Title: Evaluation of Integrals and Calculus of Variations - Lecture 3

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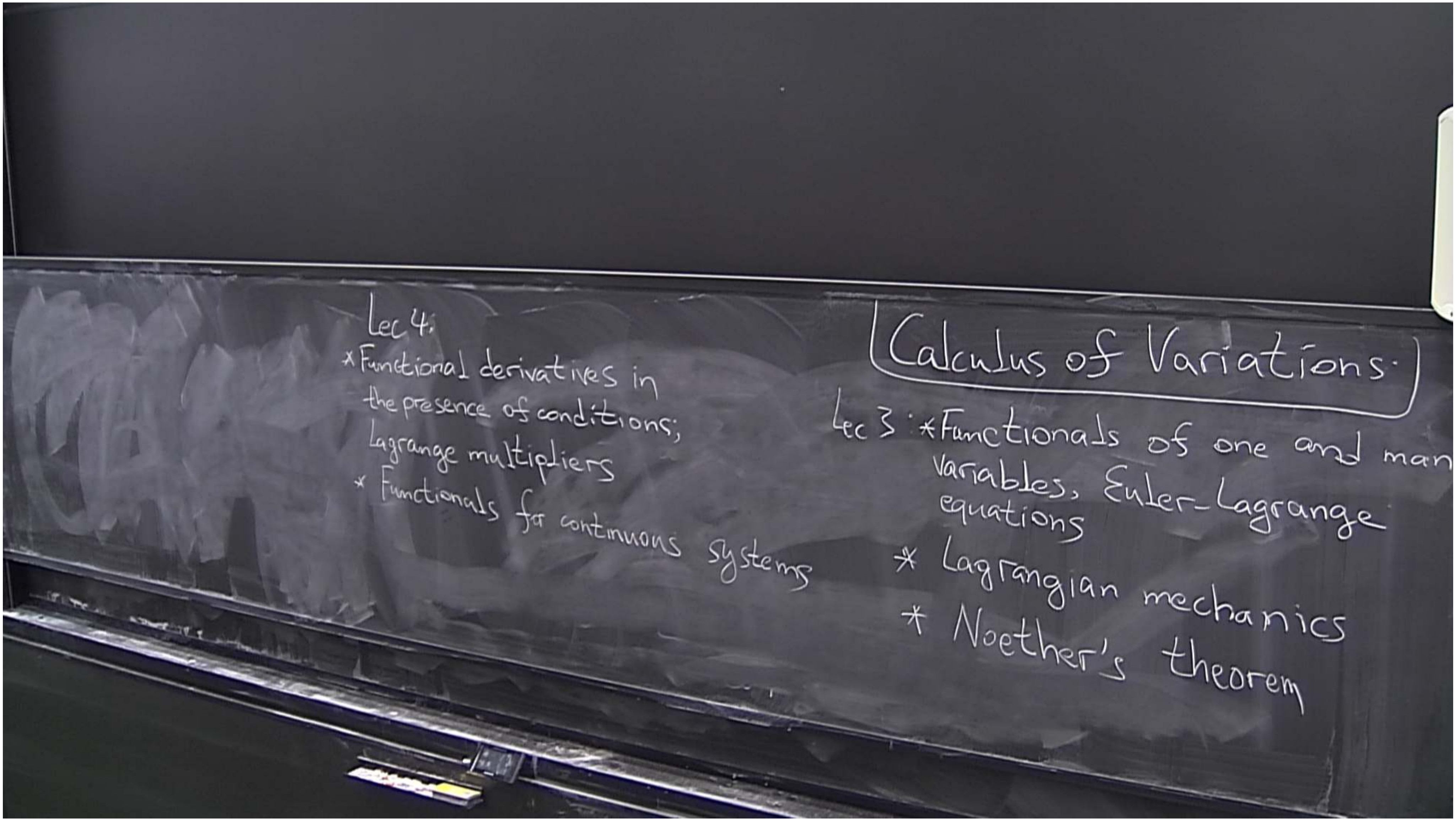
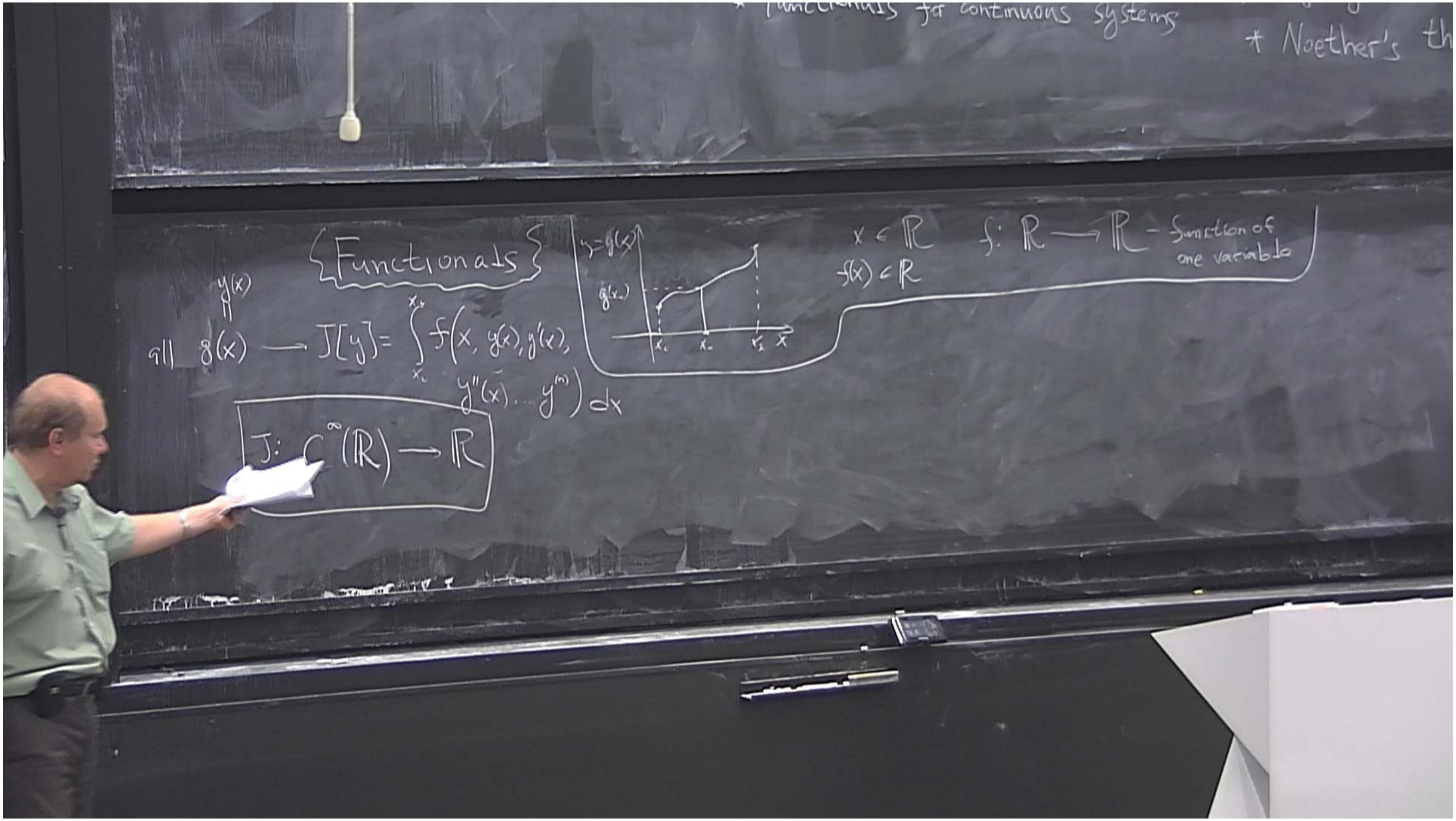
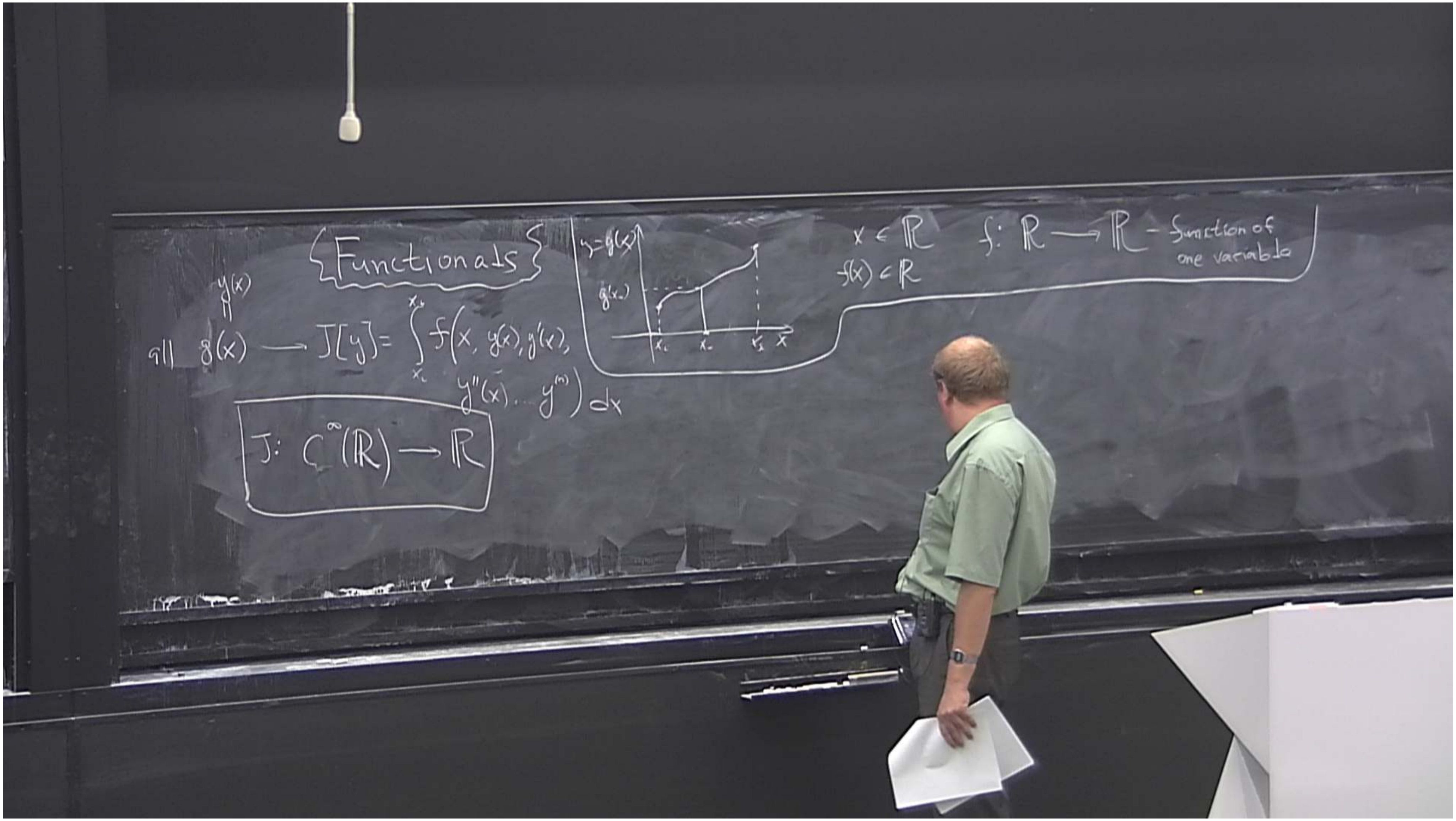
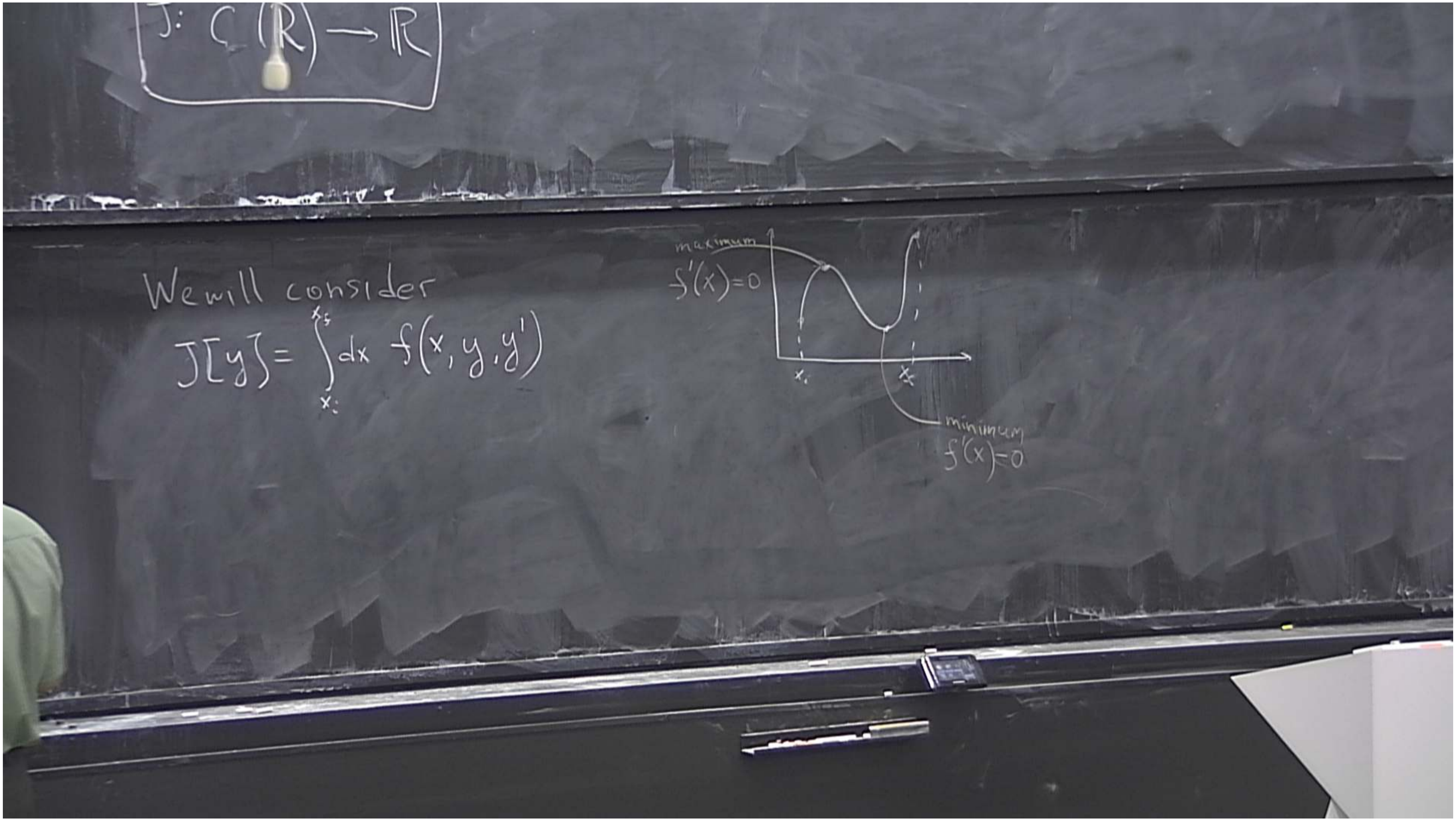
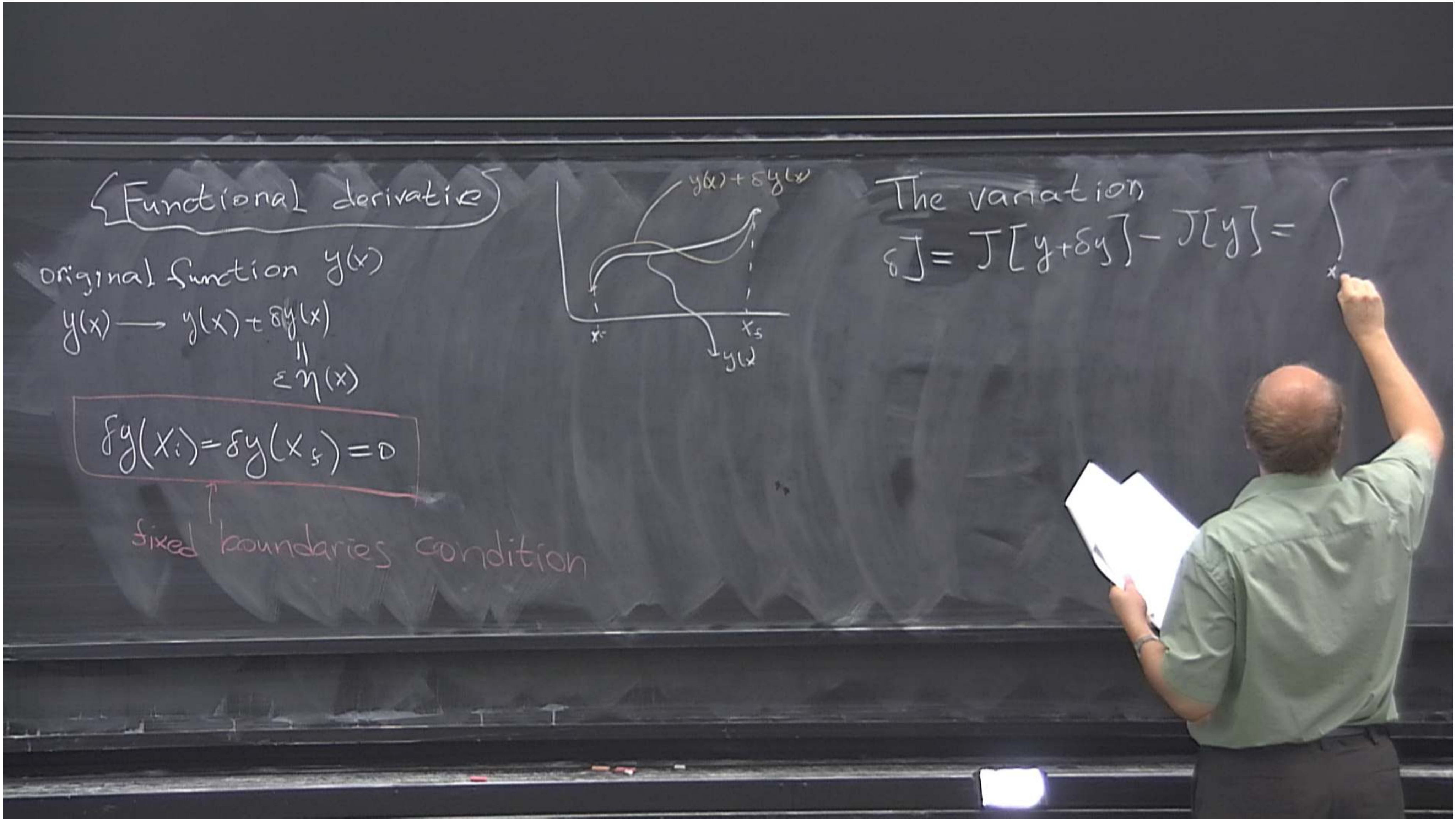
Abstract:

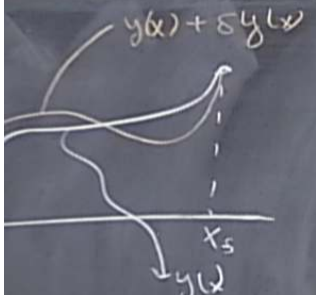

Lec 4:
* Functional derivatives in the presence of conditions;
Lagrange multipliers
* Functionals for continuous systems
Calculus of Variations.
Lec 3: * Functionals of one and man
variables, Euler-Lagrange
equations
* Lagrangian mechanics
* Noether's theorem


for continuous systems
* Noether's th
{Functionals}
y(x)
all g(x) → J[y] = ∫ f(x, y(x), y'(x), y''(x), ... y^(n)) dx
J: C^∞(ℝ) → ℝ
y=g(x)
g(x₀)
x₁ x₀ x₂ x
x ∈ ℝ
f(x) ∈ ℝ
f: ℝ → ℝ - function of one variable


Functionals
x ∈ ℝ
f(x) ∈ ℝ
f: ℝ → ℝ - function of one variable
J: C^∞(ℝ) → ℝ


J: C(R) → R
We will consider
J[y] = ∫ dx f(x, y, y')
maximum
f'(x)=0
minimum
f'(x)=0


Functional derivative
original function y(x)
y(x) ⟶ y(x) + δy(x) = εη(x)
δy(x_i) = δy(x_f) = 0
fixed boundaries condition
y(x) + δy(x)
x_i
x_f
y(x)
The variation
δJ = J[y+δy] − J[y] =



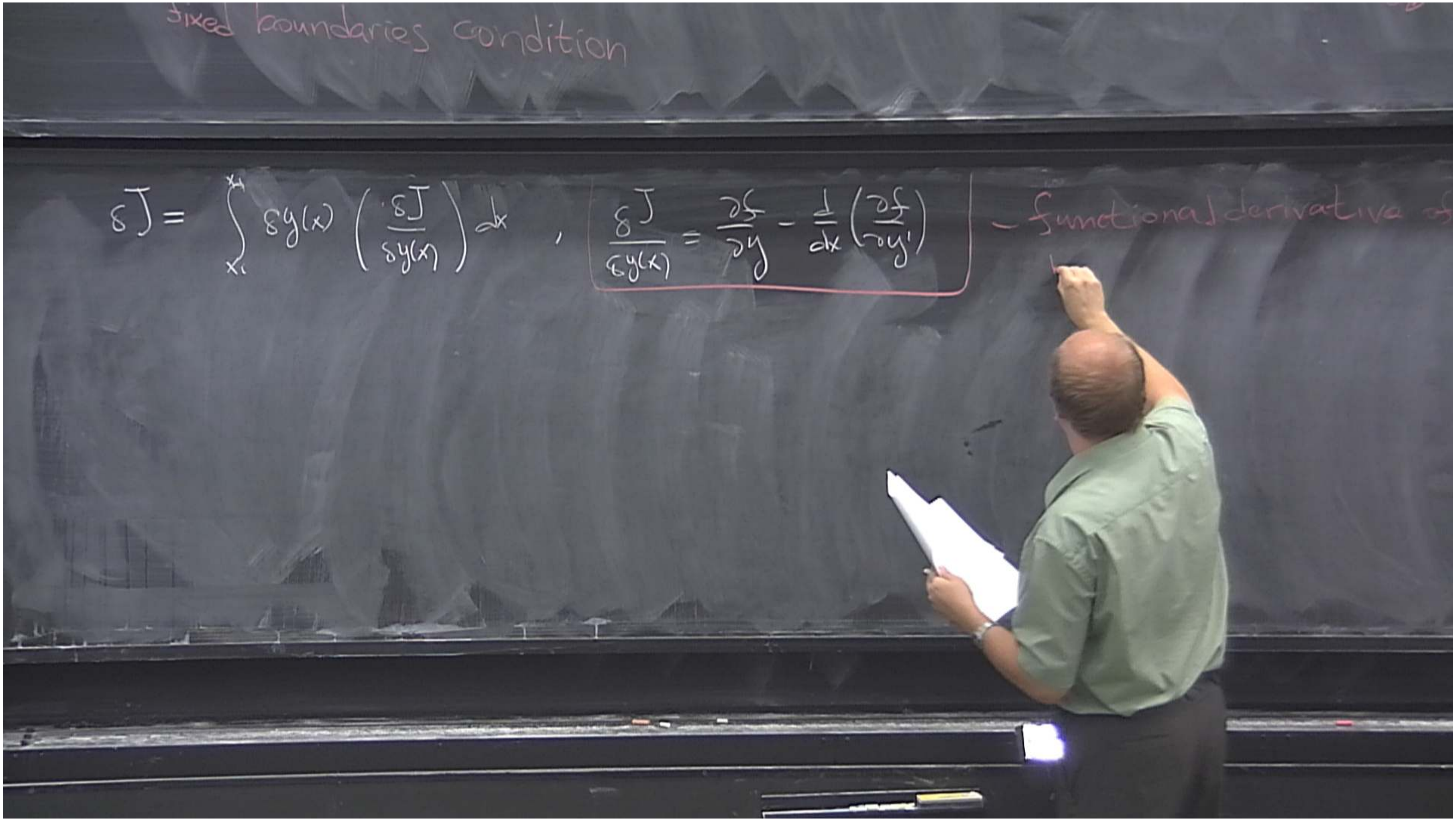
The variation

$$\delta J = J[y+\delta y] - J[y] = \int_{x_i}^{x_f} dx \left\{ f(x, y+\delta y, y'+\delta y') - f(x, y, y') \right\} =$$

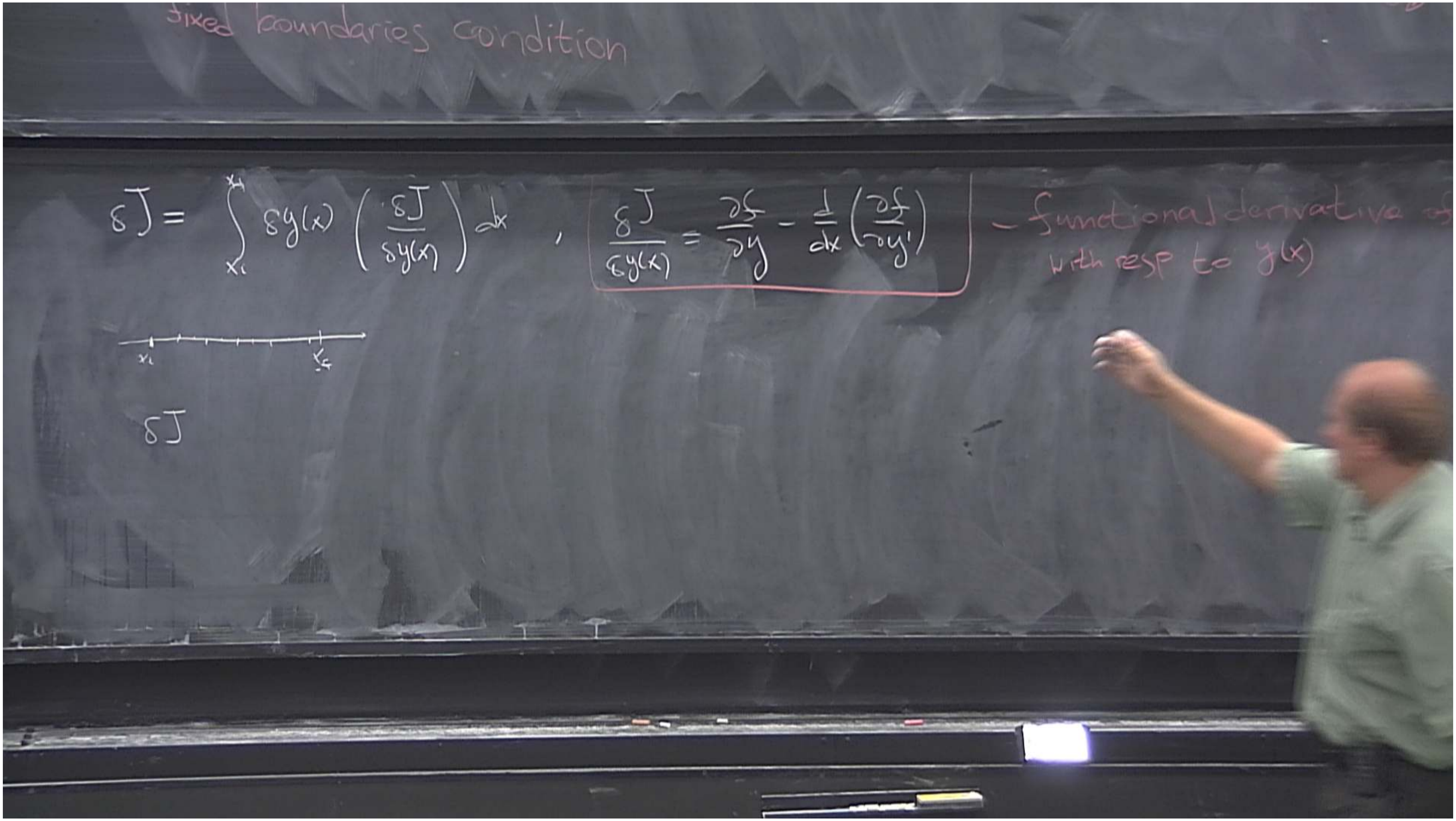
$$= \int_{x_i}^{x_f} dx \left\{ \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + O(\delta y^2) \right\} =$$

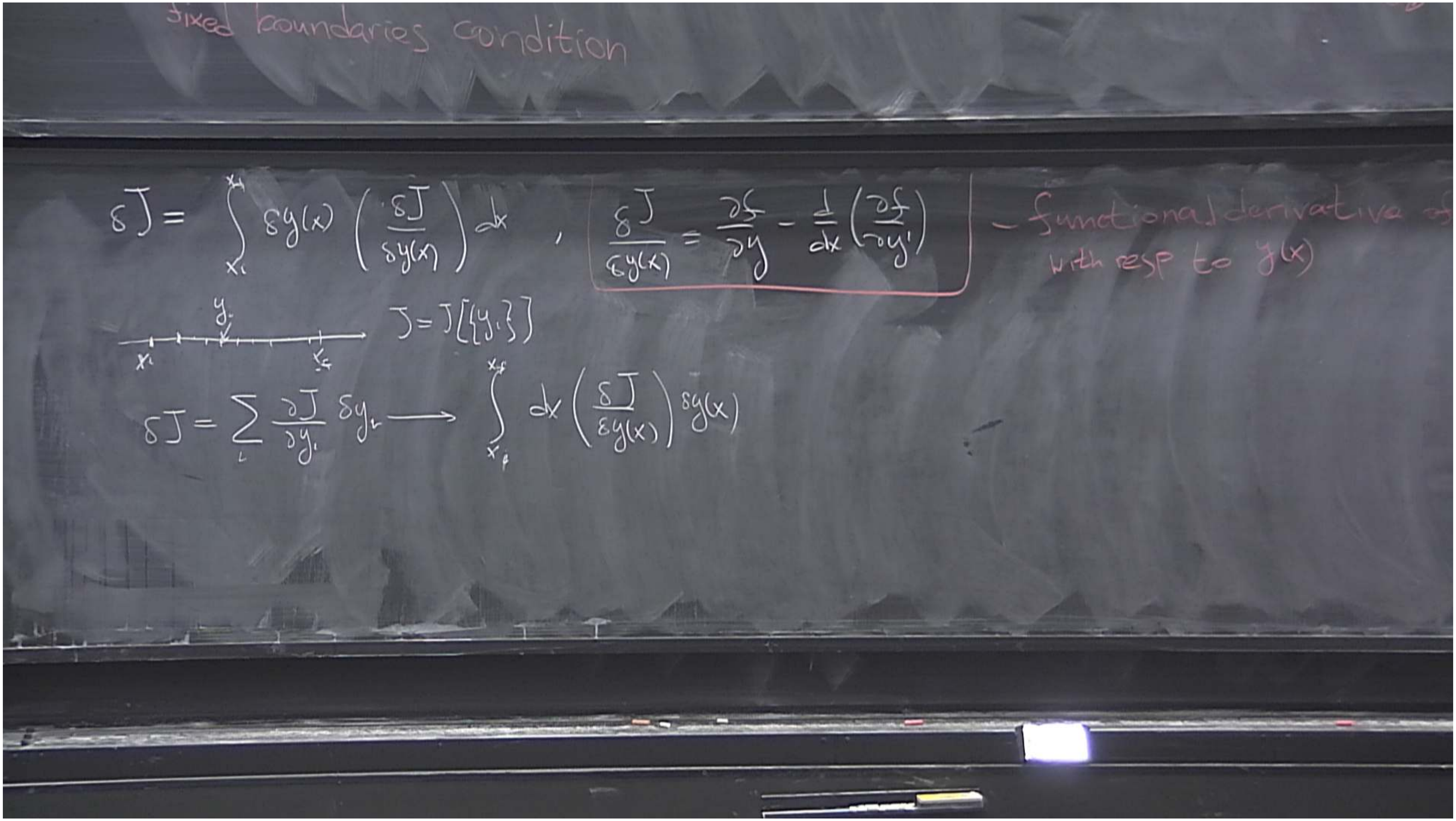
$$= \left[\delta y \cdot \frac{\partial f}{\partial y'} \right]_{x_i}^{x_f} + \int_{x_i}^{x_f} \delta y \left\{ -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y} \right\} dx =$$

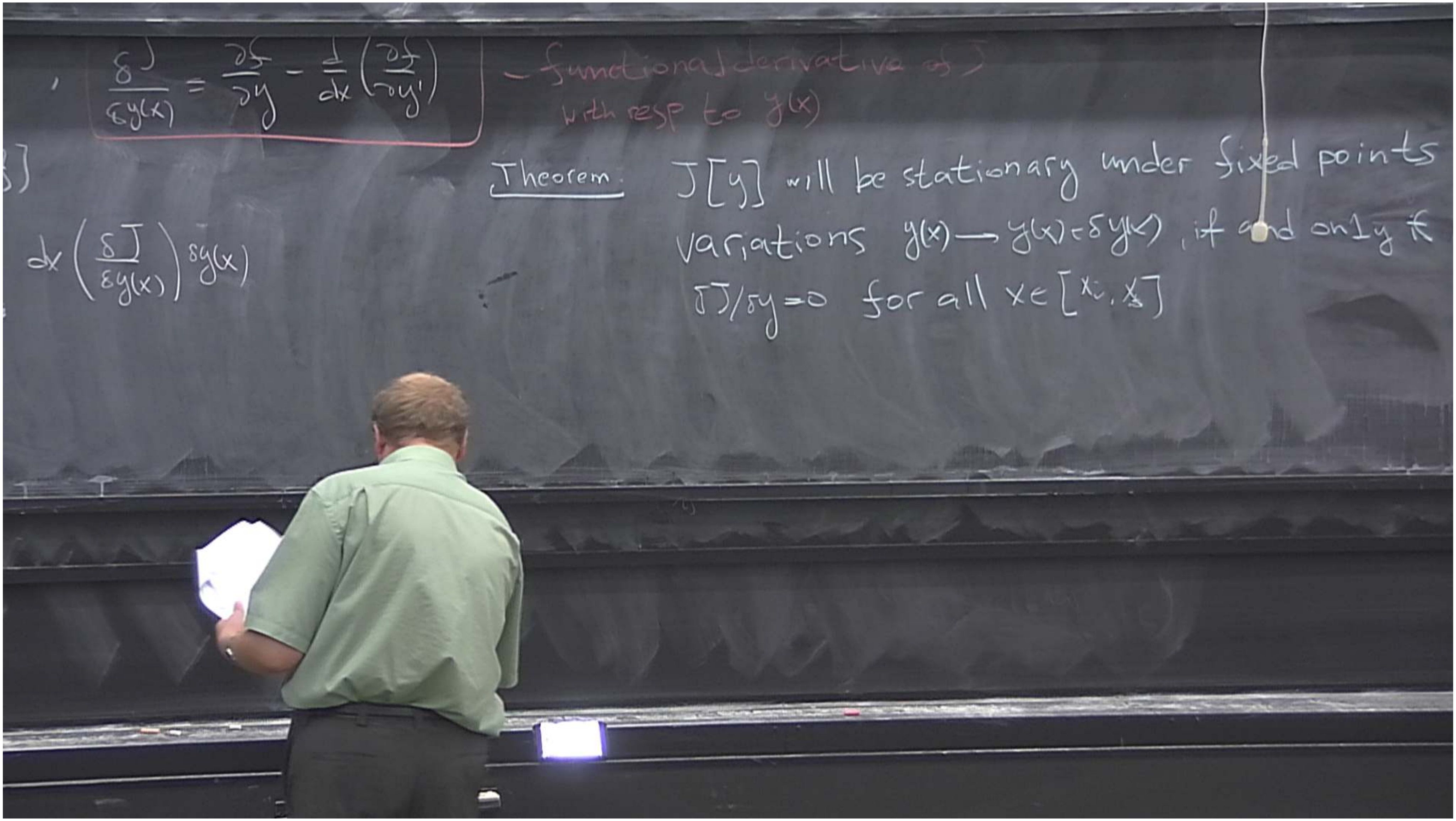
0



fixed boundaries condition
$\delta J = \int_{x_1}^{x_2} \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) dx$,
$\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$
– functional derivative of


fixed boundaries condition
$\delta J = \int_{x_i}^{x_f} \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) dx$, $\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$
– functional derivative of
with resp to y(x)
x_i
x_f
δJ


fixed boundaries condition
$\delta J = \int_{x_i}^{x_f} \delta y(x) \left(\frac{\delta J}{\delta y(x)}\right) dx$, $\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$
– functional derivative of with resp to y(x)
$J = J[\{y_i\}]$
y_i
x_i
x_f
$\delta J = \sum_i \frac{\partial J}{\partial y_i} \delta y_i \longrightarrow \int_{x_i}^{x_f} dx \left(\frac{\delta J}{\delta y(x)}\right) \delta y(x)$


δJ/δy(x) = ∂f/∂y − d/dx (∂f/∂y')
– functional derivative of J
with resp to y(x)
dx (δJ/δy(x)) δy(x)
Theorem: J[y] will be stationary under fixed points
variations y(x) → y(x)+δy(x), if and only if
δJ/δy = 0 for all x ∈ [xi, xf]

$$\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$ – functional derivative of J with resp to $y(x)$

$$dx \left(\frac{\delta J}{\delta y(x)}\right) \delta y(x)$$

Theorem. $J[y]$ will be stationary under fixed points variations $y(x) \to y(x) + \delta y(x)$, if and only if $\delta J / \delta y = 0$ for all $x \in [x_i, x_f]$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

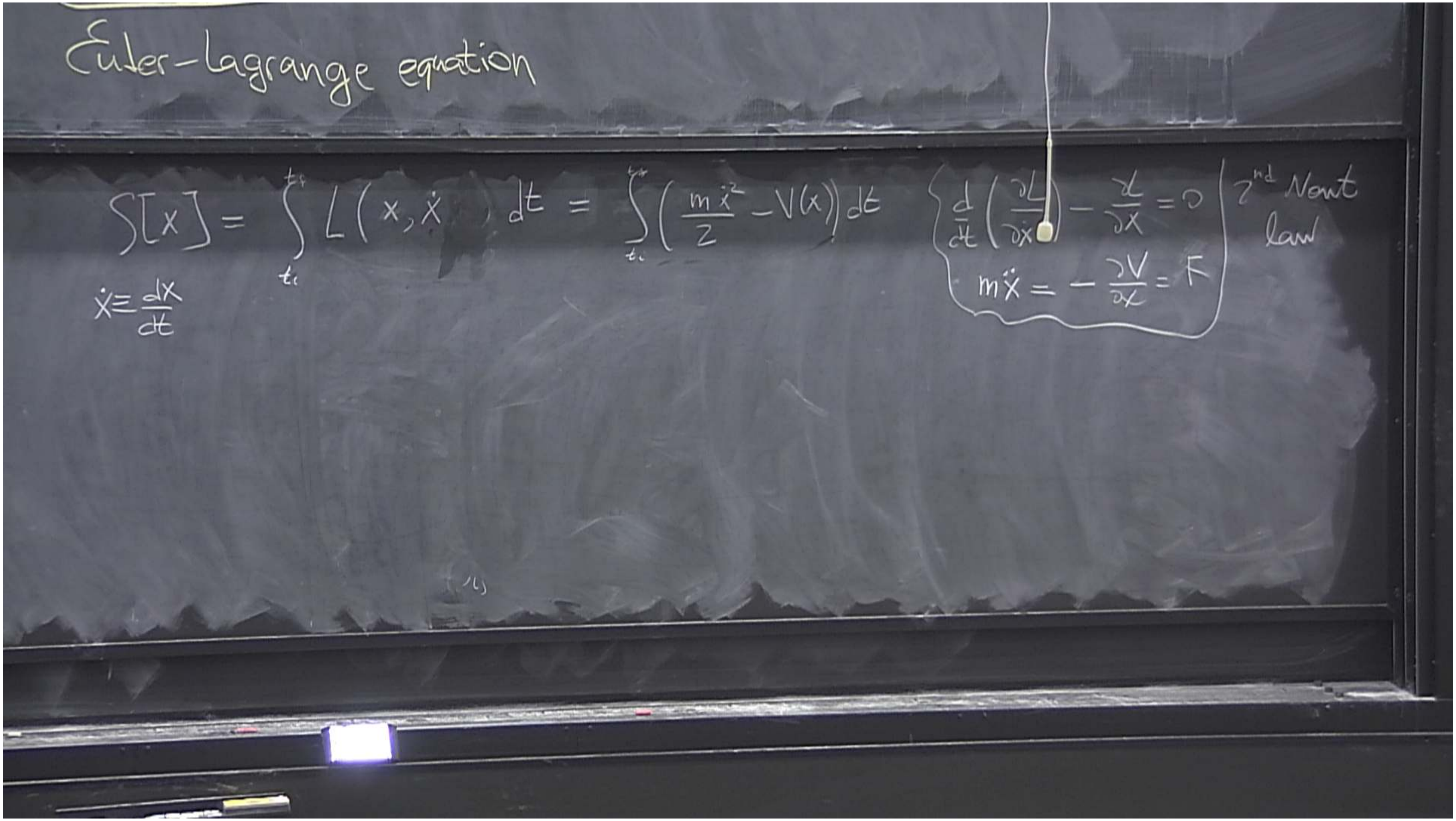
Euler–Lagrange equation

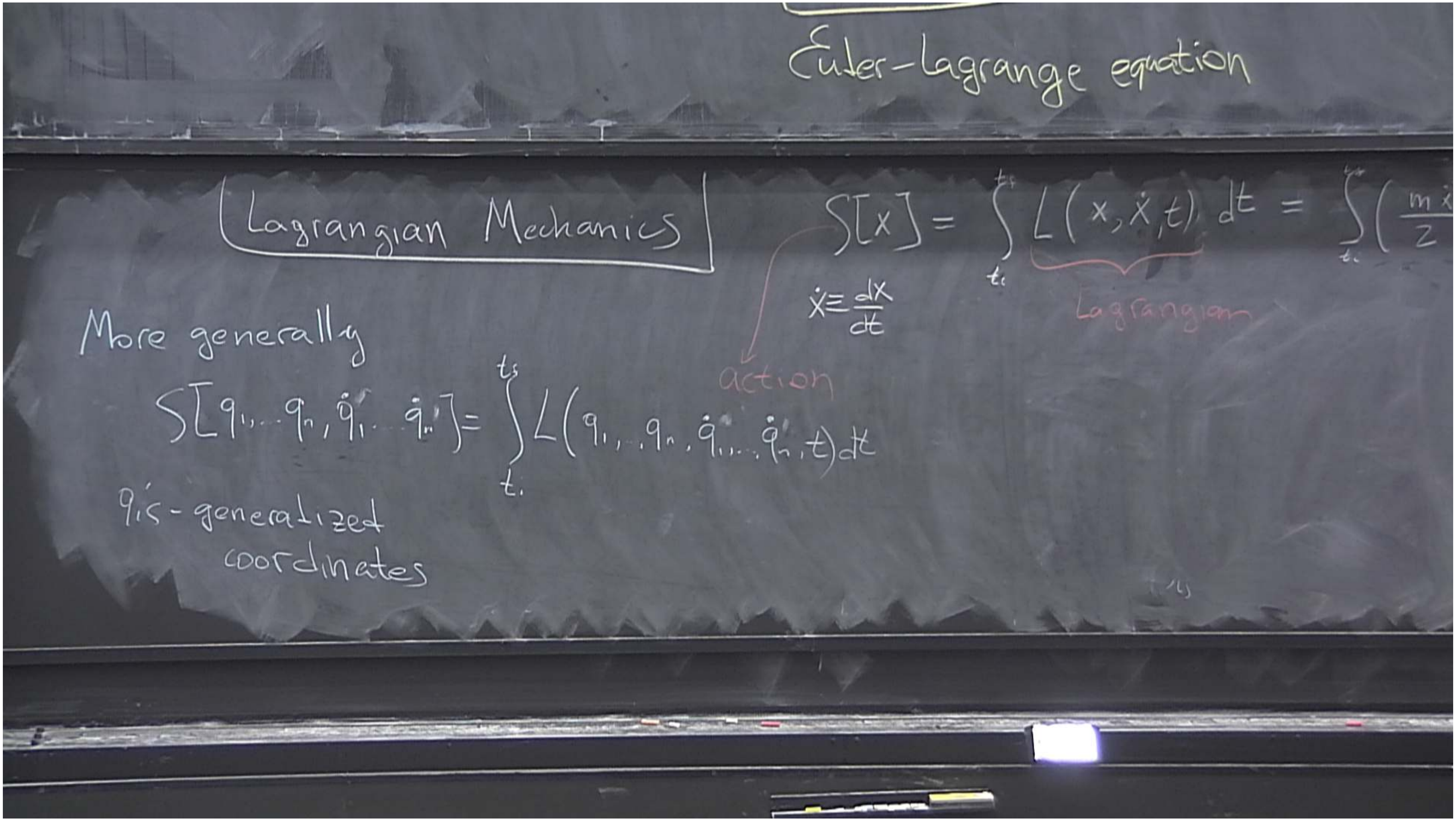
Euler-Lagrange equation

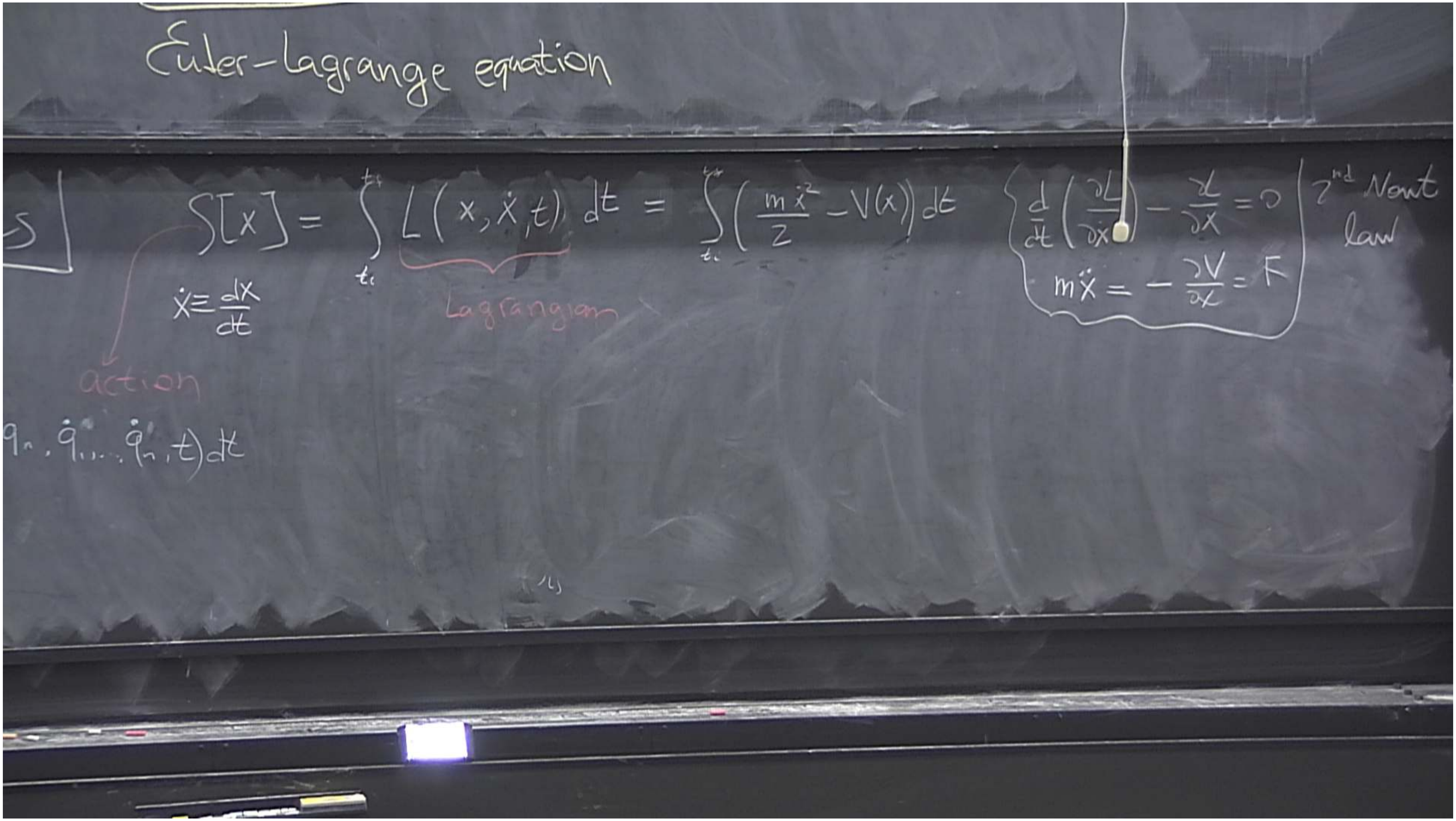
Lagrangian Mechanics

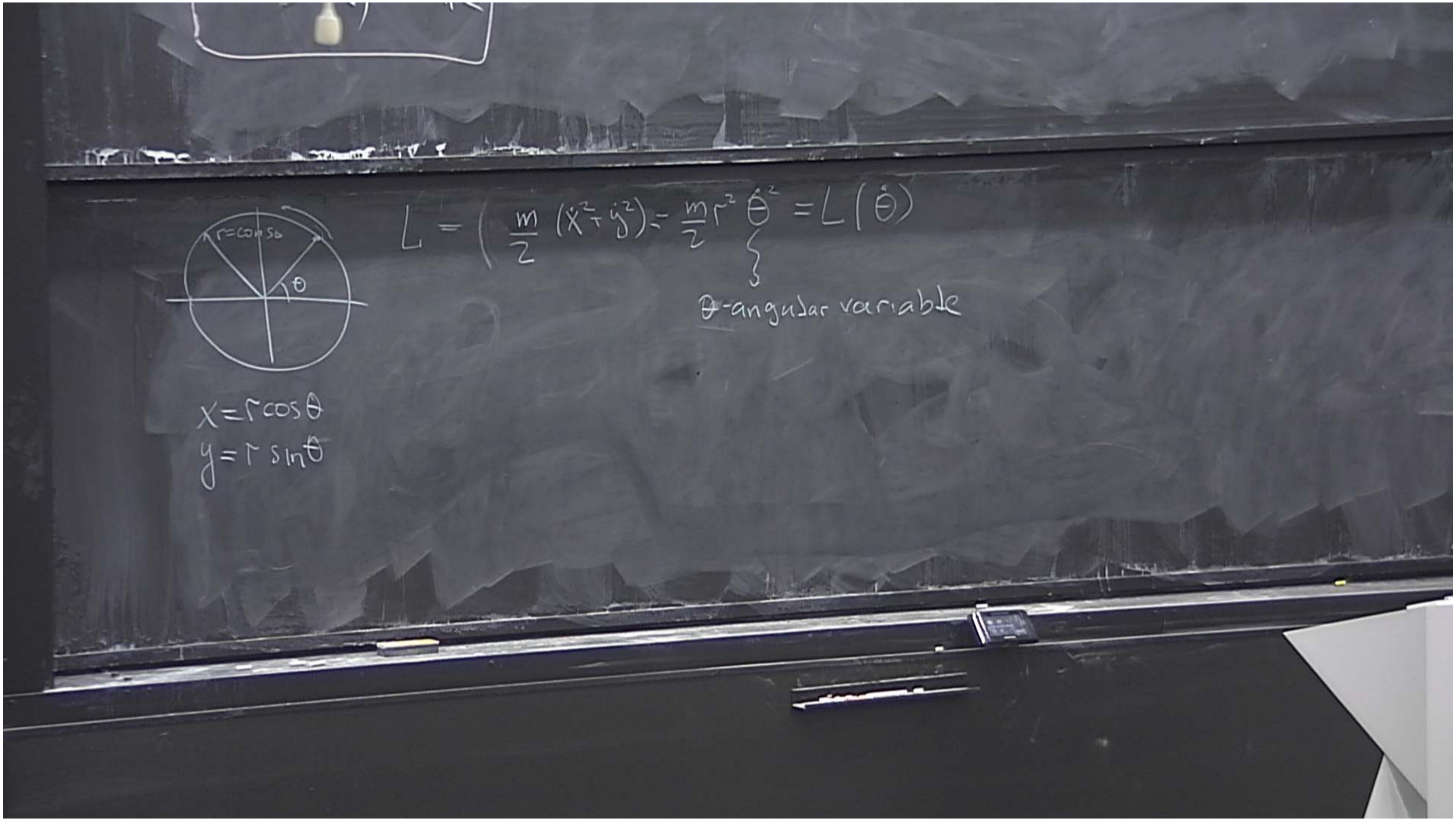
$$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt = \int_{t_i}^{t_f} \left(\frac{m\dot{x}^2}{2} - V(x) \right) dt$$

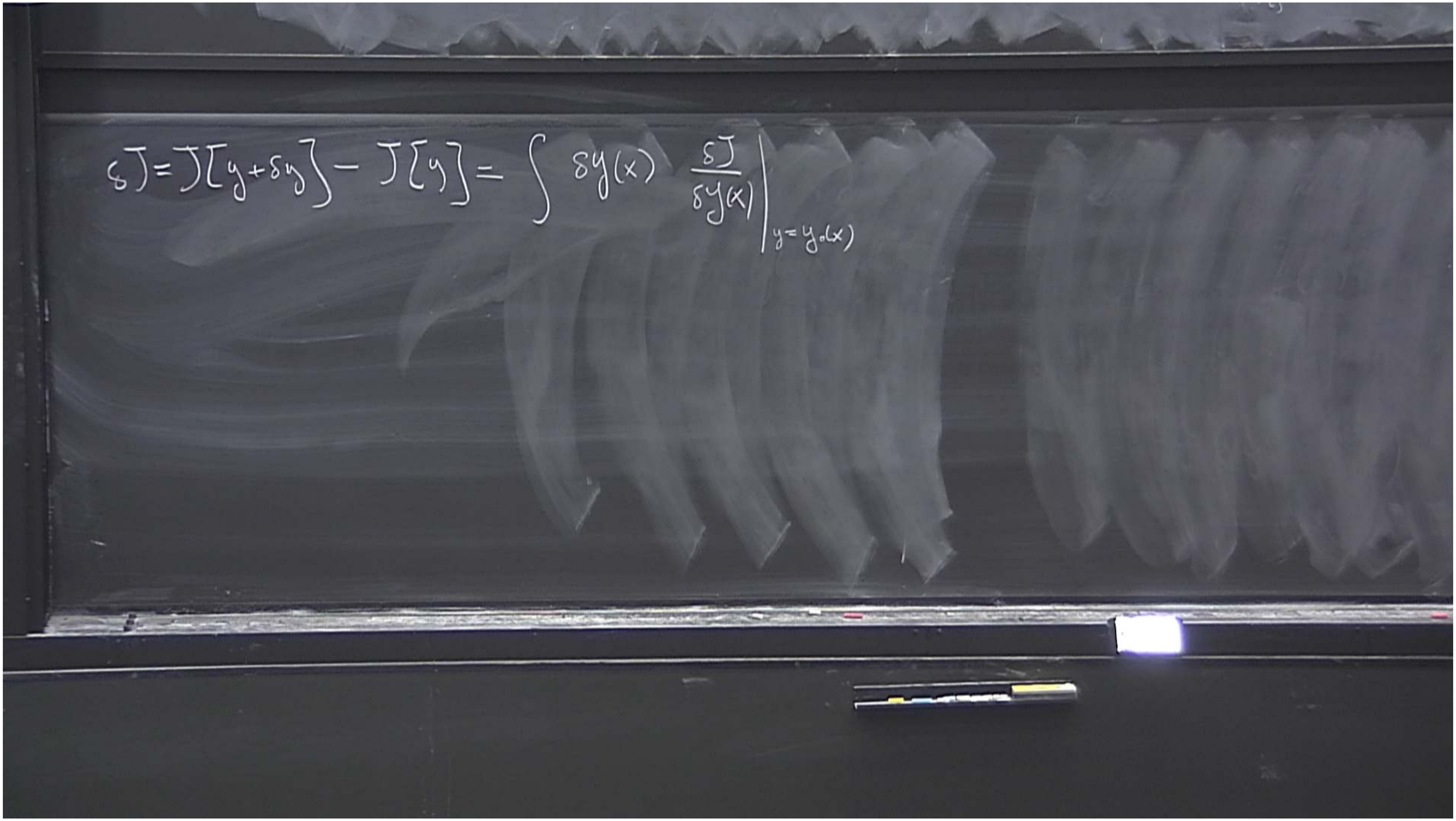
$$\dot{x} \equiv \frac{dx}{dt}$$

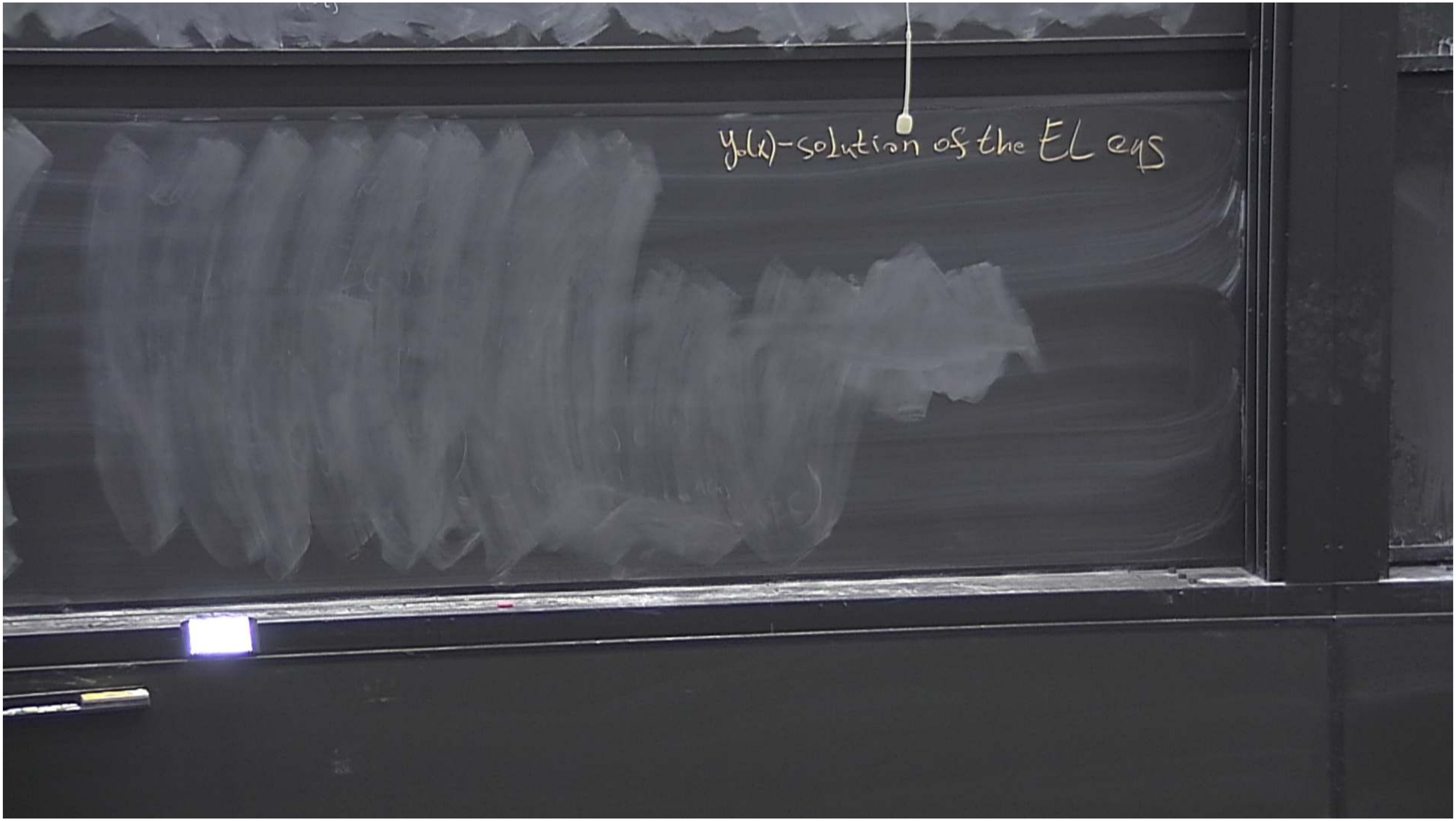

Euler-Lagrange equation
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x})\, dt = \int_{t_i}^{t_f} \left(\frac{m\dot{x}^2}{2} - V(x)\right) dt$
$\dot{x} \equiv \frac{dx}{dt}$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$
$m\ddot{x} = -\frac{\partial V}{\partial x} = F$
2nd Newt law

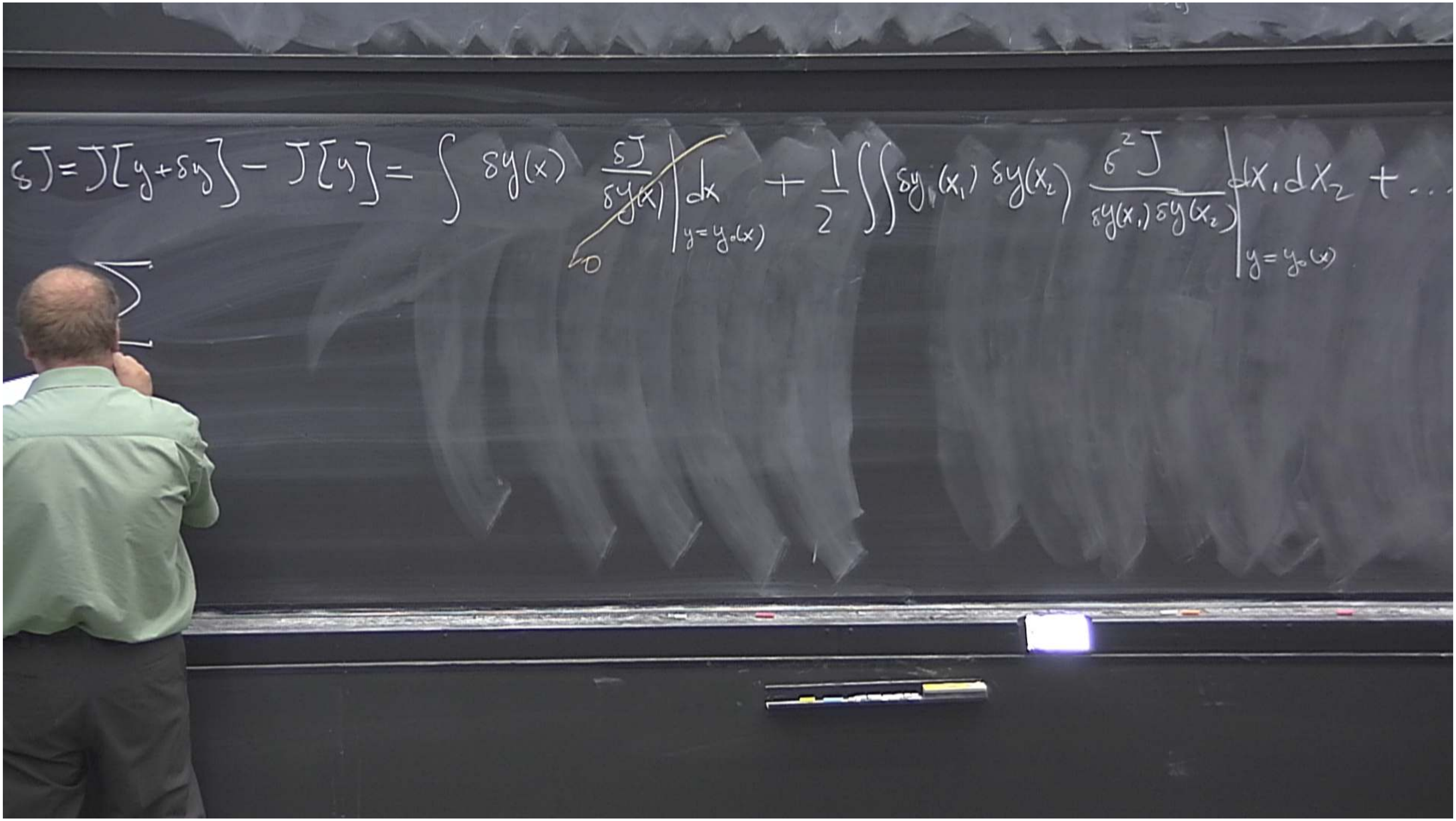

Euler-Lagrange equation
Lagrangian Mechanics
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt = \int_{t_i}^{t_f} \left(\frac{m\dot{x}}{2}\right.$
$\dot{x} \equiv \frac{dx}{dt}$
Lagrangian
action
More generally
$S[q_1, \ldots q_n, \dot{q}_1 \ldots \dot{q}_n] = \int_{t_i}^{t_f} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt$
q_i's - generalized coordinates

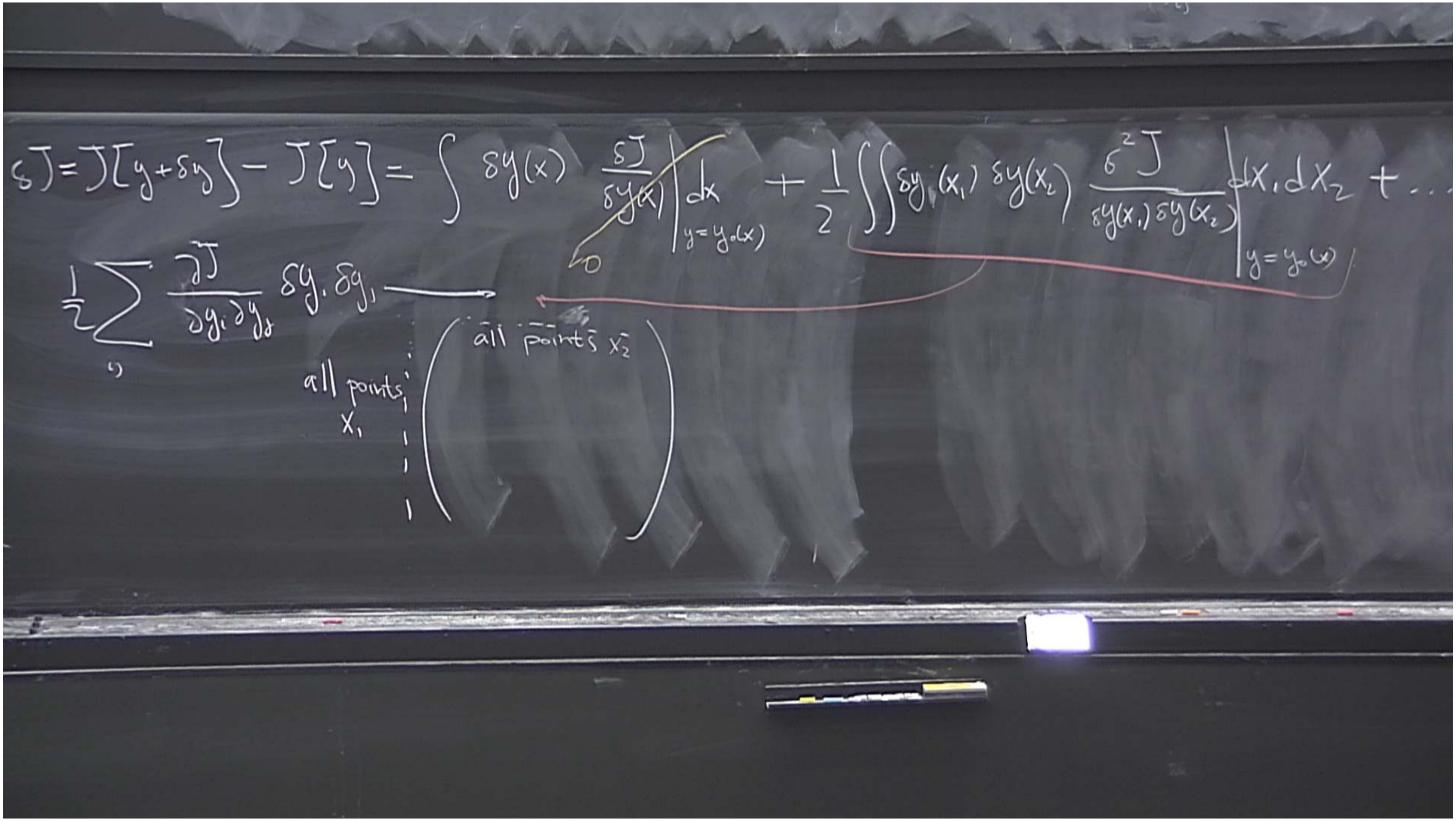

Euler-Lagrange equation
S[x] = ∫ L(x, ẋ, t) dt = ∫ (mẋ²/2 − V(x)) dt
ẋ = dx/dt
Lagrangian
action
d/dt(∂L/∂ẋ) − ∂L/∂x = 0
mẍ = − ∂V/∂x = F
2nd Newt law

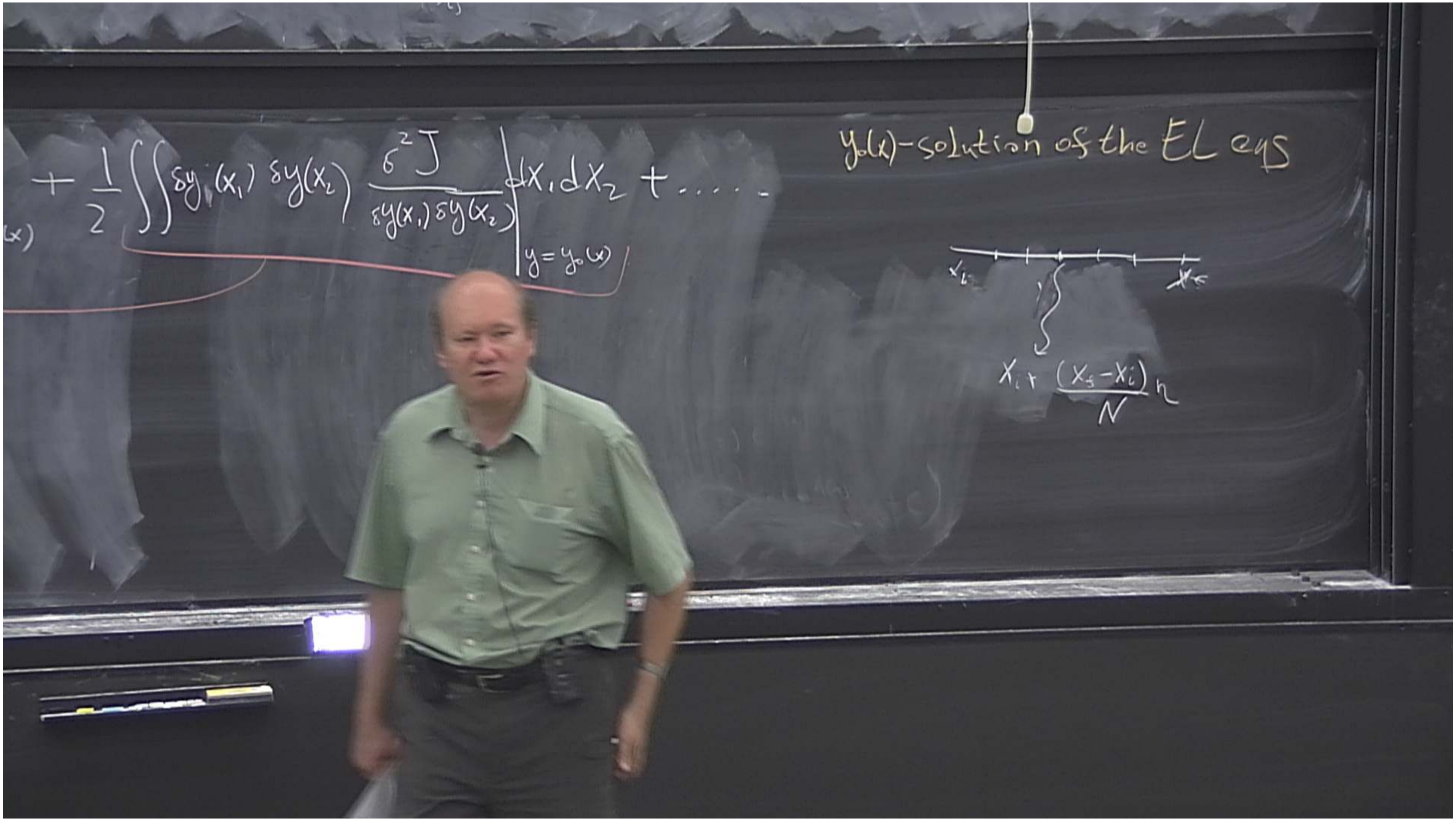

r=const
θ
L = (m/2 (ẋ² + ẏ²) = m/2 r² θ̇² = L(θ̇)
θ-angular variable
x = r cos θ
y = r sin θ


$\delta J = J[y+\delta y] - J[y] = \int \delta y(x) \left.\frac{\delta J}{\delta y(x)}\right|_{y=y_0(x)}$


$y_0(x)$ - solution of the EL eqs


δJ = J[y+δy] − J[y] = ∫ δy(x) δJ/δy(x) |y=y0(x) dx + 1/2 ∫∫ δy(x1) δy(x2) δ²J/δy(x1)δy(x2) |y=y0(x) dx1 dx2 + . . .
0
Σ


$\delta J = J[y+\delta y] - J[y] = \int \delta y(x) \frac{\delta J}{\delta y(x)}\Big|_{y=y_0(x)} dx + \frac{1}{2}\iint \delta y(x_1)\, \delta y(x_2) \frac{\delta^2 J}{\delta y(x_1)\delta y(x_2)}\Big|_{y=y_0(x)} dx_1 dx_2 + \dots$
$\frac{1}{2}\sum_{ij} \frac{\partial J}{\partial y_i \partial y_j} \delta y_i \delta y_j$
all points x_2
all points x_1


+ 1/2 ∫∫ δy(x1) δy(x2) δ²J / δy(x1) δy(x2) | y = y0(x) dx1 dx2 + ...
y0(x) - solution of the EL eqs
Xi + (Xf - Xi)/N n