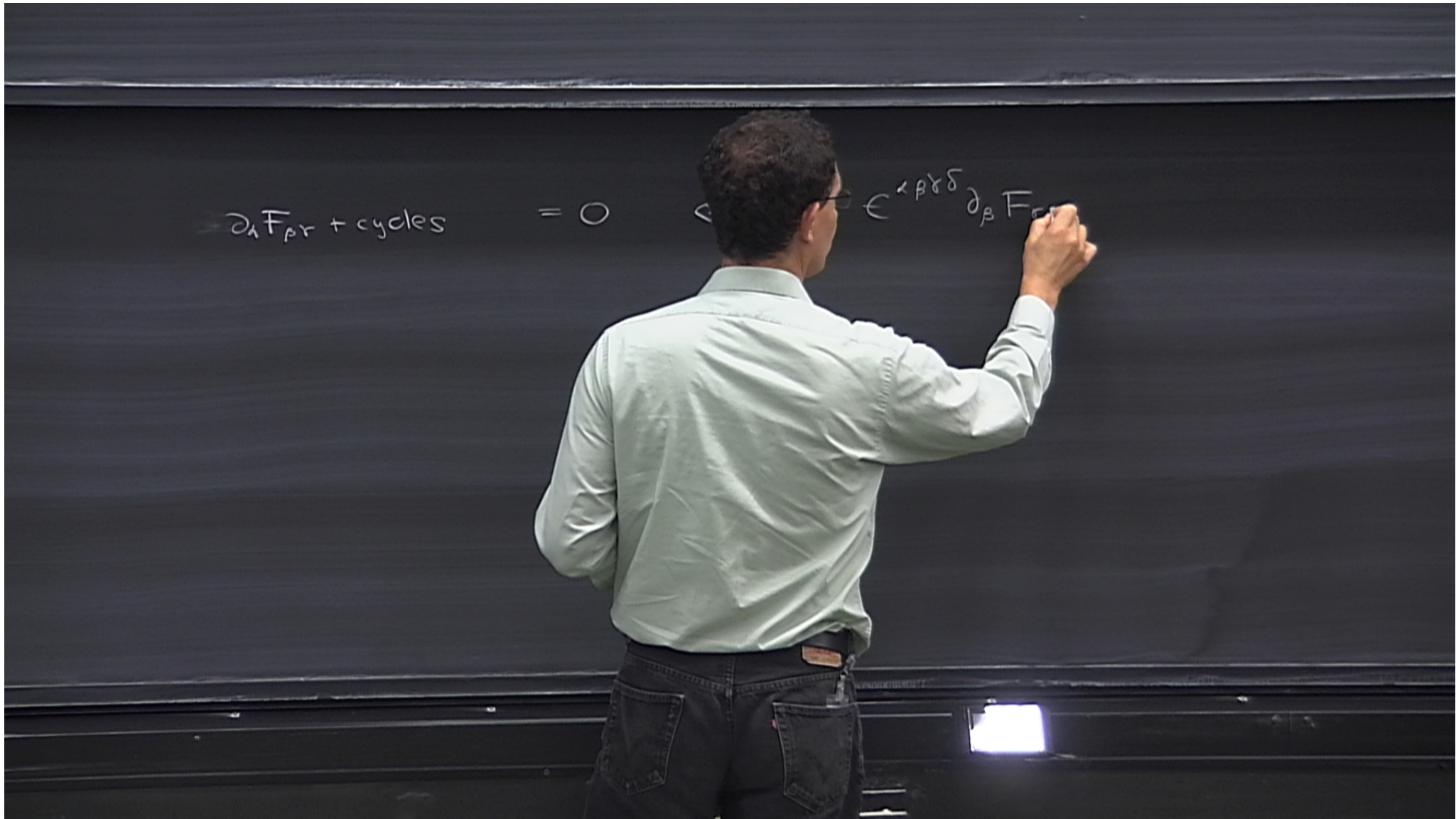


Title: Relativity - Lecture 3

Date: Sep 14, 2011 10:30 AM

URL: <http://pirsa.pi.local/11090026>

Abstract:



$\partial_\alpha F_{\beta\gamma} + \text{cycles}$

$= 0$

$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta}$

$$\partial_\alpha F_{\beta\gamma} + \text{cycles} = 0 \quad \Leftrightarrow \quad \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$$T_{(\mu\nu)} = \frac{1}{2} (T_{\mu\nu} + T_{\nu\mu})$$

$$T_{[\mu\nu]} = \frac{1}{2} (T_{\mu\nu} - T_{\nu\mu})$$

$$T_{(\mu\nu\rho)} = \frac{1}{3!} (T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu} + T_{\mu\rho\nu} + T_{\nu\mu\rho} + T_{\rho\nu\mu})$$

$$\sigma_{\rho\sigma}^{MV} = \begin{vmatrix} \delta_{\rho}^M & \delta_{\rho}^V \\ \delta_{\sigma}^M & \delta_{\sigma}^V \end{vmatrix} = \delta_{\rho}^M \delta_{\sigma}^V - \delta_{\sigma}^M \delta_{\rho}^V$$

$$\delta_{\rho\sigma}^{\mu\nu} = \begin{vmatrix} \delta_{\rho}^{\mu} & \delta_{\rho}^{\nu} \\ \delta_{\sigma}^{\mu} & \delta_{\sigma}^{\nu} \end{vmatrix} = \delta_{\rho}^{\mu} \delta_{\sigma}^{\nu} - \delta_{\sigma}^{\mu} \delta_{\rho}^{\nu}$$

$$\delta_{\rho\sigma\tau}^{\mu\nu\lambda} = \begin{vmatrix} \delta_{\rho}^{\mu} & \delta_{\rho}^{\nu} & \delta_{\rho}^{\lambda} \\ \delta_{\sigma}^{\mu} & \delta_{\sigma}^{\nu} & \delta_{\sigma}^{\lambda} \\ \delta_{\tau}^{\mu} & \delta_{\tau}^{\nu} & \delta_{\tau}^{\lambda} \end{vmatrix}$$

$$A_{[\mu\nu]} = \frac{1}{2} \delta^{\rho\sigma}_{\mu\nu} A_{\rho\sigma}$$

$$A_{[\mu\nu\lambda]} = \frac{1}{3!} \delta^{\rho\sigma\tau}_{\mu\nu\lambda} A_{\rho\sigma\tau}$$

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$$A_{[\mu\nu\lambda]} = \frac{1}{3!} \delta^{\rho\sigma\tau}_{\mu\nu\lambda} A_{\rho\sigma\tau}$$

In 4d

$$\epsilon^{\mu\nu\rho\lambda} \epsilon_{\mu\nu\rho\lambda} = -4!$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\lambda\nu\rho\sigma}$$

$$A_{[\mu\nu]} = \frac{1}{2} \delta^{\rho\sigma}_{\mu\nu} A_{\rho\sigma}$$

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In 4d

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$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\lambda\nu\rho\sigma} = -3! \delta^{\mu}_{\lambda}$$

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In 4d

$$\epsilon^{\mu\nu\rho\lambda} \epsilon_{\mu\nu\rho\lambda} = -4!$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\nu\rho\sigma} = -3! \delta^{\mu}_{\alpha}$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\rho\sigma} = -2! \delta^{\mu\nu}_{\alpha\beta}$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\sigma} = -\delta^{\mu\nu\rho}_{\alpha\beta\gamma}$$

$$A_{[\mu\nu]} = \frac{1}{2} \delta^{\rho\sigma}_{\mu\nu} A_{\rho\sigma}$$

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$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\nu\rho\sigma} = -3! \delta_{\alpha}^{\mu}$$

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$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} = \delta_{\alpha\beta\gamma\delta}^{\mu\nu\rho\sigma}$$

$$d\tau^2 = c^2 dt^2 - dx^2 \quad d\tau = c dt \underbrace{\sqrt{1 - v^2/c^2}}_{\gamma}$$

$$v = \frac{dx}{dt}$$

$$\begin{aligned} \frac{dx^\mu}{d\tau} &= \left(\frac{cdt}{d\tau}, \frac{dx}{d\tau} \right) \\ &= \left(\gamma, \underbrace{\frac{dx}{dt} \frac{dt}{d\tau}}_{\gamma v/c}, 0, 0 \right) \end{aligned}$$

$$u_{\text{obs}}^{\mu} = (\gamma, -\gamma \frac{v}{c})$$

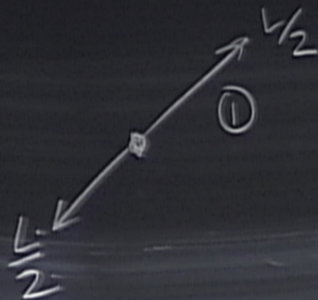
$$\eta_{\mu\nu} u^{\mu} u^{\nu} = -1$$

$$p_{\text{①}}^{\mu} = (p^0, \underline{p})$$

↑
Energy/c

$$p^{\mu} p^{\nu} \eta_{\mu\nu} = 0 = p^0{}^2 - |\underline{p}|^2$$

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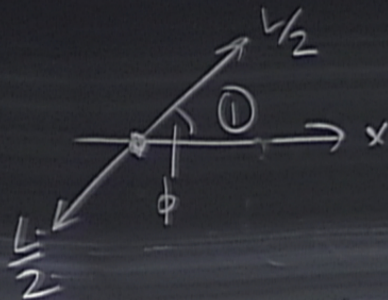
$$= \left(\gamma, \underbrace{\frac{dx}{dt} \frac{dt}{d\tau}}_{\gamma \frac{v}{c}}, 0, 0 \right)$$

$$p_{\text{1}}^\mu = (p^0, \underline{p})$$

Energy/c

$$= \frac{L}{2c} (1,$$

$$p^\mu p^\nu \eta_{\mu\nu} = 0 = p^0^2 - |\underline{p}|^2$$



$$d\tau^2 = c^2 dt^2 - dx^2 \quad d\tau = \dots$$

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$$p_{\textcircled{1}}^{\mu} = \begin{pmatrix} p^0 \\ \underline{p} \end{pmatrix}$$

↑
Energy/c

$$p^{\mu} p^{\nu} \eta_{\mu\nu} = 0 = p^0{}^2 - |\underline{p}|^2$$

$$= \frac{L}{2c} (1, \cos\phi, \sin\phi, 0)$$

$$E_{\text{of photon ①}} = - \frac{p_0^M}{c} u_M$$

in rest frame
of obs

$$u^M = \frac{dx^M}{d\tau} = (1, 0)$$

$$= - \frac{p_0^M}{c} \cdot (-1, 0)$$

in rest
frame
of obs

$\frac{L}{2}$

p_0^M
①

$$E_{\text{of photon 1}} = -\frac{p_0^M}{c} u_M$$

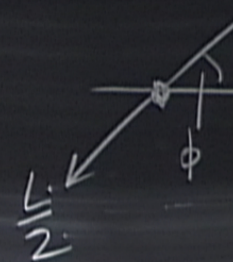
in rest frame of obs

$$= -\frac{p_0^M}{c} \cdot (-1, 0)$$

in rest frame of obs

$$u^M = \frac{dx^M}{d\tau} = (1, 0)$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



$$p_0^M =$$

$$E_{\text{of photon 1}} = -p_0^M u_M$$

in rest frame
of obs

in rest
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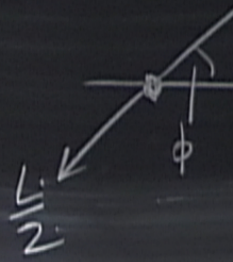
$$= -p_0^M (-1, 0)$$

$$= E$$

$$u^M = \frac{dx^M}{d\tau} = (1, 0)$$

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in rest frame of obs.

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$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= E$$

in rest frame of decaying particle,

$$= -\frac{L}{2c} (-\gamma$$

$$U_{\text{obs}}^M = \left(\gamma, \gamma \frac{v}{c} \right)$$

$$= \begin{pmatrix} \gamma \\ \gamma \frac{v}{c} \end{pmatrix}$$

$$P^M P_M = 0 = P^0^2 - \underline{P}^2$$

$$\frac{\text{Energy}}{c} = \frac{L}{2c} (1, \cos\phi, \sin\phi, 0)$$

$$E_{\text{of photon ①}} = -P_0^M U_M$$

in rest frame of obs.

$$U^M = \frac{dx^M}{d\tau} = (1, \underline{0})$$

in rest frame of obs.

$$= -P_0^M (-1, \underline{0})$$

$$= \underline{E}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

in rest frame of decaying particle,

$$= -\frac{L}{2c} (-\gamma - \gamma v \cos \phi)$$

$$= \underline{\underline{\gamma L (1 + v \cos \phi)}}$$

$$\begin{aligned} U_{\text{obs}}^M &= (\gamma, \gamma \underline{v}) \\ \eta_{\mu\nu} U^\mu U^\nu &= -1 \end{aligned}$$

$$= \begin{pmatrix} P_0^0 \\ \underline{P} \end{pmatrix} \quad P^\mu P^\nu \eta_{\mu\nu} = 0 = P_0^2 - \underline{P}^2$$

$$\begin{aligned} \text{Energy}/c &= \frac{L}{2c} (1, \cos \phi, \sin \phi, 0) \end{aligned}$$

Manifold

- natural generalisation of Minkowski (flat) spacetime

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- a topological space which is "locally M^n " dimension

Manifold

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- a topological space which is "locally" M^n

$\nwarrow$ dimension
 $\leftarrow$ Minkowski

$\mathbb{R}^n$
 $\leftarrow$ Euclidean space

Examples: $\mathbb{R}^n$

S^n : set of points in $\mathbb{R}^n$ obeying

$$x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 = 1$$

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Examples: $\mathbb{R}^n$

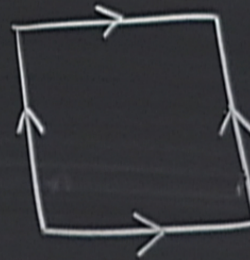
S^n : set of points in $\mathbb{R}^{n+1}$ obeying

$$x_1^2 + x_2^2 + x_3^2 + \dots + x_{n+1}^2 = 1$$

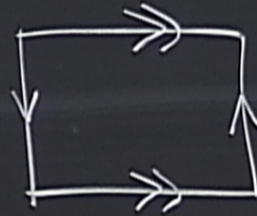


S^0

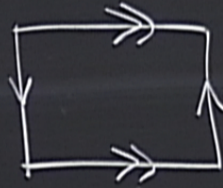
T^n, T^2



Klein bottle
(Wikipedia)



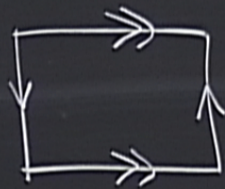
Klein bottle
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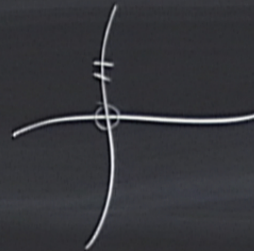
What is not a manifold?



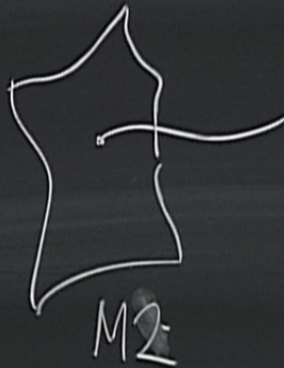
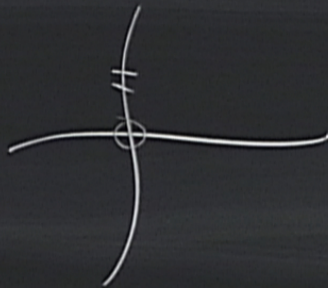
Klein bottle
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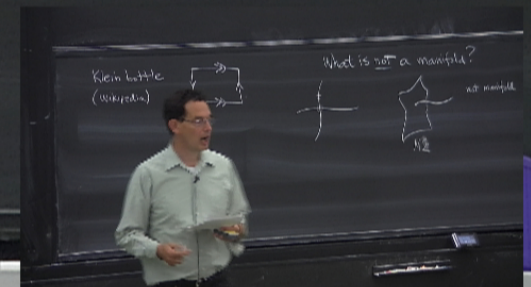
What is not a manifold?



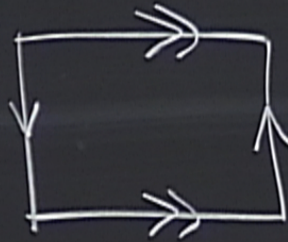
What is not a manifold?



not manifold.

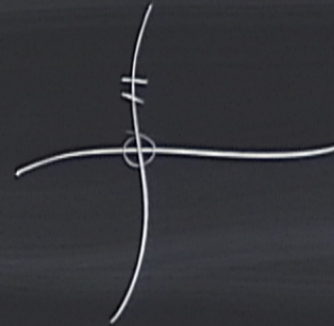


Klein bottle
(wikipedia)

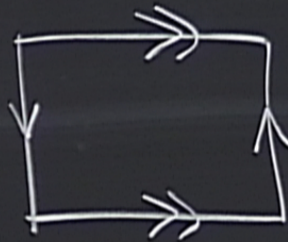


de Sitter spacetime

What is not a manifold



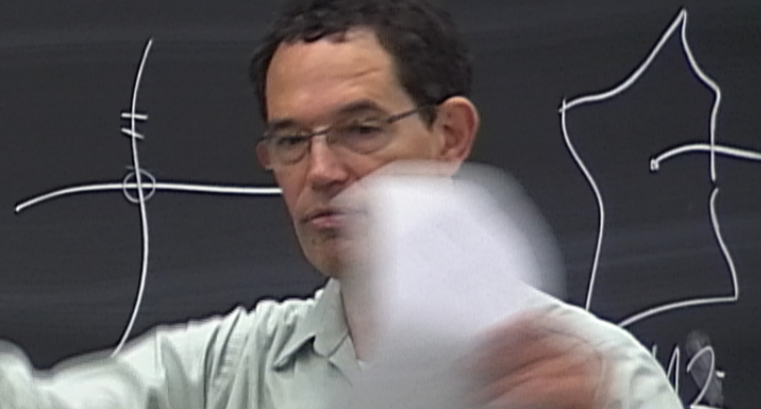
Klein bottle
(wikipedia)



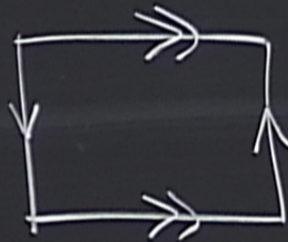
de Sitter spacetime

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \quad \subset \mathbb{M}$$

What is not a ma



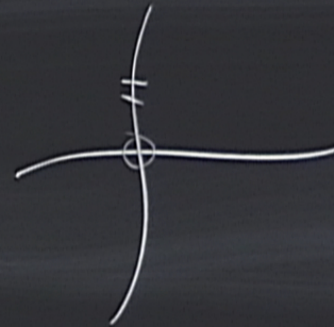
Klein bottle
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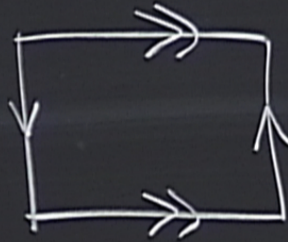
de Sitter spacetime

$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \subset M^5$$

What is not a manifold



Klein bottle
(wikipedia)



What is not a manifold



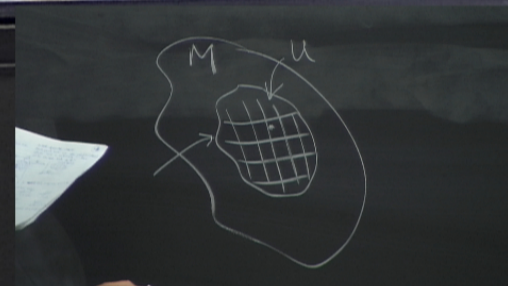
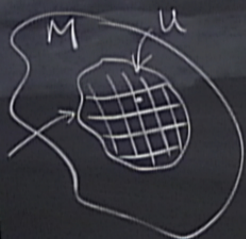
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$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1 \subset M^5$$

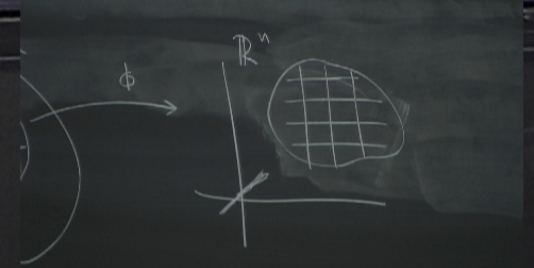
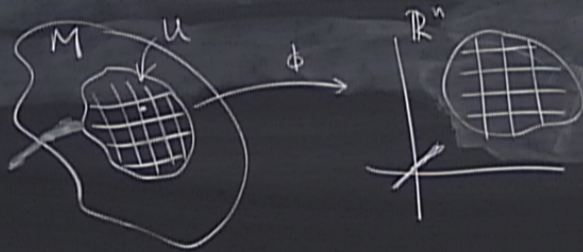
Ex: draw it, work out a nice set of 4 coordinates.

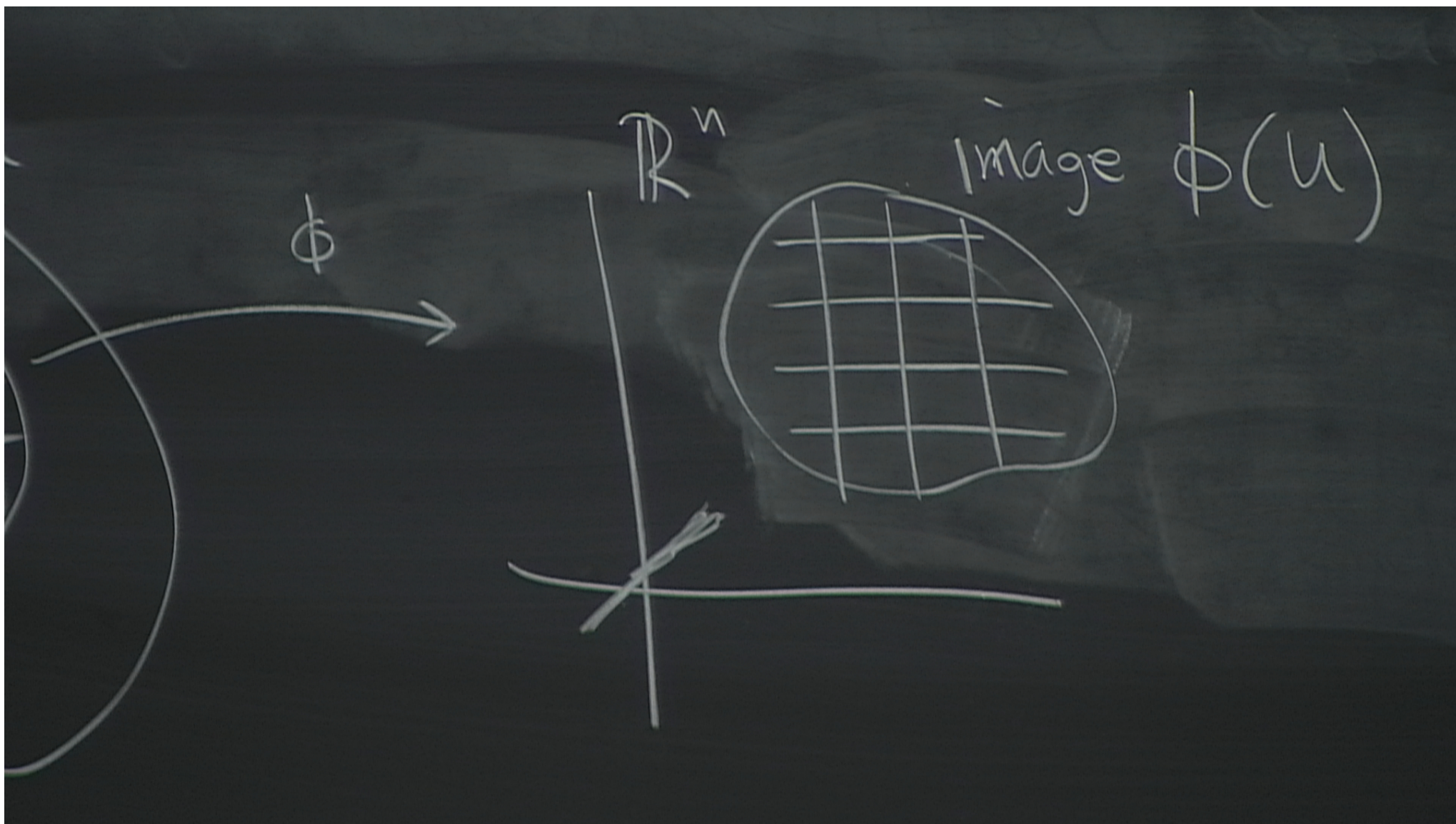
Definition: a manifold is a set of points, where every point has an open neighbourhood which has a continuous one-one map onto an open set of $\mathbb{R}^n$ (or M^n)

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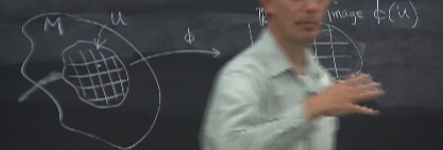


every point has an open neighbourhood
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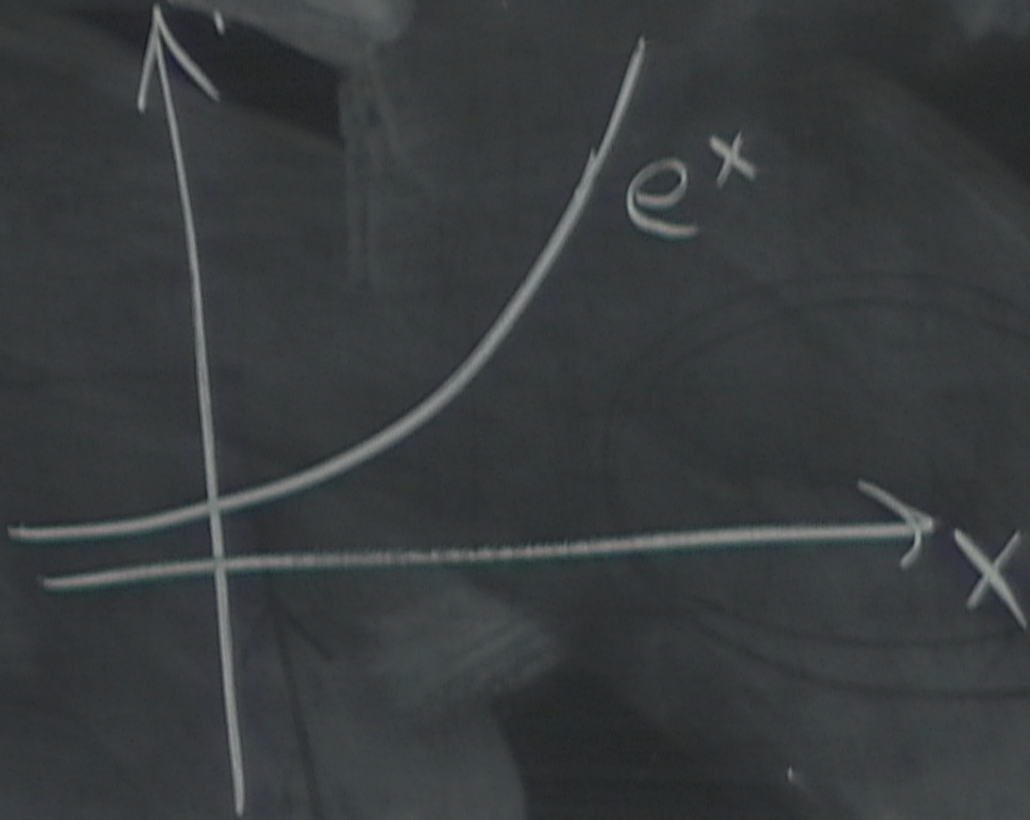
one-one : every point in $\phi(U)$ has only one point of U mapped
onto it.

onto : region is covered by map.

a manifold is a set of points, where every
which has a continuous map onto



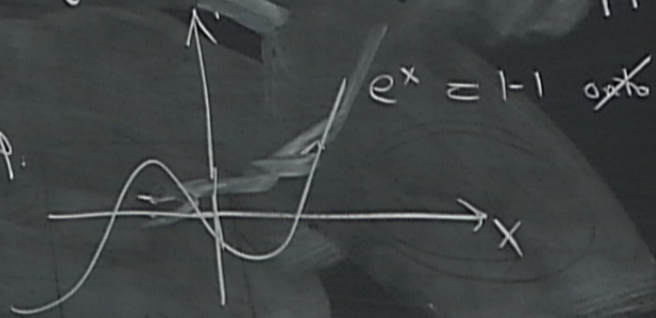
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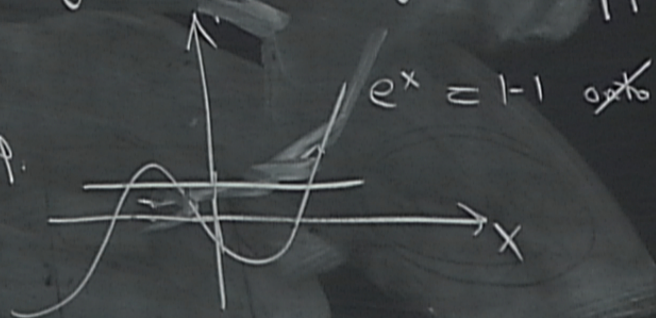
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map is invertible $\Rightarrow$ so $\exists \phi: M \rightarrow N$

$$\exists \phi^{-1}: N \rightarrow M$$



open neighbourhood:

idea: start from "open ball" in $\mathbb{R}^n$

set of x s.t. $|x - y| < r$



open neighbourhood:

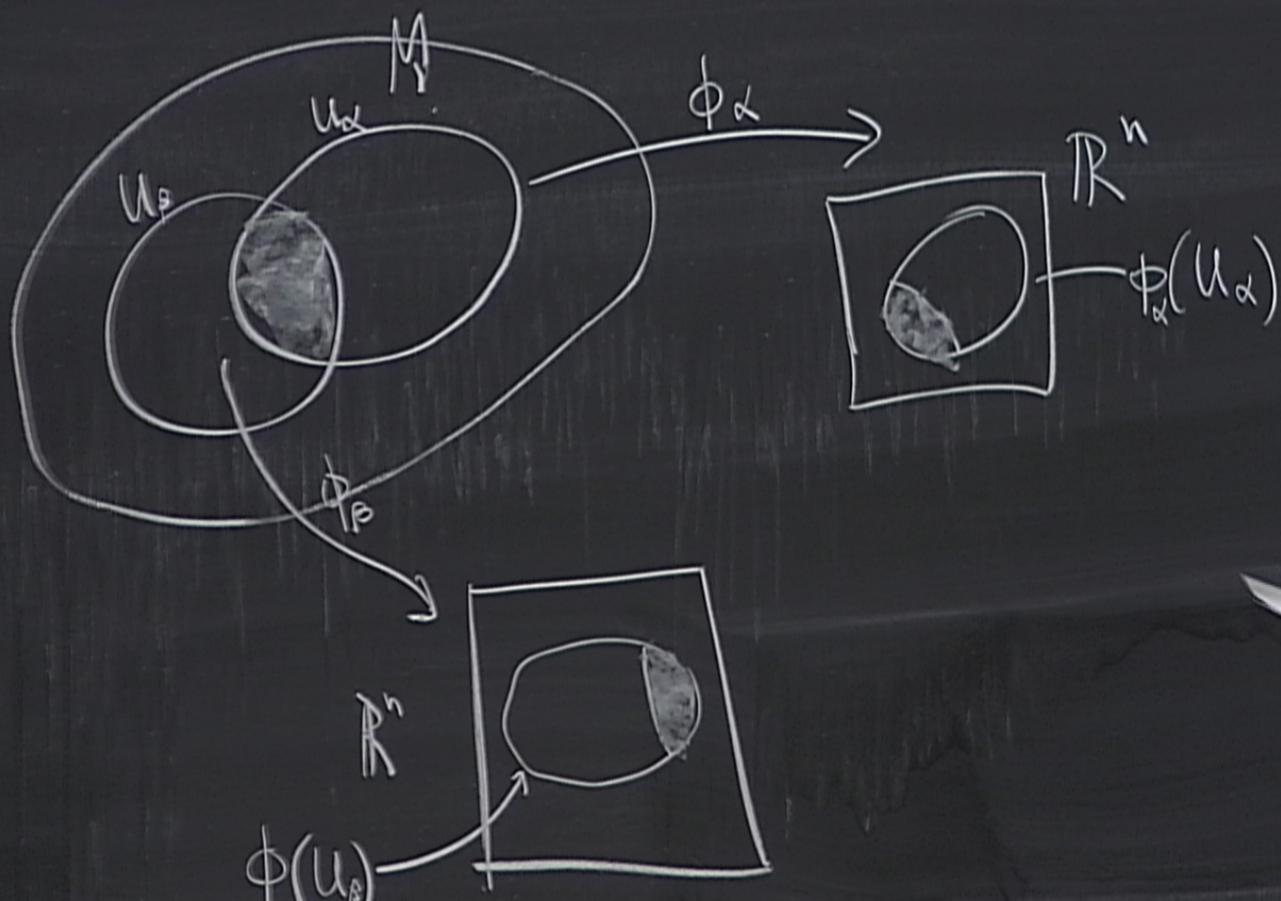
idea: start from "open ball" in $\mathbb{R}^n$

$$\text{set of } \underline{x} \text{ s.t. } |x - y| < r$$

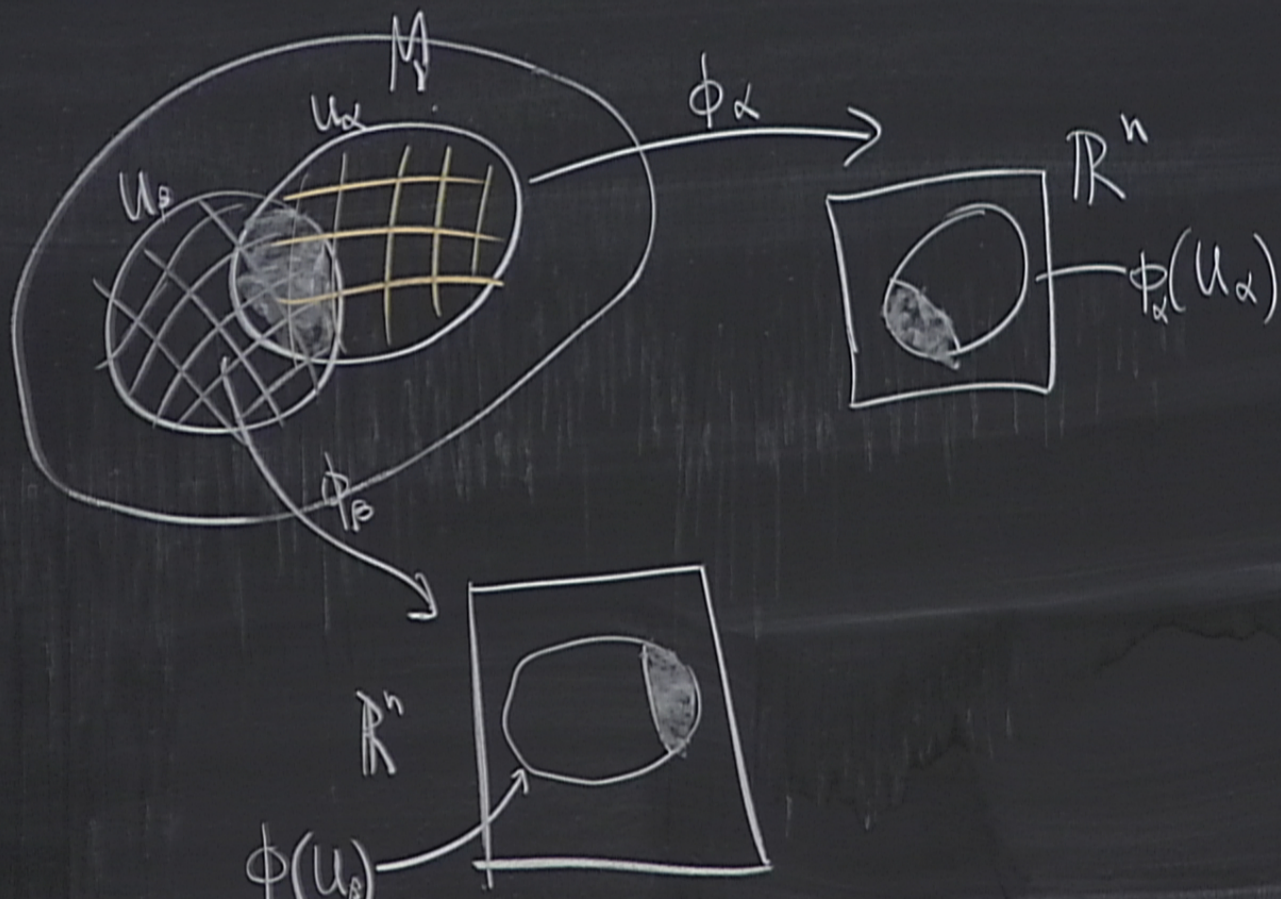
$\uparrow$



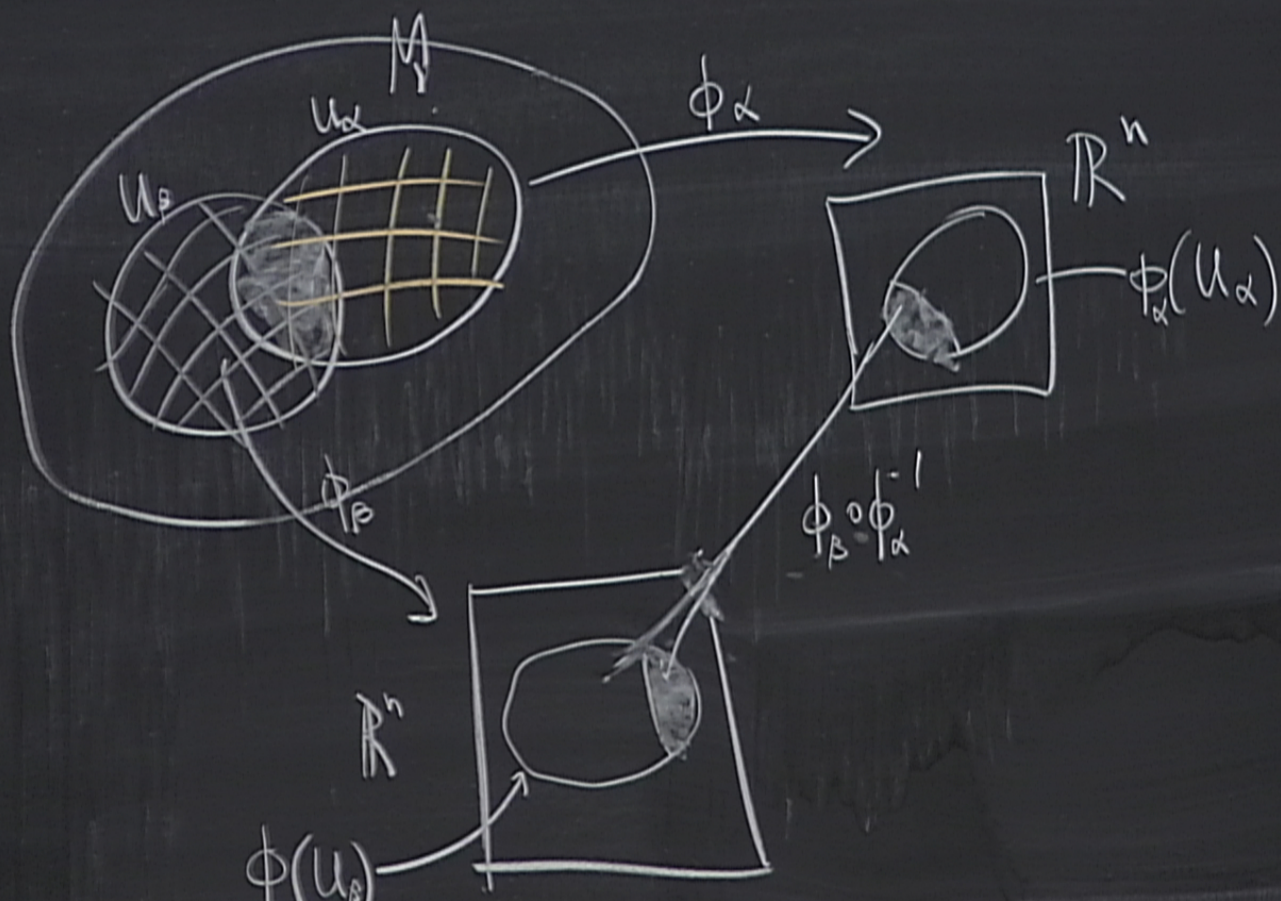
C^∞ atlas on M



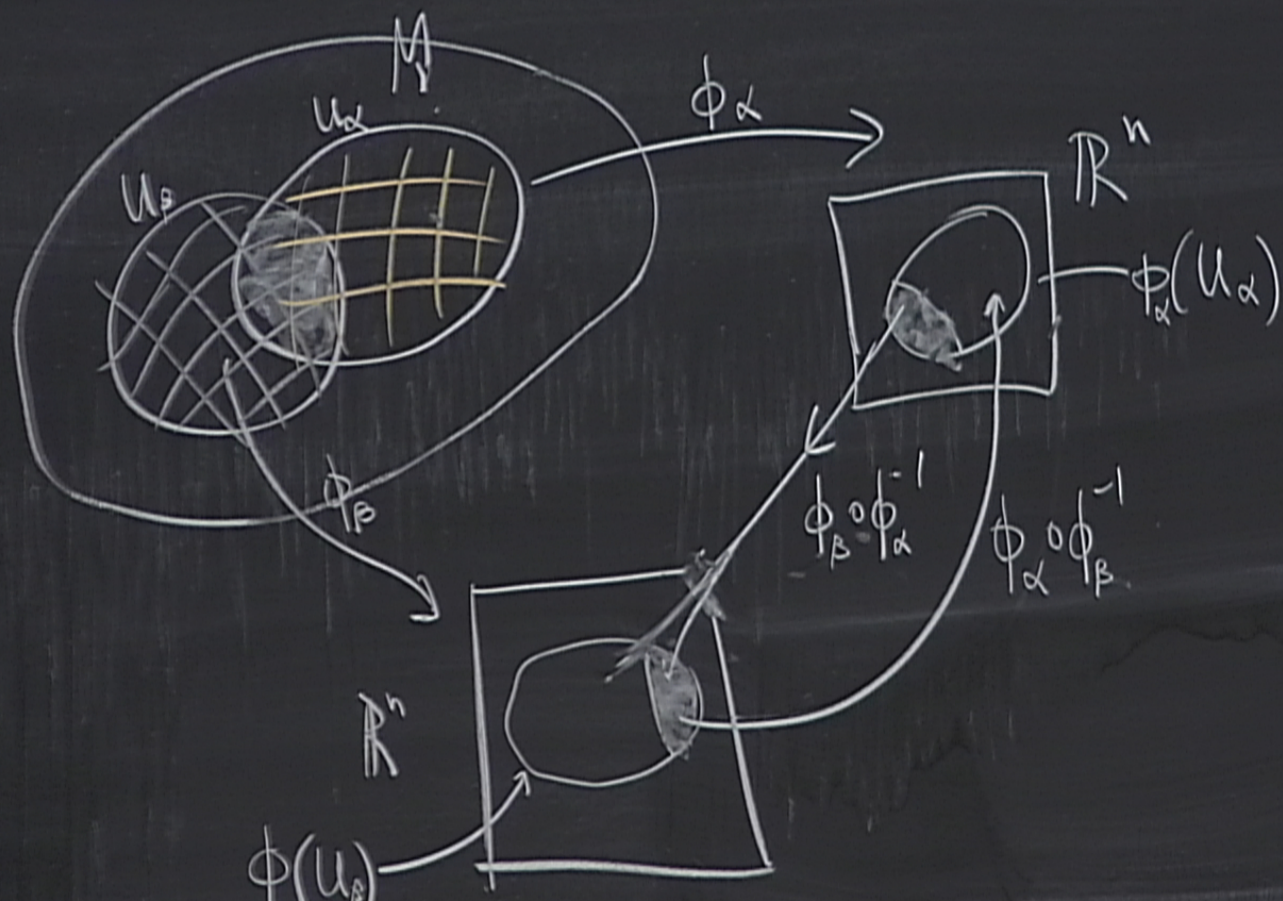
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C^∞ atlas on M

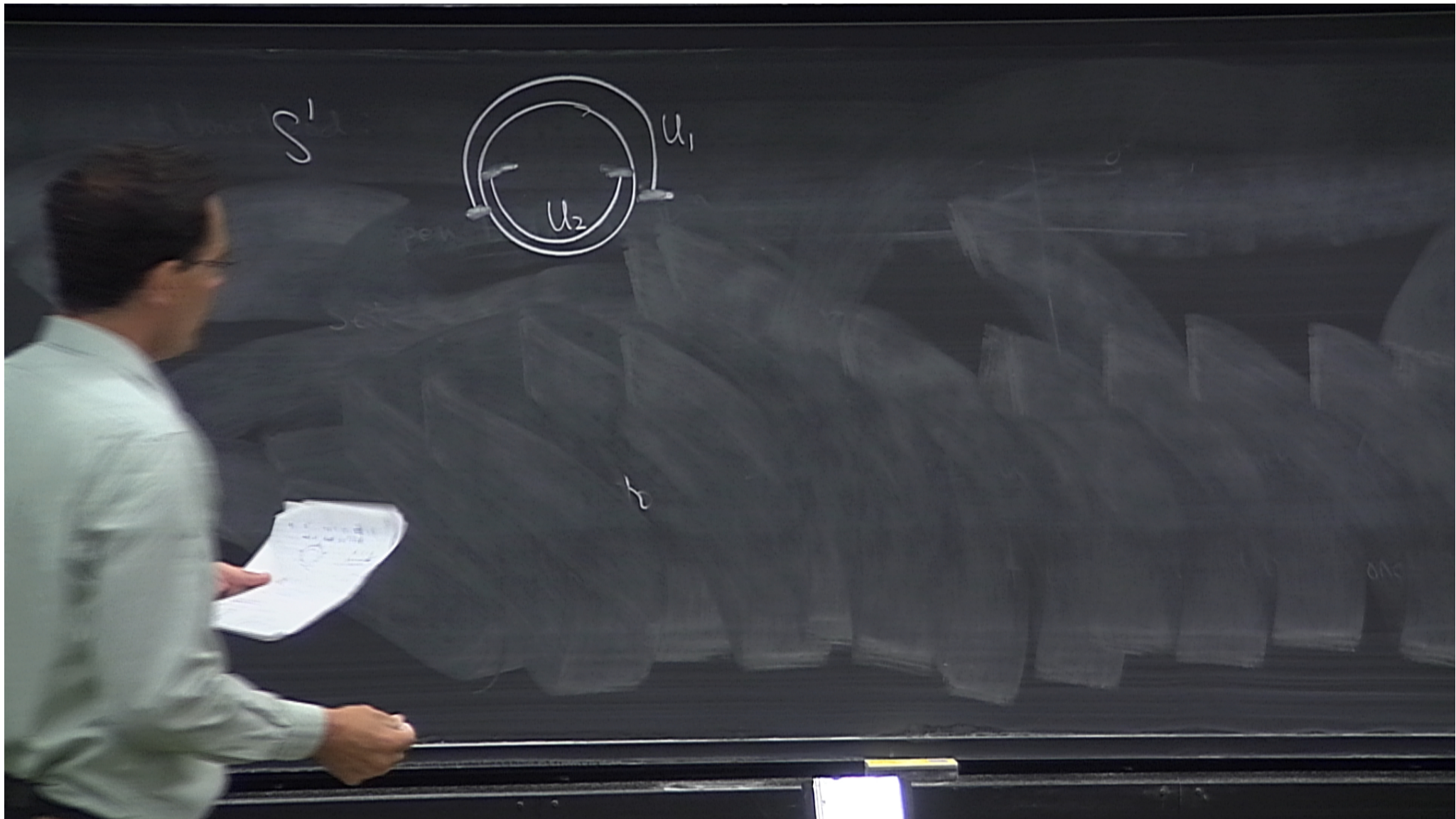


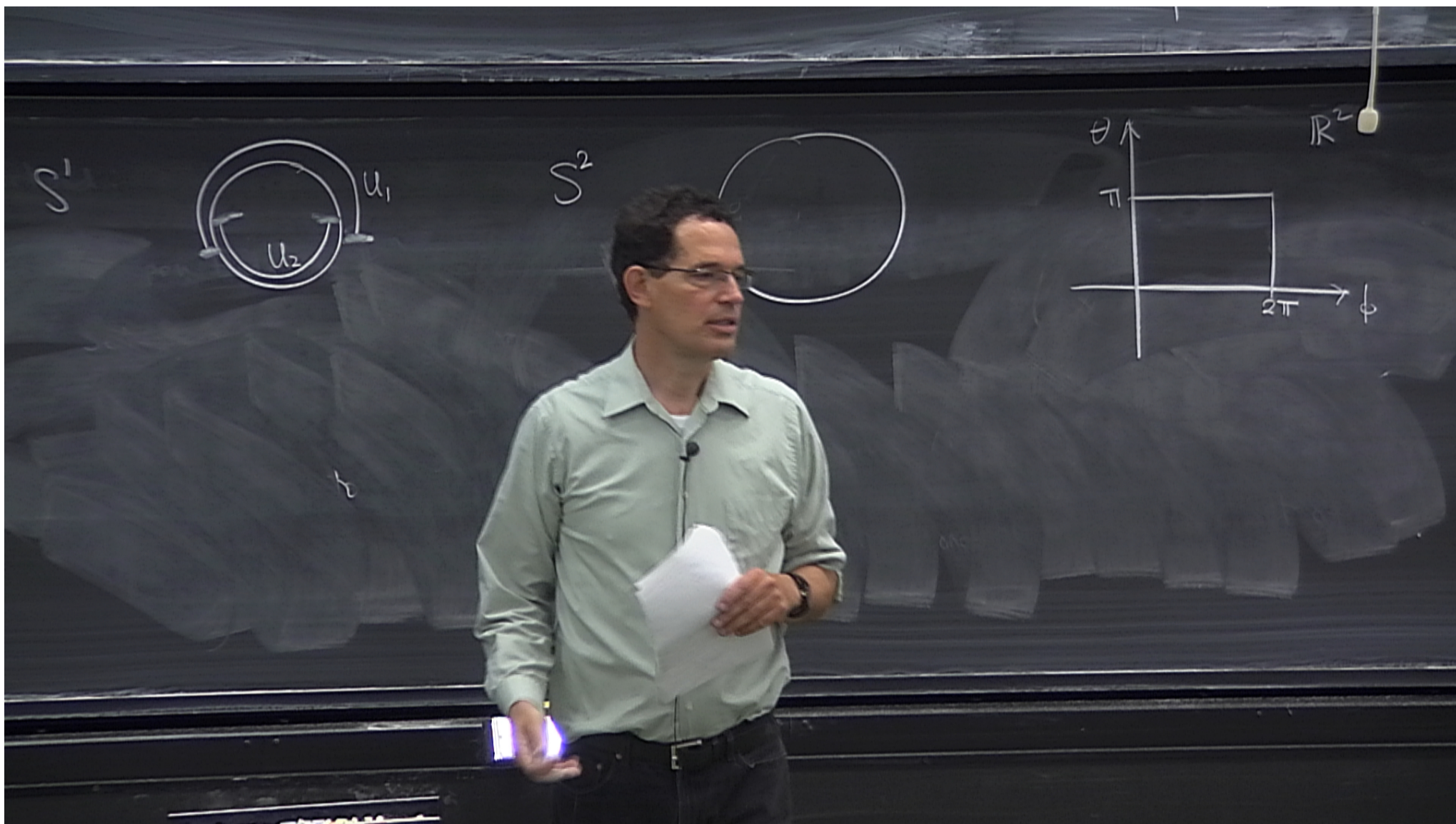
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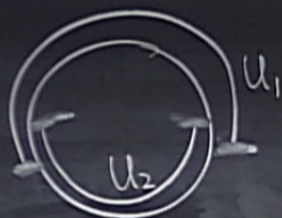
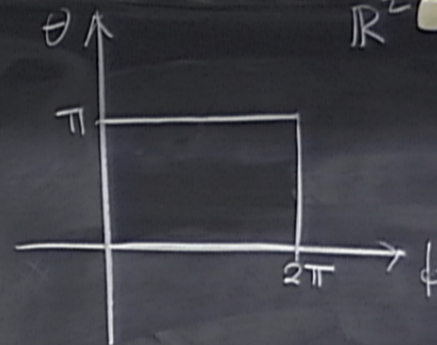


Defⁿ

Defⁿ. of C^∞ on M = collection of all neighbourhoods and maps $U, \phi(U)$,
 for which transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are C^∞ .





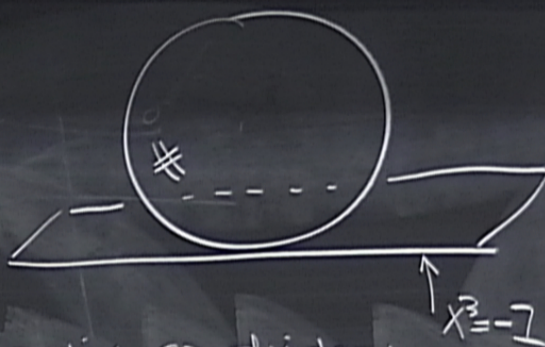
S^1  S^2 

stereographic coordinates:

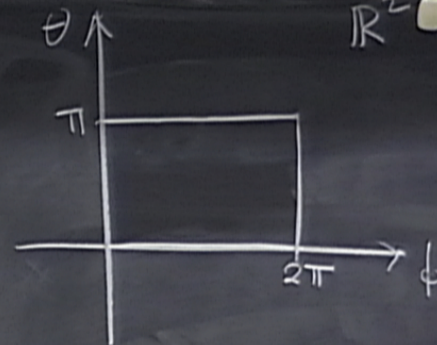
S^1

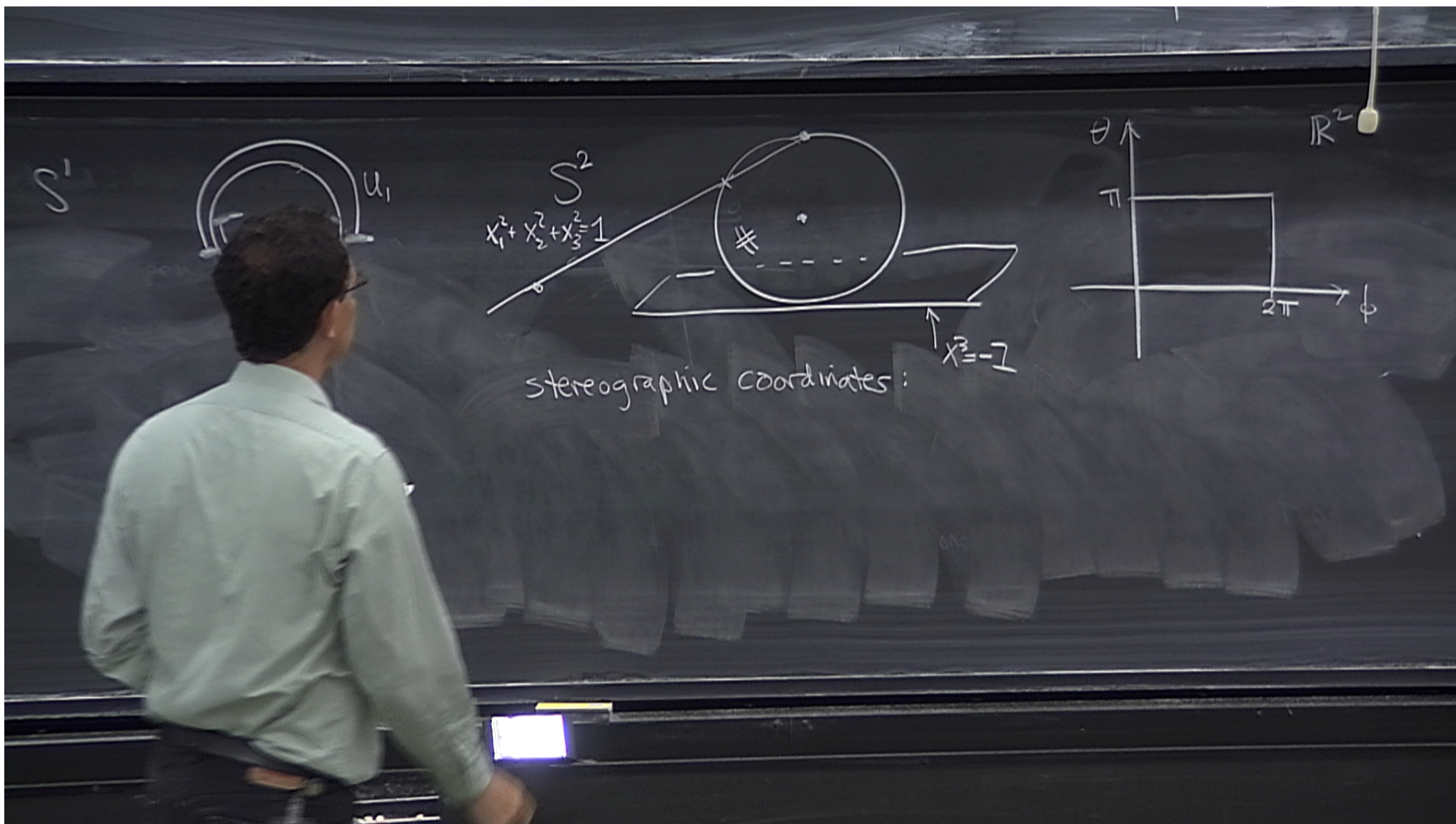


S^2
 $x_1^2 + x_2^2 + x_3^2 = 1$



stereographic coordinates:

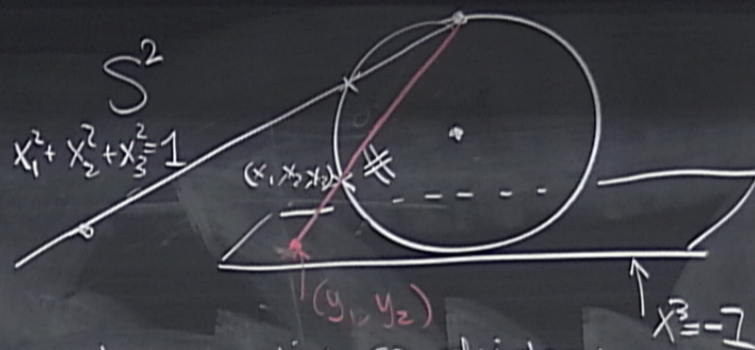




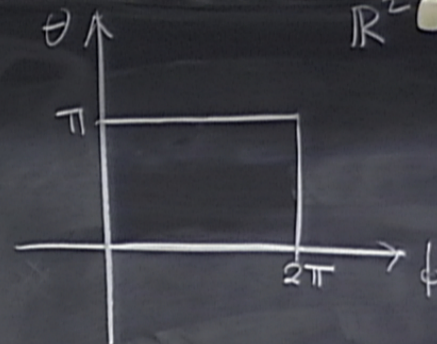
S^1



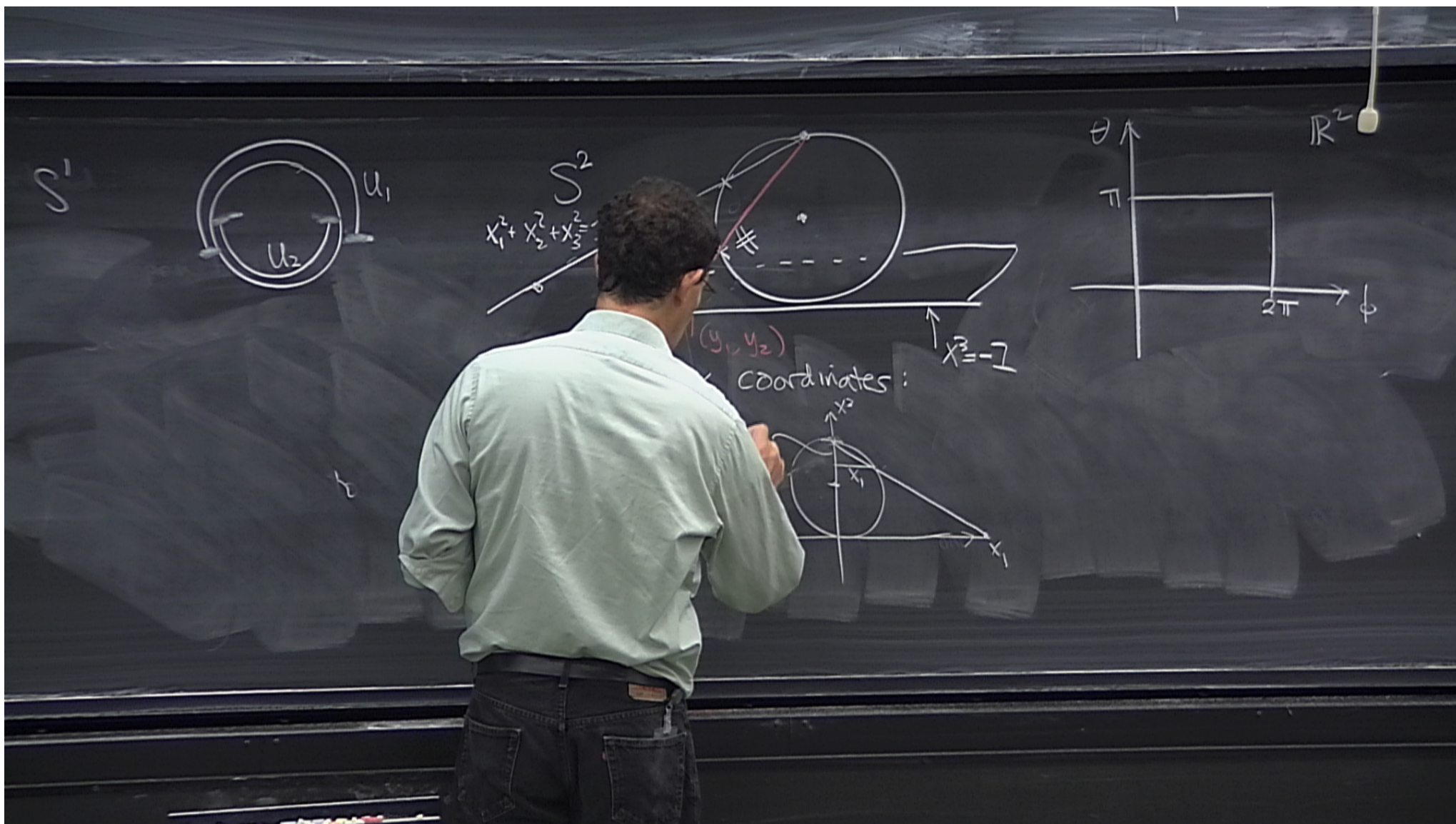
S^2
 $x_1^2 + x_2^2 + x_3^2 = 1$

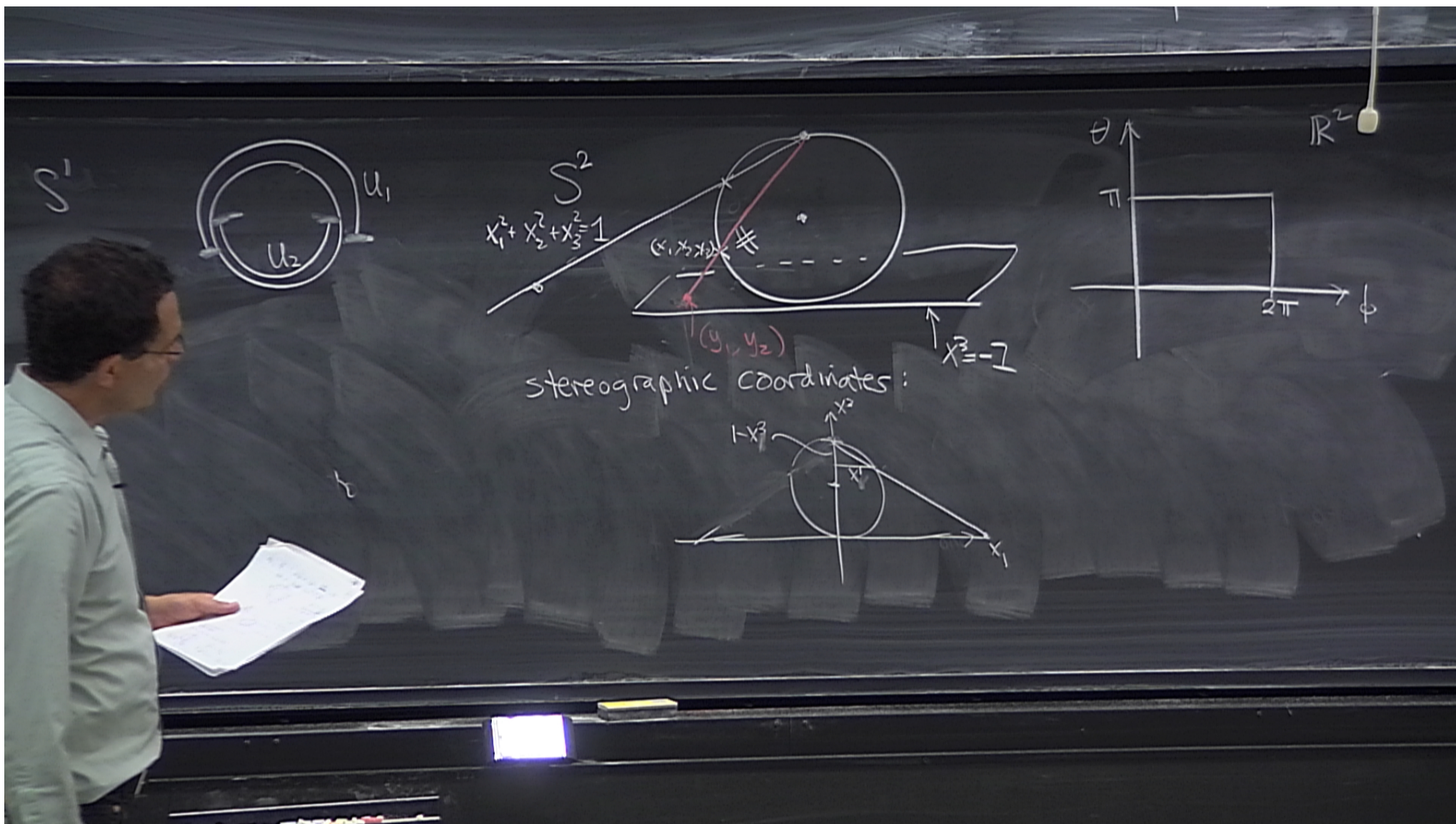


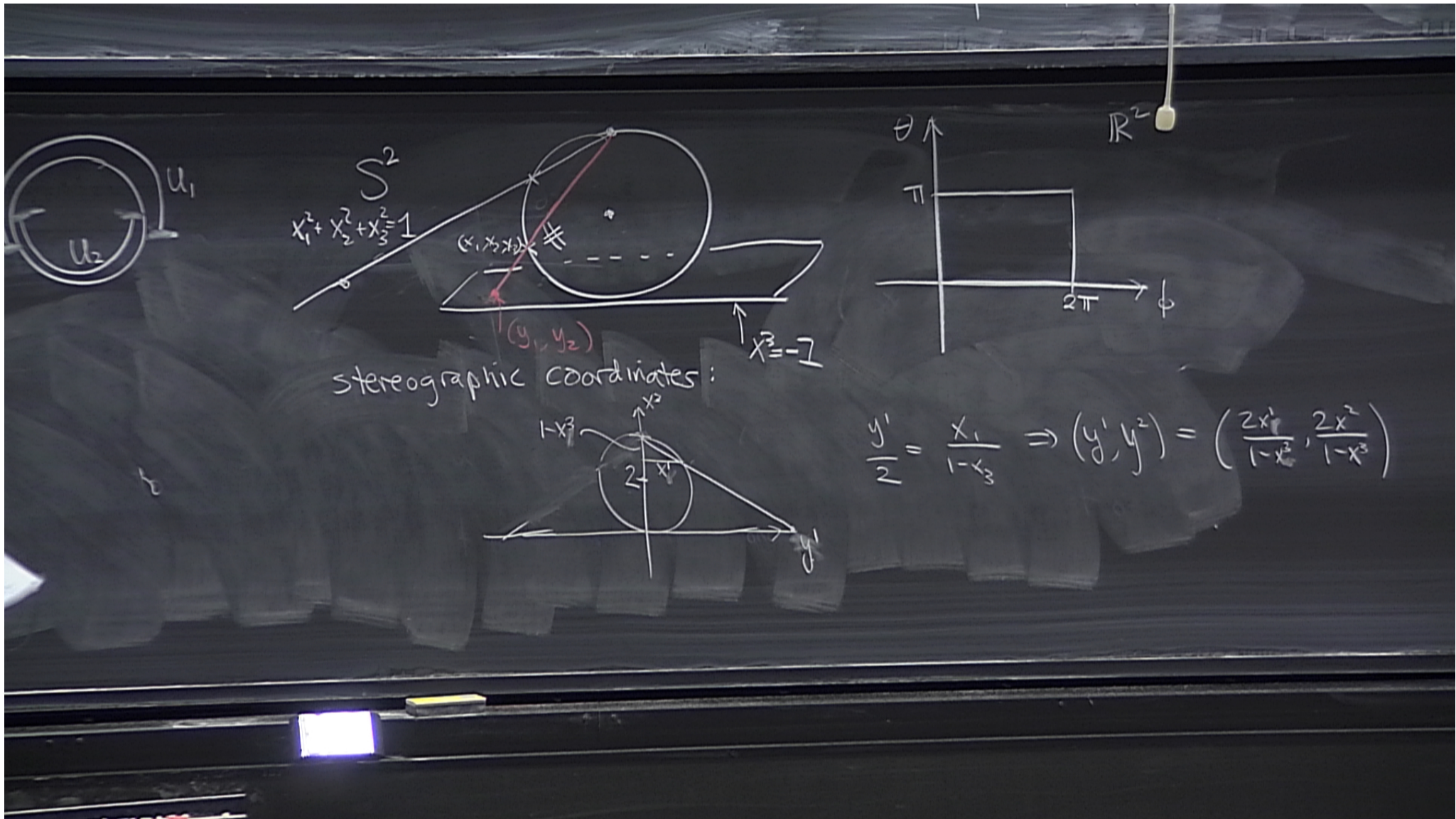
stereographic coordinates:



$\mathbb{R}^2$







Simy. S. Pole stereographic coords $(z^1, z^2) = \left(\frac{2x^1}{1+x^3}, \frac{2x^2}{1+x^3} \right)$

2 charts which together cover S^2 , overlap region is $-1 < x^3 < 1$

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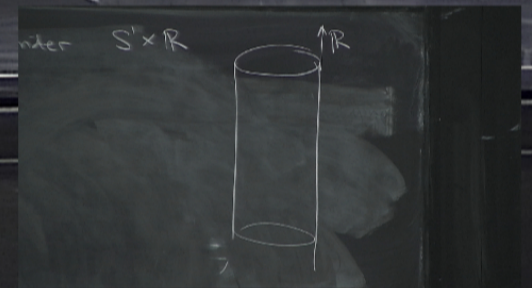
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Ex: check that transition function $\phi_{SP} \circ \phi_{NP}^{-1}$ is given by

$$z^1 = \frac{4y^1}{y_1^2 + y_2^2}, \quad z^2 = \frac{4y^2}{y_1^2 + y_2^2}, \text{ which is } C^\infty$$

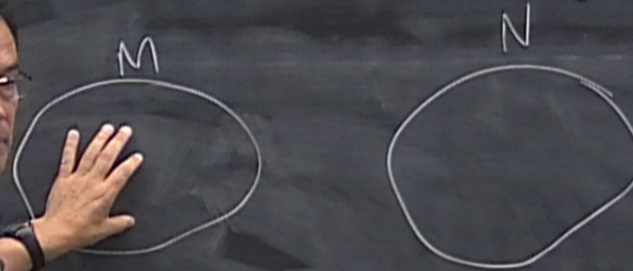
Example: find a single coordinate system (chart) covering a cylinder $S^1 \times \mathbb{R}$

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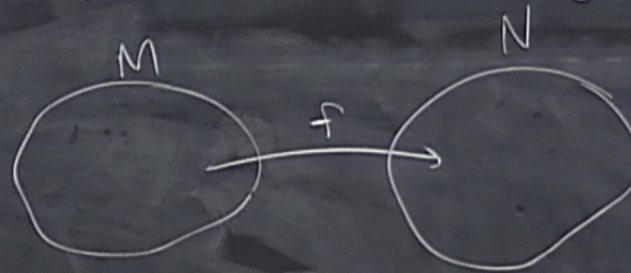
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(except at $\pm\infty$)

Calculus on manifolds:



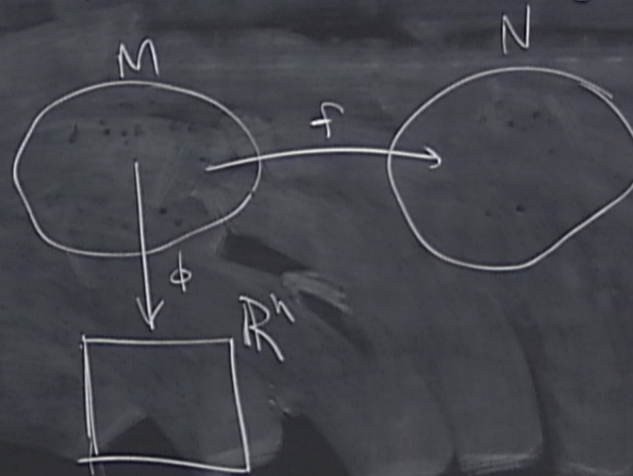
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Calculus manifolds:



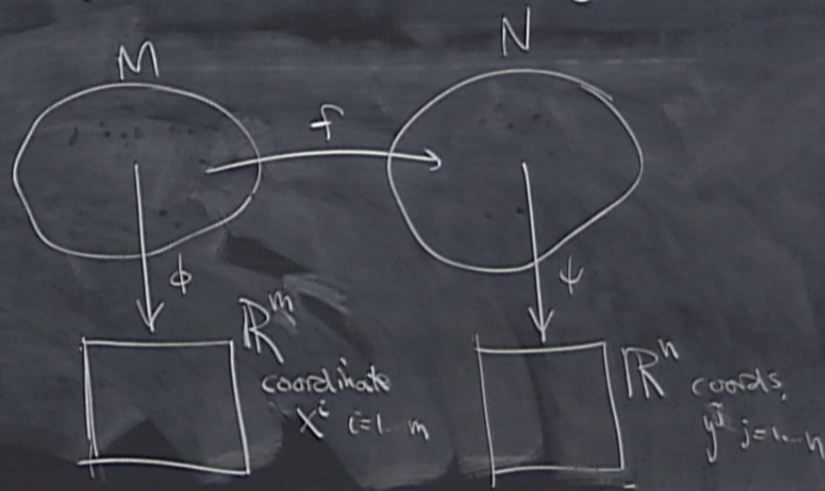
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Calculate manifolds:

