Title: Relativity - Lecture 6

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Abstract:

$= a^\mu + \left(\frac{\partial x^\mu}{\partial x'^\nu}\right)_0 (x'-a') + \frac{1}{2}\frac{\partial^2 x^\mu}{\partial x'^\alpha \partial x'^\beta}(x'-a')(x'-a') + \cdots$

$\nabla_\alpha g_{\beta\gamma} = 0$ + cycles

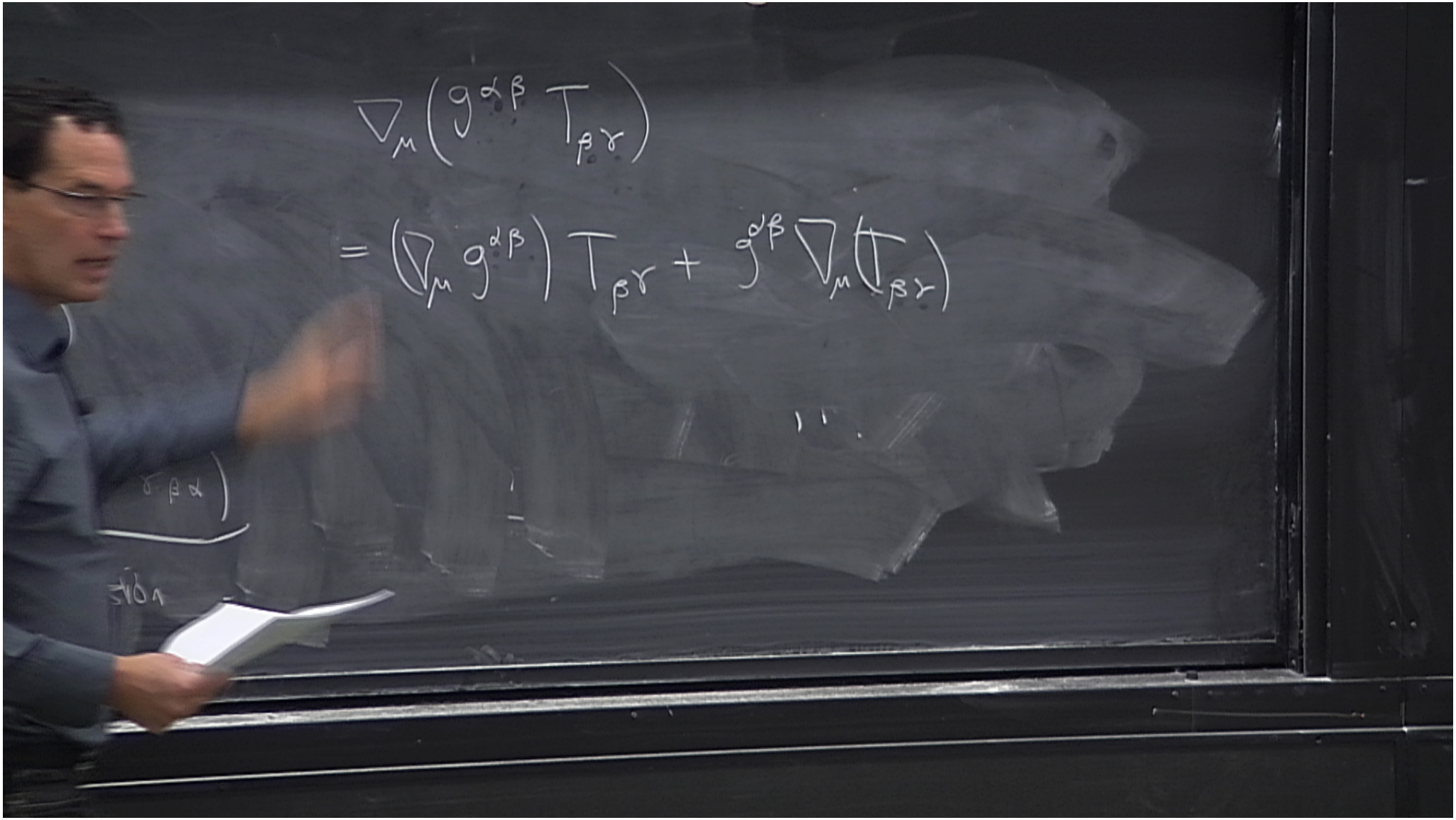
metricity

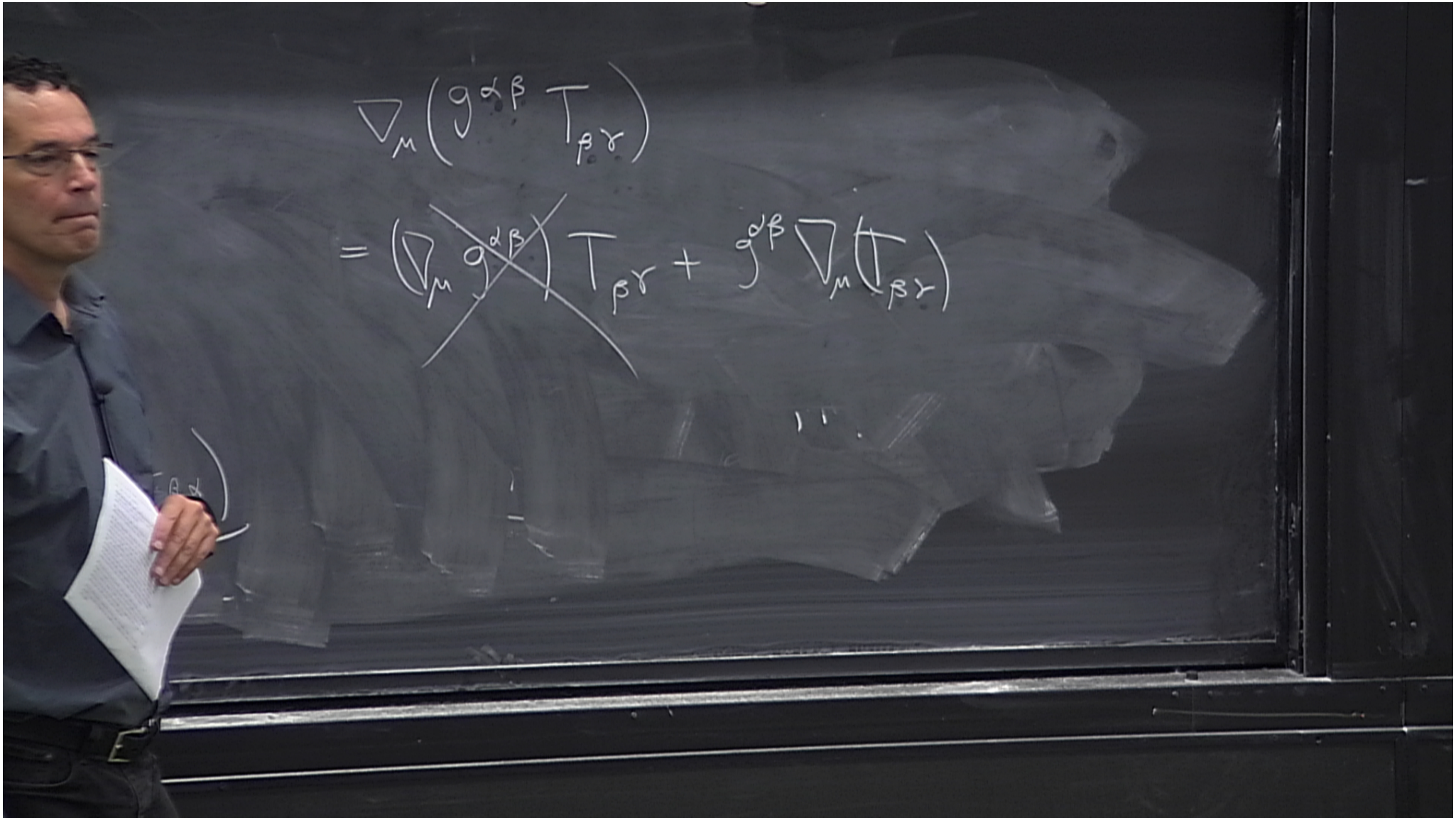
$$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$$

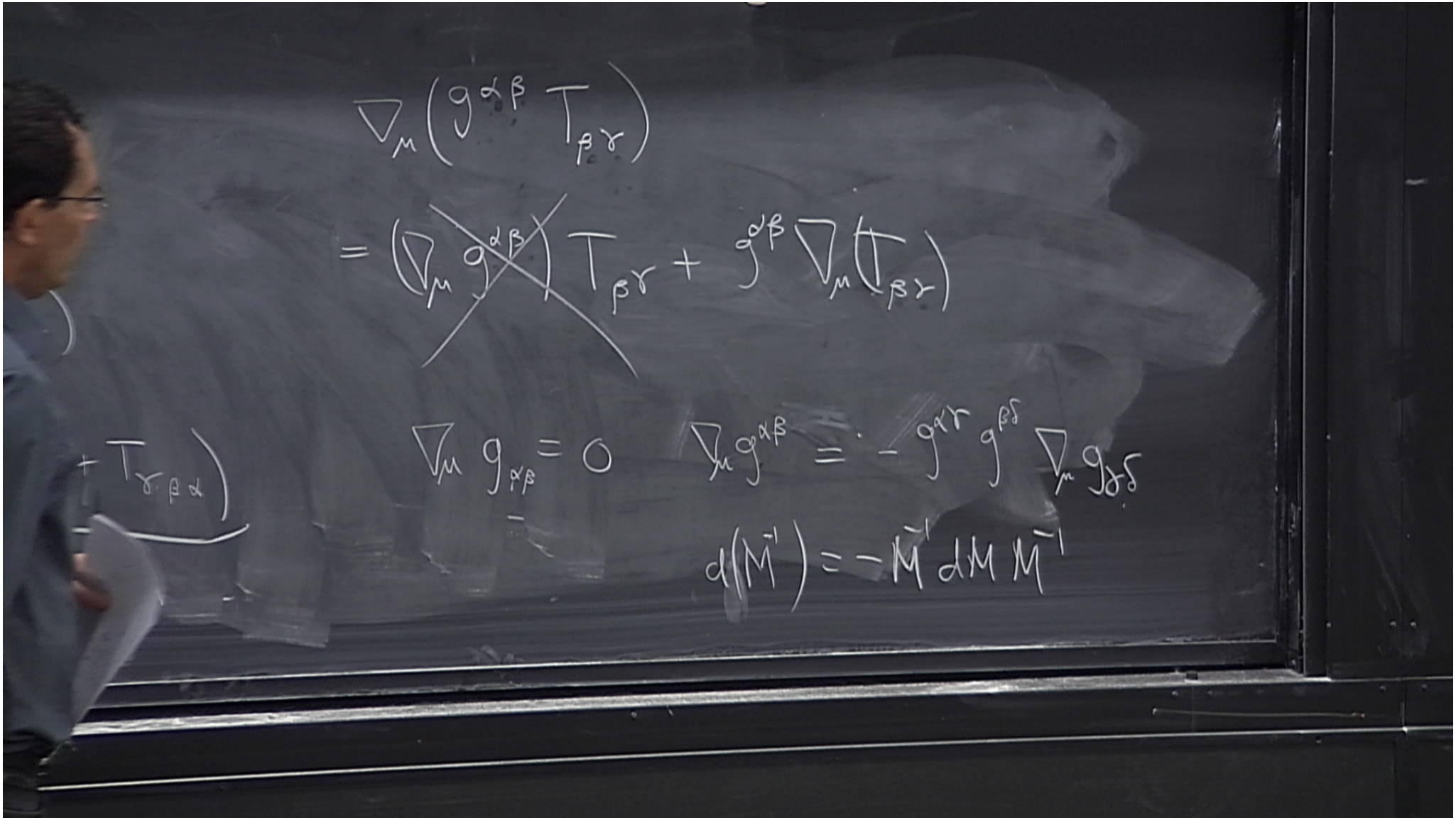
$$-\left(\Gamma_{\gamma\cdot\beta\alpha} + \Gamma_{\gamma\cdot\alpha\beta}\right) - \left(\Gamma_{\beta\cdot\gamma\alpha} - \Gamma_{\beta\cdot\alpha\gamma}\right) - \left(\Gamma_{\alpha\cdot\gamma\beta} - \Gamma_{\alpha\cdot\beta\gamma}\right) = 0$$

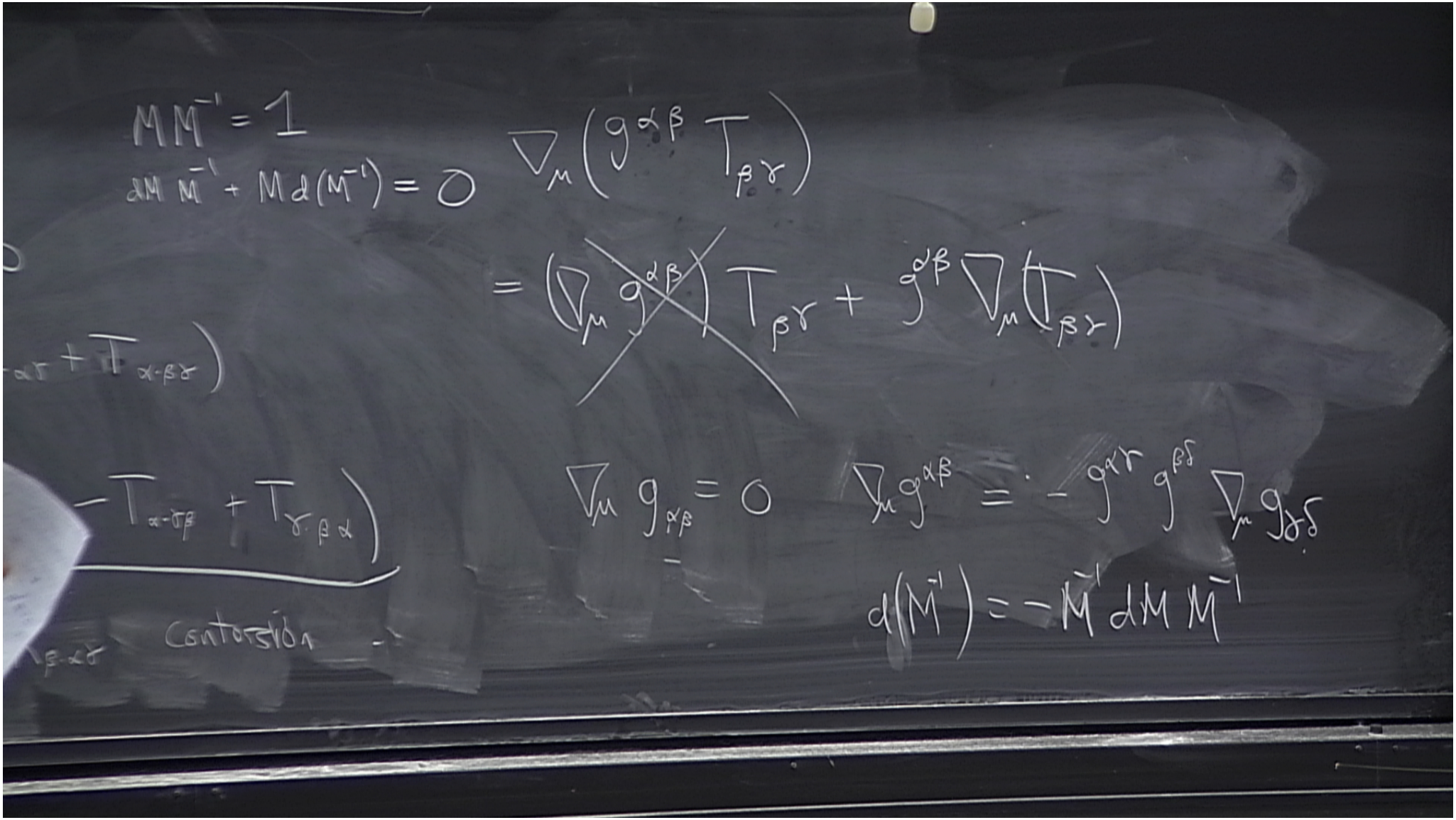
$$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2}\left(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}\right) + \frac{1}{2}\left(T_{\beta\cdot\alpha\gamma} + T_{\alpha\cdot\beta\gamma}\right)$$

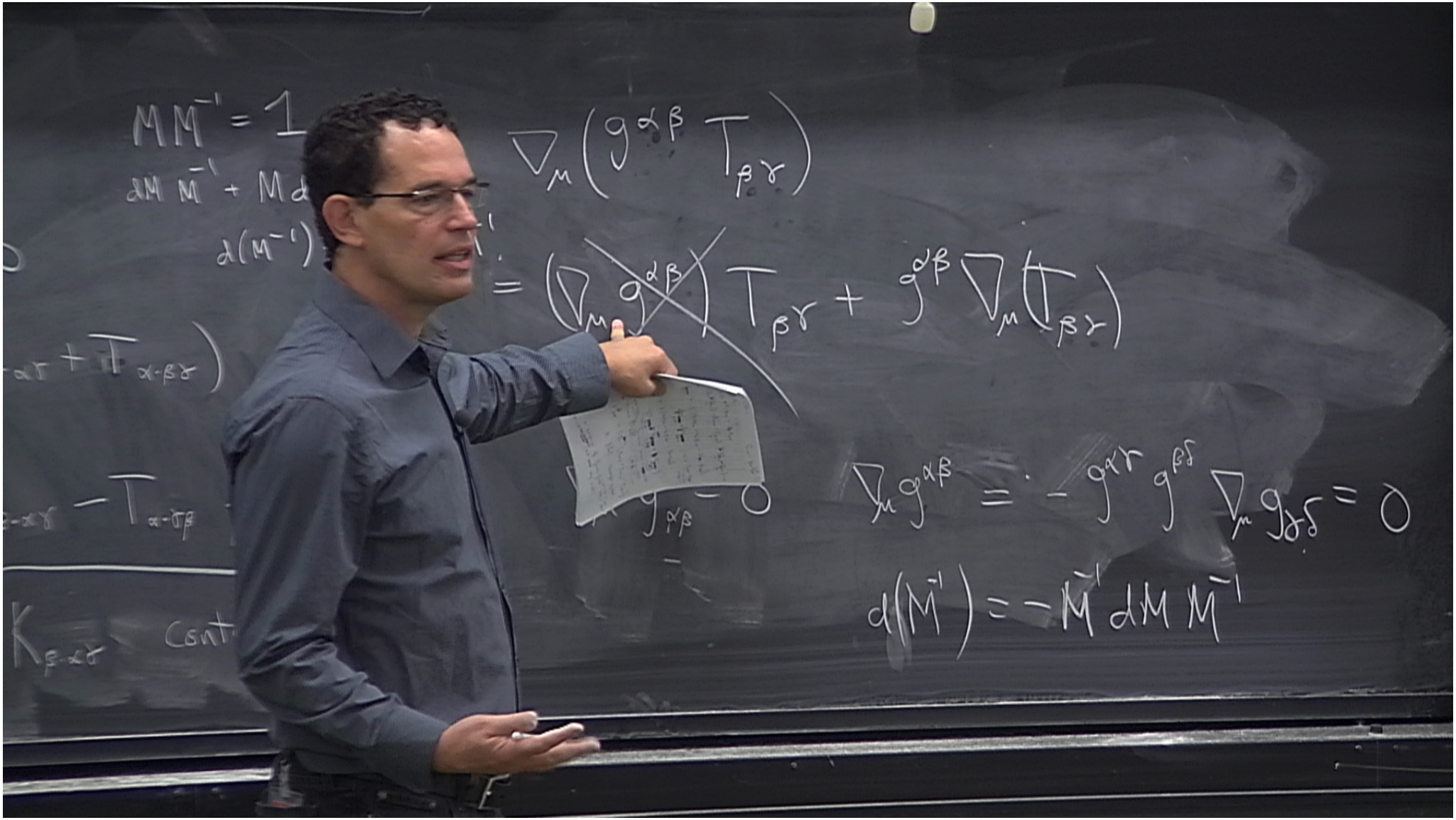
$$\Rightarrow \Gamma_{\gamma\cdot\beta\alpha} = \quad " \quad + \underbrace{\frac{1}{2}\left(T_{\beta\cdot\alpha\gamma} - T_{\alpha\cdot\gamma\beta} + T_{\gamma\beta\alpha}\right)}_{K_{\beta\cdot\alpha\gamma}\ \text{Contorsión}}$$

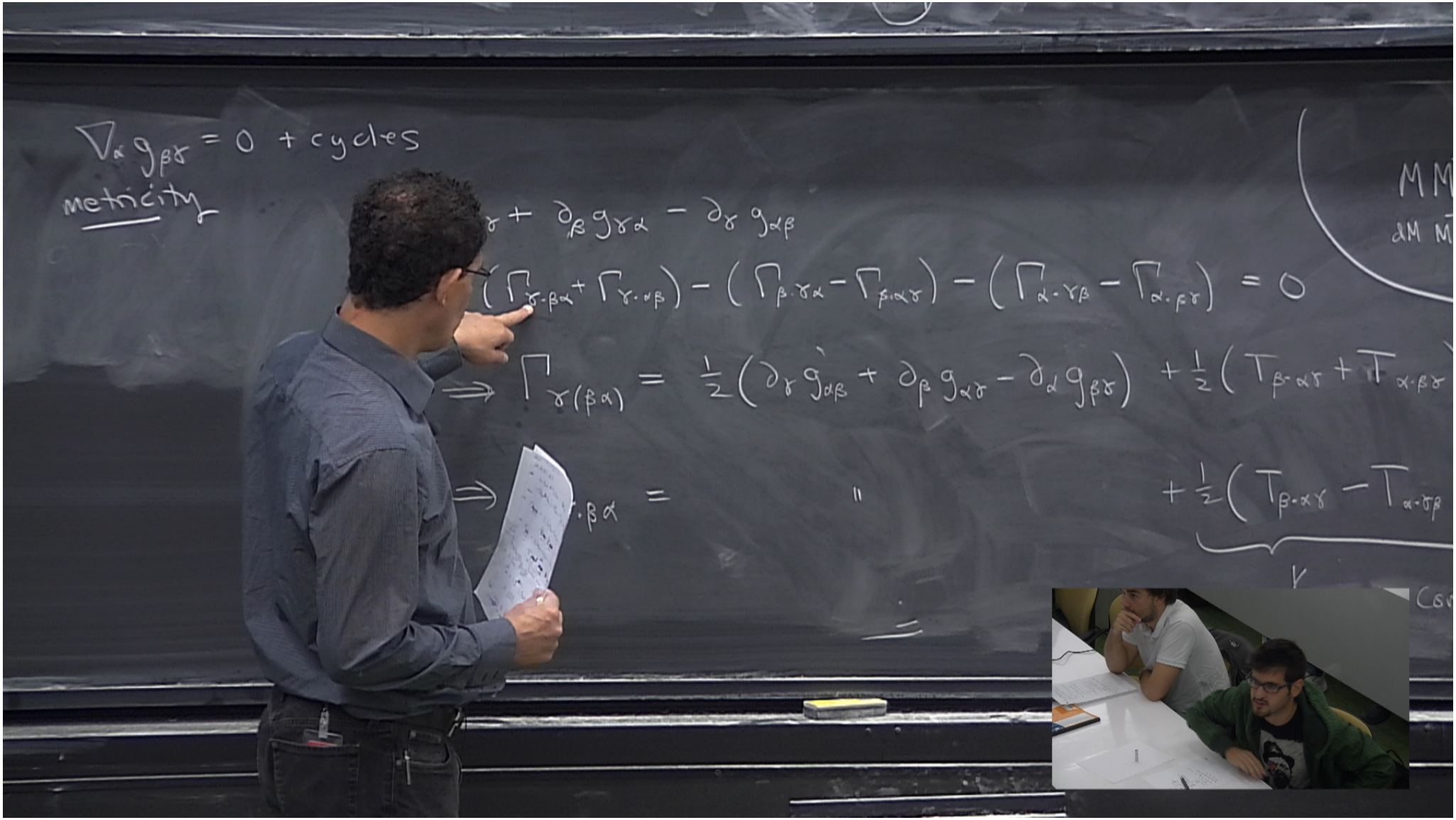

$\nabla_\mu \left(g^{\alpha\beta} T_{\beta\gamma} \right)$
$= \left(\nabla_\mu g^{\alpha\beta} \right) T_{\beta\gamma} + g^{\alpha\beta} \nabla_\mu \left(T_{\beta\gamma} \right)$

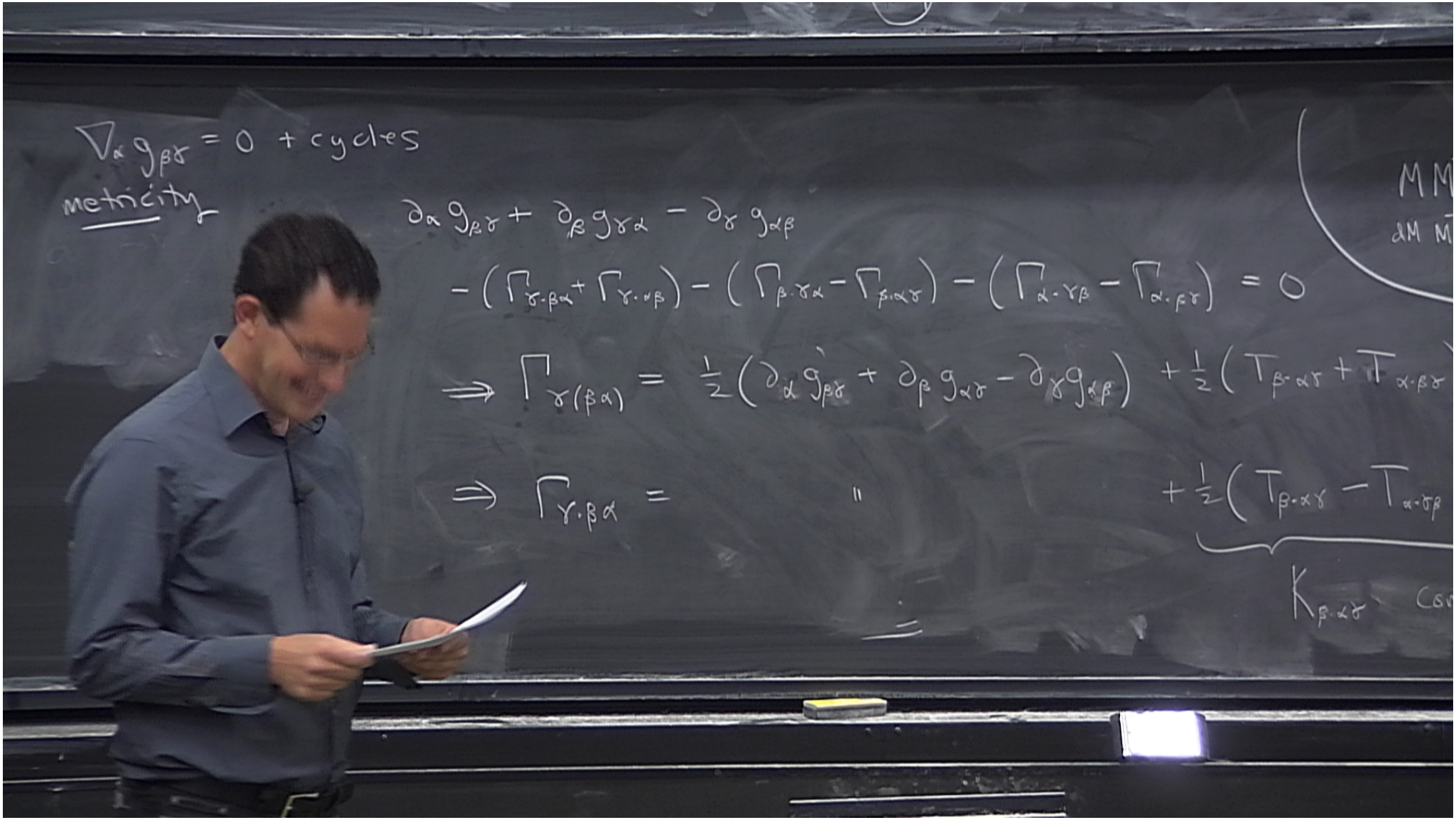

$\nabla_\mu (g^{\alpha\beta} T_{\beta\gamma})$
$= (\nabla_\mu g^{\alpha\beta}) T_{\beta\gamma} + g^{\alpha\beta} \nabla_\mu (T_{\beta\gamma})$

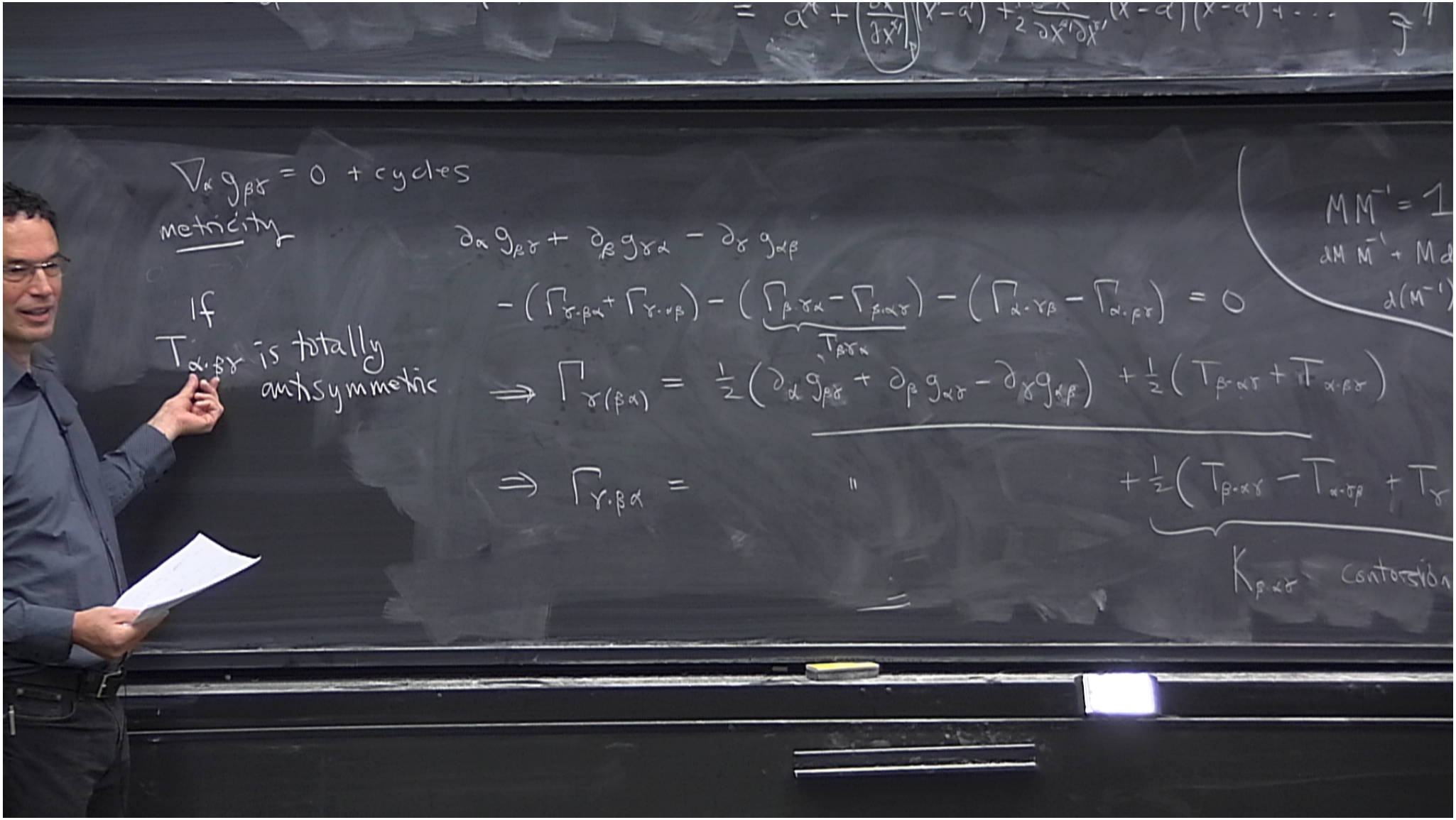

$\nabla_\mu (g^{\alpha\beta} T_{\beta\gamma})$
$= (\nabla_\mu g^{\alpha\beta}) T_{\beta\gamma} + g^{\alpha\beta} \nabla_\mu (T_{\beta\gamma})$
$+ T_{\gamma\beta\alpha})$
$\nabla_\mu g_{\alpha\beta} = 0$
$\nabla_\mu g^{\alpha\beta} = - g^{\alpha\gamma} g^{\beta\delta} \nabla_\mu g_{\gamma\delta}$
$d(M^{-1}) = - M^{-1} dM M^{-1}$


$MM^{-1} = 1$
$dM\,M^{-1} + M\,d(M^{-1}) = 0$
$\nabla_\mu (g^{\alpha\beta} T_{\beta\gamma})$
$= (\nabla_\mu g^{\alpha\beta}) T_{\beta\gamma} + g^{\alpha\beta} \nabla_\mu (T_{\beta\gamma})$
$\nabla_\mu g_{\alpha\beta} = 0$
$\nabla_\mu g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\delta} \nabla_\mu g_{\gamma\delta}$
$d(M^{-1}) = -M^{-1} dM\, M^{-1}$
Contorsión


$MM^{-1} = 1$
$\nabla_\mu (g^{\alpha\beta} T_{\beta\gamma})$
$= (\nabla_\mu g^{\alpha\beta}) T_{\beta\gamma} + g^{\alpha\beta} \nabla_\mu (T_{\beta\gamma})$
$\nabla_\mu g^{\alpha\beta} = -g^{\alpha\gamma} g^{\beta\delta} \nabla_\mu g_{\gamma\delta} = 0$
$d(M^{-1}) = -M^{-1} dM M^{-1}$


$\nabla_\alpha g_{\beta\gamma} = 0$ + cycles
metricity
$\Gamma_{\gamma(\beta\alpha)} = \frac{1}{2}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma}) + \frac{1}{2}(T_{\beta\cdot\alpha\gamma} + T_{\alpha\cdot\beta\gamma}$
$+ \frac{1}{2}(T_{\beta\cdot\alpha\gamma} - T_{\alpha\cdot\gamma\beta}$


$\nabla_\alpha g_{\beta\gamma} = 0$ + cycles
metricity
$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$
$-(\Gamma_{\gamma\cdot\beta\alpha} + \Gamma_{\gamma\cdot\alpha\beta}) - (\Gamma_{\beta\cdot\gamma\alpha} - \Gamma_{\beta\cdot\alpha\gamma}) - (\Gamma_{\alpha\cdot\gamma\beta} - \Gamma_{\alpha\cdot\beta\gamma}) = 0$
$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}) + \frac{1}{2}(T_{\beta\cdot\alpha\gamma} + T_{\alpha\cdot\beta\gamma}$
$\Rightarrow \Gamma_{\gamma\cdot\beta\alpha} =$ $+ \frac{1}{2}(T_{\beta\cdot\alpha\gamma} - T_{\alpha\cdot\gamma\beta}$
$K_{\beta\cdot\alpha\gamma}$

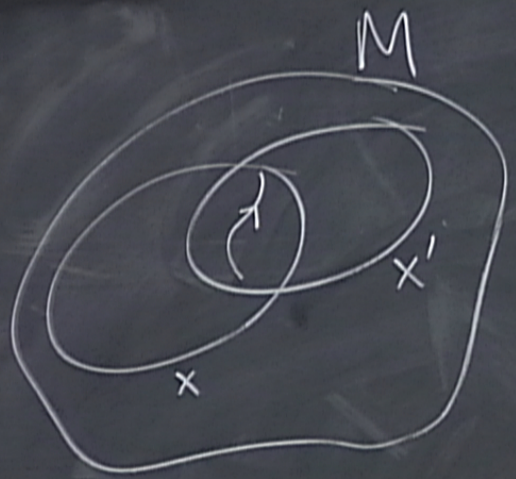

$\nabla_\alpha g_{\beta\gamma} = 0$ + cycles
metricity
$\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}$
$-(\Gamma_{\gamma\cdot\beta\alpha} + \Gamma_{\gamma\cdot\alpha\beta}) - (\underbrace{\Gamma_{\beta\cdot\gamma\alpha} - \Gamma_{\beta\cdot\alpha\gamma}}_{T_{\beta\gamma\alpha}}) - (\Gamma_{\alpha\cdot\gamma\beta} - \Gamma_{\alpha\cdot\beta\gamma}) = 0$
If
$T_{\alpha\cdot\beta\gamma}$ is totally antisymmetric
$\Rightarrow \Gamma_{\gamma(\beta\alpha)} = \frac{1}{2}(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}) + \frac{1}{2}(T_{\beta\cdot\alpha\gamma} + T_{\alpha\cdot\beta\gamma})$
$\Rightarrow \Gamma_{\gamma\cdot\beta\alpha} =$
$+\frac{1}{2}(T_{\beta\cdot\alpha\gamma} - T_{\alpha\cdot\gamma\beta} + T_{\gamma}$
$K_{\beta\cdot\alpha\gamma}$ contorsion
$MM^{-1} = 1$
$dM\,M^{-1} + M d$
$d(M^{-1})$

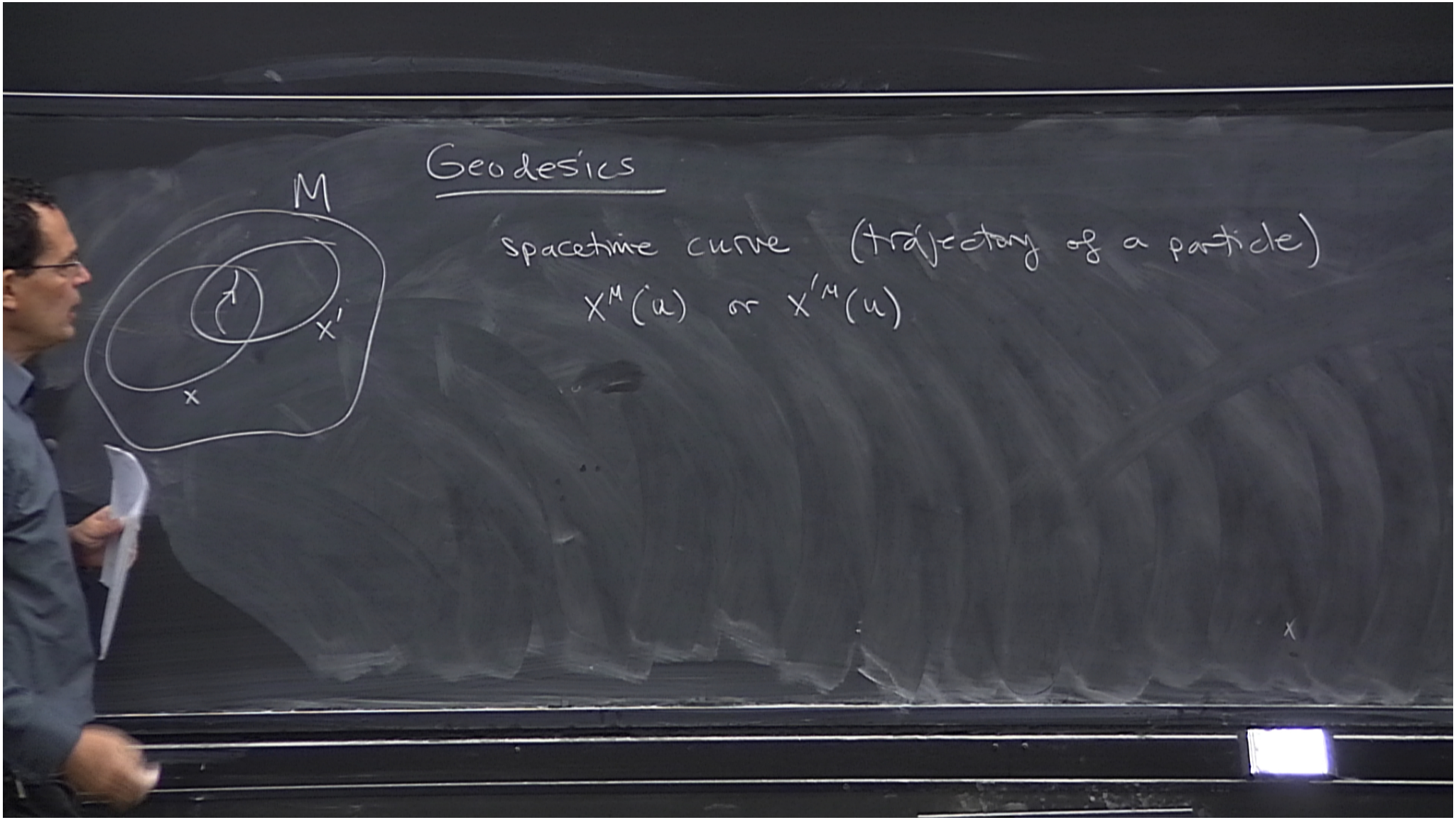
$T_{\alpha,\beta\gamma}$ is totally antisymmetric $\Rightarrow \Gamma^{\gamma}_{(\beta\alpha)}$

then

$\Gamma^{\gamma}_{(\beta\alpha)} = \{\gamma, \beta\alpha\}$ $\Rightarrow \Gamma_{\gamma}$

Geodesics





Geodesics

spacetime curve (trajectory of a particle)

$X^M(u)$ or $X'^M(u)$

coord transfm^n from x to x'.

but $x'^{\mu}(x^{\nu}(u))$ is curve in x' coords.

$\rightarrow \xi^{\mu}$

coord transfm^n: from x to x'.

but $x'^{\mu}(x^{\alpha}(u))$ is curve in x' coords.

$$\rightarrow \xi'^{\mu} = \frac{dx'^{\mu}}{du} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{dx^{\alpha}}{du} = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \xi^{\mu}$$

coord transfm: from x to x'.

but $x'^{\mu}(x^{\alpha}(u))$ is curve in x' coords.

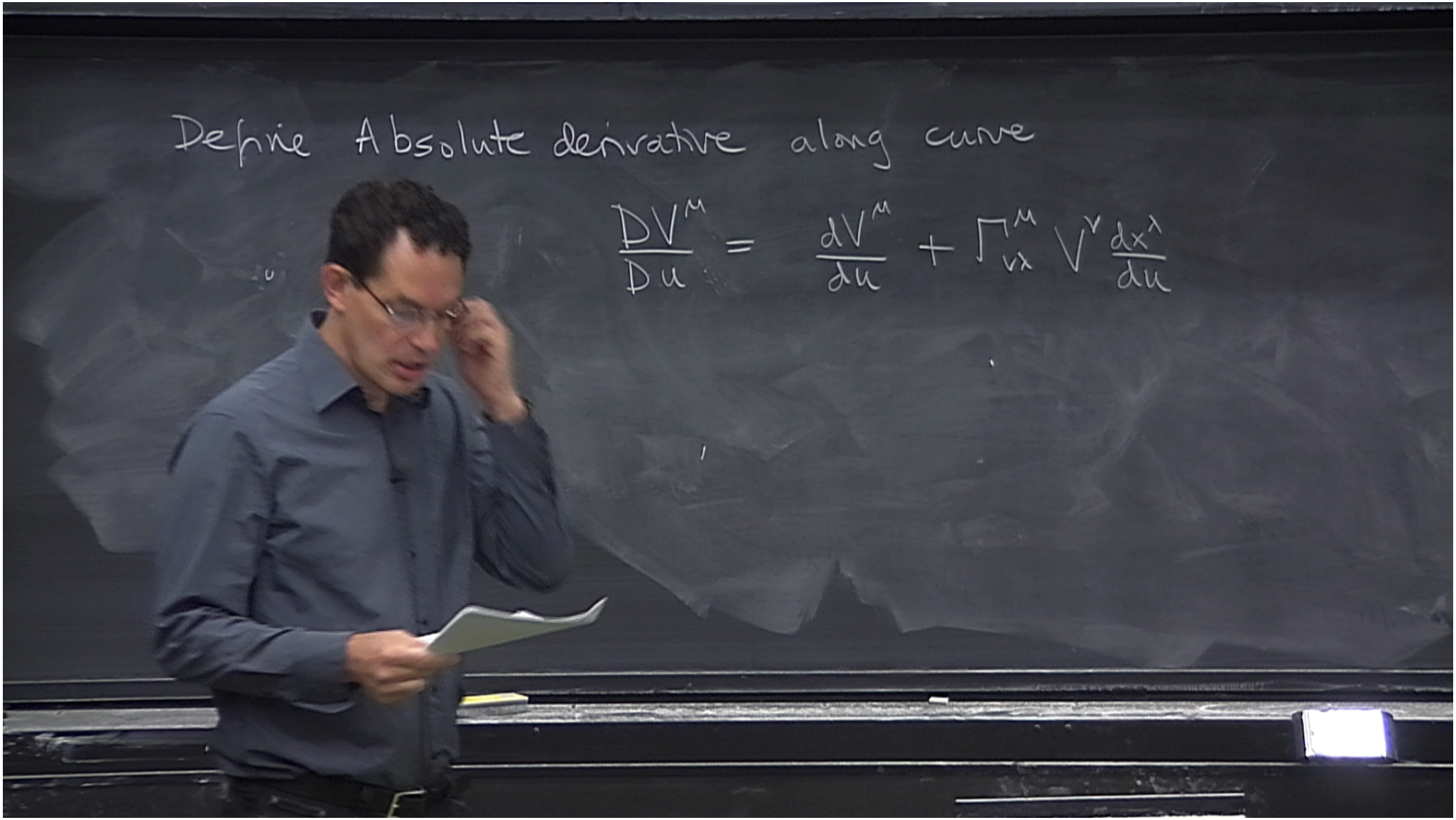
$$\rightarrow \xi'^{\mu} = \frac{dx'^{\mu}}{du} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{dx^{\alpha}}{du} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \xi^{\alpha}$$

coord transfm: ~~from~~ x to x'.

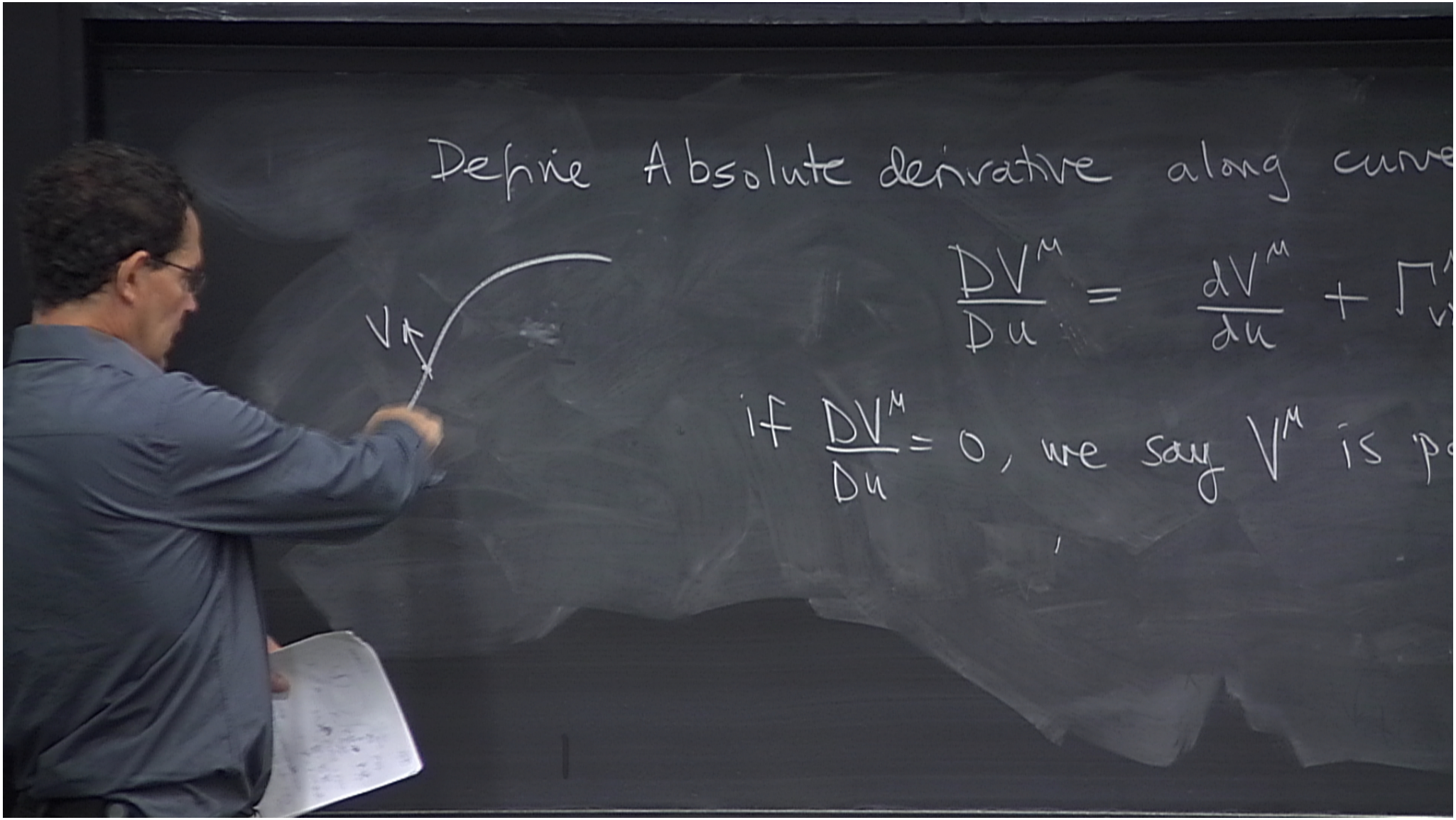
but $x'^{\mu}(x^{\nu}(u))$ is curve in x' coords.

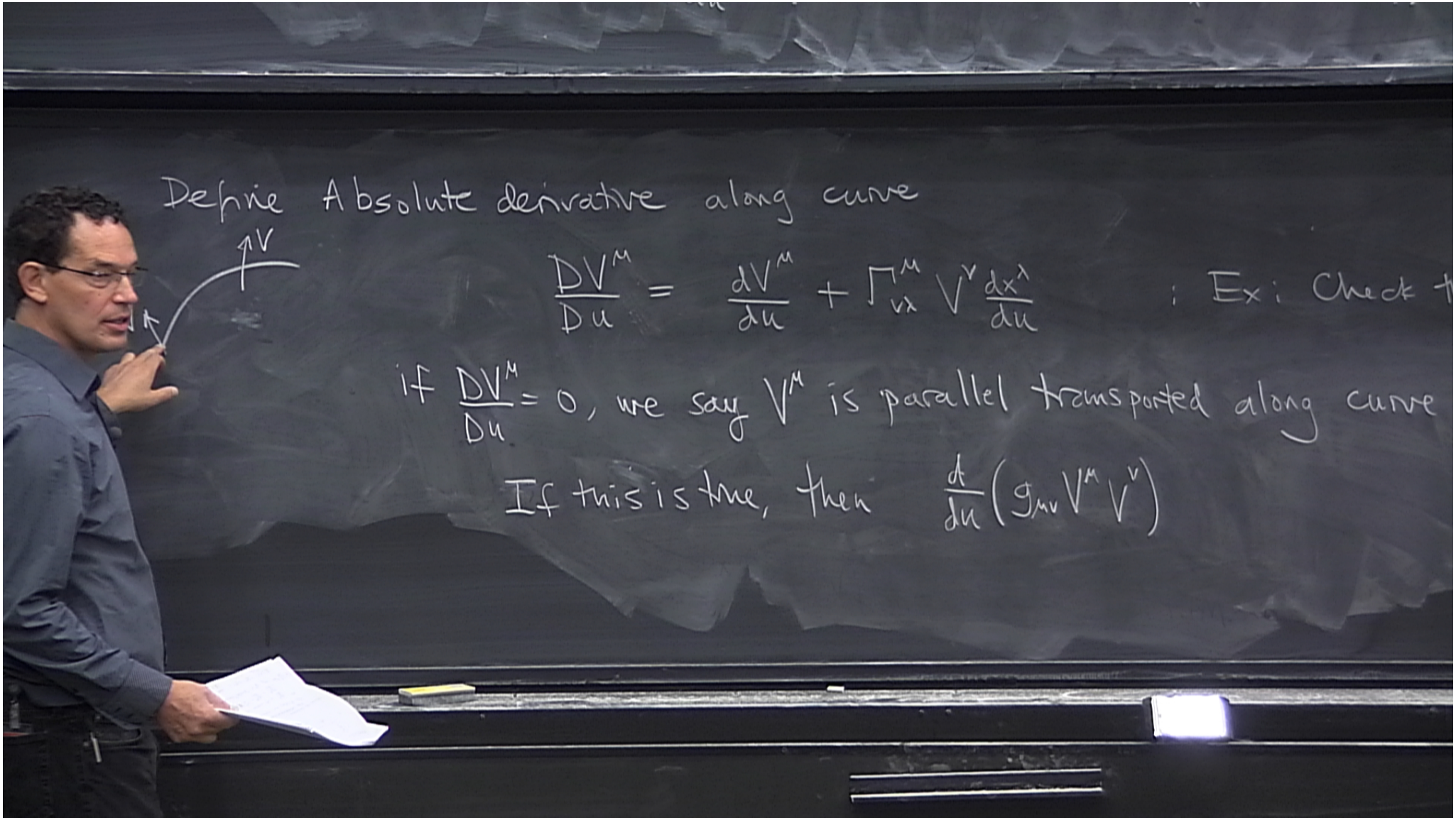
$$\rightarrow \xi'^{\mu} = \frac{dx'^{\mu}}{du} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \cdot \frac{dx^{\alpha}}{du} = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \xi^{\alpha}$$

i.e. ξ^{μ} is a contravariant vector.


Define Absolute derivative along curve
$\frac{DV^\mu}{Du} = \frac{dV^\mu}{du} + \Gamma^\mu_{\nu\lambda} V^\nu \frac{dx^\lambda}{du}$

; Ex: Check that $\frac{DV^\mu}{Du}$ is a contravariant vector.


Define Absolute derivative along curve
$\frac{DV^\mu}{Du} = \frac{dV^\mu}{du} + \Gamma$
V^μ
if $\frac{DV^\mu}{Du} = 0$, we say V^μ is pa


Define Absolute derivative along curve
V
$\frac{DV^\mu}{Du} = \frac{dV^\mu}{du} + \Gamma^\mu_{\nu\lambda} V^\nu \frac{dx^\lambda}{du}$
; Ex: Check t
if $\frac{DV^\mu}{Du} = 0$, we say V^μ is parallel transported along curve
If this is true, then $\frac{d}{du}(g_{\mu\nu} V^\mu V^\nu)$

e

$\Gamma^{\mu}_{\nu\lambda} V^{\nu} \frac{dx^{\lambda}}{du}$; Ex: Check that $\frac{DV^{\mu}}{Du}$ is a contravariant

parallel transported along curve $x^{\mu}(u)$.

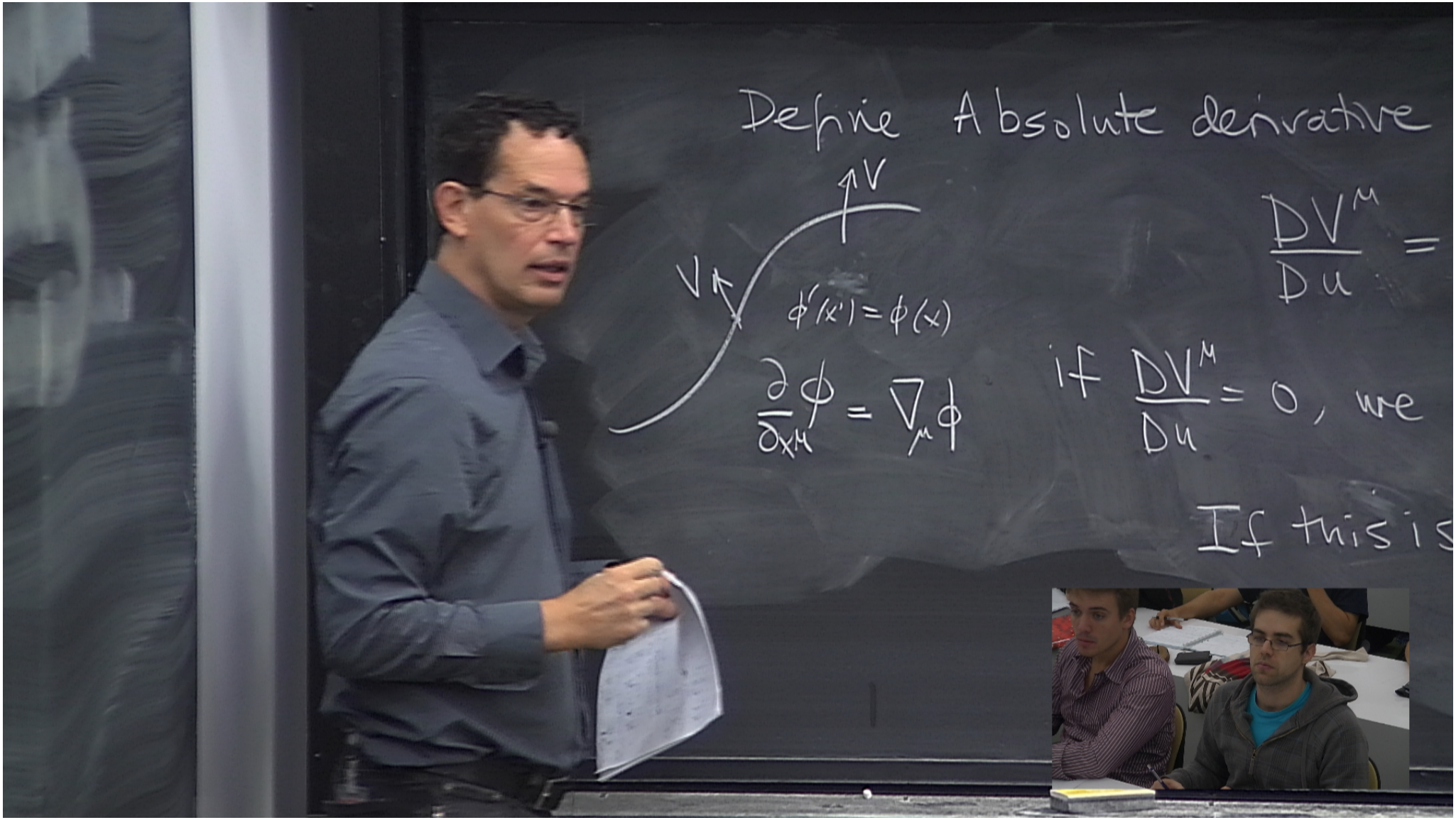
$$\underbrace{\frac{d}{du}\left(g_{\mu\nu} V^{\mu} V^{\nu}\right)}_{\text{scalar}} = \frac{D}{Du}\left(g_{\mu\nu}\right) V^{\mu} V^{\nu} + g_{\mu\nu} \frac{DV^{\mu}}{Du} V^{\nu} + g_{\mu\nu} V^{\mu} \frac{DV^{\nu}}{Du}$$

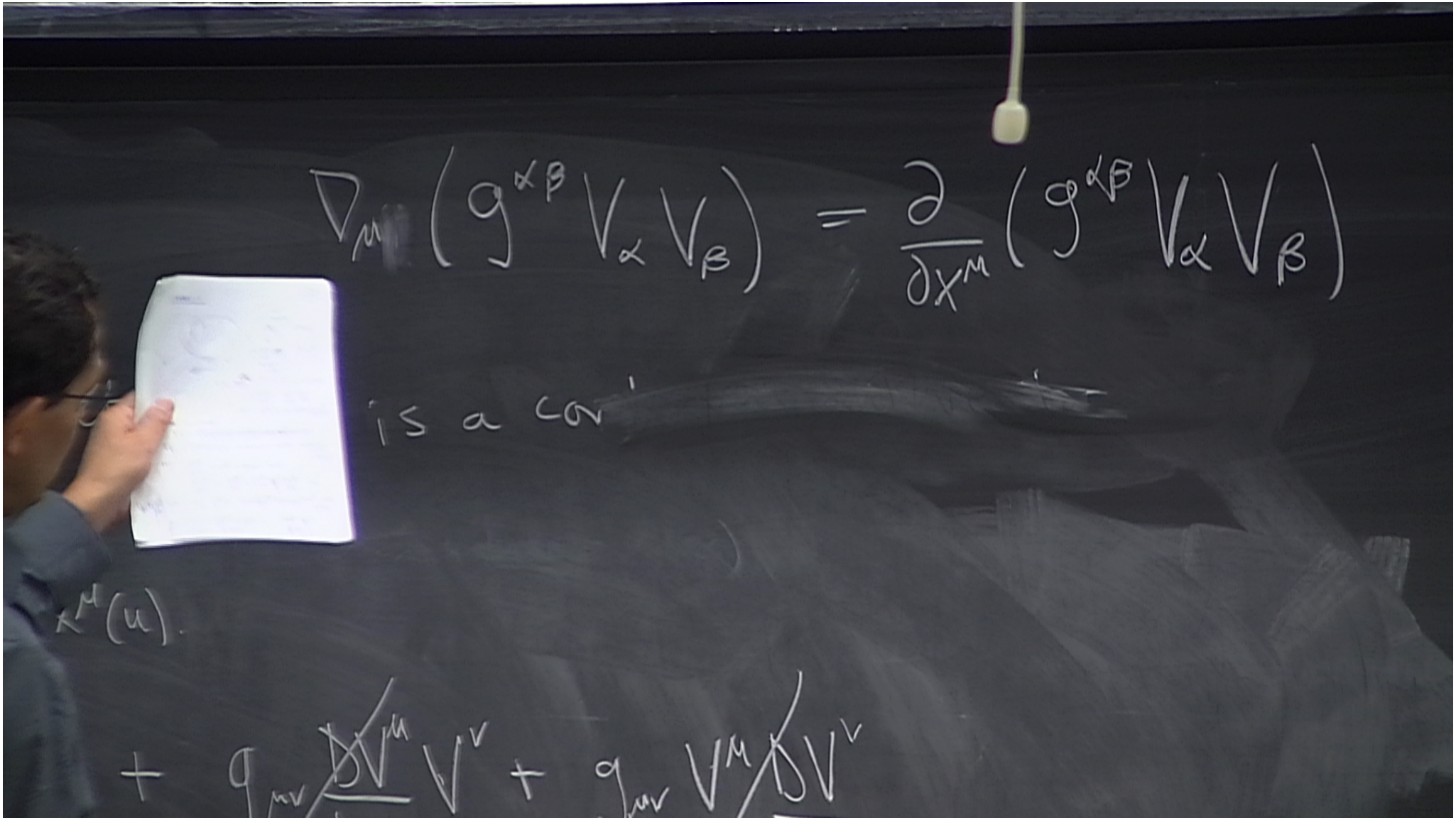
e

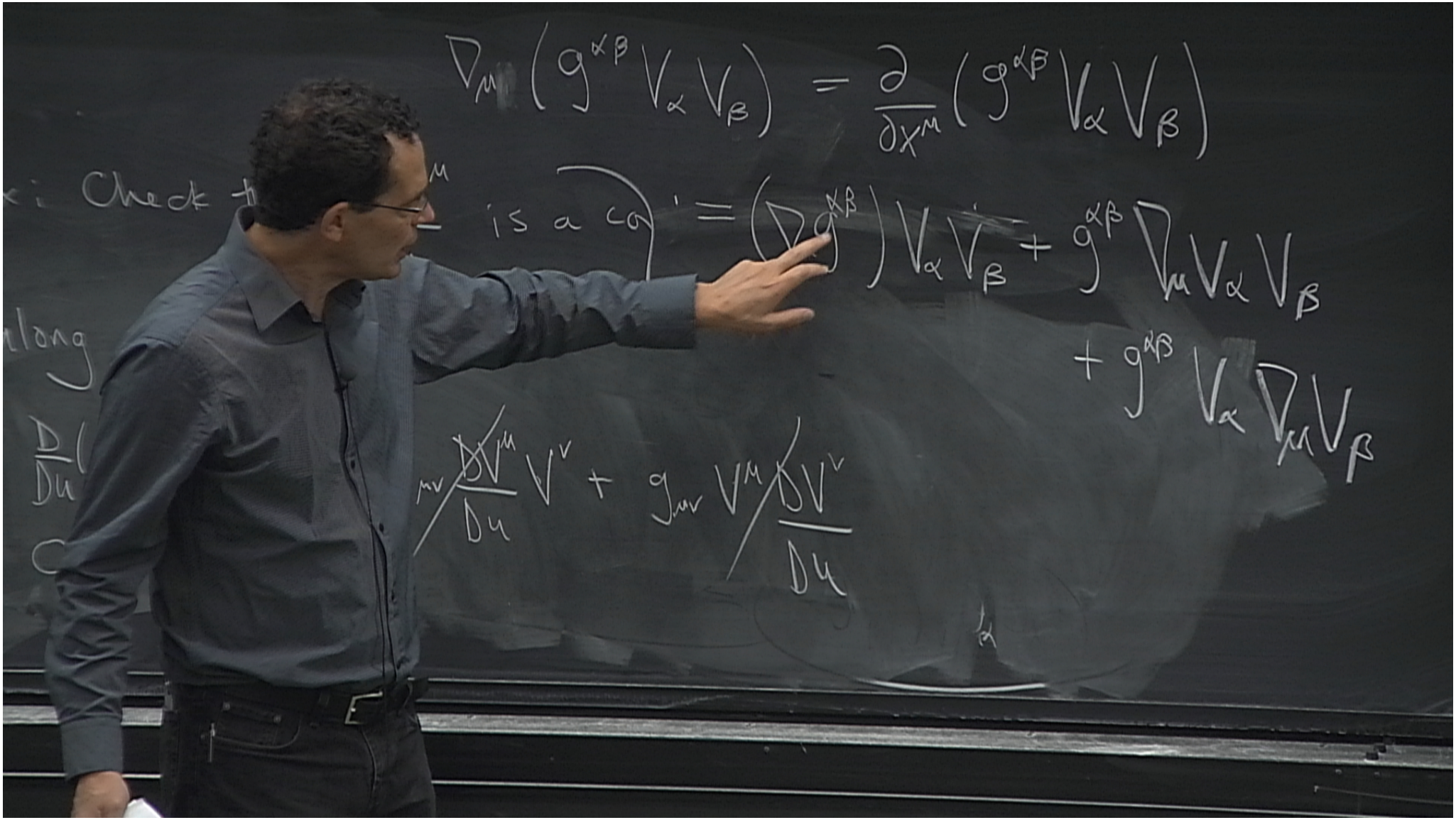
$\Gamma^{\mu}_{\nu\lambda} V^{\nu} \frac{dx^{\lambda}}{du}$; Ex: Check that $\frac{DV^{\mu}}{Du}$ is a contravariant

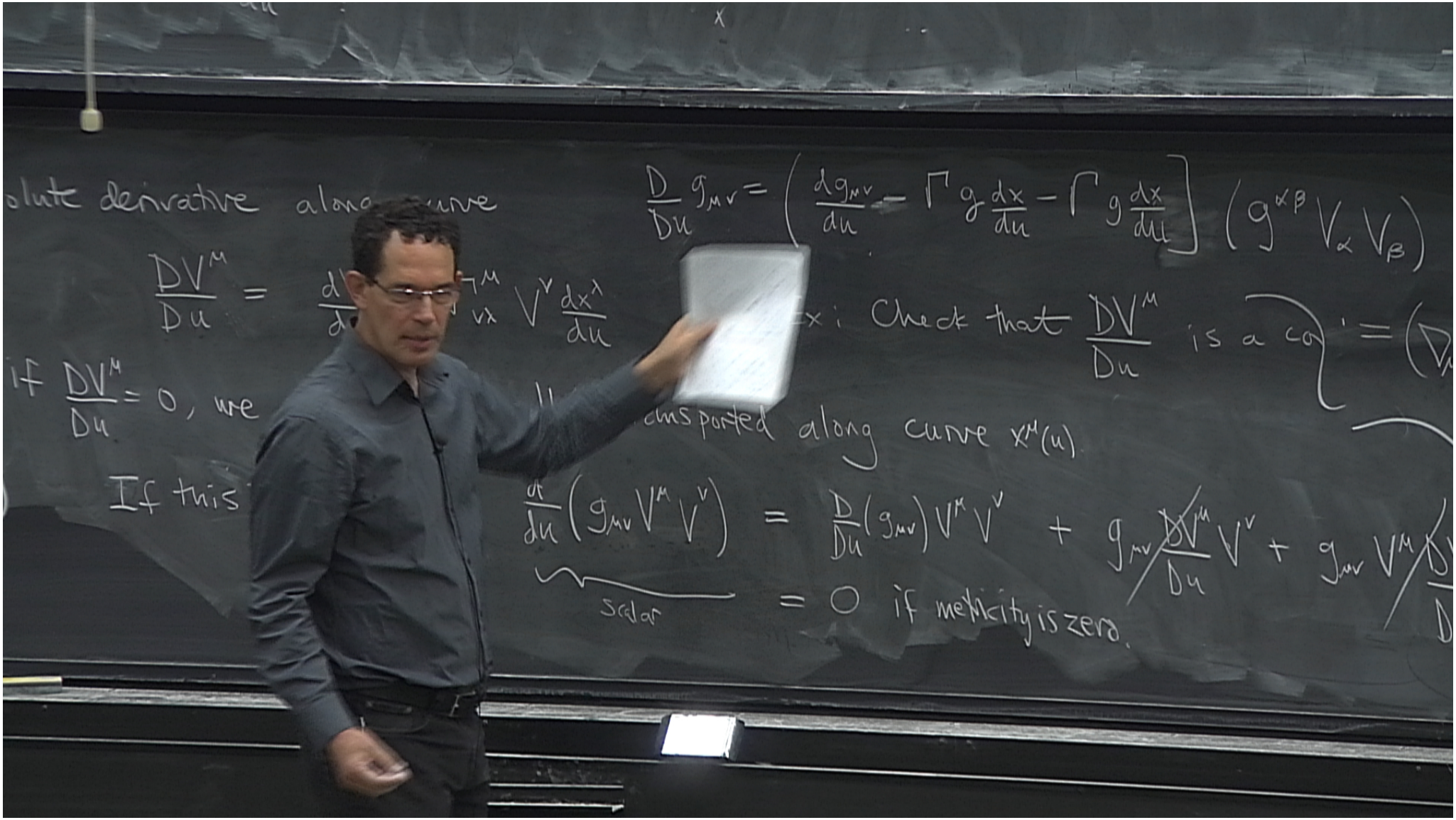
parallel transported along curve $x^{\mu}(u)$.

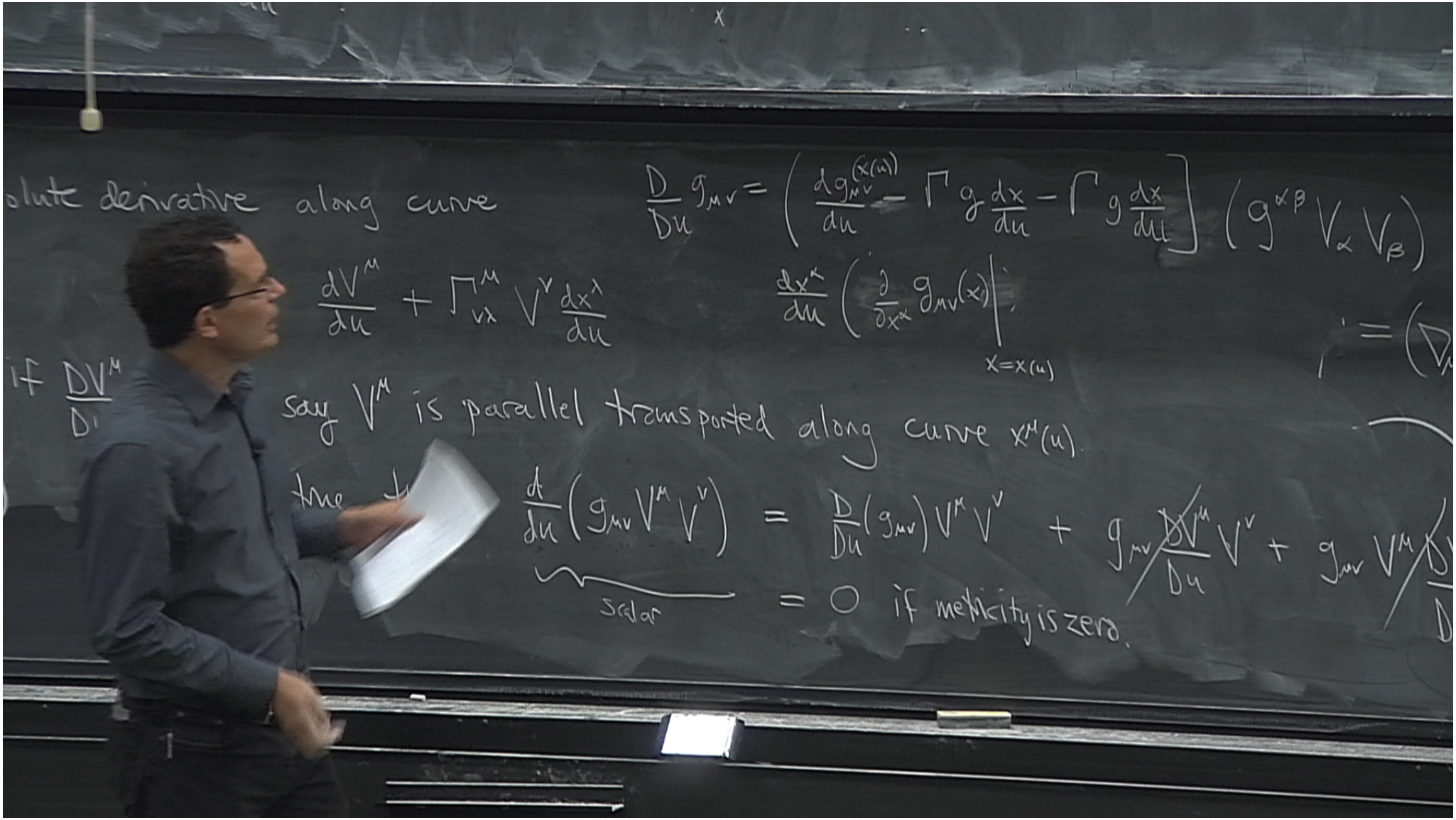
$$\underbrace{\frac{d}{du}\left(g_{\mu\nu} V^{\mu} V^{\nu}\right)}_{\text{scalar}} = \frac{D}{Du}\left(g_{\mu\nu}\right) V^{\mu} V^{\nu} + g_{\mu\nu} \frac{DV^{\mu}}{Du} V^{\nu} + g_{\mu\nu} V^{\mu} \frac{DV^{\nu}}{Du}$$

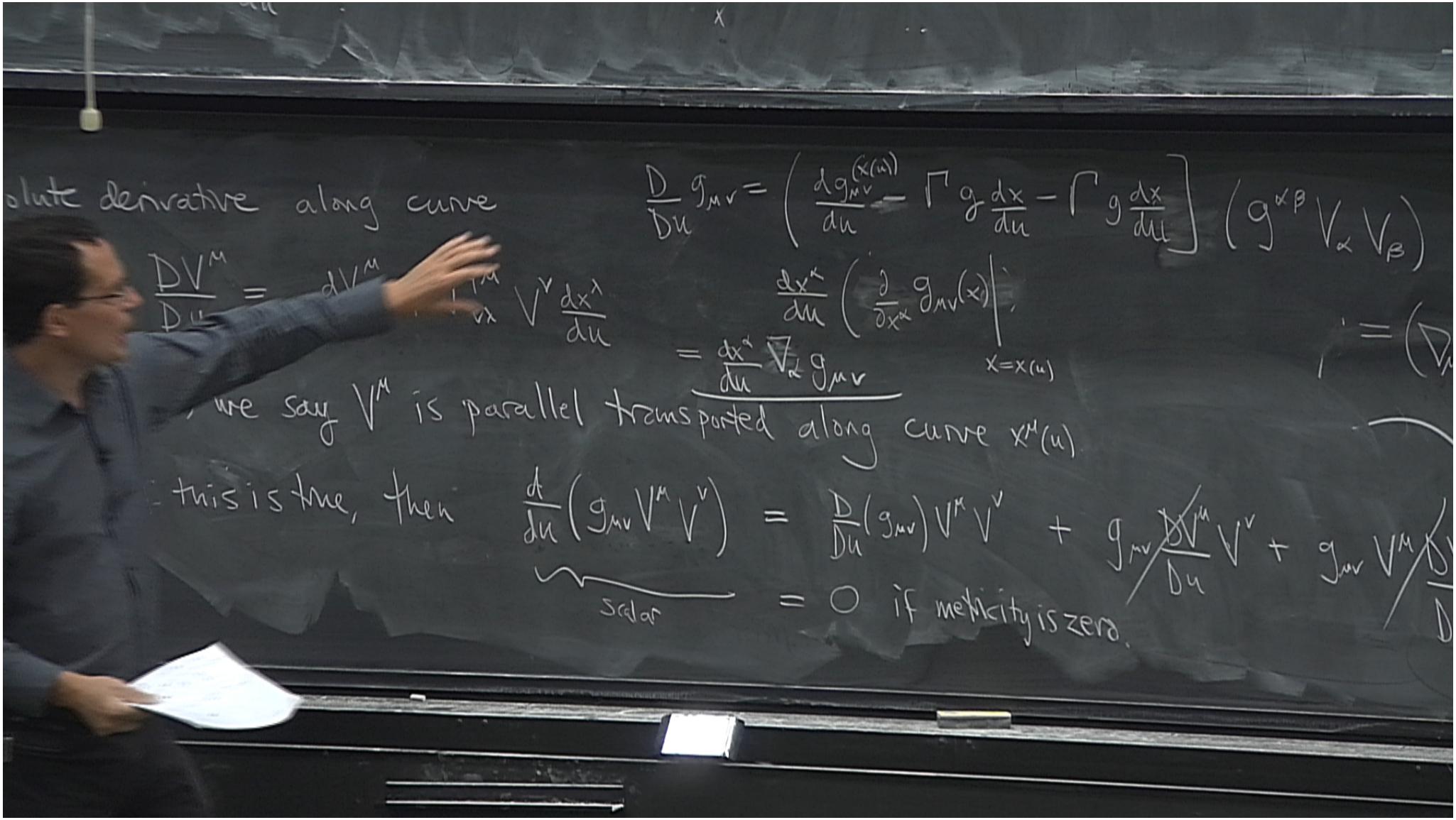

Define Absolute derivative
V
V
$\phi'(x') = \phi(x)$
$\frac{\partial}{\partial x^\mu}\phi = \nabla_\mu \phi$
$\frac{DV^\mu}{Du} =$
if $\frac{DV^\mu}{Du} = 0$, we
If this is

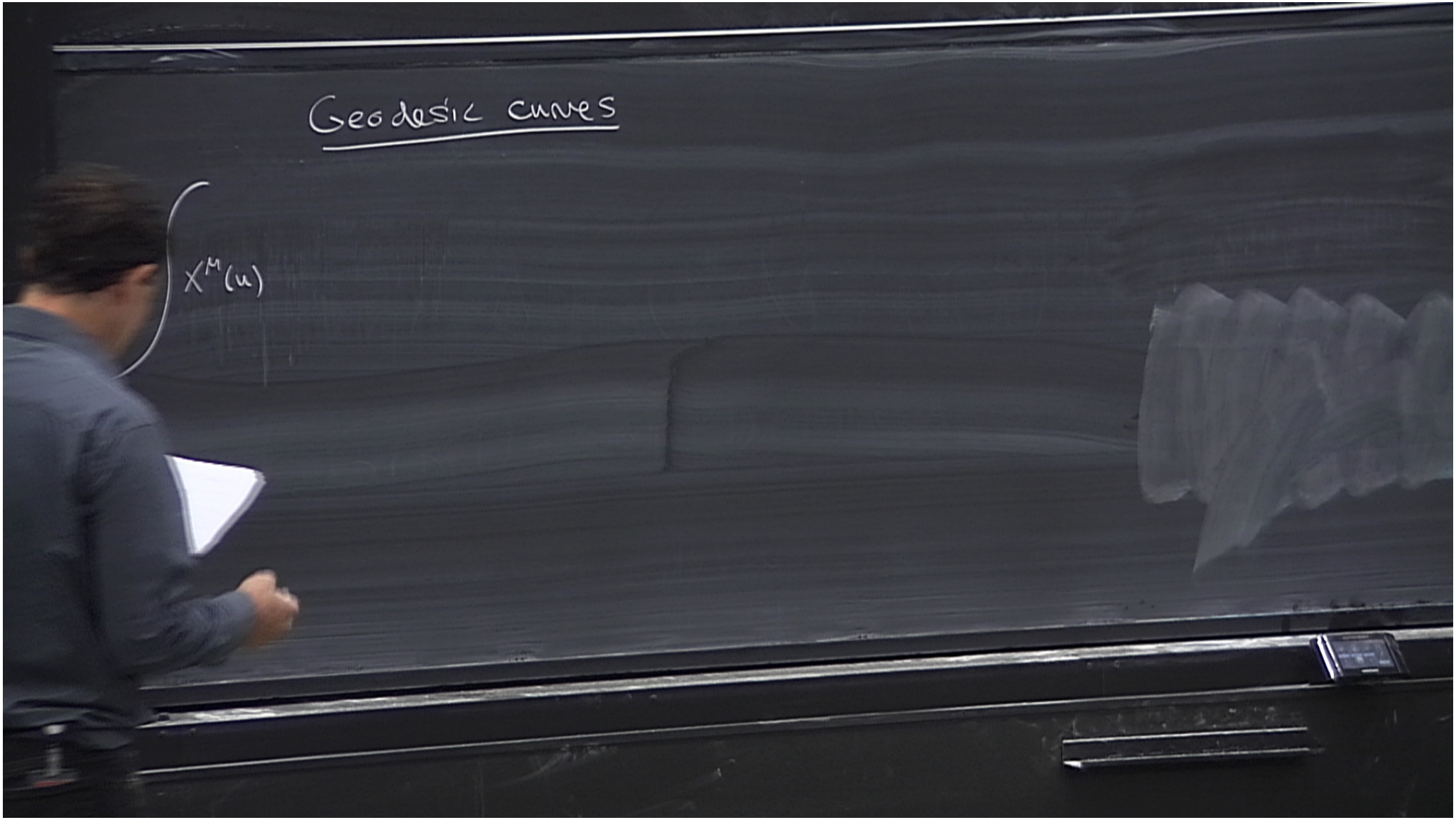

$D_\mu(g^{\alpha\beta}V_\alpha V_\beta) = \frac{\partial}{\partial x^\mu}(g^{\alpha\beta}V_\alpha V_\beta)$
is a cov
$x^\mu(u)$.
$+ g_{\mu\nu}\frac{DV^\mu}{}V^\nu + g_{\mu\nu}V^\mu\frac{DV^\nu}{}$


$\nabla_\mu (g^{\alpha\beta} V_\alpha V_\beta) = \frac{\partial}{\partial x^\mu} (g^{\alpha\beta} V_\alpha V_\beta)$
$= (\nabla_\mu g^{\alpha\beta}) V_\alpha V_\beta + g^{\alpha\beta} \nabla_\mu V_\alpha V_\beta$
$+ g^{\alpha\beta} V_\alpha \nabla_\mu V_\beta$
$g_{\nu\rho} \frac{DV^\mu}{D\lambda} V^\nu + g_{\mu\nu} V^\mu \frac{DV^\nu}{D\lambda}$


olute derivative along curve
$\frac{DV^\mu}{Du} = \frac{dV^\mu}{du} + \Gamma^\mu_{\nu\lambda} V^\nu \frac{dx^\lambda}{du}$
if $\frac{DV^\mu}{Du} = 0$, we
If this
$\frac{D}{Du} g_{\mu\nu} = \left(\frac{dg_{\mu\nu}}{du} - \Gamma g \frac{dx}{du} - \Gamma g \frac{dx}{du} \right) (g^{\alpha\beta} V_\alpha V_\beta)$
x: Check that $\frac{DV^\mu}{Du}$ is a co
nsported along curve $x^\mu(u)$
$\frac{d}{du}(g_{\mu\nu} V^\mu V^\nu) = \frac{D}{Du}(g_{\mu\nu}) V^\mu V^\nu + g_{\mu\nu} \frac{DV^\mu}{Du} V^\nu + g_{\mu\nu} V^\mu$
scalar
$= 0$ if metricity is zero.

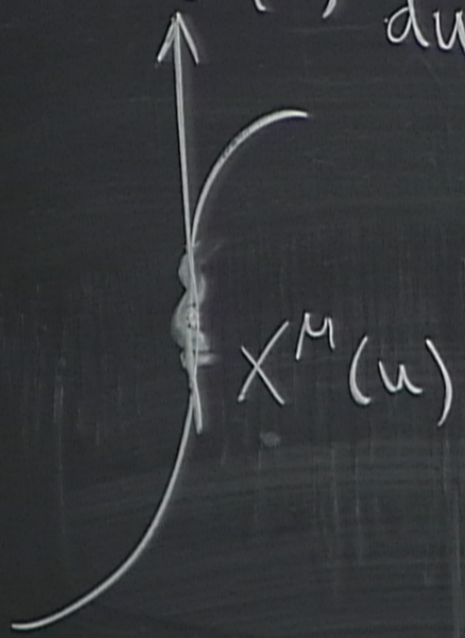

olute derivative along curve
$\frac{dV^\mu}{du} + \Gamma^\mu_{\nu\lambda} V^\nu \frac{dx^\lambda}{du}$
if $\frac{DV^\mu}{Du}$
say V^μ is parallel transported along curve $x^\mu(u)$.
$\frac{D}{Du} g_{\mu\nu} = \left(\frac{dg_{\mu\nu}^{(x(u))}}{du} - \Gamma g \frac{dx}{du} - \Gamma g \frac{dx}{du} \right) (g^{\alpha\beta} V_\alpha V_\beta)$
$\frac{dx^\kappa}{du} \left(\frac{\partial}{\partial x^\kappa} g_{\mu\nu}(x) \right|_{x=x(u)}$
$\frac{d}{du}(g_{\mu\nu} V^\mu V^\nu) = \frac{D}{Du}(g_{\mu\nu}) V^\mu V^\nu + g_{\mu\nu} \frac{DV^\mu}{Du} V^\nu + g_{\mu\nu} V^\mu \frac{DV^\nu}{Du}$
scalar
$= 0$ if metricity is zero.


olute derivative along curve
$\frac{DV^\mu}{Du} = \frac{dV^\mu}{dx^\lambda} V^\nu \frac{dx^\lambda}{du}$
$\frac{D}{Du} g_{\mu\nu} = \left(\frac{dg_{\mu\nu}^{(x(u))}}{du} - \Gamma g \frac{dx}{du} - \Gamma g \frac{dx}{du} \right) (g^{\alpha\beta} V_\alpha V_\beta)$
$\frac{dx^\alpha}{du} \left(\frac{\partial}{\partial x^\alpha} g_{\mu\nu}(x) \right|_{x=x(u)}$
$= \frac{dx^\alpha}{du} \nabla_\alpha g_{\mu\nu}$
we say V^μ is parallel transported along curve $x^\mu(u)$
this is true, then $\frac{d}{du}(g_{\mu\nu} V^\mu V^\nu) = \frac{D}{Du}(g_{\mu\nu}) V^\mu V^\nu + g_{\mu\nu} \frac{DV^\mu}{Du} V^\nu + g_{\mu\nu} V^\mu$
scalar
$= 0$ if metricity is zero.

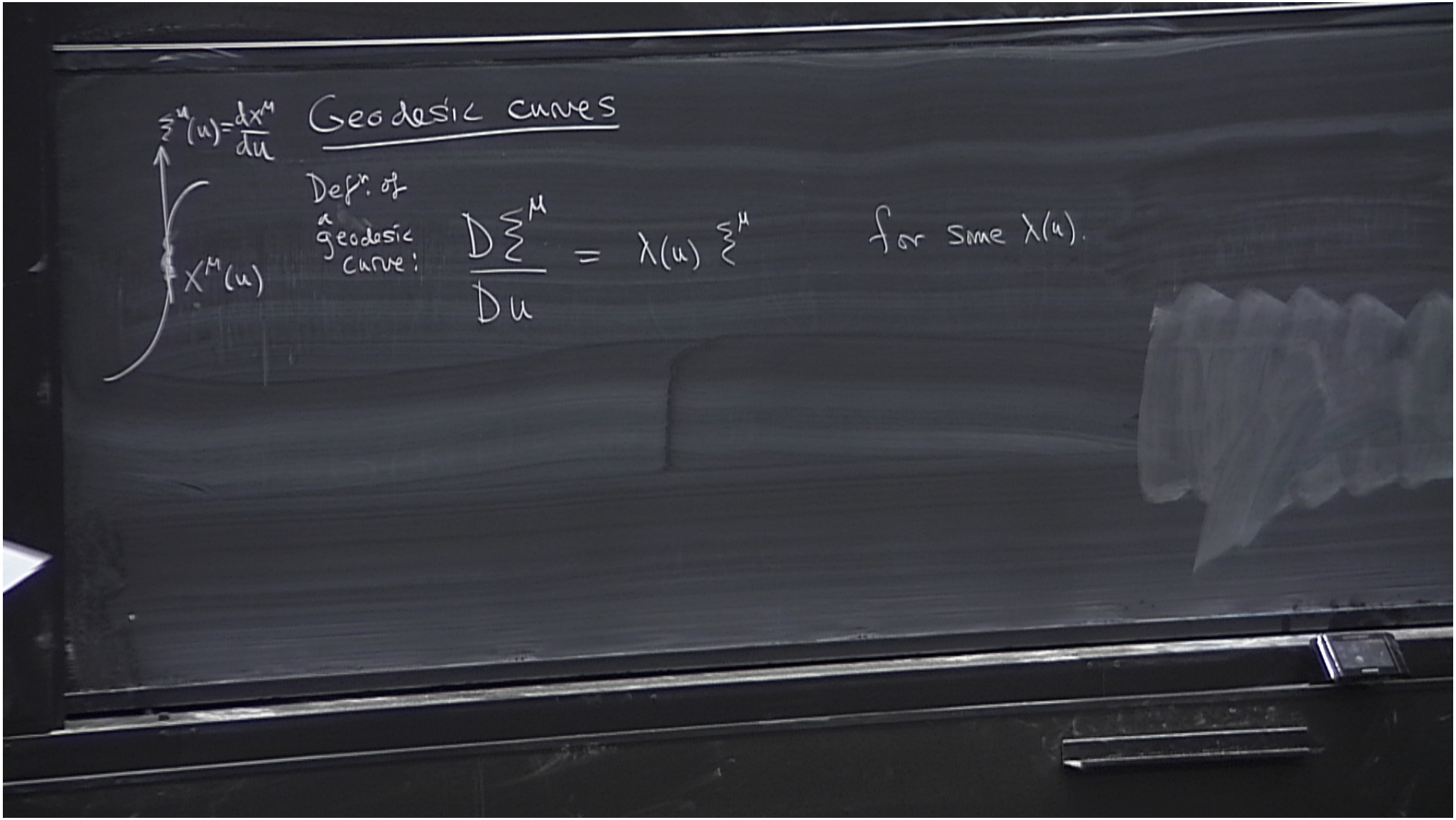

Geodesic curves
$X^M(u)$

Geodesic curves

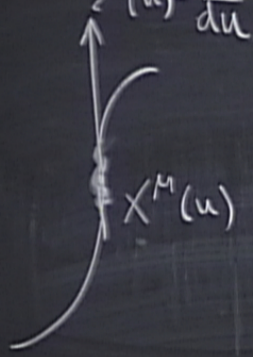
$\xi^\mu(u) = \frac{dx^\mu}{du}$



$$\frac{D\xi^\mu}{Du}$$


$\xi^\mu(u) = \frac{dx^\mu}{du}$
$x^\mu(u)$
Geodesic curves
Defn. of a geodesic curve:
$\frac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu$
for some $\lambda(u)$.

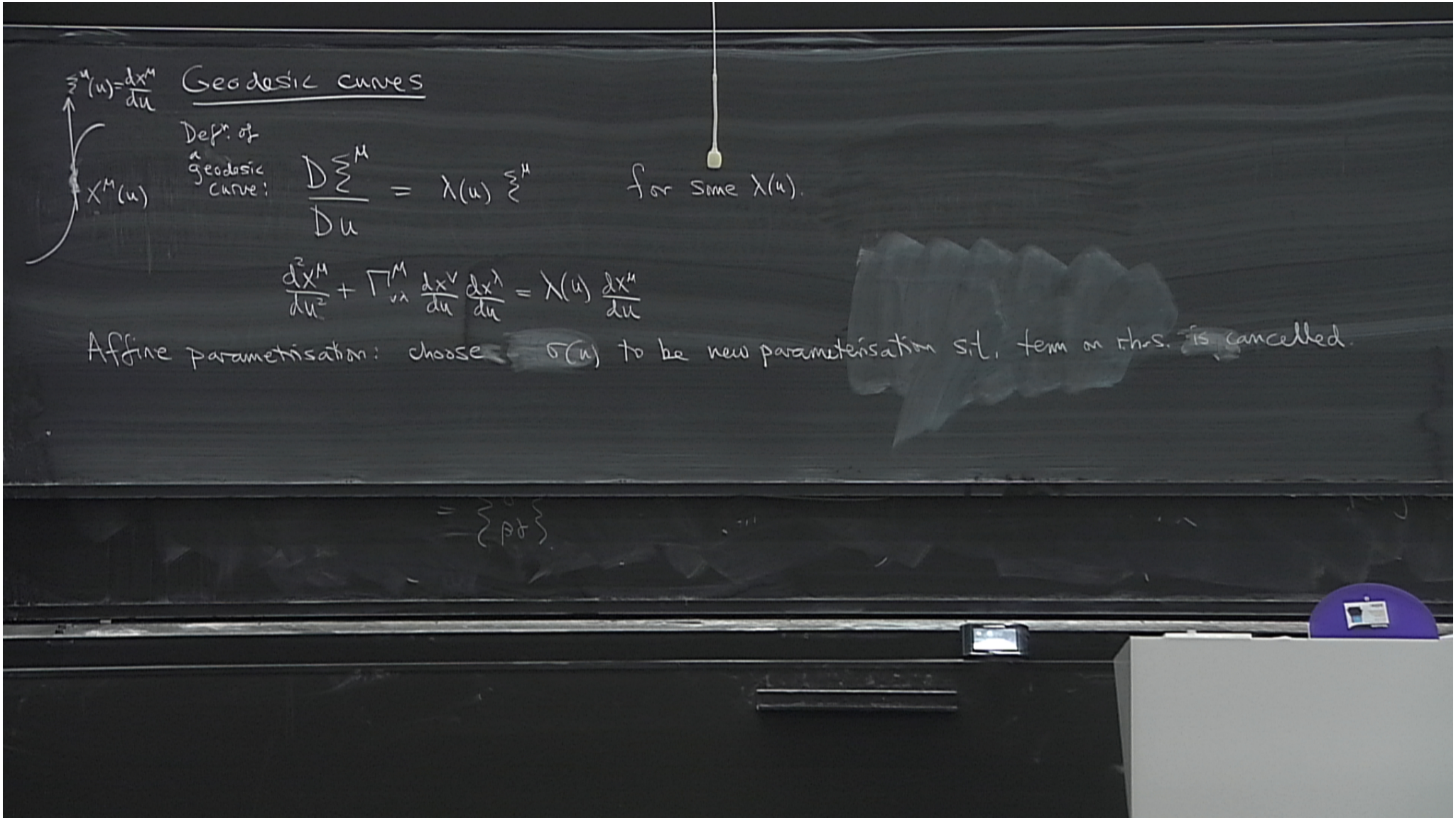
Geodesic curves



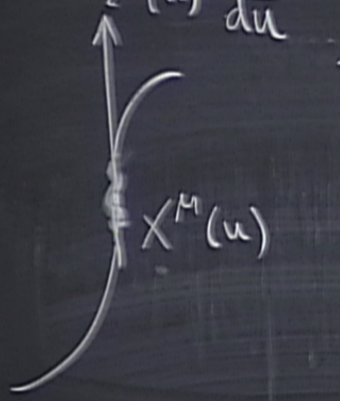
Defn. of a geodesic curve:

$$\frac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu \qquad \text{for some } \lambda(u).$$

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$$


$\xi^\mu(u)=\frac{dx^\mu}{du}$
$x^\mu(u)$
Geodesic curves
Defn. of a geodesic curve:
$\frac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu$
for some $\lambda(u)$.
$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$
Affine parametrisation: choose $\sigma(u)$ to be new parameterisation s.t. term on r.h.s. is cancelled.

Geodesic curves

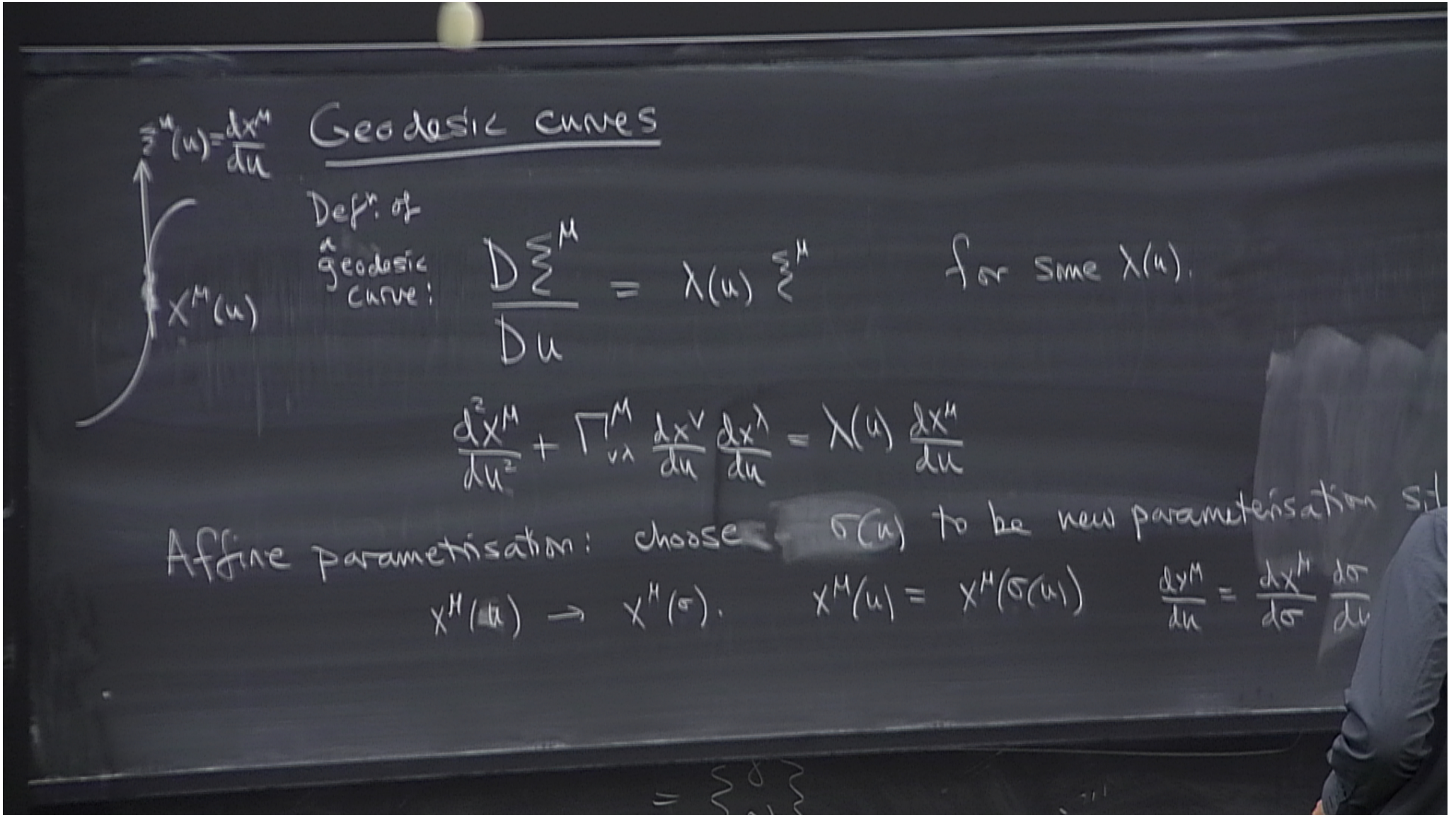


Defn. of a geodesic curve: $\dfrac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu$ for some $\lambda(u)$.

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$$

Affine parametrisation: choose $\sigma(u)$ to be new parameterisation s.t.

$x^\mu(u) \to x^\mu(\sigma)$. $\quad x^\mu(u) = x^\mu(\sigma(u))$


$\xi^\mu(u) = \frac{dx^\mu}{du}$
$x^\mu(u)$
Geodesic curves
Defn. of a geodesic curve:
$\frac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu$ for some $\lambda(u)$.
$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$
Affine parametrisation: choose $\sigma(u)$ to be new parameterisation s.t.
$x^\mu(u) \to x^\mu(\sigma)$.
$x^\mu(u) = x^\mu(\sigma(u))$
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$x^\mu(u)$

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$x^\mu(u) \to x^\mu(\sigma)$. $\quad x^\mu(u) = x^\mu(\sigma(u)) \quad \frac{dx^\mu}{du} = \frac{dx^\mu}{d\sigma}\frac{d\sigma}{du}$

$$\Rightarrow \frac{d^2\sigma}{du^2}\frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{du}\right)^2\left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right) = \lambda(u)\frac{d\sigma}{du}\frac{dx^\mu}{d\sigma}$$

$x^\mu(u)$

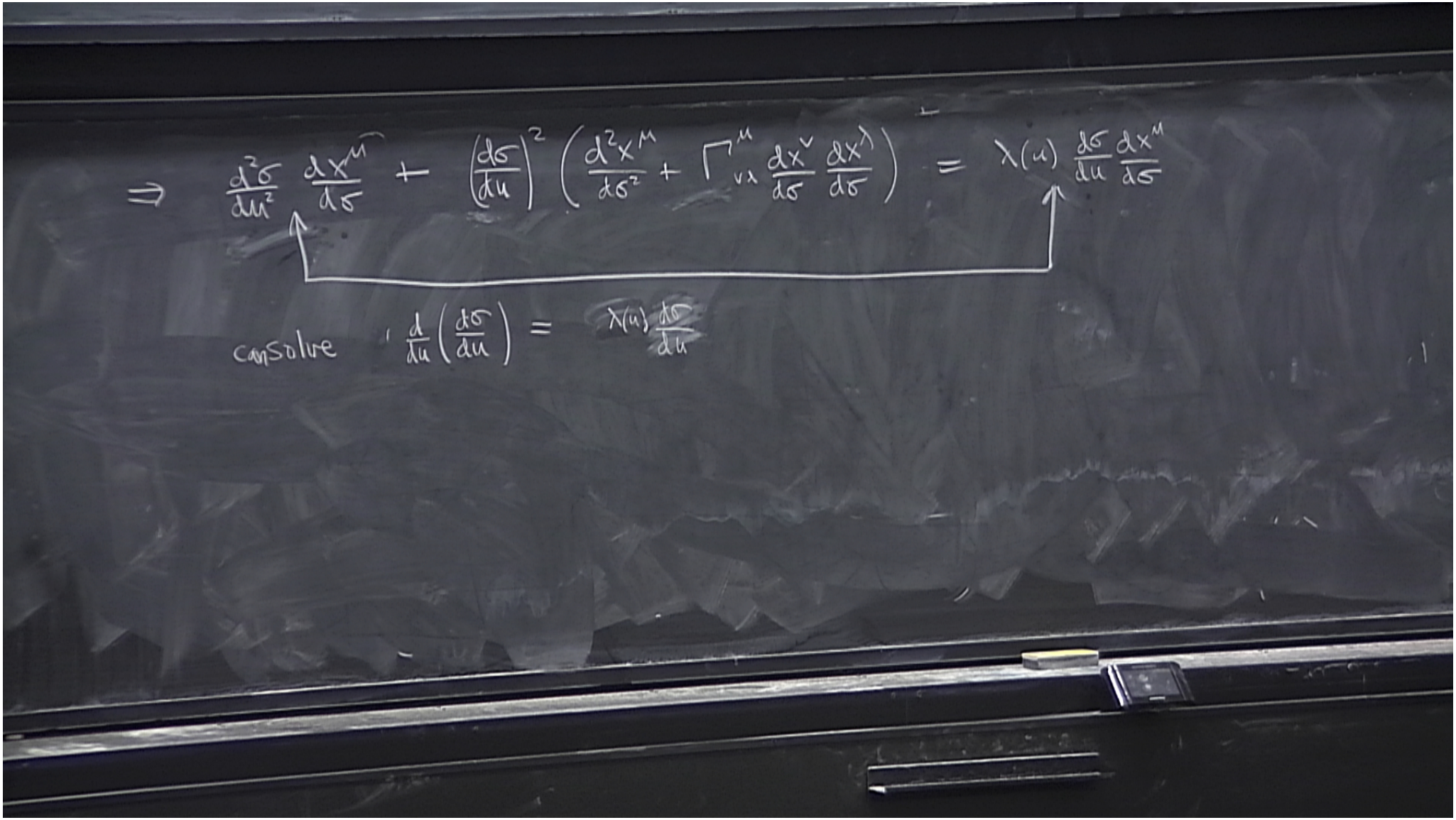
Def^n of a geodesic curve: $\dfrac{D\xi^\mu}{Du} = \lambda(u)\,\xi^\mu$ for some $\lambda(u)$.

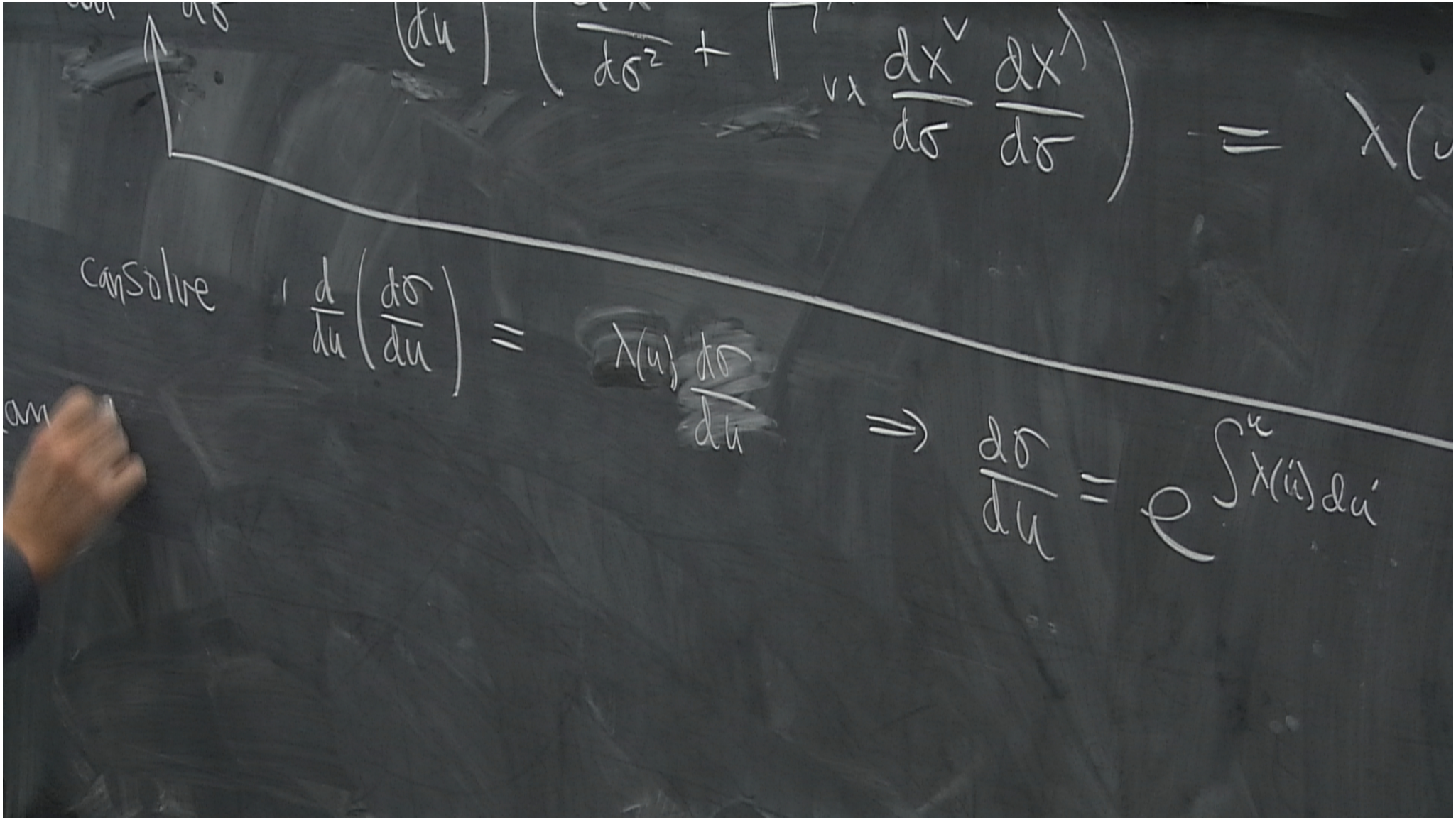
$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$$

Affine parametrisation: choose $\sigma(u)$ to be new parameterisation s.t. term on rhs. is cancelled.

$x^\mu(u) \to x^\mu(\sigma)$. $\quad x^\mu(u) = x^\mu(\sigma(u))$ $\quad \dfrac{dx^\mu}{du} = \dfrac{dx^\mu}{d\sigma}\dfrac{d\sigma}{du}$

$$\Rightarrow \frac{d^2\sigma}{du^2}\frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{du}\right)^2\left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right) = \lambda(u)\frac{d\sigma}{du}\frac{dx^\mu}{d\sigma}$$

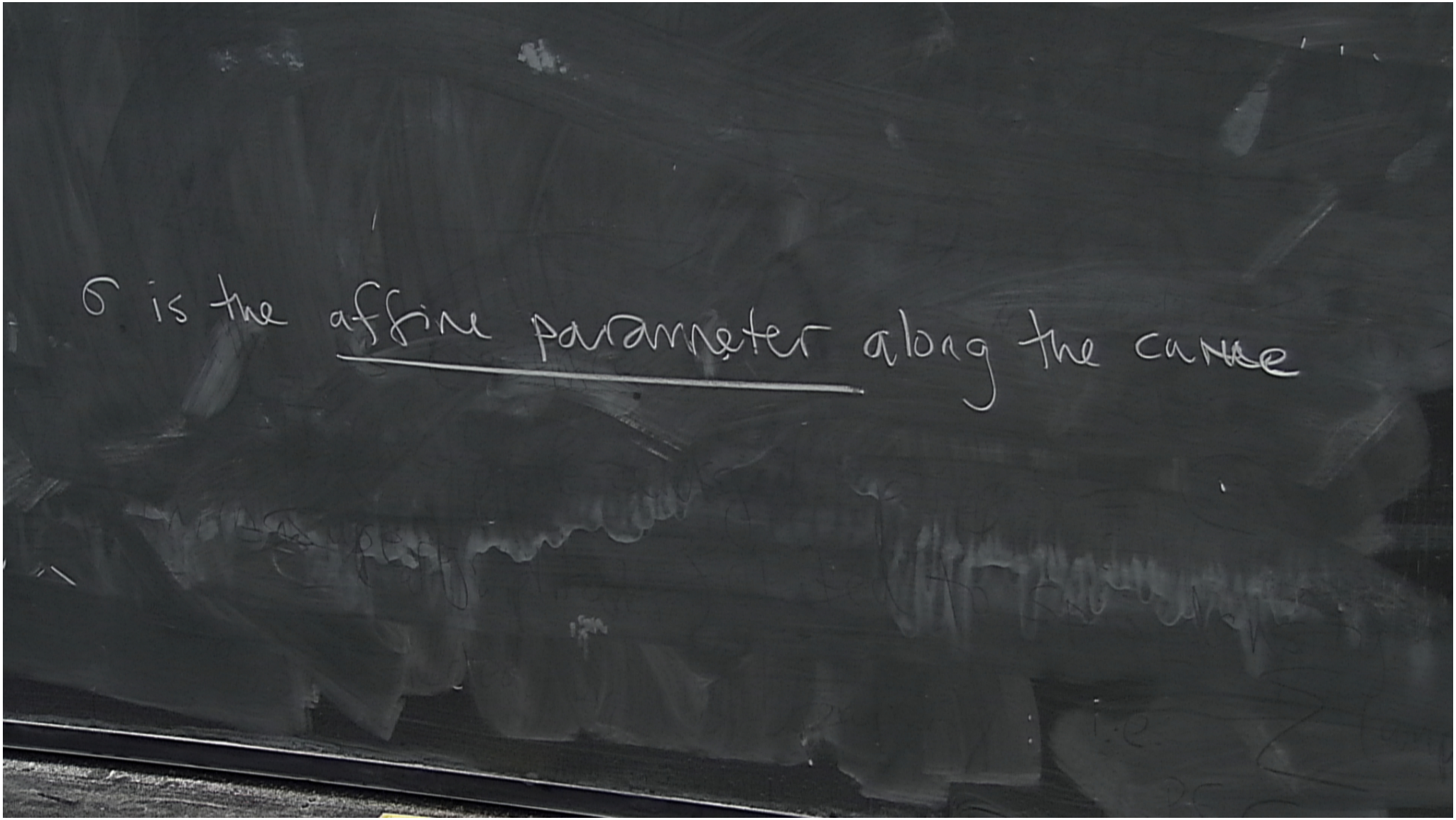

$\Rightarrow \frac{d^2\sigma}{du^2}\frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{du}\right)^2\left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right) = \lambda(u)\frac{d\sigma}{du}\frac{dx^\mu}{d\sigma}$
consolve $\frac{d}{du}\left(\frac{d\sigma}{du}\right) = \lambda(u)\frac{d\sigma}{du}$


$\left(\frac{d\mu}{d\sigma}\right)\left(\frac{d^2x^\mu}{d\sigma^2}+\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right)=\lambda(\mu)$
consolve
$\frac{d}{d\mu}\left(\frac{d\sigma}{d\mu}\right)=\lambda(\mu)\frac{d\sigma}{d\mu}$
$\Rightarrow \frac{d\sigma}{d\mu}=e^{\int^\mu \lambda(\mu')\,d\mu'}$

$$\Rightarrow \frac{d^2\sigma}{du^2}\frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{du}\right)^2\left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right) = \lambda(u)\frac{d\sigma}{du}\frac{dx^\mu}{d\sigma}$$

can solve $\frac{d}{du}\left(\frac{d\sigma}{du}\right) = \lambda(u)\frac{d\sigma}{du} \Rightarrow \frac{d\sigma}{du} = e^{\int^u \lambda(u')\,du'}$

$\Rightarrow$ can always re-parameterise the curve s.t. $\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$


σ is the affine parameter along the curve

σ is the affine parameter along the curve

Unique up to $\sigma \to \sigma' = a\sigma + b$, with a, b constants.

$$\frac{Du}{}$$

$$\frac{d^2x^\mu}{du^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{du}\frac{dx^\lambda}{du} = \lambda(u)\frac{dx^\mu}{du}$$

e parametrisation: choose $\sigma(u)$ to be new parameterisation s.t. term on rhs is cancelled.

$x^\mu(u) \to x^\mu(\sigma)$. $\quad x^\mu(u) = x^\mu(\sigma(u))$ $\quad \frac{dx^\mu}{du} = \frac{dx^\mu}{d\sigma}\frac{d\sigma}{du}$

$$\Rightarrow \frac{d^2\sigma}{du^2}\frac{dx^\mu}{d\sigma} + \left(\frac{d\sigma}{du}\right)^2\left(\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right) = \lambda(u)\frac{d\sigma}{du}\frac{dx^\mu}{d\sigma}$$

consider $\frac{d}{du}\left(\frac{d\sigma}{du}\right) = \lambda(u)\frac{d\sigma}{du} \Rightarrow \frac{d\sigma}{du} = e^{\int^u \lambda(\tilde{u})\,d\tilde{u}}$

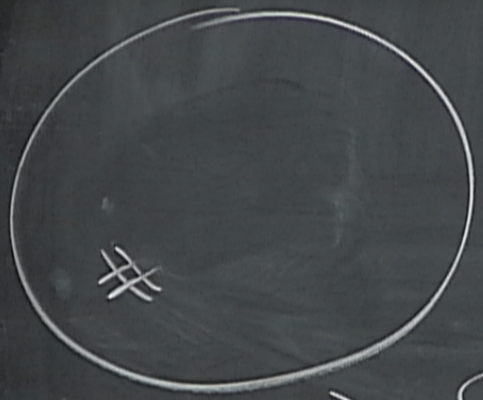
$\Rightarrow$ can always re-parameterise the curve s.t. $\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$;

in this parameterisation, $\xi^\mu(\sigma)$ is parallel transported along the curve.

σ is the affine parameter along the curve

unique up to $\sigma \to \sigma' = a\sigma + b$, with a, b

S^2



du^2 $d\sigma$

consolve $\frac{d}{du}\left(\frac{d\sigma}{du}\right)$

$\Rightarrow$ can always re-parame

$d\theta^2 + \sin^2\theta\, d\phi^2$

$g_{\theta\theta} = 1$

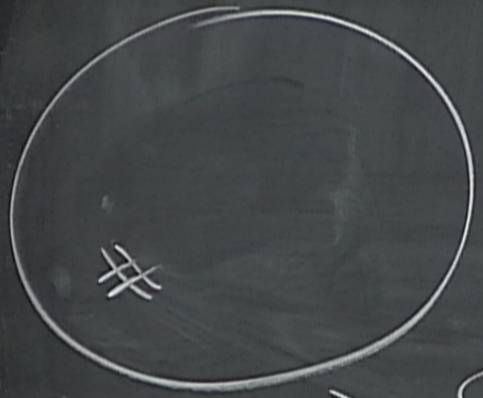
$g_{\phi\phi} = \sin^2\theta$

μ
$\nu\alpha$

in this parameterisa

S^2

du^2 $d\sigma$



consolve $\frac{d}{du}\left(\frac{d\sigma}{du}\right)$

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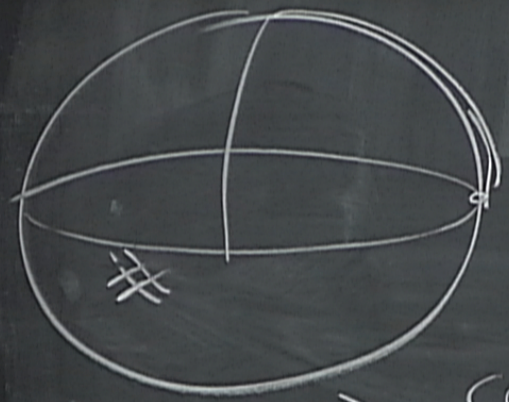
$g_{\theta\theta} = 1$

in this parameterisa

$g_{\phi\phi} = \sin^2\theta$

$\left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\}$ ← Christoffel symbol

S^2



du^2 $d\sigma$

consolve : $\frac{d}{du}\left(\frac{d\sigma}{du}\right)$

$\Rightarrow$ can always re-parame

$d\theta^2 + \sin^2\theta\, d\phi^2$

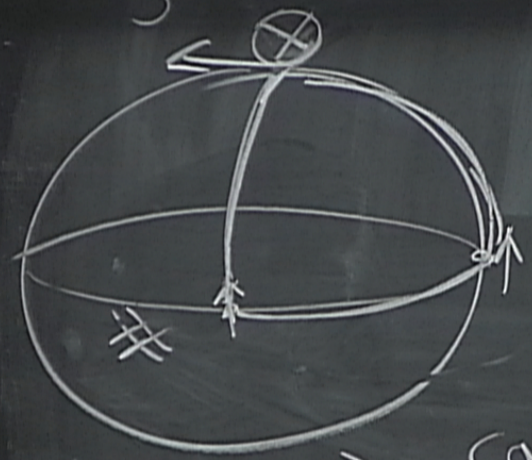
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S^2



du^2 $d\sigma$

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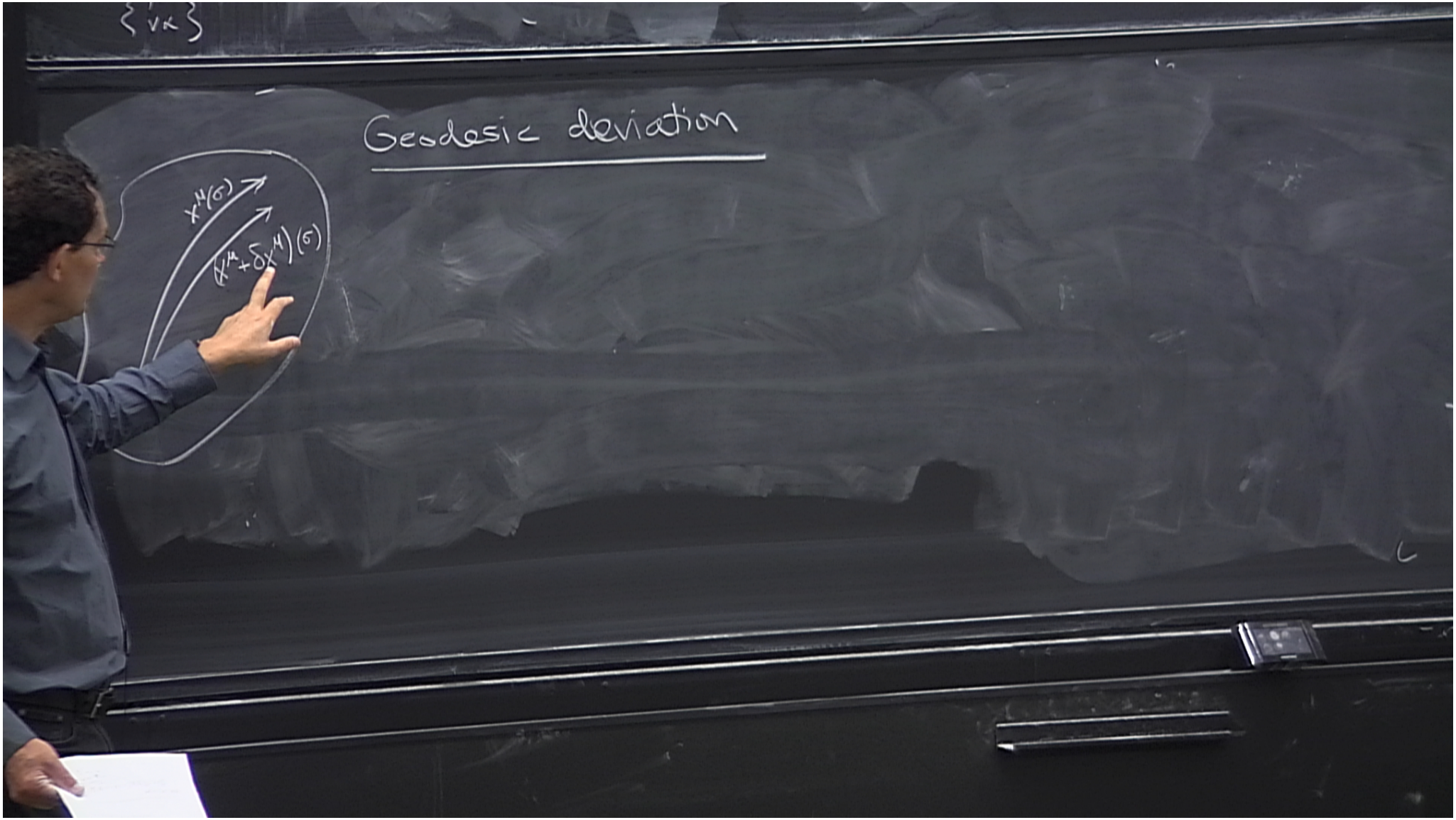
$d\theta^2 + \sin^2\theta\, d\phi^2$

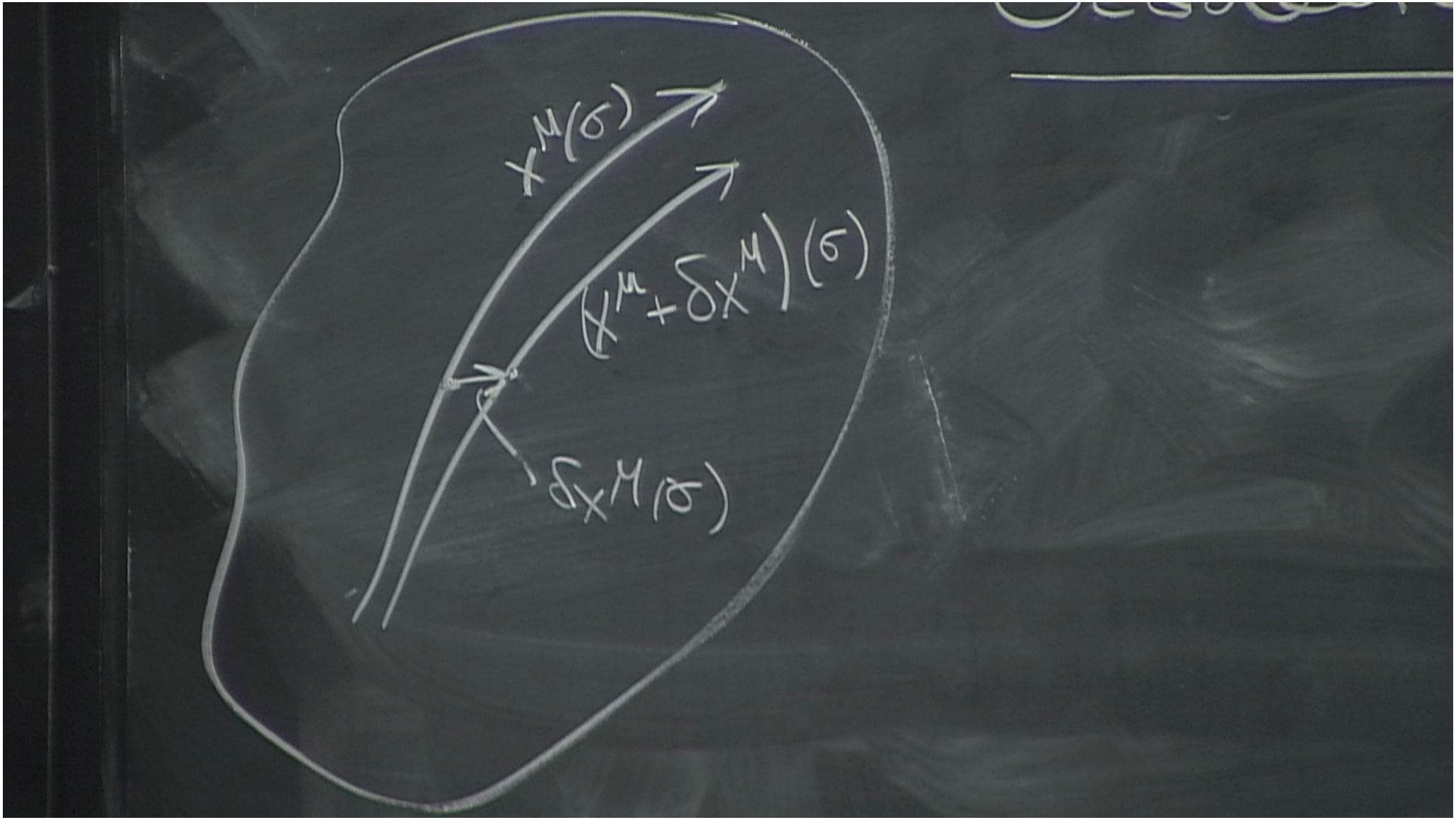
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in this parameterisa

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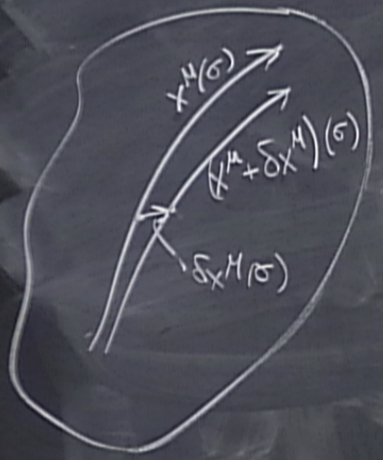
$\left\{ \begin{matrix} \mu \\ \nu\alpha \end{matrix} \right\}$ ← Christoffel symbol


Geodesic deviation


$x^\mu(\sigma)$
$(x^\mu + \delta x^\mu)(\sigma)$
$\delta x^\mu(\sigma)$

$\{v_\kappa\}$

Geodesic deviation

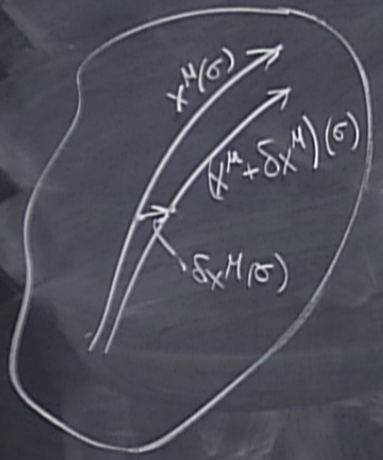


$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

$$\frac{d^2}{d\sigma^2}\left(x^\mu + \delta x^\mu\right)$$

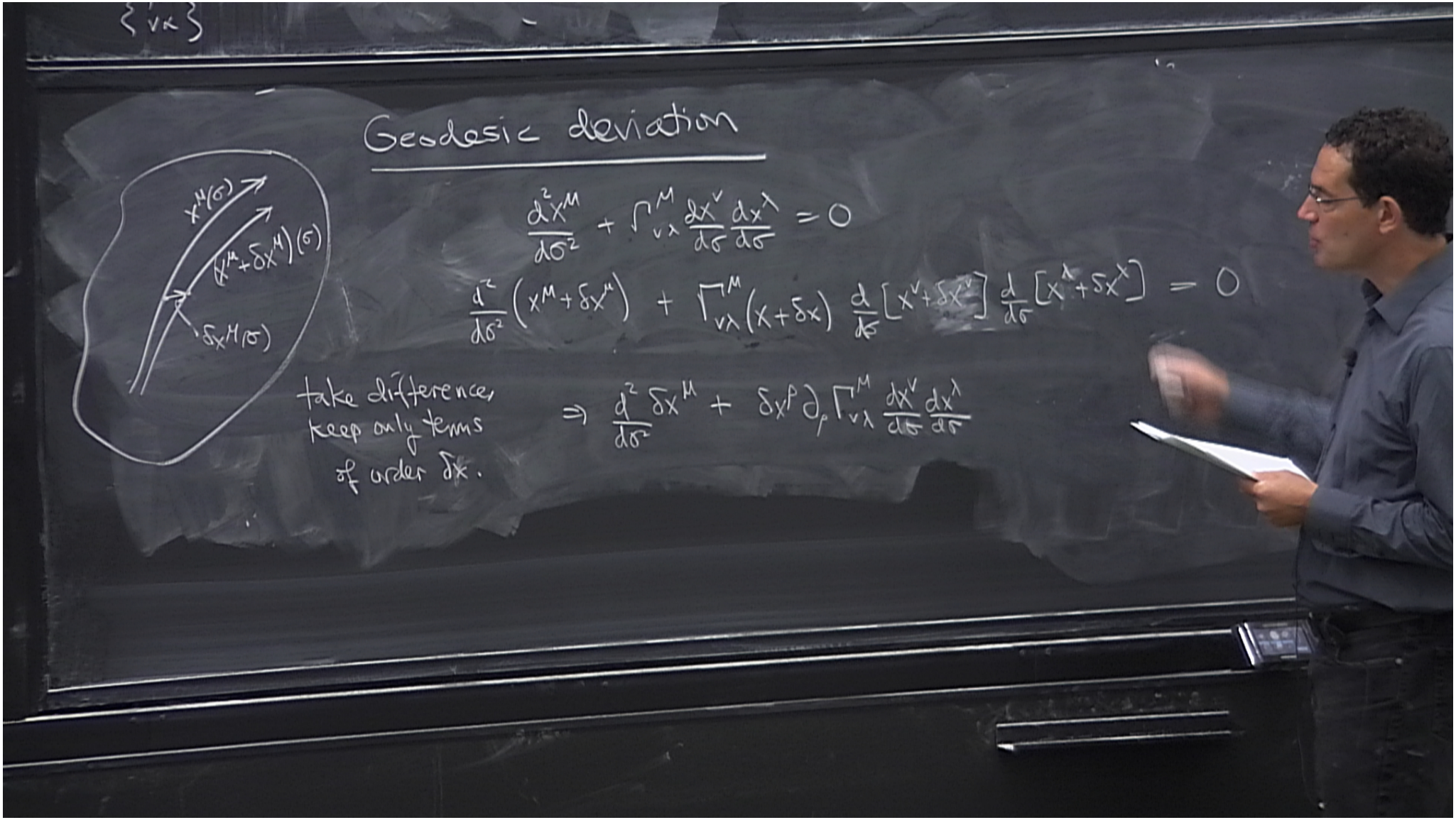
$\{v_\kappa\}$

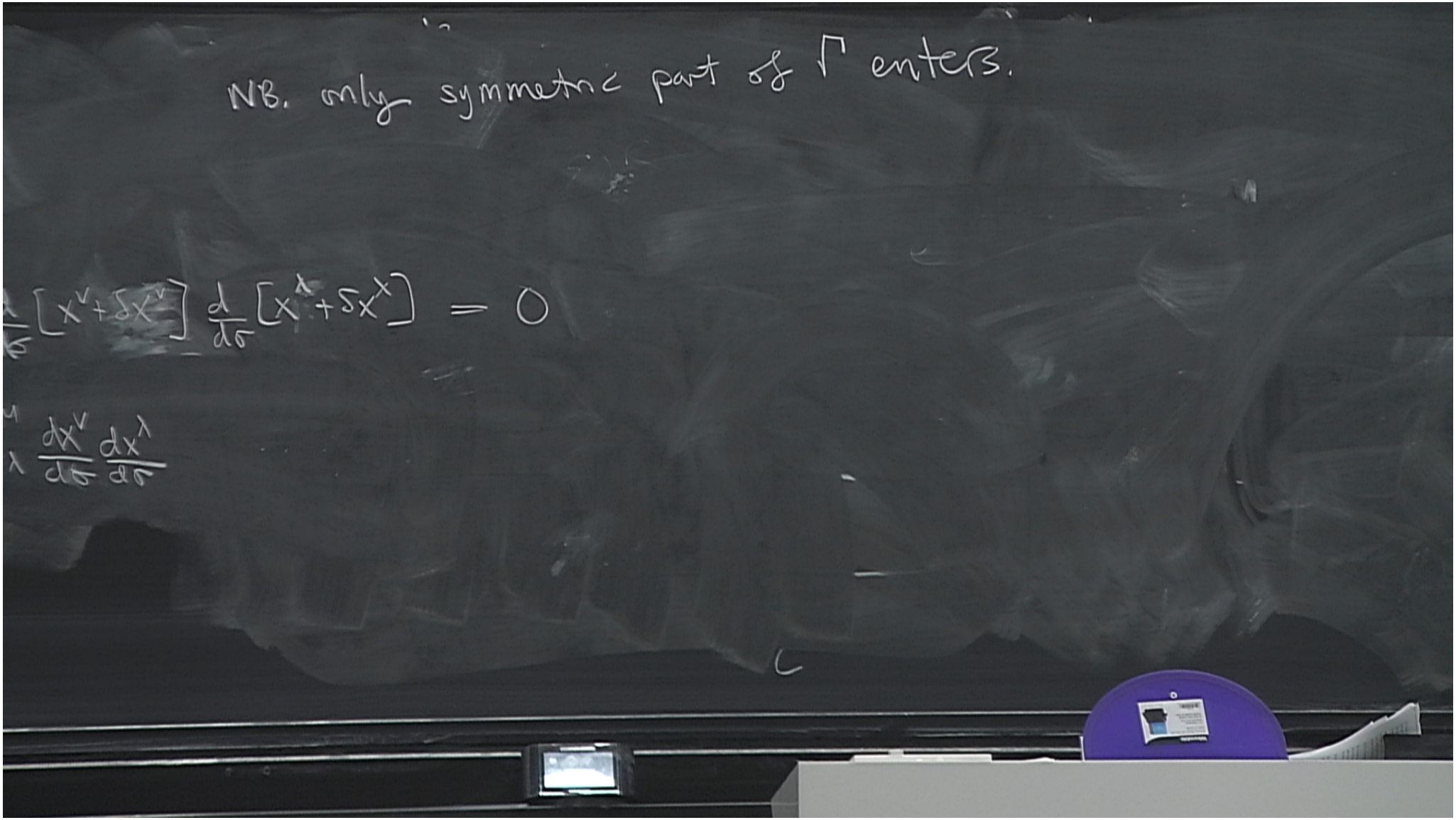
Geodesic deviation



$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

$$\frac{d^2}{d\sigma^2}\left(x^\mu+\delta x^\mu\right) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\,\frac{d}{d\sigma}\left[x^\nu+\delta x^\nu\right]\frac{d}{d\sigma}\left[x^\lambda+\delta x^\lambda\right]$$


{ νκ }
Geodesic deviation
$x^\mu(\sigma)$
$(x^\mu+\delta x^\mu)(\sigma)$
$\delta x^\mu(\sigma)$
$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$
$\frac{d^2}{d\sigma^2}(x^\mu+\delta x^\mu) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\frac{d}{d\sigma}[x^\nu+\delta x^\nu]\frac{d}{d\sigma}[x^\lambda+\delta x^\lambda] = 0$
take difference, keep only terms of order δx.
$\Rightarrow \frac{d^2}{d\sigma^2}\delta x^\mu + \delta x^\rho \partial_\rho \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}$


NB. only symmetric part of Γ enters.
$\frac{d}{d\sigma}[x^\nu + \delta x^\nu]\,\frac{d}{d\sigma}[x^\lambda + \delta x^\lambda] = 0$
$\lambda\ \frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}$

NB. only symmetric part of Γ enters.

(in all subsequent eq^{n}s. remember $\Gamma^{\alpha}_{\mu\lambda} \to \Gamma^{\alpha}_{(\mu\lambda)}$)

$$\frac{d}{d\sigma}\left[x^{\lambda} + \delta x^{\lambda}\right] = 0$$

$$+ 2\Gamma^{\mu}_{\nu\lambda}\frac{d\delta x^{\nu}}{d\sigma}\frac{dx^{\lambda}}{d\sigma} = 0$$

Geodesic deviation

NB. only symmetric Γ

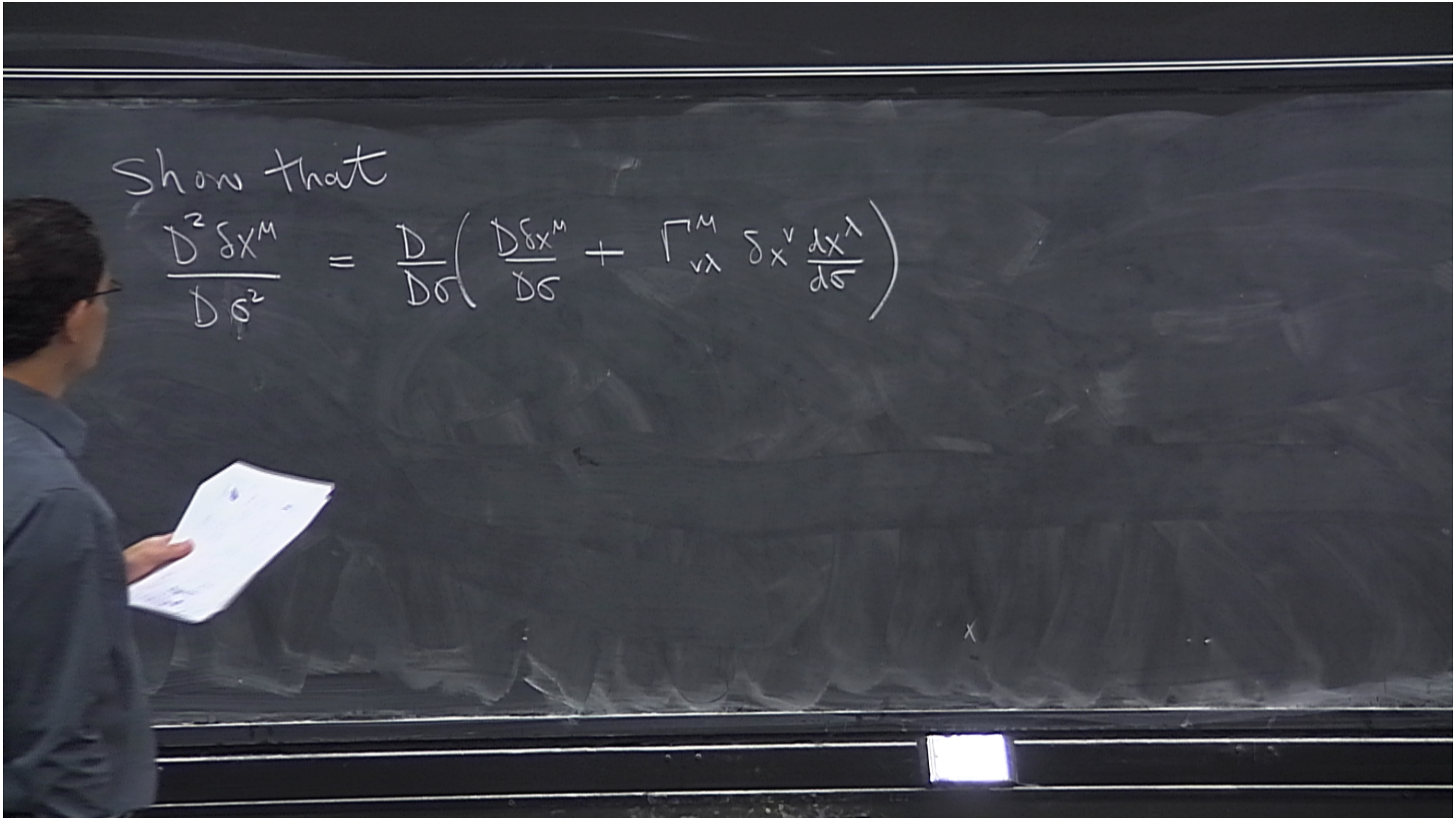
(in all subsequent eq^{n}s.

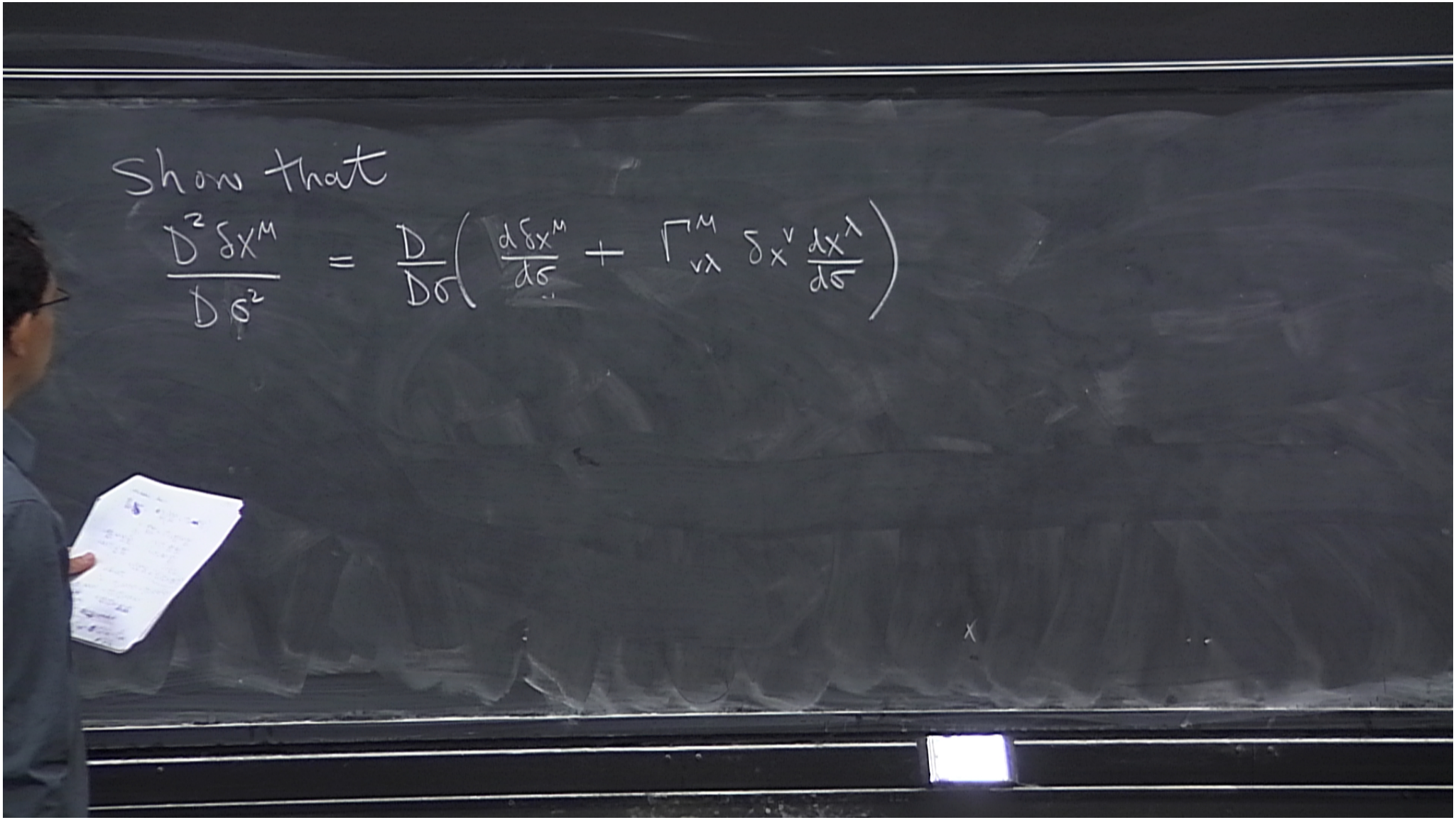
$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

$$\frac{d^2}{d\sigma^2}\left(x^\mu + \delta x^\mu\right) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\,\frac{d}{d\sigma}\left[x^\nu + \delta x^\nu\right]\frac{d}{d\sigma}\left[x^\lambda + \delta x^\lambda\right] = 0$$

e difference,
ep only terms
st order δx.

$$\Rightarrow \frac{d^2}{d\sigma^2}\delta x^\mu + \delta x^\rho \partial_\rho \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} + 2\Gamma^\mu_{\nu\lambda}\frac{d}{d\sigma}\delta x^\nu \frac{dx^\lambda}{d\sigma} = 0$$


Show that
$\frac{D^2 \delta x^\mu}{D\sigma^2} = \frac{D}{D\sigma}\left(\frac{D\delta x^\mu}{D\sigma} + \Gamma^\mu_{\nu\lambda}\, \delta x^\nu \frac{dx^\lambda}{d\sigma} \right)$


Show that
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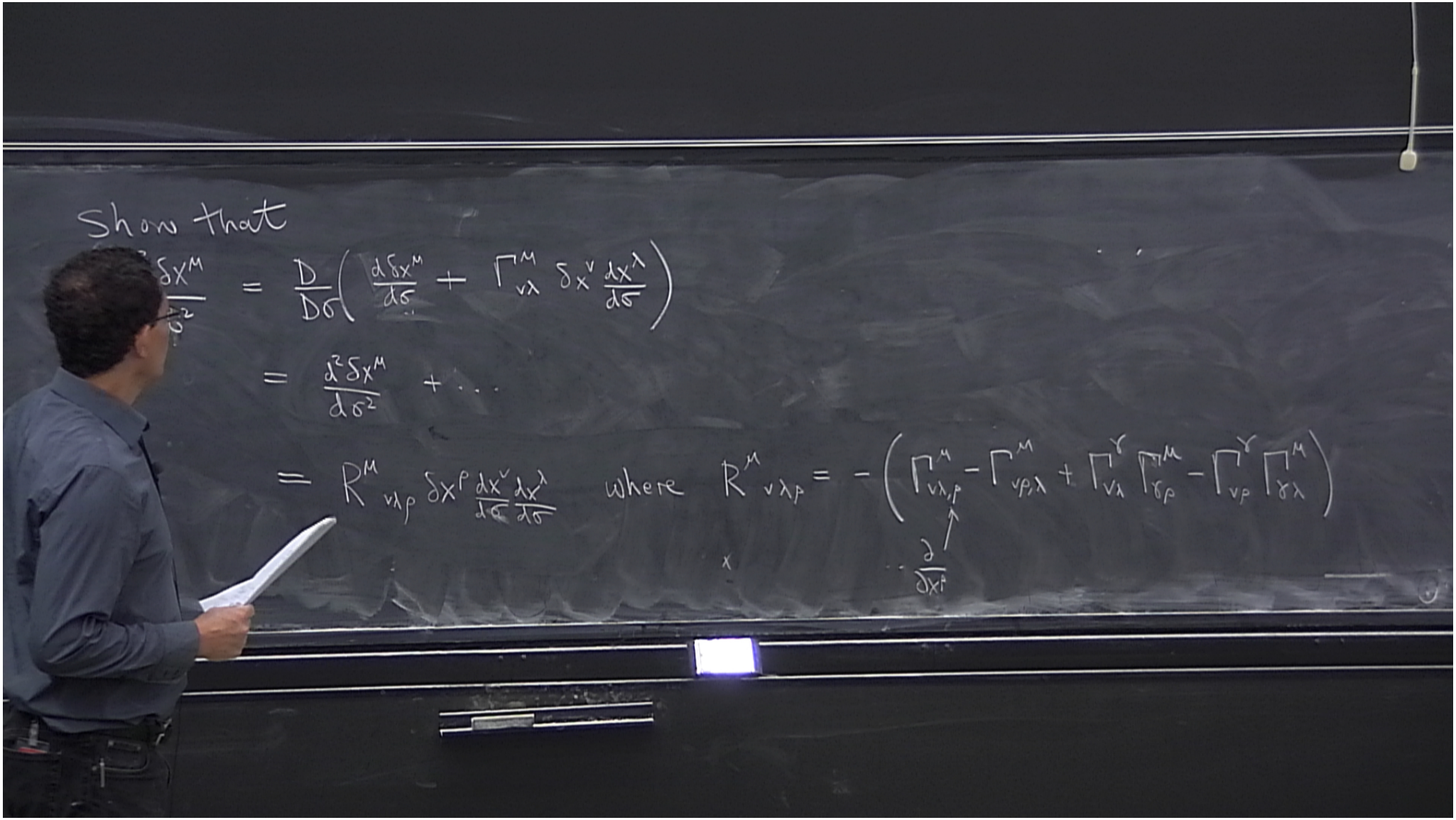
$$= \frac{d^2 \delta x^\mu}{d\sigma^2} + \dots$$

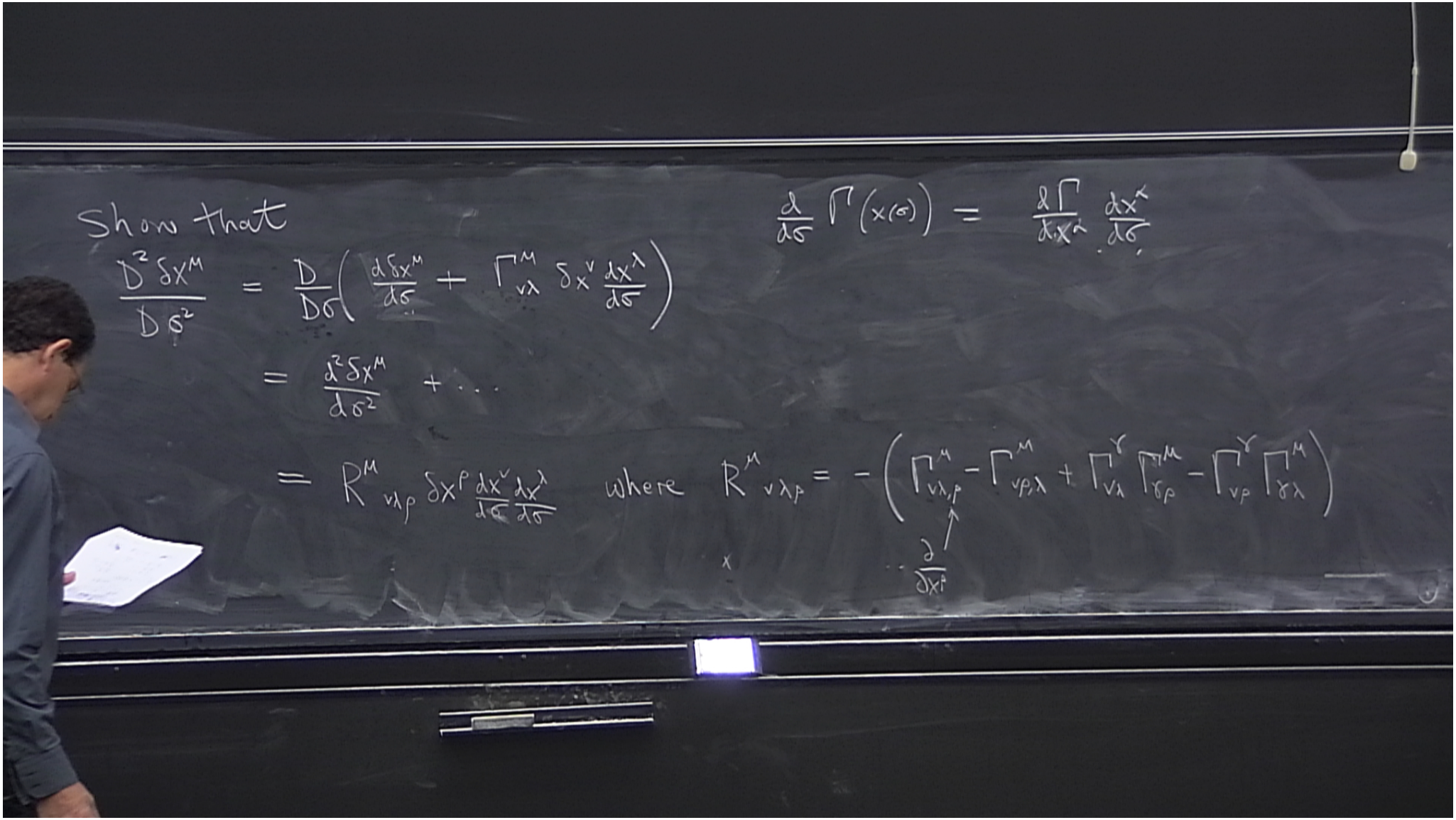
Show that

$$\frac{D^2 \delta x^\mu}{D\sigma^2} = \frac{D}{D\sigma}\left(\frac{d\delta x^\mu}{d\sigma} + \Gamma^\mu_{\nu\lambda}\, \delta x^\nu \frac{dx^\lambda}{d\sigma} \right)$$

$$= \frac{d^2 \delta x^\mu}{d\sigma^2} + \dots$$

$$= R^\mu{}_{\nu\lambda\rho}\, \delta x^\rho \frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}$$


Show that
$\frac{\delta x^\mu}{d\sigma^2} = \frac{D}{D\sigma}\left(\frac{d\delta x^\mu}{d\sigma} + \Gamma^\mu_{\nu\lambda}\,\delta x^\nu \frac{dx^\lambda}{d\sigma}\right)$
$= \frac{d^2\delta x^\mu}{d\sigma^2} + \dots$
$= R^\mu{}_{\nu\lambda\rho}\,\delta x^\rho \frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}$ where $R^\mu{}_{\nu\lambda\rho} = -\left(\Gamma^\mu_{\nu\lambda,\rho} - \Gamma^\mu_{\nu\rho,\lambda} + \Gamma^\gamma_{\nu\lambda}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\lambda}\right)$
$\frac{\partial}{\partial x^\rho}$


Show that
$\frac{D^2 \delta x^\mu}{D\sigma^2} = \frac{D}{D\sigma}\left(\frac{d\delta x^\mu}{d\sigma} + \Gamma^\mu_{\nu\lambda}\, \delta x^\nu \frac{dx^\lambda}{d\sigma} \right)$
$= \frac{d^2 \delta x^\mu}{d\sigma^2} + \dots$
$= R^\mu{}_{\nu\lambda\rho}\, \delta x^\rho \frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}$ where $R^\mu{}_{\nu\lambda\rho} = -\left(\Gamma^\mu_{\nu\lambda,\rho} - \Gamma^\mu_{\nu\rho,\lambda} + \Gamma^\gamma_{\nu\lambda}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\lambda} \right)$
$\frac{\partial}{\partial x^\rho}$
$\frac{d}{d\sigma}\Gamma(x(\sigma)) = \frac{\partial \Gamma}{\partial x^\lambda}\frac{dx^\lambda}{d\sigma}$

$$\frac{d}{d\sigma}\Gamma(x(\sigma)) = \frac{\partial\Gamma}{\partial x^\lambda}\frac{dx^\lambda}{d\sigma}$$

$$\left(\frac{dx^\lambda}{d\sigma}\right)$$

If Torsion is zero

then $\Gamma = \left\{ \right\}$

where $R^\mu{}_{\nu\lambda\rho} = -\left(\Gamma^\mu_{\nu\lambda,\rho} - \Gamma^\mu_{\nu\rho,\lambda} + \Gamma^\gamma_{\nu\lambda}\Gamma^\mu_{\gamma\rho} - \Gamma^\gamma_{\nu\rho}\Gamma^\mu_{\gamma\lambda}\right)$

($g_{\alpha\beta,\mu\nu}$ ↑ $\Gamma^\mu_{\nu\lambda,\rho}$; $\frac{\partial}{\partial x^\rho}$ ↑)

is the Riemann tensor.

is a tensor, describing the curvature.

$g_{\theta\theta} = \sin^2\theta$ ← Christoffel symbol $\left\{ {\mu \atop \nu\kappa} \right\}$

Geo

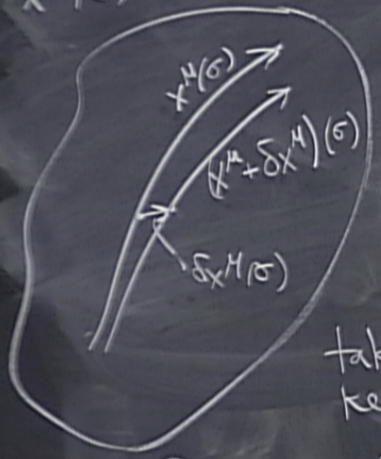
$x^\mu(\sigma)$, $(x^\mu+\delta x^\mu)(\sigma)$, $\delta x^\mu(\sigma)$

take differe
keep only ter
of order δ

$\{\nu\kappa\}$

$x^\mu(\sigma) \to x^{\bar\mu}(\sigma)$

Geodesic deviation



$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

NB. only symmetric part of Γ (in all subsequent eq^{n}s. rem

$$\frac{d^2}{d\sigma^2}\left(x^\mu+\delta x^\mu\right) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\,\frac{d}{d\sigma}\left[x^\nu+\delta x^\nu\right]\frac{d}{d\sigma}\left[x^\lambda+\delta x^\lambda\right] = 0$$

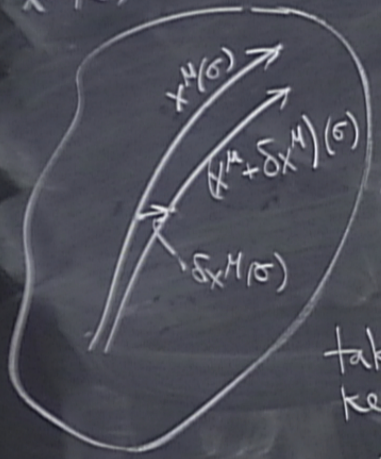
take difference, keep only terms of order δx.

$$\Rightarrow \frac{d^2\delta x^\mu}{d\sigma^2} + \delta x^\rho\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} + 2\Gamma^\mu_{\nu\lambda}\frac{d\delta x^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

$\{\nu\kappa\}$

$x^\mu(\sigma) \to x'^\mu(\sigma)$

Geodesic deviation



$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

NB. only symmetric part of Γ (in all subsequent eq^{n}s. rem

$$\frac{d^2}{d\sigma^2}\left(x^\mu+\delta x^\mu\right) + \Gamma^\mu_{\nu\lambda}(x+\delta x)\,\frac{d}{d\sigma}\left[x^\nu+\delta x^\nu\right]\frac{d}{d\sigma}\left[x^\lambda+\delta x^\lambda\right] = 0$$

take difference, keep only terms of order δx.

$$\Rightarrow \frac{d^2\delta x^\mu}{d\sigma^2} + \delta x^\rho\,\partial_\rho\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} + 2\Gamma^\mu_{\nu\lambda}\frac{d\delta x^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

$$\delta x'^\mu(\sigma) = \frac{\partial x'^\mu}{\partial x^\alpha}\delta x^\alpha$$

$$x'^\mu(\sigma) = x'^\mu\left(x^\alpha(\sigma)\right)$$