

Title: Quantum Theory - Lecture 4

Date: Sep 15, 2011 01:30 PM

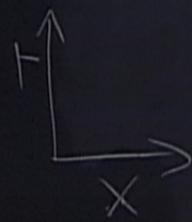
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Abstract:

$(x, t)$

$(0, 0)$

$|+|D$

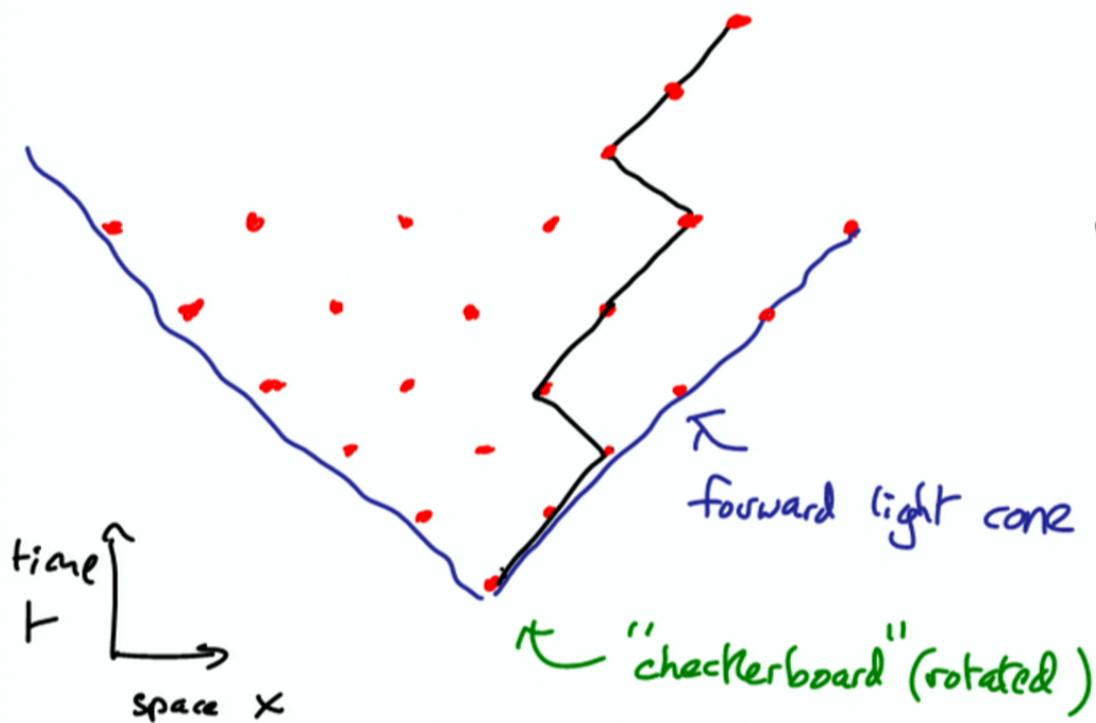


$(x, t)$



The Feynman checkerboard model

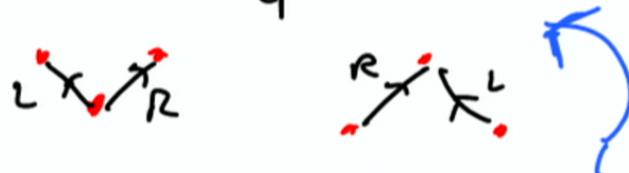
in 1+1 D Minkowski space-time



Idea: we're going to try to obtain a propagator for a mass  $m$  particle in 1+1 D by summing (only) over paths that zig-zag on the checkerboard, following forward light rays

( Why? It's just a guess – let's try it. )

We will separate out the cases where the particle starts going left or right,



(so 4 cases in total),

turns out to be natural for relativistic particles  
(+1D - see later courses.)

Consider two points A, B separated by P right steps and Q left steps.

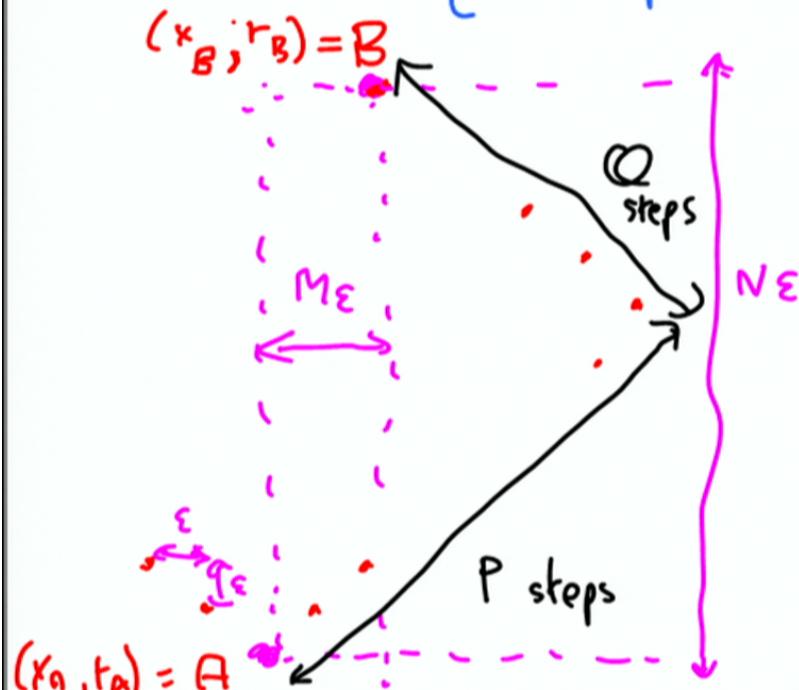
$$\text{Let } N = P + Q, M = P - Q$$

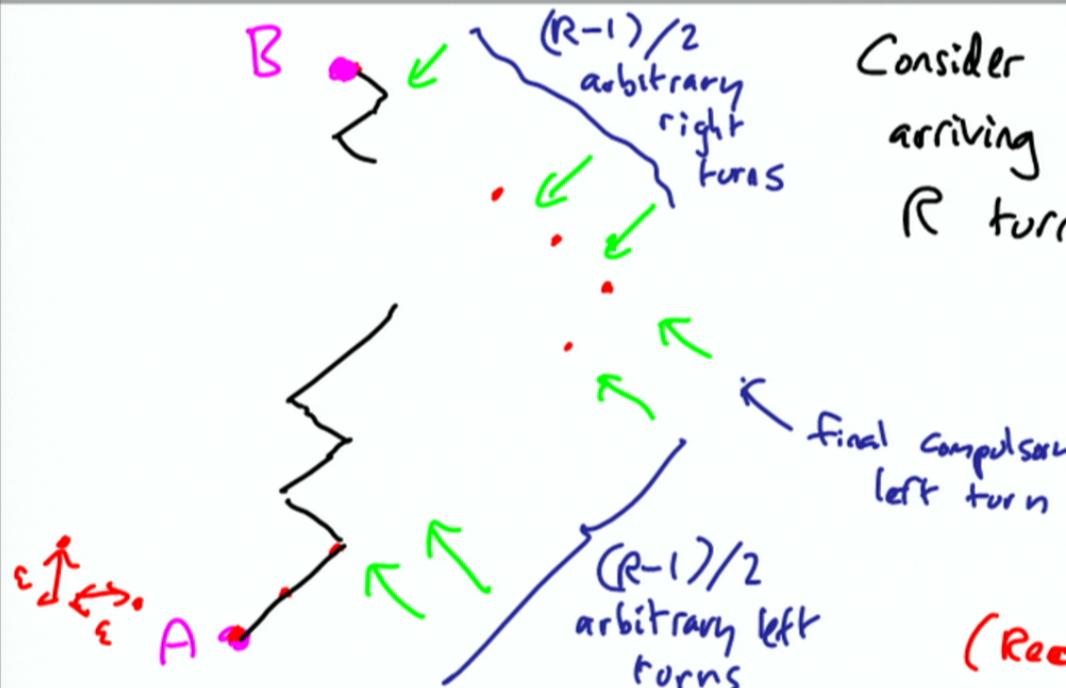
$$\text{So } P = \frac{1}{2}(N+M), Q = \frac{1}{2}(N-M)$$

$$-N \leq P - Q = M \leq N = P + Q$$

$$x = x_B - x_A = (P - Q)\varepsilon = M\varepsilon$$

$$t = t_B - t_A = (P + Q)\varepsilon = N\varepsilon$$





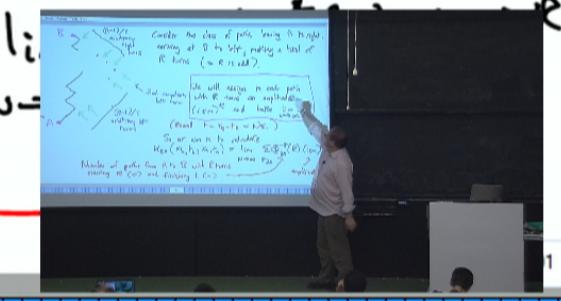
Consider the class of paths leaving A to right, arriving at B to left, making a total of R turns (so R is odd).

We will assign to each path with R turns an amplitude  $(i\varepsilon)^R$  and take  $\lim_{N \rightarrow \infty}$ .

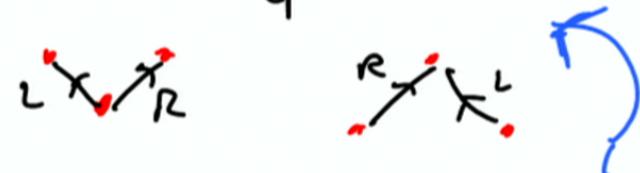
(Recall  $t = t_B - t_A = N\varepsilon.$ )

So our aim is to calculate  
 $K_{BA}(x_b, t_b; x_a, t_a) = \lim_{N \rightarrow \infty} \dots$

Number of paths from A to B with R turns  
starting R (+) and finishing L (-)



We will separate out the cases where the particle starts going left or right, and ends up going left or right (so 4 cases in total),



(this separation in

turns out to be natural for relativistic particles  
(+1D - see later courses.)

Consider two points A, B separated by P right steps and Q left steps.

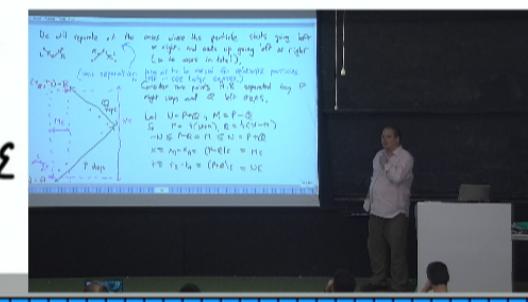
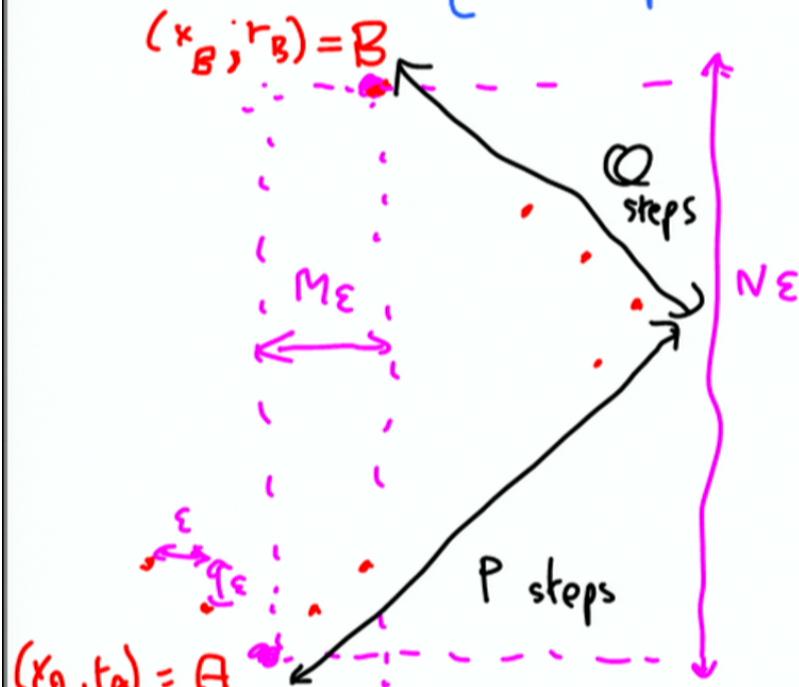
$$\text{Lef } N = P + Q, M = P - Q$$

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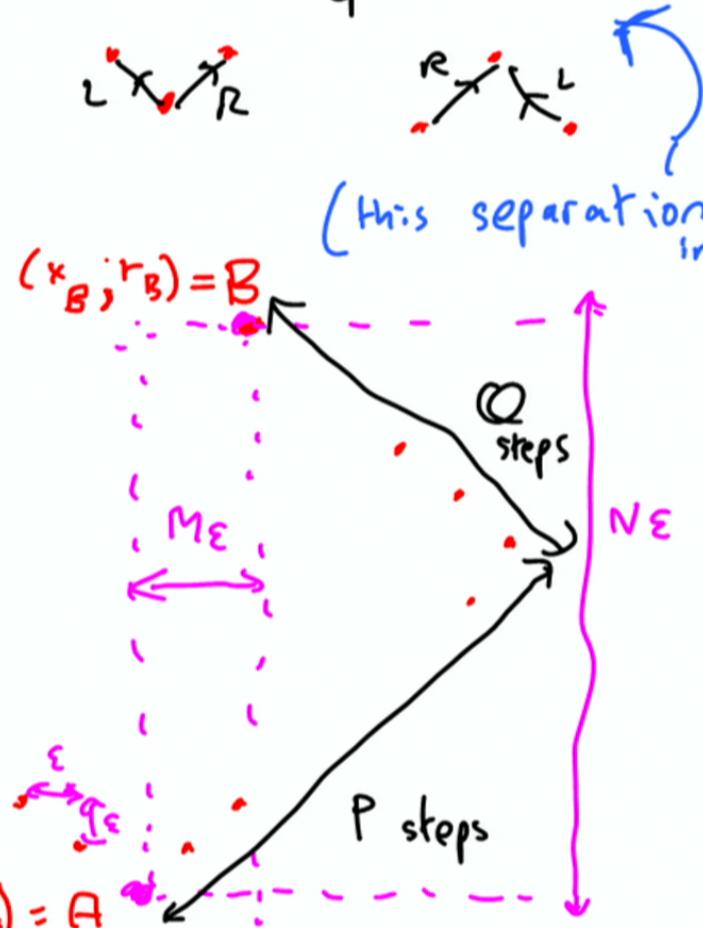
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$$t = t_B - t_A = (P + Q)\varepsilon$$



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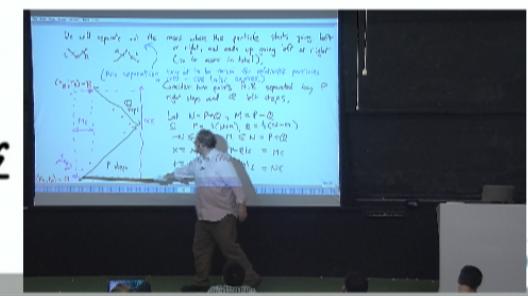
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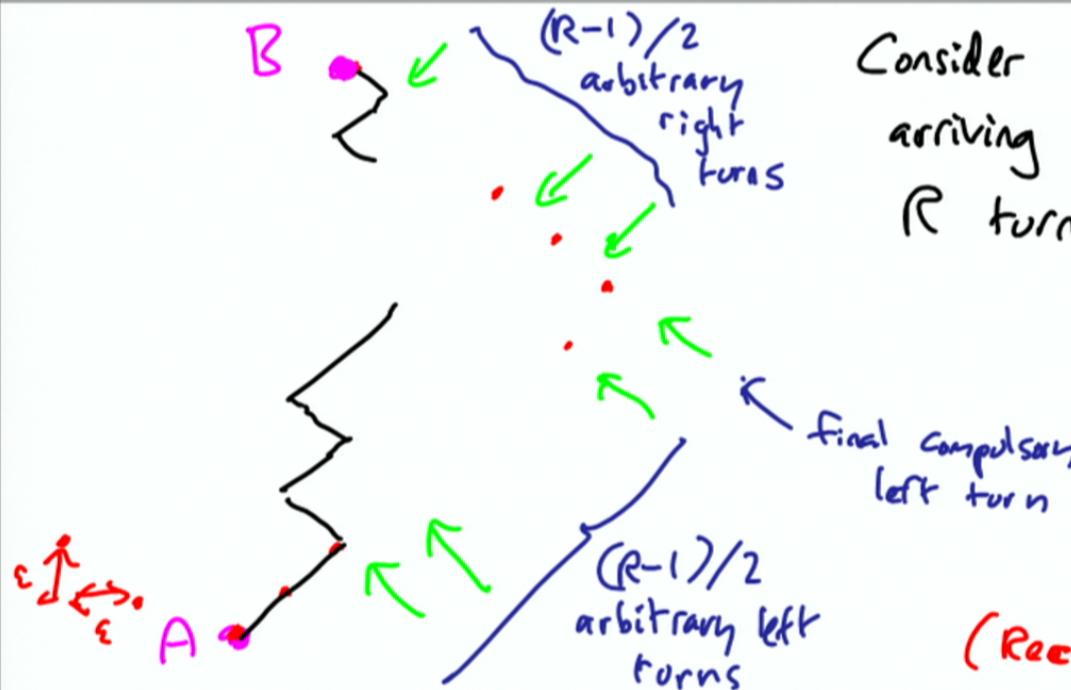
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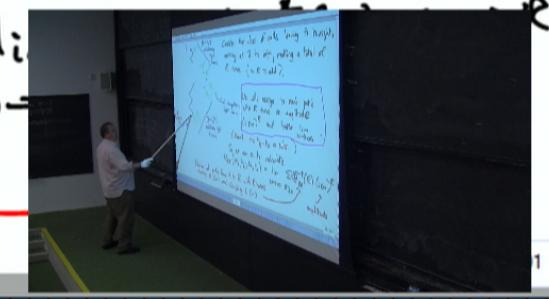
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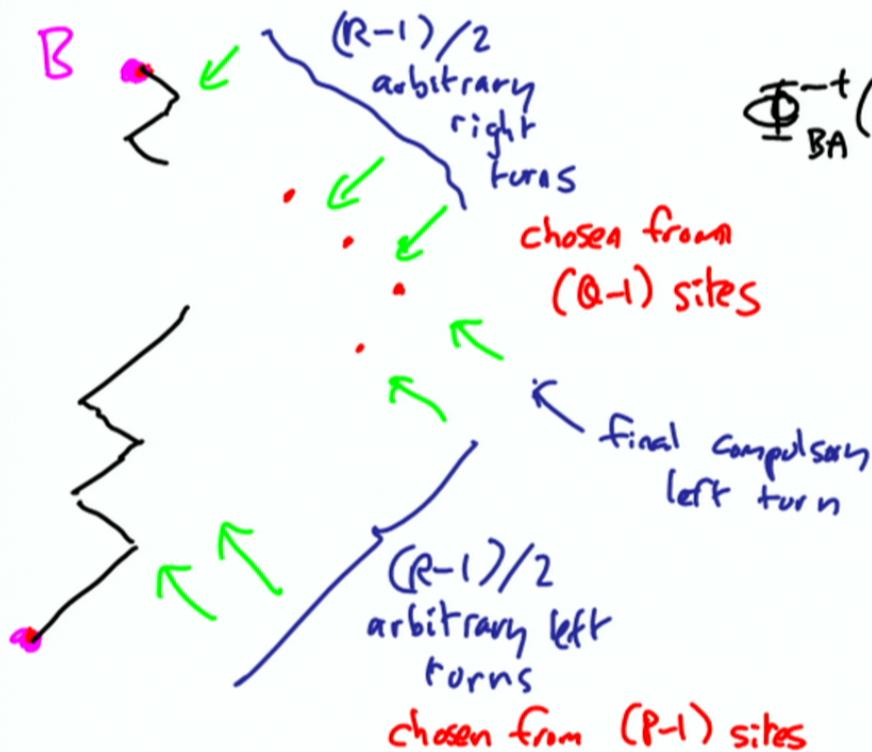
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 $K_{BA}(x_b, t_b; x_a, t_a) = \lim_{N \rightarrow \infty} \dots$

Number of paths from A to B with R turns  
 starting R (+) and finishing L (-)





Need to evaluate

$\Phi_{BA}^{-+}(R) =$  Number of paths from A to B with R turns starting  $R^+$  (+) and finishing  $L^-$  (-).

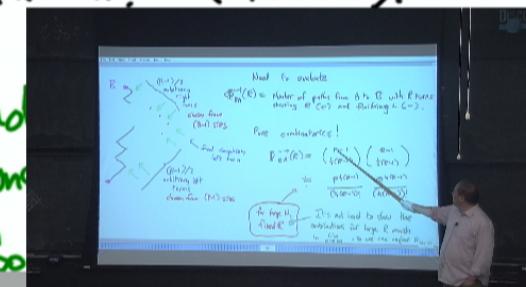
Pure combinatorics!

$$\Phi_{BA}^{-+}(R) = \binom{P-1}{\frac{1}{2}(R-1)} \binom{Q-1}{\frac{1}{2}(R-1)}$$

$$= \frac{P^{\frac{1}{2}(R-1)}}{(\frac{1}{2}(R-1))!} \frac{Q^{\frac{1}{2}(R-1)}}{(\frac{1}{2}(R-1))!}$$

for large  $N$ ,  
fixed  $R$

It's no  
contribution  
in  $\lim_{N \rightarrow \infty}$



$$\frac{(P-1)!}{\frac{1}{2}(R-1)! \left( (R-1) - \frac{1}{2}(R-1) \right)!} = \frac{(P-1)(P-2)\dots(P-\frac{1}{2}(R-1)+1)}{(-1)!}$$

=

$$\begin{aligned}
 \frac{(P-1)!}{\frac{1}{2}(R-1)! \left( (R-1) - \frac{1}{2}(R-1) \right)!} &= \frac{(P-1)(P-2)\dots(P-\frac{1}{2}(R-1)+1)}{\frac{1}{2}(R-1)!} \\
 &= \frac{P^{\frac{1}{2}(R-1)}}{\frac{1}{2}(R-1)!} \cdot \frac{Q^{\frac{1}{2}(R-1)}}{\frac{1}{2}(R-1)!}
 \end{aligned}$$

$$\Phi_{BA}^{-+}(R) \doteq \frac{P^{\frac{1}{2}(R-1)}}{(\frac{1}{2}(R-1))!} \frac{Q^{\frac{1}{2}(R-1)}}{(\frac{1}{2}(R-1))!}$$

$$K_{BA}^{-+} \doteq \sum_{R \text{ odd}} \underbrace{(i\epsilon m)^R}_{\text{amplitude}} \underbrace{(PQ)^{\frac{1}{2}(R-1)}}_{\Phi_{BA}^{-+}(R)} \underbrace{\left((\frac{1}{2}(R-1))!\right)^{-2}}$$

$x = M\varepsilon, t < N\varepsilon$   
 $\text{so } N^2 - M^2 =$   
 $N^2(1 - \frac{x^2}{t^2})$   
 gives us a  
 relativistic  
 correction factor!

$$\text{Now } PQ = \frac{1}{4}(N-M)(N+M) = \left(\frac{N}{2\varepsilon}\right)^2 \text{ where } \gamma = (1-v^2)^{-\frac{1}{2}}$$

$$v^2 = \frac{M^2}{N^2} = \frac{(x_b-x_a)^2}{(t_b-t_a)^2} = \frac{x^2}{t^2}$$

$$\text{So } K_{BA}^{-+} \doteq \frac{2\varepsilon}{N} \sum_{R \text{ odd}} \left(\frac{i\varepsilon t}{2\varepsilon}\right)^R \left(\left(\frac{R-1}{2}\right)!\right)^{-2} \quad (\text{using } t = N\varepsilon).$$

Notice  $\frac{m t}{\varepsilon} = m t (1 - \frac{x^2}{t^2})^{\frac{1}{2}} = m (t^2 - x^2)^{\frac{1}{2}} = m \gamma$

↑

relativistic  
proper time!

With a little more work we can:

- ① Verify the dominant contributions come from finite bend paths  
(#bends  $R \leq$  small multiple of  $\approx \equiv n\pi$ ) even in  $\lim_{N \rightarrow \infty}$ .  
*independent of N*

- ② Evaluate  $K_{-+}$  (and also  $K_{++} K_{+-} K_{--}$ ) exactly

Find  $K_{-+}(k, t) = i J_0(n\pi)$  in  $\lim_{N \rightarrow \infty}$   
↖ Bessel function

and ultimately (though it needs more theoretical ideas - see later courses)  
 ...

- ③ Verify the  $K$ 's are precisely the propagators for a 2-component relativistic wave equation (the Dirac equation) in 1+1D.

With a little more work we can:

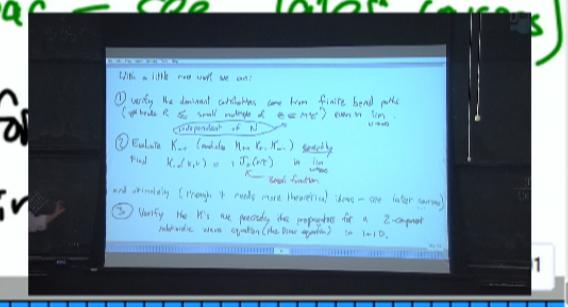
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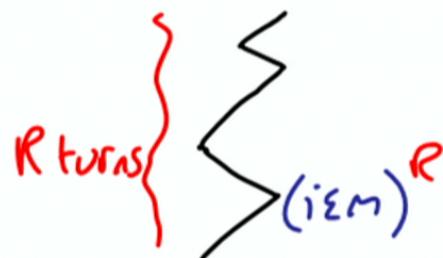
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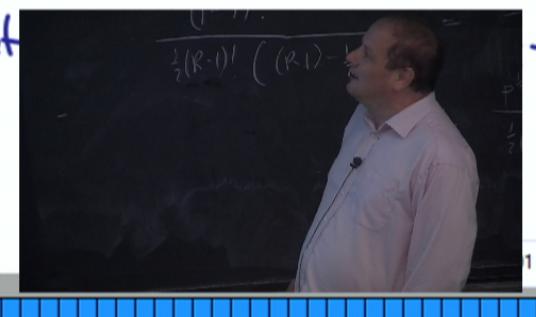


Comments

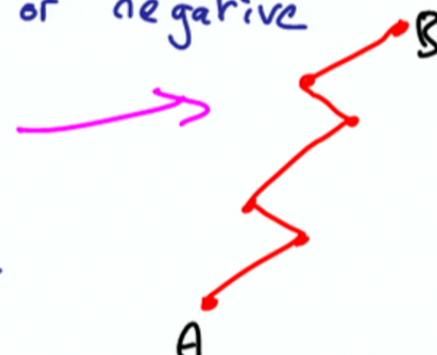
① We broke Lorentz invariance in setting up the checkerboard, but find we recover it in the continuum limit. Not guaranteed!  
Interesting! Worth keeping in mind for other contexts!

② The path amplitude rule is simple but seems quite ad hoc — intriguing that it works.

③ We have a rigorous definition of integral as a cont adequate for this 1+1D problem.



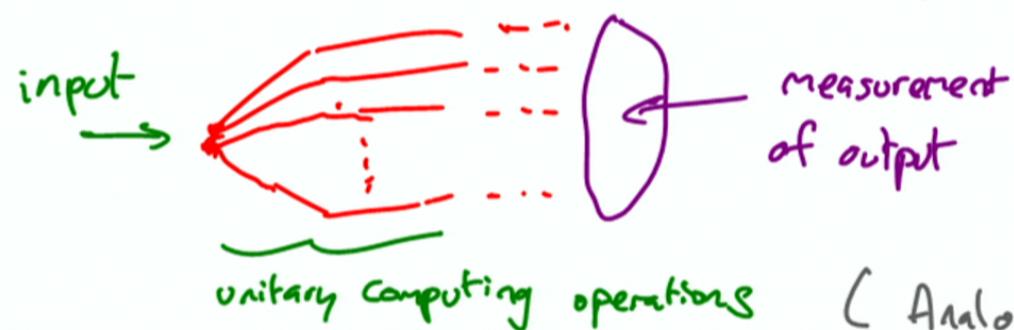
- (4) Notice again that we can't interpret the paths as real physical trajectories since (i) the amplitudes can be complex or negative  
(ii) the paths suggest (piecewise) lightlike motion but define the propagator for a massive particle.



- (5) Sadly this model seems to be special to 1+1D:  
doesn't seem to have a natural generalisation to 3+1D  
(although there are some attempts, e.g. by Jacobsen, Ord et al.)

More remarks on path integrals: ① We've looked at paths in space and time, but clearly momentum or other variables could be used instead of position.

② With this freedom in mind, the metaphor of the path integral is useful/powerful/food for intuition. For example, a quantum computer can be thought of as computing along multiple programs/pathways at once

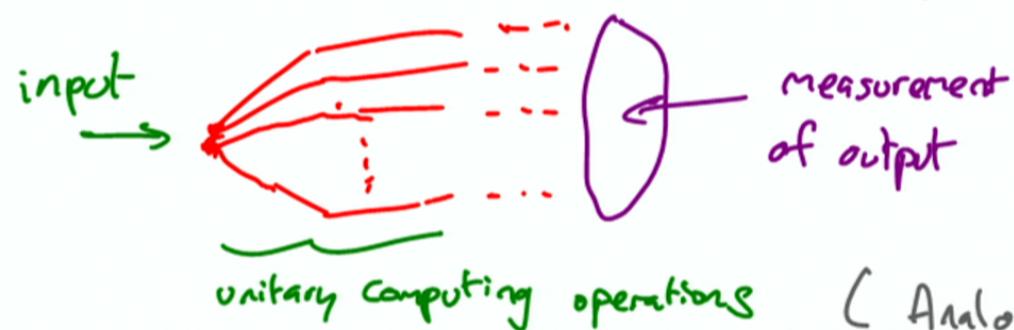


Note that we can't access all the results on all paths — we have to carry out a measurement to get output data.

(Analogy: looking at the screen in a 2-slit experiment doesn't tell us each path amplitude — only  $|A_1 + A_2|^2$ .)

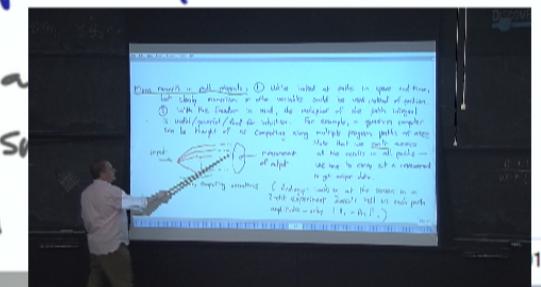
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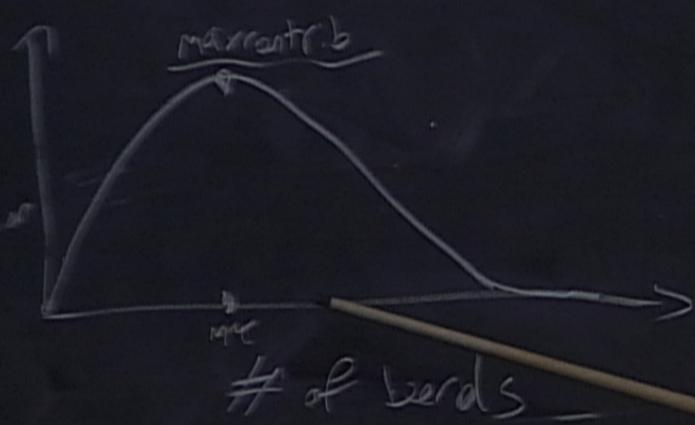


Note that we can't access all the results on all paths — we have to carry out a measurement to get output data.

(Analogy: looking at a 2-slit experiment doesn't give amplitude — only 1 A.)







## Symmetries and generators

Recall our discussion of time evolution for time-independent  $V$  and (so)  $H$

$$\text{important! } U(0) = I$$

$$U(-t) = U(t)^\dagger = U(t)^{-1}$$

$$U(t+t') = U(t)U(t')$$

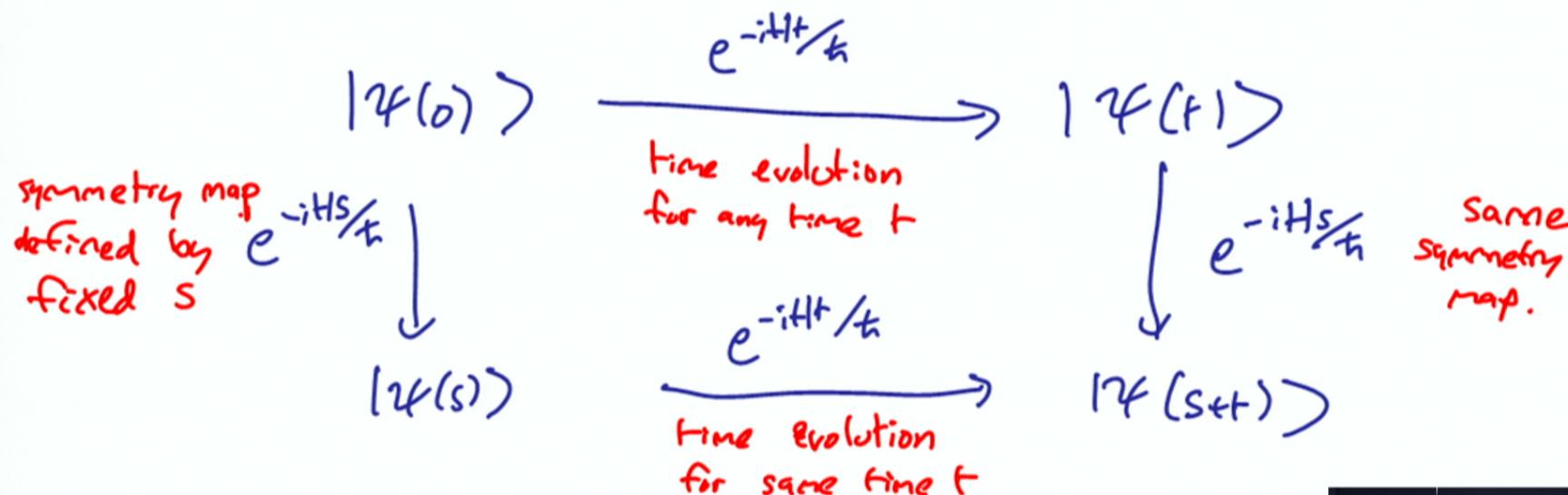
$$U(t)(U(t')U(t'')) = (U(t)U(t'))U(t'')$$

i.e. we can calculate  $H$  given  $U(\epsilon)$  for small  $\epsilon$ .

identity  
inverse  
closure  
associative

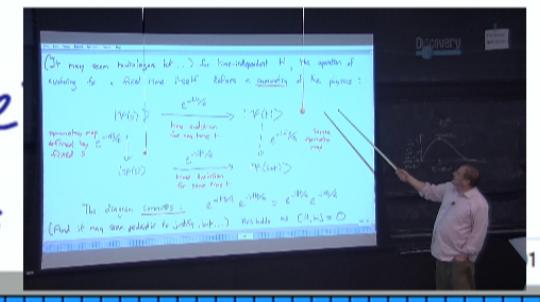
We have a one-parameter group of unitary operators parametrised by  $-\infty < t < \infty$ , with the Hamiltonian  $H$  as the "generator" of the exponential group elements.

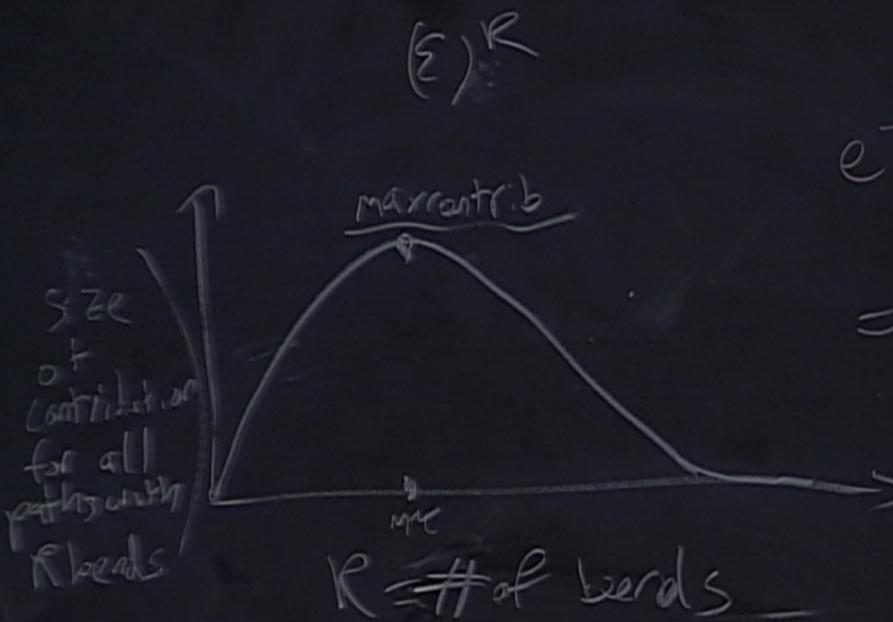
(It may seem tautologous but...) for time-independent  $H$ , the operation of evolving for a fixed time itself defines a symmetry of the physics:



The diagram commutes:  $e^{-iHs/k} e^{-iHt/k} = e^{-iH(s+t)/k}$

(And it may seem pedantic to justify, but...) this holds as





$$e^{-iHs/\hbar} e^{-iHt/\hbar} = e^{-iH(s+t)/\hbar}$$

~~$iH/\hbar$~~  stuff +  
 $(0)^0$

$$\tilde{\zeta}(R-D) \quad \tilde{\zeta}(R+D)$$

$S$  could be a discrete symmetry,  
e.g. in 1D with  $V(x) = V(-x)$   
 $S = P$ , parity:  $P \Psi(x) = \Psi(-x)$

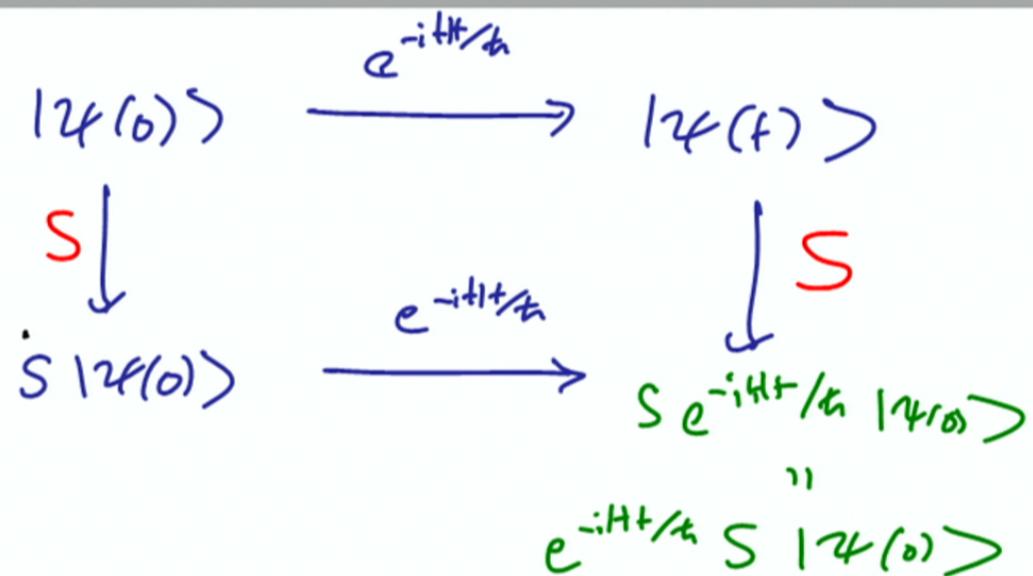
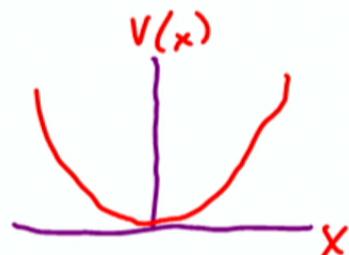
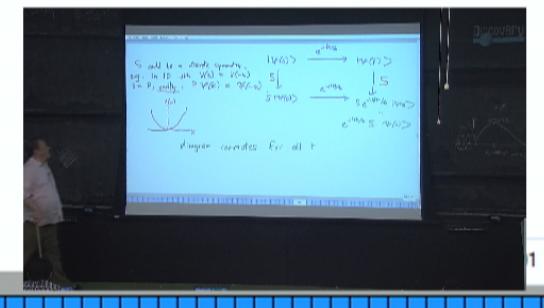
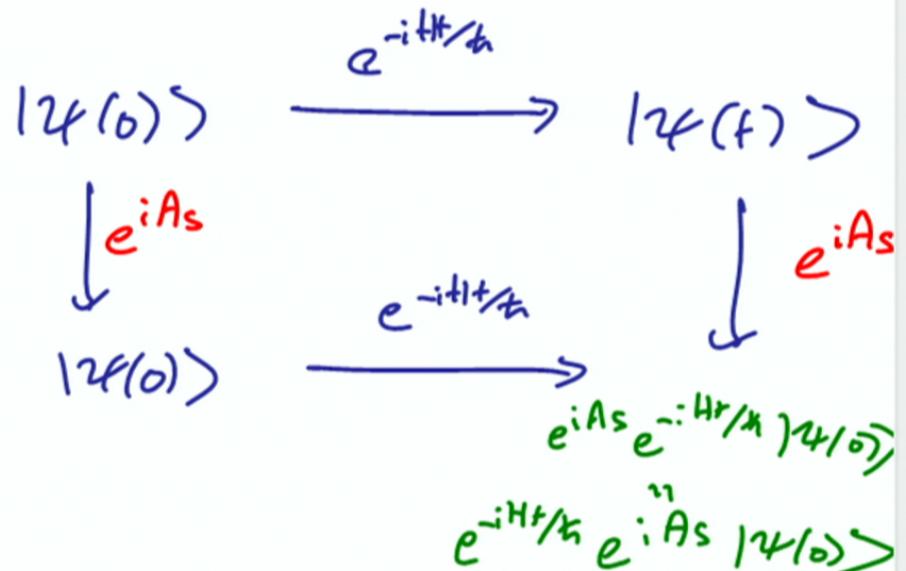


diagram commutes for all  $t$



But we're particularly interested for now in examples of continuous families of symmetries ;  $S = e^{iAs}$ ,  $A = A^\dagger$   
s real.

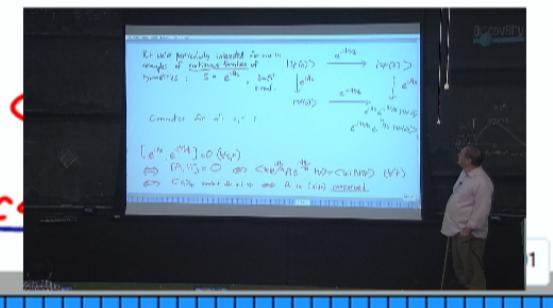
Commutes for all  $s, t$  :



$$[e^{iAs}, e^{-itH/t}] = 0 \quad (\forall s, t)$$

$$\Leftrightarrow [A, H] = 0 \Leftrightarrow \langle \psi | e^{iH/t} A e^{-iH/t} | \psi \rangle = 0$$

$$\Leftrightarrow \langle A \rangle_\psi \text{ constant for all } \psi \Leftrightarrow A \text{ is (also) } \underline{\text{c}}$$



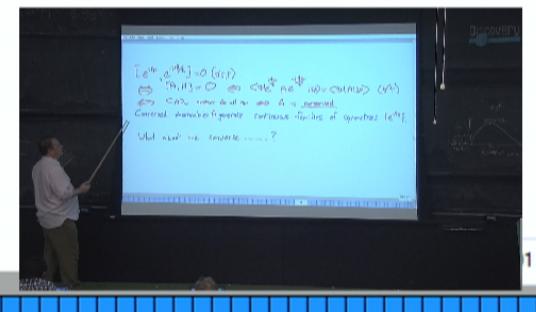
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$$\Leftrightarrow [A, H] = 0 \Leftrightarrow \langle \psi | e^{\frac{iHt}{\hbar}} A e^{-\frac{iHt}{\hbar}} | \psi \rangle = \langle \psi | A | \psi \rangle \quad (\forall t)$$

$\Leftrightarrow \langle A \rangle_{\psi}$  constant for all  $\psi \Leftrightarrow A$  is conserved

Conserved observables  $A$  generate continuous families of symmetries  $\{e^{iAs}\}$ .

What about the converse ..... ?



Conversely, if we have a continuous family  $S(s)$  defining a symmetry group,  $S(s)S(t) = S(s+t)$ , with  $S(0) = I$ , then:

$$\textcircled{1} \quad \langle \psi | S^*(s) S(s) |\psi \rangle = \langle \psi | \psi \rangle \quad \text{for all } |\psi\rangle \\ \therefore S^*(s) S(s) = I.$$

But also  $S(s) S(-s) = I$  by composition law  $S(sts') = S(s)S(s')$   
 $\therefore S^*(s) = S(-s)$ .

$$\textcircled{2} \quad \text{For } s=\varepsilon \text{ small, } S(\varepsilon) \doteq I + i\varepsilon A \quad \text{for some operator } A \text{ (continuity)} \\ \therefore S^*(\varepsilon) \doteq I - i\varepsilon A^\dagger$$

$$\text{But } S^*(\varepsilon) = S(-\varepsilon) \doteq I - i\varepsilon A. \quad \text{So } \boxed{A = A^\dagger} \text{ hermitian}$$

Conversely, if we have a continuous family  $S(s)$  defining a symmetry group,  $S(s)S(t) = S(s+t)$ , with  $S(0) = I$ , then:

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$$\text{But also } S(s) S(-s) = I \quad \text{by composition law } S(s+s') = S(s)S(s') \\ \therefore S^*(s) = S(-s).$$

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(3)

$$S(s+\varepsilon) = S(s) S(\varepsilon) \doteq S(s)(1 + i\varepsilon A). \quad \left( \lim_{\varepsilon \rightarrow 0} \left( \frac{o(\varepsilon)}{\varepsilon} \right) = 0 : \text{smaller than } \varepsilon \right)$$

$$S_0 \frac{dS(s)}{ds} = \lim_{\varepsilon \rightarrow 0} \left( \frac{S(s+\varepsilon) - S(s)}{\varepsilon} \right) = \lim_{\varepsilon \rightarrow 0} \left( \frac{S(s) + i\varepsilon A + o(\varepsilon)}{\varepsilon} \right)$$

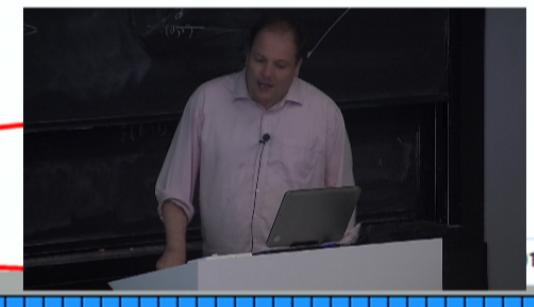
i.e.  $\frac{dS(s)}{ds} = :S(s) A$

$$\frac{dS(s)}{ds} = i s A \quad \text{gives us solution.}$$

$$S(s) = e^{As}$$

The symmetry group is generated by the hermitian  $A$  is conserved.

$\{$  Continuous symmetry groups  $\} \longleftrightarrow \{$  Conserved



$$\textcircled{3} \quad S(s+\varepsilon) = S(s)S(\varepsilon) \doteq S(s)(1+i\varepsilon A).$$

$\lim_{\varepsilon \rightarrow 0} \left( \frac{o(\varepsilon)}{\varepsilon} \right) = 0$  : smaller than  $\varepsilon$

$$S_0 \frac{dS(s)}{ds} = \lim_{\varepsilon \rightarrow 0} \left( \frac{S(s+\varepsilon) - S(s)}{\varepsilon} \right) = \lim_{\varepsilon \rightarrow 0} \left( \frac{S(s)i\varepsilon A + o(\varepsilon)}{\varepsilon} \right)$$

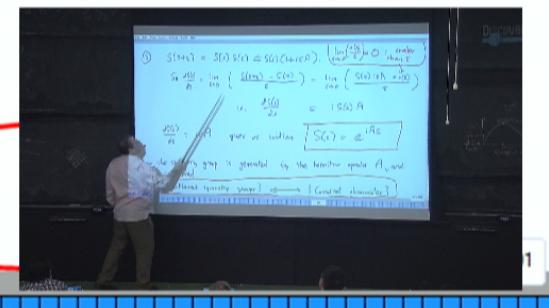
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{Continuous symmetry groups}  $\longleftrightarrow$  {Conserved}



$$A = A^+$$

Continuous families of symmetries  $\{S(s)\} \Leftrightarrow$  conserved quantities  $[A, H] = 0$   
 $S(s) = \exp(isA)$

Examples: ① already noted,  $\text{If } V \text{ time-independent}$ , time translation  $\Leftrightarrow H$ .  $[H, H] = 0$ .

② If  $V$  translation-independent (eg  $V(x) = 0$  for free particle)  
 Then  $[p, H] = 0$  and  $p$  generates symmetries  $e^{ipst}$   

$$\begin{aligned} e^{ipst} \psi(x) &= \exp\left(s \frac{d}{dx}\right) \psi(x) \\ &= \underbrace{\left(1 + s \frac{d}{dx} + \frac{1}{2}s^2 \left(\frac{d}{dx}\right)^2 + \dots\right)}_{\text{Taylor series!}} \psi(x) \\ &= \psi(x+s). \end{aligned}$$

I.e. momentum  $p$  generates translations in space.

③ In 3D, if we have a rotationally symmetric potential  $V(\underline{x}) \equiv V(|\underline{x}|)$ , then  $H$  is rotationally invariant and the rotation group  $SO(3)$  defines a continuous symmetry.

$$R \in SO(3) : \Psi(x) \rightarrow \Psi(Rx)$$

$$[H, R] = 0.$$

E.g. rotations about the  $z$  axis  $R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

For  $\alpha$  small,  $\alpha = \varepsilon$

$$R_z(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon \\ \varepsilon & 1 \\ 0 & 0 \end{pmatrix}$$



③ In 3D, if we have a rotationally symmetric potential  $V(\underline{x}) \equiv V(|\underline{x}|)$ , then  $H$  is rotationally invariant and the rotation group  $SO(3)$  defines a continuous symmetry.

$$R \in SO(3) : \Psi(\underline{x}) \rightarrow \Psi(R\underline{x})$$

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For  $\alpha$  small,  $\alpha = \varepsilon$

$$R_z(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I + \begin{pmatrix} 0 & -\varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

rotations about the z axis  $R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow R_z(\alpha) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

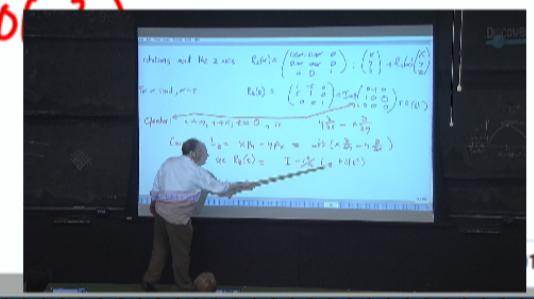
For  $\alpha$  small,  $\alpha = \varepsilon$

$$R_z(\varepsilon) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I + \varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\varepsilon^2)$$

Operator:  $x \rightarrow -y, y \rightarrow x, z = 0$ , is  $y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}$

Compare  $L_z = x p_y - y p_x = -i\hbar (x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$

We see  $R_z(\varepsilon) = I - i \frac{\varepsilon}{\hbar} L_z + O(\varepsilon^2)$



More precisely,  $R_z(\varepsilon) = \begin{pmatrix} \cos \varepsilon & -\sin \varepsilon & 0 \\ \sin \varepsilon & \cos \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \dots\right) & -\left(\varepsilon - \frac{\varepsilon^3}{3!} + \dots\right) & 0 \\ \left(\varepsilon - \frac{\varepsilon^3}{3!} + \dots\right) & \left(1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \dots\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

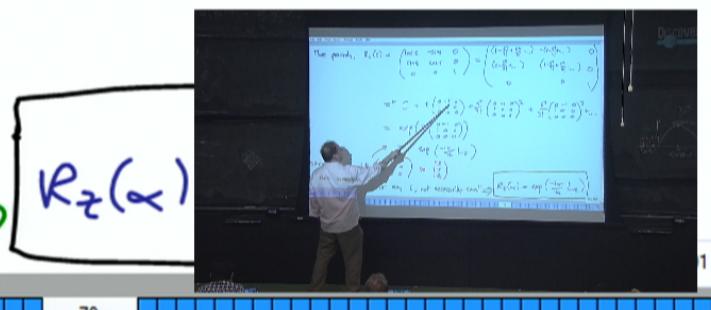
$$= I + \varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\varepsilon^2}{2!} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 + \frac{\varepsilon^3}{3!} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^3 + \dots$$

$$= \exp\left(\varepsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$\stackrel{\rightarrow}{=} \exp\left(-\frac{i\varepsilon}{\hbar} L_z\right)$$

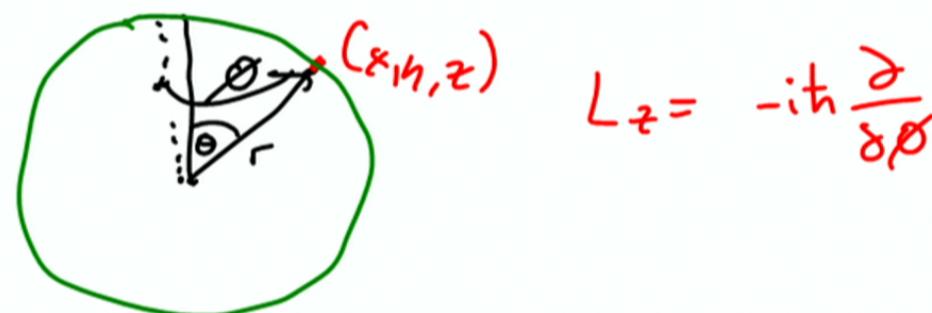
Since  $L_z = -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  on  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

\* And this converges for any  $\varepsilon$ , not necessarily small  $\Rightarrow R_z(\varepsilon)$



Comment

This is often shown by transforming to spherical polar coordinates  $(x, y, z) \rightarrow (r, \theta, \phi)$



$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Which is slick — but it's also good to see that we can work directly with rotation group generators in Cartesian coordinates  $(x, y, z)$ .



Similarly

$$R_x(\alpha) = \exp\left(-\frac{i\alpha}{\hbar} L_x\right)$$

$$R_y(\alpha) = \exp\left(-\frac{i\alpha}{\hbar} L_y\right)$$

and for any axis  $\hat{n}$ 

unit vector  $\uparrow$

$$R_{\hat{n}}(\alpha) = \exp\left(-\frac{i\alpha}{\hbar} L_{\hat{n}}\right)$$



where  $L_{\hat{n}} = \hat{n} \cdot \underline{L}$

i.e. linear combination  
of  $L_x, L_y, L_z$

$$\underline{L} = \hat{x} \wedge \underline{p} = (L_x, L_y, L_z) \\ \equiv (L_1, L_2, L_3)$$

alternative notation  $\longrightarrow$

So we can understand the action of all rotations if we know the action of the operators  $\{L_x, L_y, L_z\}$ .

Easiest to work in  $(x_1, x_2, x_3)$  notation.

$$[p_i, x_j] = -i\hbar \delta_{ij} \Rightarrow [L_i, L_j] = i\hbar \sum_{k=1}^3 L_k \quad ([L_1, L_2] = i\hbar L_3 \text{ etc})$$

Define  $L^2 = L_i L_i = L_1^2 + L_2^2 + L_3^2$  (Einstein summation convention)

$$\begin{aligned} \text{Then } [L^2, L_j] &= L_i [L_i, L_j] + [L_i, L_j] L_i \\ &= i\hbar (L_i \sum_{k=1}^3 L_k + \sum_{k=1}^3 L_k L_i) \\ &= i\hbar \underbrace{\sum_{i,j,k} L_i L_k}_{\substack{\text{antisymmetric} \\ \text{in } (i,k)}} \underbrace{\sum_{i,j,k} L_k L_i}_{\substack{\text{symmetric} \\ \text{in } (i,k)}} = 0 \end{aligned}$$

So we can find simultaneous eigenstates of  $L^2$  and one of the  $L_j$ . Conventionally, we choose  $L^2$  and  $L_3$ .

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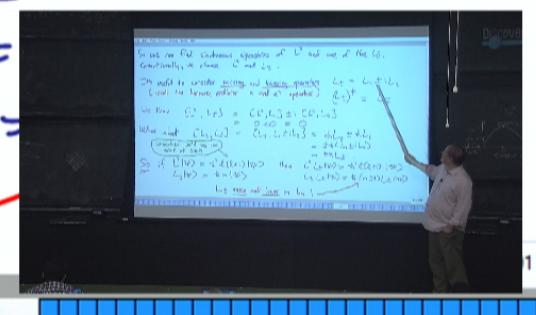
It's useful to consider raising and lowering operators  
 (recall the harmonic oscillator  $a$  and  $a^\dagger$  operators)  $L_\pm = L_1 \pm iL_2$   
 $(L_\pm)^\dagger = L_\mp$

$$\text{We know } [L^2, L_\pm] = [L^2, L_1] \pm i[L^2, L_2] \\ = 0 + 0 = 0$$

$$\text{What about } [L_3, L_\pm] = [L_3, L_1 \pm iL_2] = i\hbar L_2 \pm \hbar L_1 \\ = \pm \hbar (L_1 \pm iL_2) \\ = \pm \hbar L_\pm.$$

So if  $L^2 |l\rangle = \hbar^2 l(l+1) |l\rangle$  then  $L^2 L_\pm |l\rangle =$   
 $\qquad L_3 |l\rangle = \pm \hbar |l\rangle$   $L_3 L_\pm |l\rangle =$

$L_\pm$  raise and lower  $m$  by 1



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What about  $[L_3, L_\pm] = [L_3, L_1 \pm iL_2] = i\hbar L_2 \pm \hbar L_1 = \pm \hbar (L_1 \pm iL_2) = \pm \hbar L_\pm$ .  
 (convention we'll see the point of soon)

So if  $L^2 |l\rangle = \hbar^2 l(l+1) |l\rangle$  then  $L^2 L_\pm |l\rangle = \hbar^2 L_\pm |l\rangle$   
 $L_3 |l\rangle = \hbar m |l\rangle$   
 $L_\pm$  raise and lower m by 1

$$\text{If } L^2 |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle \text{ then } L^2 L_{\pm} |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle$$

$$L_3 |\psi\rangle = \hbar m |\psi\rangle \quad L_3 L_{\pm} |\psi\rangle = \hbar(m \pm 1) L_{\pm} |\psi\rangle$$

Now look at  $|L_+ |\psi\rangle|^2 = \langle \psi | (L_- L_+ |\psi\rangle) = \langle \psi | (L_{-iL_2})(L_{iL_2}) |\psi\rangle$

$$= \langle \psi | (L_1^2 + L_2^2 + i[L_1, L_2]) |\psi\rangle$$

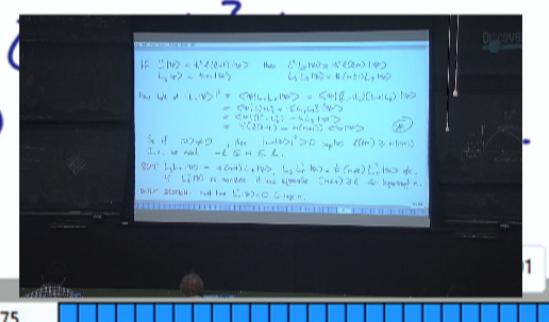
$$= \langle \psi | (L^2 - L_3^2) - \hbar L_3 |\psi\rangle$$

$$= \hbar^2 (l(l+1) - m(m+1)) \langle \psi | \psi \rangle \quad (*)$$

So if  $|\psi\rangle \neq 0$ , then  $|L_+ |\psi\rangle|^2 > 0$  implies  $l(l+1) \geq m(m+1)$   
 I.e. we need  $-l \leq m \leq l$ .

BUT  $L_3 L_+ |\psi\rangle = \hbar(m+l) L_+ |\psi\rangle$ ,  $L_3 L_+^2 |\psi\rangle = \hbar l L_+ |\psi\rangle$   
 If  $L_+^n |\psi\rangle$  is nonzero it has eigenvalue  $(m+n)$

ONLY SOLUTION: must have  $L_+^n |\psi\rangle = 0$  for large  $n$ .



$$\text{If } L^2 |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle \text{ then } L^2 L_{\pm} |\psi\rangle = \hbar^2 l(l+1) |\psi\rangle$$

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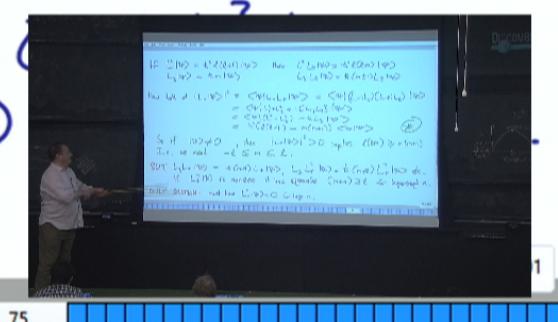
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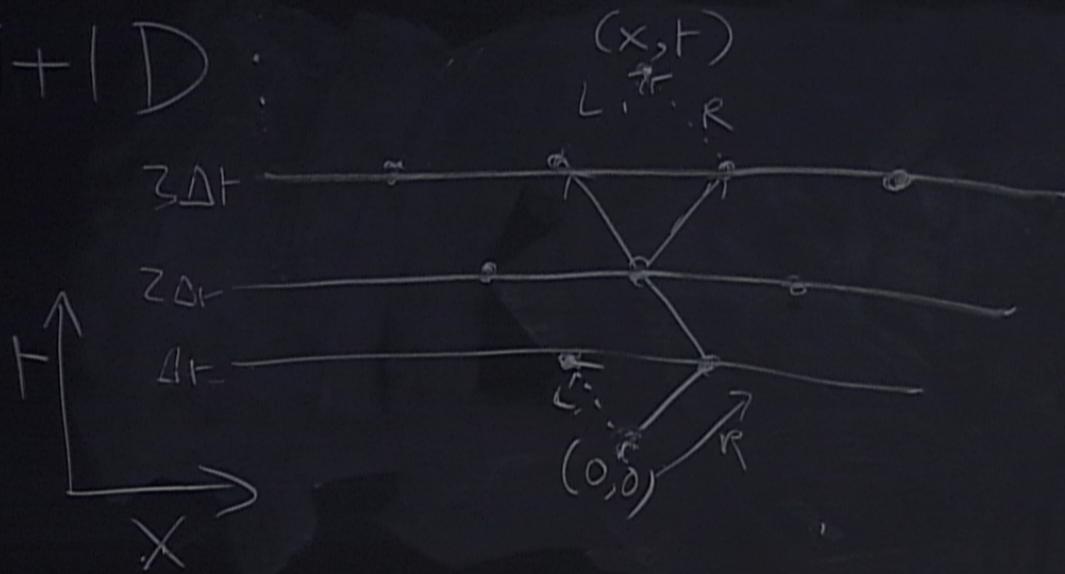
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$|+\rangle D:$



$|L^2 \rangle |2\rangle (m=2)$   
 $|L \rangle |2\rangle (m=1)$   
 $|2\rangle m$

$$(L_+ | \psi \rangle)^2 = \hbar^2 (l(l+1) - m(m+1))$$

ONLY SOLUTION: must have  $L_+^n |\psi\rangle = 0$  for large  $n$ .

\*

Can't be negative

state	$L_z$ eigenvalue
$L_+^{n+1}  \psi\rangle = 0$	$(m+n)\hbar$
$L_+^n  \psi\rangle$	$(m+n-1)\hbar$
$L_+^{n-1}  \psi\rangle$	$(m+n-2)\hbar$
$L_+^0  \psi\rangle$	$m\hbar$

From \*, this means

$$m+n = l$$

Similarly, looking at the series  $(L_-)^n |\psi\rangle$ , we see  $(L_-)^{n'} |\psi\rangle = 0$  for some  $n'$

with

$$m-n' = -l$$

CONCLUSION: the possible eigenvalues  $m$  lie in range  $-l, -l+1, \dots, l-1, l$ .

WHICH REQUIRES:  $2l$  must be positive integer, ie  $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

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**WHICH REQUIRES:**  $2l$  must be positive integer, i.e.  $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

Now if we look at the action of rotation group on position space wavefunctions we find a further constraint:

$$\psi(R_z(\alpha)x) = \exp\left(-\frac{i\alpha}{\hbar}L_z\right)\psi(x) = \exp(-i\alpha m)\psi(x)$$

if  $L_z\psi = mh\psi$

So taking  $\alpha = 2\pi$ ,  $R_z(\alpha) = I$ ,  $\psi(R_z(2\pi)x) = \psi(x)$   
and we require  $\exp(-i2\pi m) = 1$ .

That is,  $m$  must be an integer:

$$\boxed{\begin{aligned} l &= 0, 1, 2, \dots \\ -l &\leq m \leq l \end{aligned}}$$

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