Title: Evaluation of Integrals and Calculus of Variations - Lecture 3

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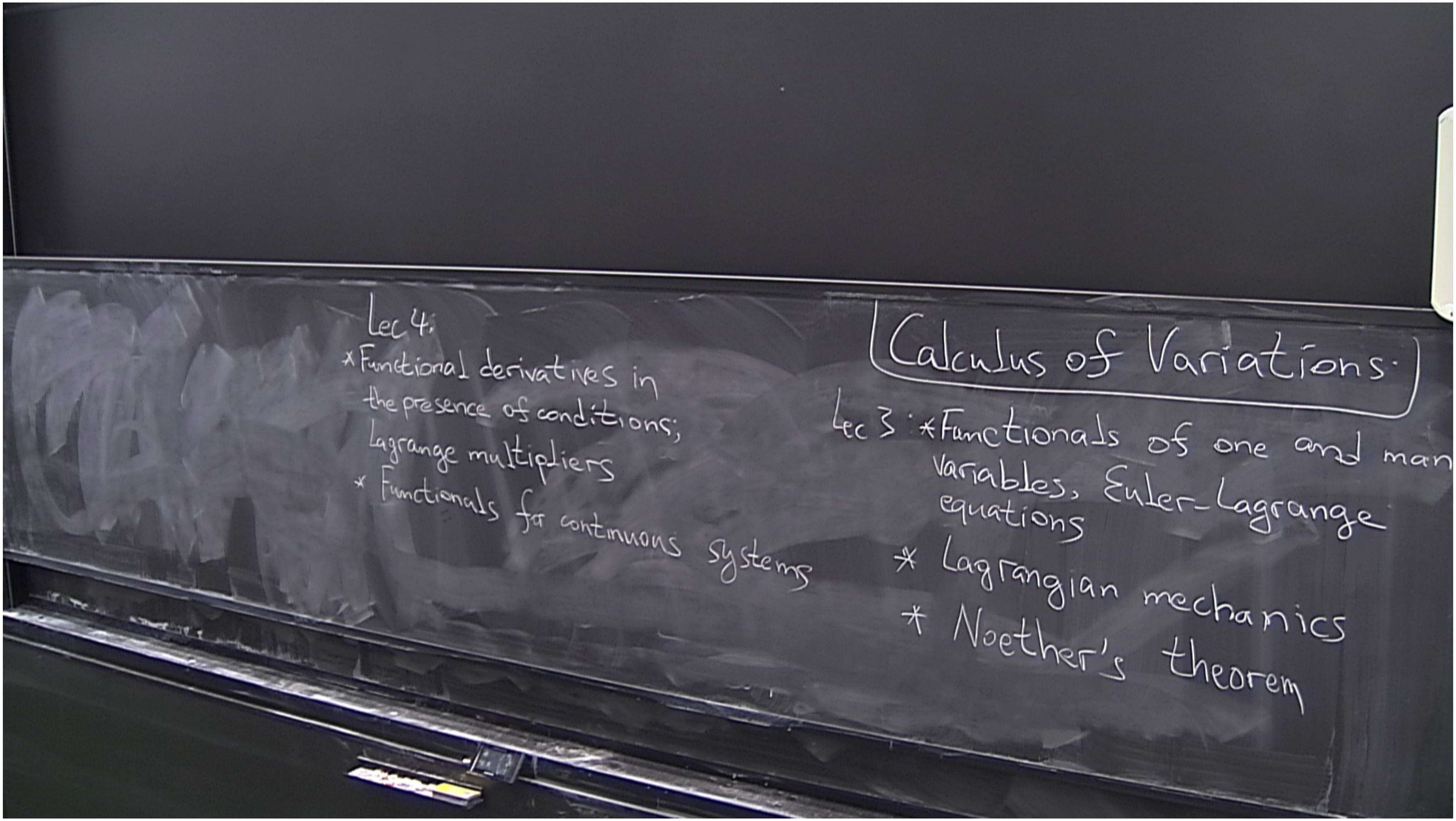
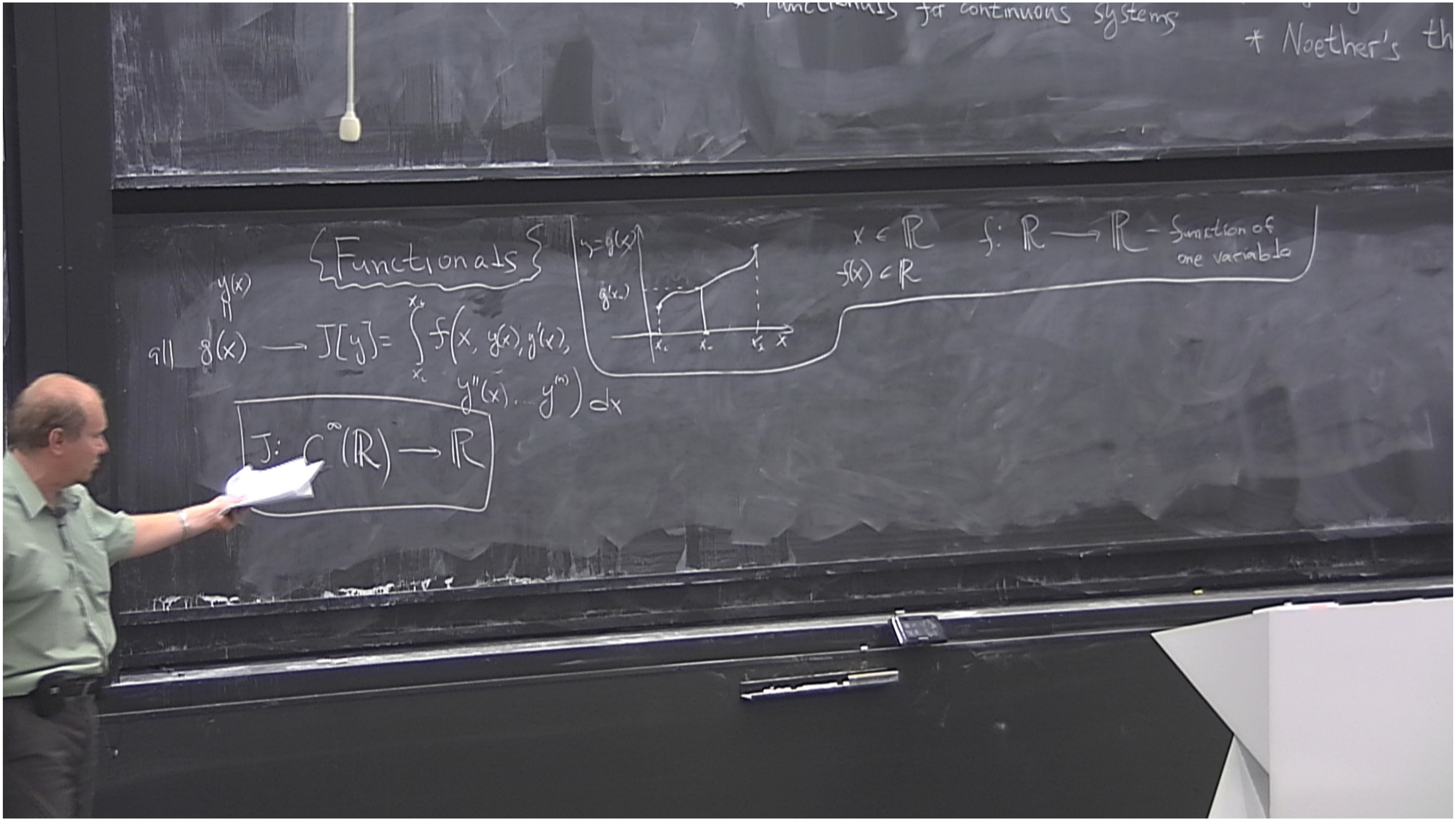
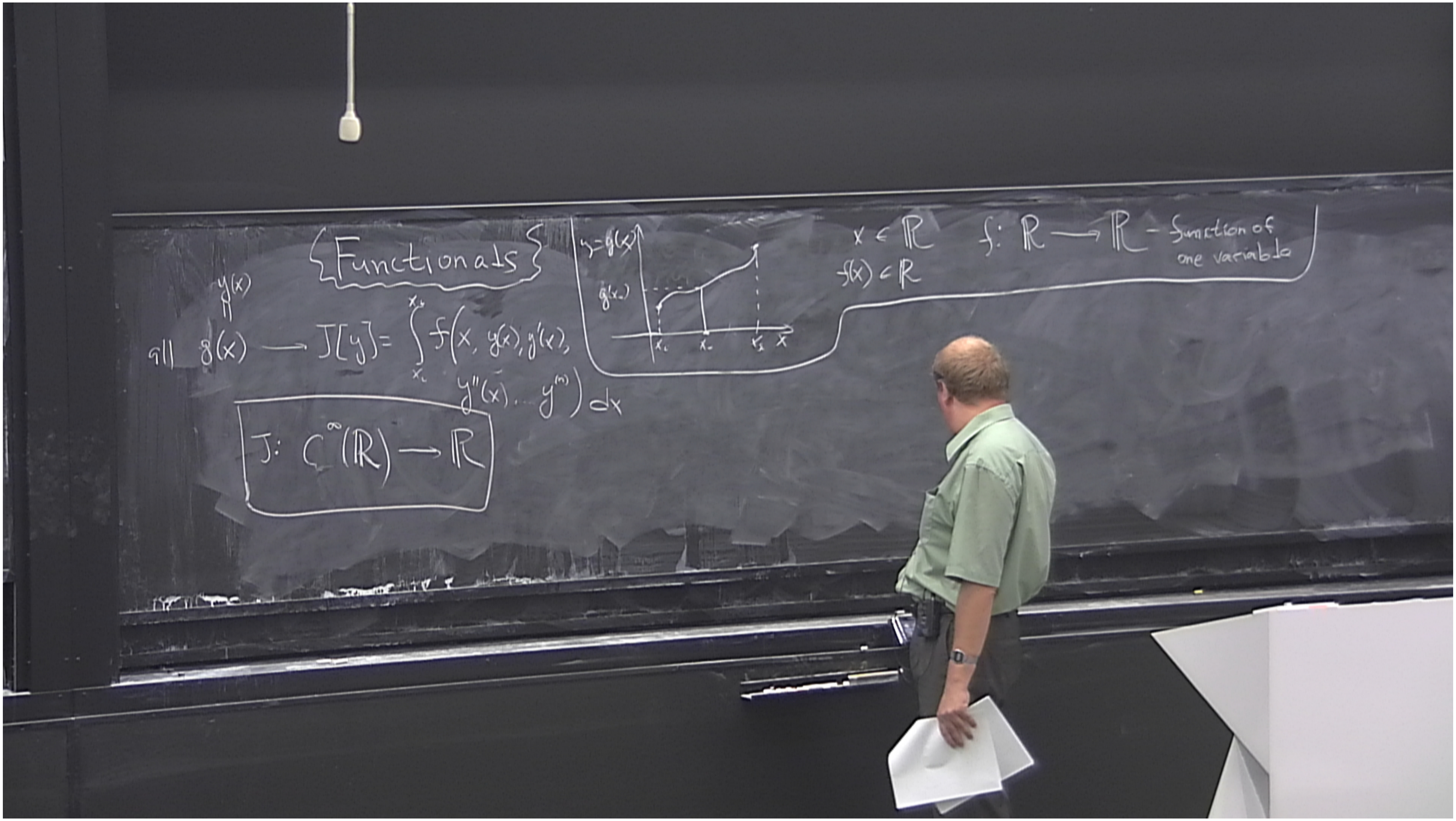
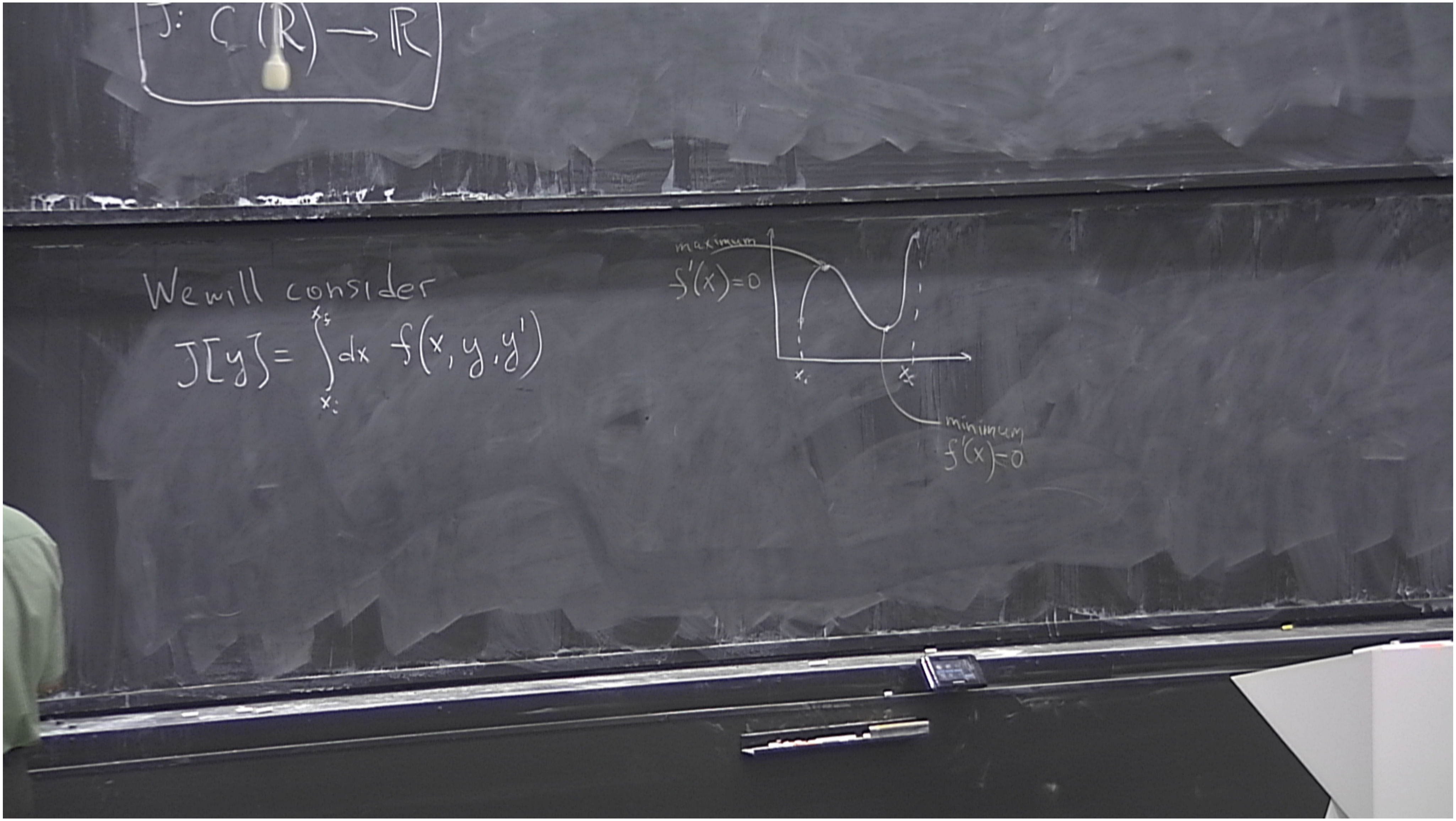
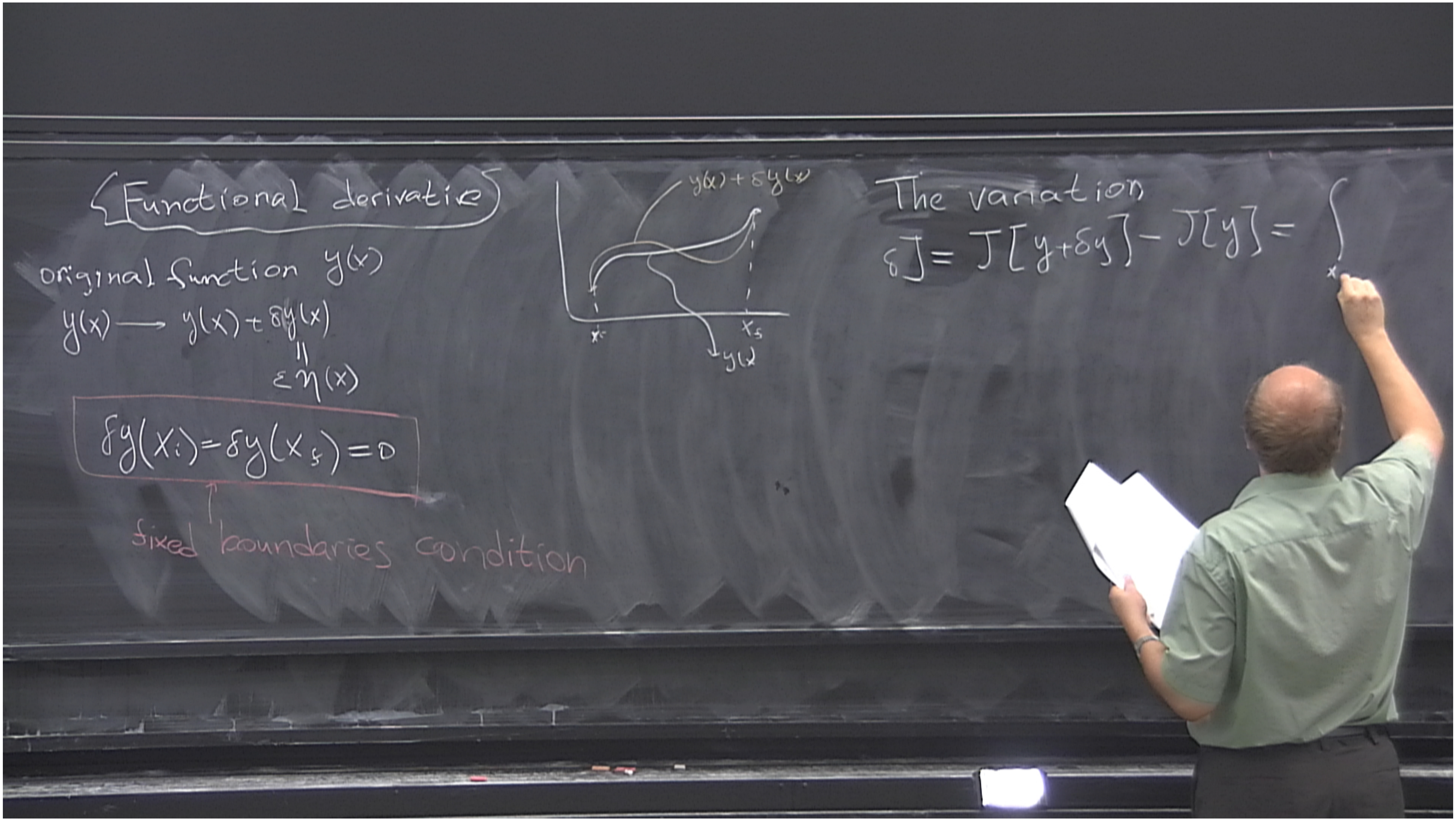
Abstract:


Lec 4:
* Functional derivatives in
the presence of conditions;
Lagrange multipliers
* Functionals for continuous systems
Calculus of Variations.
Lec 3: * Functionals of one and man
variables, Euler-Lagrange
equations
* Lagrangian mechanics
* Noether's theorem


for continuous systems
* Noether's
{Functionals}
y(x)
all g(x) → J[y] = ∫ f(x, y(x), y'(x), y''(x), ... y^(n)) dx
J: C^∞(R) → R
x ∈ R
f(x) ∈ R
f: R → R - function of one variable


{Functionals}
y(x)
all g(x) ⟶ J[y] = ∫ f(x, y(x), y'(x), y''(x) … y^(n)) dx
J: C^∞(R) ⟶ R
y=g(x)
x ∈ R
f(x) ∈ R
f: R ⟶ R - function of one variable


We will consider
$J[y] = \int_{x_i}^{x_f} dx \, f(x, y, y')$
maximum
$f'(x) = 0$
x_i
x_f
minimum
$f'(x) = 0$


{Functional derivative}
original function y(x)
y(x) ⟶ y(x) + δy(x)
= εη(x)
δy(x_i) = δy(x_f) = 0
fixed boundaries condition
y(x) + δy(x)
y(x)
x_i
x_f
The variation
δJ = J[y+δy] − J[y] = ∫

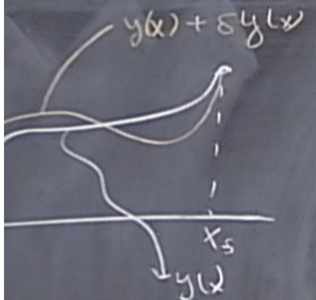
The variation

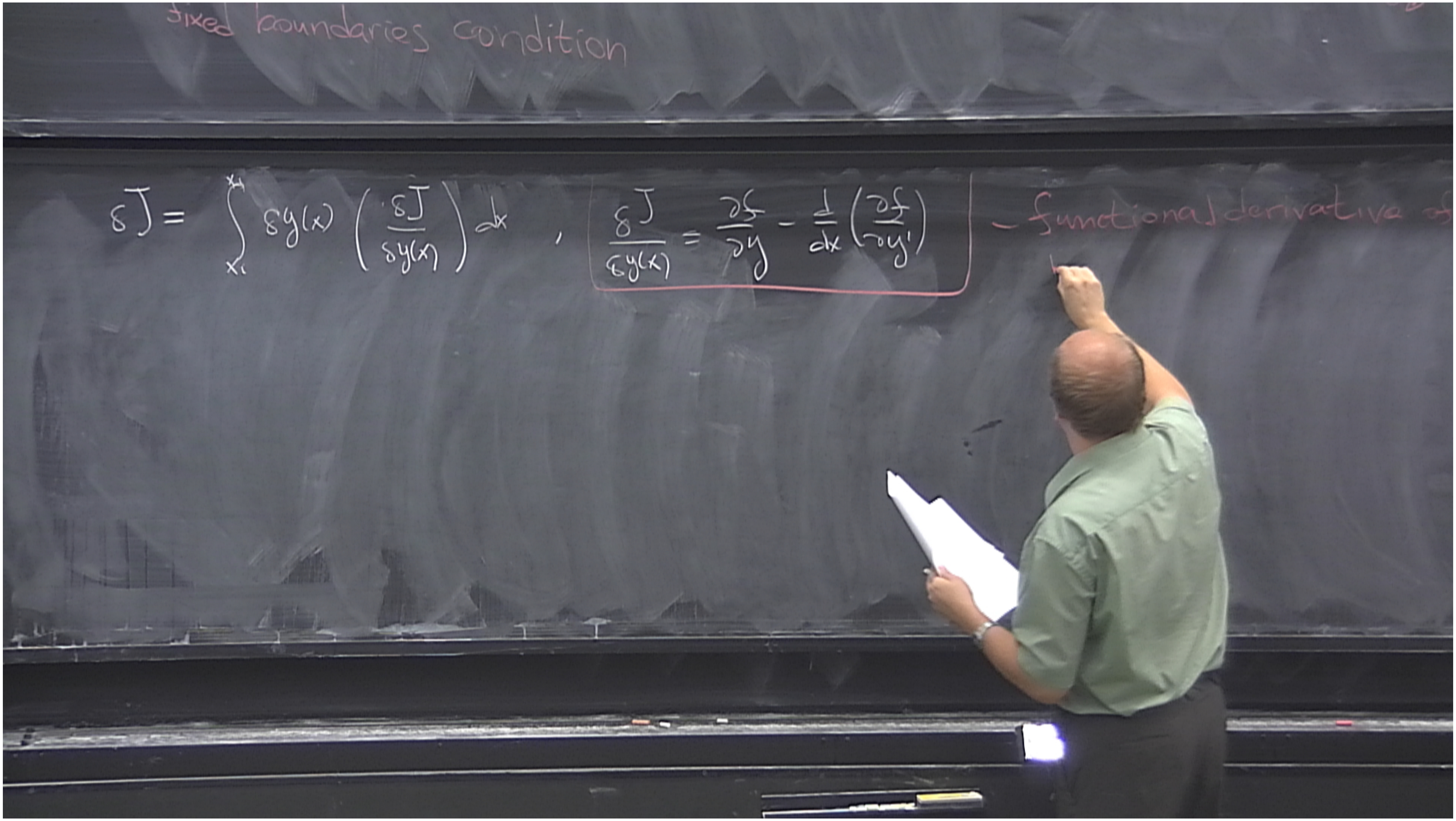
$$\delta J = J[y+\delta y] - J[y] = \int_{x_i}^{x_f} dx \left\{ f(x, y+\delta y, y'+\delta y') - f(x, y, y') \right\} =$$

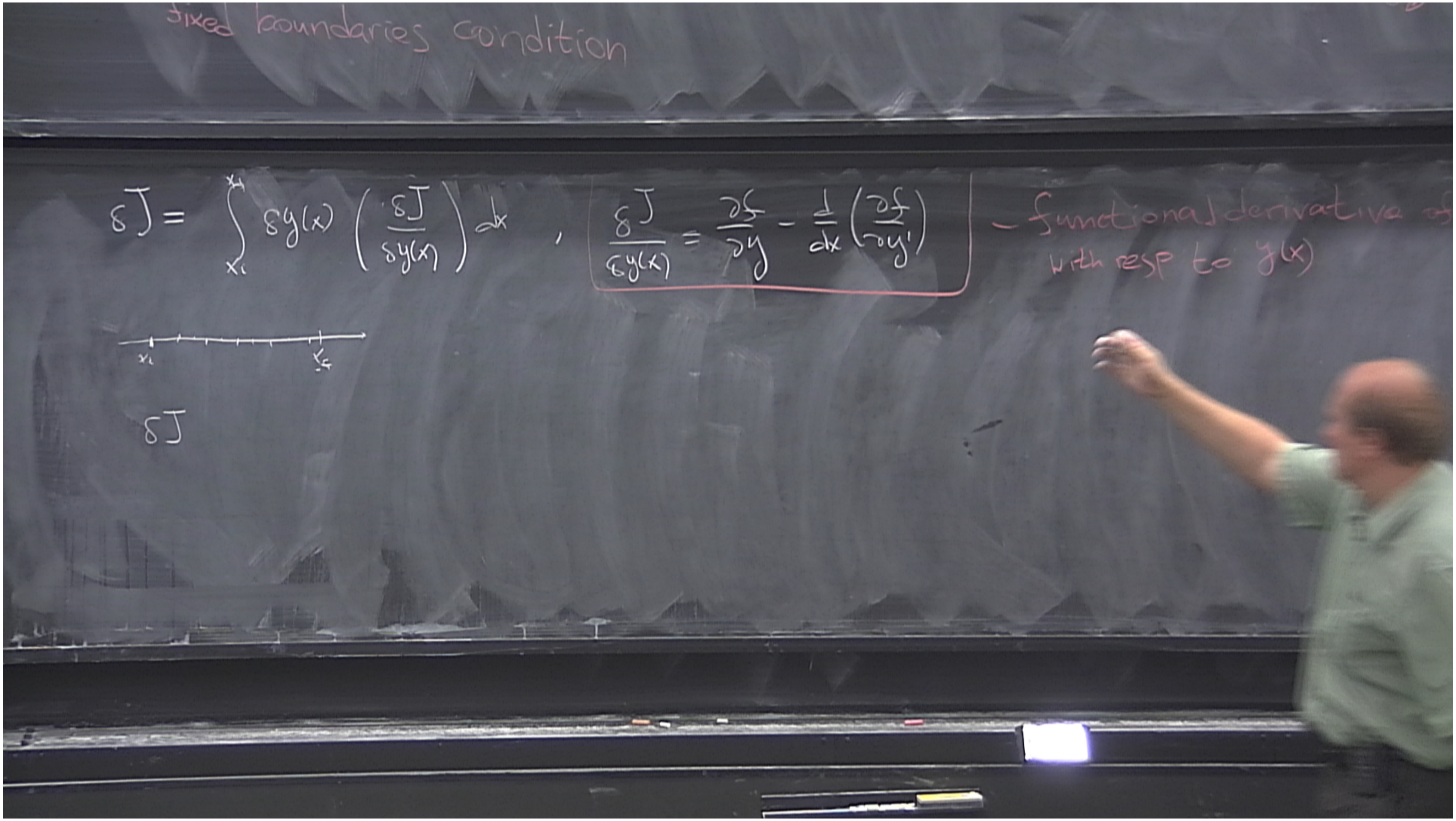
$$= \int_{x_i}^{x_f} dx \left\{ \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' + O(\delta y^2) \right\} =$$

$$= \left[\delta y \cdot \frac{\partial f}{\partial y'} \right]_{x_i}^{x_f} + \int_{x_i}^{x_f} \delta y \left\{ -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{\partial f}{\partial y} \right\} dx =$$

0



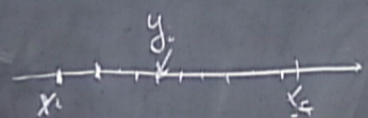

fixed boundaries condition
$\delta J = \int_{x_1}^{x_2} \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) dx$, $\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$
– functional derivative of


fixed boundaries condition
$\delta J = \int_{x_i}^{x_f} \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) dx$, $\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$
– functional derivative of
with resp to y(x)
x_i
x_f
δJ

fixed boundaries condition

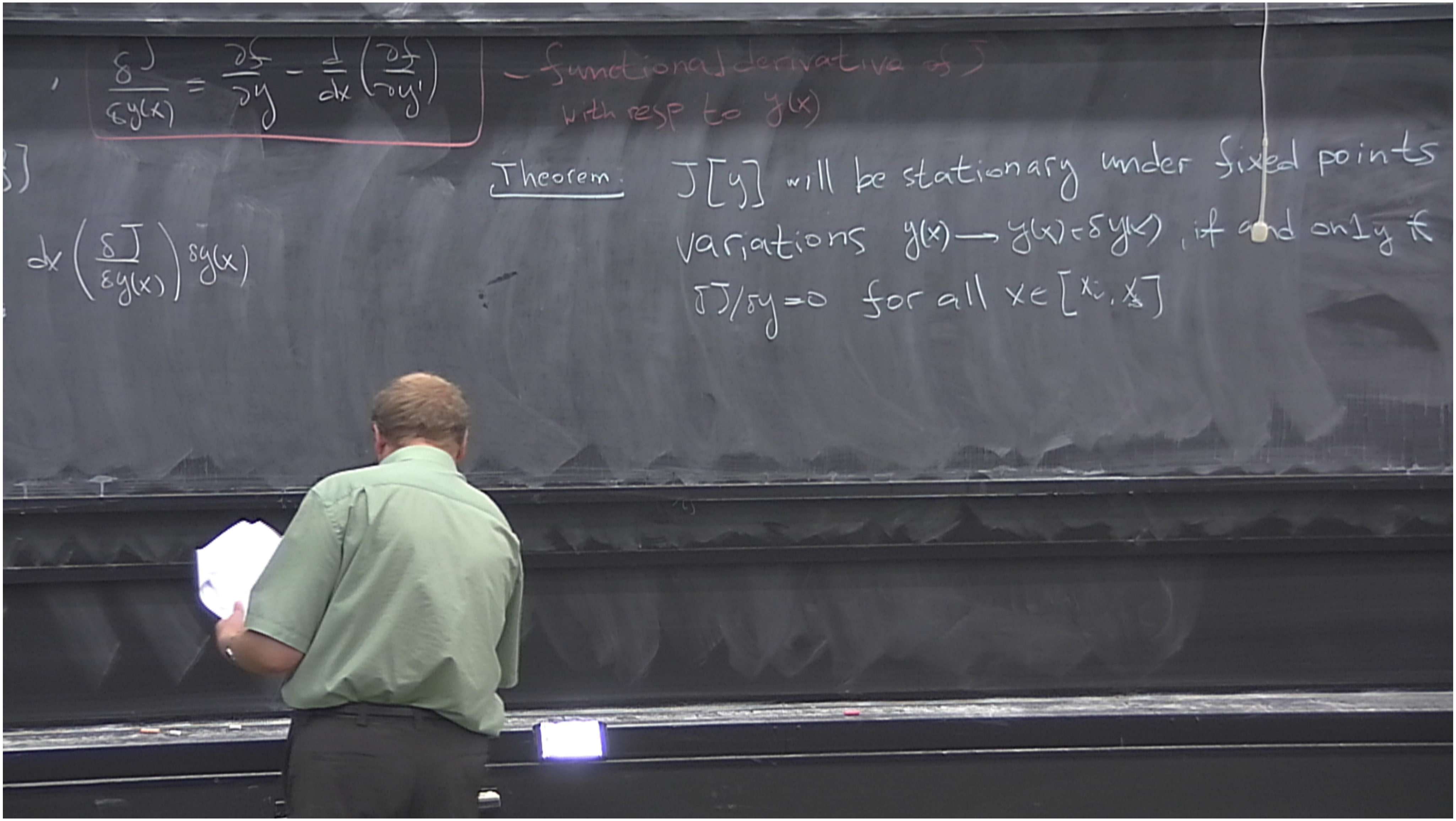
$$\delta J = \int_{x_i}^{x_f} \delta y(x) \left(\frac{\delta J}{\delta y(x)}\right) dx \;, \quad \boxed{\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)}$$

– functional derivative of
with resp to $y(x)$



$$J = J[\{y_i\}]$$

$$\delta J = \sum_i \frac{\partial J}{\partial y_i} \delta y_i \longrightarrow \int_{x_i}^{x_f} dx \left(\frac{\delta J}{\delta y(x)}\right) \delta y(x)$$


δJ/δy(x) = ∂f/∂y − d/dx (∂f/∂y')
– functional derivative of J
with resp to y(x)
dx (δJ/δy(x)) δy(x)
Theorem: J[y] will be stationary under fixed points
variations y(x) → y(x)+δy(x), if and only if
δJ/δy = 0 for all x ∈ [xi, xf]

$$\frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

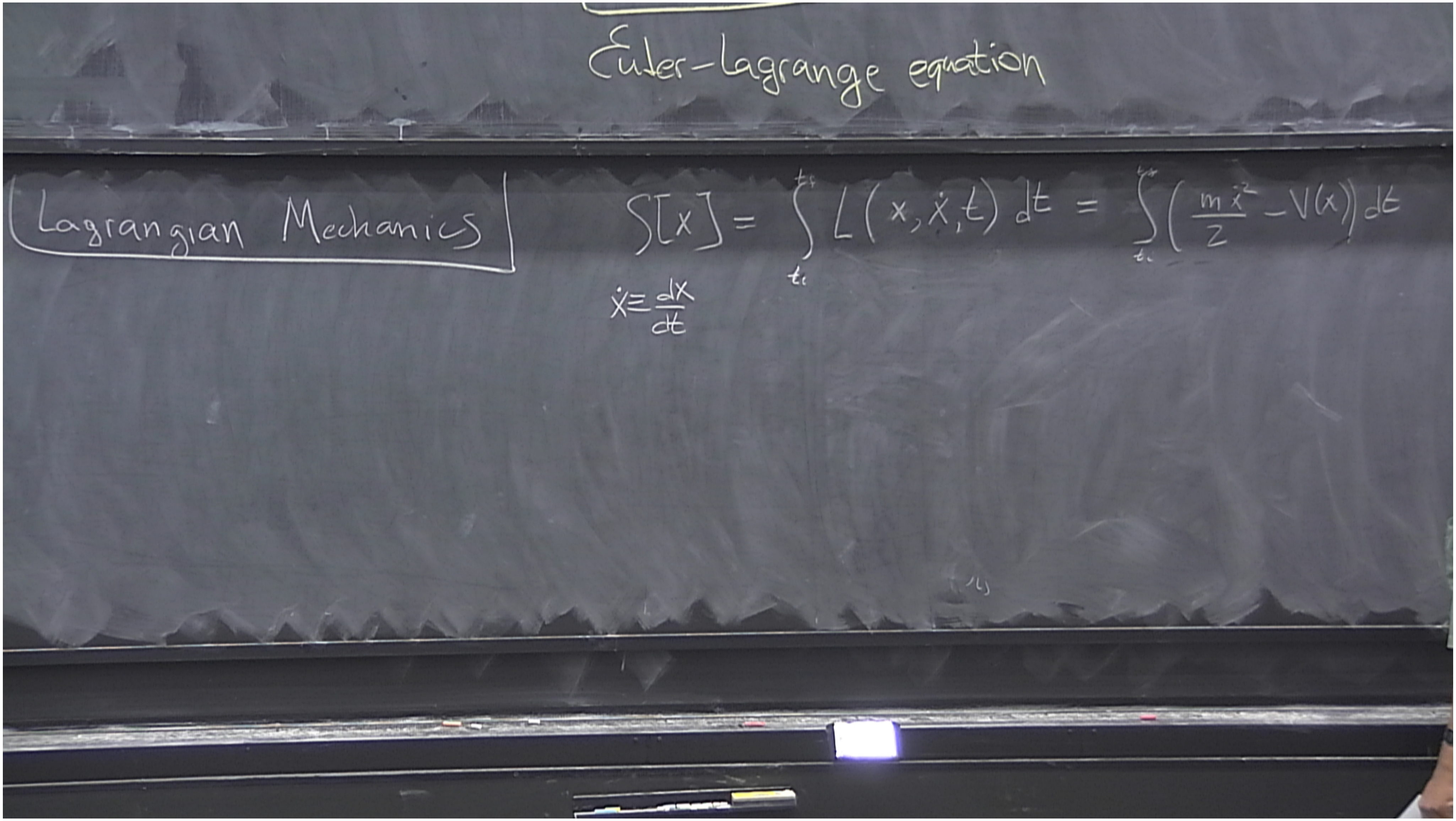
— functional derivative of J with resp to $y(x)$

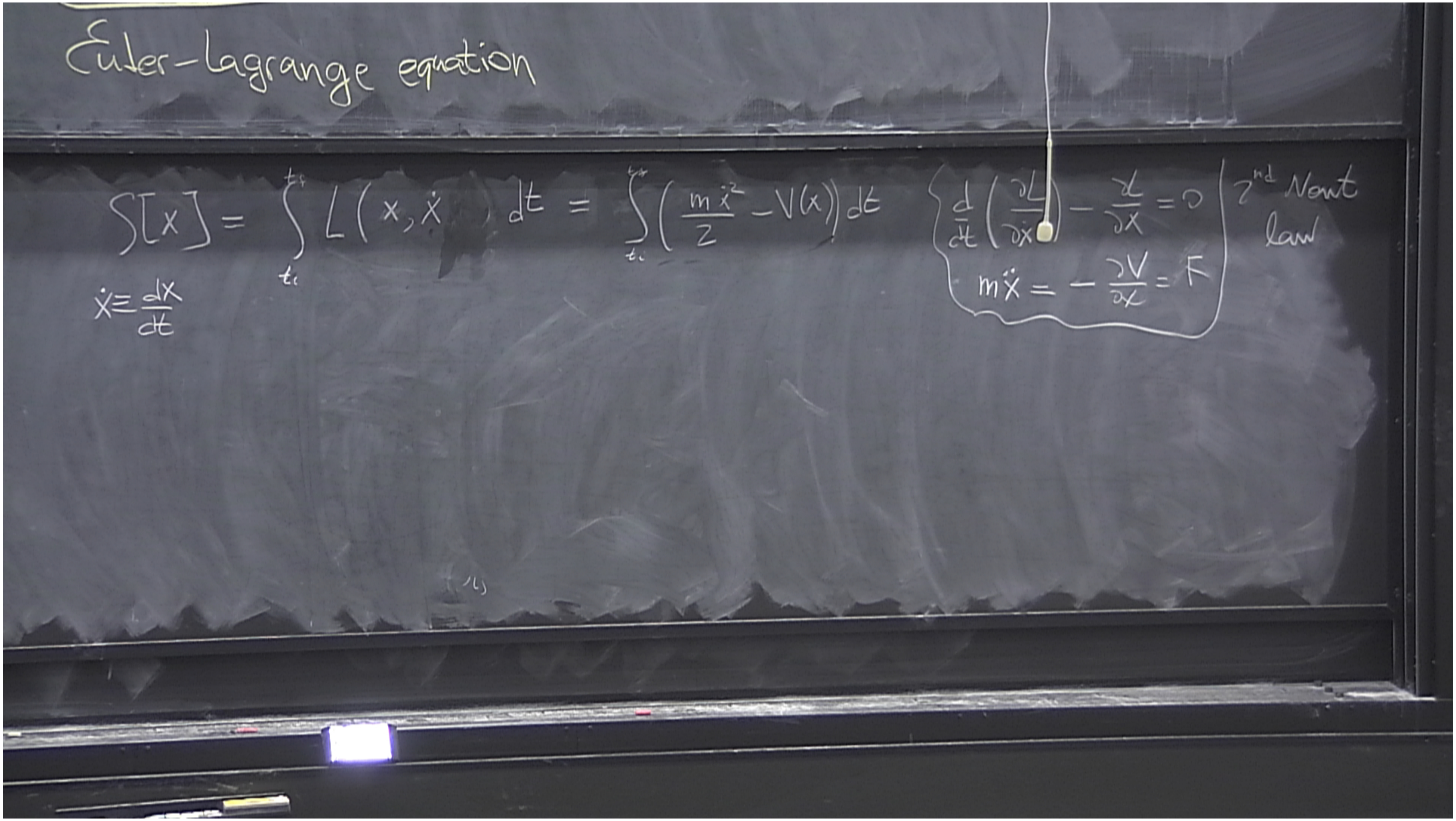
$dx \left(\frac{\delta J}{\delta y(x)}\right) \delta y(x)$

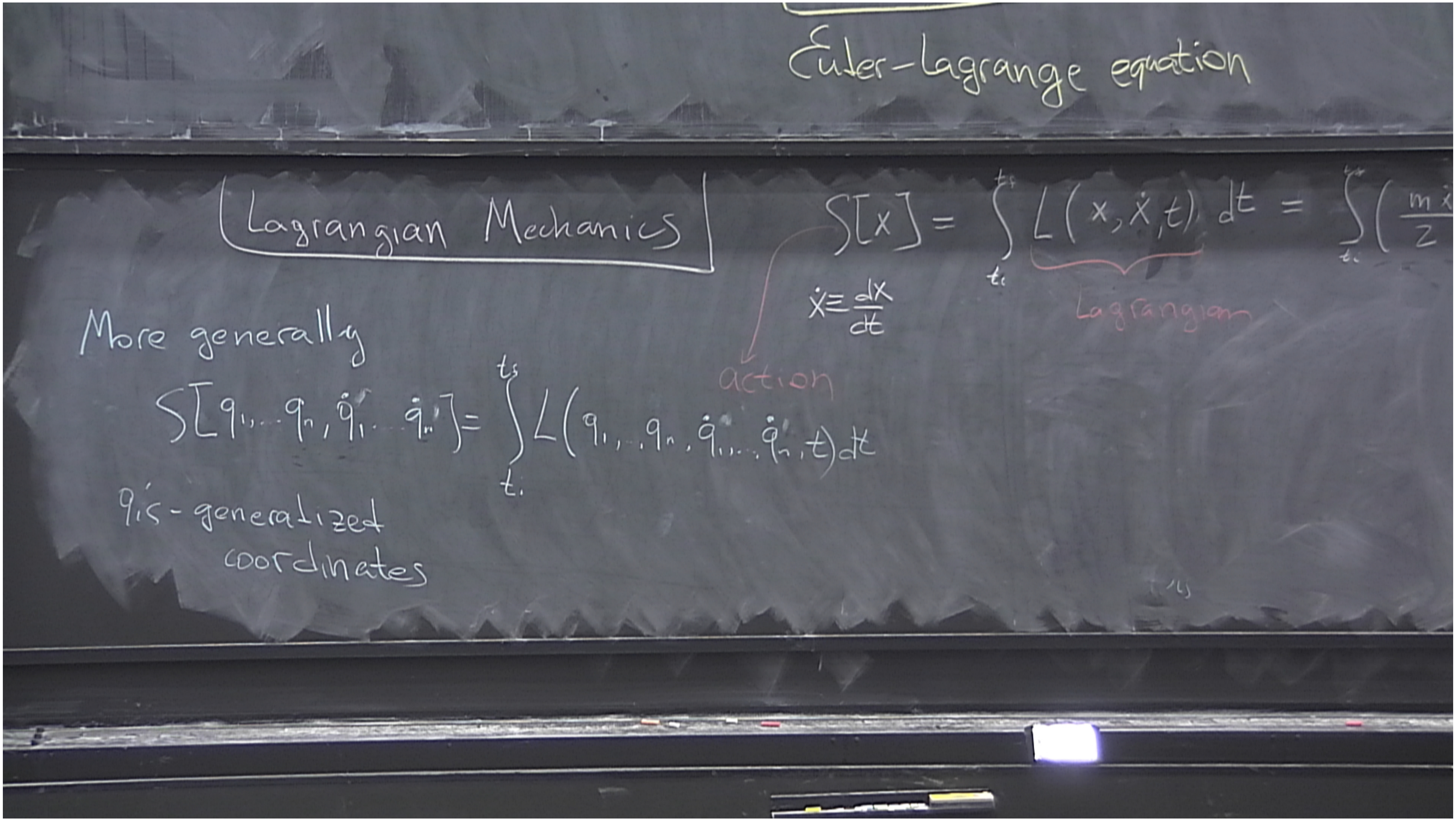
Theorem: $J[y]$ will be stationary under fixed points variations $y(x) \to y(x) + \delta y(x)$, if and only if $\delta J / \delta y = 0$ for all $x \in [x_i, x_f]$

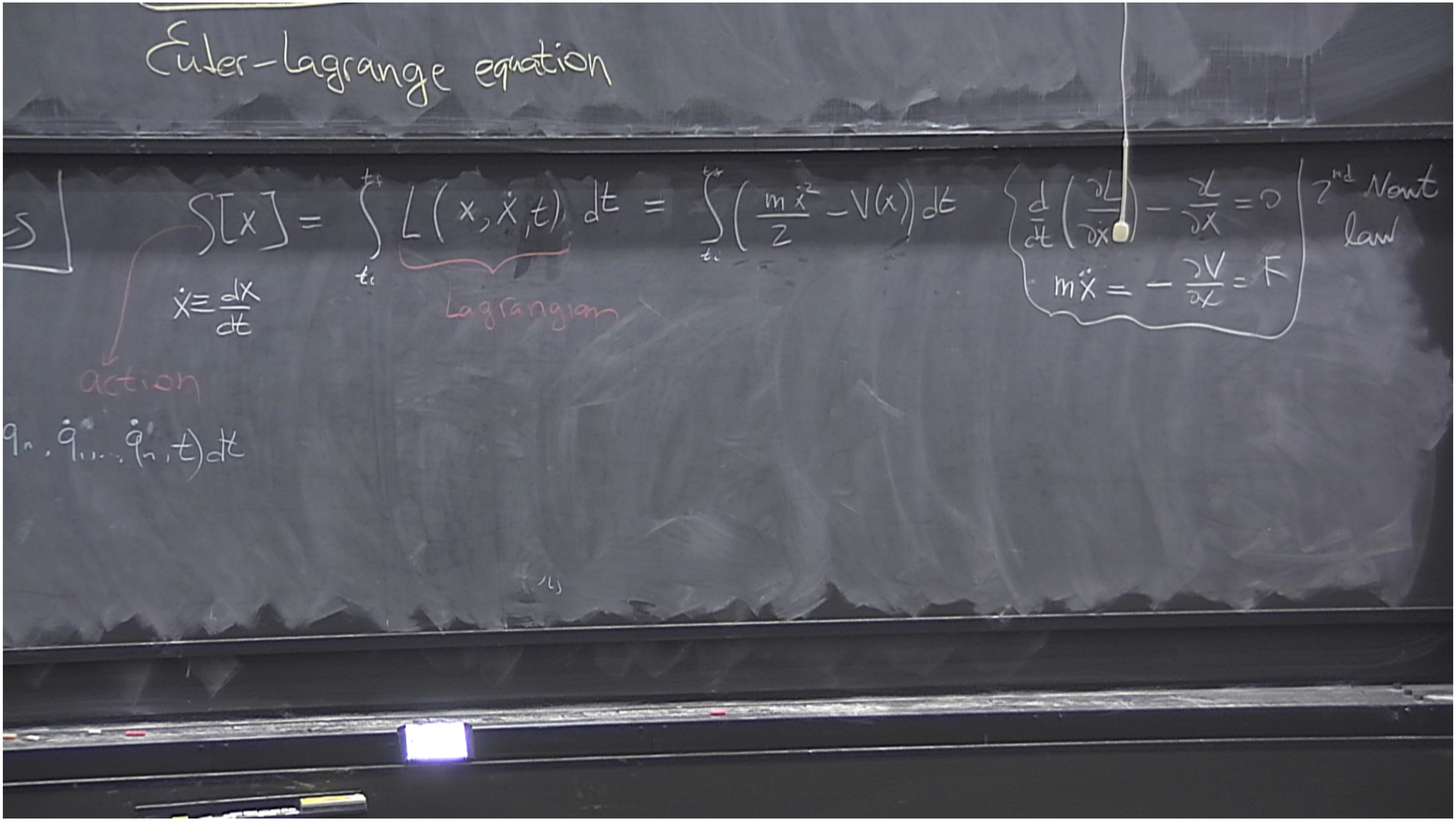
$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

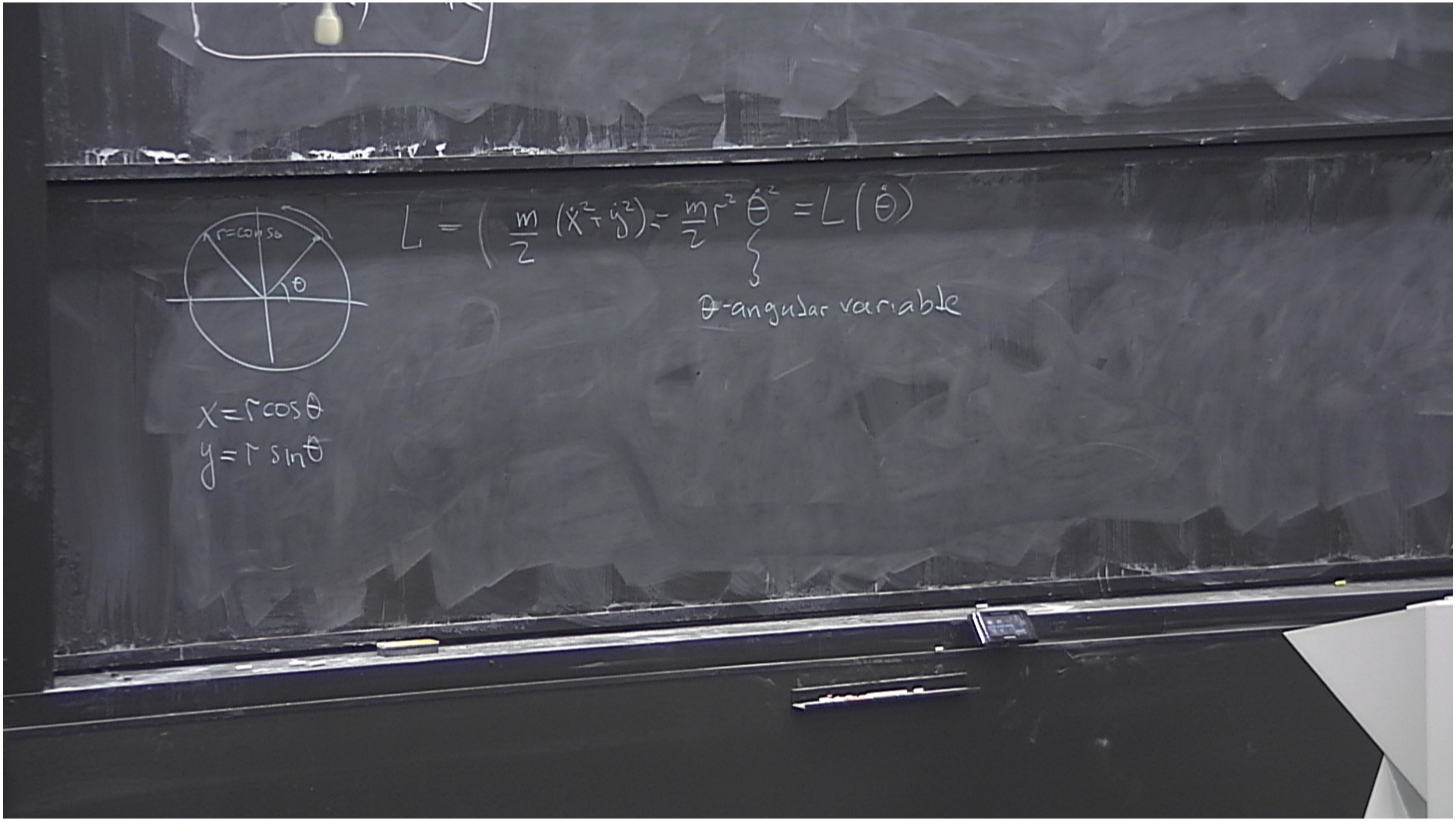
Euler-Lagrange equation


Euler-Lagrange equation
Lagrangian Mechanics
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt = \int_{t_i}^{t_f} \left(\frac{m \dot{x}^2}{2} - V(x) \right) dt$
$\dot{x} \equiv \frac{dx}{dt}$

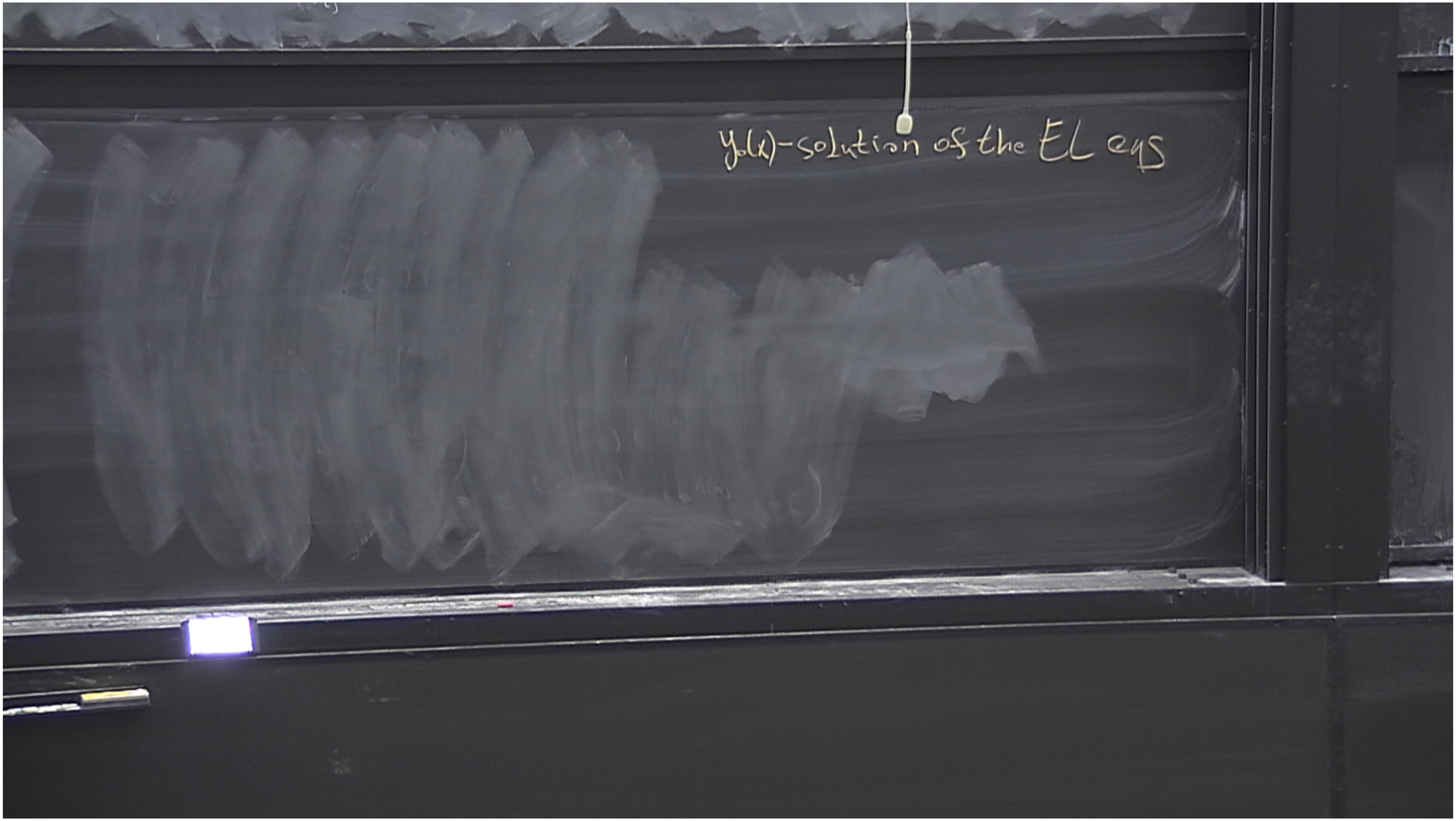

Euler-Lagrange equation
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x})\, dt = \int_{t_i}^{t_f} \left(\frac{m\dot{x}^2}{2} - V(x) \right) dt$
$\dot{x} \equiv \frac{dx}{dt}$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$
$m\ddot{x} = -\frac{\partial V}{\partial x} = F$
2nd Newt law

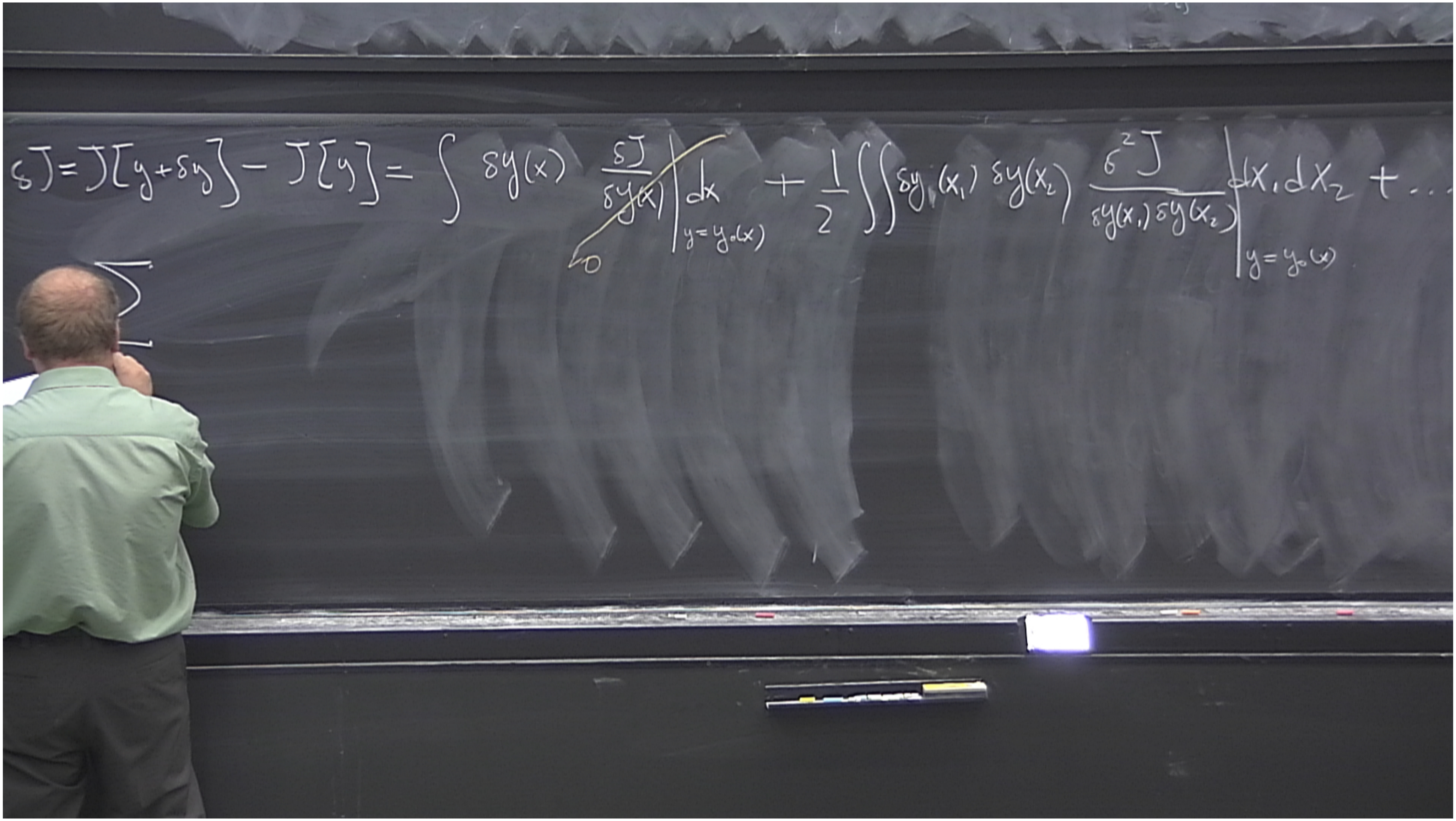

Euler-Lagrange equation
Lagrangian Mechanics
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt = \int_{t_i}^{t_f} \Big(\frac{m \dot{x}}{2}$
$\dot{x} = \frac{dx}{dt}$
Lagrangian
action
More generally
$S[q_1, \ldots q_n, \dot{q}_1 \ldots \dot{q}_n] = \int_{t_i}^{t_f} L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\, dt$
q_i's - generalized coordinates

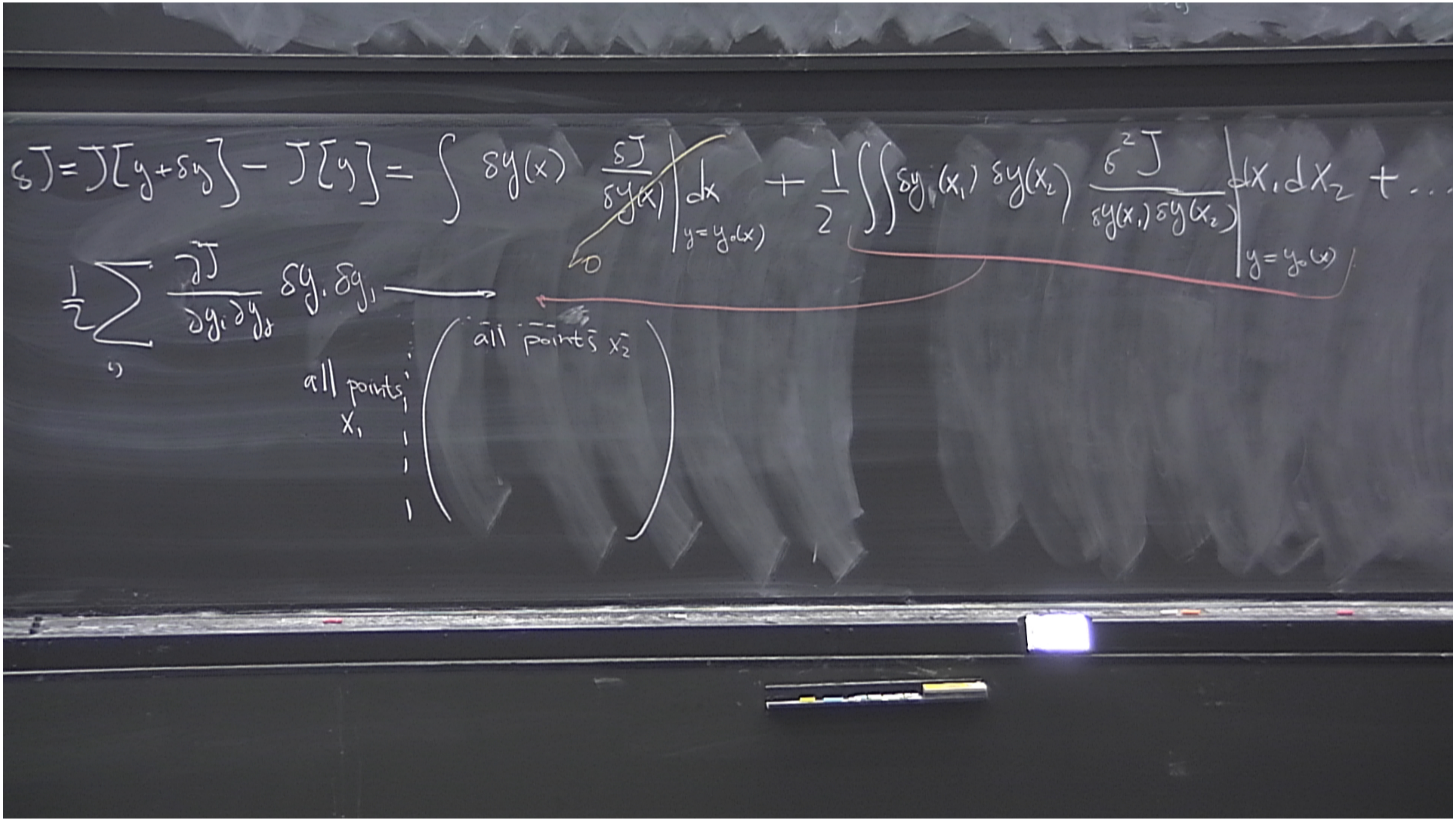

Euler-Lagrange equation
$S[x] = \int_{t_i}^{t_f} L(x, \dot{x}, t)\, dt = \int_{t_i}^{t_f} \left(\frac{m\dot{x}^2}{2} - V(x) \right) dt$
Lagrangian
$\dot{x} = \frac{dx}{dt}$
action
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$
$m\ddot{x} = -\frac{\partial V}{\partial x} = F$
2nd Newt law

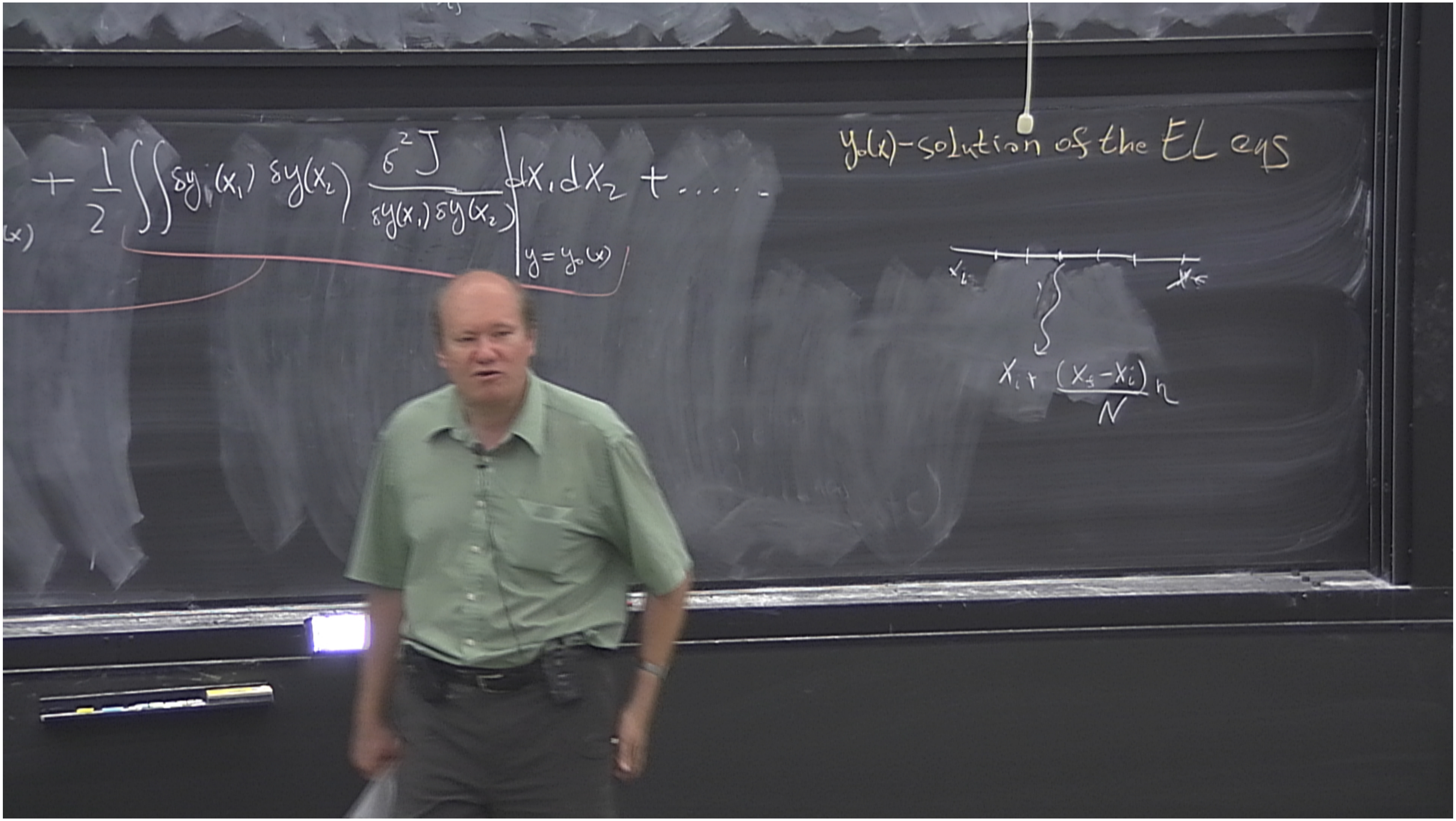

L = (m/2 (ẋ² + ẏ²) = m/2 r² θ̇² = L(θ̇)
θ - angular variable
x = r cos θ
y = r sin θ
θ

$$\delta J = J[y+\delta y] - J[y] = \int \delta y(x) \left.\frac{\delta J}{\delta y(x)}\right|_{y=y_0(x)}$$


$y_0(x)$ - solution of the EL eqs


δJ = J[y+δy] − J[y] = ∫ δy(x) δJ/δy(x)|y=y₀(x) dx + 1/2 ∫∫ δy(x₁) δy(x₂) δ²J/δy(x₁)δy(x₂)|y=y₀(x) dx₁dx₂ + . . .
0
Σ


$\delta J = J[y+\delta y] - J[y] = \int \delta y(x) \frac{\delta J}{\delta y(x)}\Big|_{y=y_0(x)} dx + \frac{1}{2}\iint \delta y(x_1)\, \delta y(x_2) \frac{\delta^2 J}{\delta y(x_1)\,\delta y(x_2)}\Big|_{y=y_0(x)} dx_1 dx_2 + \ldots$
0
$\frac{1}{2}\sum_{ij} \frac{\partial J}{\partial y_i \partial y_j} \delta y_i \delta y_j$
all points x_2
all points x_1



$+ \frac{1}{2}\iint \delta y(x_1)\,\delta y(x_2)\,\frac{\delta^2 J}{\delta y(x_1)\,\delta y(x_2)}\Big|_{y=y_0(x)} dx_1\,dx_2 + \dots$
$y_0(x)$ - solution of the EL eqs
$x_i + \frac{(x_f - x_i)}{N} n$