Title: Quantum Theory - Lecture 11

Date: Sep 26, 2011 01:30 PM

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Abstract:

Any operator A on $\mathbb{C}^2$ can be written as $A = a_0 I + \sum_{i=1}^{3} a_i \sigma_i$ $\quad (a_0, a_i \in \mathbb{C})$

For a density matrix ρ we have $\boxed{\rho = \rho^\dagger, \ \mathrm{Tr}(\rho) = 1}$.

So $\quad \rho = a_0 I + \sum a_i \sigma_i$, $\quad \rho^\dagger = a_0^* I + \sum a_i^* \sigma_i \Rightarrow \boxed{a_0, a_i \in \mathbb{R}}$

$1 = \mathrm{Tr}(\rho) = 2a_0 \qquad$ So $\boxed{a_0 = 1/2}$.

We can write $\boxed{\rho = \tfrac{1}{2}(I + \underline{a}\cdot\underline{\sigma}), \quad \text{where} \quad a_i := \mathrm{Tr}(\rho\sigma_i) \in \mathbb{R}}$

$$= \frac{1}{2}\begin{pmatrix} 1+a_3 & a_1 - i a_2 \\ a_1 + i a_2 & 1 - a_3 \end{pmatrix}$$

Eigenvalues obey $(1+a_3-2\lambda)(1-a_3-2\lambda) - a_1^2 - a_2^2 = 0 \Rightarrow (2\lambda-1)^2 = \sum a_i^2$

$$\Rightarrow \lambda = \tfrac{1}{2}(1 \pm |\underline{a}|) = \lambda_\pm$$

$\boxed{\rho \text{ is positive semidefinite}} \Rightarrow \lambda_+ \geq \lambda_- \geq 0 \Rightarrow \boxed{|\underline{a}| \leq 1}$

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Note also that if ρ is pure, $\rho = |\psi\rangle\langle\psi|$, $\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1$.

We have $\mathrm{Tr}(\rho^2) = \frac{1}{4}\mathrm{Tr}\left((1+\underline{a}.\underline{\sigma})(1+\underline{a}.\underline{\sigma})\right) = \frac{1}{2}\left(1+|\underline{a}|^2\right)$

So $\boxed{\rho \text{ is pure} \iff |\underline{a}| = 1.}$

And notice our representation is linear

$$\rho_1 = \tfrac{1}{2}(1+\underline{a}_1.\underline{\sigma})$$
$$\rho_2 = \tfrac{1}{2}(1+\underline{a}_2.\underline{\sigma})$$

$$\Rightarrow t\rho_1 + (1-t)\rho_2 = \tfrac{1}{2}\left(1 + (t\underline{a}_1 + (1-t)\underline{a}_2).\underline{\sigma}\right)$$

We have a linear representation of 2D density matrices, with pure states given by $|\underline{a}| = 1$, the surface of a sphere in 3D.

It's the Bloch sphere! The Bloch vector $\underline{a}$ gives a geometric representation of ρ on or within the sphere.

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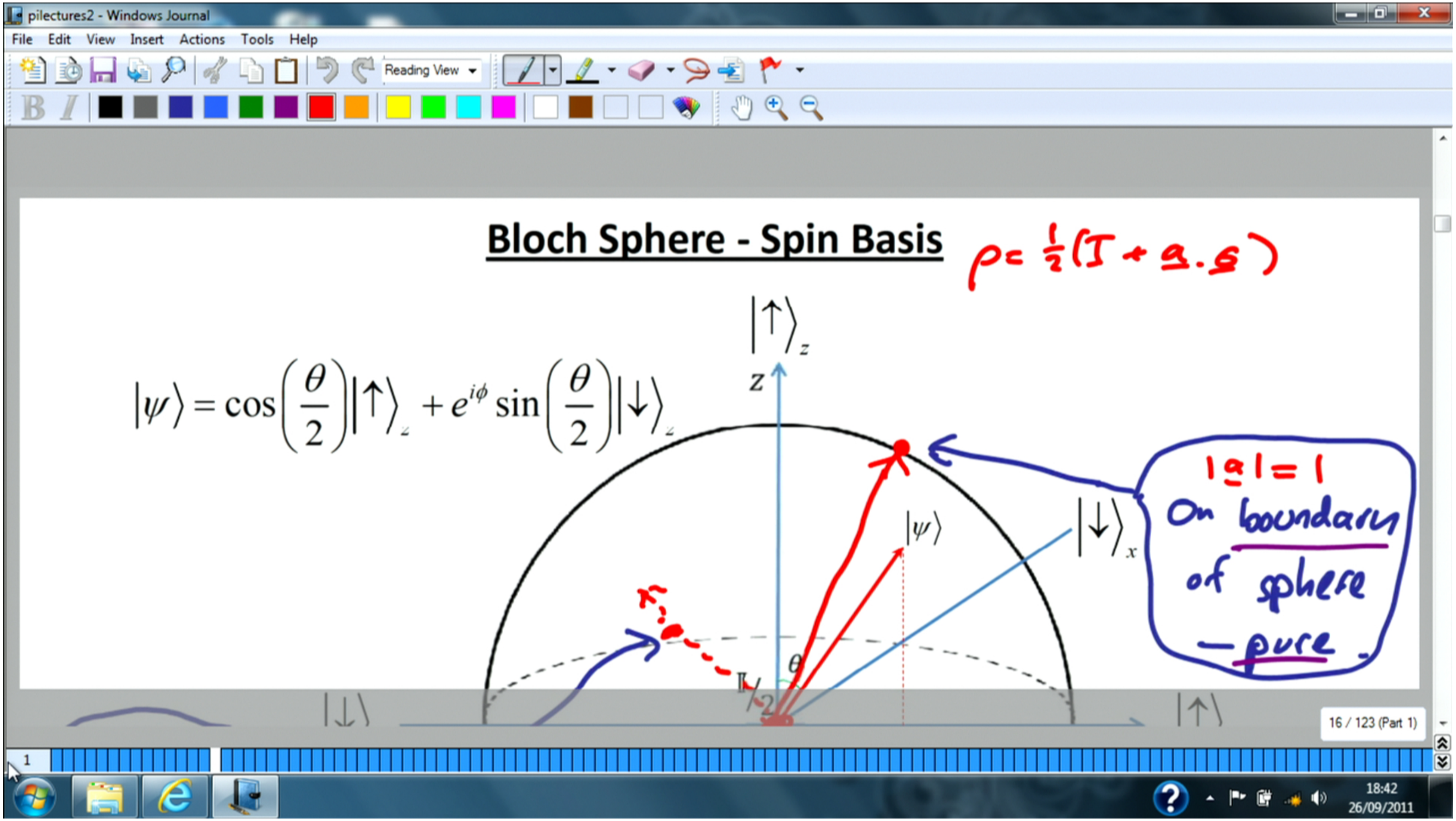
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Reading View
Bloch Sphere - Spin Basis
$\rho = \frac{1}{2}(I + \underline{a}.\underline{\sigma})$
$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle_z + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle_z$
$|\uparrow\rangle_z$
z
$|\psi\rangle$
$|\downarrow\rangle_x$
θ
$|\underline{a}| = 1$
On boundary of sphere - pure
$|\downarrow\rangle$
$|\uparrow\rangle$
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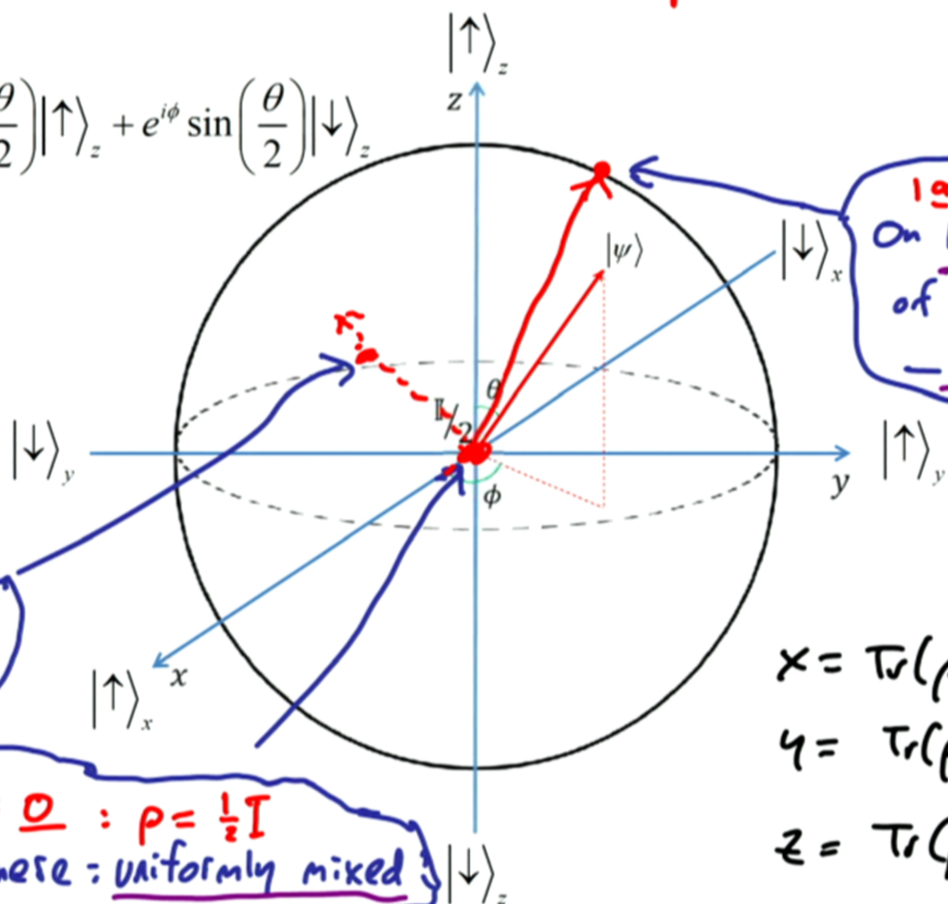
Bloch Sphere - Spin Basis

$\rho = \frac{1}{2}(I + \underline{a}\cdot\underline{\sigma})$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle_z + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle_z$$



$|\underline{a}| = 1$
On boundary of sphere – pure.

$|\underline{a}| < 1$
In interior of sphere – mixed.

$\underline{a} = \underline{0}$: $\rho = \frac{1}{2}I$
Centre of sphere: uniformly mixed

$x = Tr(\rho\sigma_x)$

$y = Tr(\rho\sigma_y)$

$z = Tr(\rho\sigma_z)$

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Can proper mixed states ever objectively describe real physical systems?

Or is probabilistic mixing always a reflection of our subjective ignorance of some objective facts? (Our colleague rolled dice: she got a definite outcome and prepared a definite state — we just don't know which.)

We don't know for sure. The density matrix evolution law and measurement postulate are self-contained (in closed form) and consistent — so it makes logical sense to postulate an initially mixed state that (depending on measurements) may stay mixed forever.

$$\rho(0) \to e^{-iHt/\hbar}\,\rho(0)\,e^{iHt/\hbar}, \ldots$$

But we don't know if nature used this possibility. (Also difficult to tell for sure whether any mixed state is proper or improper — as we'll see ...)

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To check consistency of our definition of the density matrix and the measurement postulate, we need to check this diagram also commutes

Not quite so obvious!

The a priori probabilities $\{p_i\}$ changed to $\{p_i'\}$ after a measurement with outcome j

(some states are likelier to produce j than others)

Density Matrix - Measurement

$$
\begin{tikzcd}
\boxed{\{p_i, |\psi_i\rangle\}} \arrow[r, dashed, "\text{measurement}", "\text{outcome } j"'] \arrow[d, dashed, "\text{density matrix}"] & \boxed{\left\{p_i', \frac{P_j|\psi_i\rangle}{|P_j|\psi_i\rangle|}\right\}} \arrow[d, dashed, "\text{density matrix}"] \\
\boxed{\rho = \sum p_i |\psi_i\rangle\langle\psi_i|} \arrow[r, dashed, "\text{measurement}", "\text{outcome } j"'] & \boxed{\frac{P_j \rho P_j}{Tr(P_j \rho P_j)}}
\end{tikzcd}
$$

We need to use our earlier calculation of the $\{p_i'\}$ to verify this works

OK. So, explicitly:

$$\{p_i, |\psi_i\rangle\} \xrightarrow[\text{outcome } j]{\text{measurement}} \left\{ \frac{p_i\langle\psi_i|P_j|\psi_i\rangle}{\sum_i p_i\langle\psi_i|P_j|\psi_i\rangle} , \frac{P_j|\psi_i\rangle}{|P_j|\psi_i\rangle|} \right\}$$

$\downarrow$ density matrix $\qquad\qquad$ $\downarrow$ density matrix

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$$

$$\sum_i \left(\left(\frac{p_i \cancel{\langle\psi_i|P_j|\psi_i\rangle}}{\sum_i p_i\langle\psi_i|P_j|\psi_i\rangle} \right) \frac{P_j|\psi_i\rangle\langle\psi_i|P_j}{\cancel{|P_j|\psi_i\rangle|^2}} \right)$$

$$\|$$

$$\frac{P_j \rho P_j}{\mathrm{Tr}(\rho P_j)} = \frac{P_j\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)P_j}{\sum_i p_i\langle\psi_i|P_j|\psi_i\rangle}$$

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Reading View

Mathematically, $\{p_i, |\psi_i\rangle\}$ carries more info. than ρ.

But the extra information makes no practical difference to us.

Ensemble 1: probabilities $\{\frac{1}{2}, \frac{1}{2}\}$ states $\{|\uparrow\rangle, |\downarrow\rangle\}$

Ensemble 2: probabilities $\{\frac{1}{2}, \frac{1}{2}\}$ states $\{|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),$

$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)\}$

These describe mathematically distinct ensembles, physically distinct situations.

But we can't distinguish them experimentally:

$$\rho_1 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_2 = \frac{1}{4}\left((|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + (|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)\right) = \rho$$

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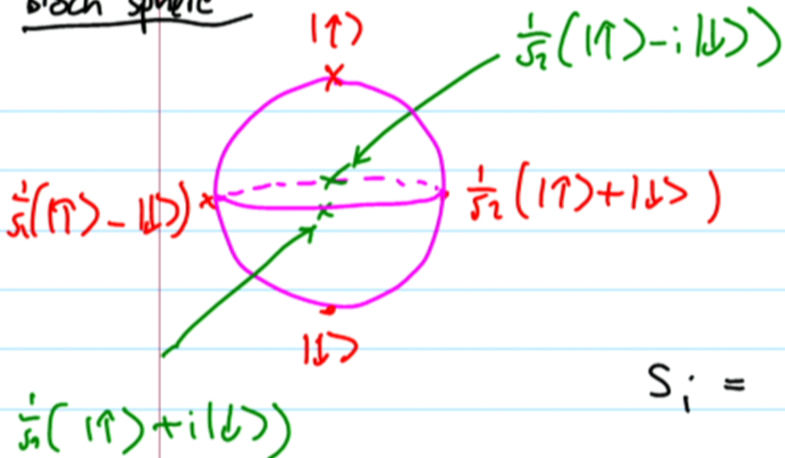
Bloch sphere



We've created a simple geometric representation of the states of a spin-½ particle, which turns out to have some very nice properties

$$S_i = \tfrac{1}{2}\hbar\,\sigma_i \qquad (i = 1,2,3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

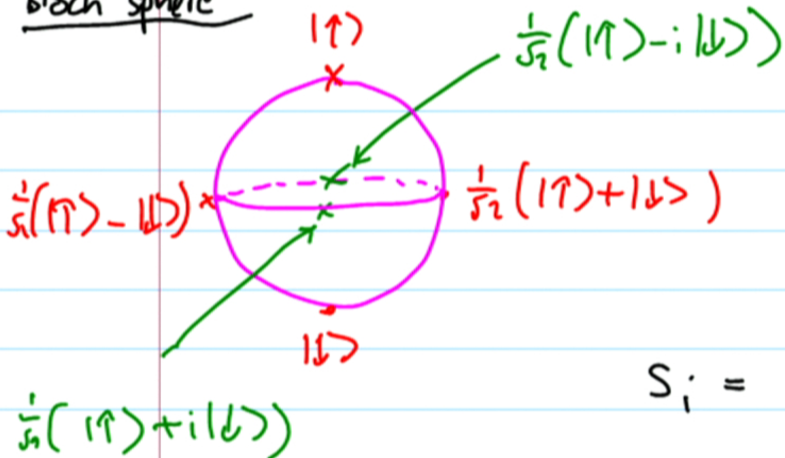
(Also called σ_x or X, σ_y or Y, σ_z or Z).

$$\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k \,.$$

Pauli matrices

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$

Bloch sphere



We've created a simple geometric representation of the states of a spin-1/2 particle, which turns out to have some very nice properties

$$S_i = \frac{1}{2}\hbar\,\sigma_i \qquad (i = 1, 2, 3)$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Reading View

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$$\rho_1 = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right)$$

$$\rho_2 = \frac{1}{4}\left((|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) + (|\uparrow\rangle - |\downarrow\rangle)(\langle\uparrow| - \langle\downarrow|)\right) = \rho$$

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Mixed States and Density Matrices

Suppose you're given a state $|\psi\rangle$

and know it's one of $|\psi_1\rangle, \ldots\ldots, |\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots, p_n$

(And this is a complete list of possibilities: $\sum_{i=1}^{n} p_i = 1$, $p_i \geq 0$.

And you can't learn anything more about the preparation of $|\psi\rangle$.)

How could this happen? Secretive colleague with a random number generator;

Imperfect preparation device with known error statistics, ...

What can you do with this information?

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Our project: (1) define an operator on H, the density matrix

$$\rho(0) = \sum_{i=1}^{n} p_i \, |\psi_i(0)\rangle\langle\psi_i(0)|$$ at time $t=0$.

(2) Introduce the evolution law $i\hbar \frac{d}{dt}\rho(t) = [H, \rho(t)]$

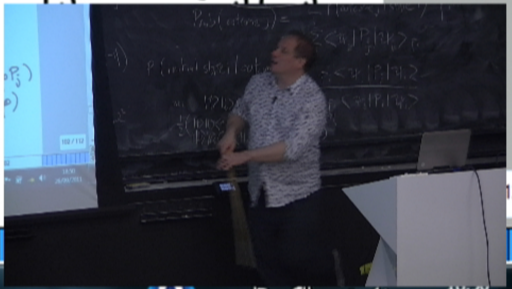
which has solution

$$\rho(t) = \exp\left(\frac{-iHt}{\hbar}\right)\rho(0)\exp\left(\frac{iHt}{\hbar}\right)$$

(3) Introduce the measurement postulate: a measurement defined by projectors $\{P_j\}$ on ρ produces outcome i with probability $p_i = \mathrm{Tr}(P_i \rho P_i)$ and post-measurement state

$$\frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho P_j)} = \frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho)}$$

(4) Check this is all consistent!



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Our project: ① define an operator on H, the density matrix

$$\rho(0) = \sum_{i=1}^{n} p_i |\psi_i(0)\rangle\langle\psi_i(0)| \quad \text{at time } t=0.$$

② Introduce the evolution law $i\hbar \frac{d}{dt}\rho(t) = [H, \rho(t)]$

which has solution $\rho(t) = \exp\left(\frac{-iHt}{\hbar}\right)\rho(0)\exp\left(\frac{iHt}{\hbar}\right)$

③ Introduce the measurement postulate: a measurement defined by projectors $\{P_j\}$ on ρ produces outcome i with probability $p_j = \mathrm{Tr}(P_j \rho P_j) = \mathrm{Tr}(P_j \rho)$

and post-measurement state $\frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho P_j)} = \frac{P_j \rho P_j}{\mathrm{Tr}(P_j \rho)}$

④ Check this is all consistent!

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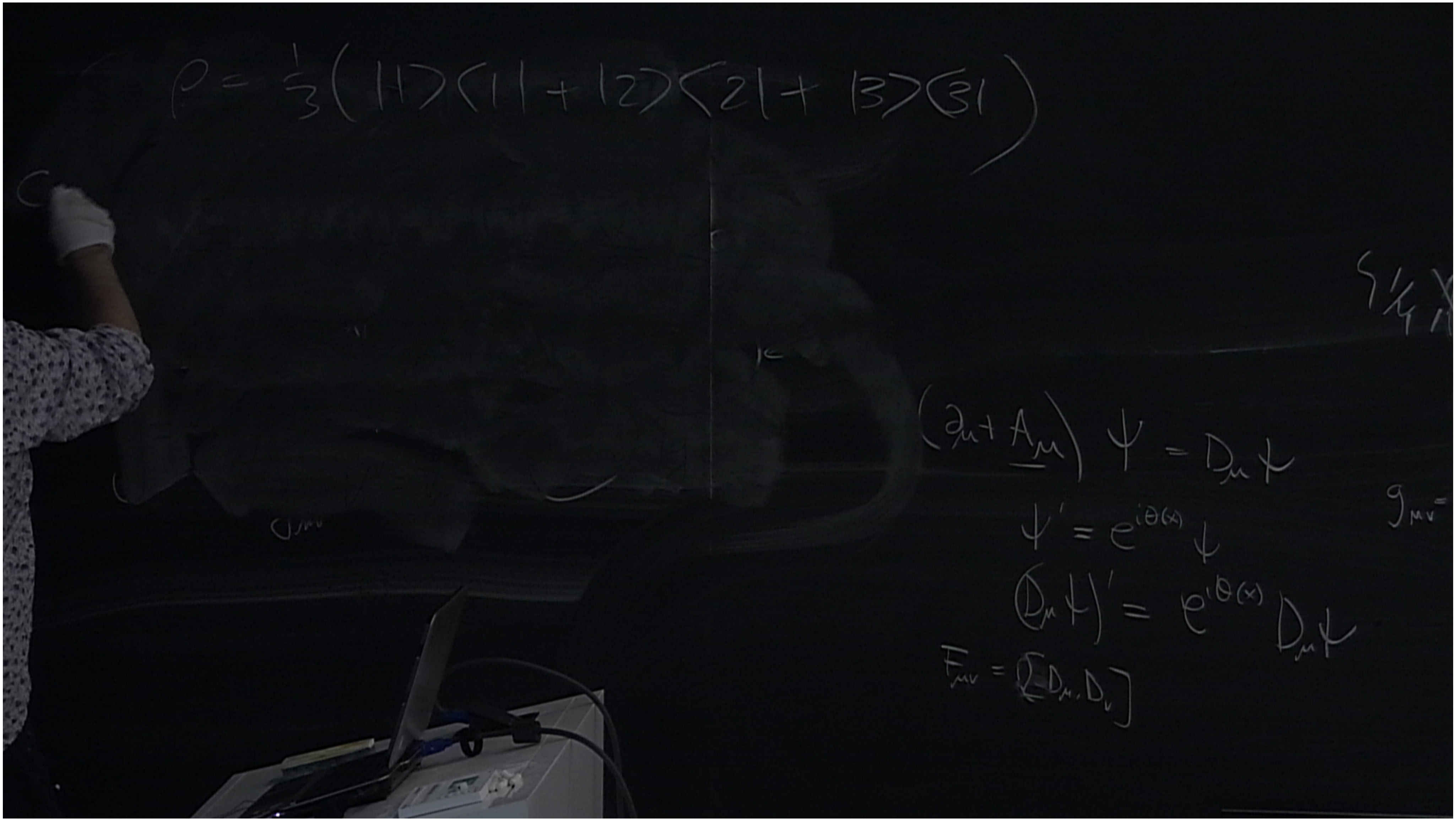
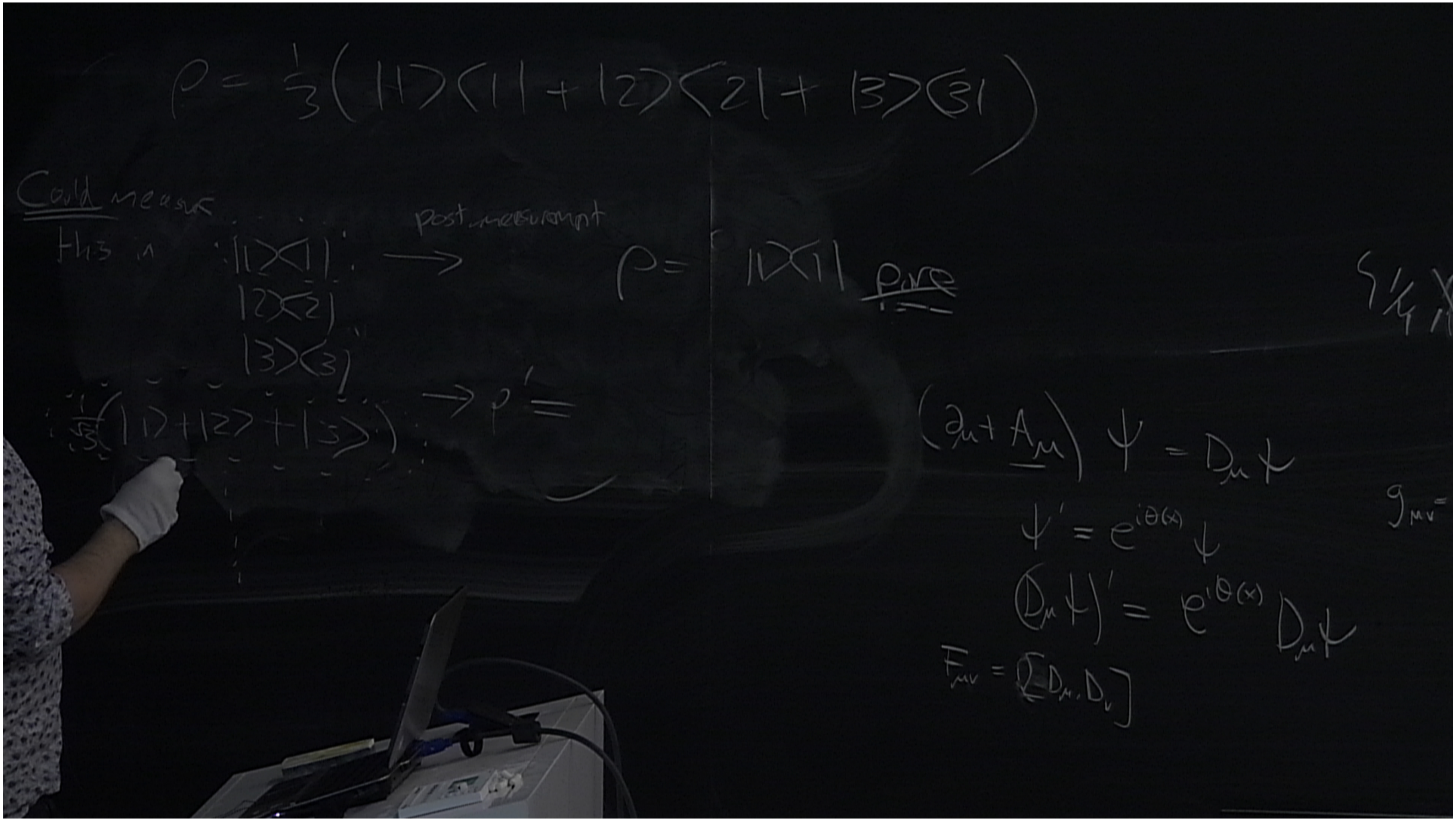

ρ = 1/3 (|1⟩⟨1| + |2⟩⟨2| + |3⟩⟨3|)
(∂μ + Aμ) ψ = Dμ ψ
ψ' = e^{iθ(x)} ψ
(Dμ ψ)' = e^{iθ(x)} Dμ ψ
Fμν = [Dμ, Dν]
gμν =


$\rho = \frac{1}{3}\left(|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| \right)$
Could measure
this in
$|1\rangle\langle 1|$
$|2\rangle\langle 2|$
$|3\rangle\langle 3|$
post measurement
$\rho = |1\rangle\langle 1|$ pure
$\frac{1}{\sqrt{3}}\left(|1\rangle + |2\rangle + |3\rangle\right) \rightarrow \rho' =$
$(\partial_\mu + A_\mu)\psi = D_\mu \psi$
$\psi' = e^{i\theta(x)}\psi$
$(D_\mu \psi)' = e^{i\theta(x)} D_\mu \psi$
$F_{\mu\nu} = [D_\mu, D_\nu]$
$g_{\mu\nu} =$

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Generally, if $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ is mixed, and we measure with a complete set of 1D projectors Q_i, ~~then~~ and get outcome j (say)

$$\rho \rightarrow \frac{Q_j \rho Q_j}{\mathrm{Tr}(Q_j \rho)} =$$

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Reading View

Generally, if $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ is mixed,

and we measure with a complete set of 1D projectors

Q_i, ~~then~~ and get outcome j (say)

$Q_j = |q_j\rangle\langle q_j|$

$$\rho \rightarrow \frac{Q_j \rho Q_j}{Tr(Q_j \rho)} = |q_j\rangle\langle q_j|$$

mixed

pure

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Reading View

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and we measure with a complete set of 1D projectors

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$$Q_j = |q_j\rangle\langle q_j|$$

$$\underbrace{\rho}_{\text{mixed}} \to \frac{Q_j \rho Q_j}{Tr(Q_j \rho)} = \underbrace{|q_j\rangle\langle q_j|}_{\text{pure}}$$

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Reading View

Generally, if $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ is mixed,

and we measure with a complete set of 1D projectors

Q_i, ~~then~~ and get outcome j (say)

$$\boxed{Q_j = |a_j\rangle\langle a_j|}$$

$$\rho \rightarrow \frac{Q_j \rho Q_j}{Tr(Q_j \rho)} = |a_j\rangle\langle a_j|$$

mixed

pure

$$Q_j \rho Q_j = |a_j\rangle\langle a_j| \sum_{p_i} |\psi_i\rangle\langle\psi_i| a_j\rangle\langle a_j|$$

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Generally, if $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ is mixed,

and we measure with a complete set of 1D projectors

Q_i, ~~then~~ and get outcome j (say)

$$\boxed{Q_j = |q_j\rangle\langle q_j|}$$

$$\underset{\text{mixed}}{\rho} \rightarrow \frac{Q_j \rho Q_j}{Tr(Q_j \rho)} = \underset{\text{pure}}{|q_j\rangle\langle q_j|}$$

$$Q_j \rho Q_j = |q_j\rangle\langle q_j| \sum p_i |\psi_i\rangle\langle\psi_i| q_j\rangle\langle q_j|$$

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Generally, if $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ is mixed,

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Q_i, ~~then~~ and get outcome j (say)

$$\boxed{Q_j = |a_j\rangle\langle a_j|}$$

$$\rho \rightarrow \frac{Q_j \rho Q_j}{\mathrm{Tr}(Q_j \rho)} = |a_j\rangle\langle a_j|$$

mixed

pure

$$Q_j \rho Q_j = |a_j\rangle \langle a_j | \sum p_i |\psi_i\rangle\langle\psi_i| a_j\rangle \langle a_j|$$

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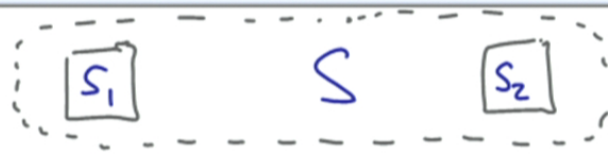
Composite Systems and Entanglement



We're interested in systems comprising two or more identifiable subsystems $S = S_1 + S_2$. Maybe separated, maybe occupying the same region; maybe different particle types, maybe same.

Quantum theory tells us the space of states of S is $H = H_1 \otimes H_2$ — tensor product space.

Linear algebra tells us: $\dim(H) = \dim(H_1) . \dim(H_2)$

If $\{ |e_1\rangle \ldots |e_{n_1}\rangle \}$ orthonormal basis of H_1

$\{ |f_1\rangle \ldots |f_{n_2}\rangle \}$ orthonormal basis of H_2

Then $\{ |e_i\rangle \otimes |f_j\rangle \}_{i=1, j=1}^{n_1, n_2}$ orthonormal basis of H.

General state in H: $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} |e_i\rangle \otimes |f_j\rangle$

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If $\{ |e_1\rangle .. |e_{n_1}\rangle \}$ orthormal basis of H_1

$\{ |f_1\rangle ... |f_{n_2}\rangle \}$ orthonormal basis of H_2

Then $\{ |e_i\rangle \otimes |f_j\rangle \}_{i=1}^{n_1}{}_{j=1}^{n_2}$ orthonormal basis of H.

General state in H : $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} |e_i\rangle \otimes |f_j\rangle$

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General state in $\mathcal{H}$: $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} \, |e_i\rangle \otimes |f_j\rangle$

We can single out a special class : <u>product states</u>

$$\begin{array}{ccccc} |\psi\rangle & = & |\psi_1\rangle & \otimes & |\psi_2\rangle \\ \cap & & \cap & & \cap \\ H & & H_1 & & H_2 \end{array}$$

(Example : our chosen basis vectors $|e_i\rangle \otimes |f_j\rangle$

But also : $\left(\sum a_i |e_i\rangle\right) \otimes \left(\sum b_j |f_j\rangle\right)$.)

Note: almost all states are <u>not</u> product states. (The set of product states is measure zero in the set of all pure states in H.)

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General state in H : $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} |e_i\rangle \otimes |f_j\rangle$

general state parametrised.

We can single out a special class : product states

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$\begin{array}{ccc} \cap & \cap & \cap \\ H & H_1 & H_2 \end{array}$$

product states

complex parameters

$(n_1 - 1)$

$+ (n_2 - 1)$

(Example : our chosen basis vectors $|e_i\rangle \otimes |f_j\rangle$

But also : $\left(\sum a_i |e_i\rangle\right) \otimes \left(\sum b_j |f_j\rangle\right)$.)

Note : almost all states are not product states. (The set of product states is measure zero in the set of all pure states in H.)

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General state in $\mathcal{H}$: $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} \, |e_i\rangle \otimes |f_j\rangle$

general state parametrised by $m_1 m_2 - 1$ parameters

We can single out a special class : product states

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$\begin{array}{ccc} \cap & \cap & \cap \\ \mathcal{H} & \mathcal{H}_1 & \mathcal{H}_2 \end{array}$$

product states complex parameters $(m_1 - 1) + (m_2 - 1)$

(Example : our chosen basis vectors $|e_i\rangle \otimes |f_j\rangle$

But also : $\overline{\left(\sum a_i |e_i\rangle\right) \otimes \left(\sum b_j |f_j\rangle\right)}$.)

Note: almost all states are <u>not</u> product states. (The set of product states is measure zero in the set of all pure states in $\mathcal{H}$.)

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Comments on notation (1) We will sometimes suppress the $\otimes$ symbol, when it's (hopefully!) clear from the context that it's implicit.

E.g. we write $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} |e_i\rangle \otimes |f_j\rangle$ as $|\psi\rangle = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{ij} |e_i\rangle |f_j\rangle$

(2) We define density matrices on product spaces $H_1 \otimes H_2$ with the convention that the H_1 vectors stay on the left for both bra and ket states.

So $$|\psi\rangle\langle\psi| = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{i'=1}^{n_1} \sum_{j'=1}^{n_2} a_{ij} a^*_{i'j'} |e_i\rangle |f_j\rangle \langle e_{i'}| \langle f_{j'}|$$

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Physical significance of product states I

Quantum theory tells us observables on either subsystem correspond to hermitian operators on the corresponding factor:

Observable A on S_1 $\longleftrightarrow$ $A \otimes I$ acting on $H = H_1 \otimes H_2$.

If we measure A on S_1, when S is in a state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$

we see
$$\langle A \rangle = \langle \psi | A \otimes I | \psi \rangle = \langle \psi_1 | \otimes \langle \psi_2 | \, A \otimes I \, |\psi_1\rangle \otimes |\psi_2\rangle$$
$$= \langle \psi_1 | A | \psi_1 \rangle . \langle \psi_2 | \psi_2 \rangle$$
$$= \langle \psi_1 | A | \psi_1 \rangle$$

i.e. same calculation as if we ignored S_2 and took $|\psi_1\rangle$ as the state of S_1. We can (for this calculation) treat S_1 and S_2 as separate systems with individual states $|\psi_1\rangle$, $|\psi_2\rangle$.

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Physical significance of product states II : isolated subsystems

We say S_1 and S_2 are isolated if there is no interaction term in their joint Hamiltonian:

$$H = H_1 \otimes I + I \otimes H_2 \cancel{+ H_I}$$

and thus the evolution operator factorizes:

$$e^{-iHt/\hbar} = e^{-i\frac{H_1 \otimes I t}{\hbar}} \, e^{-i\frac{I \otimes H_2 t}{\hbar}} = \underbrace{e^{-iH_1t/\hbar}}_{U_1(t)} \otimes \underbrace{e^{-iH_2t/\hbar}}_{U_2(t)}$$

$$U(t) = U_1(t) \otimes U_2(t)$$

If two isolated systems are initially in a product state $|\psi(0)\rangle = |\psi_1(0)\rangle \otimes |\psi_2(0)\rangle$ they remain so under unitary evolution: $|\psi(t)\rangle = U_1(t) \otimes U_2(t) \, |\psi_1(0)\rangle \otimes |\psi_2(0)\rangle$

$$= |\psi_1(t)\rangle \otimes |\psi_2(t)\rangle$$

where $|\psi_i(t)\rangle = U_i(t)\,|\psi_i(0)\rangle$.

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$$= |\psi_1(t)\rangle \otimes |\psi_2(t)\rangle$$

where $|\psi_i(t)\rangle = U_i(t) \, |\psi_i(0)\rangle$.

And also after subsystem measurements:

$$|\psi(t)\rangle \rightarrow \frac{P \otimes I \, |\psi(t)\rangle}{|P \otimes I \, |\psi(t)\rangle|} = \frac{P|\psi_1(t)\rangle}{|P|\psi_1(t)\rangle|} \otimes |\psi_2(t)\rangle$$
$$= |\psi_1'(t)\rangle \otimes |\psi_2(t)\rangle$$

after a measurement on S_1 with outcome corresponding to $P \otimes I$. (Similarly S_2.)

For isolated systems in a product state, the individual states $|\psi_i(t)\rangle$ characterize evolutions + measurements, i.e. all of quantum physics. They are the states of S_i in the same sense that $|\psi_1(t)\rangle \otimes |\psi_2(t)\rangle$ is the state of S

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For isolated systems in a product state, the individual states $|\psi_i(t)\rangle$ characterize evolutions + measurements, i.e. all of quantum physics. They are the states of S_i in the same sense that $|\psi_1(t)\rangle \otimes |\psi_2(t)\rangle$ is the state of S.

It's then just a matter of convention whether we treat $S = S_1 + S_2$ a composite system or S_1 and S_2 two separate systems: nothing depends on it.

But (1) almost all states of $S_1 + S_2$ are not product states
(2) and systems that are isolated now may not always have been (could e.g. have a common source in an experiment, or star, or.....big bang)

What can we say about the physics of $S_1 + S_2$ if they're not in a product state? Let's start by giving these states a name:

Def^n. A pure state $|\psi\rangle \in H_1 \otimes H_2$ is entangled if it can't be written as a product state: $|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$.

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Schrödinger coined the term 'entanglement' to describe this peculiar connection between quantum systems (Schrödinger, 1935; p. 555):

> When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled.

He added (Schrödinger, 1935; p. 555):

> Another way of expressing the peculiar situation is: the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of all its *parts,* even though they may be entirely separate and therefore virtually capable of being 'best possibly known,' i.e., of possessing, each of them, a representative of its own. The lack of knowledge is by no means due to the interaction being insufficiently known — at least not in the way that it could possibly be known more completely — it is due to the interaction itself.

Schrödinger, E. (1935) "Discussion of Probability Relations Between Separated Systems,," *Proceedings of the Cambridge Philosophical Society*, 31: 555–563; 32 (1936): 446–451.

Text from Stanford Encyclopaedia of Philosophy article on "Quantum Information and Entanglement" by Jeffrey Bub.

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How can we tell? ① In principle, linear algebra $|\psi\rangle = \sum a_{ij} |e_i\rangle \otimes |f_j\rangle$

$$\stackrel{?}{=} \sum a_i |e_i\rangle \otimes \sum b_j |f_j\rangle$$

$\Rightarrow$ $a_{ij} = a_i b_j$: check if this is solvable.

② Slicker + more illuminating : consider $|\psi\rangle\langle\psi|$ and its <u>partial traces</u>

$$\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) = \sum_i \langle e_i|\psi\rangle\langle\psi|e_i\rangle$$

$\in H_1$ $\in H_1 \otimes H_2$

NB <u>partial inner product</u> – these aren't numbers but <u>bra and ket</u> states in H_2 :

$$\mathrm{Tr}_{H_2}(|\psi\rangle\langle\psi|) = \sum_j \langle f_j|\psi\rangle\langle\psi|f_j\rangle$$ ← Similarly, <u>bra and ket</u> in H_1.

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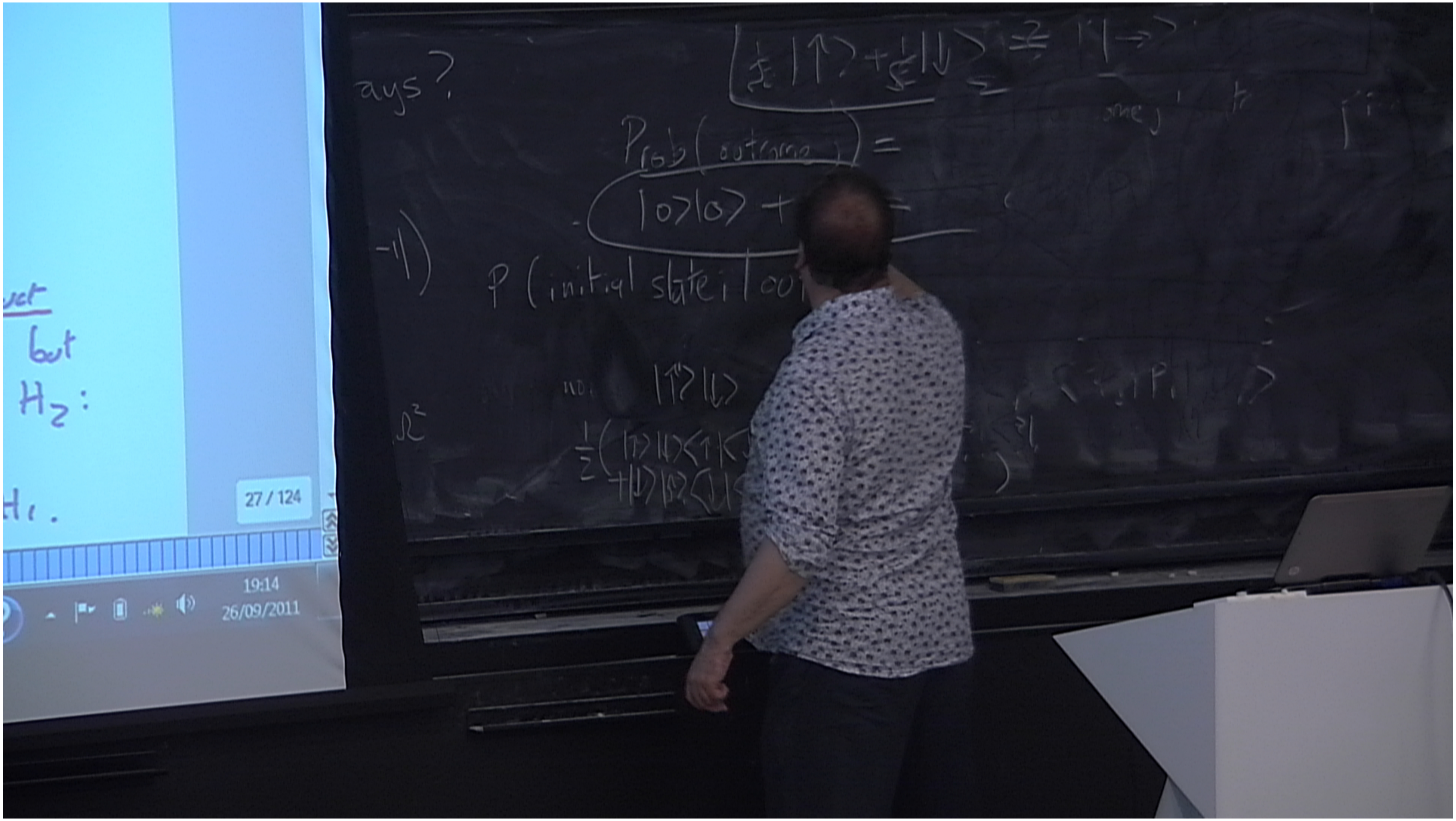
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ays ?
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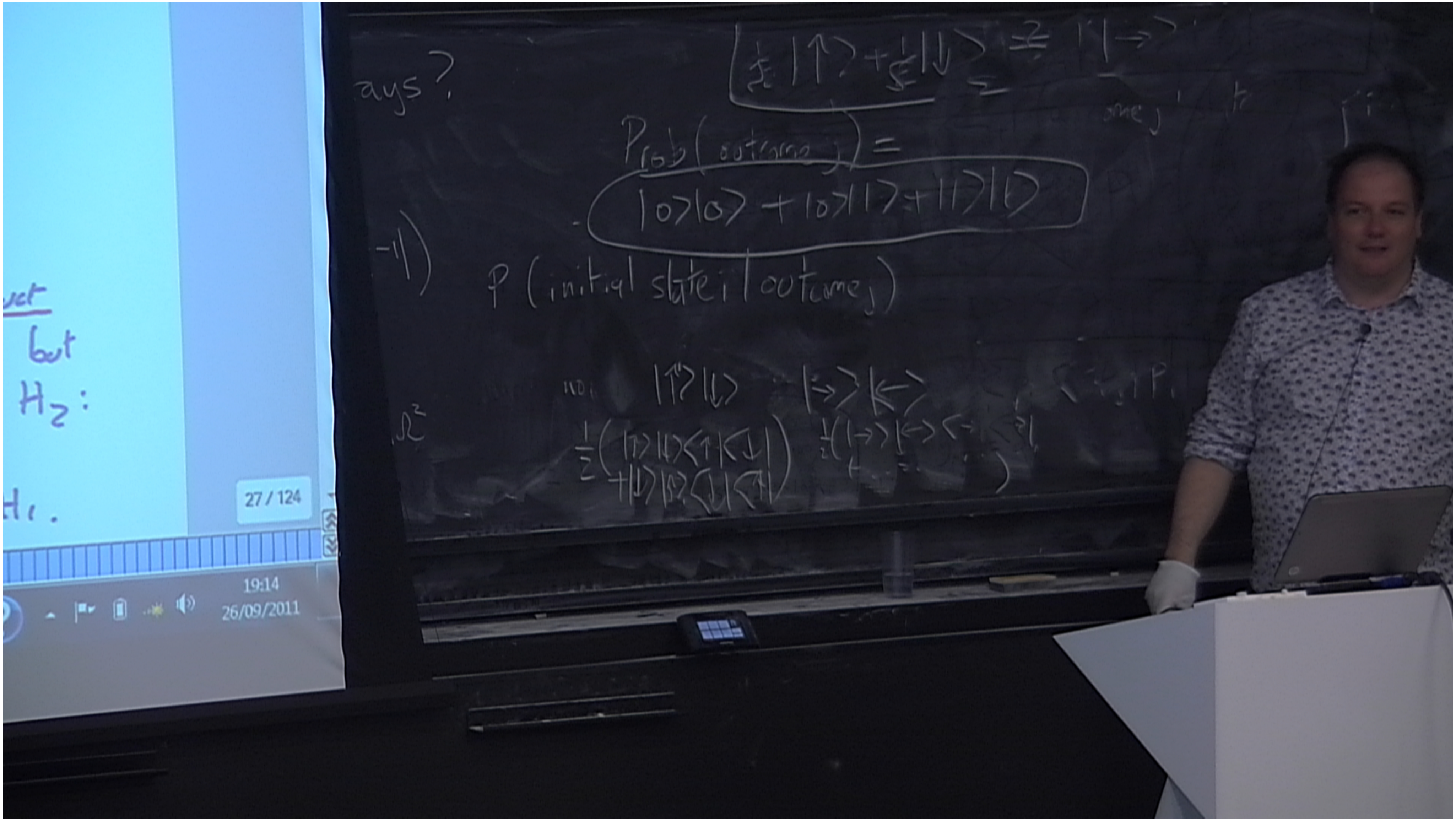
P (initial state | out



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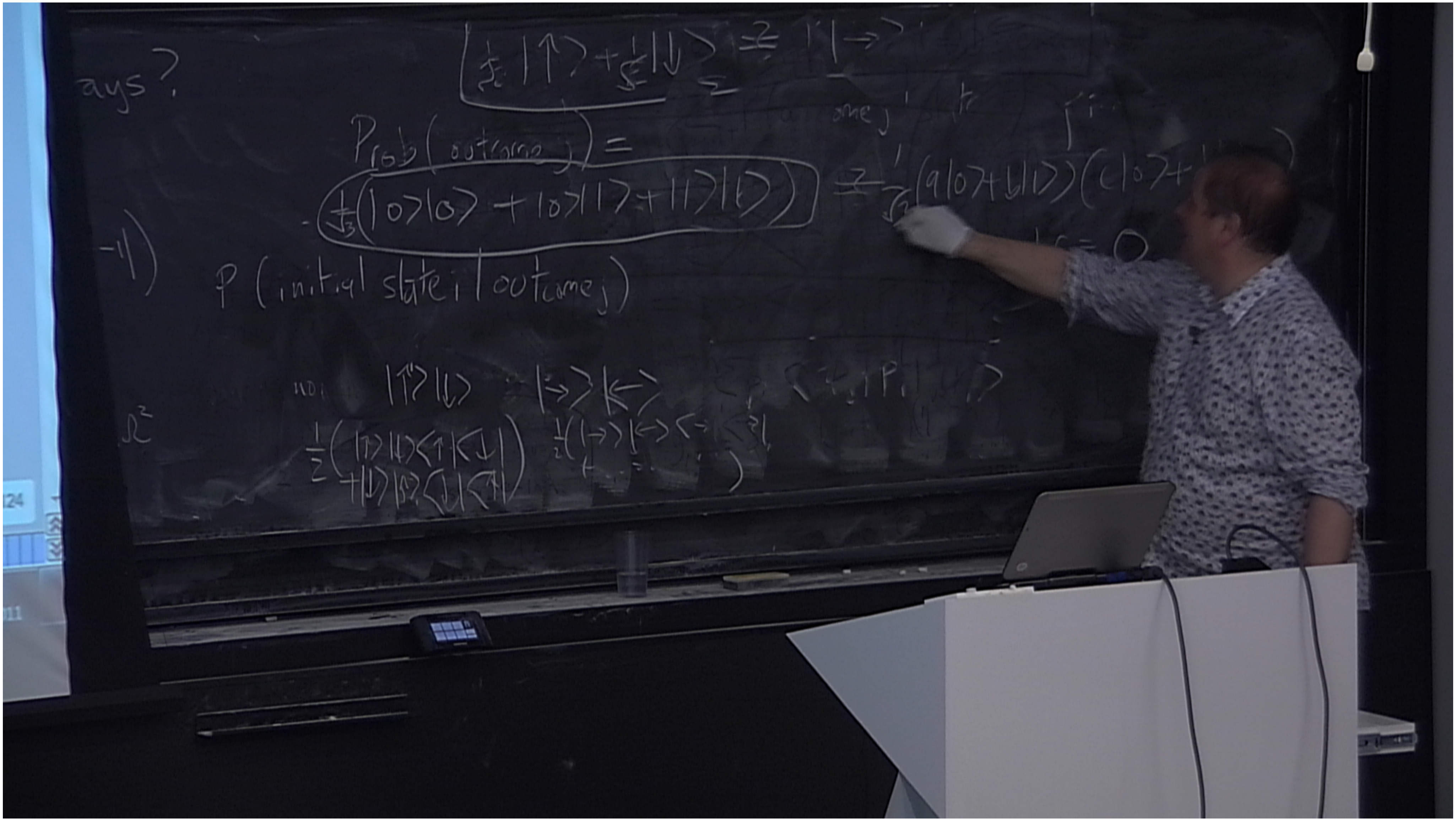
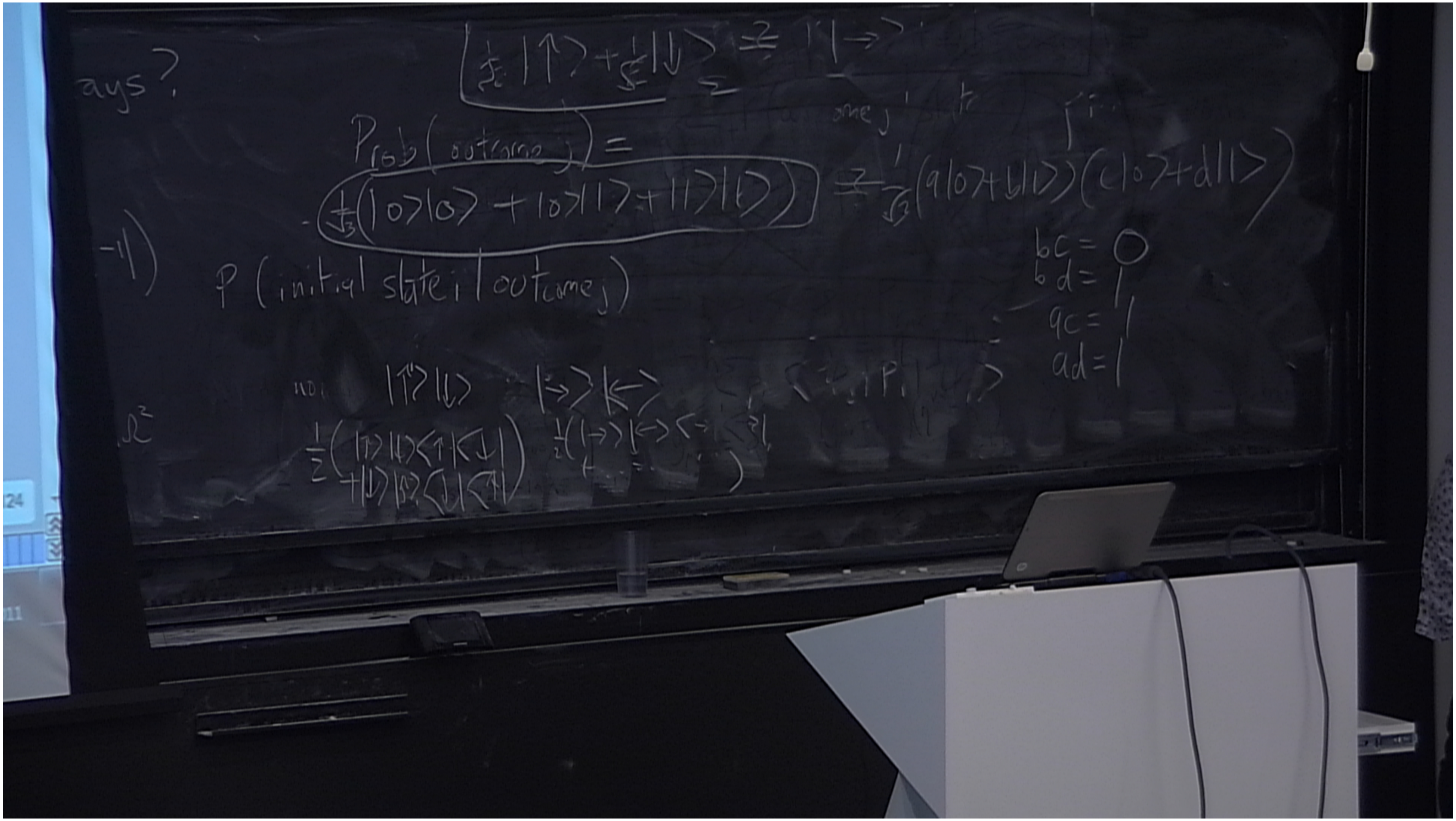
P (initial state_i | outcome_j)


Prob(outcome_j) =
P (initial state_i | outcome_j)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P(initial state_i | outcome_j)
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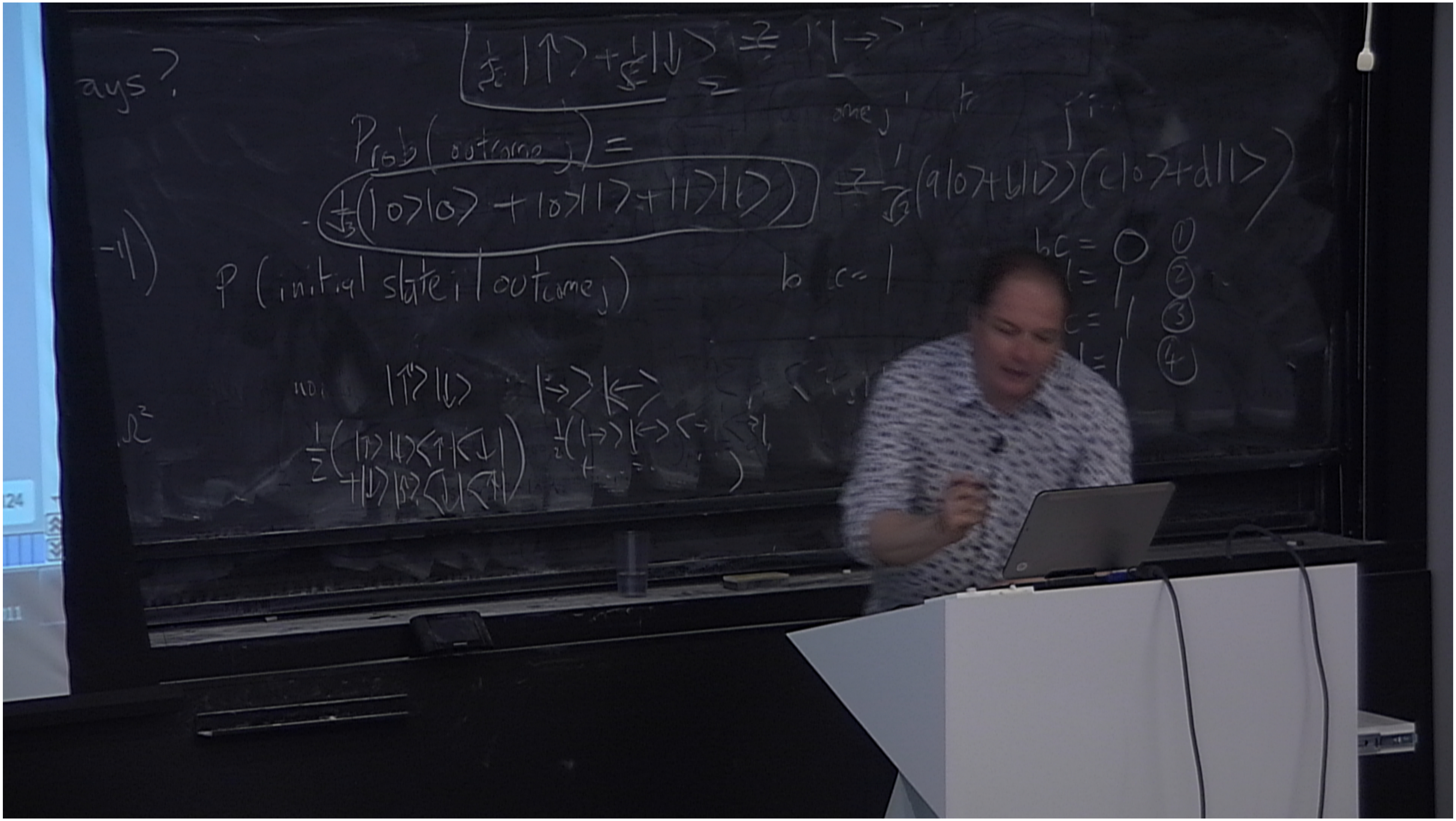
bd = 1
ac = 1
ad = 1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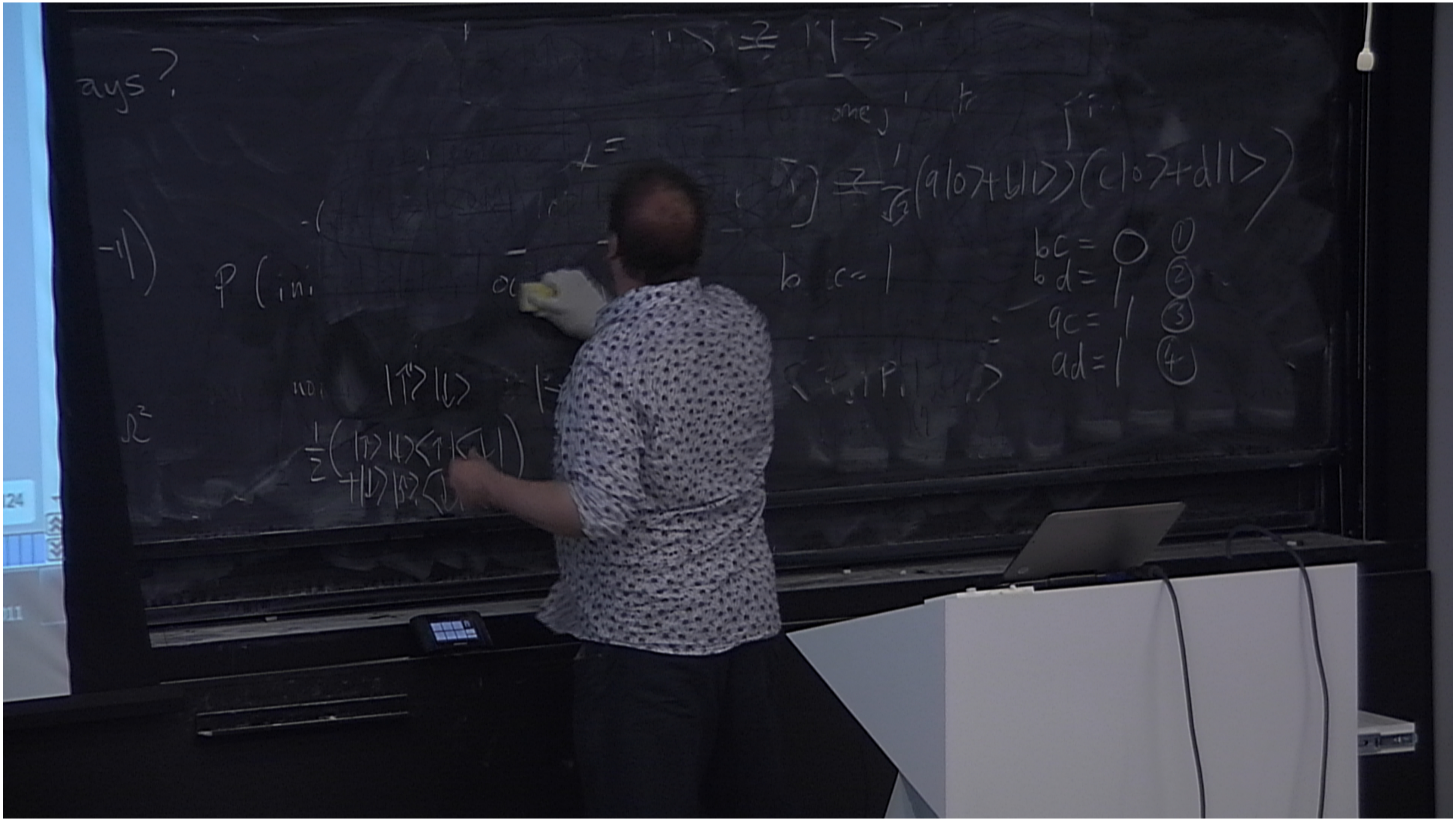
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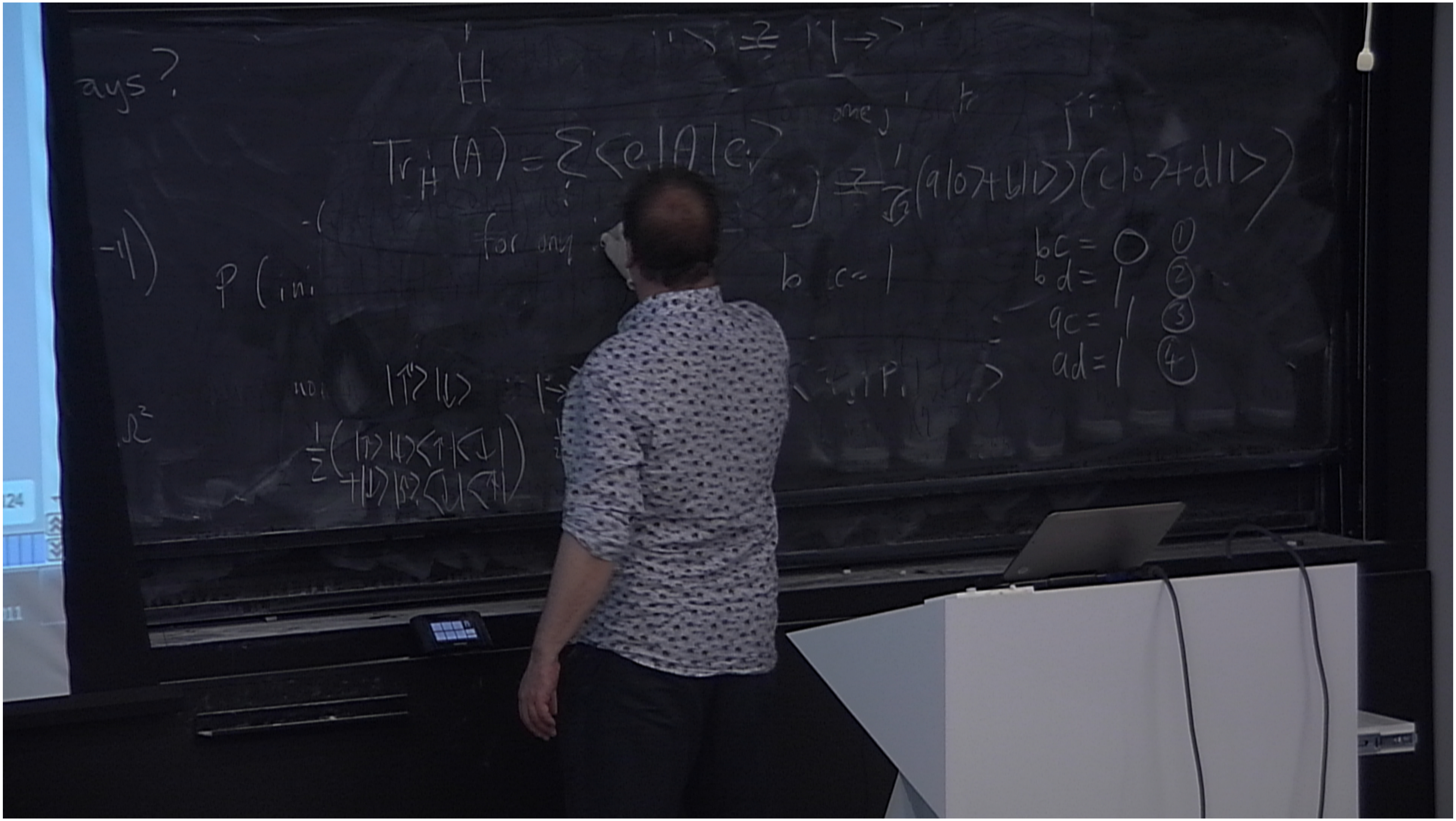
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What's the partial inner product   A natural sesquilinear map from

$$\{ |\phi\rangle \in H_1 , |\psi\rangle \in H_1 \otimes H_2 \} \longrightarrow \langle \phi | \psi \rangle \in H_2 .$$

We can define it for basis vectors of H_1 and product basis vectors of $H_1 \otimes H_2$

$$\langle e_i | \, |e_j\rangle \otimes |f_k\rangle = \delta_{ij} |f_k\rangle$$

and then extend the definition by linearity in $|\psi\rangle$, antilinearity in $|\phi\rangle$.

$$\{ \sum_i a_i |e_i\rangle , \sum_{jk} b_{jk} |e_j\rangle \otimes |f_k\rangle \} \rightarrow \sum_{ijk} a_i^* b_{jk} \delta_{ij} |f_k\rangle$$

(Check the definition is basis-independent.)

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What's the partial inner product A natural sesquilinear map from
antilinear in 1st part, linear in 2nd

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$\in H_1$ $\quad H_1 \otimes H_2$

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(**Check** the definition is basis-independent.)

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$$\boxed{\langle e_i | \underbrace{|e_j\rangle \otimes |f_k\rangle}_{H_1 \otimes H_2} = \delta_{ij} |f_k\rangle}$$

$\in H_1$

and then extend the definition by linearity in $|\psi\rangle$, antilinearity in $|\phi\rangle$.

$$\{ \underbrace{\sum_i a_i |e_i\rangle}_{\phi} , \underbrace{\sum_{jk} b_{jk} |e_j\rangle \otimes |f_k\rangle}_{\psi} \} \rightarrow \sum_{ijk} a_i^* b_{jk} \delta_{ij} |f_k\rangle$$

antilinear linear

(Check the definition is basis-independent.)

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antilinear linear

(Check the definition is basis-independent.)

$$\langle \phi | \psi \rangle$$

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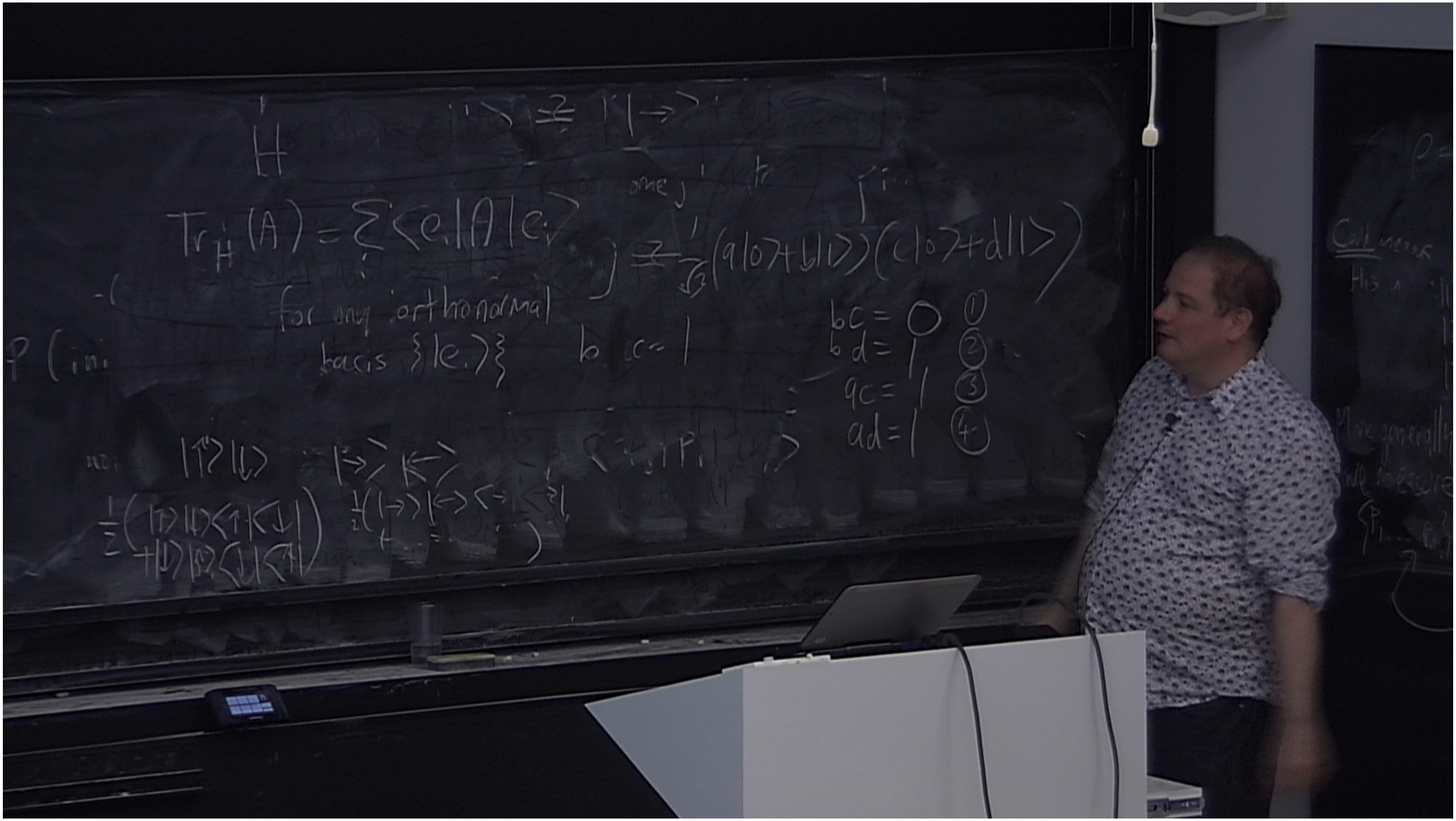

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What's the partial trace?

Tr_{H_2} : {operators on $H_1 \otimes H_2$} $\rightarrow$ {operators on H_1}

linear map.

We can define Tr_{H_2} by its action on a basis of operators,

eg (1) $\mathrm{Tr}_{H_2}\left(|e_{i_1}\rangle \otimes |f_{j_1}\rangle\langle e_{i_2}| \otimes \langle f_{j_2}| \right) = \delta_{j_1 j_2} |e_{i_1}\rangle\langle e_{i_2}|$

and extend by linearity.

Or equivalently, (2) $\mathrm{Tr}_{H_2}(A) = \sum_j \langle f_j | A | f_j \rangle$ any orthonormal basis $\{|f_j\rangle\}$ of H_2.

Is this equivalent? Yes; take $A = |e_{i_1}\rangle|f_{j_1}\rangle\langle e_{i_2}|\langle f_{j_2}|$, (2) gives same as (1), both are linear, ok.

Does it matter which basis $\{|f_j\rangle\}$?? No. Check $\sum_j \langle f_j' | A | f_j' \rangle = \sum_j \langle f_j | A | f_j \rangle$

(It's enough to check for 'our basis operators' A.)

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(It's enough to check for our basis operators A.)

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$$\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) = \sum_i \langle e_i|\psi\rangle\langle\psi|e_i\rangle$$

$\in H_1$ $\in H_1 \otimes H_2$

Remember, these aren't numbers but bra and ket states in H_2

$$\boxed{\text{So } \rho_2 = \mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) \text{ is an operator on } H_2 .}$$

$$\rho_2 = \rho_2^\dagger \quad , \quad \mathrm{Tr}_{H_2}(\rho_2) = \mathrm{Tr}_{H_2}(\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|)) = \mathrm{Tr}_{H_1\otimes H_2}(|\psi\rangle\langle\psi|)$$

$$= 1 .$$

If $|\phi\rangle \in H_2$, then $\langle\phi|\rho_2|\phi\rangle = \langle\phi|\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|)|\phi\rangle$

$$= \sum_i \langle e_i| \otimes \langle\phi| (|\psi\rangle\langle\psi|) |e_i\rangle \otimes |\phi\rangle$$

$$\geq 0$$

$$\boxed{\text{So.. } \rho_2 \text{ is a density matrix on } H_2 . \ (\text{Similarly } \rho_1 \text{ on } H_1 .)}$$

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$$\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) = \sum_i \underbrace{\langle e_i|}_{\in H_1}\underbrace{\psi\rangle\langle\psi}_{\in H_1\otimes H_2}|e_i\rangle$$

Remember, these aren't numbers but bra and ket states in H_2

$\boxed{\text{So } \rho_2 = \mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) \text{ is an } \underline{\text{operator}} \text{ on } H_2.}$

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If $|\phi\rangle \in H_2$, then
$$\langle\phi|\rho_2|\phi\rangle = \langle\phi|\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|)|\phi\rangle$$
$$= \sum_i \langle e_i|\otimes\langle\phi|\,(|\psi\rangle\langle\psi|)\,|e_i\rangle\otimes|\phi\rangle$$
$$\geq 0$$

$\boxed{\text{So.. } \rho_2 \text{ is a } \underline{\text{density matrix}} \text{ on } H_2. \text{ (Similarly } \rho_1 \text{ on } H_1.)}$

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Now if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ is a product state,

then $\rho_2 = \mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) = \mathrm{Tr}_{H_1}(|\psi_1\rangle \otimes |\psi_2\rangle \langle\psi_1| \otimes \langle\psi_2|)$

$$= \langle\psi_1|\psi_1\rangle \, |\psi_2\rangle\langle\psi_2|$$

$$= |\psi_2\rangle\langle\psi_2|$$

i.e. ρ_2 is pure (and so is $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and we can obtain the factors $|\psi_1\rangle, |\psi_2\rangle$ directly from ρ_1, ρ_2)

Conversely, if $\rho_2 = |\psi_2\rangle\langle\psi_2|$ is pure, then $\sum_i \langle e_i|\psi\rangle\langle\psi|e_i\rangle = |\psi_2\rangle\langle\psi_2|$

$\Rightarrow \langle e_i|\psi\rangle = a_i|\psi_2\rangle$ (some $a_i \in \mathbb{C}$)

$\Rightarrow |\psi\rangle = \underbrace{\left(\sum a_i |e_i\rangle\right) \otimes |\psi_2\rangle}_{\text{product state!}}$

Summary: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ product state

$\Longleftrightarrow$ $\mathrm{Tr}_{H_1}(|\psi\rangle\langle\psi|) = |\psi_2\rangle\langle\psi_2|$ pure states

$\mathrm{Tr}_{H_2}(|\psi\rangle\langle\psi|) = |\psi_1\rangle\langle\psi_1|$

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More about the partial trace

A remarkable fact ("the "Schmidt decomposition").

If $|\psi\rangle \in H_1 \otimes H_2$, and its partial traces $\rho_1 = Tr_{H_2}(|\psi\rangle\langle\psi|)$, $\rho_2 = Tr_{H_1}(|\psi\rangle\langle\psi|)$ have nonzero eigenvalues and eigenvectors $\{\lambda_i\}, \{|e_i\rangle\}$ and $\{\mu_i\}, \{|f_i\rangle\}$ respectively

then ① $\{\lambda_i\} = \{\mu_i\}$

② We can write $|\psi\rangle = \sum_{i=1}^{n} \lambda_i |e_i\rangle \otimes |f_i\rangle$

notice $n \leq \min(n_1, n_2)$ — a single sum over orthnormal basis product vectors

ρ_1 and ρ_2 are essentially the same matrix (up to number of zero eigenvalues).

Proof more or less directly verifiable from statement (see tutorial example/handout)

Note By seeing "how mixed" ρ is — how many nonzero eigenvalues, how far from 1 or 0 they are — we can gauge "how entangled" $|\psi\rangle$ is.

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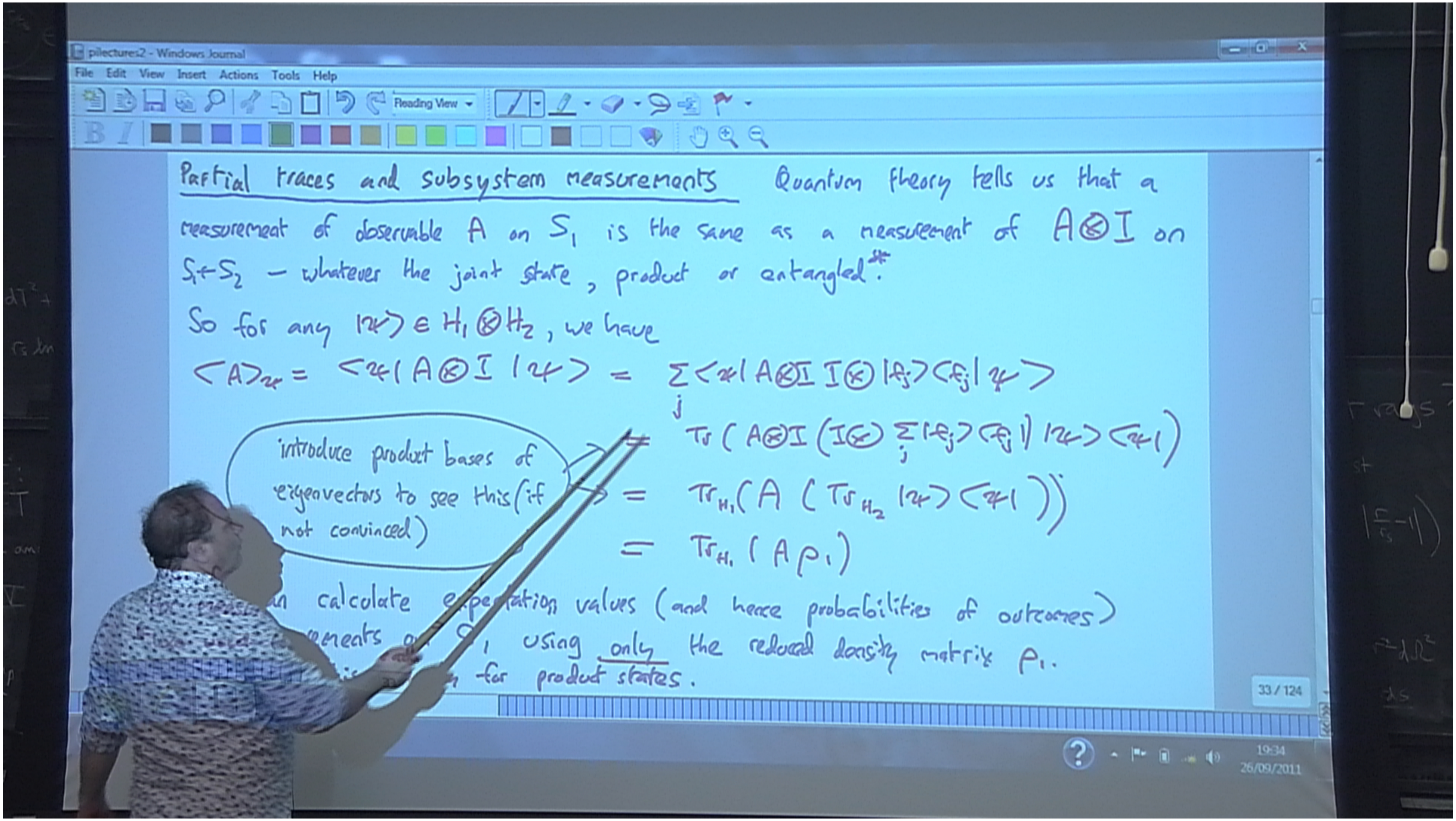
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Partial traces and subsystem measurements
Quantum theory tells us that a measurement of observable A on S_1 is the same as a measurement of $A \otimes I$ on $S_1 + S_2$ — whatever the joint state, product or entangled*.
So for any $|\psi\rangle \in H_1 \otimes H_2$, we have
$\langle A \rangle_\psi = \langle \psi | A \otimes I | \psi \rangle = \sum_j \langle \psi | A \otimes I \; I \otimes |f_j\rangle\langle f_j| \psi \rangle$
$= \mathrm{Tr}(A \otimes I (I \otimes \sum_j |f_j\rangle\langle f_j|) |\psi\rangle\langle\psi|)$
$= \mathrm{Tr}_{H_1}(A (\mathrm{Tr}_{H_2} |\psi\rangle\langle\psi|))$
$= \mathrm{Tr}_{H_1}(A \rho_1)$
introduce product bases of eigenvectors to see this (if not convinced)
calculate expectation values (and hence probabilities of outcomes)
ements of S_1 using only the reduced density matrix ρ_1.
for product states.
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Partial traces and subsystem measurements Quantum theory tells us that a measurement of observable A on S_1 is the same as a measurement of $A \otimes I$ on $S_1 + S_2$ — whatever the joint state, product or entangled*.

So for any $|\psi\rangle \in H_1 \otimes H_2$, we have

$$\langle A \rangle_\psi = \langle \psi | A \otimes I | \psi \rangle = \sum_j \langle \psi | A \otimes I \; I \otimes |f_j\rangle\langle f_j | \psi \rangle$$

$$= \mathrm{Tr}\left(A \otimes I \left(I \otimes \sum_j |f_j\rangle\langle f_j| \right) |\psi\rangle\langle\psi| \right)$$

$$= \mathrm{Tr}_{H_1}\left(A \left(\mathrm{Tr}_{H_2} |\psi\rangle\langle\psi| \right) \right)$$

$$= \mathrm{Tr}_{H_1} (A \rho_1)$$

introduce product bases of eigenvectors to see this (if not convinced)

We can calculate expectation values (and hence probabilities of outcomes) for measurements on S_1 using only the reduced density matrix ρ_1.

*We used this already for product states.

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We can calculate expectation values (and hence probabilities of outcomes) for measurements on S_1 using only ρ_1 via $\langle A \rangle_{\psi} = \; = \mathrm{Tr}_{H_1}(A\rho_1)$

And it's precisely the same expression we obtained for proper mixed state density matrices.

Suggestive .... can we verify that ρ_1 evolves in exactly the same way as a proper mixed state density matrix and encodes all the physics of S_1?

Clearly this part can't work if $S_1 + S_2$ interact: ok, what if they are isolated?

$$|\psi\rangle \rightarrow e^{-iH_1t/\hbar} \otimes e^{-iH_2t/\hbar} |\psi\rangle$$

$$\mathrm{Tr}_{H_2}(|\psi_t\rangle\langle\psi_t|) = \mathrm{Tr}_{H_2}\left(e^{-iH_1t/\hbar} \otimes \cancel{e^{-iH_2t/\hbar}} |\psi\rangle\langle\psi| e^{iH_1t/\hbar} \otimes \cancel{e^{iH_2t/\hbar}}\right)$$

$$= e^{-iH_1t/\hbar} \mathrm{Tr}_{H_2}(|\psi\rangle\langle\psi|) e^{iH_1t/\hbar} = e^{-iH_1t/\hbar} \rho_1 e^{iH_1t/\hbar}$$

Looks very promising – but what about measurements on S_2?

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We can calculate expectation values (and hence probabilities of outcomes) for measurements on S_1 using only ρ_1 via $\langle A\rangle_\psi = \; = \mathrm{Tr}_{H_1}(A\rho_1)$

And it's precisely the same expression we obtained for proper mixed state density matrices.

Suggestive .... can we verify that ρ_1 evolves in exactly the same way as a proper mixed state density matrix and encodes all the physics of S_1?

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$$|\psi\rangle \to e^{-iH_1t/\hbar}\otimes e^{-iH_2t/\hbar}|\psi\rangle$$

$$\mathrm{Tr}_{H_2}(|\psi_t\rangle\langle\psi_t|) = \mathrm{Tr}_{H_2}\left(e^{-iH_1t/\hbar}\otimes \cancel{e^{-iH_2t/\hbar}}\,|\psi\rangle\langle\psi|\, e^{iH_1t/\hbar}\otimes \cancel{e^{iH_2t/\hbar}}\right)$$

$$= e^{-iH_1t/\hbar}\,\mathrm{Tr}_{H_2}(|\psi\rangle\langle\psi|)\,e^{iH_1t/\hbar} = e^{-iH_1t/\hbar}\rho_1 e^{iH_1t/\hbar}$$

Looks very promising — but what about measurements on S_2?

Measurements on composite systems Given $|\psi\rangle \in H_1 \otimes H_2$, what happens if we measure operator B on S_2, i.e. $I \otimes B$ on $S_1 + S_2$?

After measurement we have a state $|\psi'\rangle = \dfrac{I \otimes P_i |\psi\rangle}{|I \otimes P_i |\psi\rangle|}$ some eigenprojection of B

So $\rho_1' = Tr_{H_2}(|\psi'\rangle\langle\psi'|) = Tr_{H_2}\left(\dfrac{I\otimes P_i |\psi\rangle\langle\psi| I \otimes P_i}{|I \otimes P_i|\psi\rangle|^2}\right)$

$\rho_1 = Tr_{H_2}(|\psi\rangle\langle\psi|)$ hmm, these don't look generally equal

Try an example: $|\psi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle \otimes |f_1\rangle + |e_2\rangle \otimes |f_2\rangle)$, $P_i = |f_i\rangle\langle f_i|$.

$|\psi'\rangle = |e_i\rangle \otimes |f_i\rangle$ (NB no summation convention here)

$\rho_1' = |e_i\rangle\langle e_i| \neq \rho_1 = \frac{1}{2}(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)$

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In general, this suggests, measurements on one entangled subsystem change the reduced density matrix of the other, $\rho_1' \neq \rho_1$, even if they're isolated.

Interesting. Worrying? Let's think. In the case of proper mixed states, the density matrix represented all the information we had about the system: more info. existed (in the memory of our colleague or the internal state of our misfiring device)

Here we see something analogous: information, not initially available, about S_1 can be obtained (created?) by measurements on S_2.

But what if we focus on what we can learn about S_1 only by experiments on S_1? We can't assume S_2 isn't being measured, but we must now assume we don't learn the measurement outcomes for S_2.

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But what if we focus on what we can learn about S_1 only by experiments on S_1? We can't assume S_2 isn't being measured, but we must assume we don't learn the measurement outcomes for S_2.

So we need to do a different calculation.

After measurement on S_2 of (eigen)projectors $\{P_i\}$ we get one of the states

$$|\psi_i'\rangle = \frac{I \otimes P_i |\psi\rangle}{|I \otimes P_i |\psi\rangle|}$$ with probabilities $p_i = |I \otimes P_i |\psi\rangle|^2$. We don't know which.

So given our ignorance the relevant post-measurement density matrix is

$$\rho_1' = \mathrm{Tr}_{H_2}\left(\sum p_i |\psi_i'\rangle\langle\psi_i'|\right) = \mathrm{Tr}_{H_2}\left(\sum_i \cancel{|I \otimes P_i |\psi\rangle|^2} \cdot \frac{I \otimes P_i |\psi\rangle\langle\psi| I \otimes P_i}{\cancel{|I \otimes P_i |\psi\rangle|^2}}\right)$$

$$= \mathrm{Tr}_{H_2}\left(\cancel{(I \otimes \sum P_i)} |\psi\rangle\langle\psi|\right) = \rho_1 .$$

YES!

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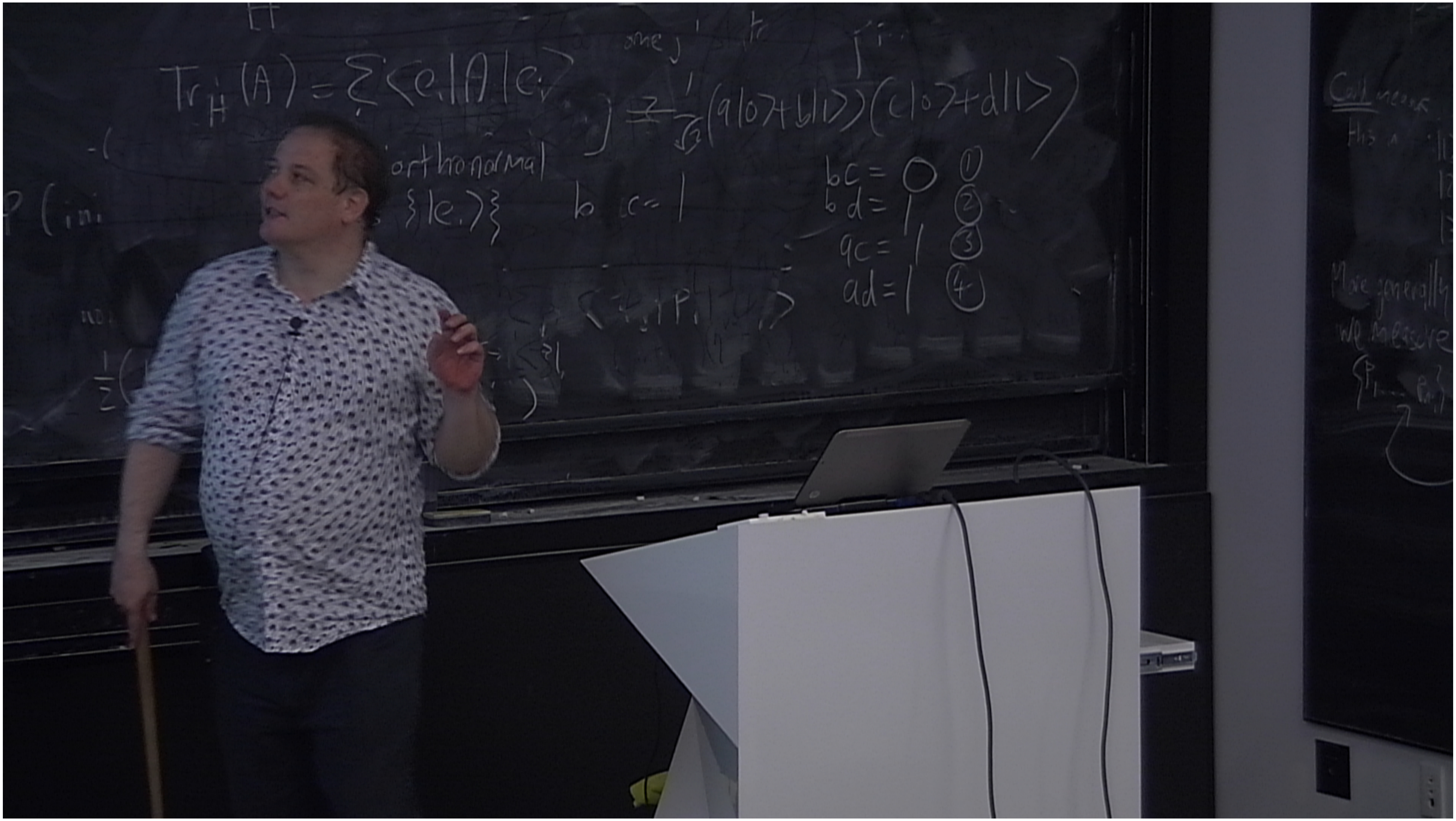

orthonormal
bc = 0
bd = 1
ac = 1
ad = 1
More generally
we measure

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Putting this all together: If $S_1 + S_2$ are isolated, $H = H_1 \otimes I + I \otimes H_2$,
initial state $|\psi(0)\rangle = \sum a_{ij} |e_i\rangle \otimes |f_j\rangle$,
we define the reduced density matrix $\rho_1(t) = Tr_{H_2}(|\psi(t)\rangle\langle\psi(t)|)$,
and update $\rho_1(t)$ by measurement outcomes on S_1 (but not S_2).

Then $\rho_1(t) \longrightarrow e^{-iH_1t/\hbar} \rho_1(0) e^{iH_1t/\hbar}$ under unitary evolution

$\rho_1(t) \longrightarrow \dfrac{P \rho_1(t) P}{Tr(P\rho_1(t))}$ under measurement on S_1 with outcome $\Leftrightarrow P$

$\rho_1(t) \rightarrow \rho_1(t)$ unchanged by unitary evolution or measurements on S_2, given our ignorance of the outcomes.

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