R^\gamma_{\alpha\mu\nu} V_\gamma \\ ( \quad ) V^\alpha &= T_{\mu\nu}^\beta \nabla_\beta V^\alpha + R^\alpha_{\gamma\mu\nu} V^\gamma \end{aligned} \quad \left. \vphantom{\begin{aligned} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\alpha \\ ( \quad ) V^\alpha \end{aligned}} \right\} \text{general rule}$$



$$W_\mu^{ab} \sum_{ab}$$

$$= \partial_\mu + W_\mu \quad [D_\mu, D_\nu] \propto R_{\mu\nu}^{ab} = R_{\mu\nu\rho\lambda} e^{\rho a} e^{\lambda b}$$

Riemann curvature:

$$R^{\mu}_{\nu\lambda\rho} = - \left( \Gamma^{\mu}_{\nu\lambda, \rho} - \Gamma^{\mu}_{\nu\rho, \lambda} + \Gamma^{\sigma}_{\nu\lambda} \Gamma^{\mu}_{\sigma\rho} - \Gamma^{\sigma}_{\nu\rho} \Gamma^{\mu}_{\sigma\lambda} \right)$$

conventions:



$$\Gamma_{\nu\rho,\lambda}^{\mu} + \Gamma_{\nu\lambda}^{\sigma} \Gamma_{\sigma\rho}^{\mu} - \Gamma_{\nu\rho}^{\sigma} \Gamma_{\sigma\lambda}^{\mu})$$

conventions: Signature of  $g_{\mu\nu}$

$(-+++)$   
time space



$$R_{\mu\nu\rho\lambda} e^{a\rho} e^{b\lambda}$$

$$\Gamma_{\nu\lambda,\rho}^{\mu} - \Gamma_{\nu\rho,\lambda}^{\mu} + \Gamma_{\nu\lambda}^{\delta} \Gamma_{\delta\rho}^{\mu} - \Gamma_{\nu\rho}^{\delta} \Gamma_{\delta\lambda}^{\mu}$$

conventions: Signature of  $g_{\mu\nu}$   $(-+++)$   
 time space

Sign of  $R^{\mu}_{\nu\lambda\rho}$ : chosen so that  
 curvature of sphere is positive  
 (Rici scalar)



$$W_\mu = W_\mu^{ab} \sum_{ab}$$

$$D_\mu = \partial_\mu + W_\mu \quad [D_\mu, D_\nu] \propto R_{\mu\nu}^{ab} = R_{\mu\nu\rho\lambda} e^{a\rho} e^{b\lambda}$$

Riemann curvature:

$$\text{If } T_{\mu\nu}^\lambda = 0, \quad R^\mu_{\nu\lambda\rho} = - \left( \Gamma^\mu_{\nu\lambda, \rho} - \Gamma^\mu_{\nu\rho, \lambda} + \Gamma^\delta_{\nu\lambda} \Gamma^\mu_{\delta\rho} - \Gamma^\delta_{\nu\rho} \Gamma^\mu_{\delta\lambda} \right)$$

$$R_{\mu\nu\lambda\rho} = g_{\mu\alpha} R^\alpha_{\nu\lambda\rho} = \frac{1}{2} (g_{\mu\rho, \nu\lambda} - g_{\mu\lambda, \nu\rho} - g_{\nu\rho, \mu\lambda} + g_{\lambda\nu, \rho\mu}) + g_{\alpha\beta}$$



$$R_{\mu\nu\rho\lambda} e^{a\rho} e^{b\lambda}$$

$$\left( \Gamma_{\nu\lambda,\rho}^M - \Gamma_{\nu\rho,\lambda}^M + \Gamma_{\nu\lambda}^\gamma \Gamma_{\gamma\rho}^M - \Gamma_{\nu\rho}^\gamma \Gamma_{\gamma\lambda}^M \right)$$

$$2(g_{\mu\rho, \nu\lambda} - g_{\mu\lambda, \nu\rho} - g_{\nu\rho, \mu\lambda} + g_{\nu\lambda, \mu\rho}) + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha} \Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha} \Gamma_{\nu\rho}^{\beta})$$

conventions: Signature of  $g_{\mu\nu}$   $(-+++)$   
time space

Sign of  $R^M_{\alpha\beta}$  : chosen so that curvature of sphere is positive (Ricci scalar)



$$U_{\mu}^{ab} \sum_{ab}$$

$$D_{\mu} + W_{\mu} \quad [D_{\mu}, D_{\nu}] \propto R_{\mu\nu}^{ab} = R_{\mu\nu\rho\lambda} e^{a\rho} e^{b\lambda}$$

Riemann curvature:

conventions:

$$\chi_{\nu} = 0, \quad R^{\mu}_{\nu\lambda\rho} = - \left( \Gamma^{\mu}_{\nu\lambda, \rho} - \Gamma^{\mu}_{\nu\rho, \lambda} + \Gamma^{\sigma}_{\nu\lambda} \Gamma^{\mu}_{\sigma\rho} - \Gamma^{\sigma}_{\nu\rho} \Gamma^{\mu}_{\sigma\lambda} \right)$$

$$R_{\mu\nu\rho} = g_{\mu\alpha} R^{\alpha}_{\nu\lambda\rho} = \frac{1}{2} (g_{\mu\rho, \nu\lambda} - g_{\mu\lambda, \nu\rho} - g_{\nu\rho, \mu\lambda} + g_{\lambda\nu, \rho\mu}) + g_{\alpha\beta} (\Gamma^{\alpha}_{\mu\rho} \Gamma^{\beta}_{\nu\lambda} - \Gamma^{\alpha}_{\mu\lambda} \Gamma^{\beta}_{\nu\rho})$$

- can choose a coordinate system st. at some point,  $\Gamma = 0$ .



... system s.t. at some point,  $t = 0$ .

Symmetries: a) Symmetric  
b) Antisymmetric

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\nu\mu\rho\lambda}$$



the system is at some point,  $t = 0$ .

- Symmetries:
- a) Symmetric  $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$
  - b) Antisymmetric  $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\nu\mu\rho\lambda}$
  - c)  $R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$
- (can derive from Jacobi  $\nabla_\mu \nabla_\nu \nabla_\lambda \psi =$



from some point,  $\Gamma = 0$ ,

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$

symmetric

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\nu\mu\rho\lambda}$$

$$R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} = 0$$

(can derive from Jacobi

$$\nabla_{[\mu} \nabla_{\nu} \nabla_{\lambda]} \psi = -\frac{1}{2} R^{\gamma}_{[\mu\nu\lambda]} \nabla_{\gamma} \psi$$

$= 0$  identically

$$[\nabla_{\mu}, [\nabla_{\nu}, \nabla_{\lambda}]] + [\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]] + [\nabla_{\nu}, [\nabla_{\lambda}, \nabla_{\mu}]] = 0$$



at some point,  $\Gamma = 0$ ,

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$

symmetric

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\nu\mu\rho\lambda}$$

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$$\nabla_{[\mu} \nabla_{\nu} \nabla_{\lambda]} \psi = -\frac{1}{2} R^{\gamma}_{[\mu\nu\lambda]} \nabla_{\gamma} \psi$$

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$$[\nabla_{\mu}, [\nabla_{\nu}, \nabla_{\lambda}]] + [\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]] + [\nabla_{\nu}, [\nabla_{\lambda}, \nabla_{\mu}]] = 0$$



from some point,  $\Gamma = 0$ ,

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$$

symmetric

$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\nu\mu\rho\lambda}$$

$$R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$$

$$= R_{\mu[\rho\nu\lambda]}$$

(can derive from Jacobi

$$\nabla_{[\mu} \nabla_{\nu} \nabla_{\lambda]} \psi = -\frac{1}{2} R^{\gamma}_{[\mu\nu\lambda]} \nabla_{\gamma} \psi$$

$= 0$  identically

$$[\nabla_{\mu}, [\nabla_{\nu}, \nabla_{\lambda}]] + [\nabla_{\lambda}, [\nabla_{\mu}, \nabla_{\nu}]] + [\nabla_{\nu}, [\nabla_{\lambda}, \nabla_{\mu}]] = 0$$







Indep components of Riemann?

$R$  in  $N$  dim's.

$\begin{matrix} \text{Symm.} \\ \curvearrowright \\ \text{---} \end{matrix}$

$\downarrow$

takes  
 $\frac{N(N-1)}{2}$   
values  
(antisymm)

$$: \frac{1}{2} \left( \frac{N(N-1)}{2} \right) \cdot \left( \frac{N(N-1)}{2} + 1 \right) = \frac{1}{8} N(N-1)(N^2 - N + 2)$$



Symmetries:

- a) Symmetric  $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$
- b) Antisymmetric  $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} = +R_{\rho\lambda\mu\nu}$
- c) Cyclicity  $R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$  (can derive)

$$\partial_\alpha F_{\beta\gamma} + \partial_\sigma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha}$$

$$= R_{\mu[\rho\nu\lambda]}$$

(in fact, antisymmetric in all four indices)



$R$    
 takes  
 $\frac{N(N-1)}{2}$   
 values  
 (antisymm)

$$\frac{1}{2} \left( \frac{N(N-1)}{2} \right) \cdot \left( \frac{N(N-1)}{2} + 1 \right) = \frac{1}{8} N(N-1)(N^2 - N + 2) = A$$

cyclicality imposes  $\frac{N(N-1)(N-2)(N-3)}{4!} = B$  constraint.

$$\Rightarrow \# \text{ indep components} = A - B = \frac{N^2(N-1)}{12}$$

$N$   
 $1$   
 $2$



$$\left(\frac{N(N-1)}{2} + 1\right) = \frac{1}{8}N(N-1)(N^2-N+2) = A$$

my imposes  $\frac{N(N-1)(N-2)(N-3)}{4!} = B$  constraint.

independent components =  $A - B = \frac{N^2}{12}(N^2-1)$

| N | #  |            |
|---|----|------------|
| 1 | 0  |            |
| 2 | 1  | $R_{1212}$ |
| 3 | 6  |            |
| 4 | 20 | ///        |



- can choose a coordinate system st. at some point,  $\Gamma = 0$ .

- Symmetries:
- a) Symmetric  $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$
  - b) Antisymmetric  $R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} =$
  - c) Cyclicity  $R_{\mu\nu\lambda\rho} + R_{\mu\rho\nu\lambda} + R_{\mu\lambda\rho\nu} = 0$  (can
- $$= R_{\mu[\rho\nu\lambda]}$$
- (In fact, antisymmetric in all four indices)

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$$







$$\Rightarrow \# \text{ indep components} = A - B = \frac{N}{12}(N-1)$$

is flat (ie.  $g_{\mu\nu} = \eta_{\mu\nu}$  everywhere) in some coordinate system iff  $R^M_{\nu\lambda\rho} = 0$  everywhere.

$$\text{flat} \Leftrightarrow R^M_{\nu\lambda\rho} = 0$$

$$\text{curved} \Leftrightarrow R^M_{\nu\lambda\rho} \neq 0$$

  $R=0$

  $R \neq 0$



Ricci Tensor

$$R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta} = R_{\beta\alpha}$$



Ricci Tensor

$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} = R_{\beta\alpha}$$

10 components in 4 dim's

Ricci Scalar (Curvature Scalar)  $R = g^{\alpha\beta} R_{\alpha\beta}$

Weyl Tensor

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2} g_{\alpha\mu} R_{\beta\nu} - \frac{1}{2} g_{\beta\nu} R_{\alpha\mu} + \frac{1}{2} g_{\alpha\nu} R_{\beta\mu} + \frac{1}{2} g_{\beta\mu} R_{\alpha\nu}$$



$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} = R_{\beta\alpha} \quad 10 \text{ components in 4 dim's}$$

(Curvature Scalar)  $R = g^{\alpha\beta} R_{\alpha\beta}$

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2} g_{\alpha\mu} R_{\nu\beta} + \frac{1}{2} g_{\alpha\nu} R_{\mu\beta} + \frac{1}{2} g_{\beta\mu} R_{\nu\alpha} - \frac{1}{2} g_{\beta\nu} R_{\mu\alpha} - \frac{1}{6} g_{\alpha\nu} g_{\mu\beta} R + \frac{1}{6} g_{\alpha\mu} g_{\nu\beta} R$$



$$R_{\alpha\beta} = R_{\beta\alpha} \quad 10 \text{ components in 4 dim's}$$

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

$$g_{\mu\alpha} R_{\nu\beta} + \frac{1}{2} g_{\alpha\nu} R_{\mu\beta} + \frac{1}{2} g_{\beta\mu} R_{\nu\alpha} - \frac{1}{2} g_{\beta\nu} R_{\mu\alpha} - \frac{1}{6} g_{\alpha\nu} g_{\mu\beta} R + \frac{1}{6} g_{\alpha\mu} g_{\nu\beta} R$$

i.e.  $C_{\alpha\beta\mu\nu}$  is traceless in any pair of indices. : 10 indept components in 4 dim's



Theorem:  $g_{\mu\nu}(x)$  is flat (i.e.  $g = \eta$ ) everywhere in some coordinate system

Ricci Tensor

$$R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} = R_{\beta\alpha}$$

10 components in 4 dim<sup>n</sup>s

Ricci Scalar

(Curvature Scalar)

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

Weyl Tensor

$$C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2} g_{\alpha\mu} R_{\beta\nu} + \frac{1}{2} g_{\alpha\nu} R_{\beta\mu} + \frac{1}{2} g_{\beta\mu} R_{\alpha\nu} - \frac{1}{2} g_{\beta\nu} R_{\alpha\mu} - \frac{1}{6} g_{\alpha\nu} g_{\beta\mu} R$$

$$R_{\alpha\beta\mu\nu} = 0$$

$$\text{So } g^{\alpha\mu} C_{\alpha\beta\mu\nu} = 0$$

i.e.  $C_{\alpha\beta\mu\nu}$  is traceless in any pair of indices



Theorem:  $g_{\mu\nu}(x)$  is flat (i.e.  $g = \eta$ ) everywhere in some coordinate system

Ricci Tensor  $R_{\alpha\beta} = R^{\lambda}_{\alpha\lambda\beta} = R_{\beta\alpha}$  10 components in 4 dim<sup>n</sup>s

Ricci Scalar (Curvature Scalar)  $R = g^{\alpha\beta} R_{\alpha\beta}$

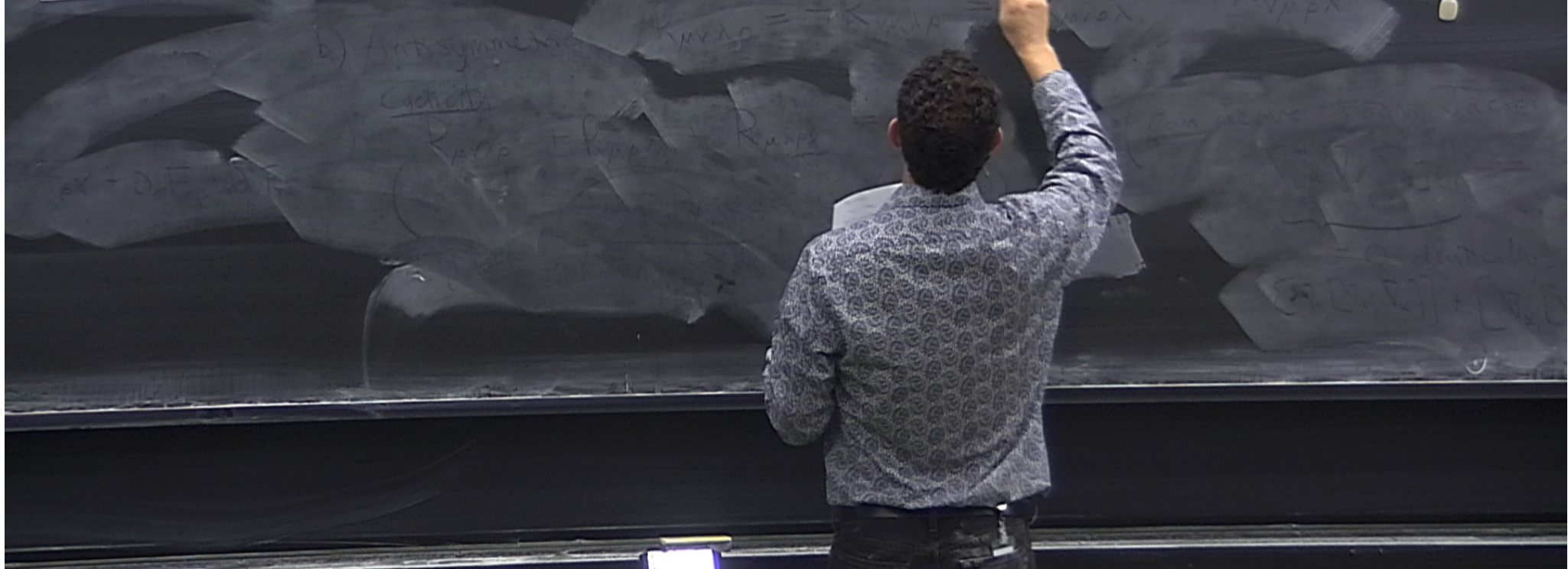
Weyl Tensor  $C_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{2} g_{\mu\alpha} R_{\nu\beta} + \frac{1}{2} g_{\alpha\nu} R_{\mu\beta} + \frac{1}{2} g_{\beta\mu} R_{\nu\alpha} - \frac{1}{2} g_{\beta\nu} R_{\mu\alpha} - \frac{1}{6} g_{\alpha\nu} g_{\mu\beta}$   
 So  $g^{\alpha\mu} C_{\alpha\beta\mu\nu} = 0$  i.e.  $C_{\alpha\beta\mu\nu}$  is traceless in any pair of indices  
 i.e.  $Riemann = Ricci + Weyl$



$$2(g_{\mu\rho,\nu\lambda} - g_{\mu\lambda,\nu\rho} - g_{\nu\rho,\mu\lambda} + g_{\nu\lambda,\mu\rho}) + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha}\Gamma_{\nu\rho}^{\beta})$$

- can choose a coordinate system st. at some point,  $\Gamma = 0$ .

Theorem 5:  $C_{\alpha\beta\mu\nu} = 0 \Rightarrow g_{\mu\nu} = \underbrace{\Omega^2(x)}_{\text{conformal factor}} \eta_{\mu\nu}$  i.e. metric is conformally flat





$\Gamma^{\lambda}_{\mu\rho} \Gamma^{\rho}_{\nu\lambda} - 2(\Gamma^{\mu\rho}_{\nu\lambda} - \Gamma^{\mu\lambda}_{\nu\rho} - \Gamma^{\rho\mu}_{\nu\lambda} + \Gamma^{\rho\lambda}_{\nu\mu}) + g_{\alpha\beta}(\Gamma^{\alpha}_{\mu\rho} \Gamma^{\beta}_{\nu\lambda} - \Gamma^{\alpha}_{\nu\lambda} \Gamma^{\beta}_{\mu\rho})$   
 - can choose a coordinate system st. at some point,  $\Gamma = 0$ .

Theorem:  $C_{\alpha\beta\mu\nu} = 0 \Rightarrow$  there exists a coordinate system in which  
 $g_{\mu\nu} = \underbrace{\Omega^2(x)}_{\text{metric}} \eta_{\mu\nu}$  i.e. metric

$$T_{\alpha\beta;\mu} = \partial_\mu T_{\alpha\beta} = \frac{\partial}{\partial x^\mu} T_{\alpha\beta}$$

$$T_{\alpha\beta;\mu} \equiv \nabla_\mu T_{\alpha\beta}$$

Bianchi identities

$$R_{\alpha\beta\mu\nu;\kappa} + R_{\alpha\beta\kappa\mu;\nu} + R_{\alpha\beta\nu\kappa;\mu} = 0$$

totally antisymmetric on  $\mu\nu\kappa$



$2(\gamma_{\mu\rho,\nu\lambda} - \gamma_{\mu\lambda,\nu\rho} - \gamma_{\nu\rho,\mu\lambda} + \gamma_{\nu\lambda,\mu\rho}) + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha} \Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha} \Gamma_{\nu\rho}^{\beta})$   
 use a coordinate system s.t. at some point,  $\Gamma = 0$ .

curvature of sphere is  $\neq 0$   
 (Ricci scalar)

$C_{\alpha\beta\mu\nu} = 0 \Rightarrow$  there exists a coordinate system in which  
 $g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu}$  i.e. metric is conformally flat

Identities

$$R_{\alpha\beta\mu\nu;\kappa} + R_{\alpha\beta\kappa\mu;\nu} + R_{\alpha\beta\nu\kappa;\mu} = 0$$

totally antisymmetric on  $\mu\nu\kappa$

Ex: Derive from Jacobi  $([\nabla_\mu, \nabla_\nu] \nabla_\alpha) + ([\nabla_\alpha, \nabla_\mu] \nabla_\nu) + ([\nabla_\alpha, \nabla_\nu] \nabla_\mu) = 0$

$\Rightarrow$  contract indices to obtain  
 $\nabla^\rho (R_{\rho\nu} - \frac{1}{2} g_{\rho\nu} R) = 0$



$$2(\gamma_{\mu\rho,\nu\lambda} - \gamma_{\mu\lambda,\nu\rho} - \gamma_{\nu\rho,\mu\lambda} + \gamma_{\nu\lambda,\mu\rho}) + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha} \Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha} \Gamma_{\nu\rho}^{\beta})$$

use a coordinate system s.t. at some point,  $\Gamma = 0$ .

curvature of sphere is  $\neq 0$   
(Ricci scalar)

$$C_{\alpha\beta\mu\nu} = 0 \Rightarrow g_{\mu\nu} = \underbrace{\Omega^2(x)}_{\text{there exists a coordinate system in which}} \eta_{\mu\nu} \quad \text{i.e. metric is conformally flat}$$

Identities

$$R_{\alpha\beta\mu\nu;\kappa} + R_{\alpha\beta\kappa\mu;\nu} + R_{\alpha\beta\kappa\nu;\mu} = 0$$

totally antisymmetric  $\mu\nu\kappa$

Ex: contract indices to obtain

$$\nabla^{\rho}(R_{\rho\nu} - \frac{1}{2}g_{\rho\nu}R) = 0$$

Ex: Derive from Jacobi  $([\nabla_{\mu}, \nabla_{\nu}]\nabla_{\lambda}) + ([\nabla_{\nu}, \nabla_{\lambda}]\nabla_{\mu}) + ([\nabla_{\lambda}, \nabla_{\mu}]\nabla_{\nu}) = 0$



$$J_{\mu\lambda,\nu\rho} + J_{\nu\rho,\mu\lambda} + J_{\lambda\nu,\rho\mu} + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha}\Gamma_{\nu\rho}^{\beta})$$

system st. at some point,  $\Gamma = 0$ .

curvature of sphere is positive  
(Ricci scalar)

there exists a coordinate system in which

$$\Rightarrow g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu} \quad \text{i.e. metric is conformally flat}$$

$$R_{\mu\nu\lambda\rho} = -R_{\lambda\rho\mu\nu} = -R_{\nu\rho\lambda\mu} = +R_{\rho\mu\lambda\nu}$$

$$+ R_{\alpha\beta\nu\kappa;\mu} = 0$$

tric on  $\mu\nu\kappa$

Ex: contract indices to obtain

$$\nabla^{\rho} \left( R_{\rho\nu} - \frac{1}{2} g_{\rho\nu} R \right) = 0$$

$G_{\rho\nu}$  Einstein tensor

$$([\nabla_{\mu}, \nabla_{\nu}]\nabla_{\lambda}) + ([\nabla_{\rho}, \nabla_{\mu}]\nabla_{\lambda}) + ([\nabla_{\nu}, \nabla_{\rho}]\nabla_{\lambda}) = 0.$$



$$J_{\mu\lambda,\nu\rho} + J_{\nu\rho,\mu\lambda} + J_{\lambda\nu,\rho\mu} + g_{\alpha\beta}(\Gamma_{\mu\rho}^{\alpha}\Gamma_{\nu\lambda}^{\beta} - \Gamma_{\mu\lambda}^{\alpha}\Gamma_{\nu\rho}^{\beta})$$

system st. at some point,  $\Gamma = 0$ .

curvature of sphere is positive  
(Ricci scalar)

there exists a coordinate system in which  
 $\Rightarrow g_{\mu\nu} = \Omega^2(x) \eta_{\mu\nu}$  i.e. metric is conformally flat

$$R_{\mu\nu} = \text{const } T_{\mu\nu}$$

Ex: contract indices to obtain

$$\nabla^{\rho} (R_{\rho\nu} - \frac{1}{2} g_{\rho\nu} R) = 0$$

$G_{\rho\nu}$  Einstein tensor

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\gamma\beta} + R_{\alpha\gamma\beta\delta} = 0$$

antisymmetric on  $\mu\nu$  &  $\kappa$

$$\text{Jacobi } ([\nabla_{\mu}, \nabla_{\nu}]\nabla_{\lambda}) + ([\nabla_{\nu}, \nabla_{\lambda}]\nabla_{\mu}) + ([\nabla_{\lambda}, \nabla_{\mu}]\nabla_{\nu}) = 0.$$



st. at some point,  $\Gamma = 0$ .

$\Rightarrow$  there exists a coordinate system in which  
 $g_{\mu\nu} = \underbrace{\Omega^2(x)}_{\text{2}} \eta_{\mu\nu}$  i.e. metric is conformally flat Poisson

$$\frac{d^2 x}{dt^2} = -\nabla\phi : \nabla^2\phi = 4\pi G\rho$$

$$\underbrace{\nabla^2\phi}_{\text{R}} = \underbrace{4\pi G\rho}_{\text{GT}}$$

$$D_{\mu\nu} = -\text{st } T_{\mu\nu}$$

Ex: contract ind

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$$

$$\underbrace{G_{\mu\nu}}_{\text{Einstein tensor}}$$

$$R_{\alpha\beta\gamma\delta;\mu} = 0$$

$$(\nabla_\mu \nabla_\nu \nabla_\rho + [\nabla_\rho \nabla_\mu] \nabla_\nu + [\nabla_\nu \nabla_\rho] \nabla_\mu) V_\lambda = 0.$$