Title: Quantum Theory - Lecture 5

Date: Sep 16, 2011 01:30 PM

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Abstract:

That is, m must be an integer $\quad \ell = 0, 1, 2, \ldots$

$$-\ell \le m \le \ell$$

These are the possible eigenvalues for

$$L^2 = \ell(\ell+1)\hbar^2$$
$$L_3 = m\hbar$$

for position space wave functions – e.g. electrons orbiting in a central potential.

They characterize the orbital angular momentum.

But nature makes use of another possibility: particles can have internal degrees of freedom that are also affected by rotations

Metaphor (don't take it literally!) – like spin about some axis.

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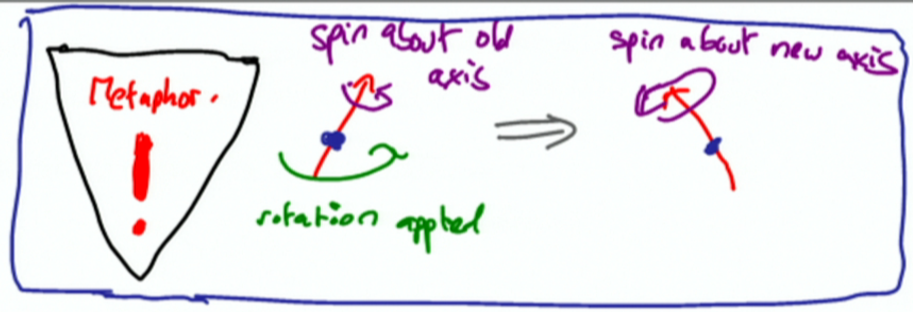
Our earlier analysis also applies to the action of the rotation group on internal (spin) degrees of freedom.



Write the action of a rotation on internal states as: $R_x(\alpha) = e^{iS_x\alpha/\hbar}$, $R_y(\alpha)$, $R_z(\alpha)$ similarly.

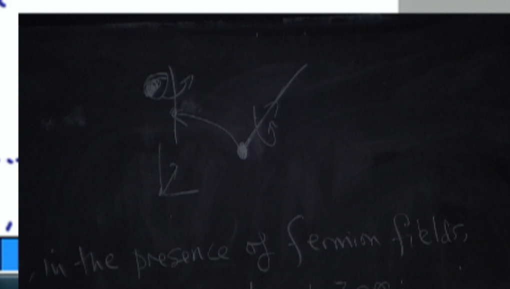
generators

We're dealing with the same group, so the generators must have the same commutation relations as L_i, i.e. $[S_i, S_j] = i\epsilon_{ijk}\hbar S_k$

$[S_i, S^2] = 0$

CONCLUSION: the possible eigenvalues m lie in range $-s, -s+1, \ldots$

WHICH REQUIRES: $2s$ must be positive integer, ie $s = 0, \frac{1}{2}, 1, \ldots$



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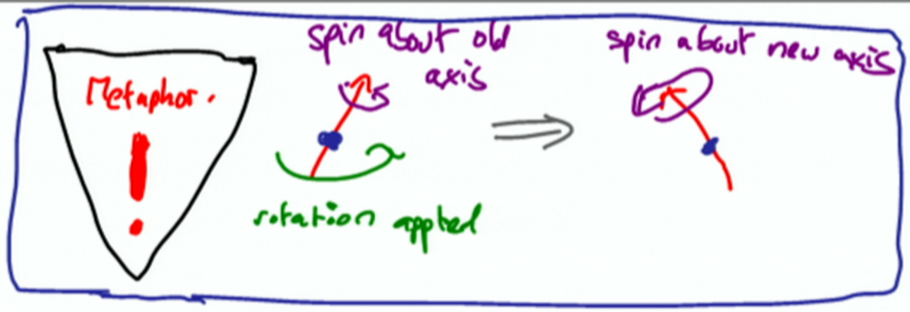

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But note that the S_i are acting on some "internal" Hilbert space, not on position space wave functions

$$H = H_{spin} \otimes H_{position}$$

So we don't have an a priori expression for S_z (etc) analogous to

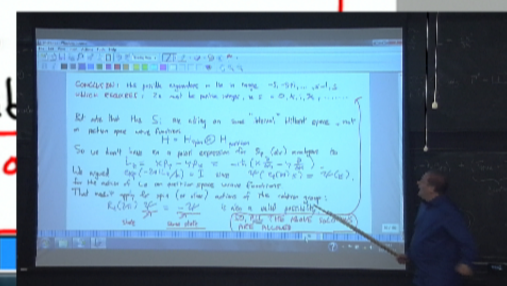
$$L_z = x p_y - y p_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

We argued $\exp(-2\pi i L_z/\hbar) = I$ since $\psi(R_z(2\pi)\underline{x}) = \psi(\underline{x})$.
for the action of L_z on position space wave functions.
That needn't apply for spin (or other) actions of the rotation

$R_z(2\pi)\psi = -\psi$ is also a valid possib

state — same state

SO, ALL THE ABO
ARE ALLOWED



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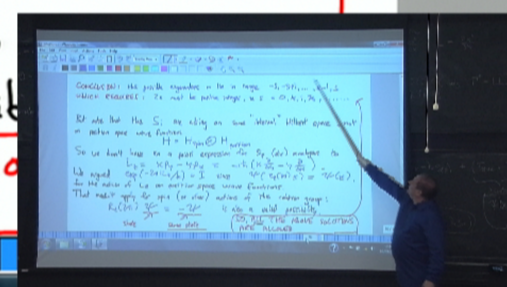

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WHICH REQUIRES: $2s$ must be positive integer, ie $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots\dots$

half-odd-integer values ALLOWED

Nature indeed uses this freedom. It turns out (spin-statistics theorem):

$s = 0, 1, 2, \dots$ integer spin $\longleftrightarrow$ bosons (photon, W, Z, Higgs?, graviton?)

$s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ $\frac{1}{2}$(odd integer) spin $\longleftrightarrow$ fermions (electron, quark, neutrino ...)

Given $H = H_{spin} \otimes H_{position}$, we should write spin operators as $S_i \otimes I$

orbital angular momentum $I \otimes L_i$

To understand the full action of the rotation group we need to look at the action of total angular momentum $J_i = S_i \otimes I + I \otimes L_i$. (spin-orbital coupling)

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Spin-½ particles and the Bloch sphere

$\mathcal{H}_{spin}$ for $s = \frac{1}{2}$ has two independent states, $|m=\frac{1}{2}\rangle$ and $|m=-\frac{1}{2}\rangle$.

Call them $|\uparrow\rangle$ and $|\downarrow\rangle$.

(Think of spins about z axis.)

We can write a general state as

$$a|\uparrow\rangle + b|\downarrow\rangle = ab^{-1}|\uparrow\rangle + |\downarrow\rangle \qquad (a, b \in \mathbb{C})$$

up to normalisation factor

But note $ab^{-1} = \infty$ must be allowed in this convention ($b = 0$ is possible.)

(complex projective plane)

I.e. {spin states of spin ½ particle} $\Leftrightarrow$ $\mathbb{C} \cup \{\infty\}$.

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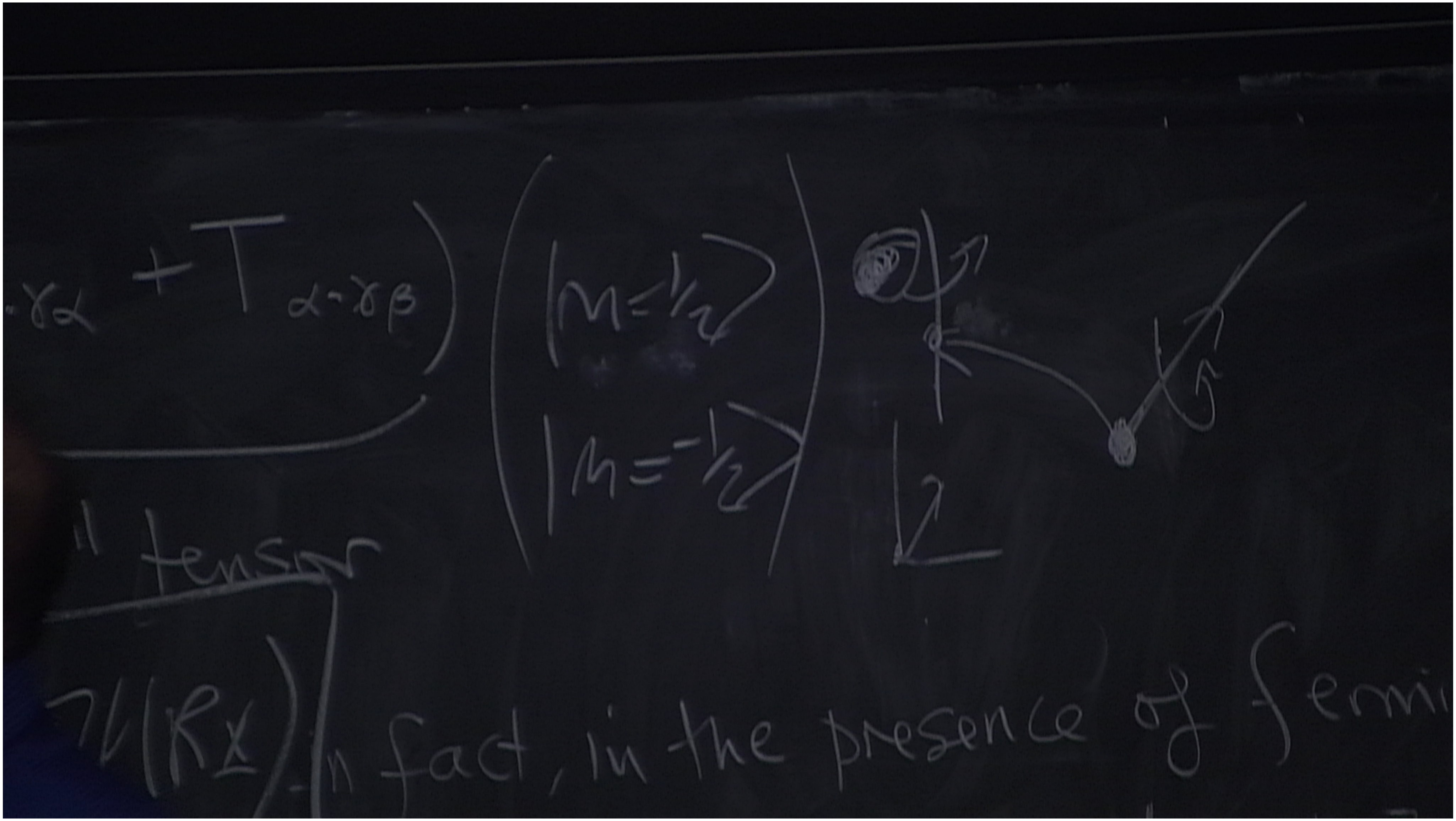
|m=1/2⟩
|m=−1/2⟩
tensor
In fact, in the presence of fermi

Spin-½ particles and the Bloch sphere

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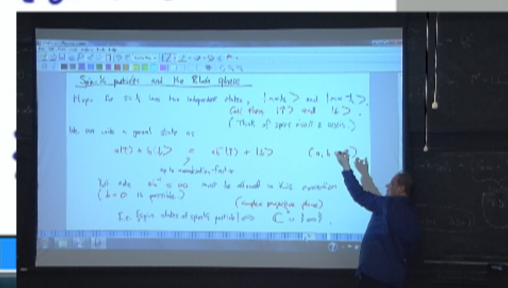
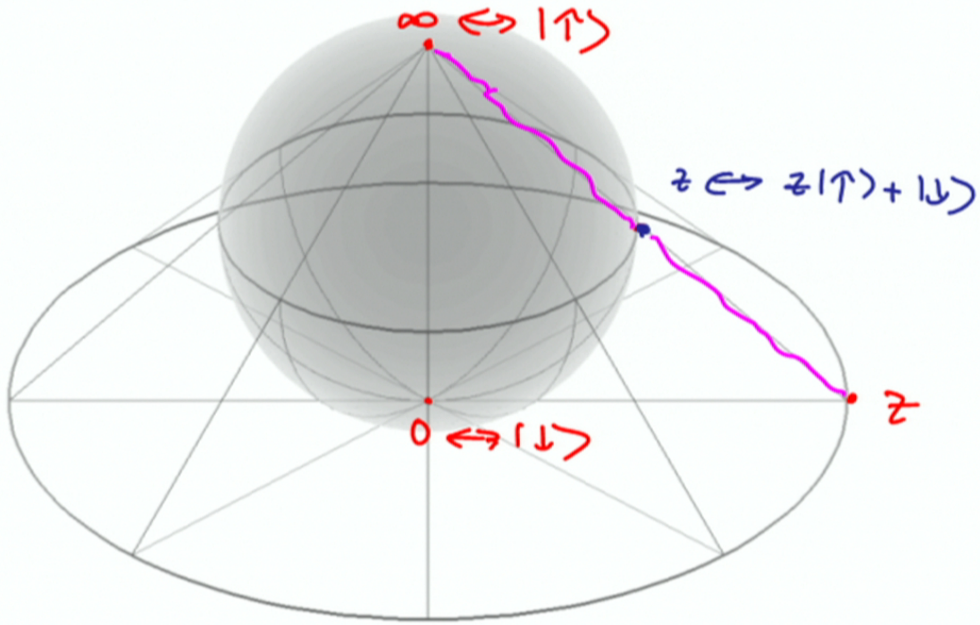


$\{\text{spin states of spin } \tfrac{1}{2} \text{ particle}\} \Leftrightarrow \mathbb{C} \cup \{\infty\}$.

$(z\,|\uparrow\rangle + |\downarrow\rangle) \Leftrightarrow z \in \mathbb{C} \cup \{\infty\}$

N.B. unnormalized

We can map $\mathbb{C} \cup \{\infty\}$ by stereographic projection onto the Riemann sphere



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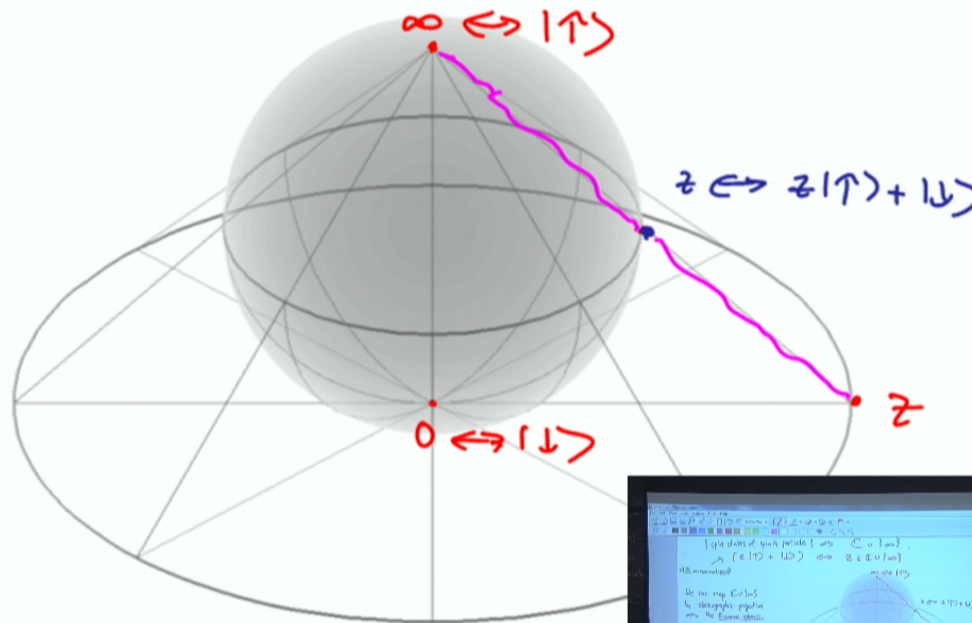

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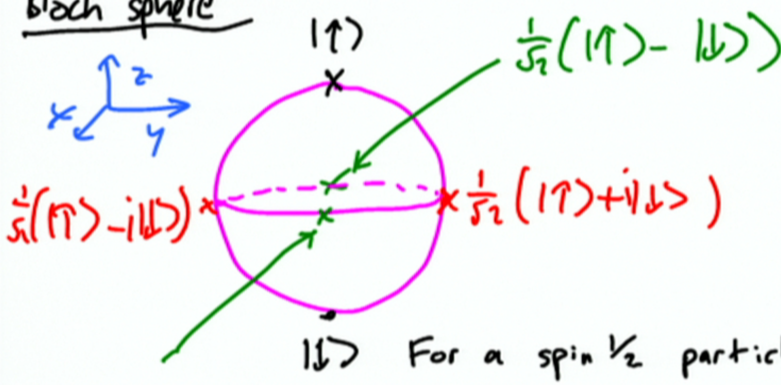
Bloch sphere



$\boxed{\text{now normalizing our states}}$

We've created a simple geometric representation of the states of a spin-½ particle, which turns out to have some very nice properties.

For a spin ½ particle the rotation group generators act on $\begin{pmatrix}|\uparrow\rangle\\ |\downarrow\rangle\end{pmatrix}$ as:

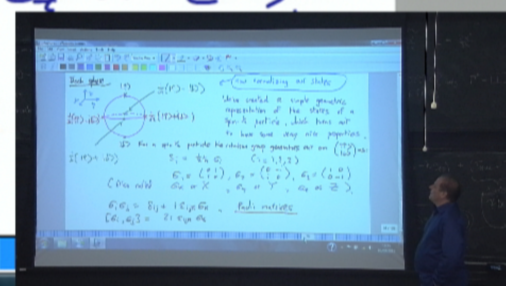
$$S_i = \tfrac{1}{2}\hbar\,\sigma_i \qquad (i = 1,2,3)$$

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

(Also called σ_x or X, σ_y or Y, σ_z or Z).

$$\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k\,. \qquad \text{Pauli matrices}$$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\,\sigma_k$$



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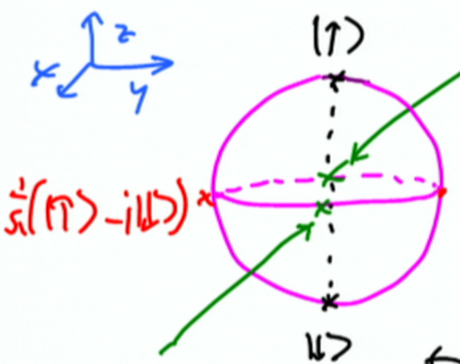

$\frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ ← σ_1 eigenvectors

$\frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$ $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$ ← σ_2 eigenvectors

$|\uparrow\rangle$, $|\downarrow\rangle$ ← σ_3 eigenvectors

$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$

Rotations

$R_z(\alpha) = \exp(-i\frac{\alpha}{2}\sigma_z)$	rotates Bloch sphere through α about z axis
$R_x(\alpha) = \exp(-\frac{i\alpha}{2}\sigma_x)$	rotates Bloch sphere through α about x axis.
Etc.	

Similarly

Notice that $R_z(2\pi) = \exp(-i\pi\sigma_z) = -I$: pick up phase -1 but state unchanged.

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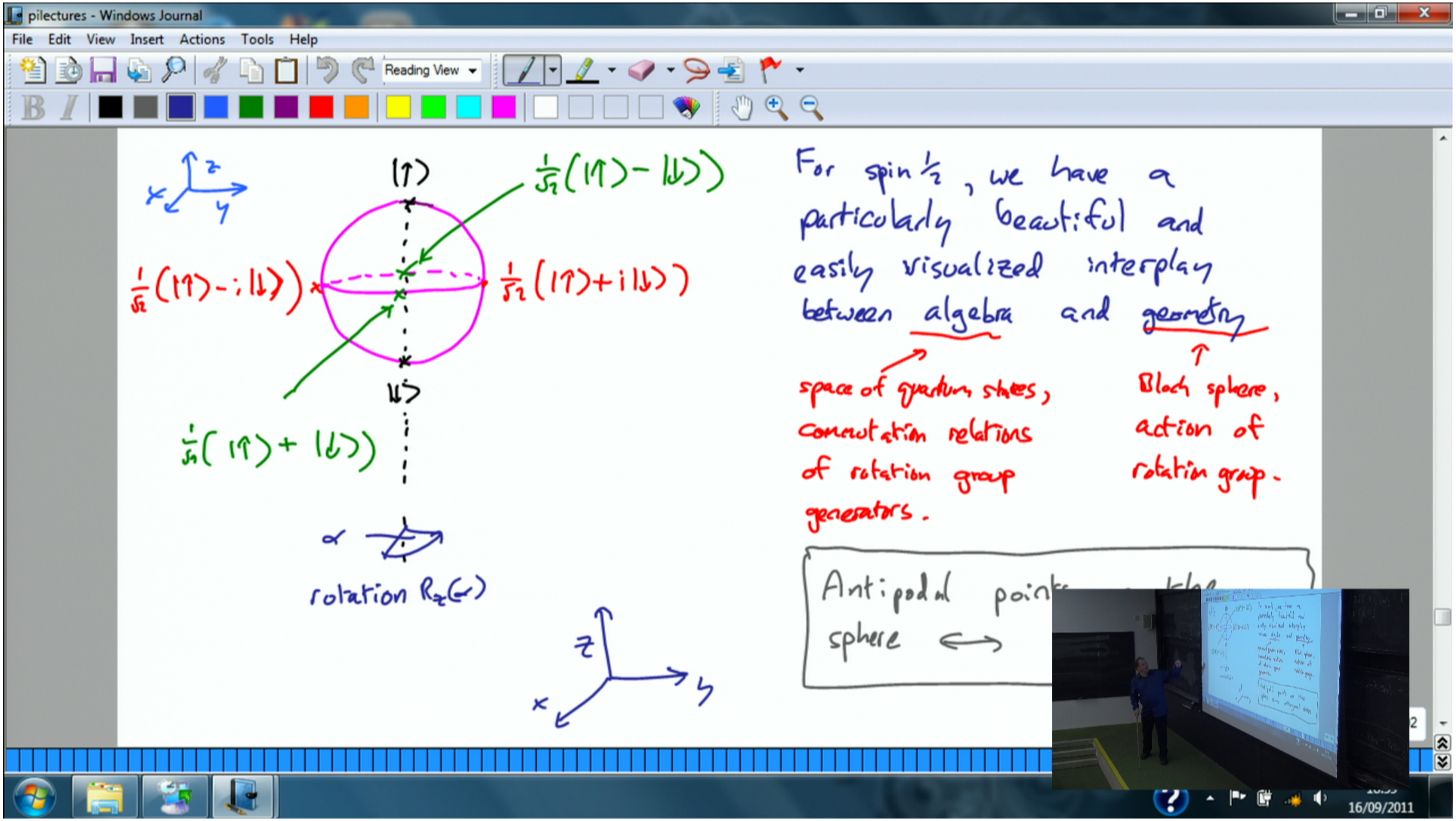
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1/√2 (|↑⟩ + i|↓⟩)
1/√2 (|↑⟩ + |↓⟩)
α
rotation R_z(α)
For spin ½, we have a particularly beautiful and easily visualized interplay between algebra and geometry
space of quantum states, commutation relations of rotation group generators.
Bloch sphere, action of rotation group.
Antipodal points ... the sphere ⟷
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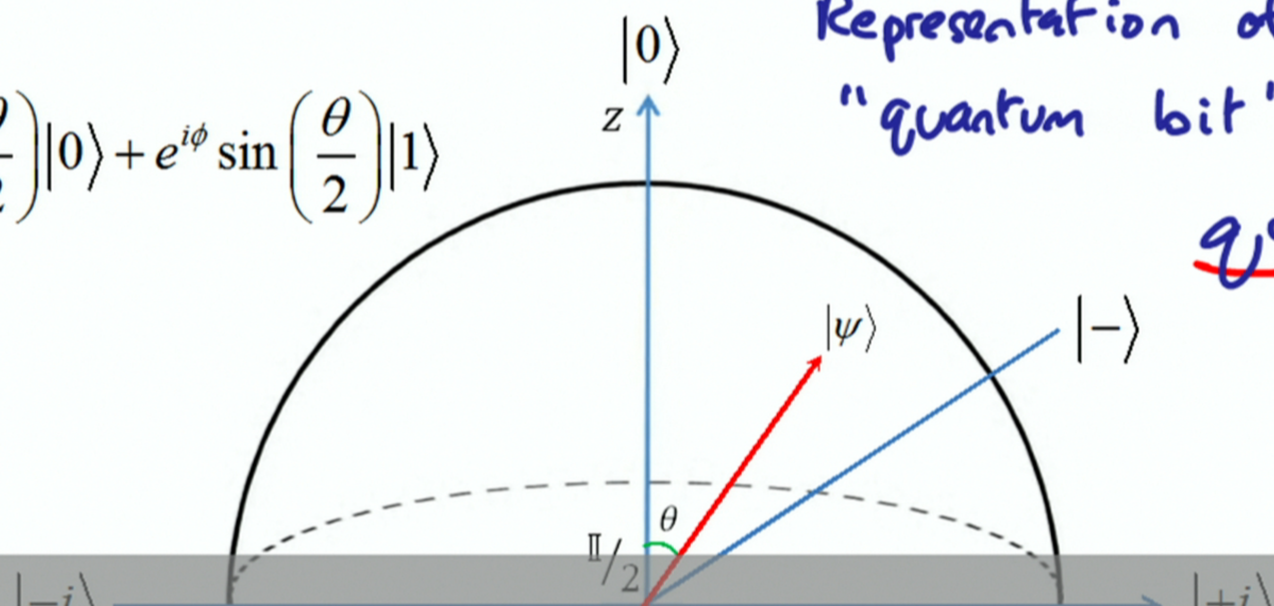
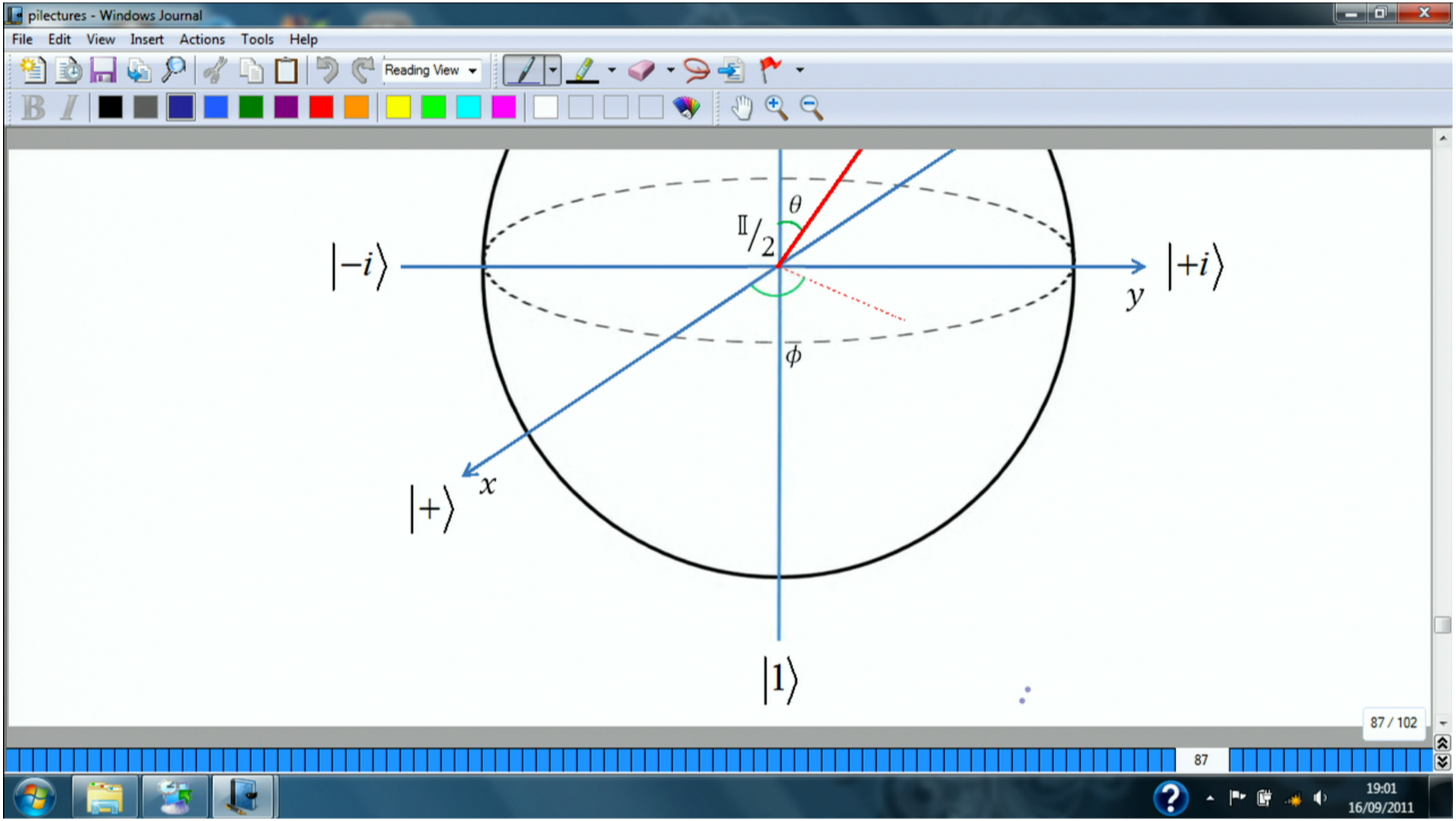
Bloch Sphere - Computational Basis

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Representation of a "quantum bit" or qubit.

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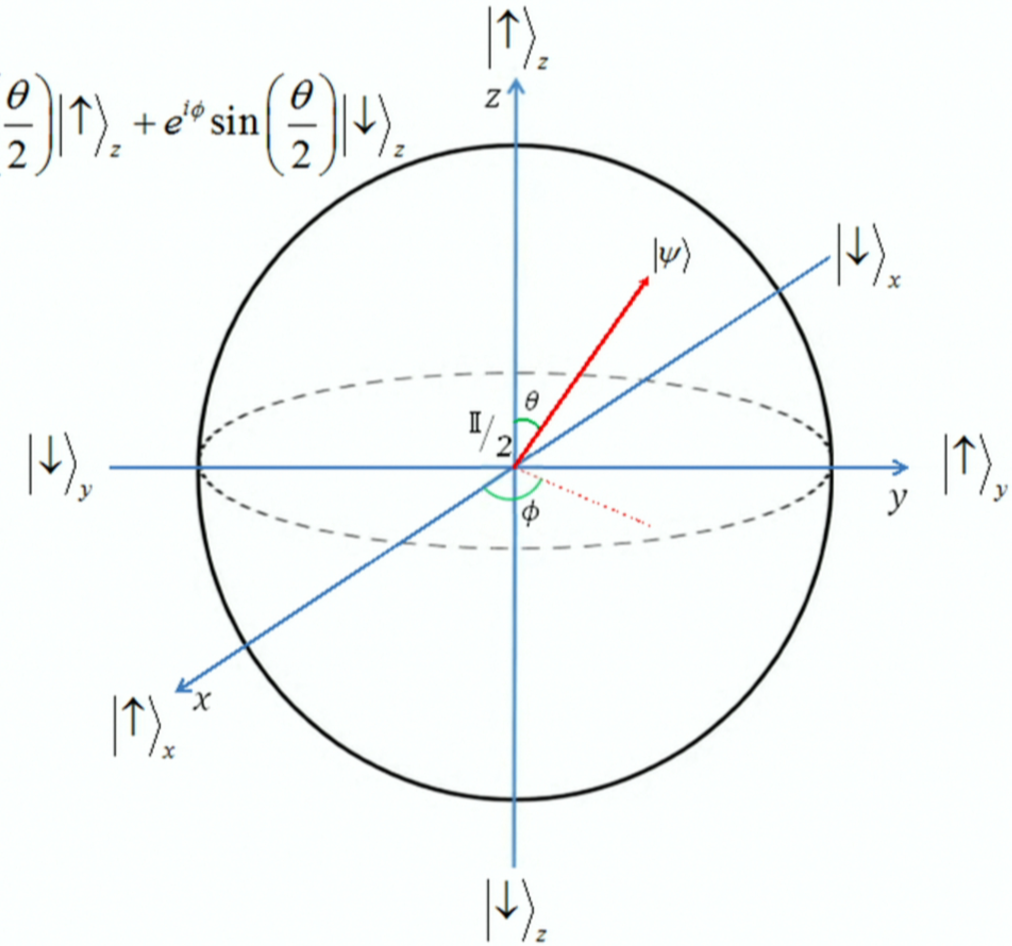
Bloch Sphere - Spin Basis

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle_z + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle_z$$



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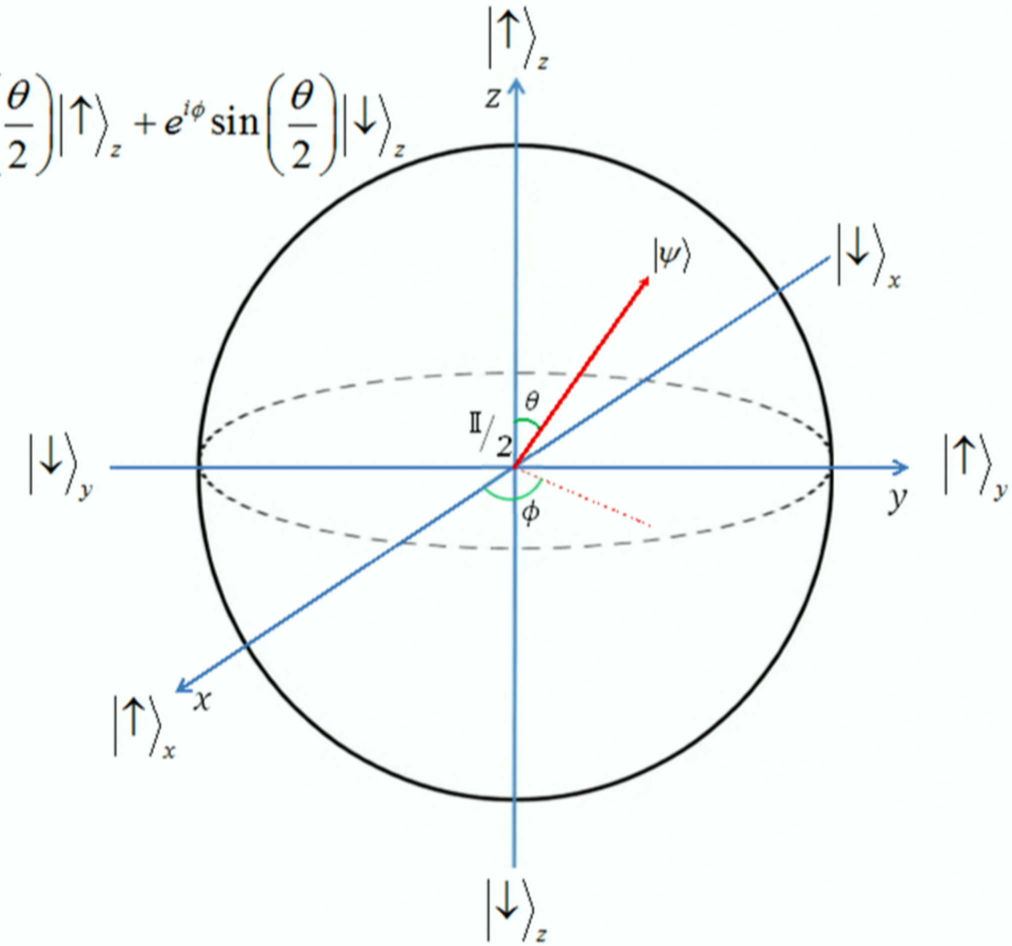

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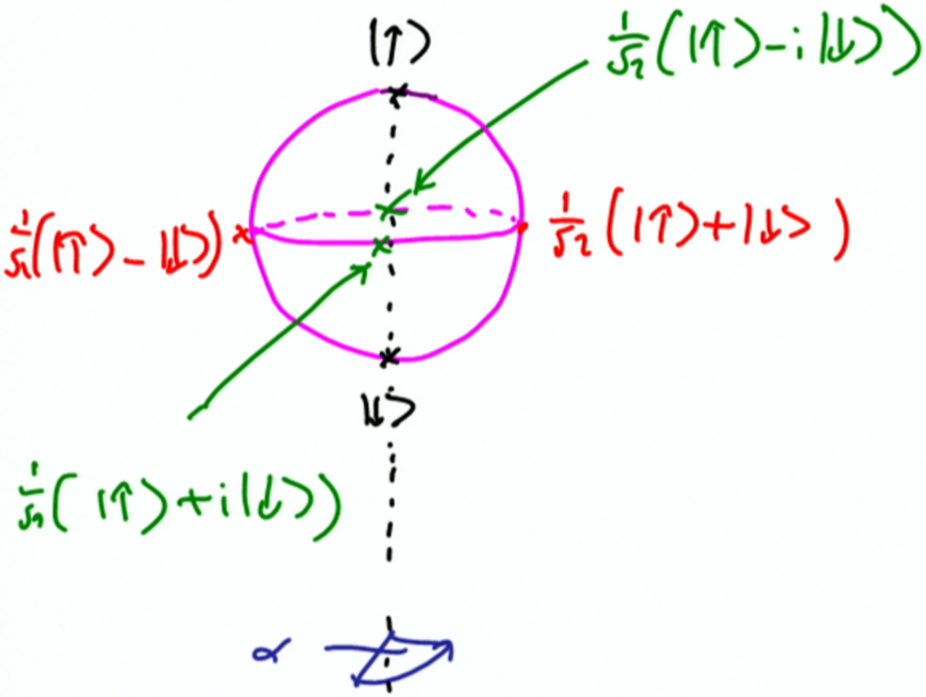

One way to move a quantum state around the Bloch sphere: apply a Hamiltonian $H = \lambda \underline{n} \cdot \underline{\sigma}$

E.g. if $H = \sigma_z$

$$e^{-iHt/\hbar} = e^{-i\sigma_z t/\hbar} = R_z(2t/\hbar)$$

rotation

$$R_z(\alpha) = \exp(-i \tfrac{\alpha}{2} \sigma_z)$$

rotates state on Bloch sphere through α about z axis

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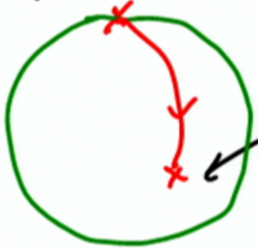
The Quantum Zeno effect: Another way of manipulating states on the Bloch sphere* is by an appropriate sequence of measurements.



Suppose we have some Hamiltonian H acting as a Bloch sphere rotation.

And suppose after a short time Δt we measure the evolved state using projections $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

$$\text{Prob}(\text{outcome } 0) = |\langle 0| e^{-iH\Delta t/\hbar}|0\rangle|^2 = \left|1 - i\frac{\Delta t}{\hbar}\langle 0|H|0\rangle - \frac{\Delta t^2}{\hbar^2}\langle 0|H^2|0\rangle\right|^2$$
$$= 1 - \left(\frac{\Delta t}{\hbar}\right)^2 (\Delta H)^2 + O(\Delta t^3)$$

where $(\Delta H)^2 = \langle 0|H^2|0\rangle - \langle 0|H|0\rangle^2$.

* actually in any Hilbert space of any dimension: check it out!

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The Quantum Zeno effect: Another way of manipulating states on the Bloch sphere*
is by an appropriate sequence of measurements.

$|\psi(0)\rangle = |0\rangle$

$|\psi(t)\rangle = e^{-iHt/\hbar}|0\rangle$

Suppose we have some Hamiltonian H acting as a Bloch sphere rotation.

And suppose after a short time Δt we measure the evolved state using projections $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

$$\text{Prob (outcome 0)} = |\langle 0|e^{-iH\Delta t/\hbar}|0\rangle|^2 = |1 - i\tfrac{\Delta t}{\hbar}\langle 0|H|0\rangle$$

$$= 1 - (\Delta t/\hbar)^2 ($$

where $(\Delta H)^2 = \langle 0|H^2|0\rangle - \langle 0|H|0\rangle^2$.

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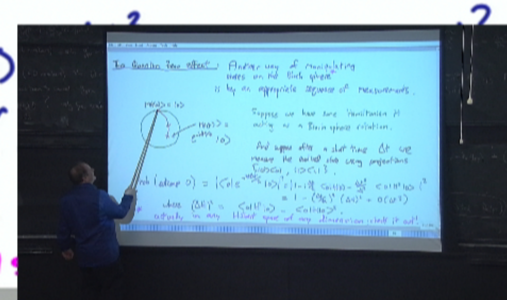

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$$\text{Prob}(\text{outcome } 0) = 1 - \left(\tfrac{\Delta t}{\hbar}\right)^2 (\Delta H)^2$$

i.e. $1 - O((\Delta t)^2)$

So if we fix a time interval T, take $\Delta t = T/N$, iterate N times:

$$\boxed{\text{Prob}(\text{outcome } 0 \text{ all } N \text{ times}) \doteq (1 - cN^{-2})^N \doteq 1 - cN^{-1}} .$$

$$\boxed{\longrightarrow 1 \quad \text{as } N \to \infty.}$$

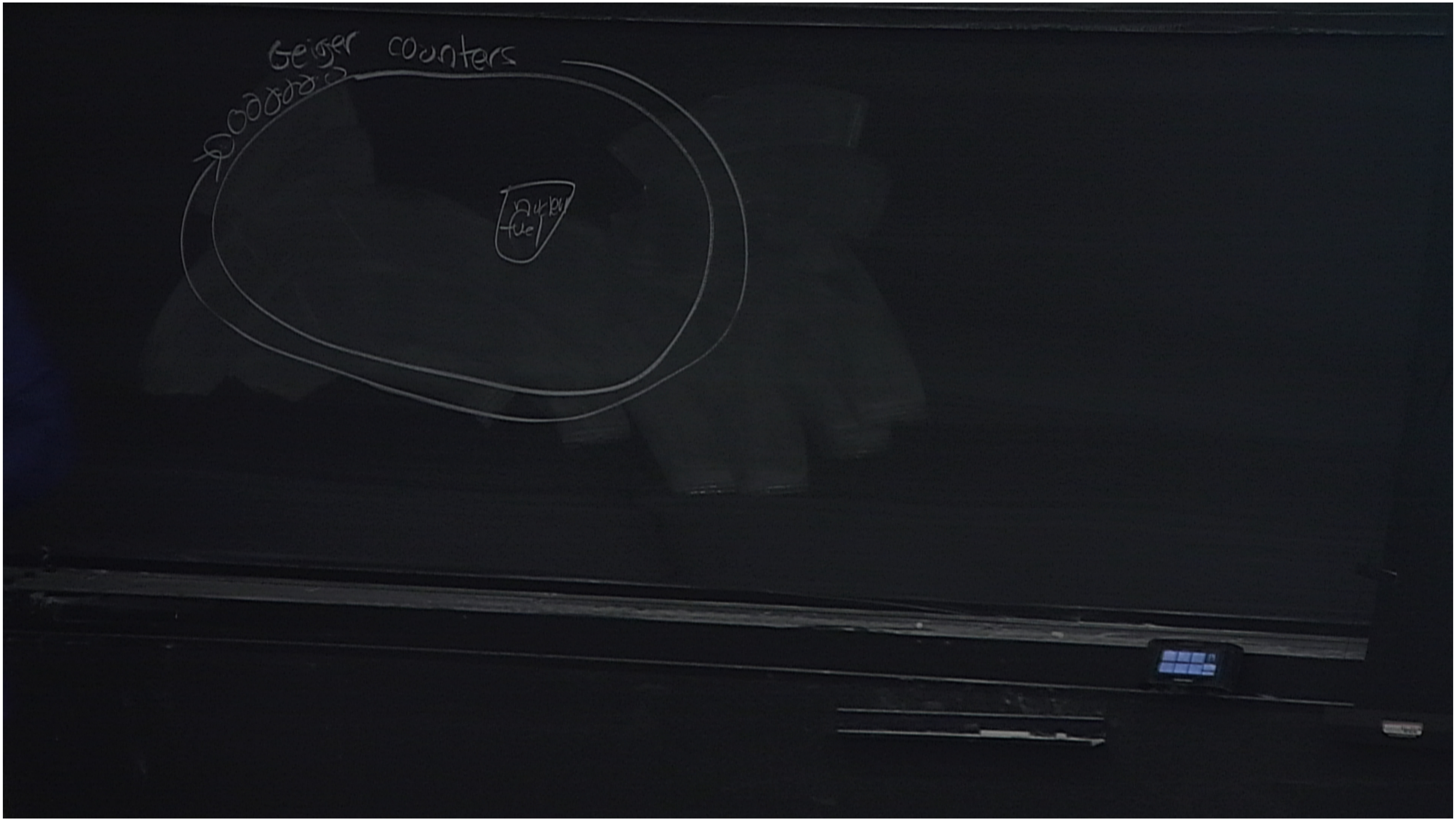
N.B.

However large (in operator norm) H is, however long T is, sufficiently fast repeated measurements effectively "freeze" the system in state $|0\rangle$.

(In this precise sense) a quantum watched pot really does never boil!

! Be careful of over-extrapolating this idea, though!


Geiger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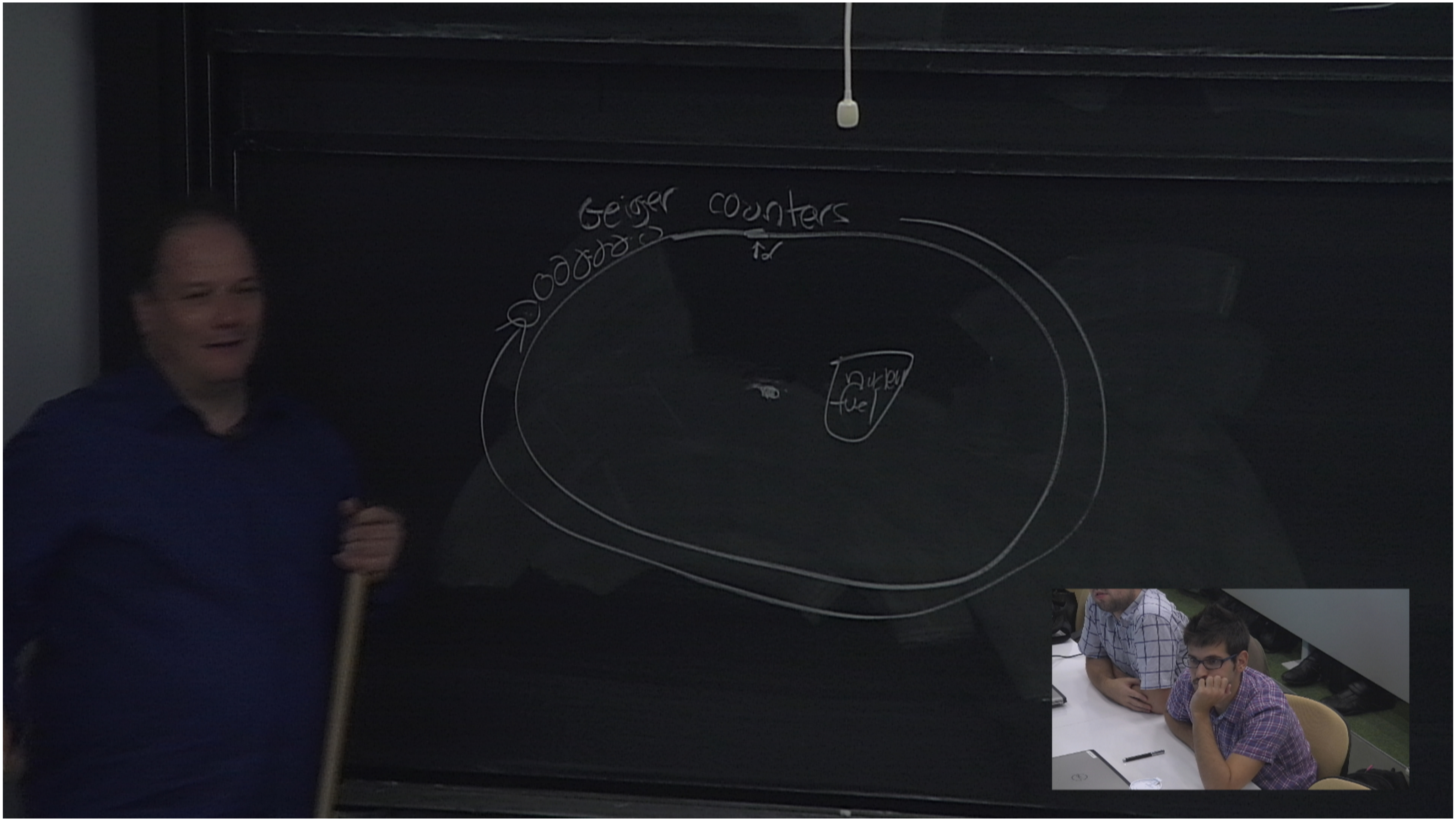
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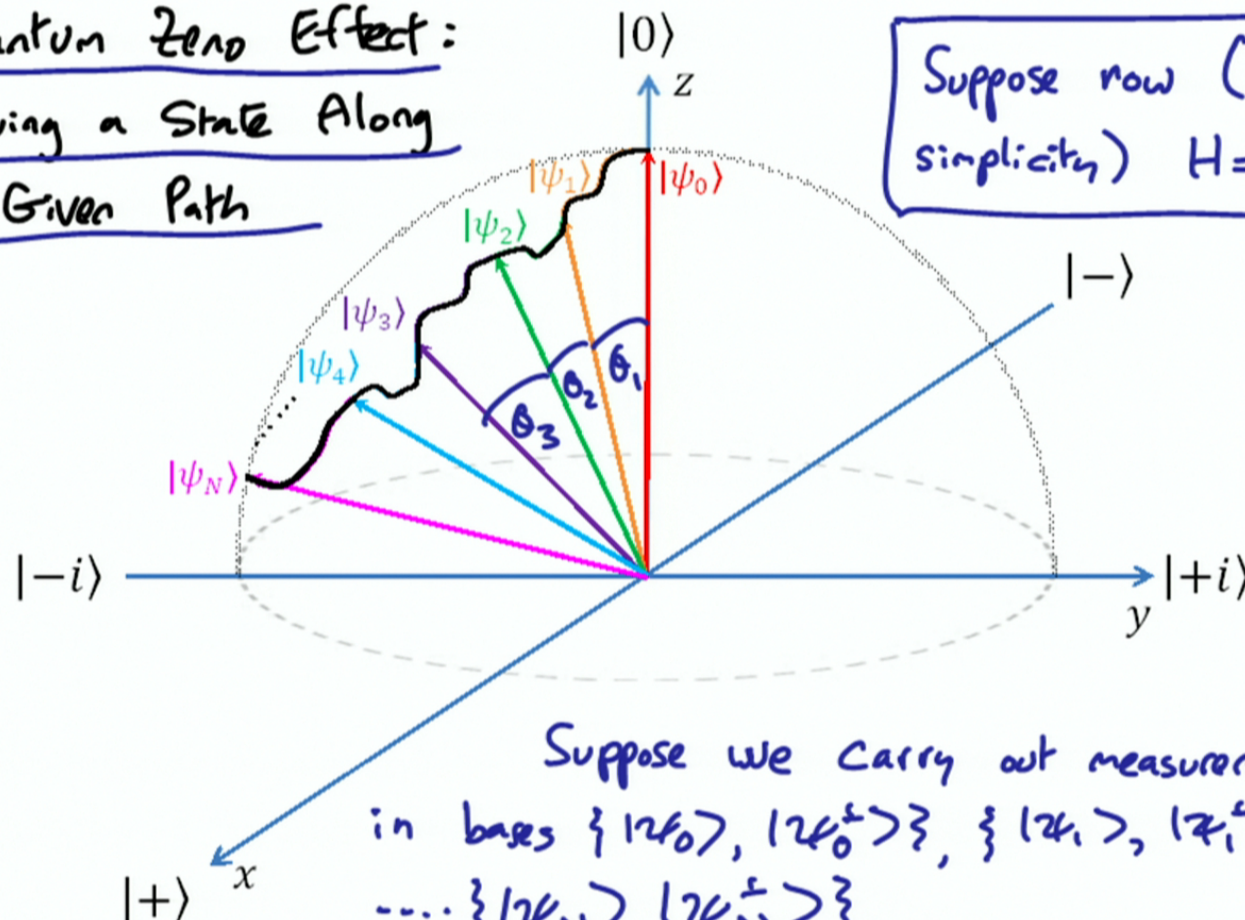
Quantum Zeno Effect: Moving a State Along a Given Path



Suppose now (for simplicity) $H = 0$

Suppose we carry out measurements in bases $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$, $\{|\psi_1\rangle, |\psi_1^\perp\rangle\}$, ... $\{|\psi_N\rangle, |\psi_N^\perp\rangle\}$

where $|\langle\psi_i|\psi_{i+1}\rangle| = \cos\left(\theta_i/2\right)$, $\theta_i \leq \theta$.

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where $|\langle\psi_i|\psi_{i+1}\rangle| = \cos\theta_i/2$, $\theta_i \leq \theta$.

$$\text{Prob}(\text{outcome } |\psi_0\rangle) = 1$$

$$\text{Prob}(\text{outcome } |\psi_1\rangle) = \cos^2\left(\theta_1/2\right) \geq \cos^2(\theta/2)$$

$$\text{Prob}(\text{outcome } |\psi_2\rangle \text{ given outcome } |\psi_1\rangle) = \cos^2(\theta_2/2) \geq \cos^2(\theta/2)$$

. . . . . .

So $\text{Prob}(\text{outcomes } |\psi_0\rangle, |\psi_1\rangle, \ldots, |\psi_N\rangle) \geq (\cos^2(\theta/2))^N = \cos^{2N}(\theta/2)$

If $\theta = \pi/N$, $\cos^{2N}(\theta/2) = \cos^{2N}\left(\frac{\pi}{2N}\right) \geq \left(1 - \frac{\pi^2}{8N^2}\right)^{2N}$

$\longrightarrow 1$ as $N \to \infty$.

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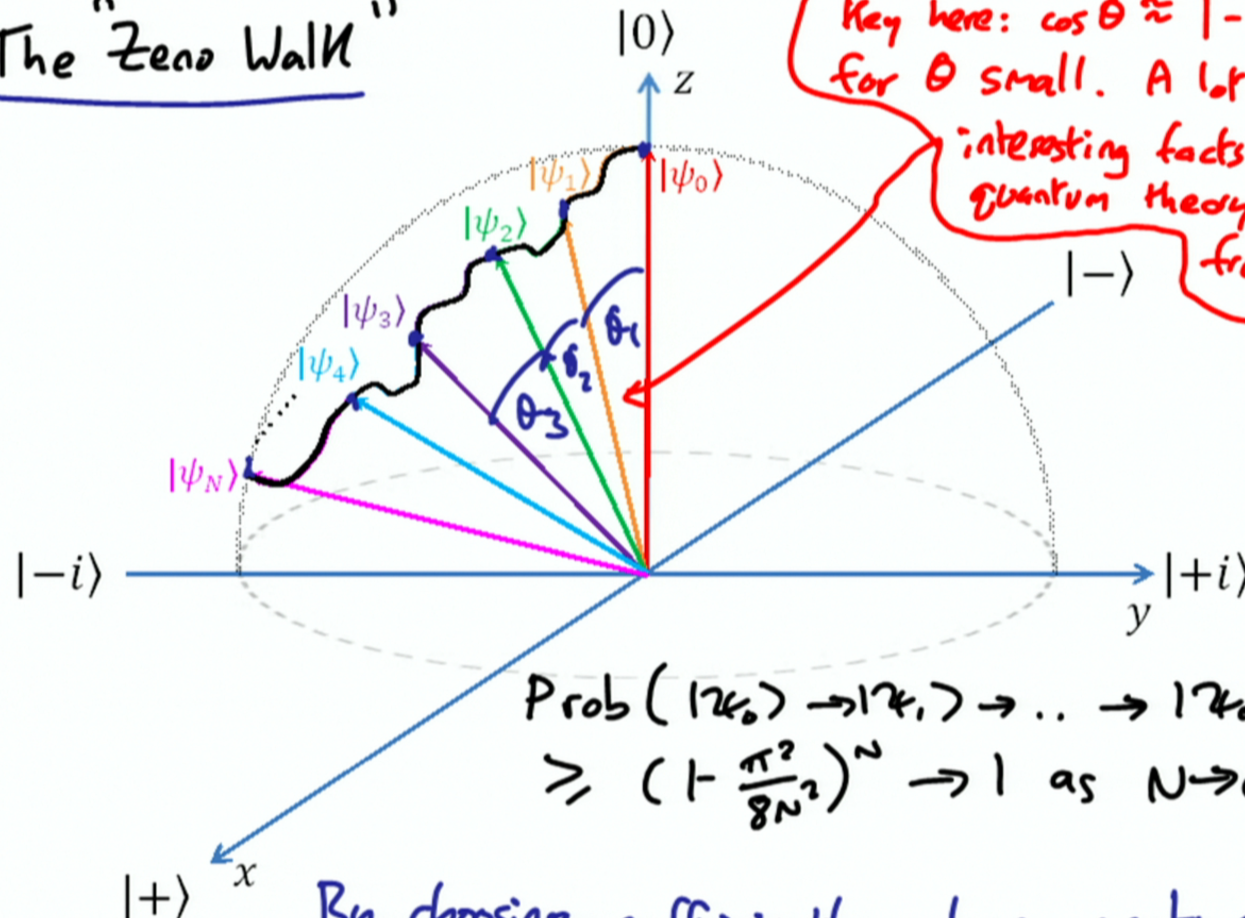
The "Zeno Walk"

Key here: $\cos\theta \approx 1 - O(\theta^2)$ for θ small. A lot of interesting facts about quantum theory follow from this!



$$\text{Prob}\left(|\psi_0\rangle \to |\psi_1\rangle \to \dots \to |\psi_N\rangle\right) \geq \left(1 - \frac{\pi^2}{8N^2}\right)^N \to 1 \text{ as } N \to \infty$$

By choosing sufficiently close points en route we can (almost) guarantee to "walk" the state along any given path.

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"The only reasons for the spin rotation are those measurements. This result is quite surprising because of the accepted assumption that if the outcome of a measurement of some dynamical variable is certain (i.e with probability one), then the state of the system was not disturbed."

(Aharonov and Vardi, 1980)

Yakir Aharonov

Measurement - even with (almost) certain outcome - seems here to be an active intervention, not a passive process revealing a pre-existing system state.

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Mixed States and Density Matrices

Suppose you're given a state $|\psi\rangle$

and know it's one of $|\psi_1\rangle, \ldots\ldots, |\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots, p_n$

(And this is a complete list of possibilities : $\sum_{i=1}^{n} p_i = 1, \; p_i \geq 0$.
And you can't learn anything more about the preparation of $|\psi\rangle$.)

How could this happen? Secretive colleague with a random number generator;

Imperfect preparation device with known error statistics, ...

What can you do with this information?

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$|\psi\rangle$ is one of $|\psi_1\rangle, \ldots\ldots, |\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots, p_n$

What can you do with this information?

One option: Keep this list of states and probabilities, and keep track of how it changes as you apply Hamitonian evolution or make measurements.

Apply $e^{-iHt/\hbar}$: $|\psi\rangle$ is one of $e^{-iHt/\hbar}|\psi_1\rangle, \ldots\ldots, e^{-iHt/\hbar}|\psi_n\rangle$

with respective probabilities $p_1, \ldots\ldots\ p_n$

Maybe not too cumbersome? – though it means solving the Schrödinger equation n times (and n may be big and the solutions may be hard to calculate).

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$$\text{Prob}(\text{outcome } j) = \sum_i \text{Prob}(\text{outcome } j \mid \text{state } i)\,\text{Prob}(\text{state } i)$$

$$= \sum_i \langle \psi_i | P_j | \psi_i \rangle\, p_i \,.$$

State after outcome j? It's one of $\dfrac{P_j|\psi_1\rangle}{|P_j|\psi_1\rangle|}, \ldots, \dfrac{P_j|\psi_j\rangle}{|P_j|\psi_j\rangle|}$.

Which one? We don't know, but we can work out probabilities:

$$\text{Prob}(\text{initial state } i \mid \text{outcome } j) = \frac{\text{Prob}(\text{state } i \text{ and outcome } j)}{\text{Prob}(\text{outcome } j)}$$

$$= \frac{p_i \langle \psi_i | P_j | \psi_i \rangle}{\sum_i p_i \langle \psi_i | P_j | \psi_i \rangle}$$

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So after measurement we have a new list of possible states to keep track of :

$|\psi\rangle$ is one of $\dfrac{P_j|\psi_1\rangle}{|P_j|\psi_1\rangle|}, \ldots\ldots, \dfrac{P_j|\psi_j\rangle}{|P_j|\psi_j\rangle|}$.

with probabilities $\dfrac{\langle\psi_1|P_i|\psi_1\rangle\, p_1}{\sum_j \langle\psi_j|P_i|\psi_j\rangle\, p_j}, \ldots\ldots\ldots$

This is
(given outcome j : we get a different list of states and probabilities for each j.)

It's not impossible, to work this way - at least for finite n — but it's cumbersome! We'd like something better.

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