

Title: Relativity - Lecture 2

Date: Sep 13, 2011 10:30 AM

URL: <http://pirsa.pi.local/11090025>

Abstract:

Please do
NOT erase

Lorentz transform

Books:

1. S. Carroll
Maxwell

$$\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$$

- Space and geometry
on arxiv

$$\epsilon_{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$$

$\eta^{\alpha\beta}$
 $\eta_{\alpha\beta} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$
- more advanced

charge conservation
 $\nabla \cdot \mathbf{J} = 0$

860s Maxwell

E.M.
(SI units)

$$\mathbf{E} = -\nabla \phi$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Lorentz transformations

Maxwell's equations possess an amazing spacetime symmetry.

consider linear transformation

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

Lorentz transformations

Maxwell's equations possess an amazing spacetime symmetry.

consider linear transformation

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

rows α' columns β

Maxwell's eqs
 $\nabla \cdot \mathbf{E} = \rho$
 $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
 $\nabla \cdot \mathbf{B} = 0$
 $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$
 consider linear transformation

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

rows α'
 columns β

$$(ct', \mathbf{x}')$$

prime always means "in the primed coord. system"

Lorentz transformations

Maxwell's equations possess an amazing spacetime symmetry.

consider linear transformation

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

rows α'
columns β

$(ct', \underline{x}')$

prime always means "in the primed coord. system"

$$\text{Similarly, } A^{\alpha'} = \Lambda^{\alpha'}_{\beta} A^{\beta}$$

Lorentz transformations

Maxwell's equations possess an amazing spacetime symmetry.

consider linear transformation

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

rows α'
columns β

$(ct', \underline{x}')$

prime always means "in the primed coord. system"

Similarly,

$$A^{\alpha'} = \Lambda^{\alpha'}_{\beta} A^{\beta}$$

$$J^{\alpha'} = \Lambda^{\alpha'}_{\beta} J^{\beta} \text{ etc.}$$

Assume Λ invertible, so

$$x^\beta = \tilde{\Lambda}^\beta_{\alpha'} x^{\alpha'}$$

$\tilde{\Lambda}$ is matrix inverse
of Λ

$\hat{\Lambda}$ is matrix inverse
of Λ

$$\Lambda_{\alpha'}^{\beta} \hat{\Lambda}_{\beta}^{\gamma} = \delta_{\alpha'}^{\gamma}$$

$$\hat{\Lambda}_{\alpha'}^{\beta} \Lambda_{\beta}^{\gamma} = \delta_{\alpha'}^{\gamma}$$

$$x^\beta = \tilde{\Lambda}^\beta_{\alpha'} x^{\alpha'}$$

$$\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \frac{\partial}{\partial x^\beta} = \tilde{\Lambda}^\beta_{\alpha'} \frac{\partial}{\partial x^\beta}$$

Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

OR $\eta = \Lambda^T \eta \Lambda$

Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

c. for orthogonal transformations

OR $\eta = \Lambda^T \eta \Lambda$ $O^T O = 1$

Why? because then, any tensor equation constructed by raising and lowering indices with η is Lorentz invariant.

Why? because then, any tensor equation constructed by raising and lowering indices with η is Lorentz invariant, i.e. if it is valid in one coordinate system, then it is valid in all coordinate systems.

Example: wave equation

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta$$

Example: wave equation

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F$$

Example: wave equation

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F'$$

Example: wave equation

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \eta^{\alpha'\beta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'} \partial_\alpha \partial_\beta F$$

Example: wave equation ($F' = F$)

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

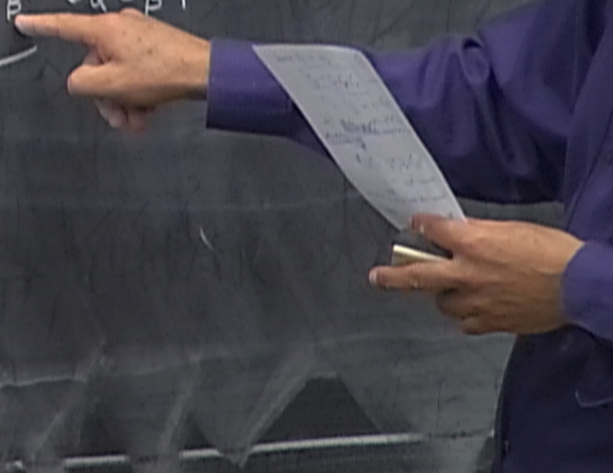
Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \eta^{\alpha'\beta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'} \partial_\alpha \partial_\beta F$$

Example: wave equation ($F' = F$)

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \underbrace{\eta^{\alpha'\beta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'}}_{\tilde{\Lambda} \eta \tilde{\Lambda}^T} \partial_\alpha \partial_\beta F$$


Example: wave equation ($F' = F$)

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

$$\Lambda^T \eta \Lambda = \eta$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \underbrace{\eta^{\alpha'\beta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'}}_{\tilde{\Lambda} \eta \tilde{\Lambda}^T = \eta} \partial_\alpha \partial_\beta F$$

Example: wave equation ($F' = F$)

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

$$\Lambda^T \eta \Lambda = \eta$$
$$\Rightarrow \hat{\Lambda} \eta \hat{\Lambda}^T = \eta$$

Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \underbrace{\eta^{\alpha'\beta'} \tilde{\Lambda}^{\alpha'}_{\alpha} \tilde{\Lambda}^{\beta'}_{\beta}}_{\hat{\Lambda} \eta \hat{\Lambda}^T = \eta} \partial_\alpha \partial_\beta F$$

Example: wave equation ($F' = F$)

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Lorentz inv:

$$\eta^{\alpha'\beta'} \partial_{\alpha'} \partial_{\beta'} F' = \underbrace{\eta^{\alpha'\beta'} \tilde{\Lambda}_{\alpha'}^{\alpha} \tilde{\Lambda}_{\beta'}^{\beta}}_{\hat{\Lambda} \eta \hat{\Lambda}^T = \eta} \partial_\alpha \partial_\beta F = \eta^{\alpha\beta} \partial_\alpha \partial_\beta F = 0$$

Please do
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Books:

1. S. Carroll

Maxwell $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$

- Spacetime and geometry
for arch. $\partial_\beta F_{\alpha\beta} = 0$

$$\eta^{\alpha\beta} \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What about $\epsilon^{\alpha\beta\gamma\delta}$?

1860s Maxwell

Maxwell's equations

$$\vec{E} = -\vec{\nabla}\phi$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

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$$\vec{B} = \vec{\nabla}\times\vec{A}$$

① What about $\epsilon^{\alpha\beta\gamma\delta}$?

$$\epsilon^{\alpha'\beta'\gamma'\delta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'} \tilde{\Lambda}^{\gamma}_{\gamma'} \tilde{\Lambda}^{\delta}_{\delta'} = \epsilon^{\alpha\beta\gamma\delta} \det \Lambda$$

(Signs) linear $\nabla \cdot \underline{B} = 0$ magnetic

charge conservation

$$\nabla \cdot \underline{E} = -\frac{\partial \rho}{\partial t}$$

$$\underline{E} = -\nabla \phi$$

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

$$\epsilon_0 \mu_0 \frac{\partial \underline{E}}{\partial t} = \underline{J}$$

Ampere + Maxwell

prime always means

Maxwell's correction
(in the primed (rest) system)

What about $\epsilon^{\alpha\beta\gamma\delta}$?

$$\epsilon^{\alpha'\beta'\gamma'\delta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'} \tilde{\Lambda}^{\gamma}_{\gamma'} \tilde{\Lambda}^{\delta}_{\delta'} = \epsilon^{\alpha\beta\gamma\delta}$$

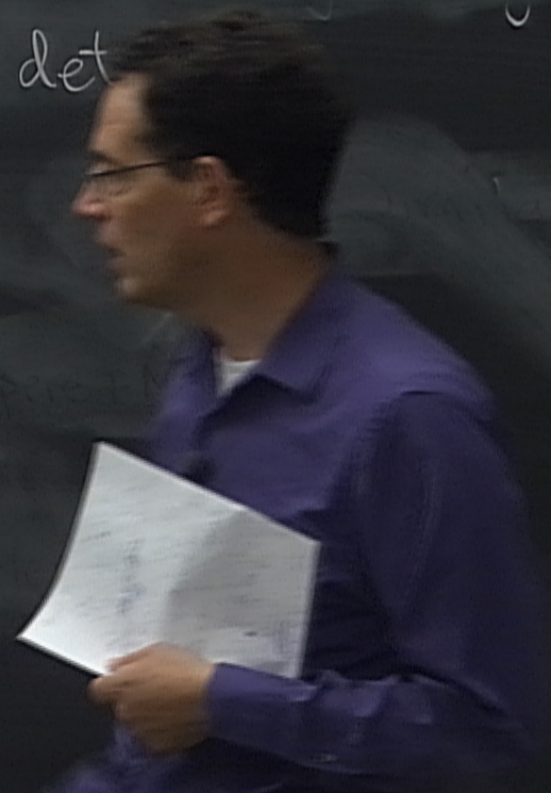
$$\det A = \epsilon_{abcd} A_{1a} A_{2b} A_{3c} A_{4d}$$

What about $\epsilon^{\alpha\beta\gamma\delta}$?

$$\epsilon^{\alpha'\beta'\gamma'\delta'} \tilde{\Lambda}^{\alpha}_{\alpha'} \tilde{\Lambda}^{\beta}_{\beta'} \tilde{\Lambda}^{\gamma}_{\gamma'} \tilde{\Lambda}^{\delta}_{\delta'} = \epsilon^{\alpha\beta\gamma\delta} \det$$

$$\det A = \epsilon_{abcd} A_{1a} A_{2b} A_{3c} A_{4d}$$

prime always means in the primed coord



Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$\eta = \Lambda^T \eta \Lambda$$

$$1 = -1$$

c.f. orthogonal
transfm's

$$O^T O = 1$$

Restrict Λ 's to obey $j=1,2,3$

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

$$\begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

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c.f. orthogonal
transfmⁿs

$$O^T O = 1$$

$$\det \eta = -1 = -1 (\det \Lambda)^2$$

Restrict Λ 's to obey

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c.f. orthogonal
transfmⁿs

$$O^T O = 1$$

$$\det -1 = -1 (\det \Lambda)^2 \Rightarrow \det \Lambda = \pm 1$$

Restrict Λ 's to obey

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OR

$$\eta = \Lambda^T \eta \Lambda$$

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proper

c.f. orthogonal
transfm's

$$O^T O = 1$$

Restrict Λ 's to obey

$$\eta_{\alpha\beta} = \eta_{\alpha'\beta'} \Lambda^{\alpha'}_{\alpha} \Lambda^{\beta'}_{\beta}$$

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OR $\eta = \Lambda^T \eta \Lambda$

c.f. orthogonal
transfmⁿs

$$O^T O = 1$$

$$\det \eta = -1 = -1 (\det \Lambda)^2 \Rightarrow \det \Lambda = \pm 1$$

proper
improper

Please do
NOT erase

General Relativity

Maxwell $\partial_\alpha F^{\alpha\beta} = -\mu_0 J^\beta$

$\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = 0$

$\eta^{\alpha\beta} \quad \eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

What about $\epsilon^{\alpha\beta\gamma\delta}$?

$\epsilon^{\alpha'\beta'\gamma'\delta'} \Lambda^\alpha_{\alpha'} \Lambda^\beta_{\beta'} \Lambda^\gamma_{\gamma'} \Lambda^\delta_{\delta'} = \epsilon^{\alpha\beta\gamma\delta} \det \Lambda$

$\epsilon^{\alpha\beta\gamma\delta} \det A = \epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} A_{\alpha_1}^{\alpha} A_{\alpha_2}^{\beta} A_{\alpha_3}^{\gamma} A_{\alpha_4}^{\delta}$

Examples of L.T.s

EM POTENTIAL $\Lambda_{\beta}^{\alpha'}$ = rotation $\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$

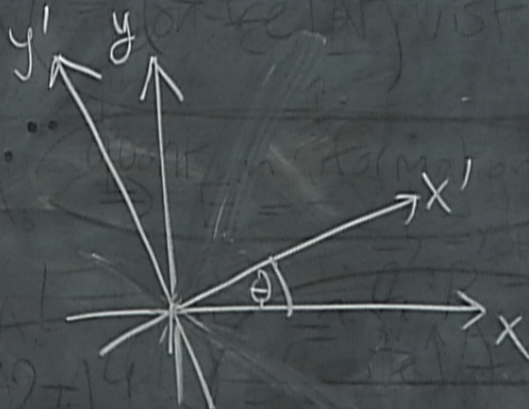
$\square G(R, T) = \square H(R, T) \quad \text{if } F_{0i} = \partial_i A_0 - \partial_0 A_i$

$\frac{1}{4} \int d\vec{x} d\omega G_{\mu\nu}$

$$\text{rotation} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x' = x \cos\theta + y \sin\theta$$

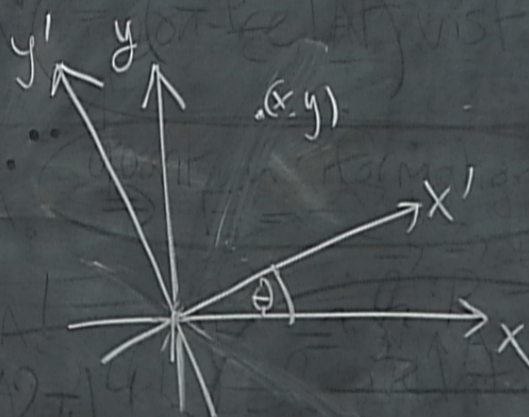
$$y' = y \cos\theta - x \sin\theta$$

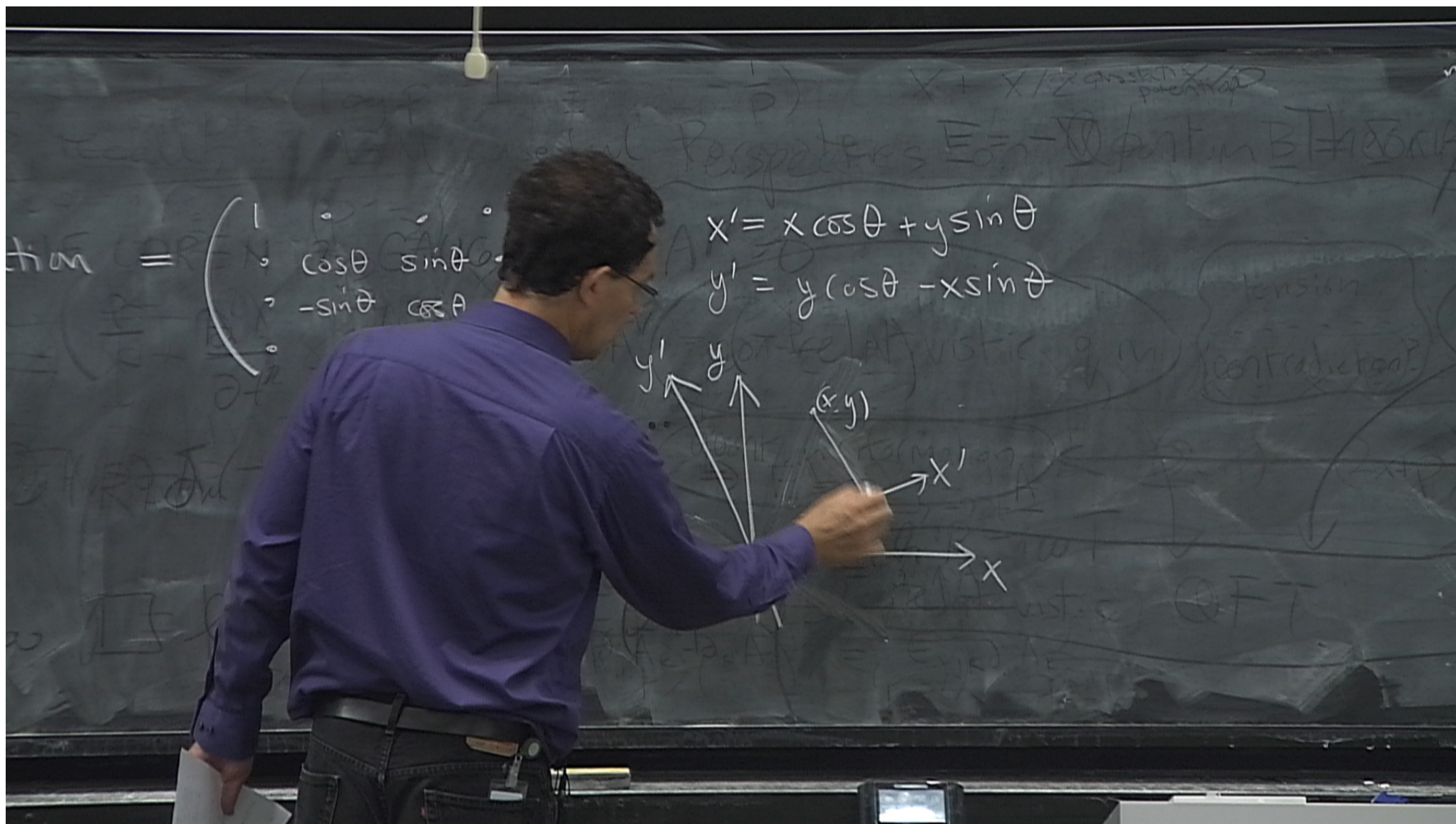


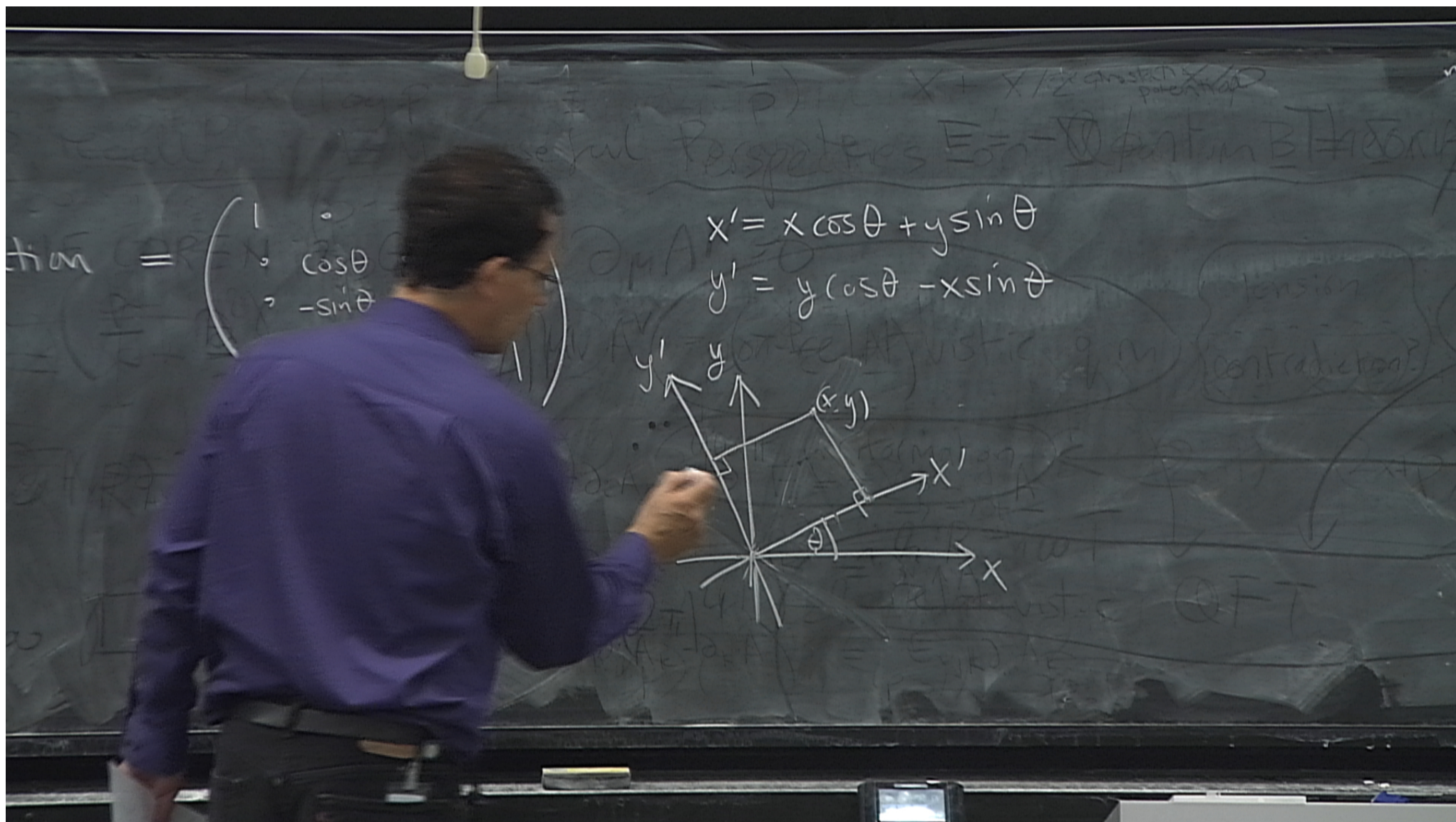
$$\text{rotation} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x' = x \cos\theta + y \sin\theta$$

$$y' = y \cos\theta - x \sin\theta$$





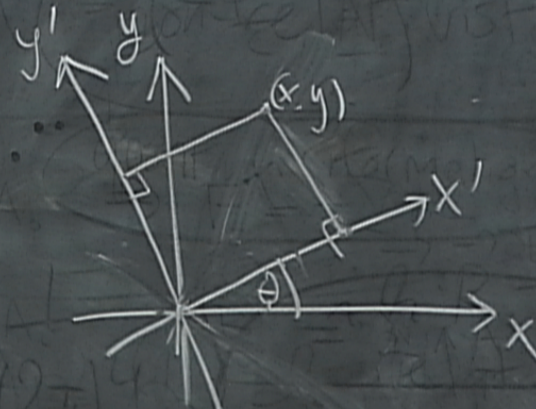


of L.T.s

= rotation $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$x'^2 + y'^2 = x^2 + y^2$$

$$\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ y' &= y \cos\theta - x \sin\theta \end{aligned}$$



More interesting: boost

$$\Lambda_{\beta}^{\alpha'} = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 & 0 \\ -\sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1. S. Carroll

More interesting: boost

$$\Lambda_{\beta}^{\alpha} = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 & 0 \\ -\sinh \theta & \cosh \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{\eta} \Lambda^T = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c & -s \\ -s & c \end{pmatrix}$$

boost

$$\Lambda_{\beta}^{\alpha'} = \begin{pmatrix} \cosh \theta & -\sinh \theta & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda_{\eta} \Lambda^T = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} -c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. $ct' = ct \cosh \theta - x \sinh \theta$
 $x' = x \cosh \theta - ct \sinh \theta$

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^{\beta}$$

i.e.

$$ct' = ct \cosh \theta - x \sinh \theta$$

$$x' = x \cosh \theta - ct \sinh \theta$$

$$y' = y$$

$$z' = z$$

$$x' = 0$$

$$x' = 0 \rightarrow x = \tanh \theta c t \equiv v t$$

$$x' = 0 \rightarrow x = \tanh \theta ct \equiv vt$$

i.e. ct', x', y', z' are the coordinates for an observer who moves at velocity $\underline{v} = (v, 0, 0)$ w.r.t. the unprimed coord. system.

$$x'=0 \rightarrow x = \tanh\theta ct \equiv vt$$

i.e. ct', x', y', z' are the coordinates for an observer who moves at velocity $\underline{v} = (v, 0, 0)$ w.r.t. the unprimed coord. system.

$$v = c \tanh\theta \quad -\infty < \theta < \infty$$

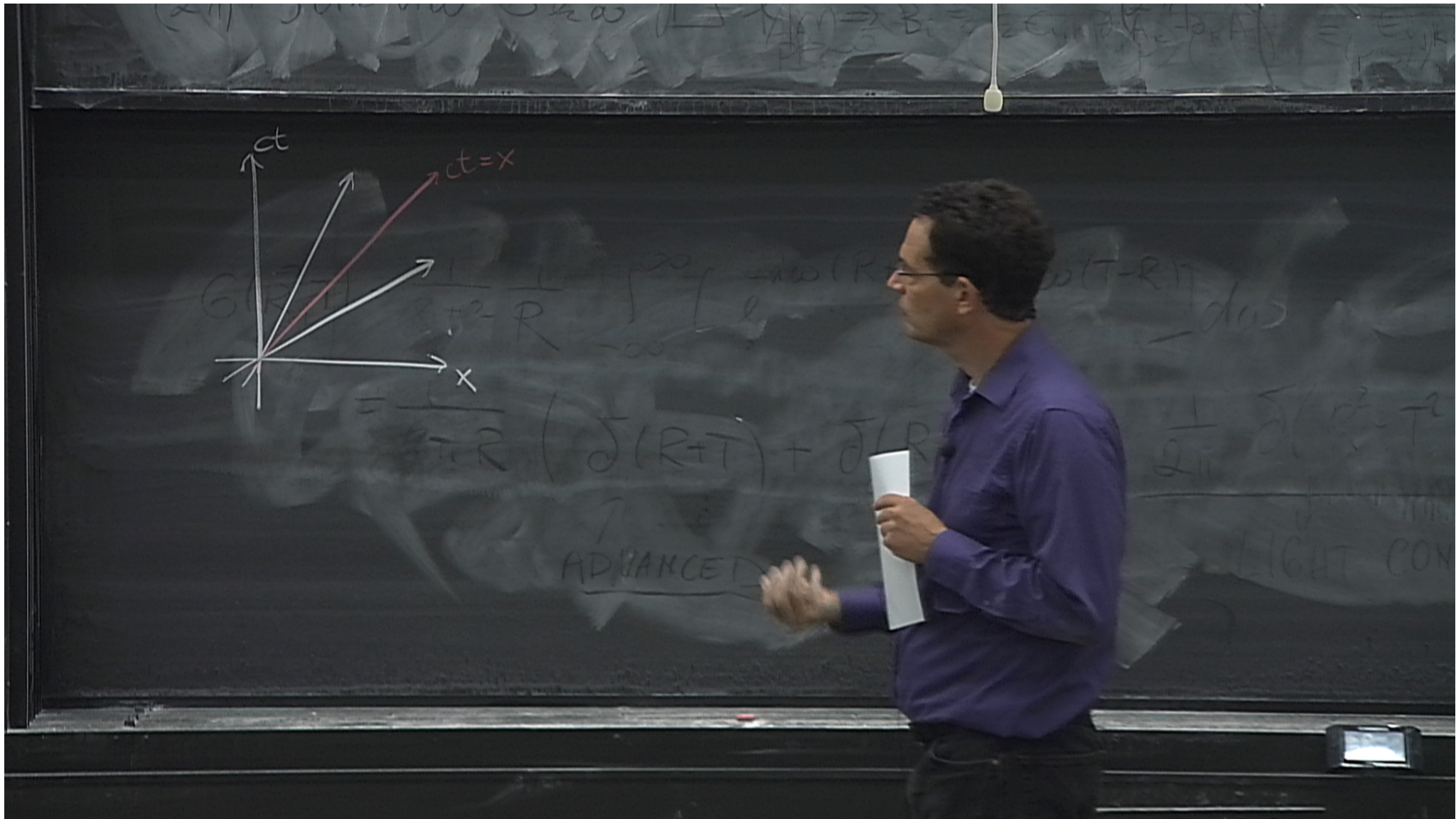
$$-1 < v < 1$$

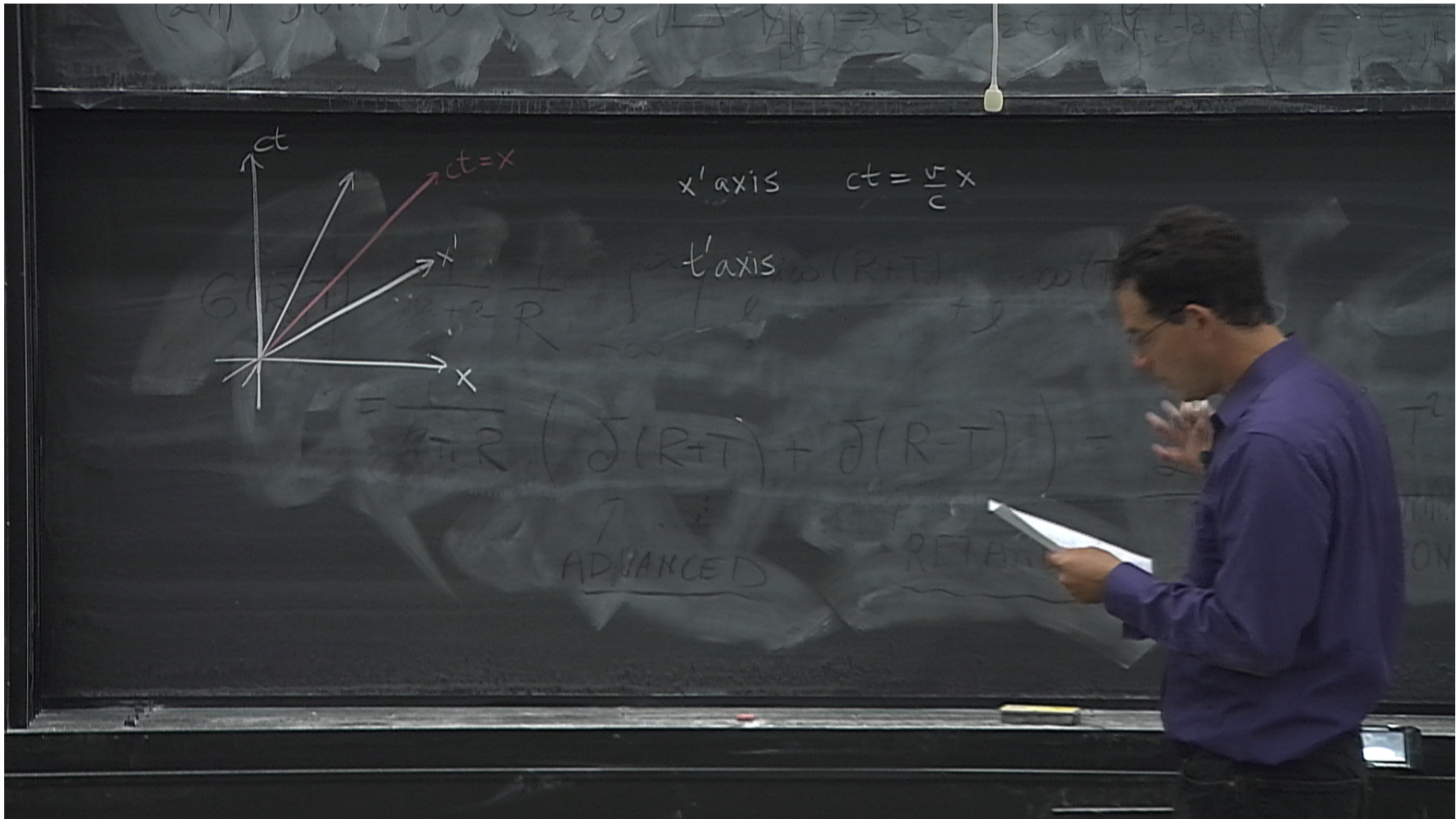
$$x'=0 \rightarrow x = \tanh\theta ct \equiv v t$$

i.e. ct', x', y', z' are the coordinates for an observer who moves at velocity $\underline{v} = (v, 0, 0)$ w.r.t. the unprimed coord. system.

$$v = c \tanh\theta \quad -\infty < \theta < \infty$$

$$-1 < v/c < 1$$





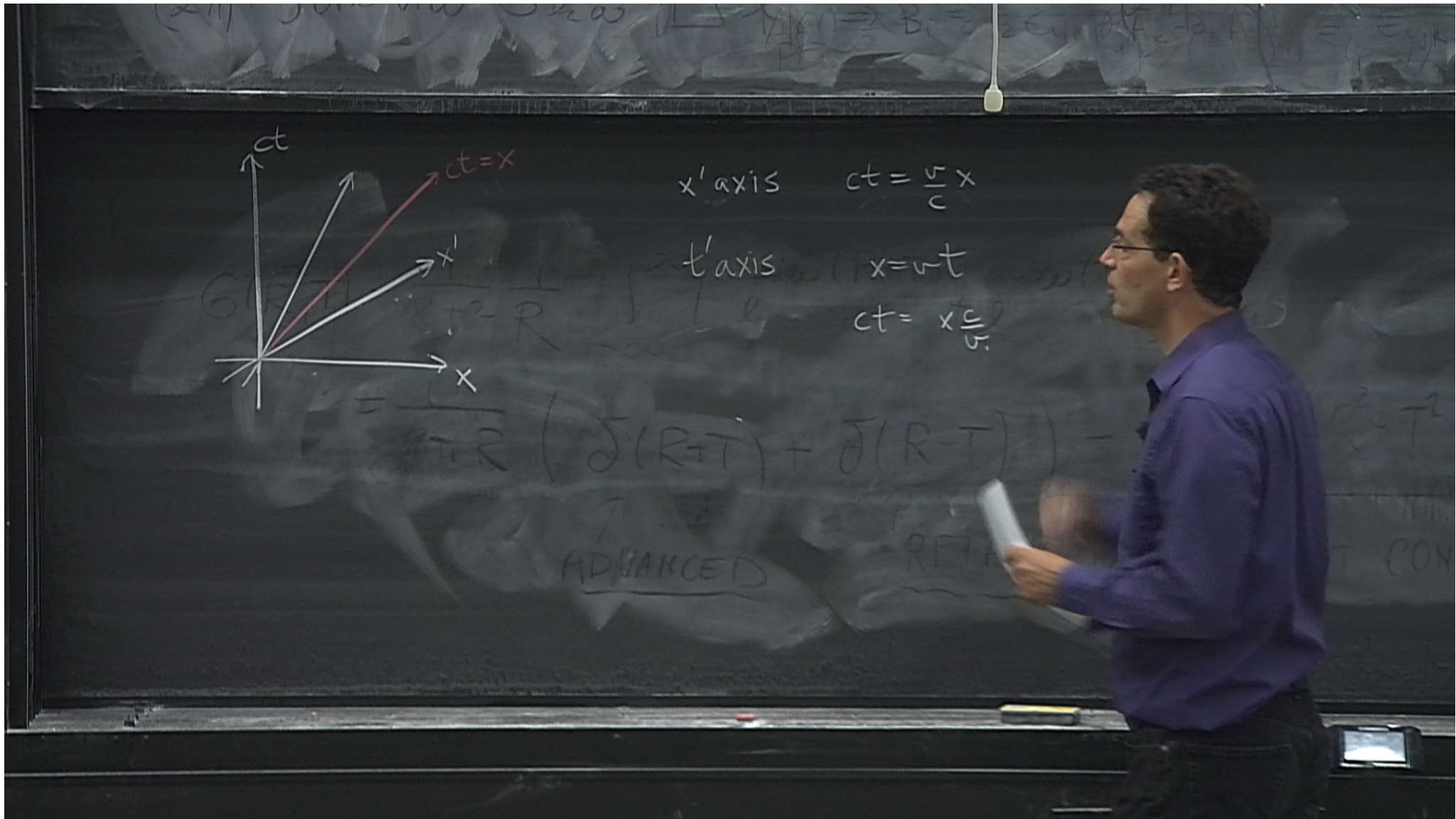
x' axis

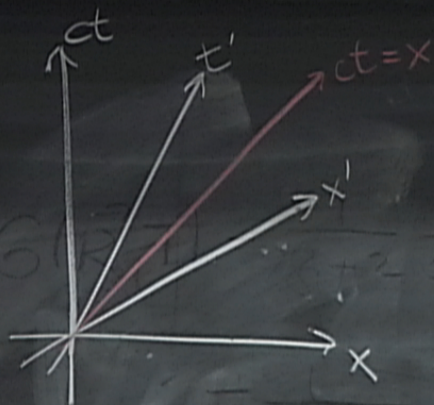
$$ct = \frac{v}{c} x$$

t' axis

ADVANCED

RETARD





x' axis $ct = \frac{v}{c} x$

ct' axis $x = vt$

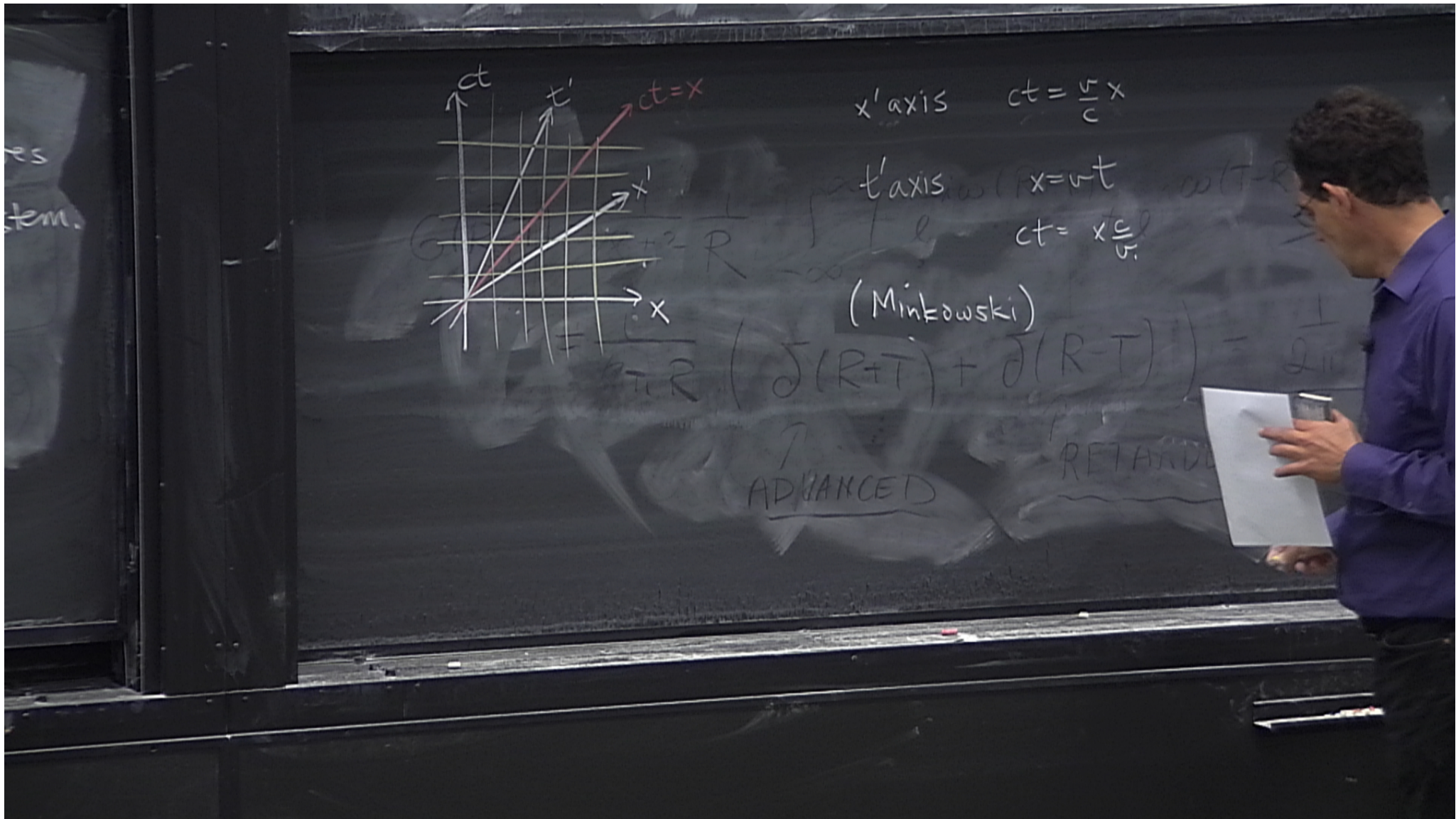
$ct = x \frac{c}{v}$

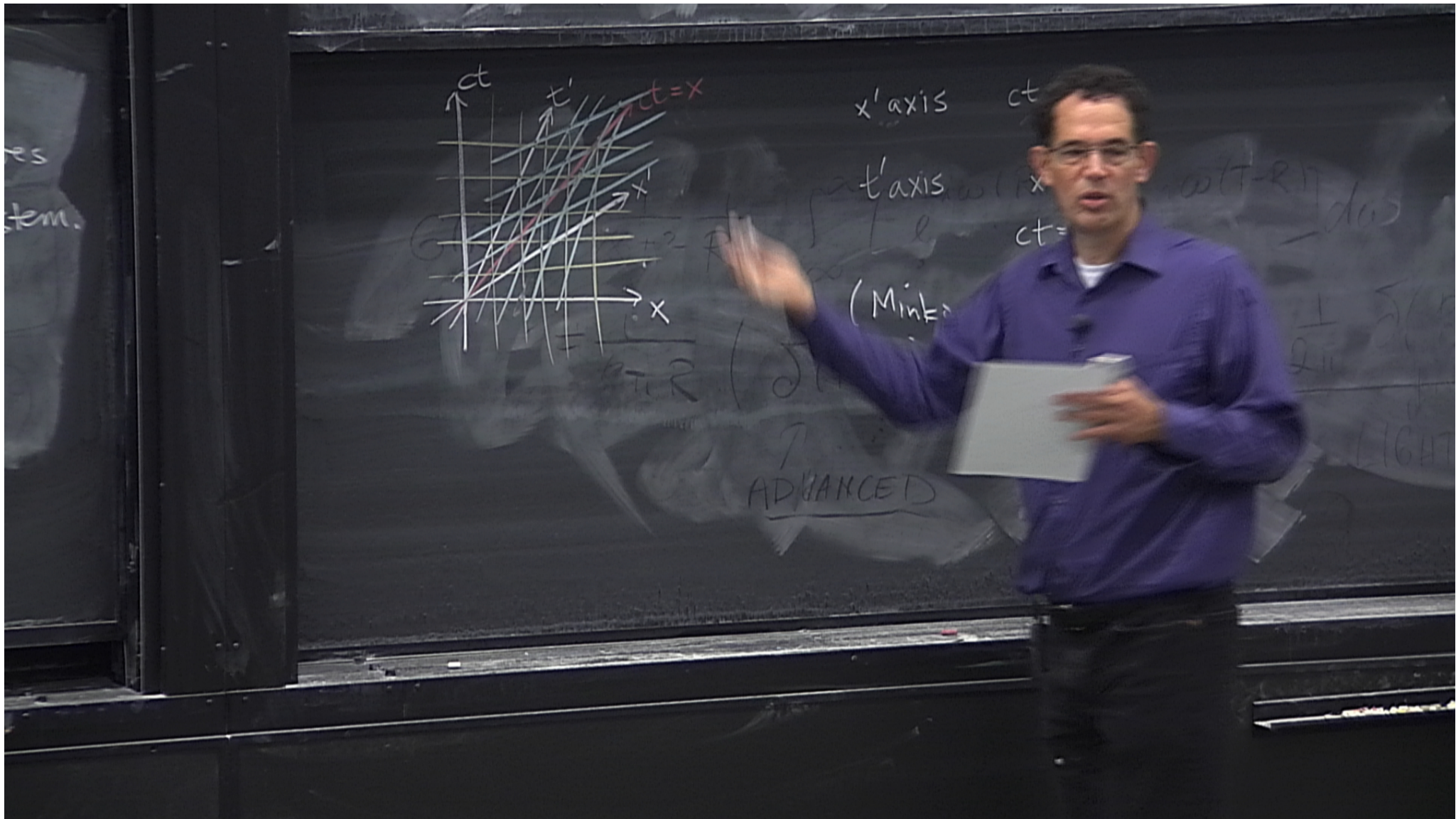
(Minkowski)

ADVANCED

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LIGHT CON





$$t' = \gamma \left(t - \frac{v}{c} x \right)$$

$$x' = \gamma (x - vt)$$

$$\cosh \theta = \frac{1}{\sinh \theta}$$

$$\cosh^2 \theta = 1 + \sinh^2 \theta$$

at velocity

$$v = c \tanh \theta$$

at velocity

$$v = c \tanh \theta$$

$$t' = \gamma \left(t - \frac{v}{c} x \right)$$

$$x' = \gamma (x - vt)$$

$$\cosh \theta = \frac{1}{\operatorname{sech} \theta} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma$$

$$\begin{aligned} \operatorname{sech}^2 \theta &= 1 - \tanh^2 \theta \\ &= 1 - \frac{v^2}{c^2} \end{aligned}$$

at velocity

$$v = c \tanh \theta$$

$$t' = \gamma \left(t - \frac{v}{c^2} x \right)$$

$$x' = \gamma (x - vt)$$

$$\cosh \theta = \frac{1}{\operatorname{sech} \theta} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \gamma$$

$$\begin{aligned} \operatorname{sech}^2 \theta &= 1 - \tanh^2 \theta \\ &= 1 - \frac{v^2}{c^2} \end{aligned}$$

$$\eta_{\mu\nu} X^\mu X^\nu = \eta_{\mu'\nu'} X^{\mu'} X^{\nu'}$$

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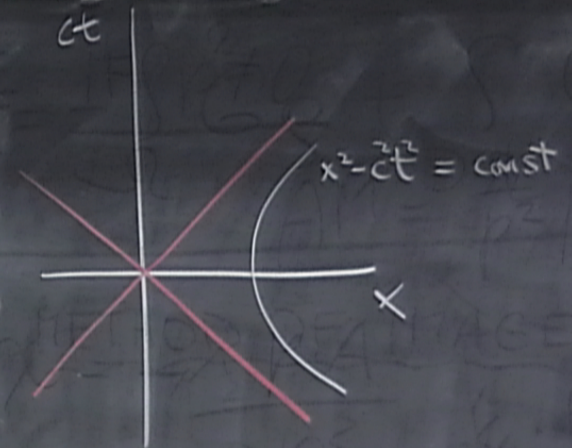
$$\underline{x^2 - c^2 t^2 = x'^2 - c^2 t'^2}$$

Lorentz-invariant
interval

$$\eta_{\mu\nu} X^\mu X^\nu = \eta_{\mu'\nu'} X^{\mu'} X^{\nu'}$$

$$\underline{x^2 - c^2 t^2 = x'^2 - c^2 t'^2}$$

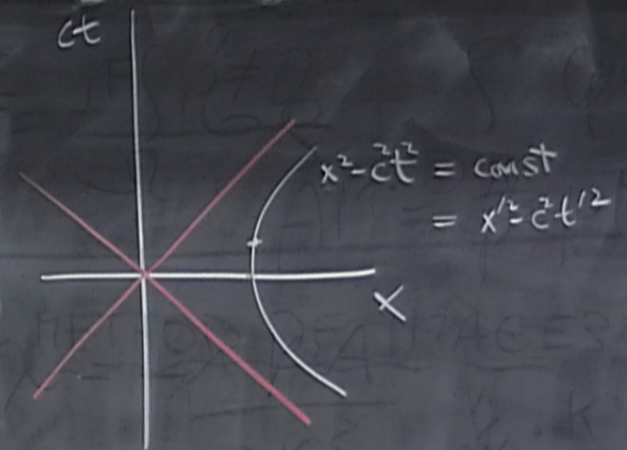
Lorentz-invariant
interval



$$\eta_{\mu\nu} X^\mu X^\nu = \eta_{\mu'\nu'} X^{\mu'} X^{\nu'}$$

$$\underline{x^2 - c^2 t^2 = x'^2 - c^2 t'^2}$$

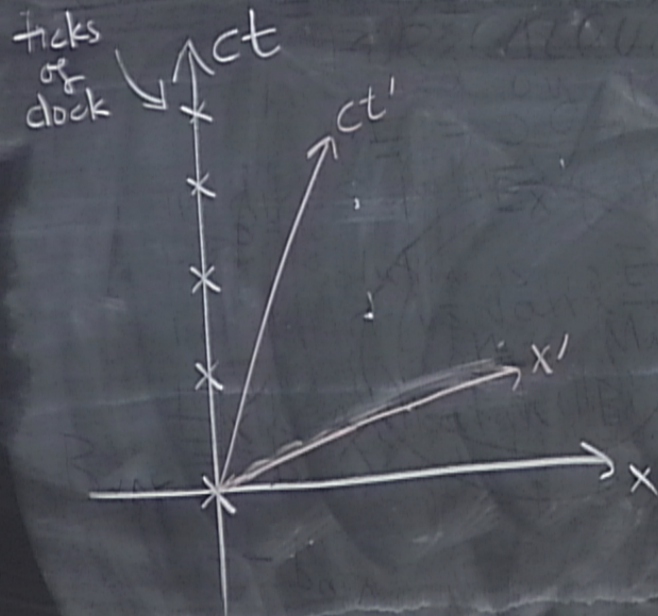
Lorentz-invariant
interval



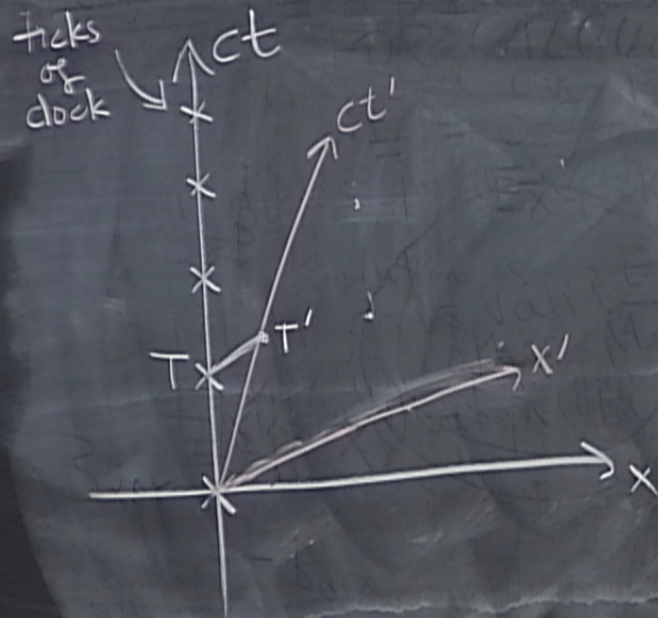
Time dilation

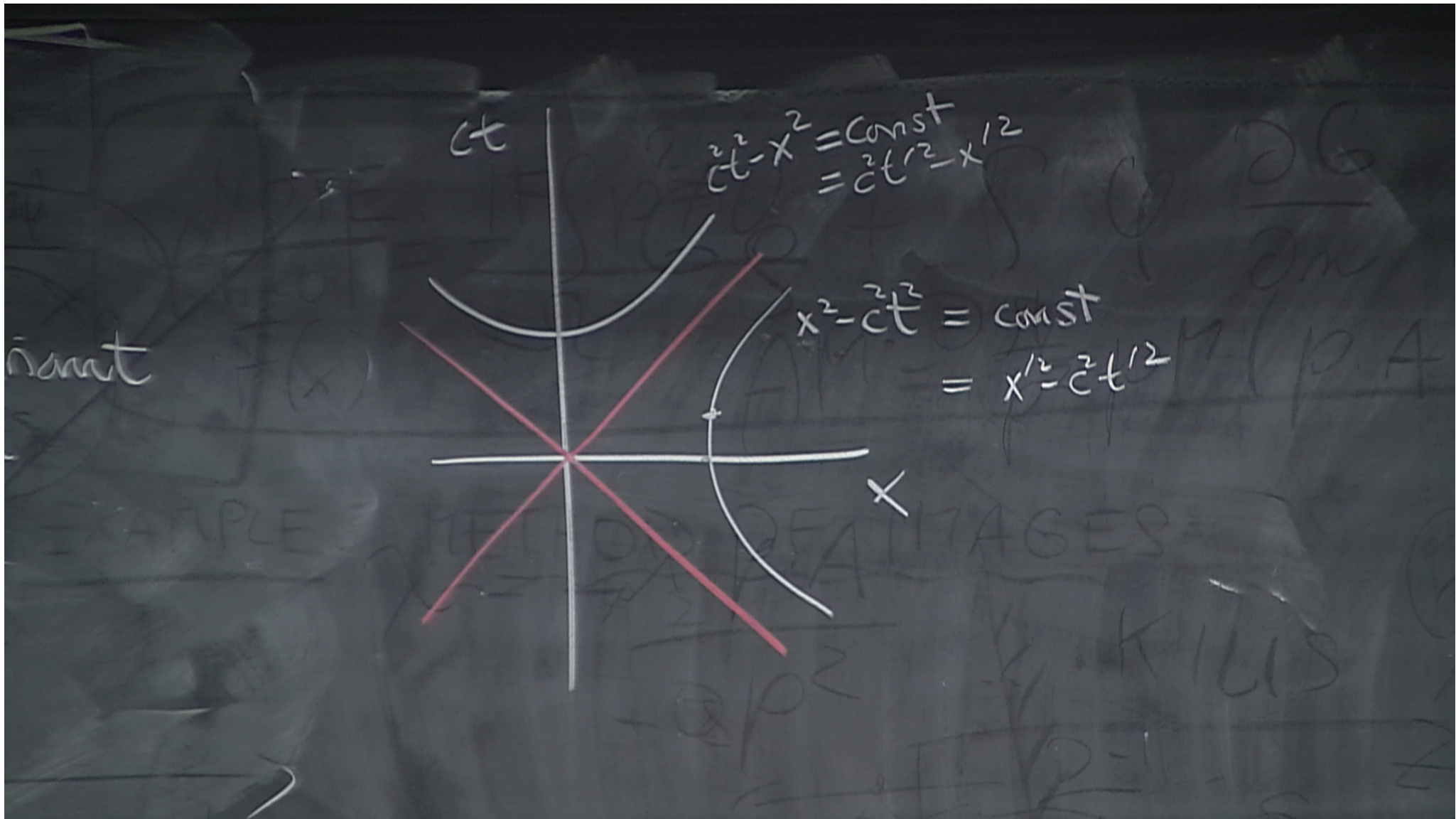
 $\uparrow ct$ $\rightarrow x$

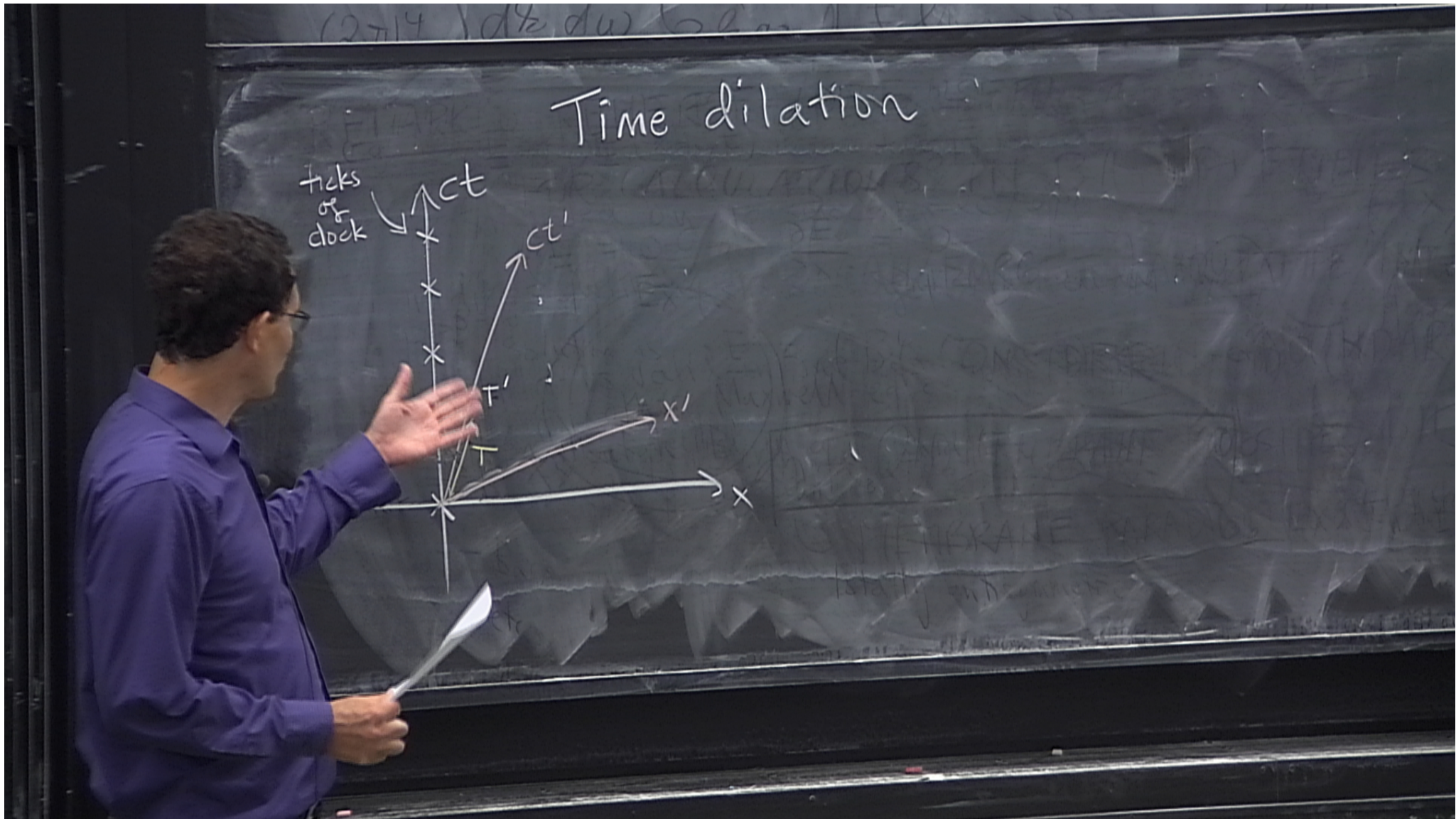
Time dilation



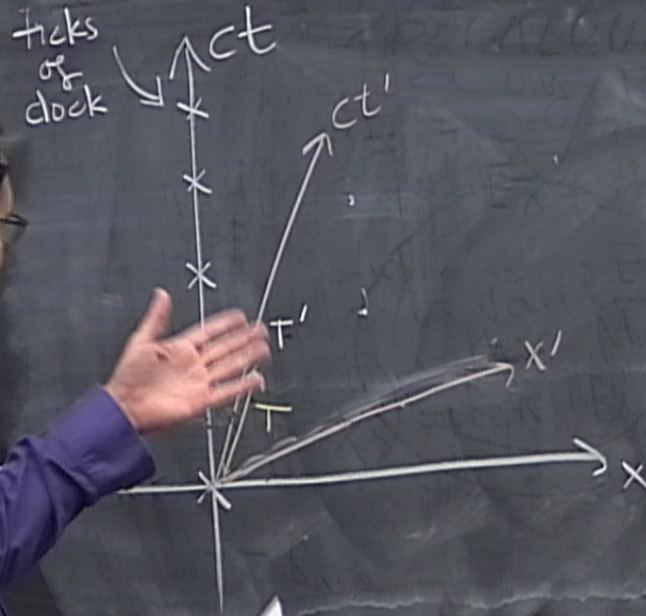
Time dilation



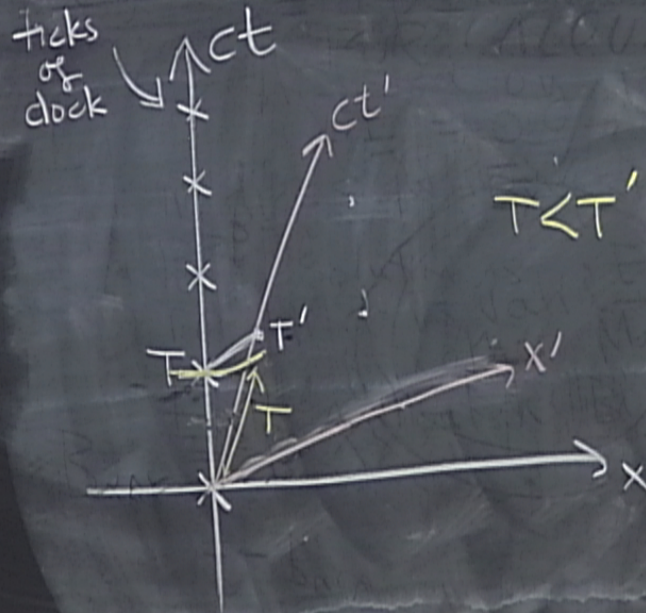




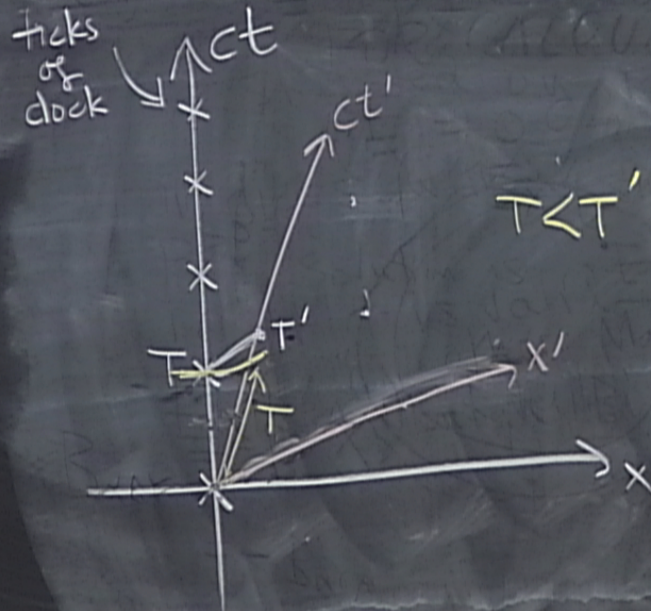
Time dilation



Time dilation



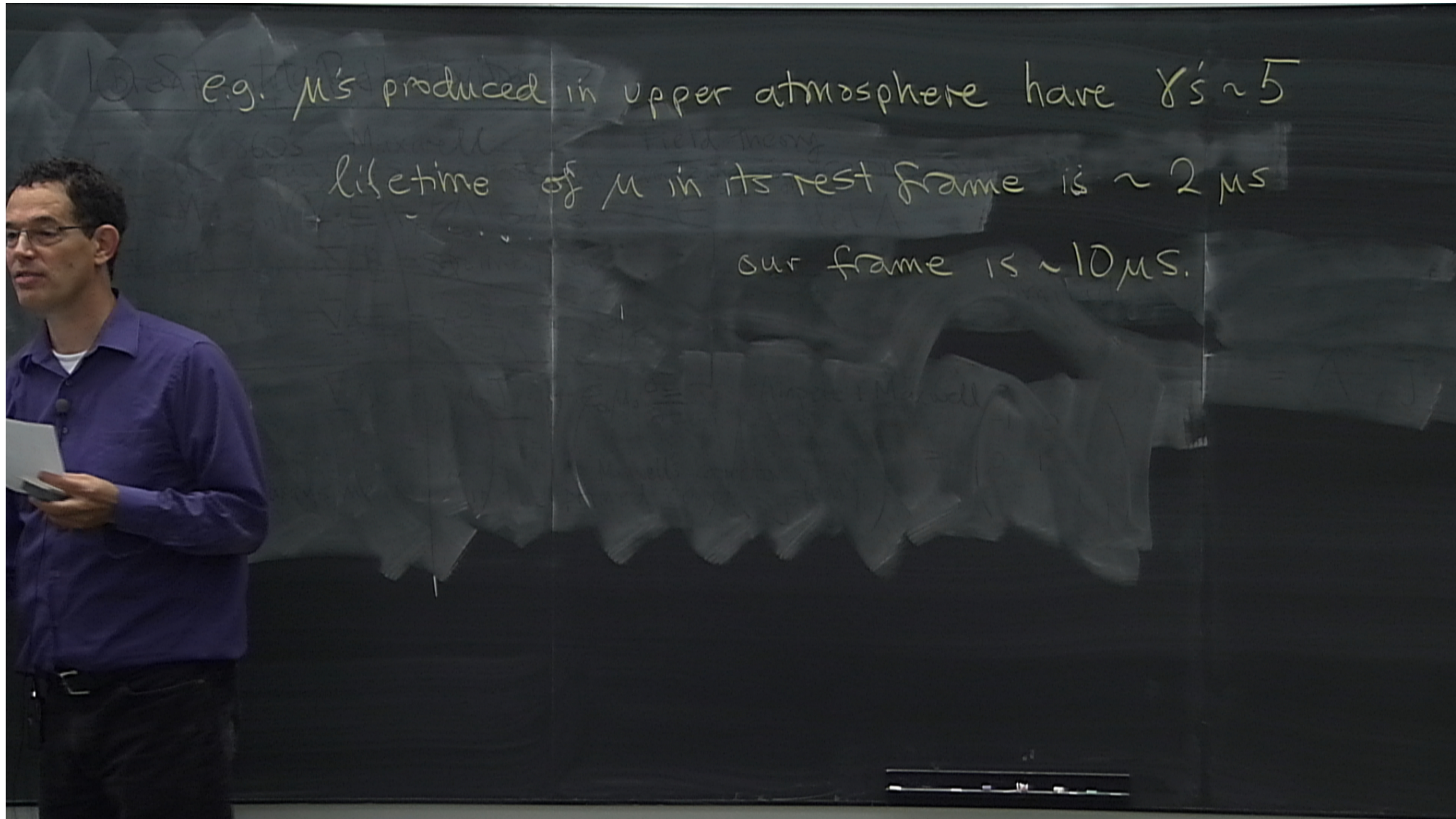
Time dilation



$$(ct, x) = (0, 0)$$

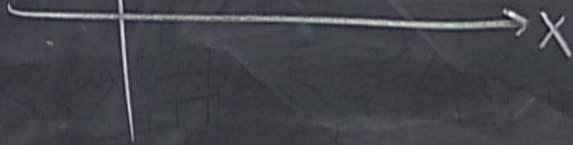
$$(cT, 0)$$

$$\begin{aligned}
 (ct, x) &= (0, 0) \\
 (cT, 0) &\longleftrightarrow (ct', x') = (0, 0) \\
 &= (\gamma cT, -\gamma vT) \\
 \Rightarrow T' &= \gamma T > T
 \end{aligned}$$

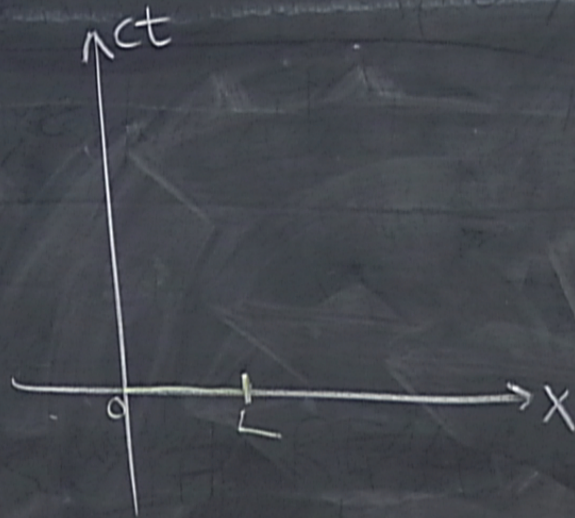


Lorentz contraction =

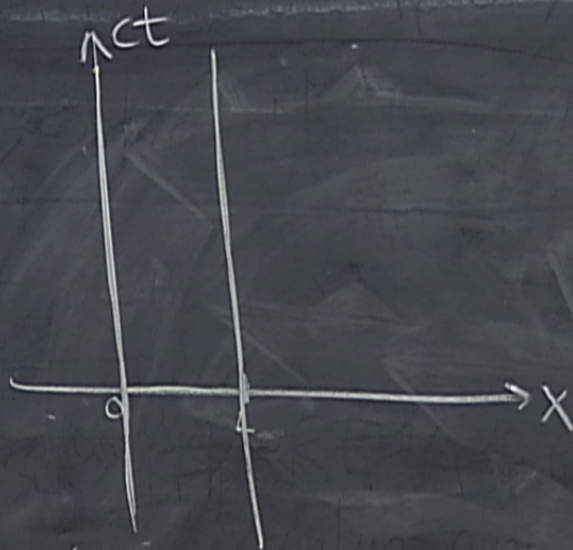
ct



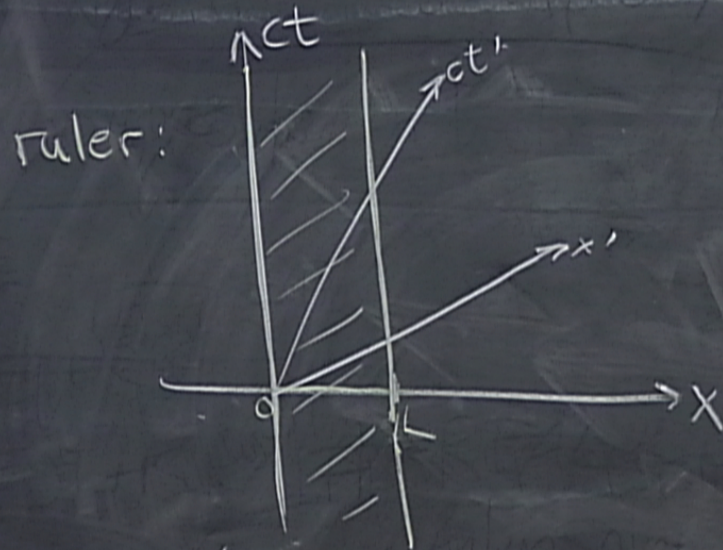
Lorentz contraction =



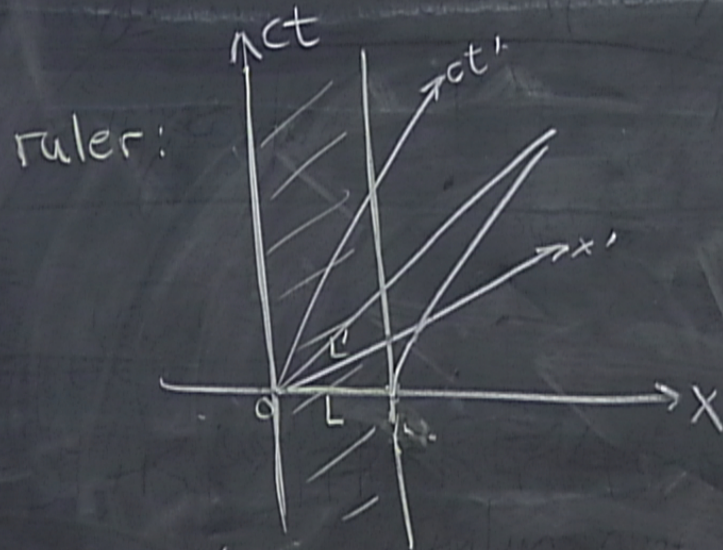
Lorentz contraction :-



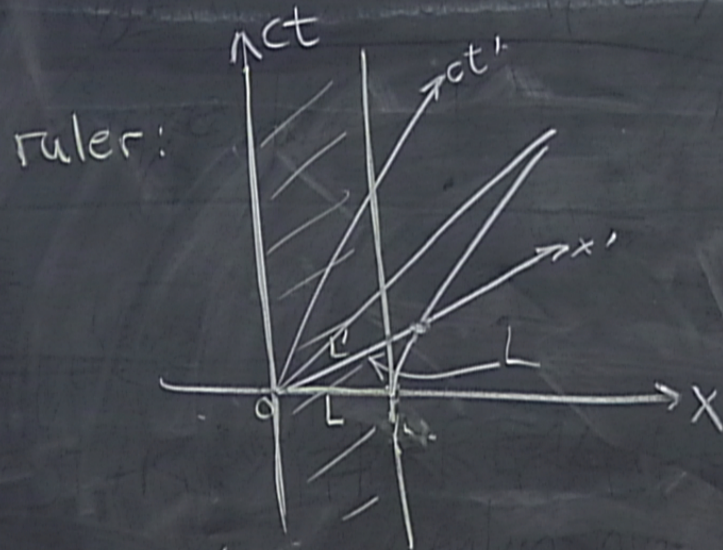
Lorentz contraction =



Lorentz contraction :-

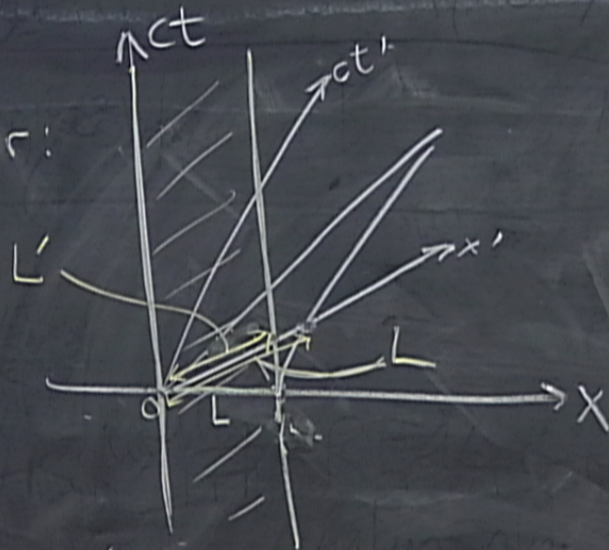


Lorentz contraction =



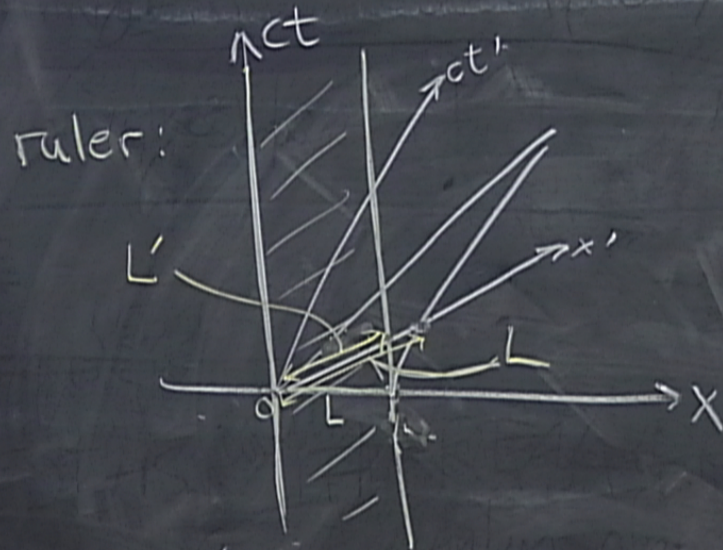
Lorentz contraction

ruler:



$$L' < L$$

Lorentz contraction :-

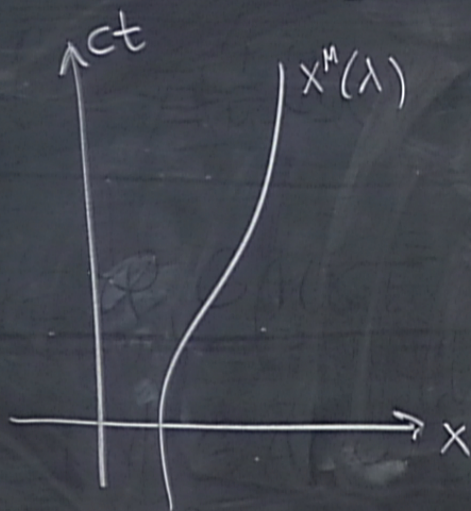


in fact

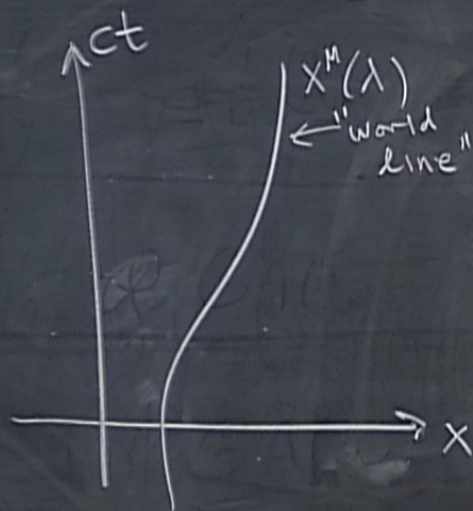
$$L' = \frac{L}{\gamma}$$

$$L' < L$$

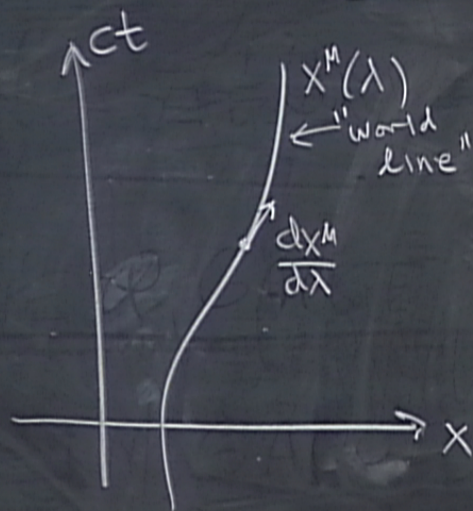
Particles



Particles



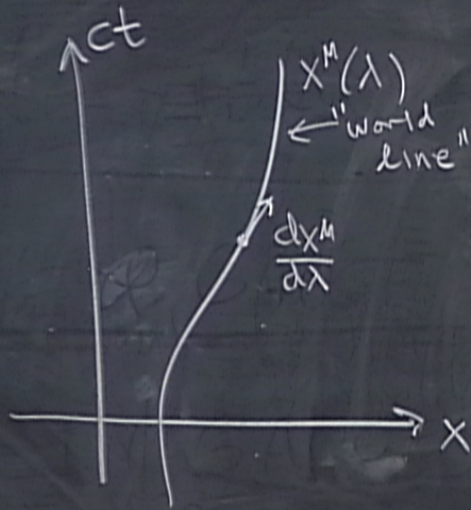
Particles



(parametrised curve)

$$I \int \eta_{\mu\nu}$$

Particles

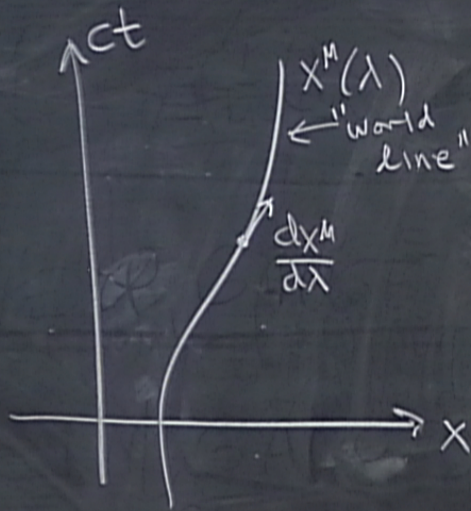


(parametrised curve)

$$\text{If } \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} > 0$$

world line is
spacelike

Particles



(parametrised curve)

$$\text{If } \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} > 0$$

world line is

spacelike

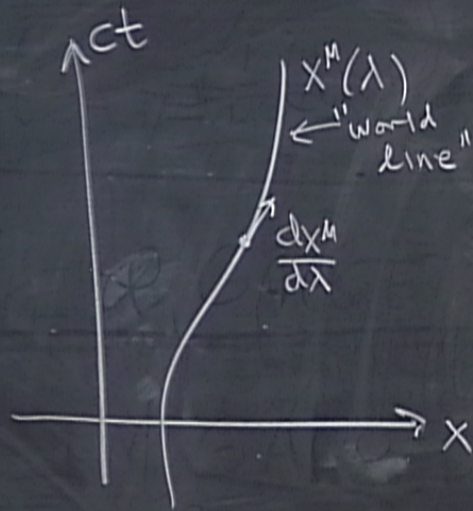
$$= 0$$

null

$$< 0$$

timelike

Particles



(parametrised curve)

$$\text{If } \eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} > 0$$

$$= 0$$

$$< 0$$

world line is

spacelike

null

timelike

← physical (massive) particles

Define proper time by $d\tau^2 = c^2 dt^2 - dx^2$

"
tachyons
(don't exist)

physical
(massive)
particles

Define proper time by $d\tau^2 = c^2 dt^2 - dx^2$

$$\text{i.e. } \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$$

ticks
of
clock

at exist)

ical
sire)
ticer



Define proper time by $d\tau^2 = c^2 dt^2 - dx^2$

(τ measured in m)

$$\text{ie. } \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$$

Let $u^\mu = \frac{dx^\mu}{d\tau}$ be the 4-velocity

$$\eta_{\mu\nu} u^\mu u^\nu = -1$$

ticks
of
clock



4-momentum $p^\mu = m_0 c$ rest mass

~~$G(R) = \frac{1}{4\pi^2 R} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left[\frac{1}{2} (\dot{R} + T) + \frac{1}{2} (\dot{R} - T) \right]$~~

$= \frac{1}{4\pi^2 R} \left(\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left[\frac{1}{2} (\dot{R} + T) + \frac{1}{2} (\dot{R} - T) \right] \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left[\frac{1}{2} (\dot{R} + T) + \frac{1}{2} (\dot{R} - T) \right]$

ADVANCED

RETARDED

LIGHT CONE

$\delta(R-T)$



$$= 1 - v^2/c^2$$

Define proper time by $d\tau^2 = c^2 dt^2 - dx^2$

(τ measured in m)

$$\text{i.e. } \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$$

world line is

spacelike

← "tachyons"
(don't exist)

null

timelike

← physical
(massive)
particle

Let $u^\mu = \frac{dx^\mu}{d\tau}$ be the 4-velocity

u^μ dimⁿless

$$\eta_{\mu\nu} u^\mu u^\nu = -1$$



4-momentum $p^M = m_0 c u^M = \frac{v}{c} \times$ rest mass

obeys $p^2 = \eta_{\mu\nu} p^\mu p^\nu = m_0^2 c^2$
 $= -p^0^2 + p^2$

$p^0 = \sqrt{p^2 + m_0^2 c^2} \approx m_0 c$

ADVANCED

RETARDED

LIGHT CONE



4-momentum $p^M = m_0 c u^M$ rest mass

obeys $p^2 = \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$
 $= -p^0^2 + p^2$

$p^0 = \sqrt{p^2 + m_0^2 c^2} \approx m_0 c$

ADVANCED

RETARDED

LIGHT CONE



4-momentum $p^\mu = m_0 c u^\mu = \frac{m_0 v}{c} \gamma$

obeys $p^\mu p_\mu = -m_0^2 c^2$
 $= -p^0^2 + \mathbf{p}^2$

$p^0 = \sqrt{\mathbf{p}^2 + m_0^2 c^2} \approx m_0 c + \frac{\mathbf{p}^2}{2m_0 c} + \dots$

ADVANCED

RETARDED

LIGHT CONE

4-momentum $p^\mu = m_0 c u^\mu$ rest mass

obeys $p^\mu p_\mu = \eta_{\mu\nu} p^\mu p^\nu = -m_0^2 c^2$
 $= -p^0^2 + \underline{p}^2$

$p^0 = \sqrt{\underline{p}^2 + m_0^2 c^2} \approx m_0 c + \frac{\underline{p}^2}{2m_0 c} + \dots$

$\underline{v} = \frac{\underline{p}}{p^0} = \frac{\underline{p}}{\sqrt{\underline{p}^2 + m_0^2 c^2}}$

$|\underline{v}| < c$

CREATED

LIGHT CONE

$$p^0 c \approx m_0 c^2 + \frac{p^2}{2m}$$

↑
KE

$$p^0 c \approx m_0 c^2 + \frac{p^2}{2m} + \dots$$

$\uparrow$ rest mass $\uparrow$ KE

$$\Rightarrow E = m_0 c^2$$

?

$$p^0 \approx m_0 c^2 + \frac{p^2}{2m} + \dots$$

$\uparrow$ rest mass $\uparrow$ KE

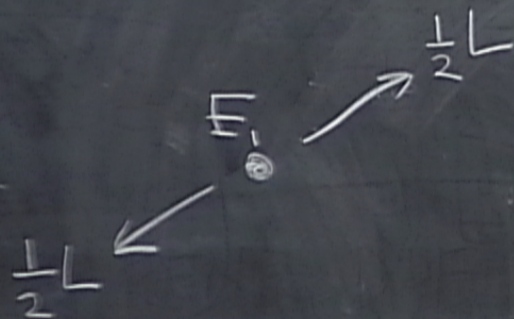
Einstein

$$\Rightarrow E = m_0 c^2$$

?

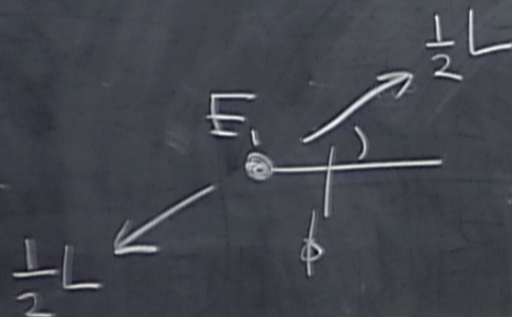
Einstein

E_0



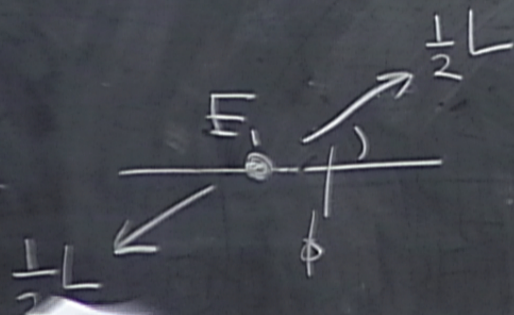
Einstein

E_0



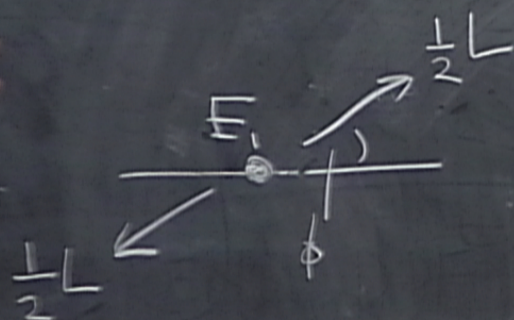
Einstein
rest frame
 E_0

moving frame
 $E_0 \rightarrow$



Einstein
rest frame
 E_0

moving frame
 $E_0 \rightarrow$



$$\frac{\frac{1}{2}L(1 + \frac{v}{c} \cos \phi)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\frac{\frac{1}{2}L(1 - \frac{v}{c} \cos \phi)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p^0 c \approx m_0 c^2 + \frac{p^2}{2m} + \dots$$

$\uparrow$ rest mass $\uparrow$ KE

$$\Rightarrow E = m_0 c^2$$

?

Einstein
rest frame
 E_0

before

after

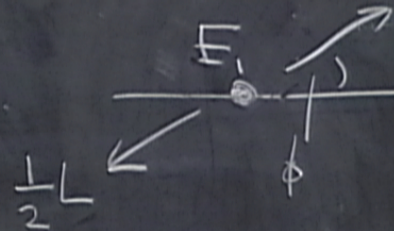
rest
frame

$$E_0$$

$$E_1 + \frac{1}{2}L + \frac{1}{2}L$$

moving
frame

$$H_0$$



moving frame

H_0


$$\frac{\frac{1}{2}L(1 - \frac{v}{c} \cos \phi)}{\sqrt{1 - \frac{v^2}{c^2}}}$$



$$\frac{\frac{1}{2}L(1 + \frac{v}{c} \cos \phi)}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow E = m_0 c^2$$

before

after

$$E_0$$

$$E_1 + \frac{1}{2}L + \frac{1}{2}L$$

$$H_0$$

$$H_1 + \frac{L}{\sqrt{1 - (v/c)^2}}$$

$$K_0 = H_0 - E_0$$

$$K_0 = H_0 - E_0, \quad K_1 = H_1 - E_1$$

$$K_0 - K_1 = L \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \approx \frac{1}{2} L \frac{v^2}{c^2} \quad v \ll c.$$