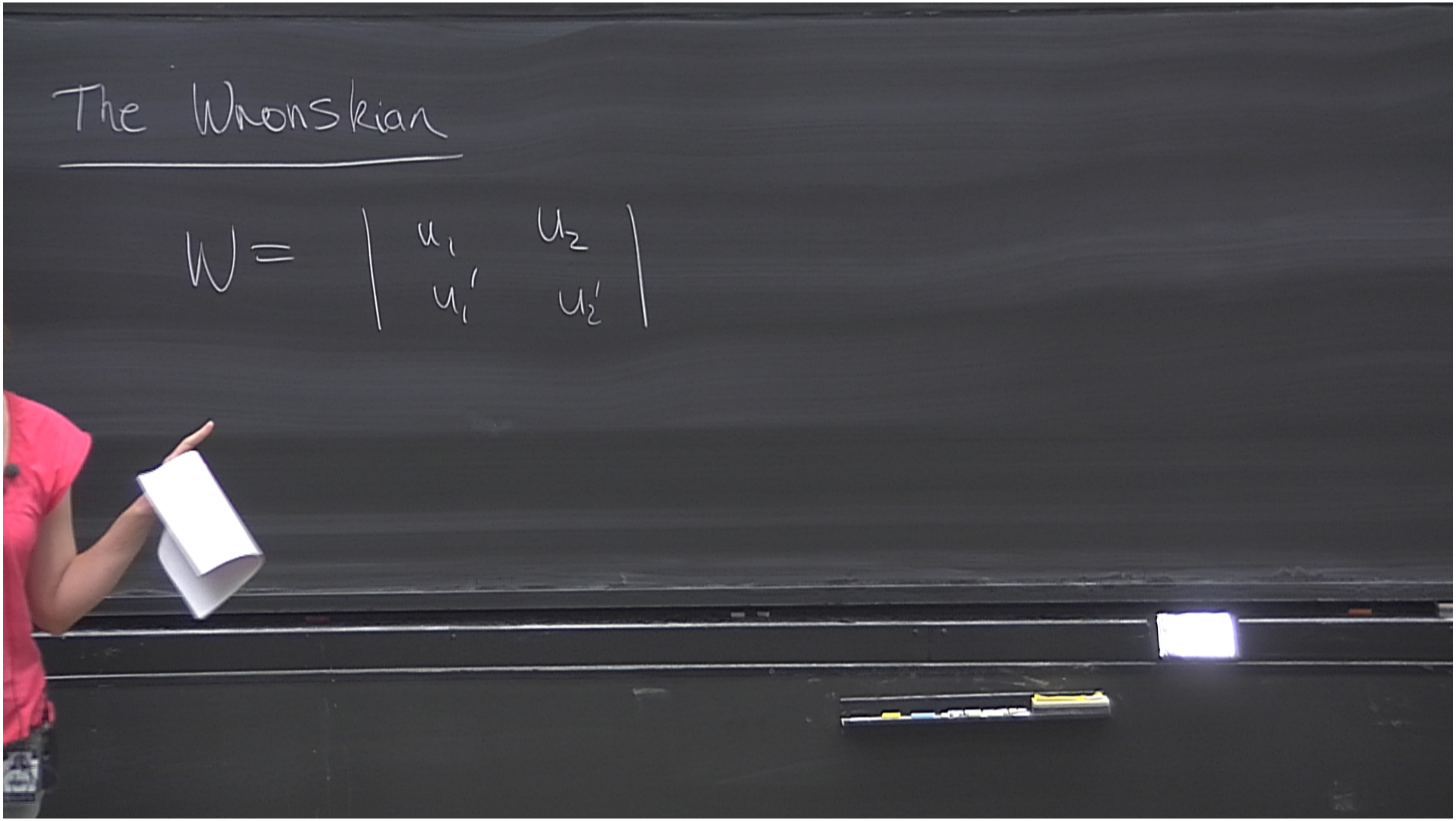
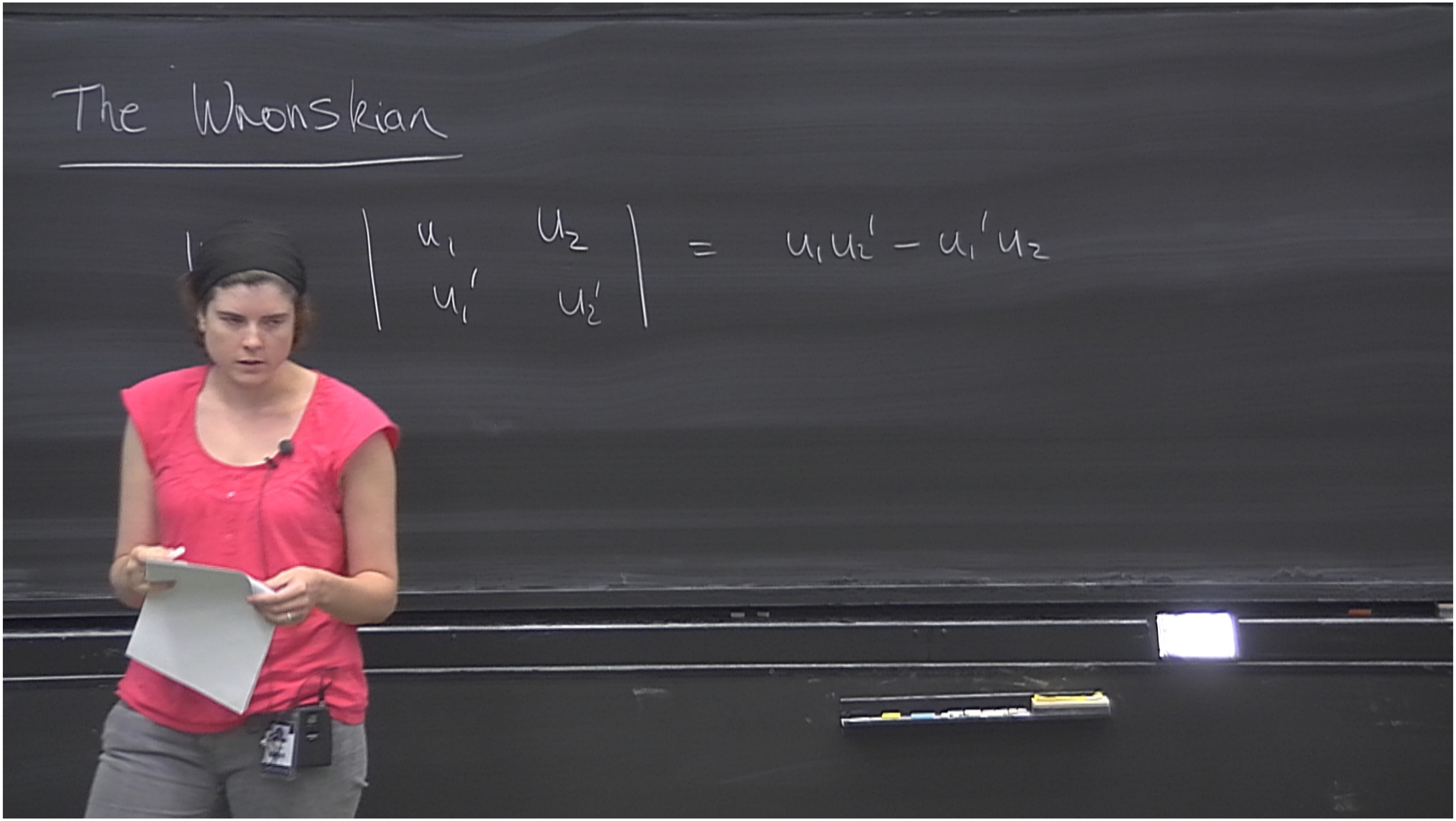
Title: Differential Equations - Lecture 3

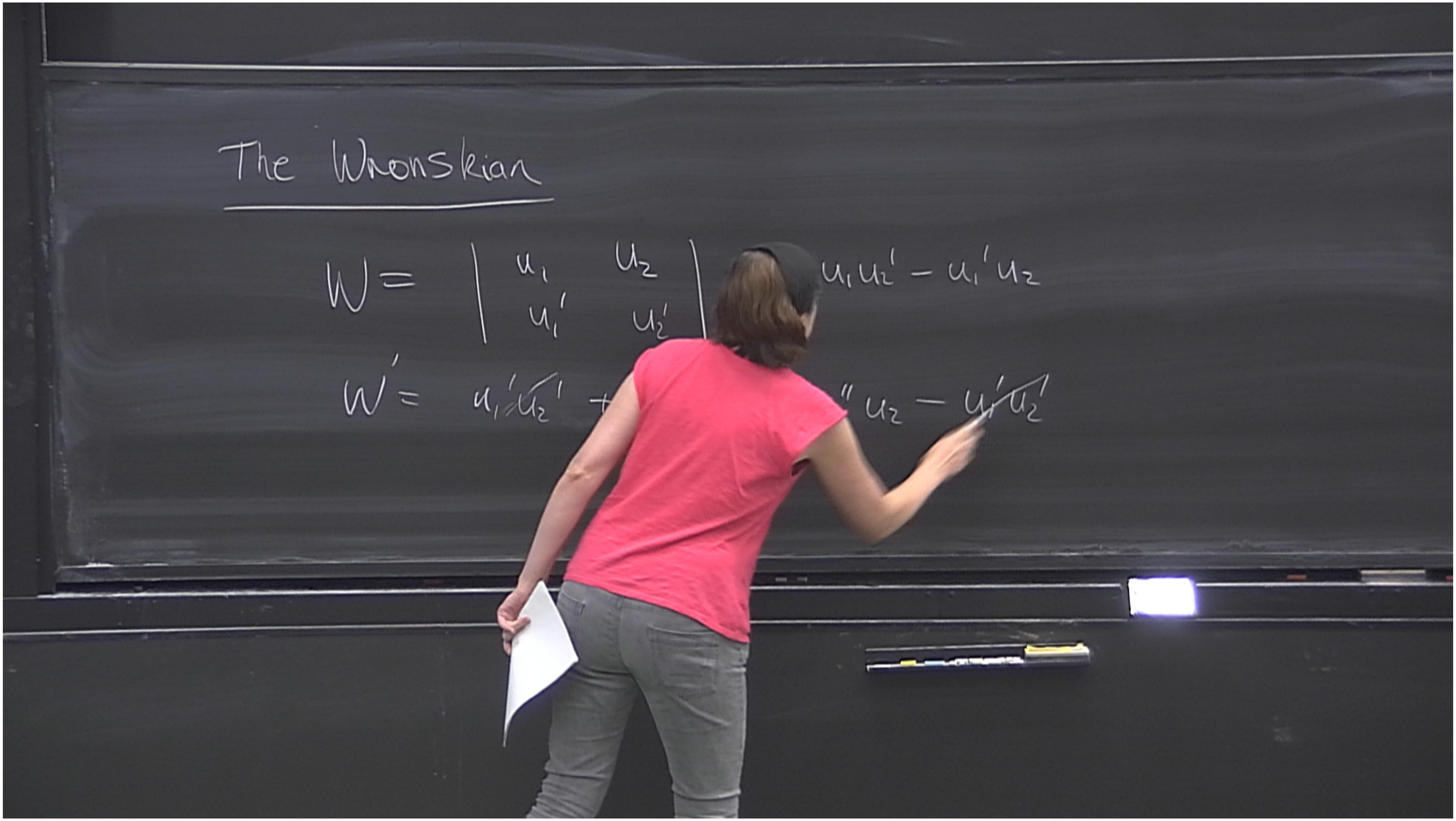
Date: Aug 31, 2011 09:00 AM

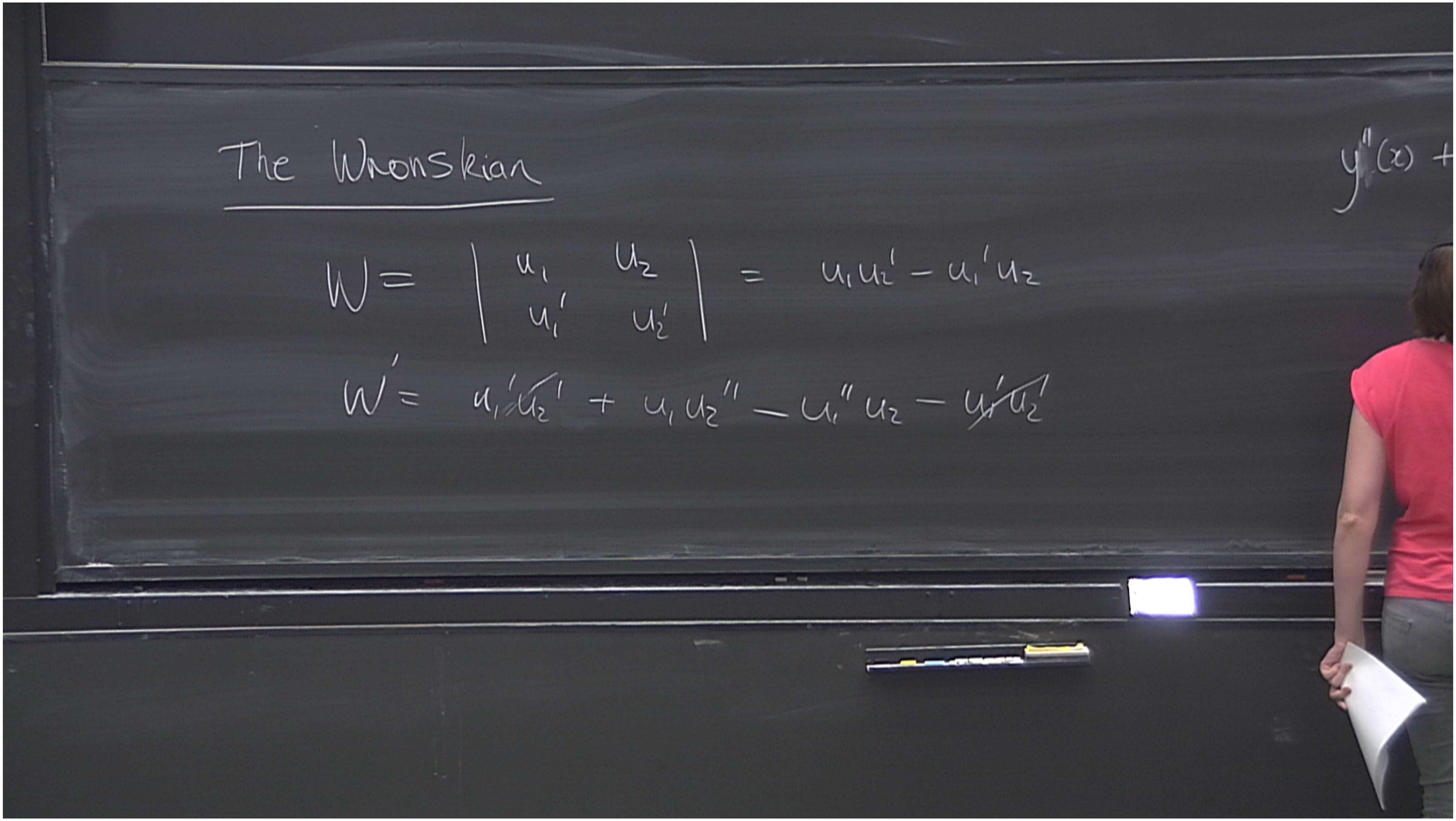
URL: http://pirsa.org/11080142

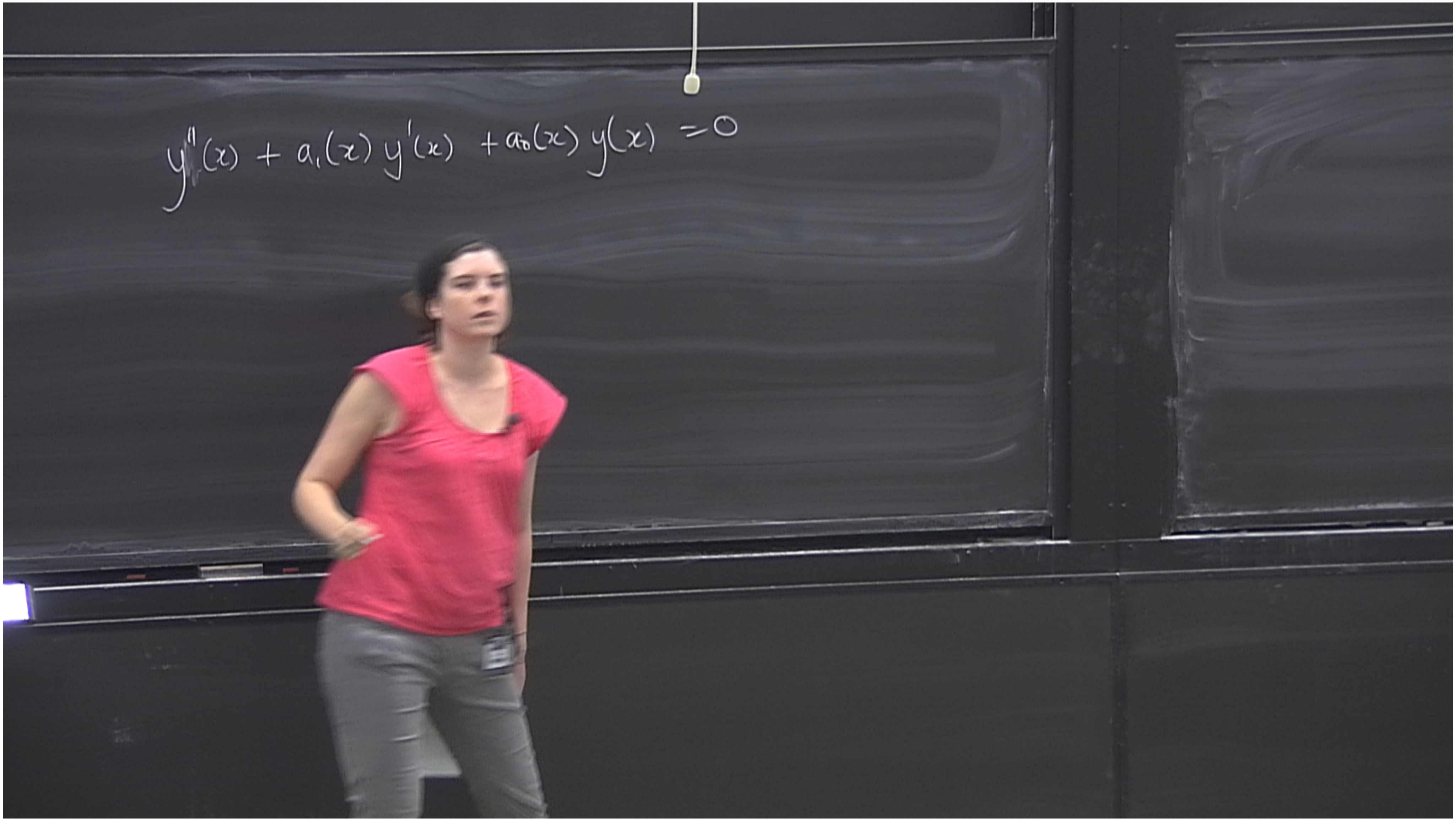
Abstract:

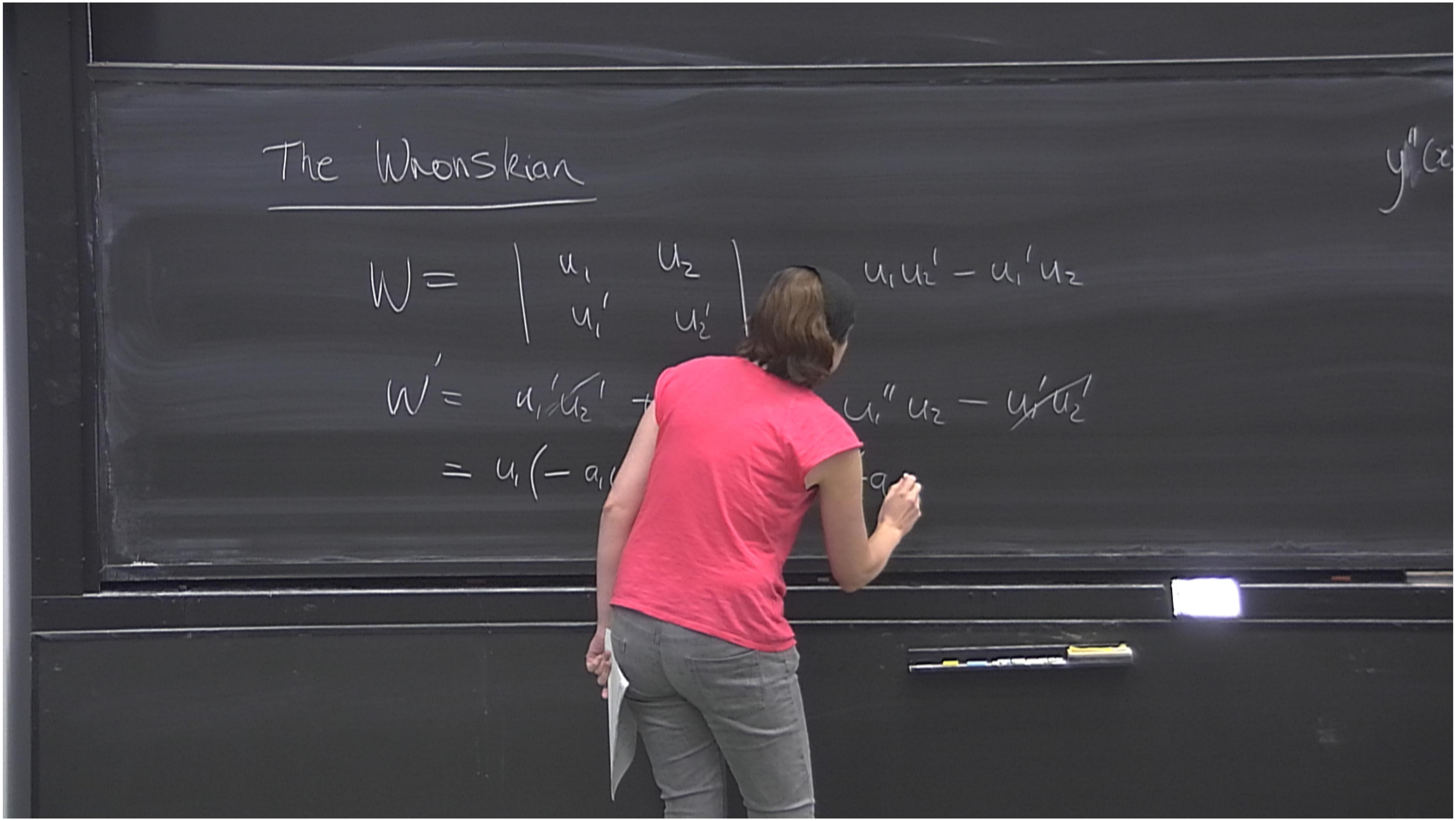

The Wronskian
$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

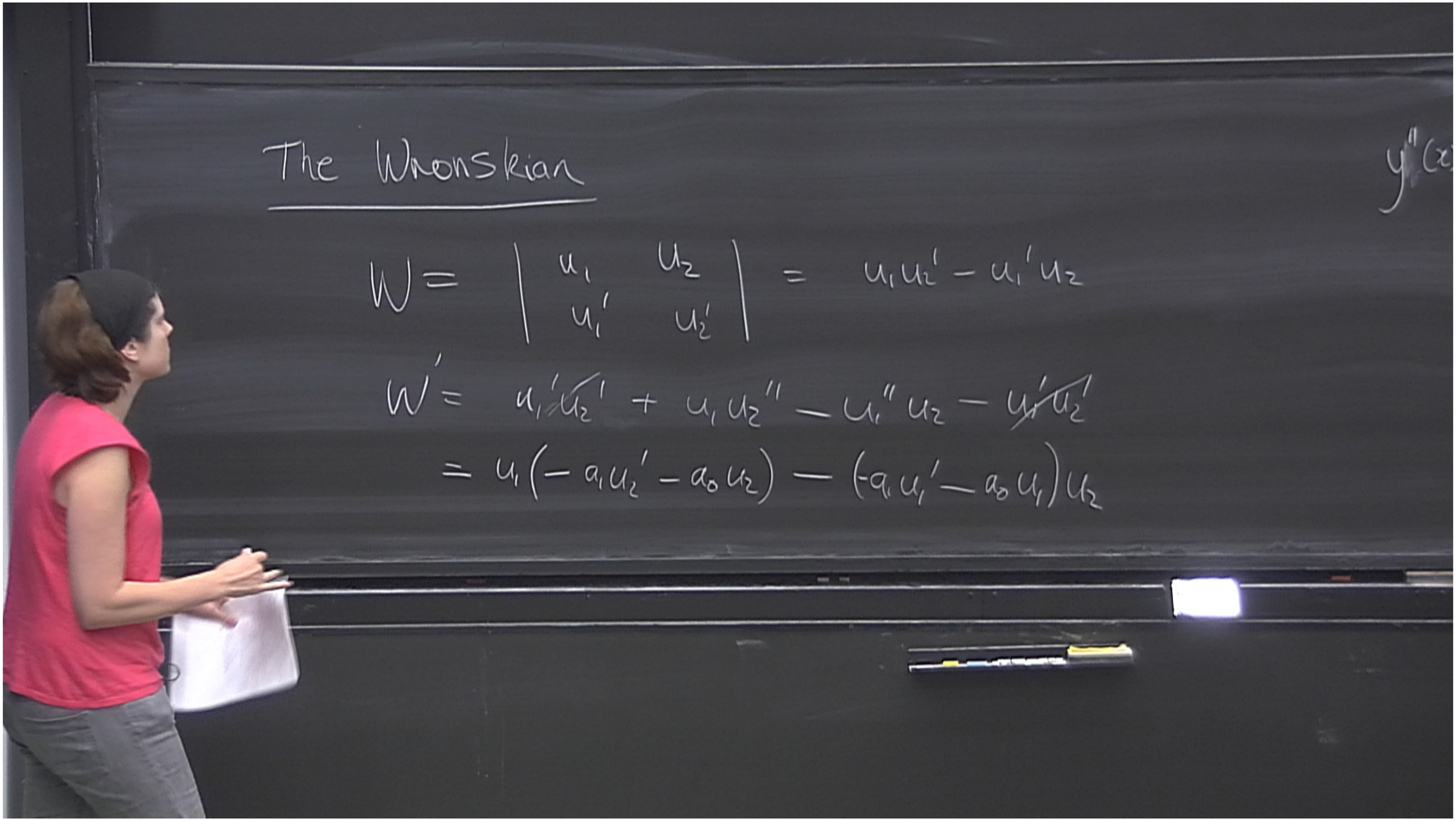

The Wronskian
$\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$

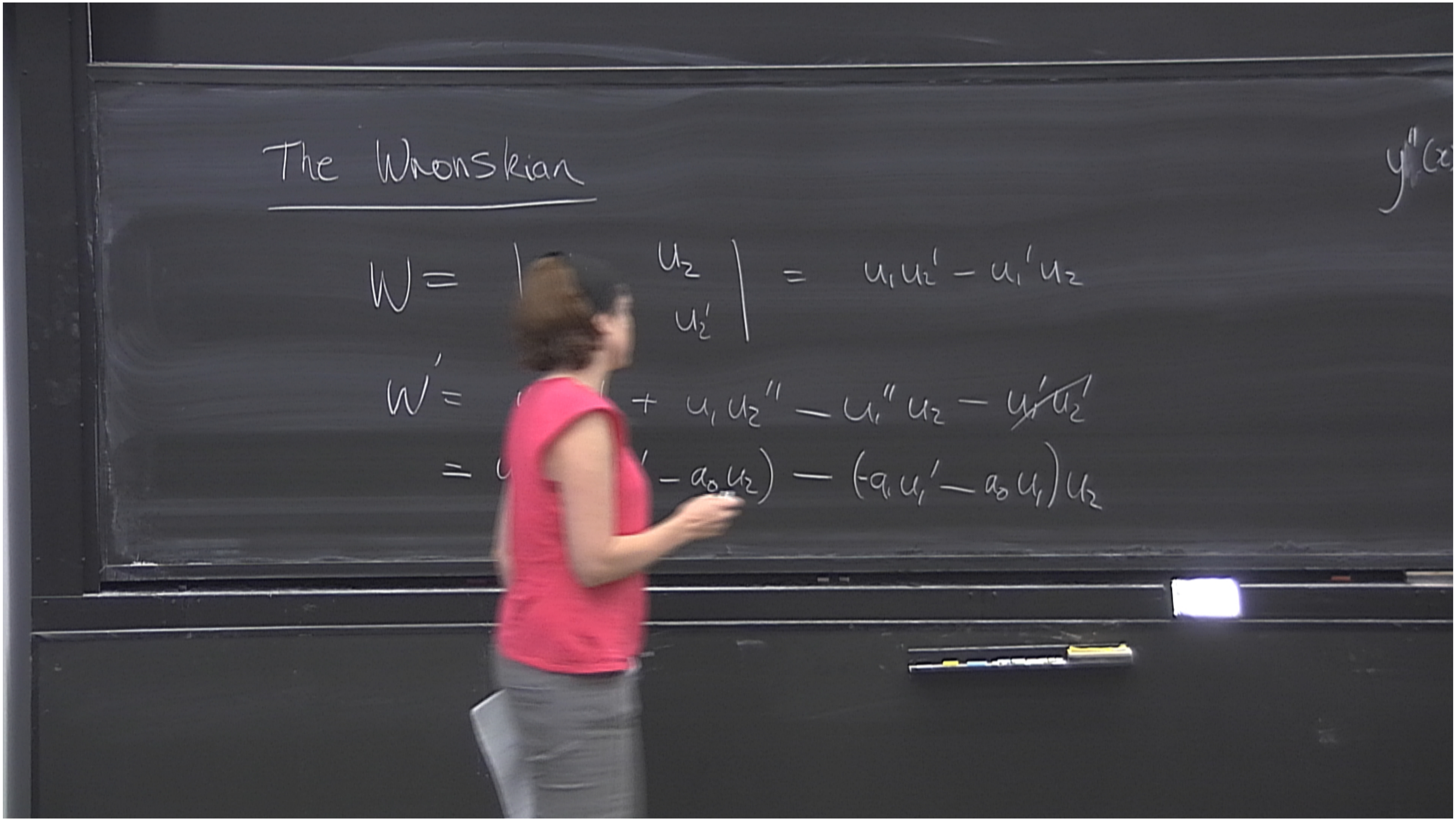

The Wronskian
W = | u1 u2 / u1' u2' | = u1u2' − u1'u2
W' = u1'u2' + … "u2 − u1'u2'

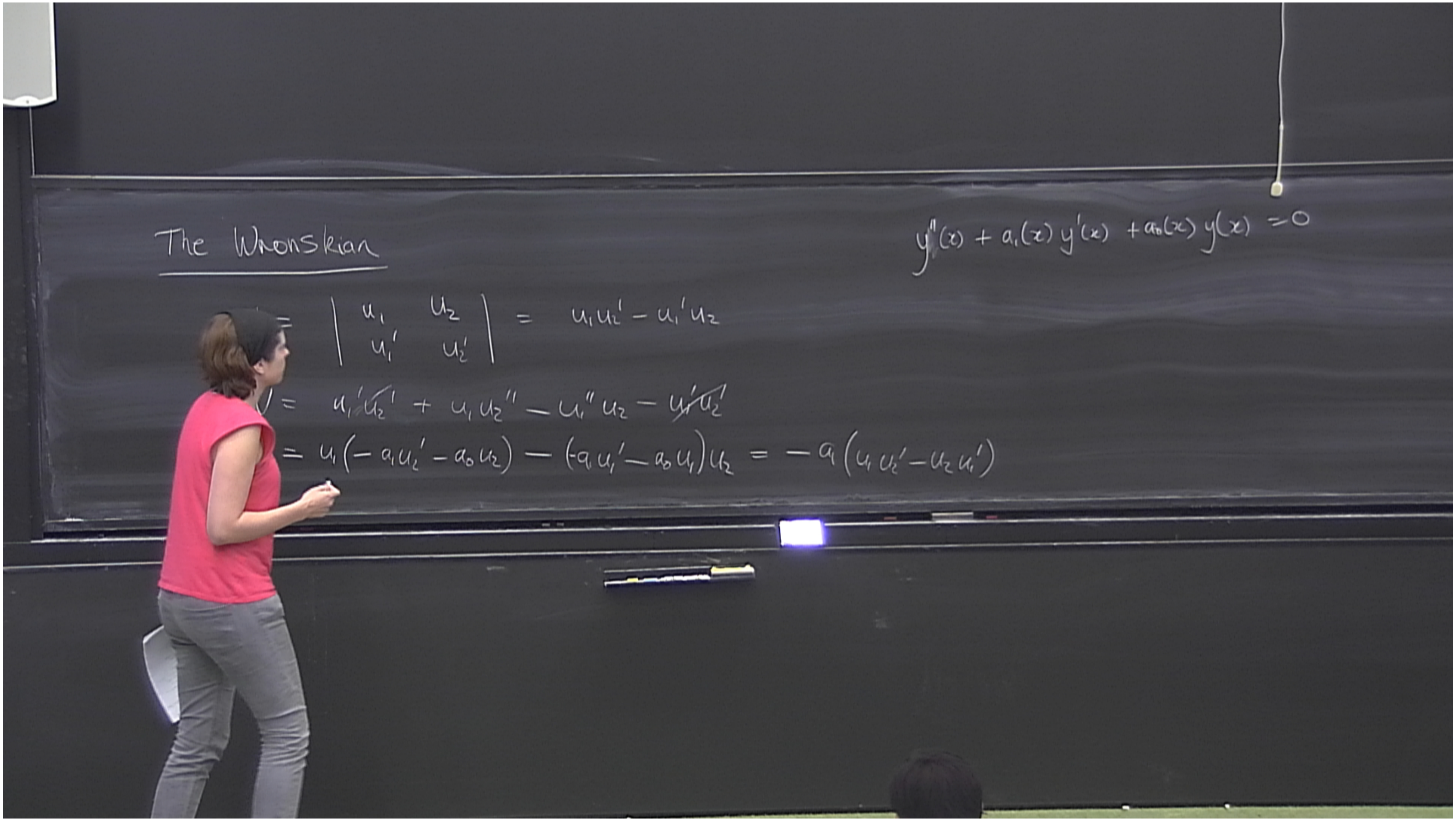

The Wronskian
$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$
$W' = \cancel{u_1' u_2'} + u_1 u_2'' - u_1'' u_2 - \cancel{u_1' u_2'}$
$y''(x) +$


$y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = 0$


The Wronskian
$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$
$u_1 u_2' - u_1' u_2$
$W' = u_1' \cancel{u_2'} +$
$u_1'' u_2 - \cancel{u_1' u_2'}$
$= u_1 (- a_1$
$y''(x$


The Wronskian
$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$
$W' = \cancel{u_1' u_2'} + u_1 u_2'' - u_1'' u_2 - \cancel{u_1' u_2'}$
$= u_1(-a_1 u_2' - a_0 u_2) - (-a_1 u_1' - a_0 u_1) u_2$
$y''(x$


The Wronskian
W = | u2 u2' | = u1 u2' − u1' u2
W' = + u1 u2'' − u1'' u2 − u1' u2'
= − a0 u2) − (−a1 u1' − a0 u1) u2


The Wronskian
y''(x) + a₁(x) y'(x) + a₀(x) y(x) = 0
= | u1 u2 ; u1' u2' | = u1u2' − u1'u2
= u1'u2' + u1u2'' − u1''u2 − u1'u2'
= u1(−a1u2' − a0u2) − (−a1u1' − a0u1)u2 = −a1(u1u2' − u2u1')

The Wronskian

$$y''(x) + a_1(x)\,y'(x) + a_0(x)\,y(x) = 0$$

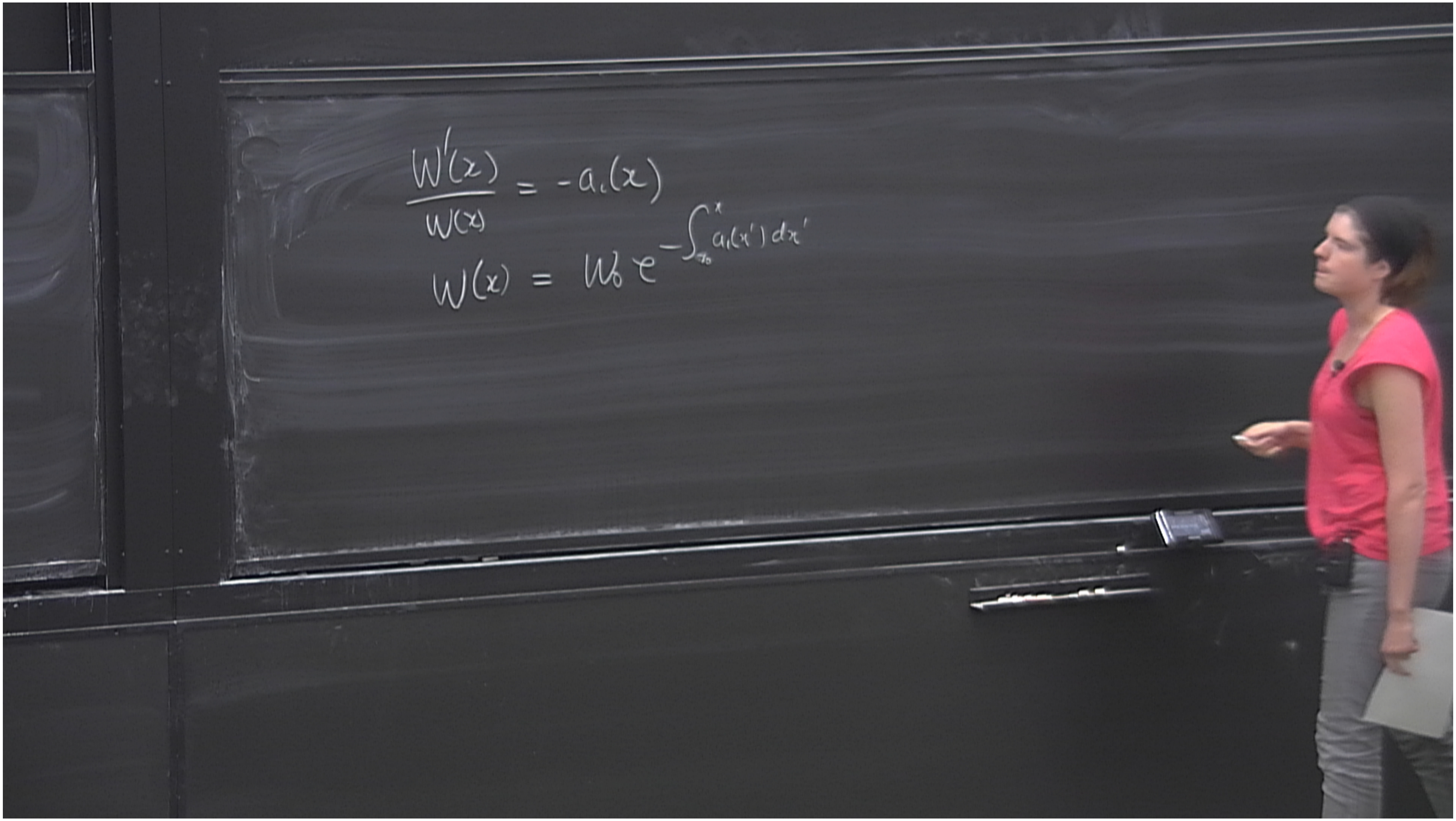
$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$$

$$W' = u_1'\cancel{u_2'} + u_1 u_2'' - u_1'' u_2 - \cancel{u_1' u_2'}$$

$$= u_1(-a_1 u_2' - a_0 u_2) - (-a_1 u_1' - a_0 u_1)u_2 = -a_1(u_1 u_2' - u_2 u_1') = -a_1(x)\,W(x)$$

$$\frac{W'(x)}{W(x)} = -a_1(x)$$

$$W(x) = W_0\, e^{-\int_{x_0}^{x} a_1(x')\,dx'}$$


$\frac{W'(x)}{W(x)} = -a_1(x)$
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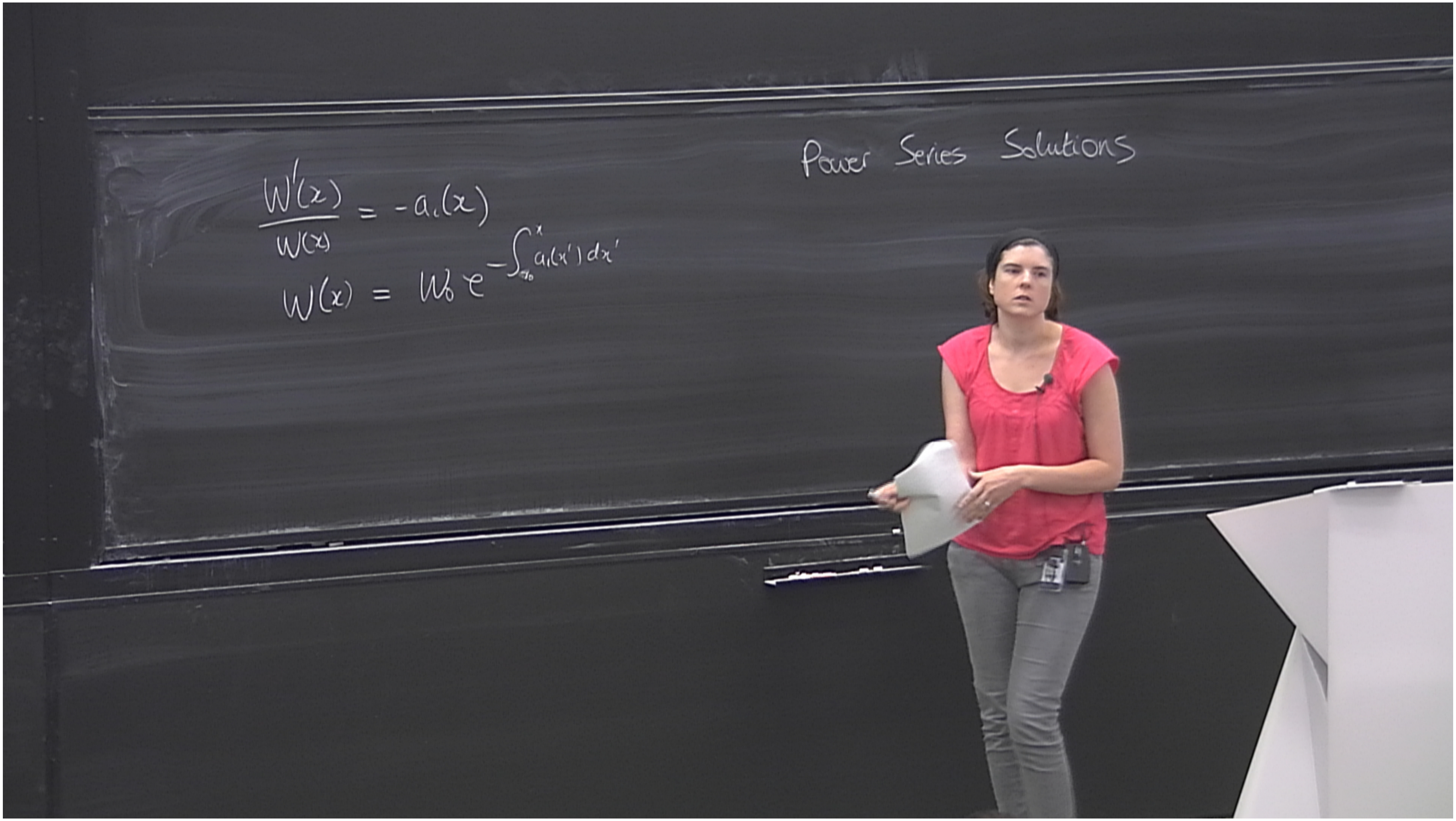
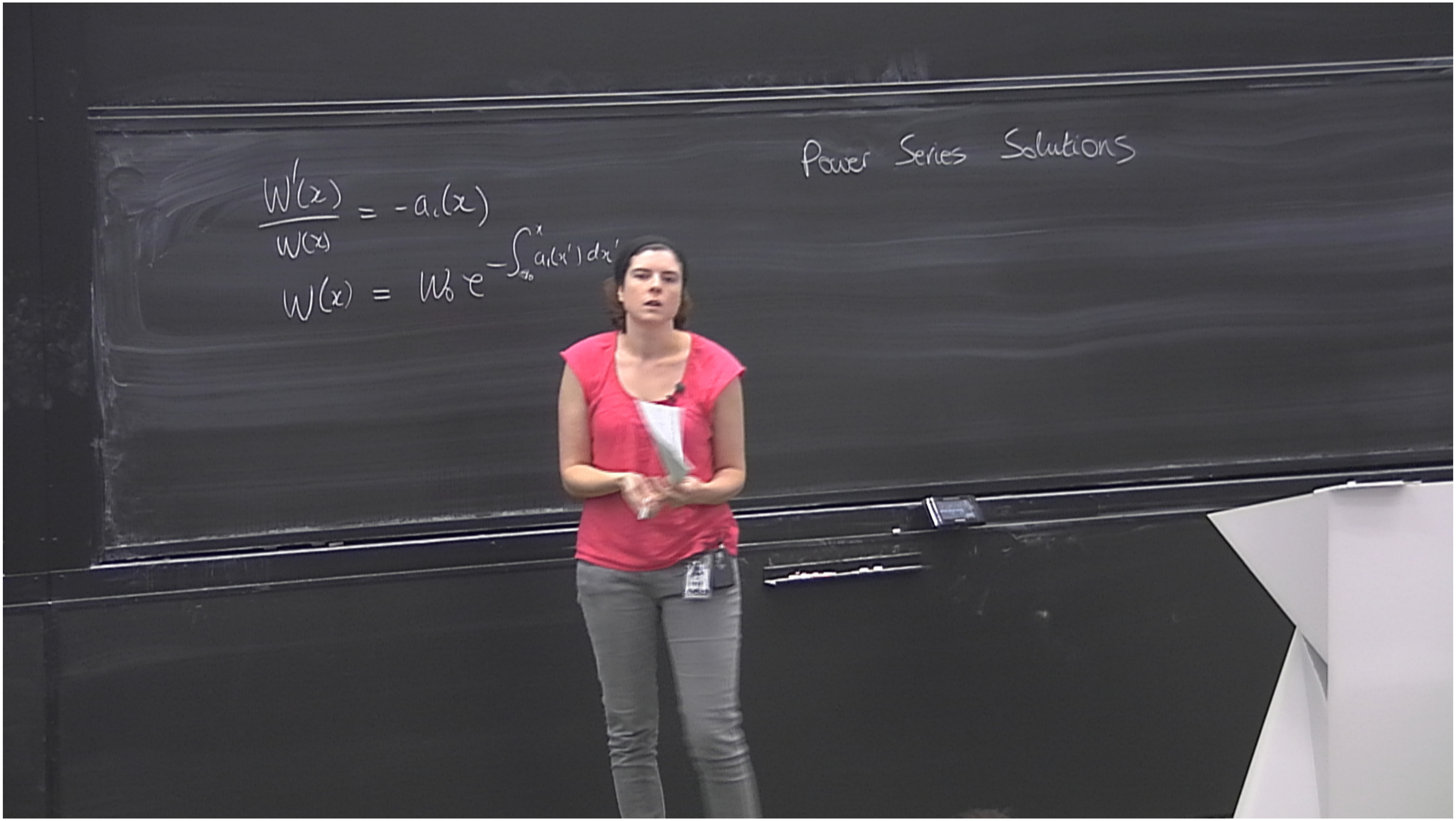

Power Series Solutions
$\frac{W'(x)}{W(x)} = -a_1(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x')\,dx'}$



Power Series Solutions
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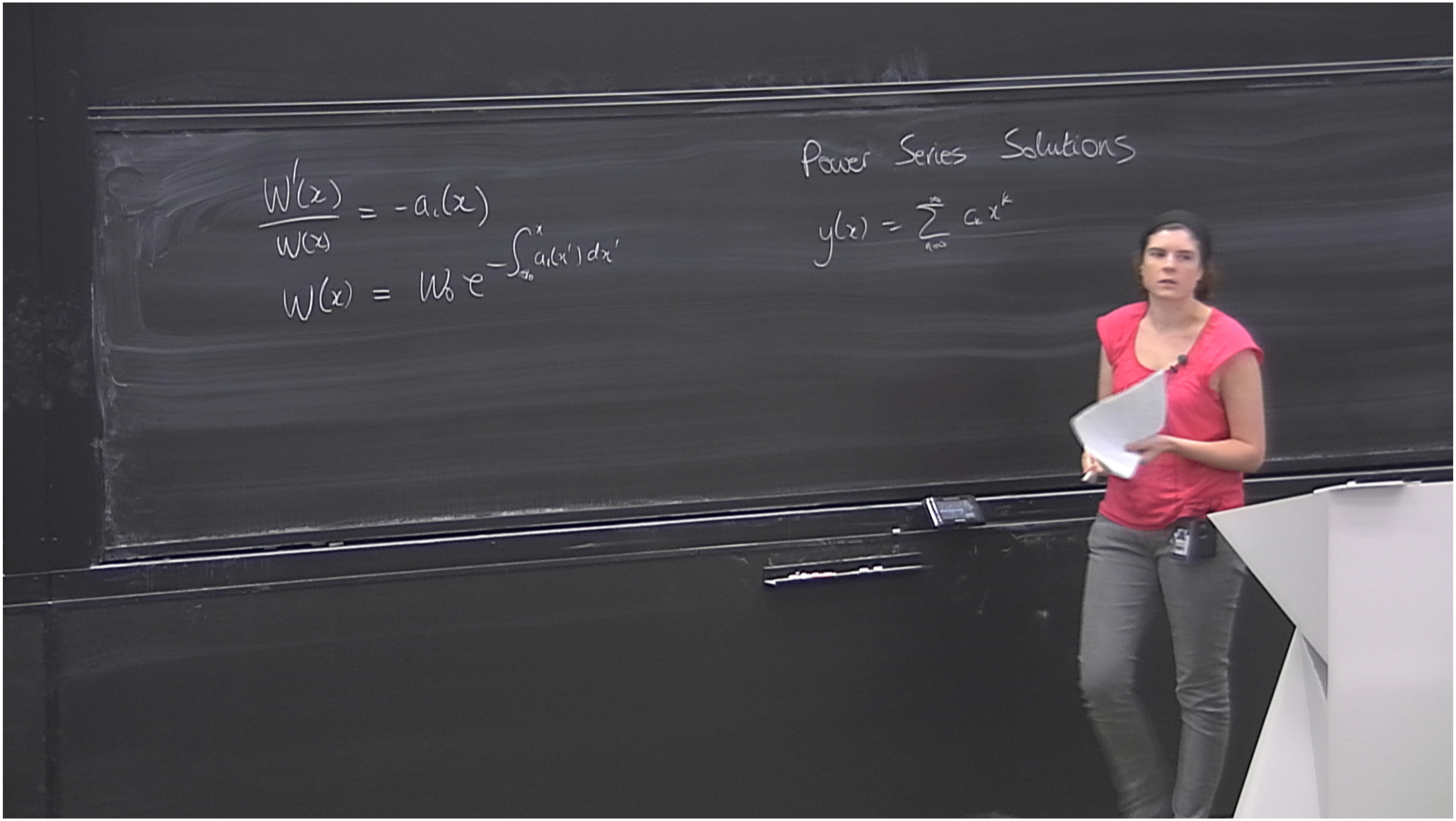
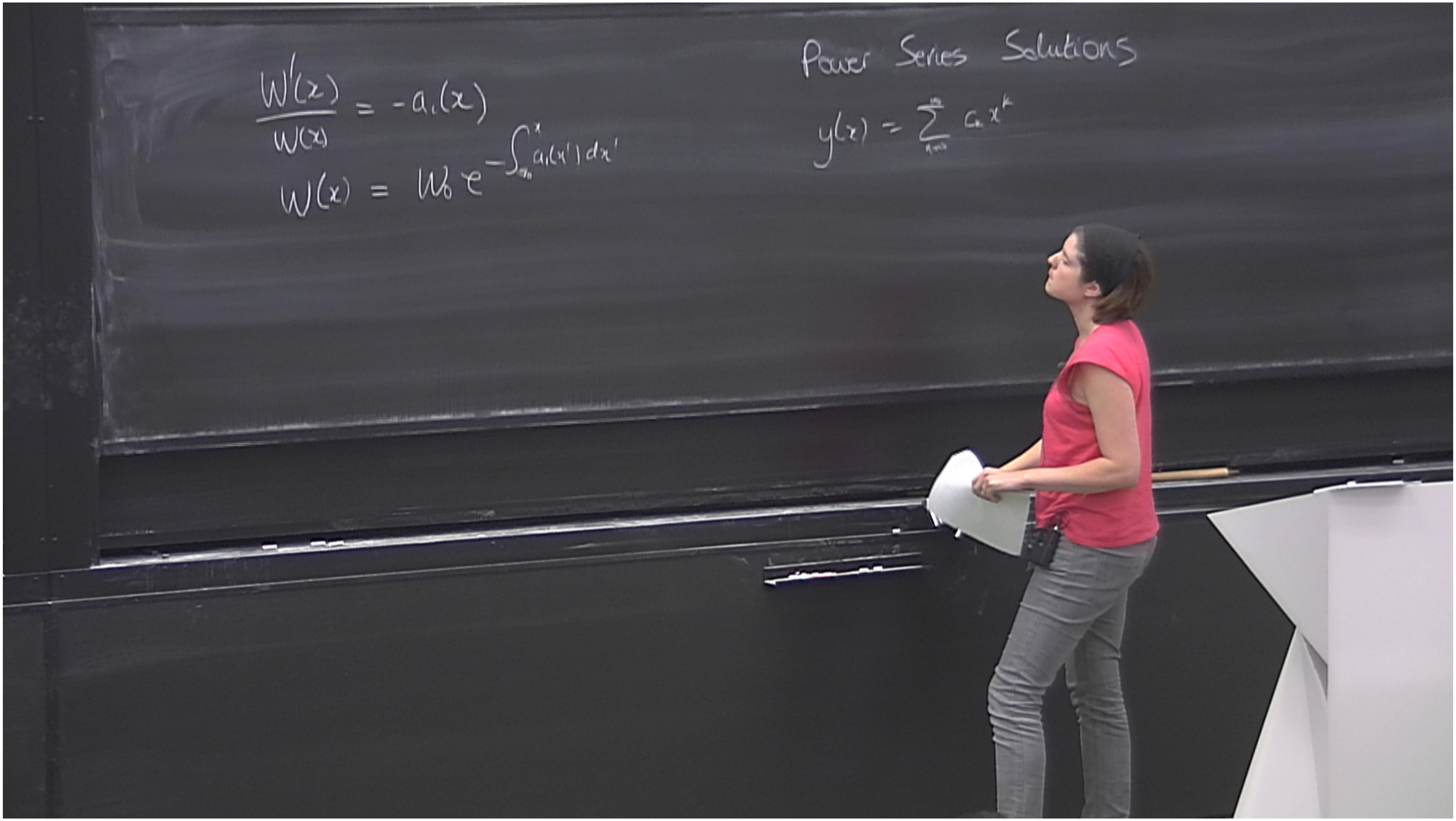

Power Series Solutions
$\frac{W'(x)}{W(x)} = -a_1(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x')dx'}$
$y(x) = \sum_{n=0}^{\infty} c_k x^k$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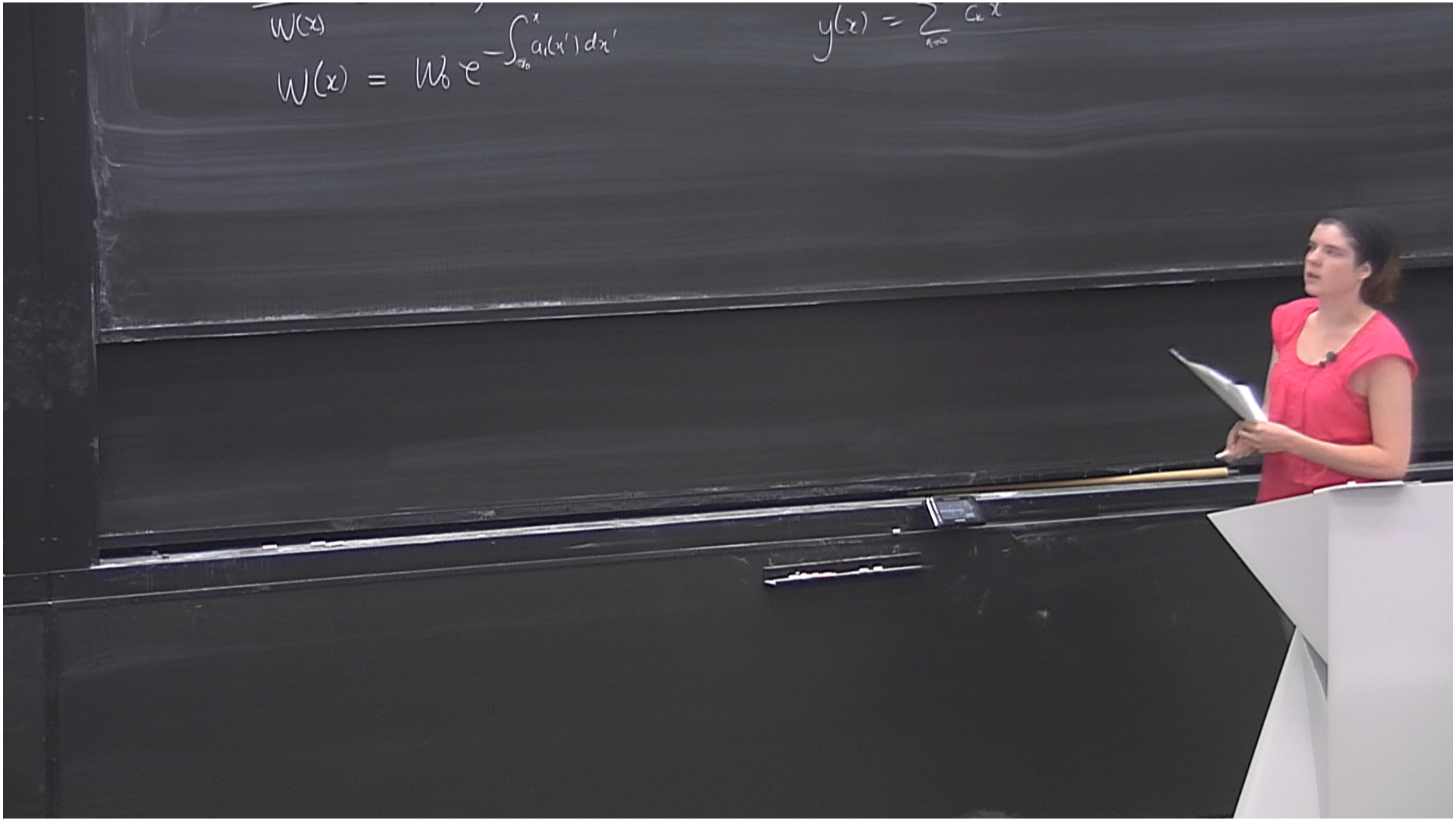
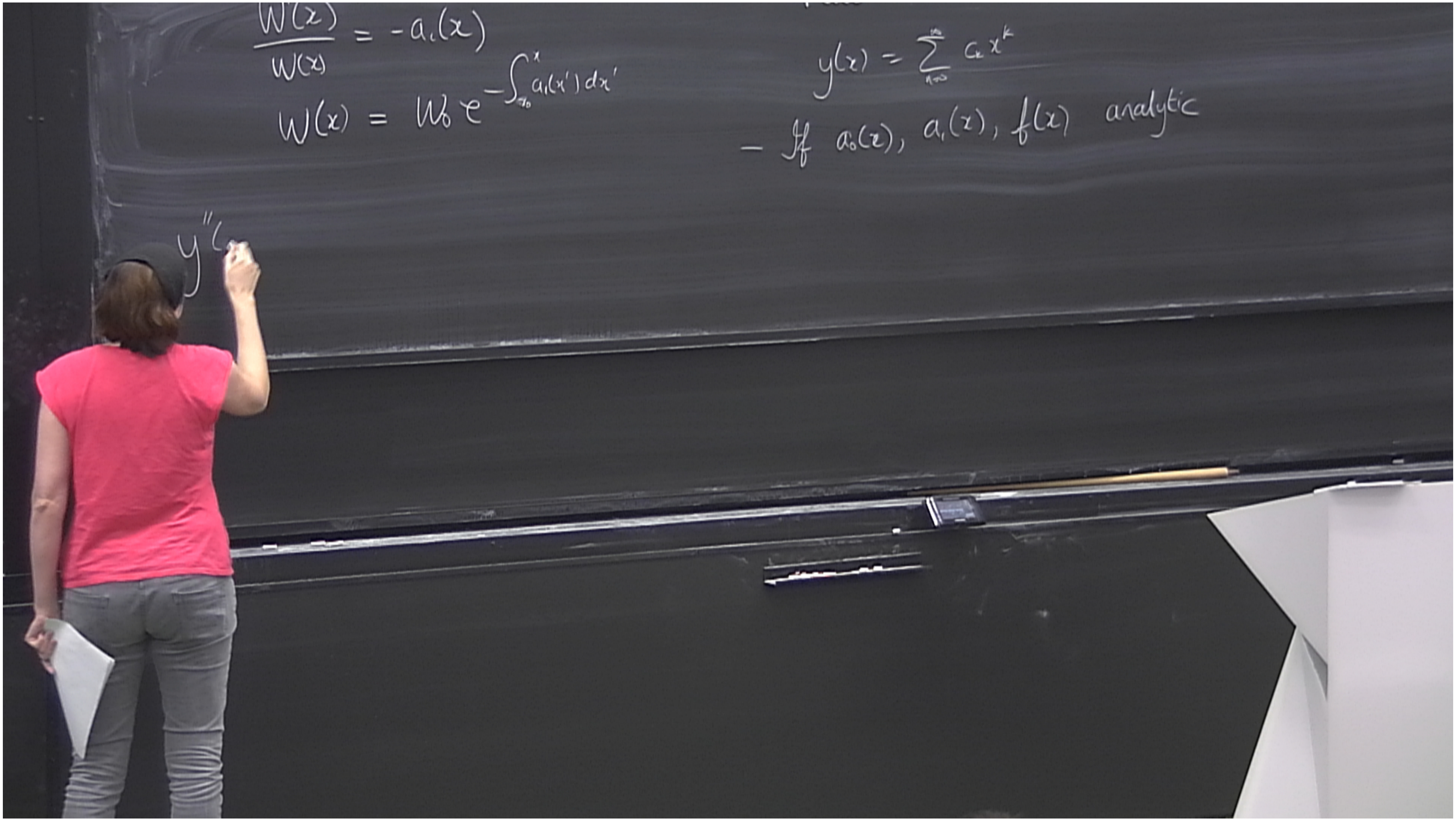
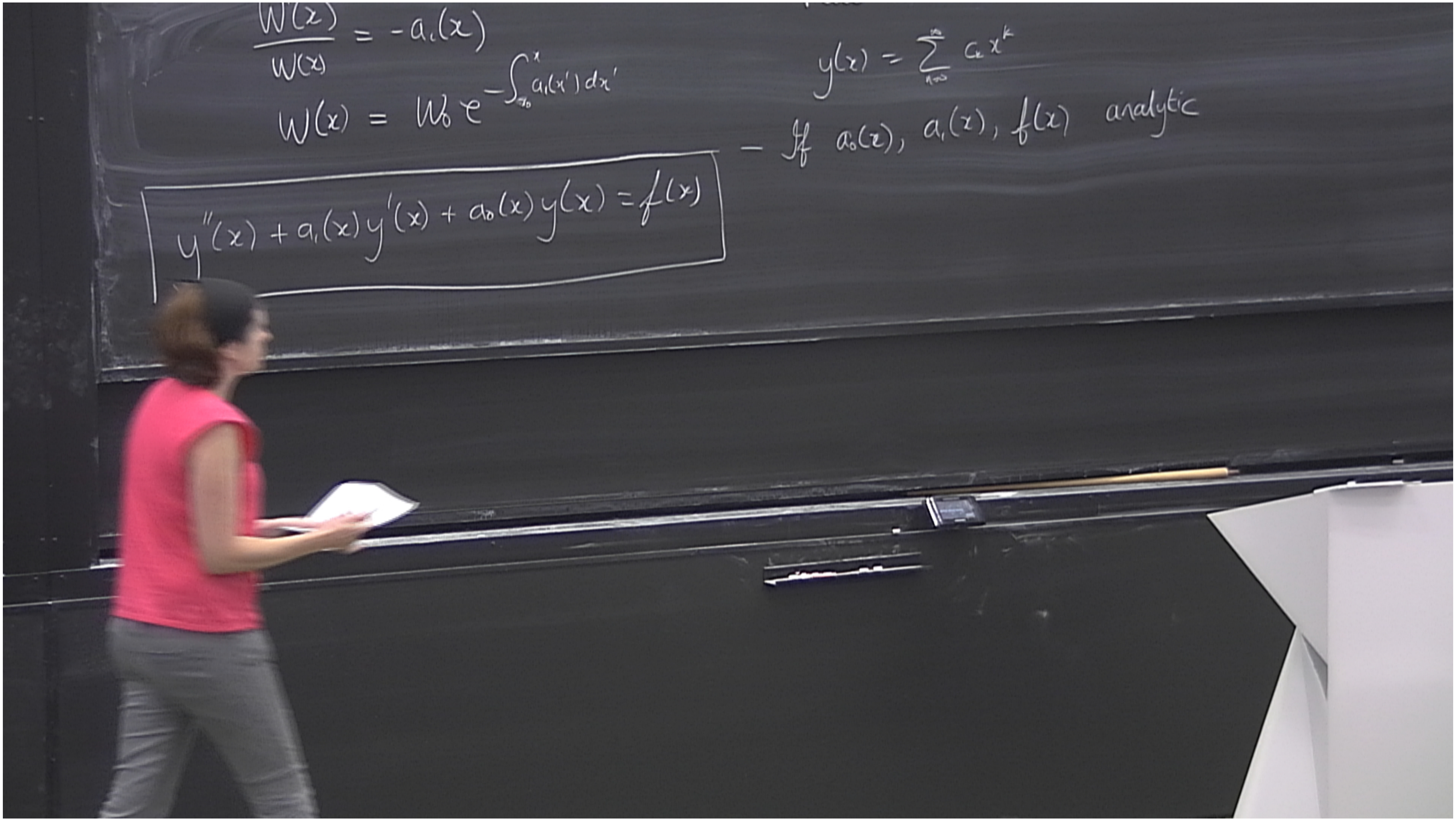
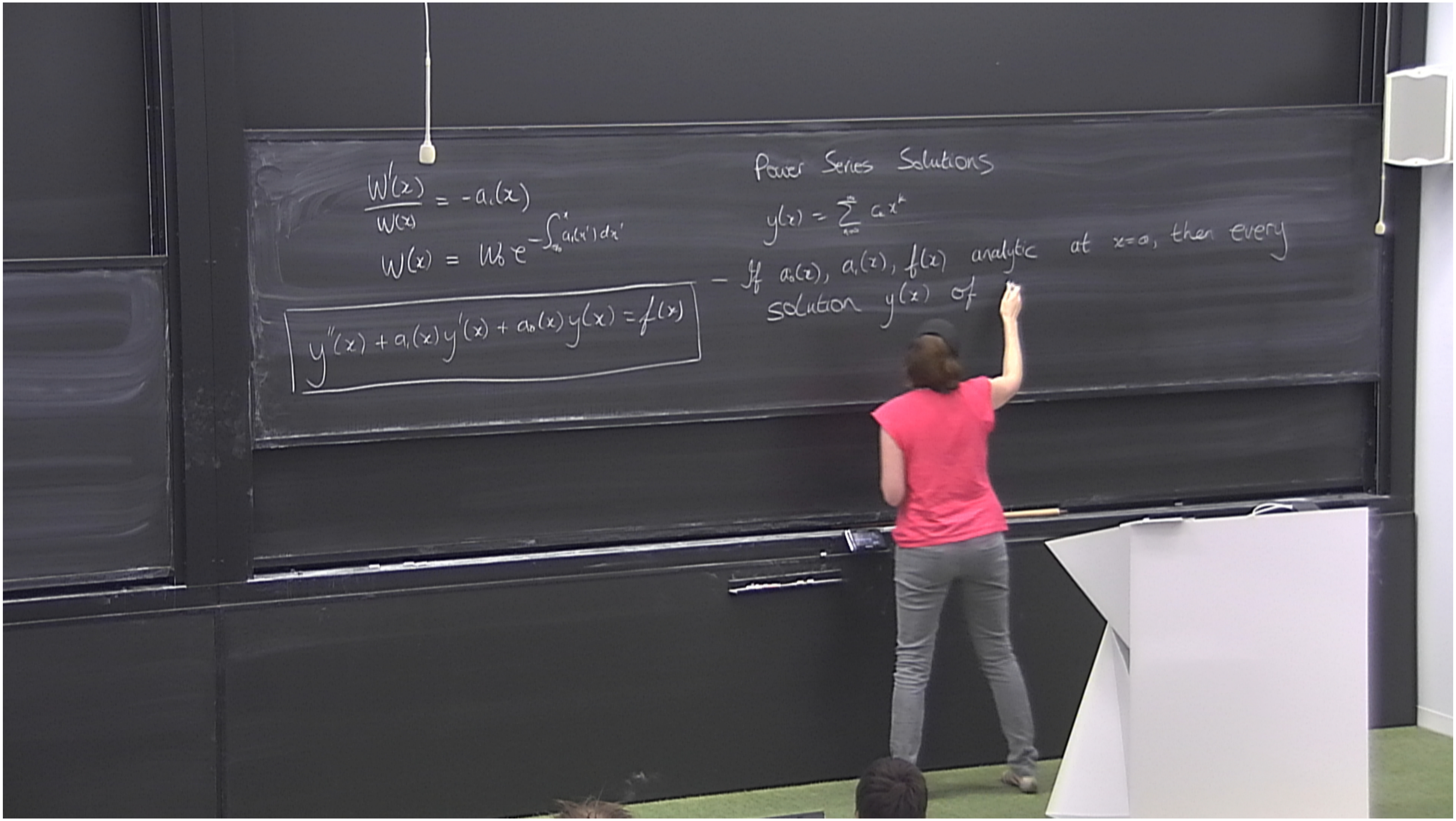
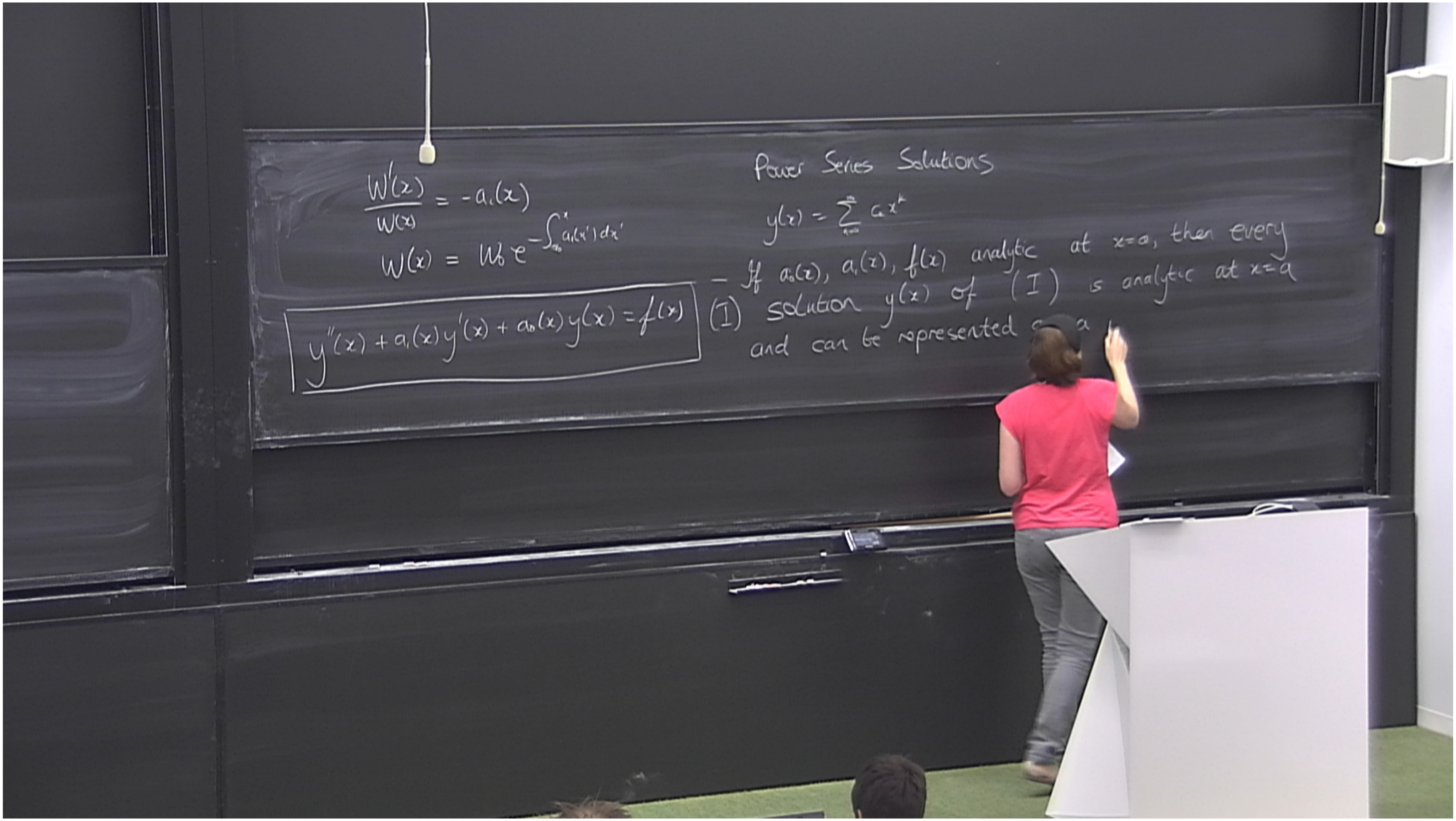

$W(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x') dx'}$
$y(x) = \sum_{n=0} c_n x$

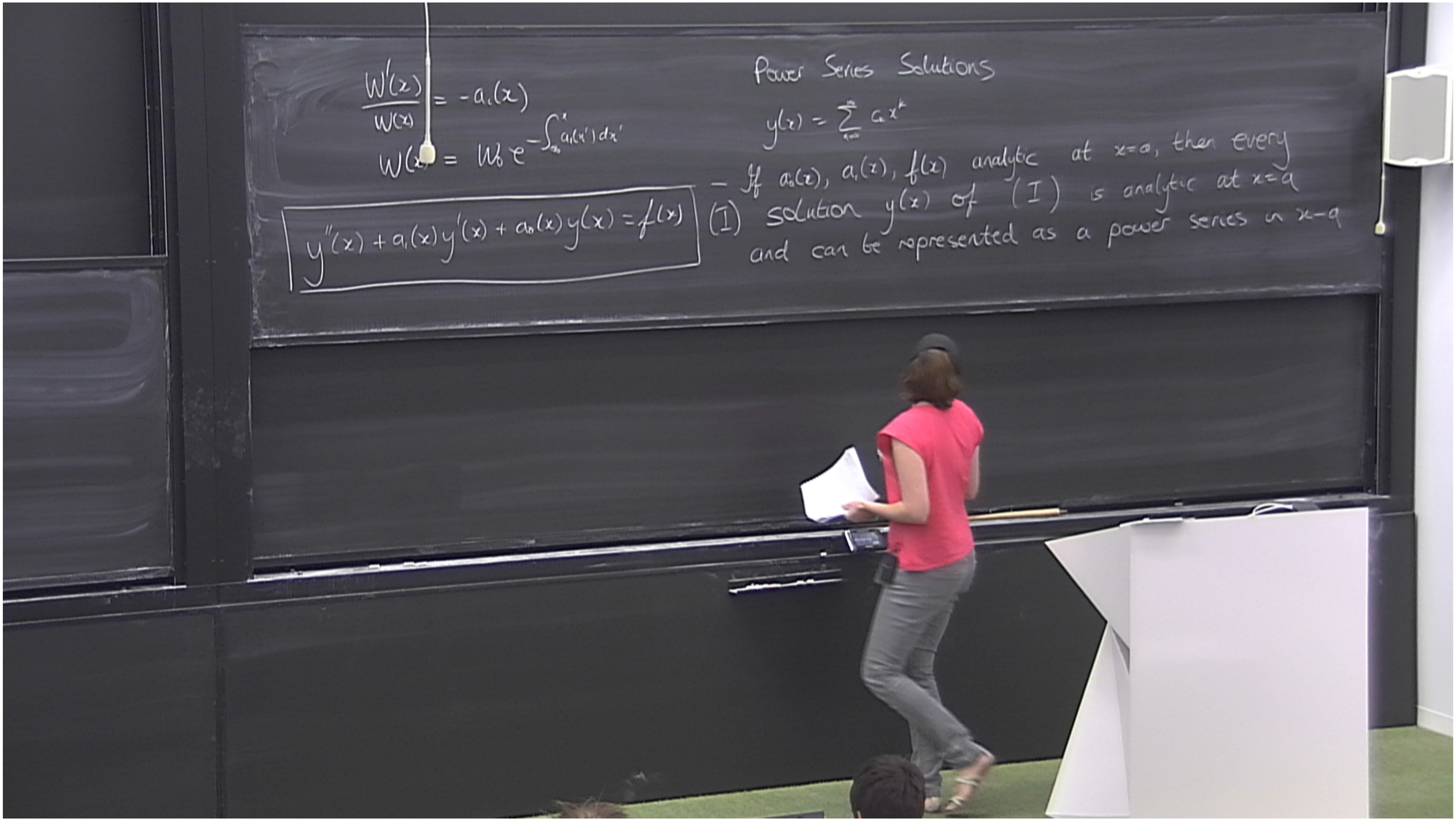

W'(x) / W(x) = -a1(x)
W(x) = W0 e^(-∫ a1(x') dx')
y(x) = Σ ck x^k
- If a0(x), a1(x), f(x) analytic
y''(



W'(x) / W(x) = -a_1(x)
W(x) = W_0 e^{-∫_{x_0}^{x} a_1(x') dx'}
y''(x) + a_1(x) y'(x) + a_0(x) y(x) = f(x)
y(x) = Σ_{n=0}^{∞} c_n x^k
- If a_0(x), a_1(x), f(x) analytic


$\frac{W'(x)}{W(x)} = -a_1(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x')dx'}$
$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$
Power Series Solutions
$y(x) = \sum_{k=0}^{\infty} c_k x^k$
– If $a_0(x)$, $a_1(x)$, $f(x)$ analytic at $x=0$, then every
solution $y(x)$ of


$\frac{W'(x)}{W(x)} = -a_1(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x')dx'}$
$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$
(I)
Power Series Solutions
$y(x) = \sum_{k=0}^{\infty} c_k x^k$
– If $a_0(x)$, $a_1(x)$, $f(x)$ analytic at $x=a$, then every
solution $y(x)$ of (I) is analytic at $x=a$
and can be represented



$\frac{W'(x)}{W(x)} = -a_1(x)$
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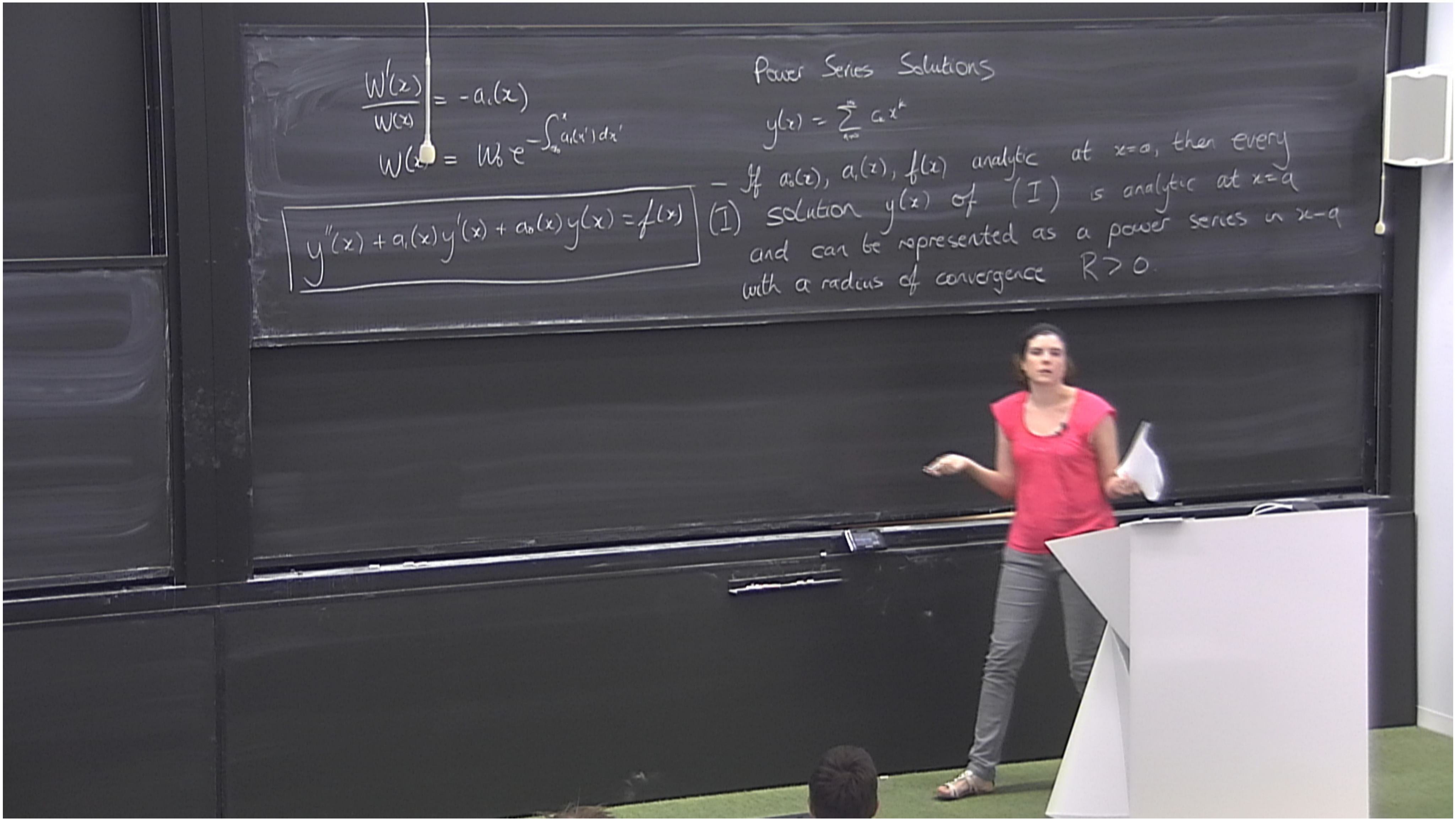
Power Series Solutions
$y(x) = \sum_{k=0}^{\infty} c_k x^k$
– If $a_0(x)$, $a_1(x)$, $f(x)$ analytic at $x=a$, then every
solution $y(x)$ of (I) is analytic at $x=a$
and can be represented as a power series in $x-a$


Power Series Solutions
$\frac{W'(x)}{W(x)} = -a_1(x)$
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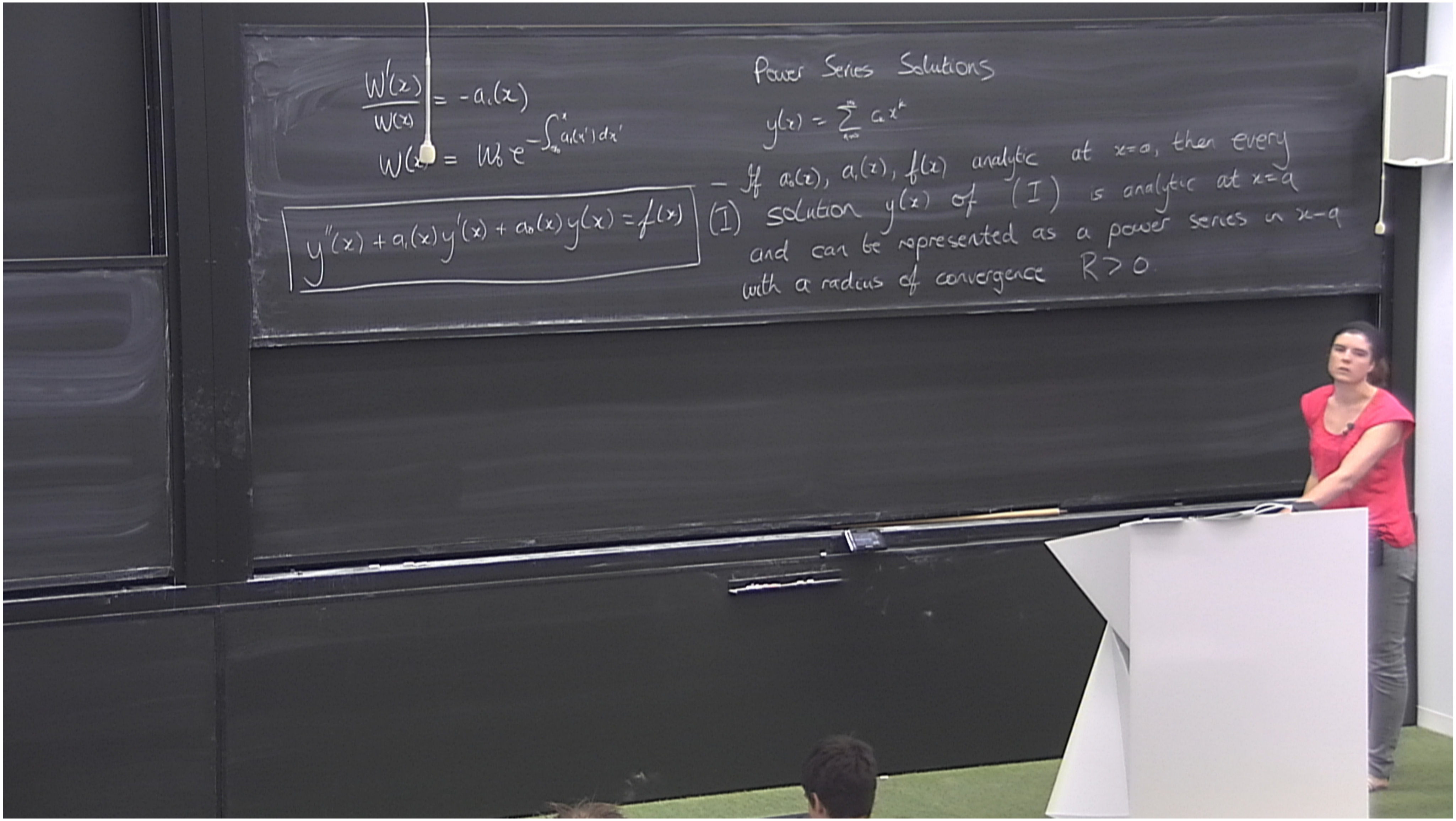
$y(x) = \sum_{k=0}^{\infty} c_k x^k$
$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$ (I)
– If $a_0(x)$, $a_1(x)$, $f(x)$ analytic at $x=a$, then every solution $y(x)$ of (I) is analytic at $x=a$ and can be represented as a power series in $x-a$ with a radius of convergence $R > 0$.



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Power Series Solutions
y(x) = Σ ck x^k
- If a0(x), a1(x), f(x) analytic at x=a, then every
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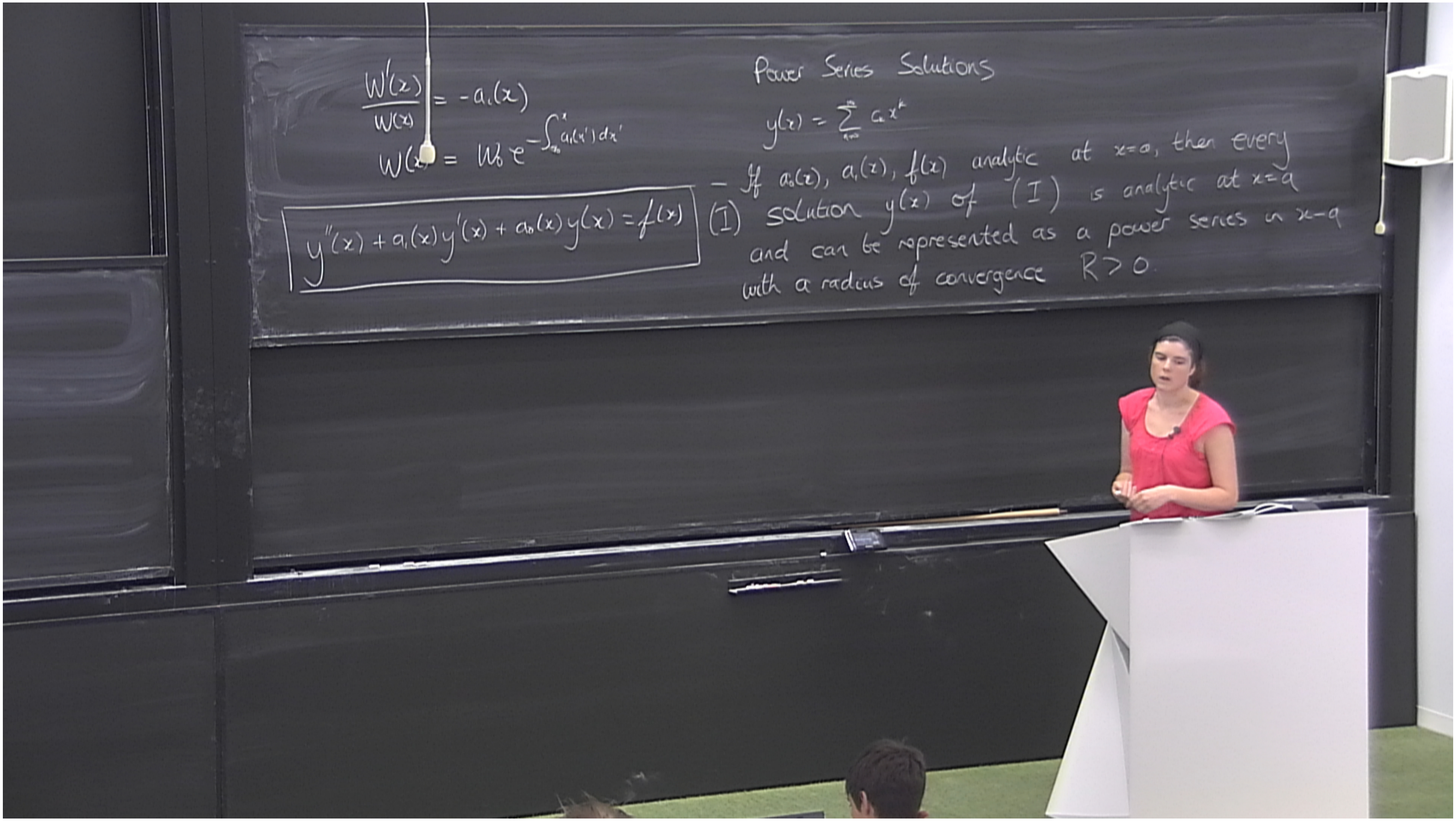
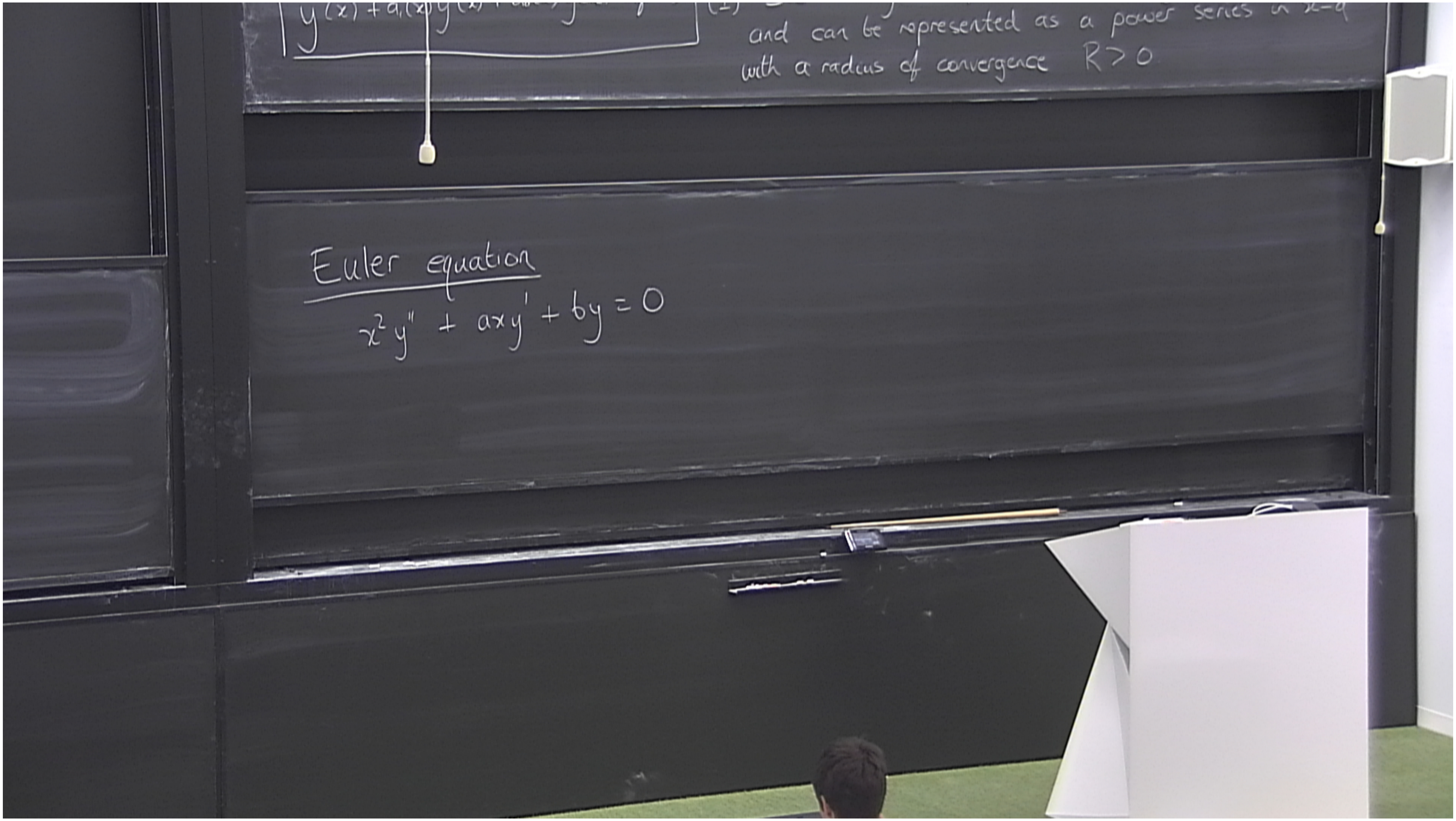
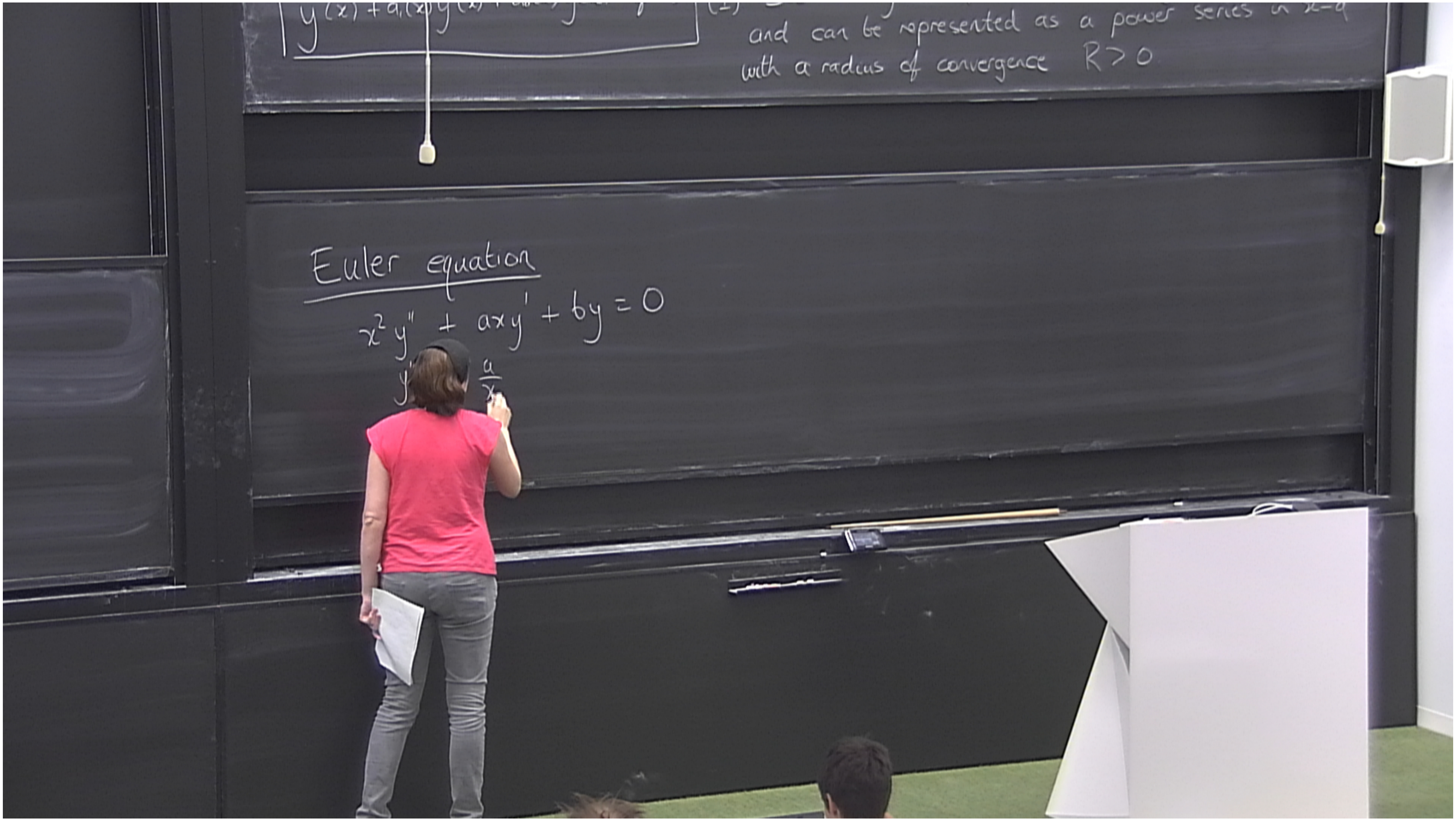
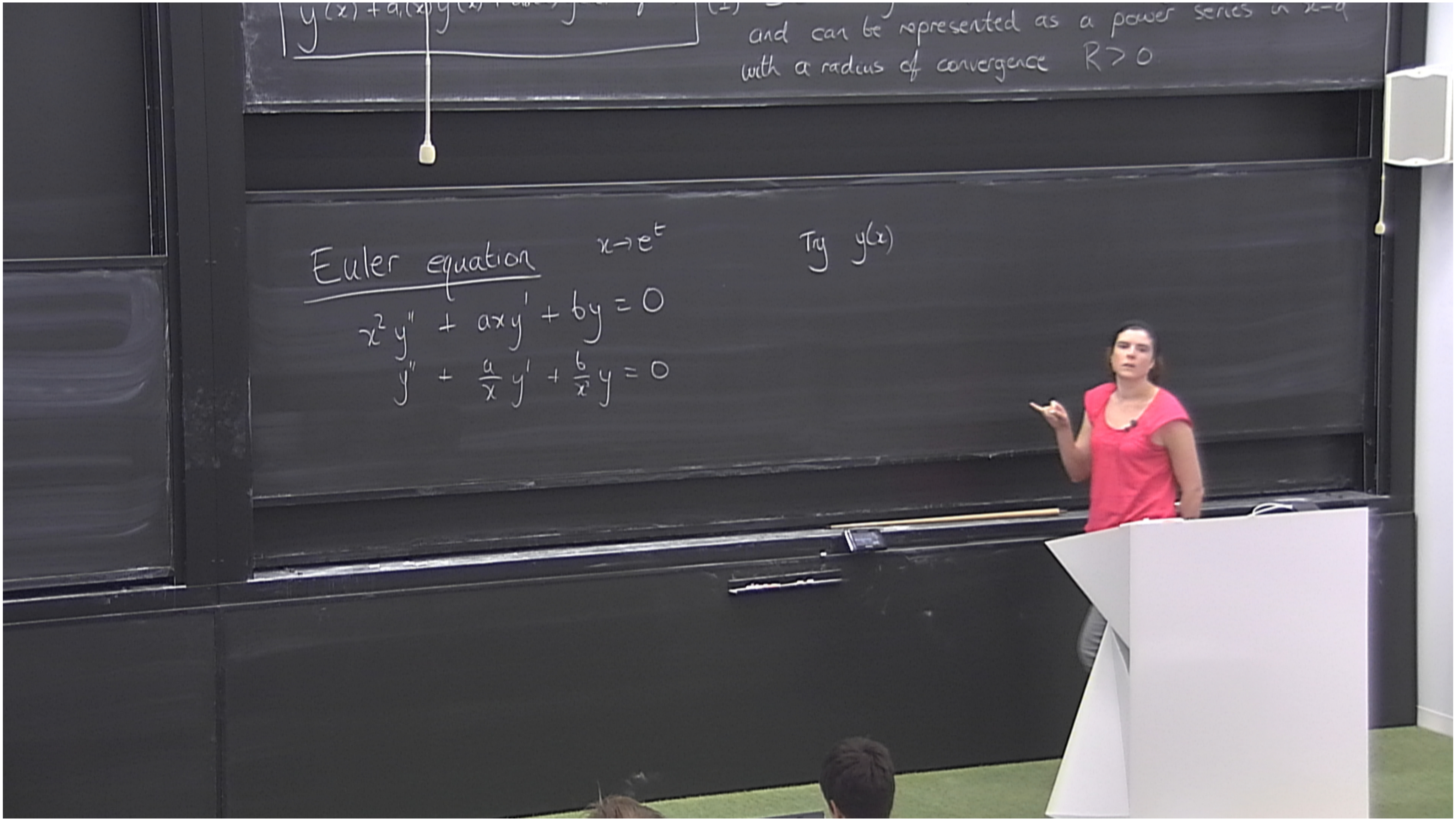
and can be represented as a power series in x-a
with a radius of convergence R > 0.

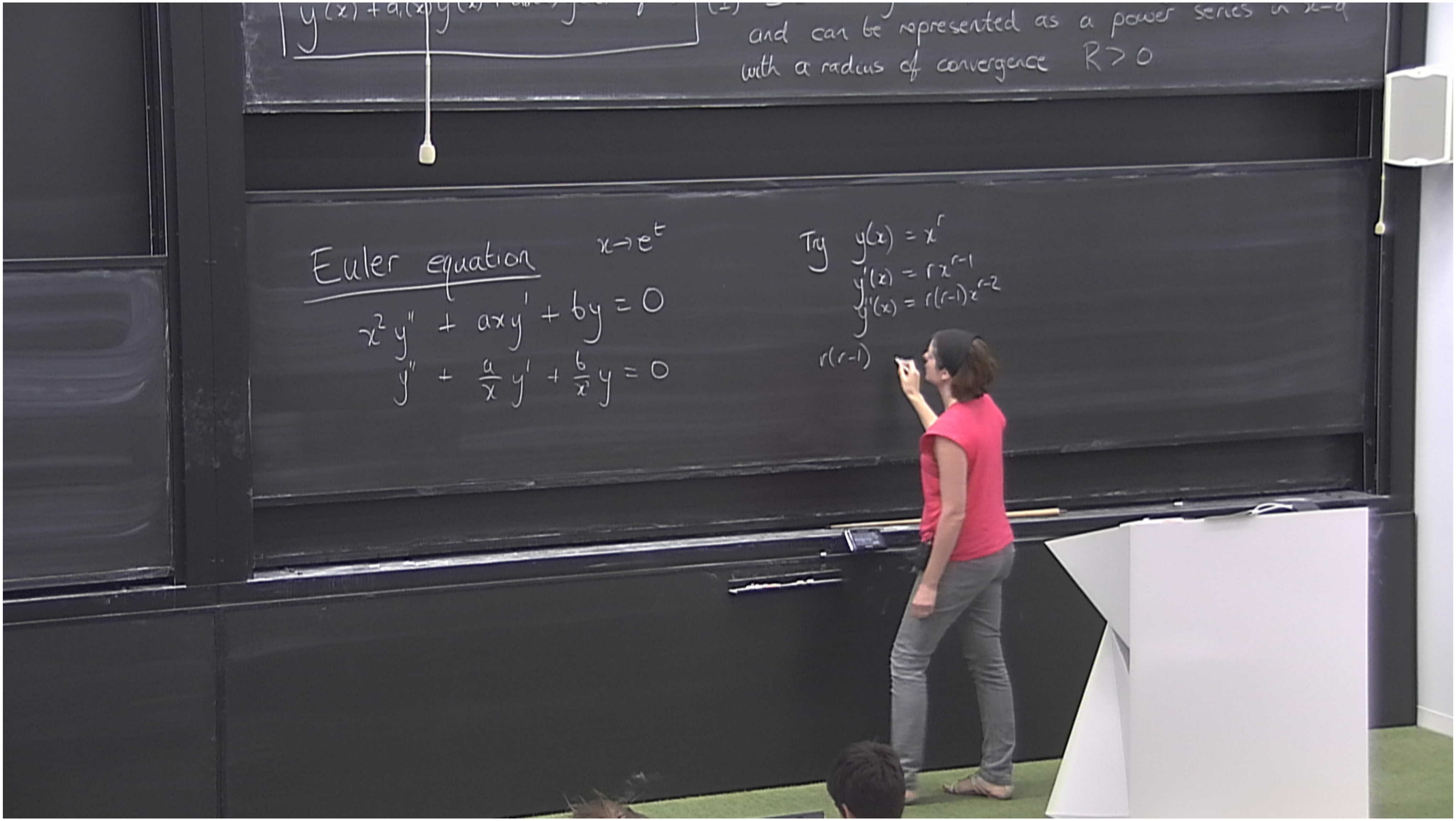


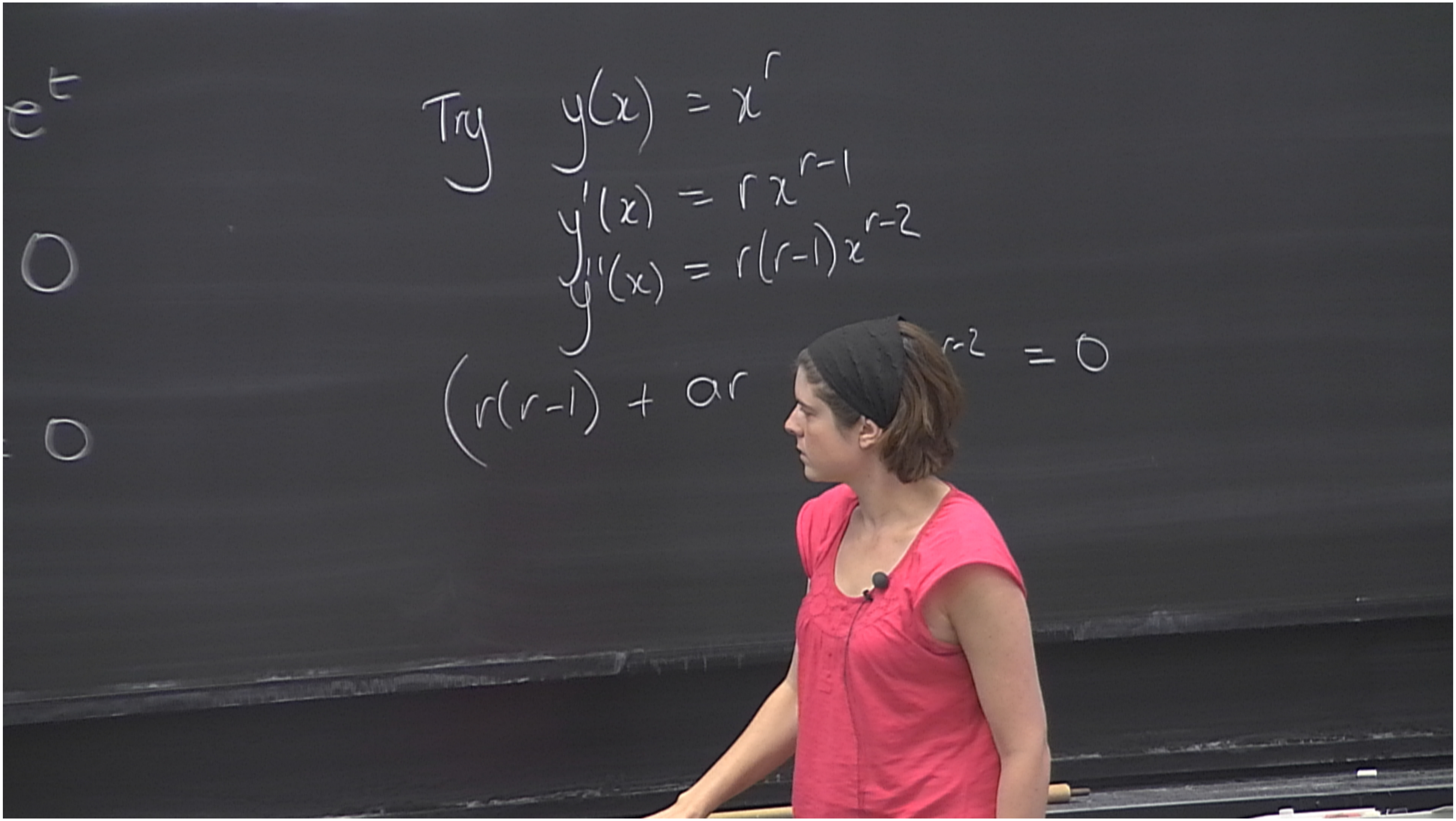
$\frac{W'(x)}{W(x)} = -a_1(x)$
$W(x) = W_0 e^{-\int_{x_0}^{x} a_1(x')dx'}$
$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x)$ (I)
Power Series Solutions
$y(x) = \sum_{n=0}^{\infty} c_n x^n$
– If $a_0(x)$, $a_1(x)$, $f(x)$ analytic at $x=a$, then every
solution $y(x)$ of (I) is analytic at $x=a$
and can be represented as a power series in $x-a$
with a radius of convergence $R>0$.

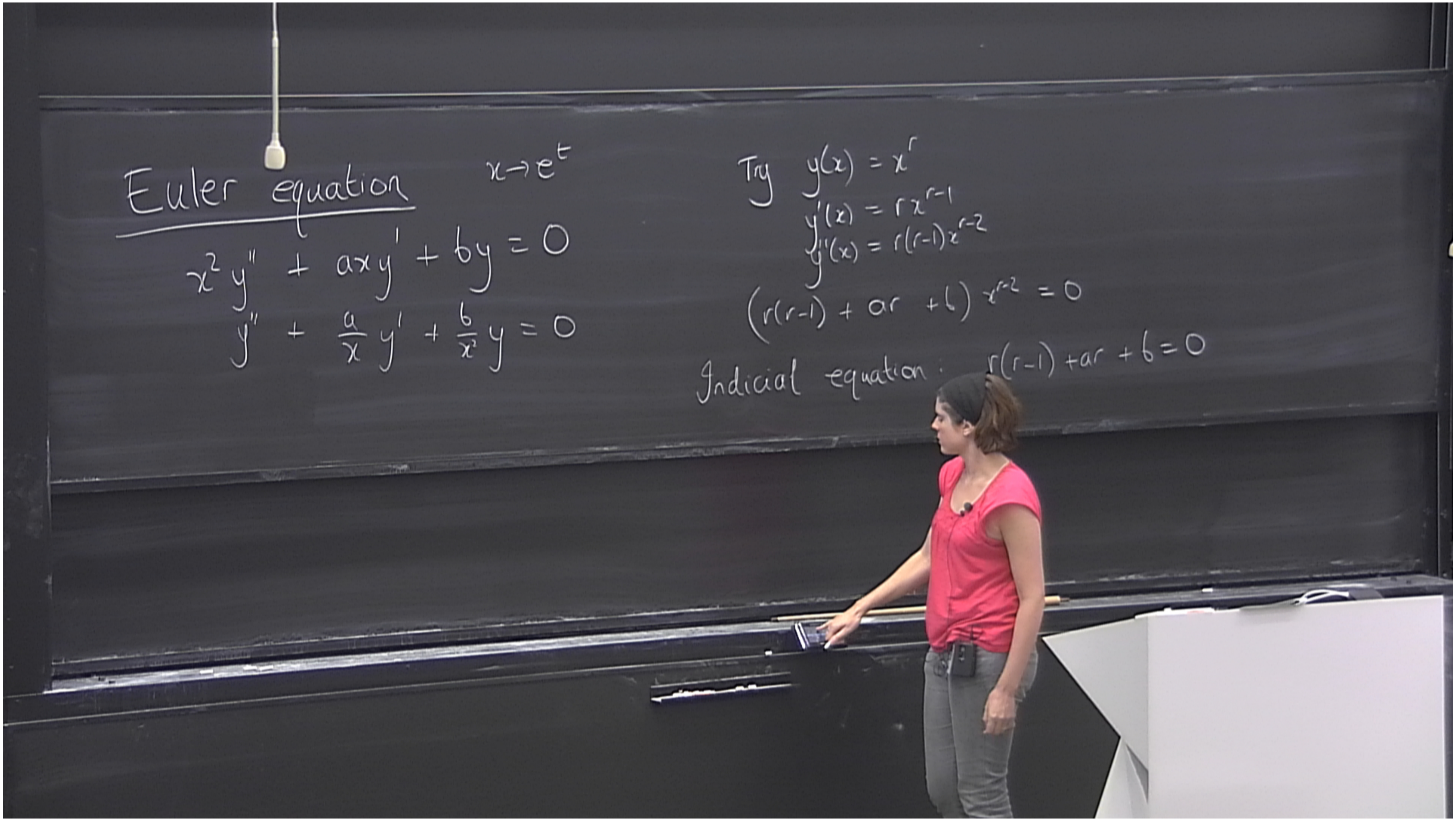

and can be represented as a power series in
with a radius of convergence $R > 0$.
Euler equation
$x^2 y'' + axy' + by = 0$

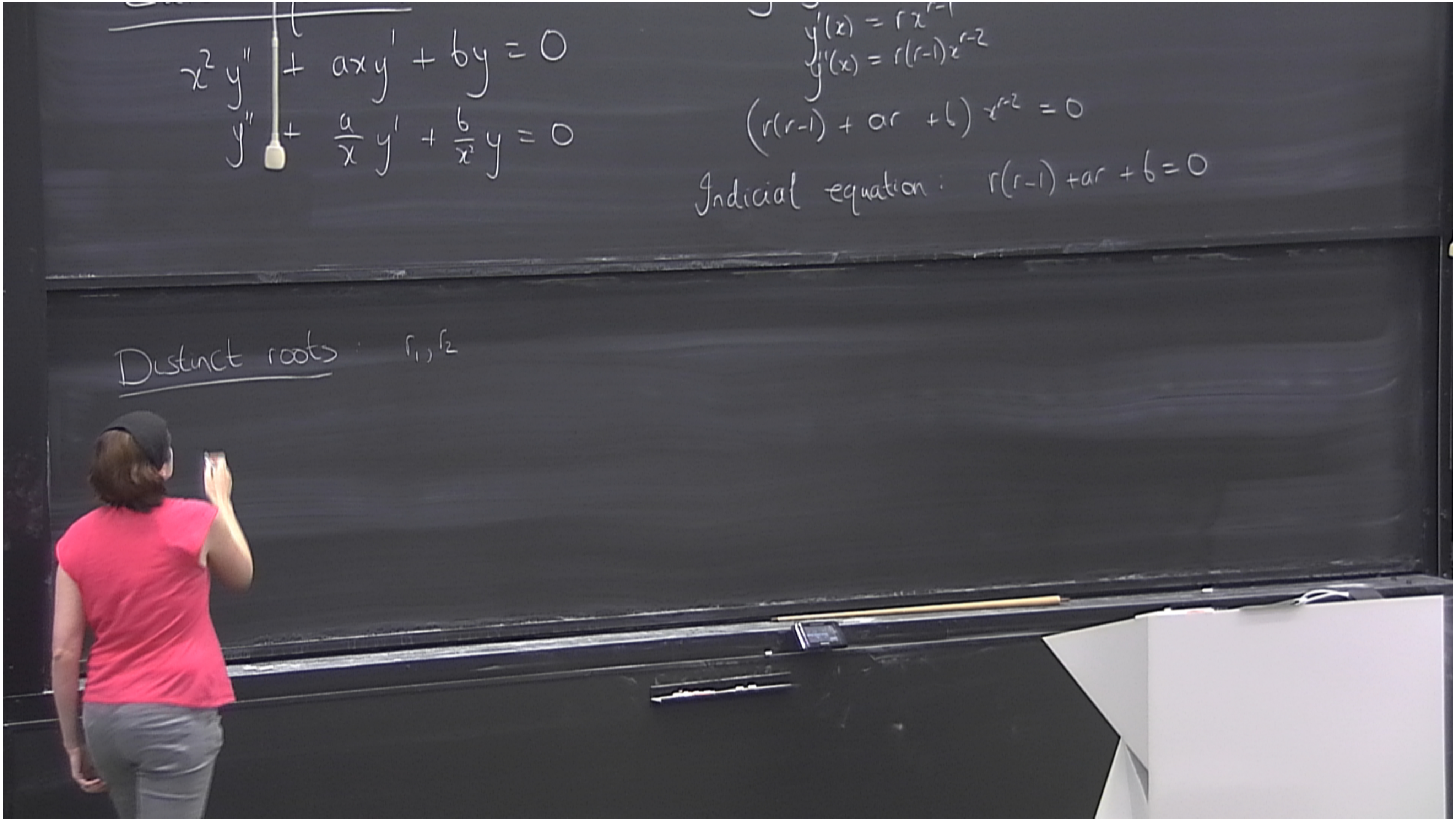

and can be represented as a power series in x−a
with a radius of convergence R > 0
Euler equation
x² y'' + axy' + by = 0

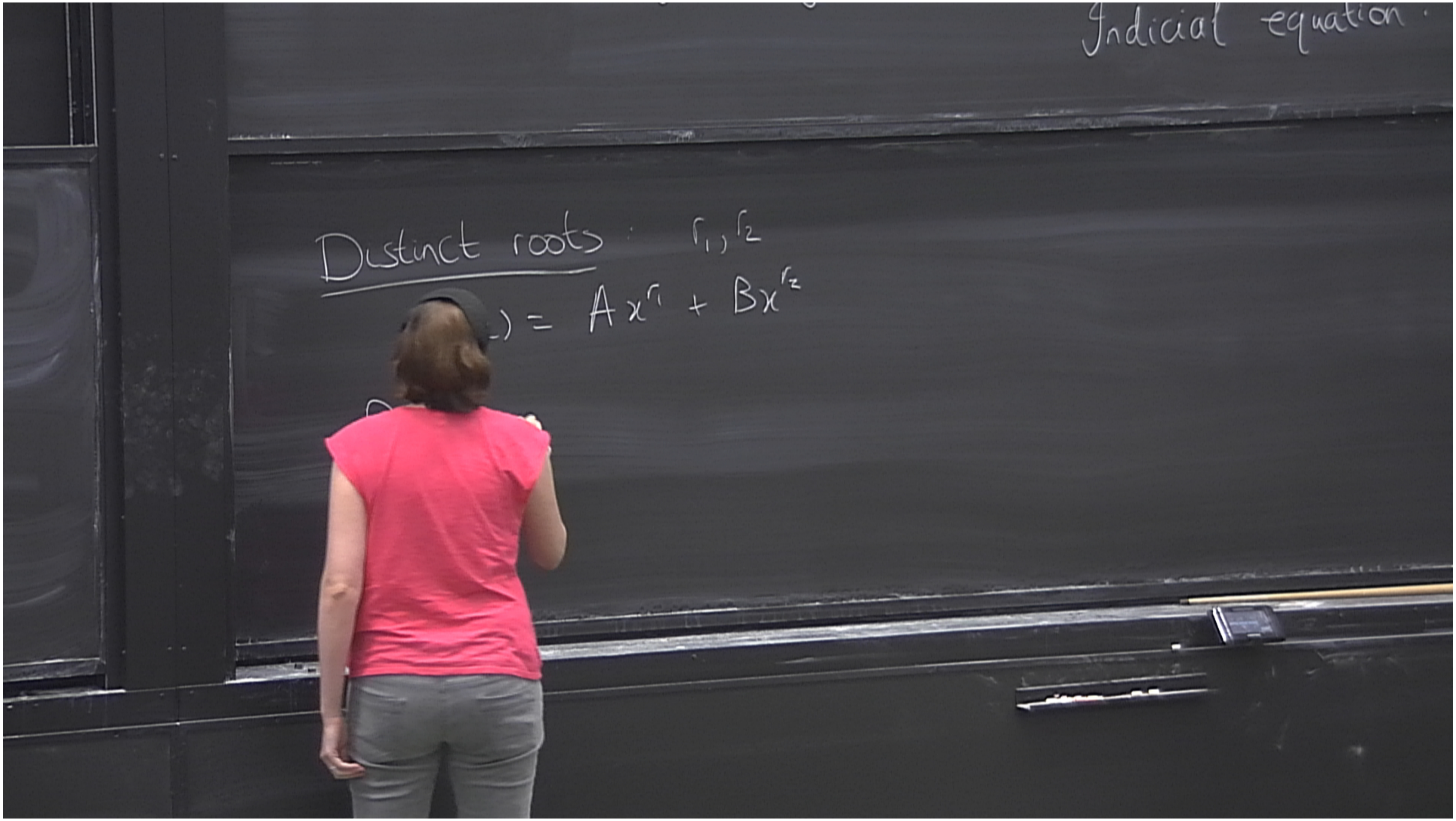

and can be represented as a power series in x−a
with a radius of convergence R > 0.
Euler equation
x → e^t
Try y(x)
x²y'' + axy' + by = 0
y'' + (a/x) y' + (b/x²) y = 0


and can be represented as a power series in $x-a$
with a radius of convergence $R > 0$
Euler equation
$x \to e^t$
$x^2 y'' + axy' + by = 0$
$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0$
Try $y(x) = x^r$
$y'(x) = r x^{r-1}$
$y''(x) = r(r-1)x^{r-2}$
$r(r-1)$


e^t
0
0
Try $y(x) = x^r$
$y'(x) = r x^{r-1}$
$y''(x) = r(r-1)x^{r-2}$
$(r(r-1) + ar$
$^{r-2} = 0$


Euler equation
$x \to e^t$
$x^2 y'' + a x y' + b y = 0$
$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0$
Try $y(x) = x^r$
$y'(x) = r x^{r-1}$
$y''(x) = r(r-1) x^{r-2}$
$(r(r-1) + ar + b)\, x^{r-2} = 0$
Indicial equation: $r(r-1) + ar + b = 0$


$x^2 y'' + a x y' + b y = 0$
$y'' + \frac{a}{x} y' + \frac{b}{x^2} y = 0$
$y'(x) = r x^{r-1}$
$y''(x) = r(r-1) x^{r-2}$
$(r(r-1) + ar + b) x^{r-2} = 0$
Indicial equation: $r(r-1) + ar + b = 0$
Distinct roots: r_1, r_2


Indicial equation.
Distinct roots : r_1, r_2
$) = Ax^{r_1} + Bx^{r_2}$

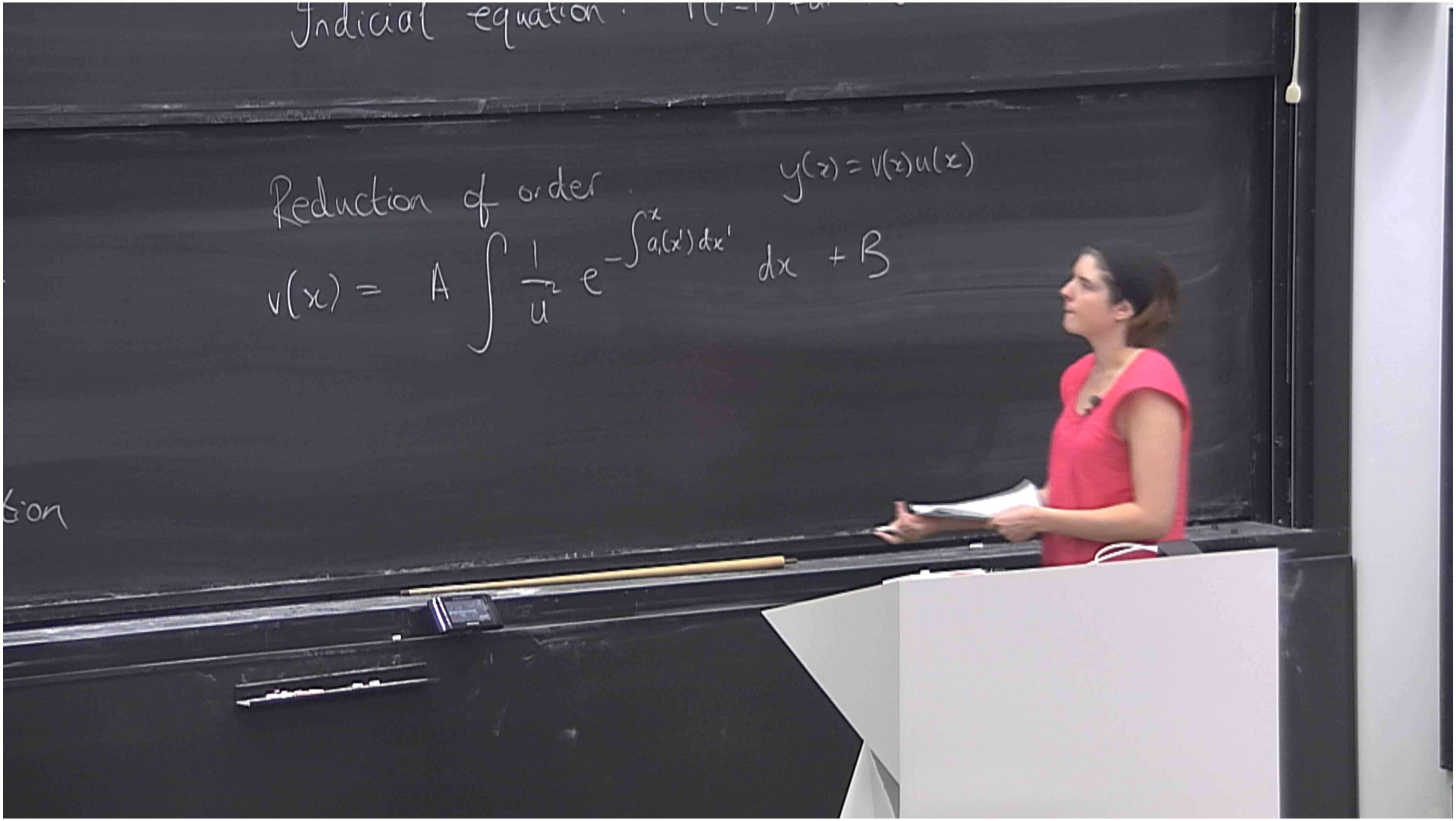
Indicial equation:

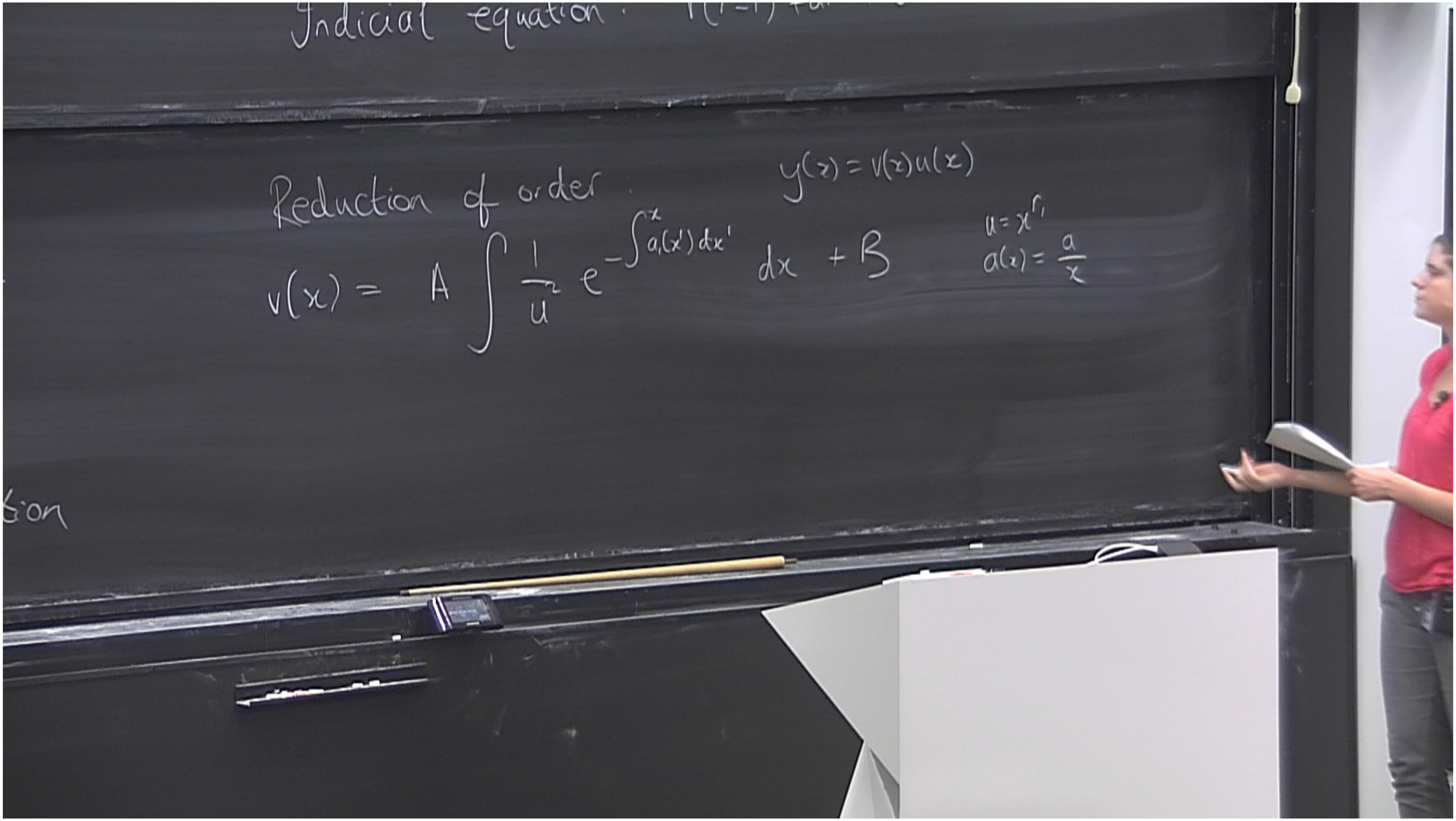
Distinct roots : r_1, r_2

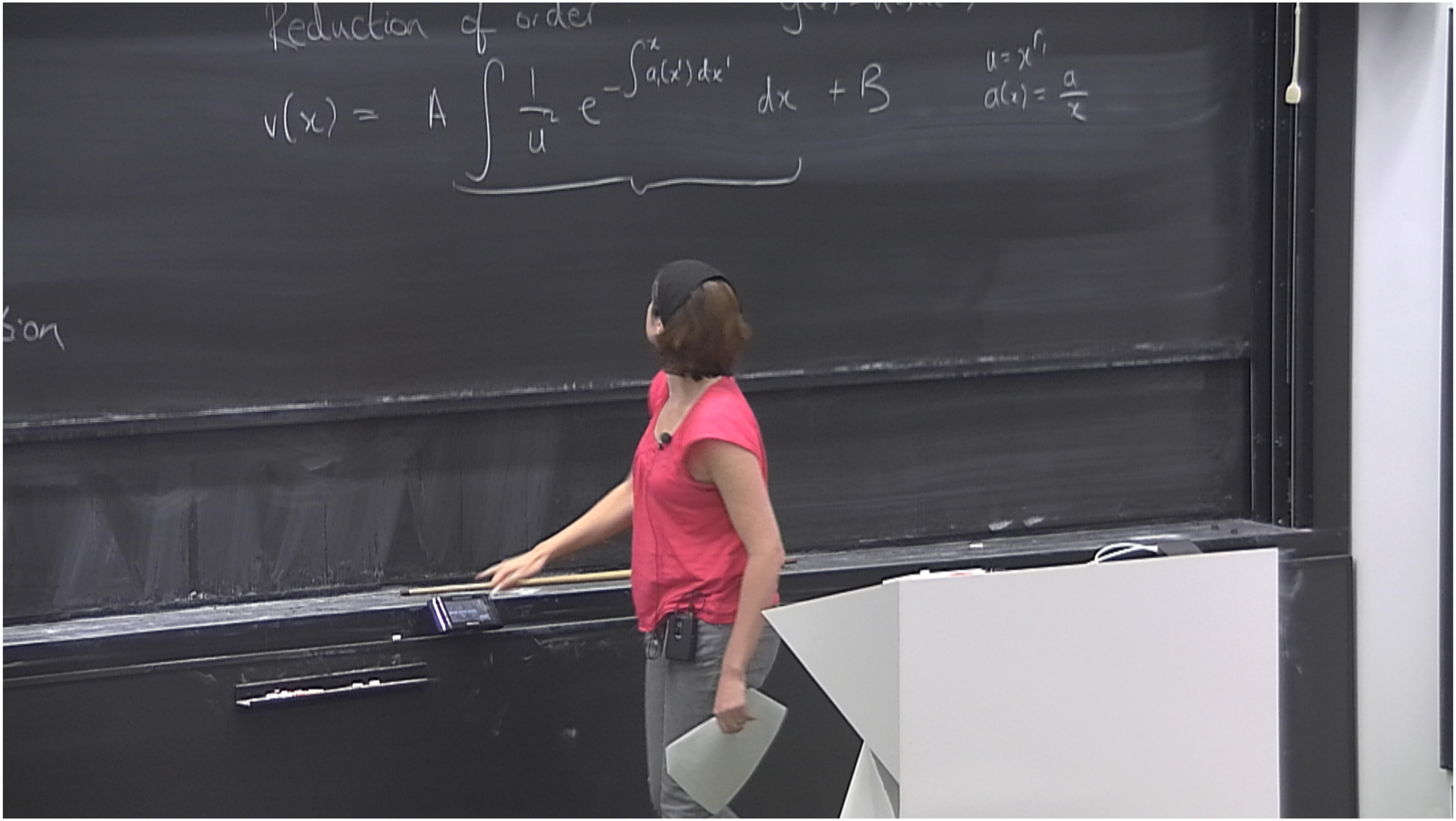
$$y(x) = A x^{r_1} + B x^{r_2}$$

Double root $r = r_1$

$u_1(x) = x^{r_1}$ is a solution


Indicial equation:
Reduction of order
y(x) = v(x)u(x)
v(x) = A ∫ 1/u² e^(−∫^x a₁(x') dx') dx + B


Indicial equation:
Reduction of order
$y(x) = v(x)u(x)$
$v(x) = A \int \frac{1}{u^2} e^{-\int^x a_1(x')\,dx'} dx + B$
$u = x^{r_1}$
$a(x) = \frac{a}{x}$


Reduction of order
$v(x) = A \int \frac{1}{u^2} e^{-\int^x a_1(x')\,dx'}\,dx + B$
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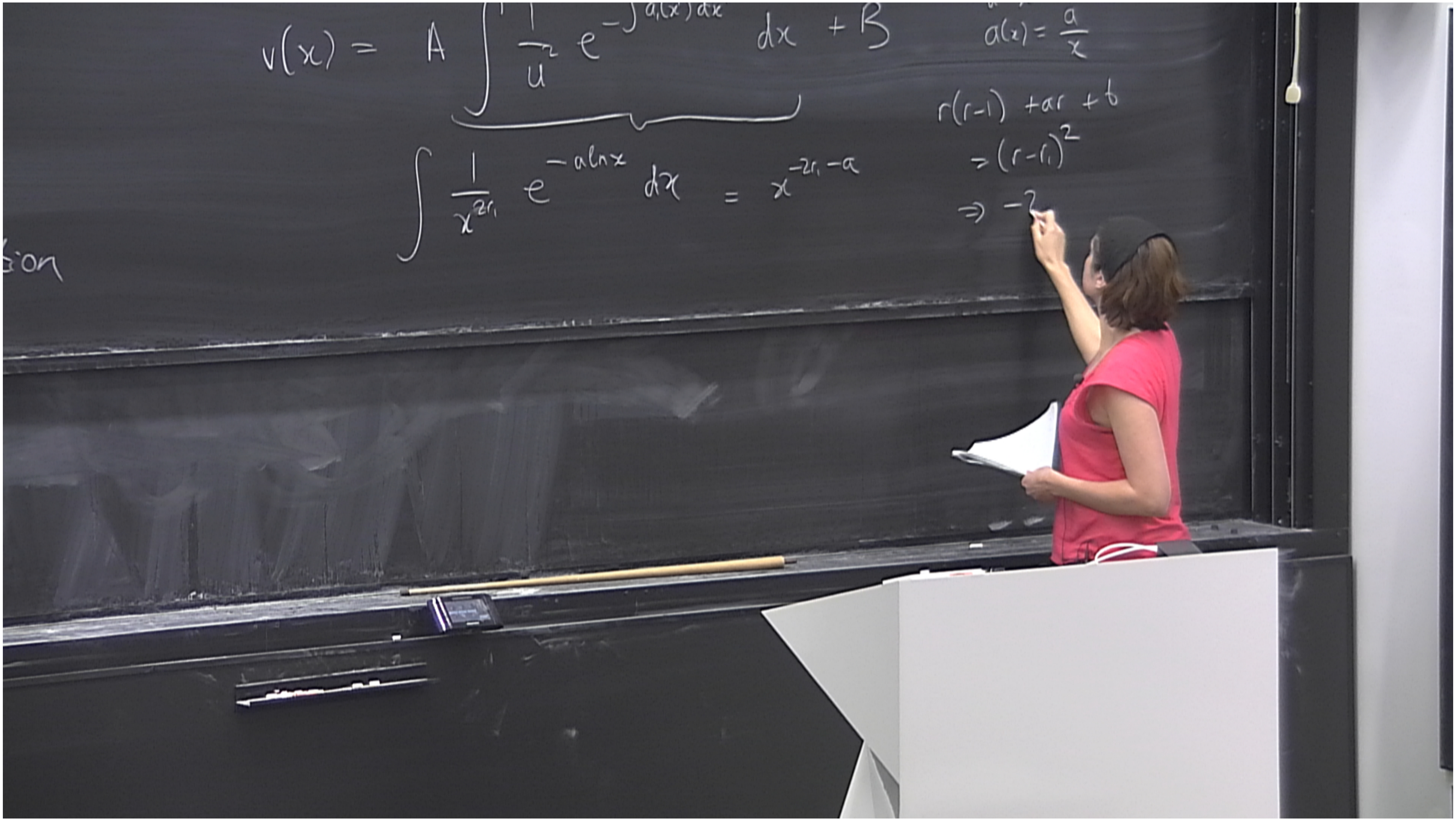
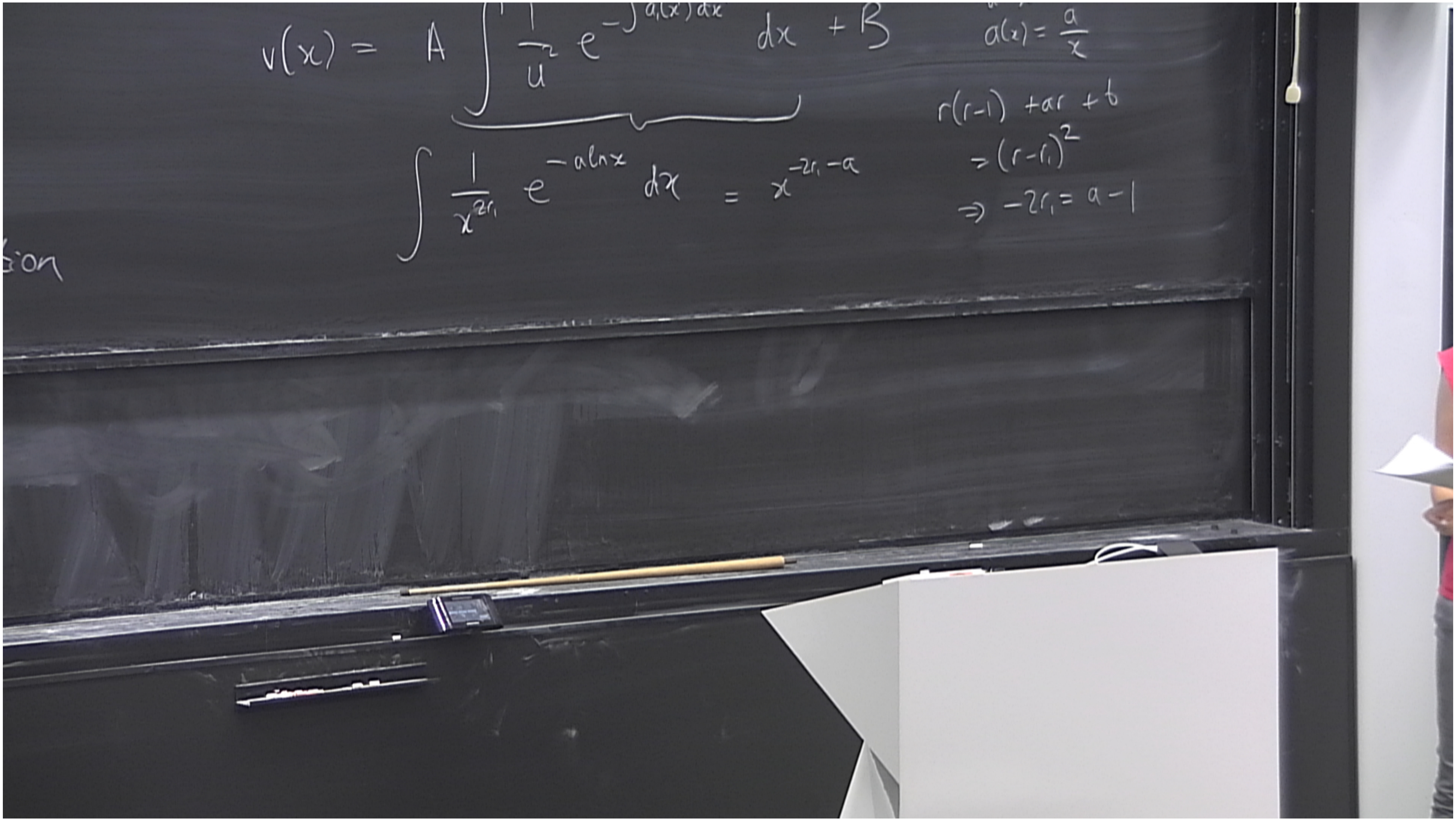
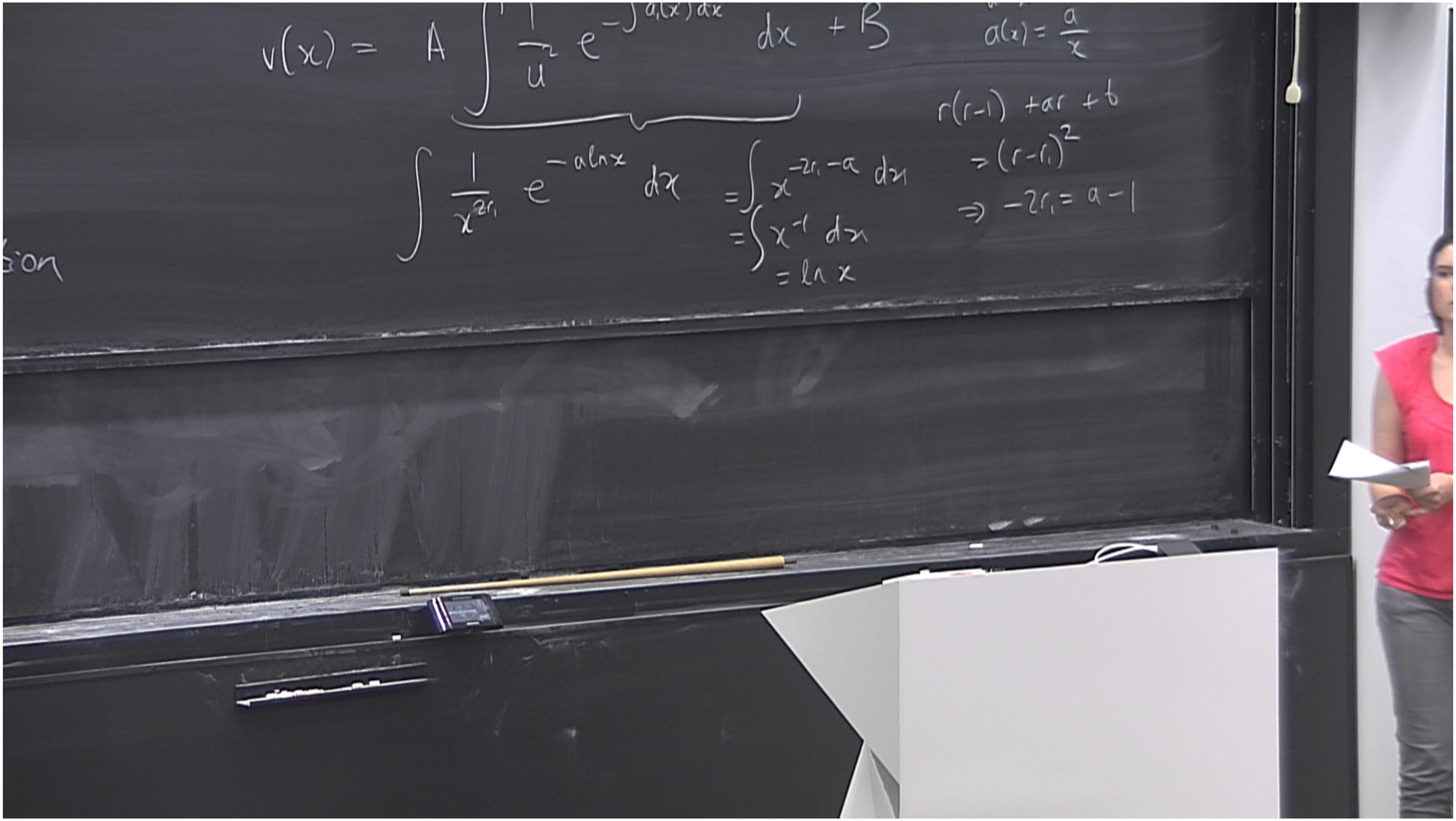
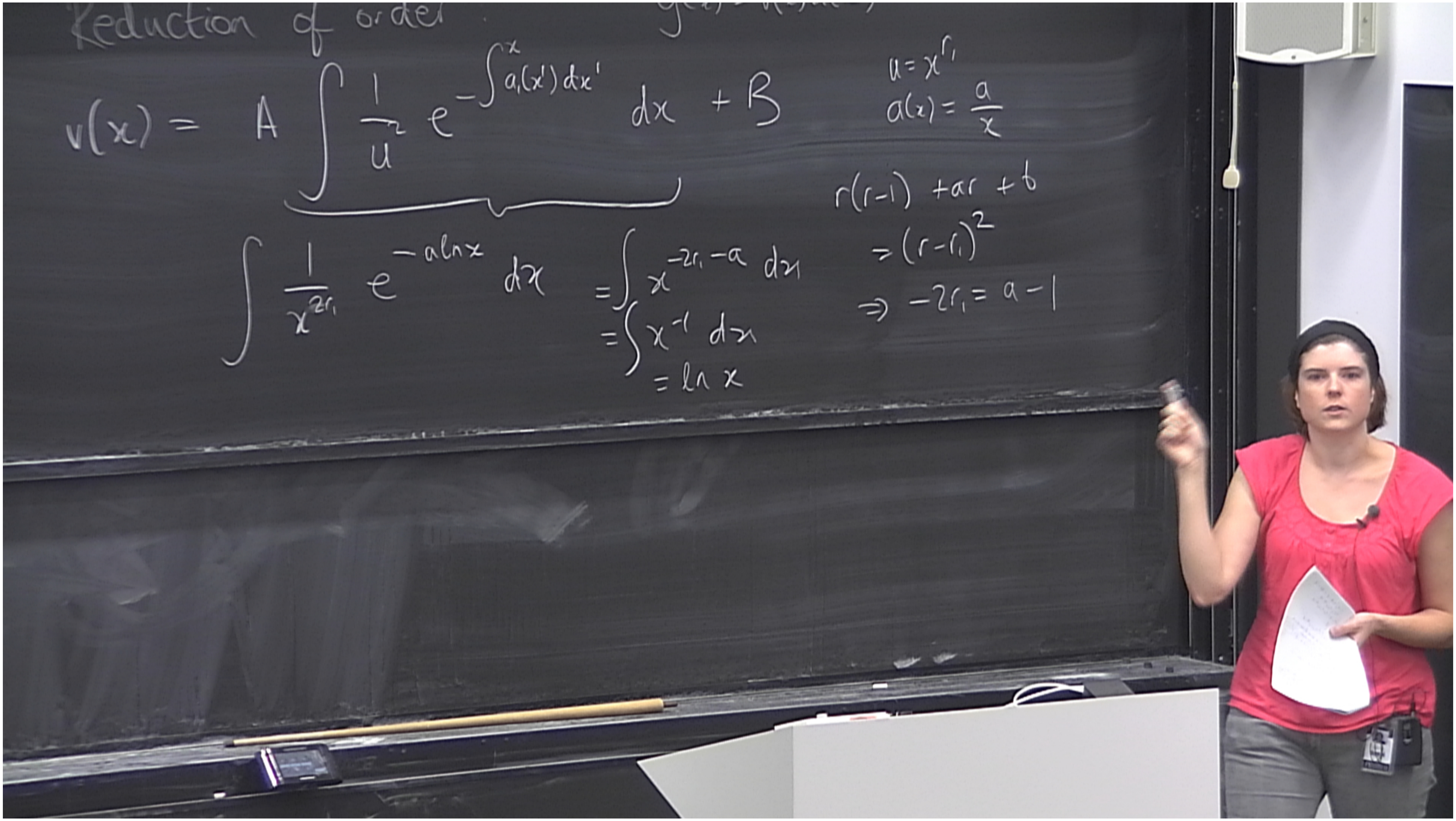
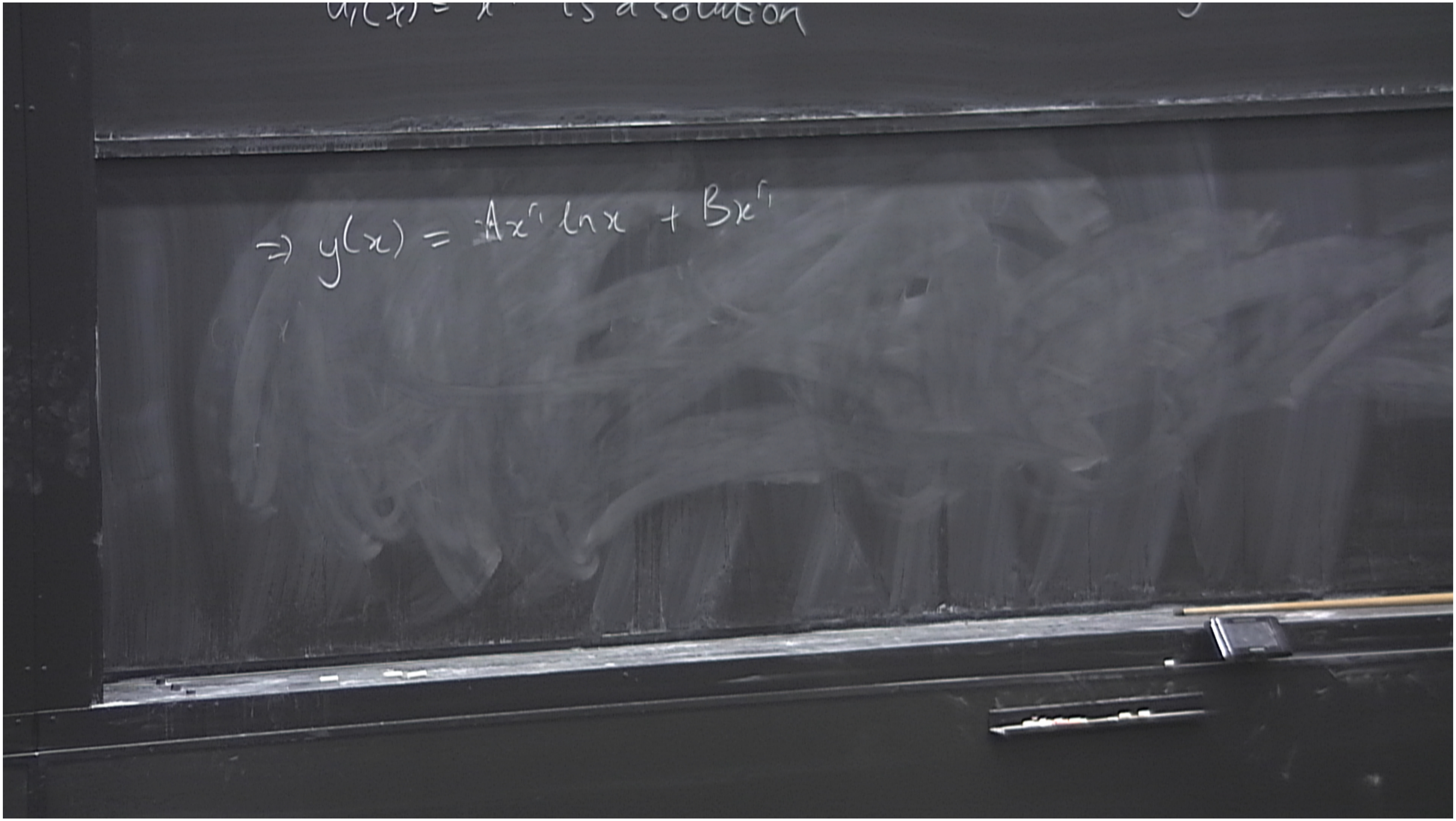

v(x) = A ∫ 1/u² e^(−∫a₁(x)dx) dx + B
a(x) = a/x
r(r−1) + ar + b
⇒ (r−r₁)²
∫ 1/x^(2r₁) e^(−a ln x) dx = x^(−2r₁−a)
⇒ −2
tion

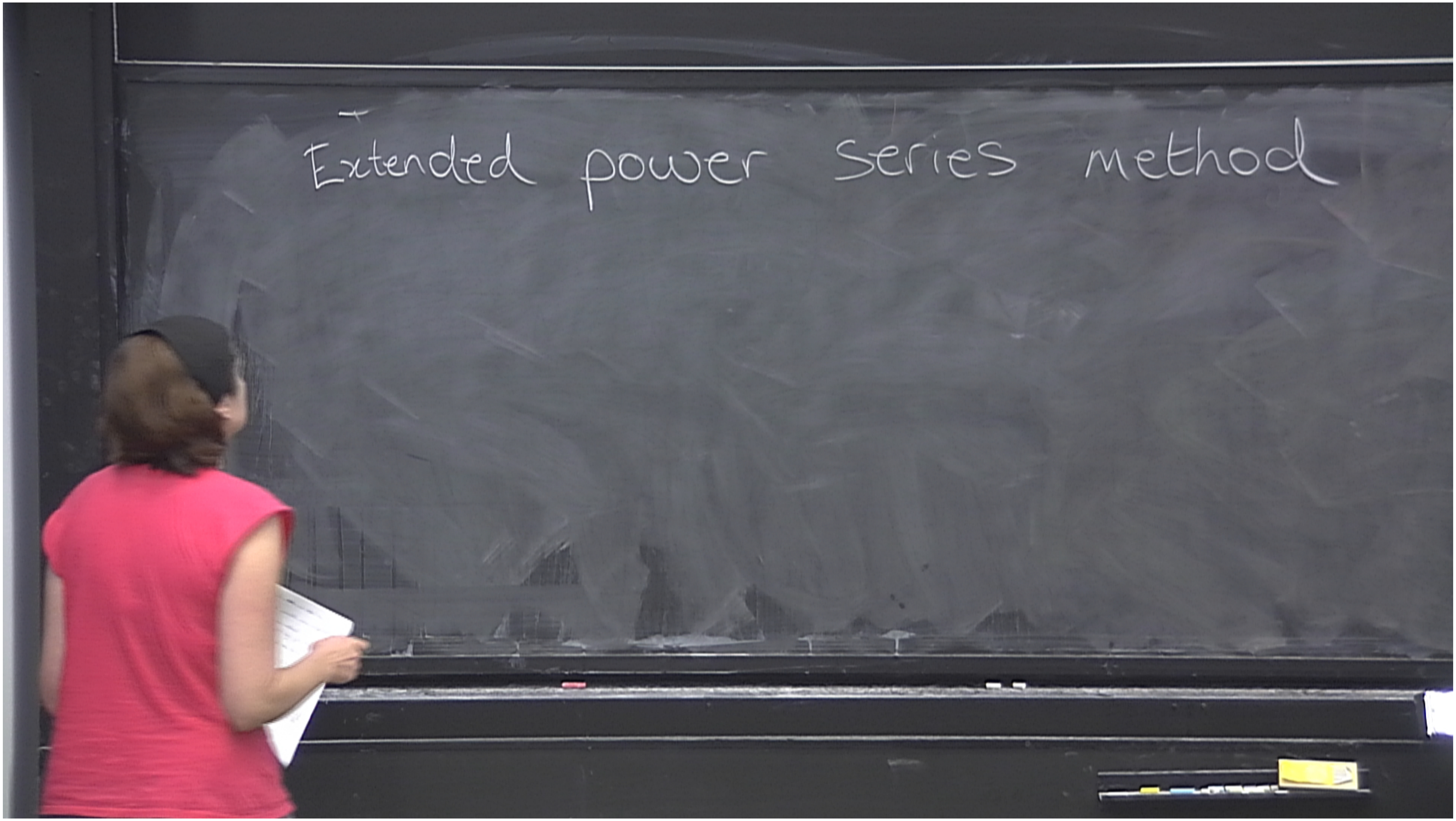

$v(x) = A \int \frac{1}{u^2} e^{-\int a_1(x)\,dx}\, dx + B$
$a(x) = \frac{a}{x}$
$r(r-1) + ar + b$
$\Rightarrow (r-r_1)^2$
$\int \frac{1}{x^{2r_1}} e^{-a \ln x}\, dx = x^{-2r_1 - a}$
$\Rightarrow -2r_1 = a - 1$

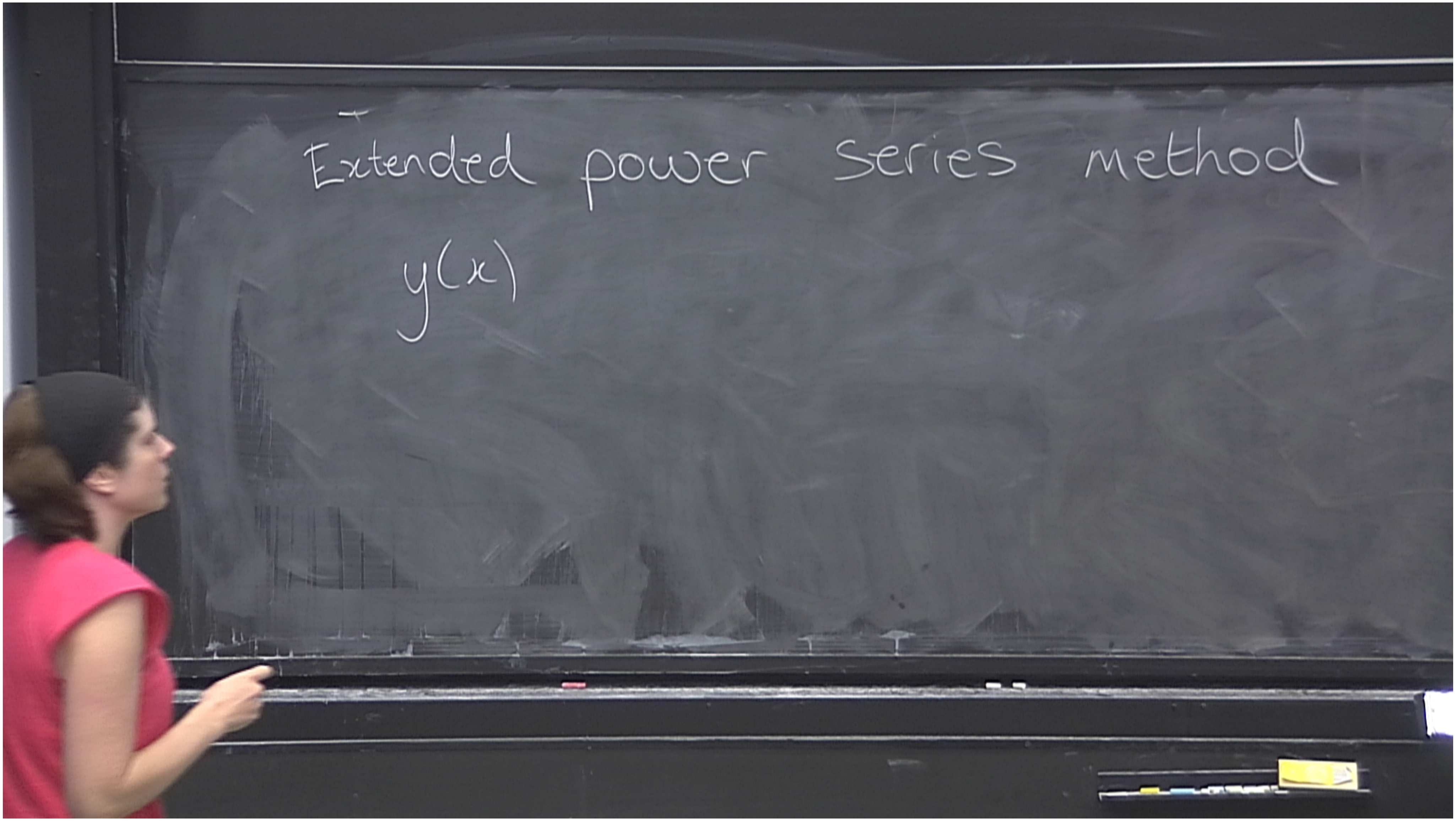


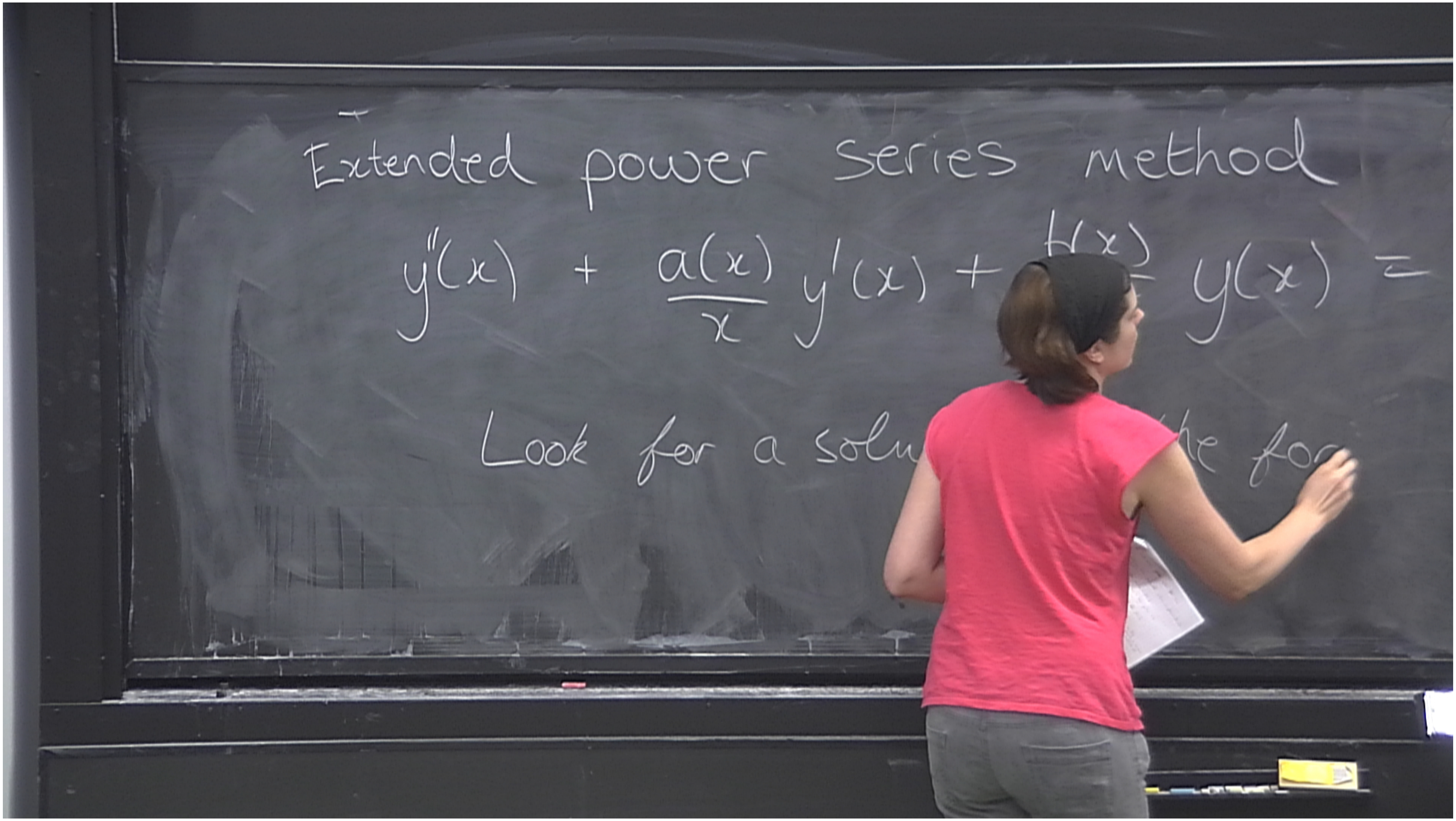
$v(x) = A \int \frac{1}{u^2} e^{-\int a_1(x)\,dx}\,dx + B$
$a(x) = \frac{a}{x}$
$r(r-1) + ar + b$
$\Rightarrow (r-r_1)^2$
$\Rightarrow -2r_1 = a-1$
$\int \frac{1}{x^{2r_1}} e^{-a\ln x}\,dx = \int x^{-2r_1-a}\,dx = \int x^{-1}\,dx = \ln x$
tion

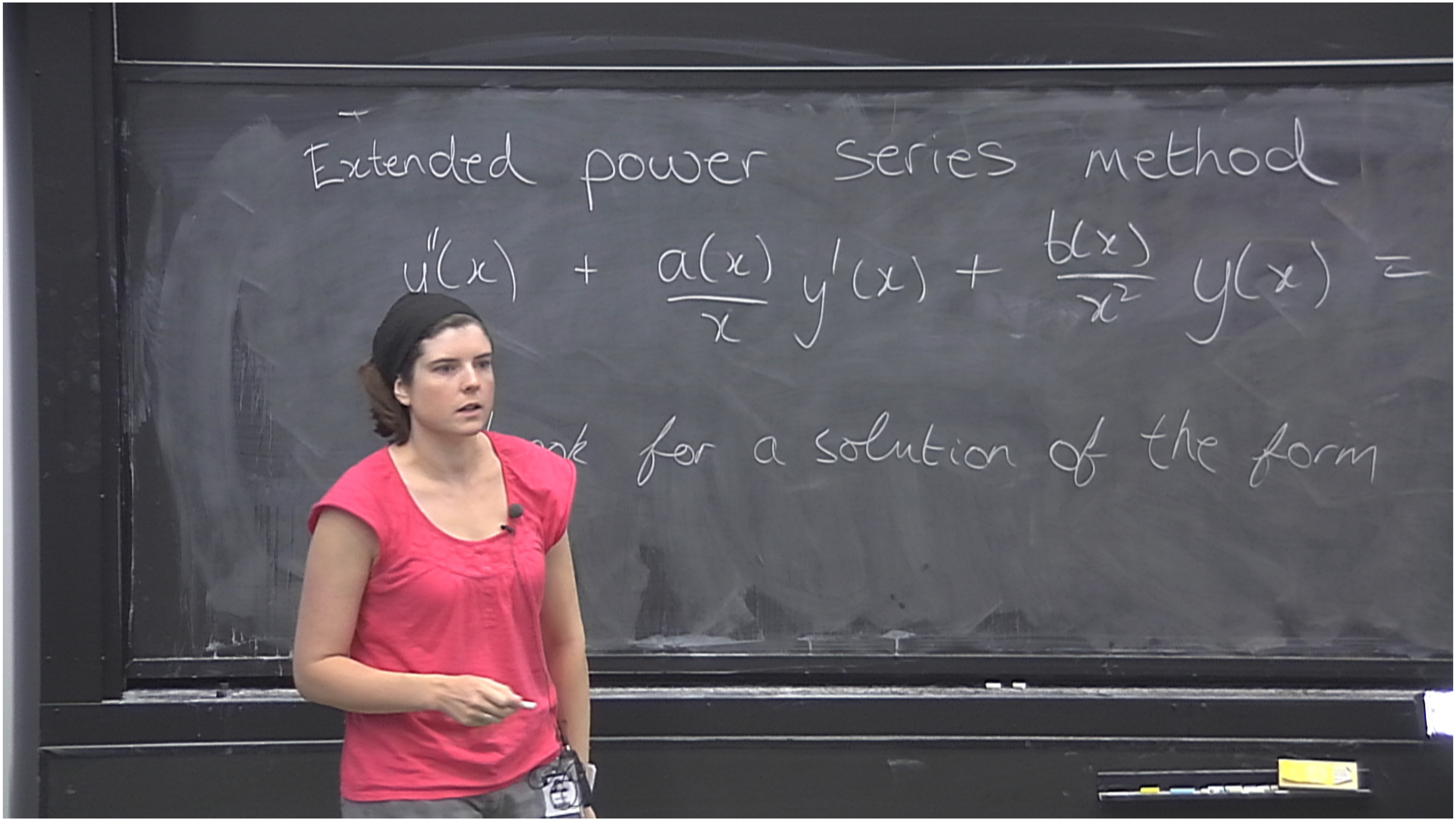

Reduction of order
$v(x) = A \int \frac{1}{u^2} e^{-\int^x a_1(x')\,dx'} dx + B$
$u = x^{r_1}$
$a(x) = \frac{a}{x}$
$r(r-1) + ar + b$
$= (r-r_1)^2$
$\Rightarrow -2r_1 = a-1$
$\int \frac{1}{x^{2r_1}} e^{-a \ln x} dx$
$= \int x^{-2r_1 - a} dx$
$= \int x^{-1} dx$
$= \ln x$

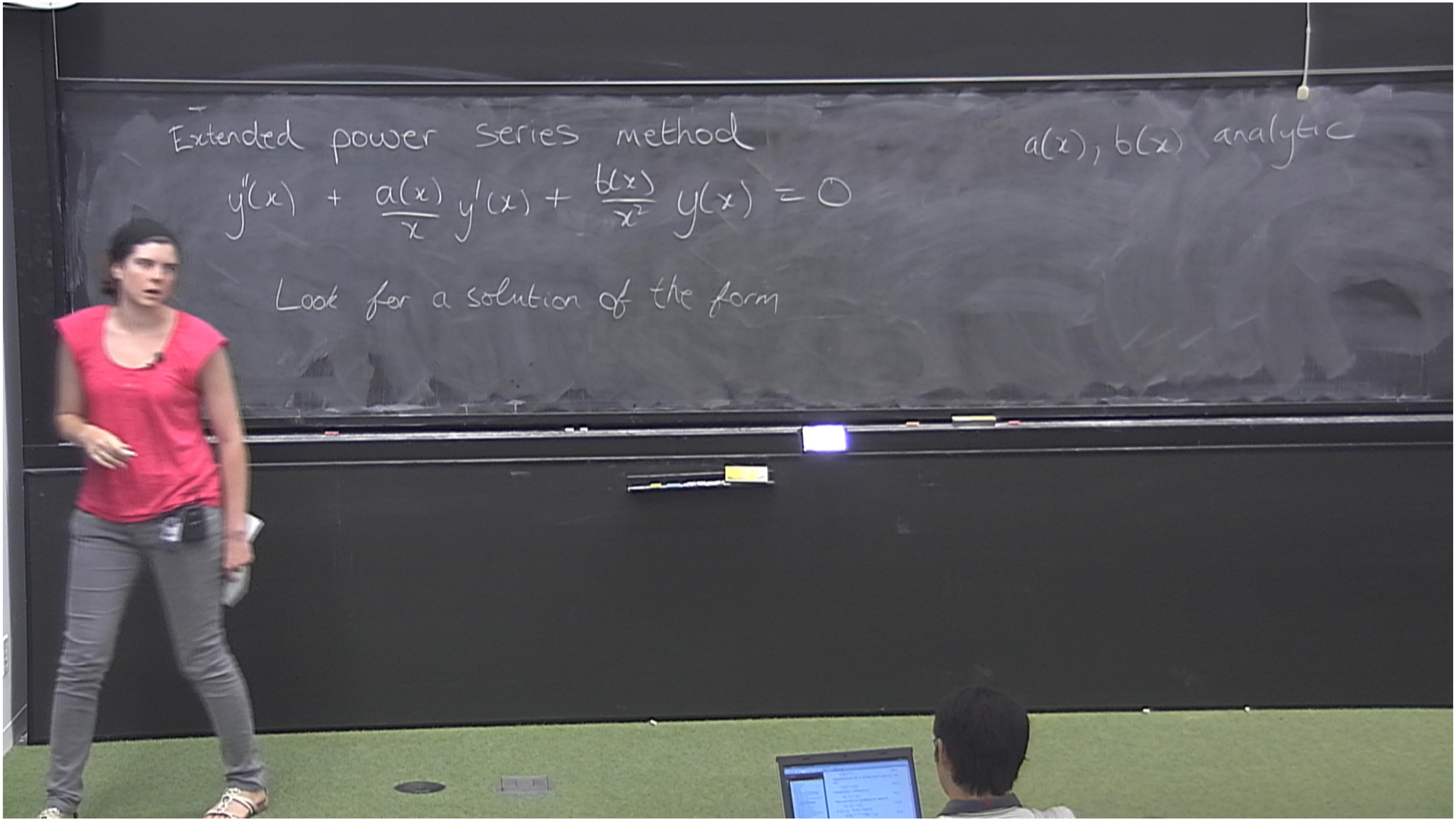

is a solution
⇒ y(x) = Ax^n ln x + Bx^n

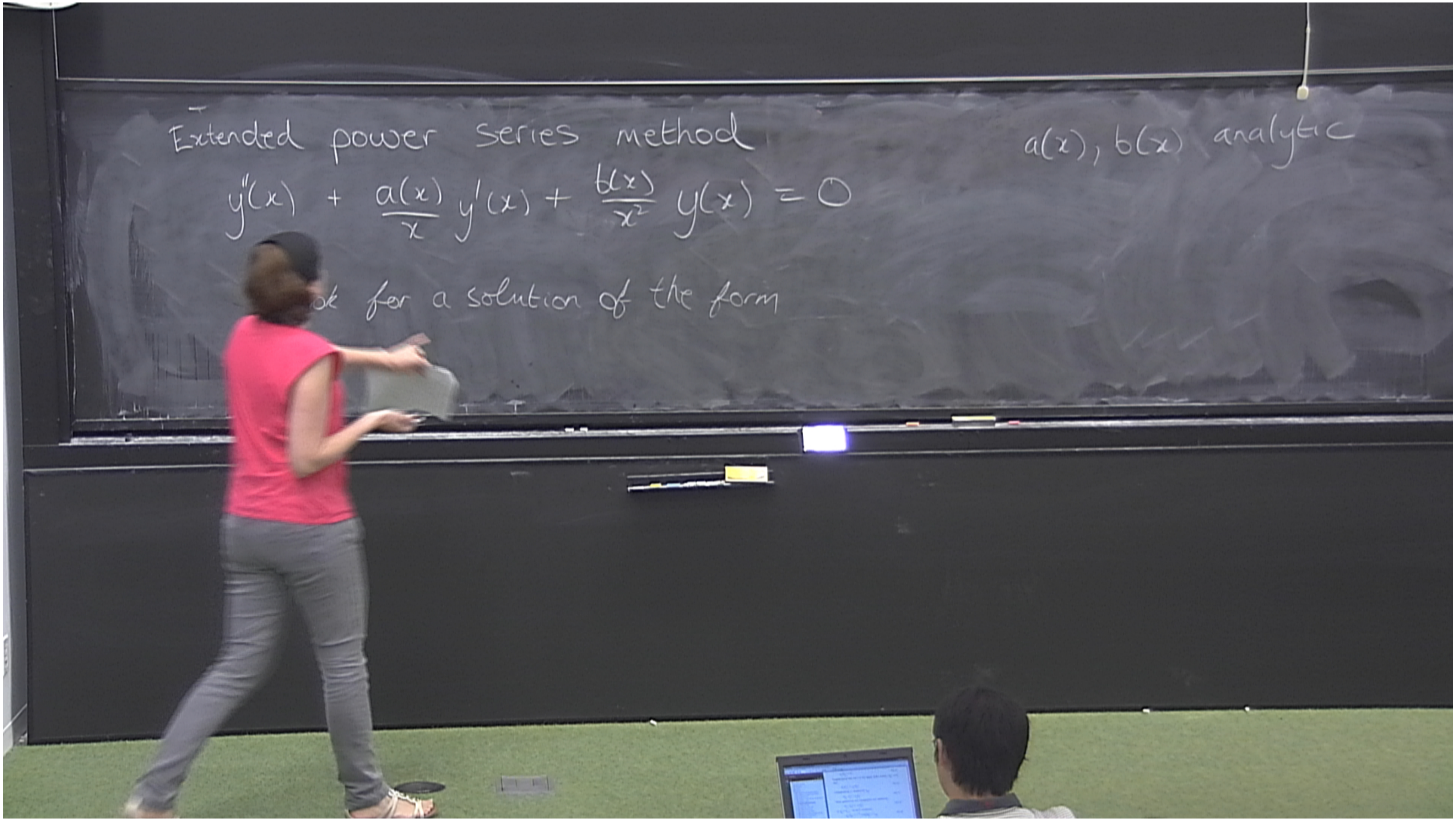

Extended power series method

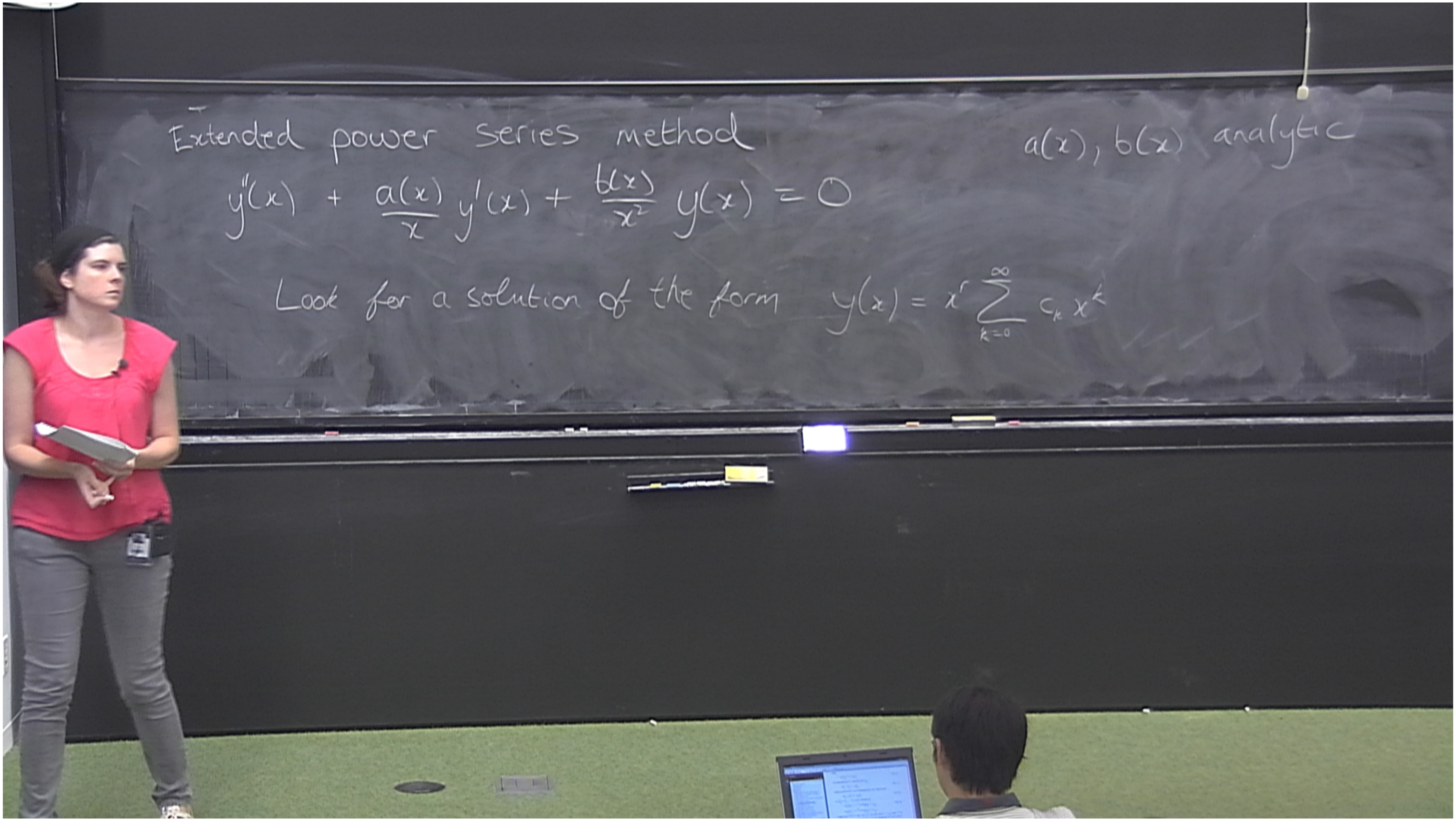

Extended power series method
y(x)


Extended power series method
$y''(x) + \frac{a(x)}{x} y'(x) +$
$y(x) =$
Look for a

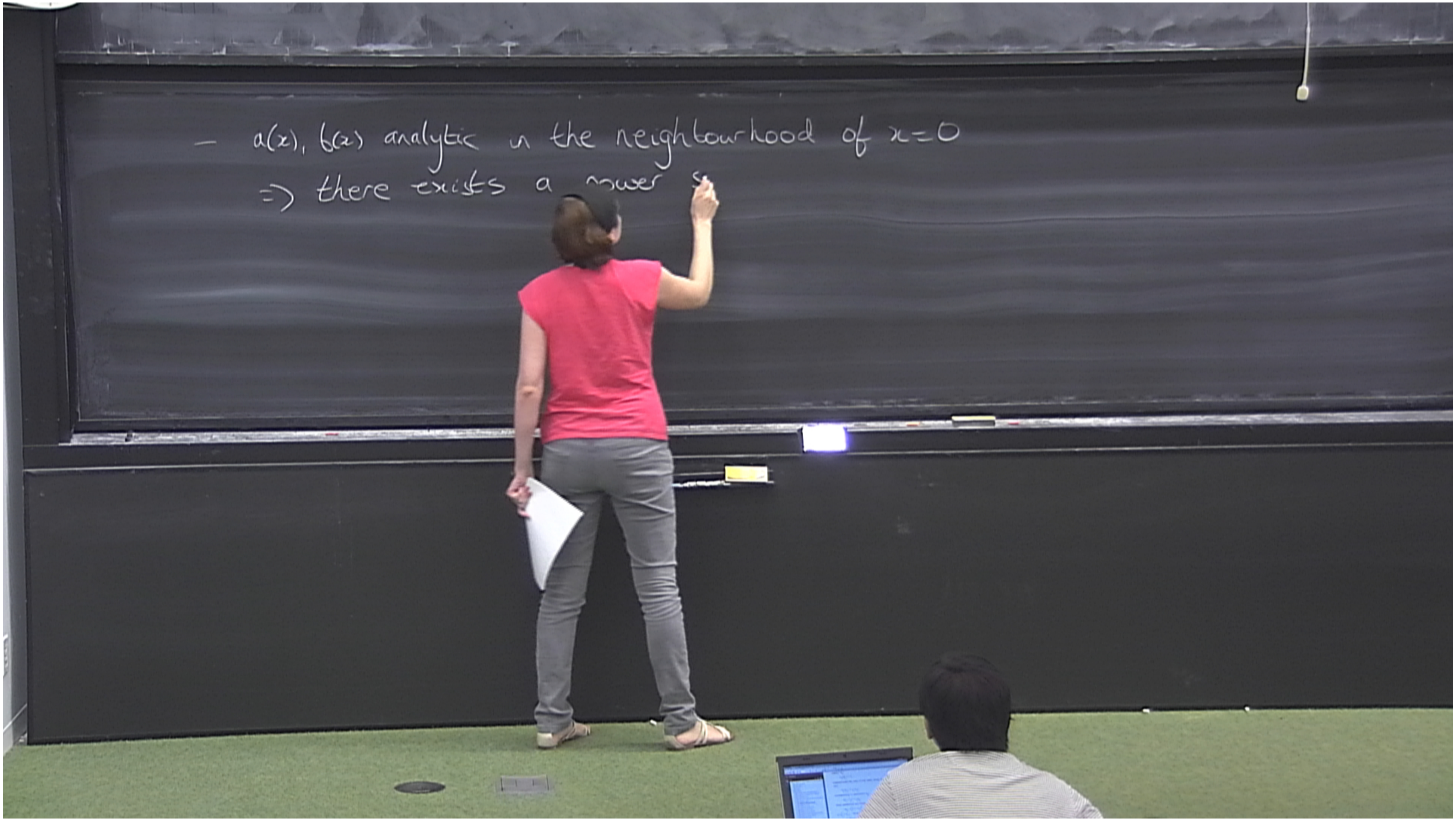

Extended power series method
$y''(x) + \frac{a(x)}{x} y'(x) + \frac{b(x)}{x^2} y(x) =$
for a solution of the form

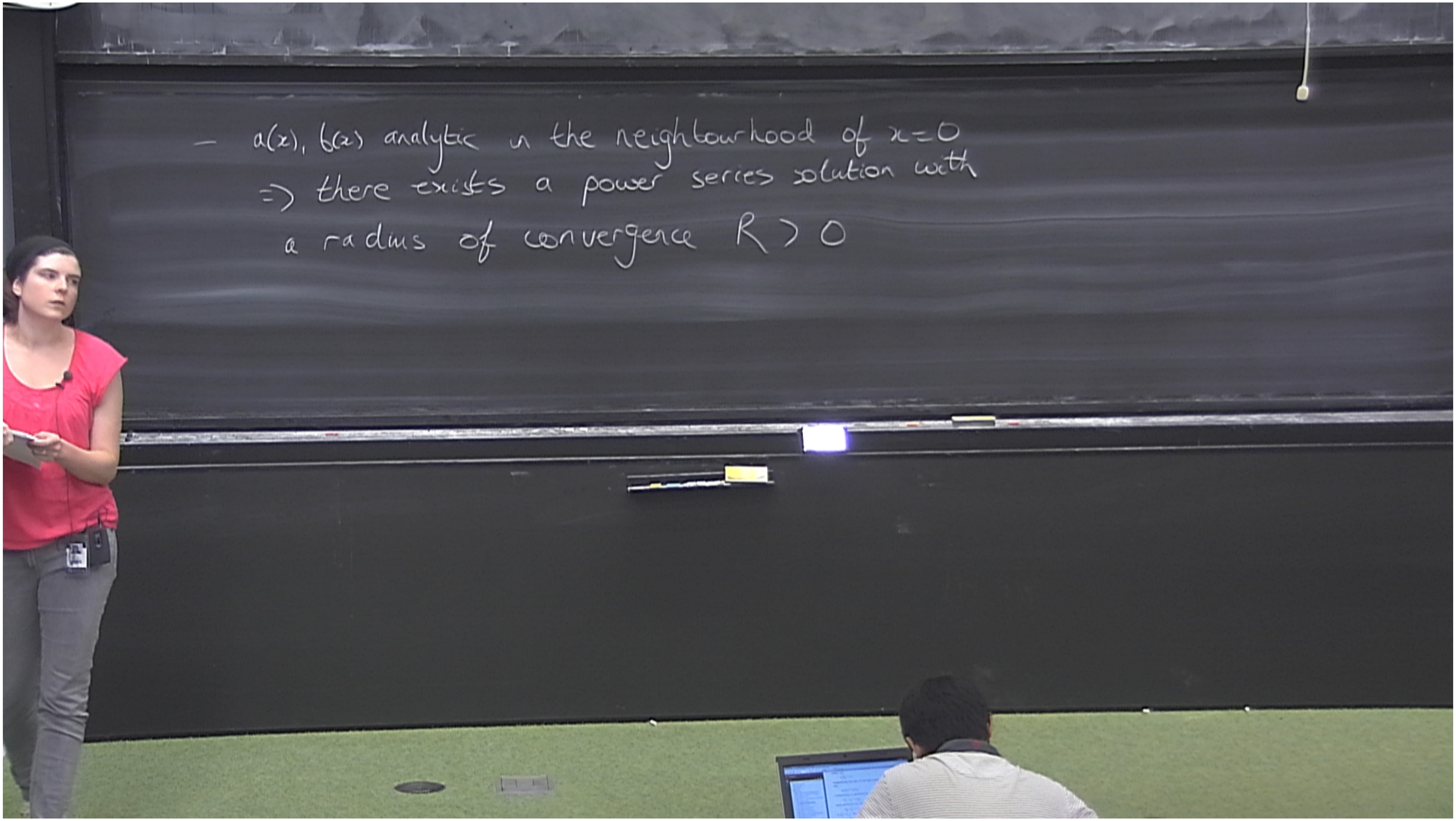

Extended power series method
a(x), b(x) analytic
$y''(x) + \frac{a(x)}{x} y'(x) + \frac{b(x)}{x^2} y(x) = 0$
Look for a solution of the form

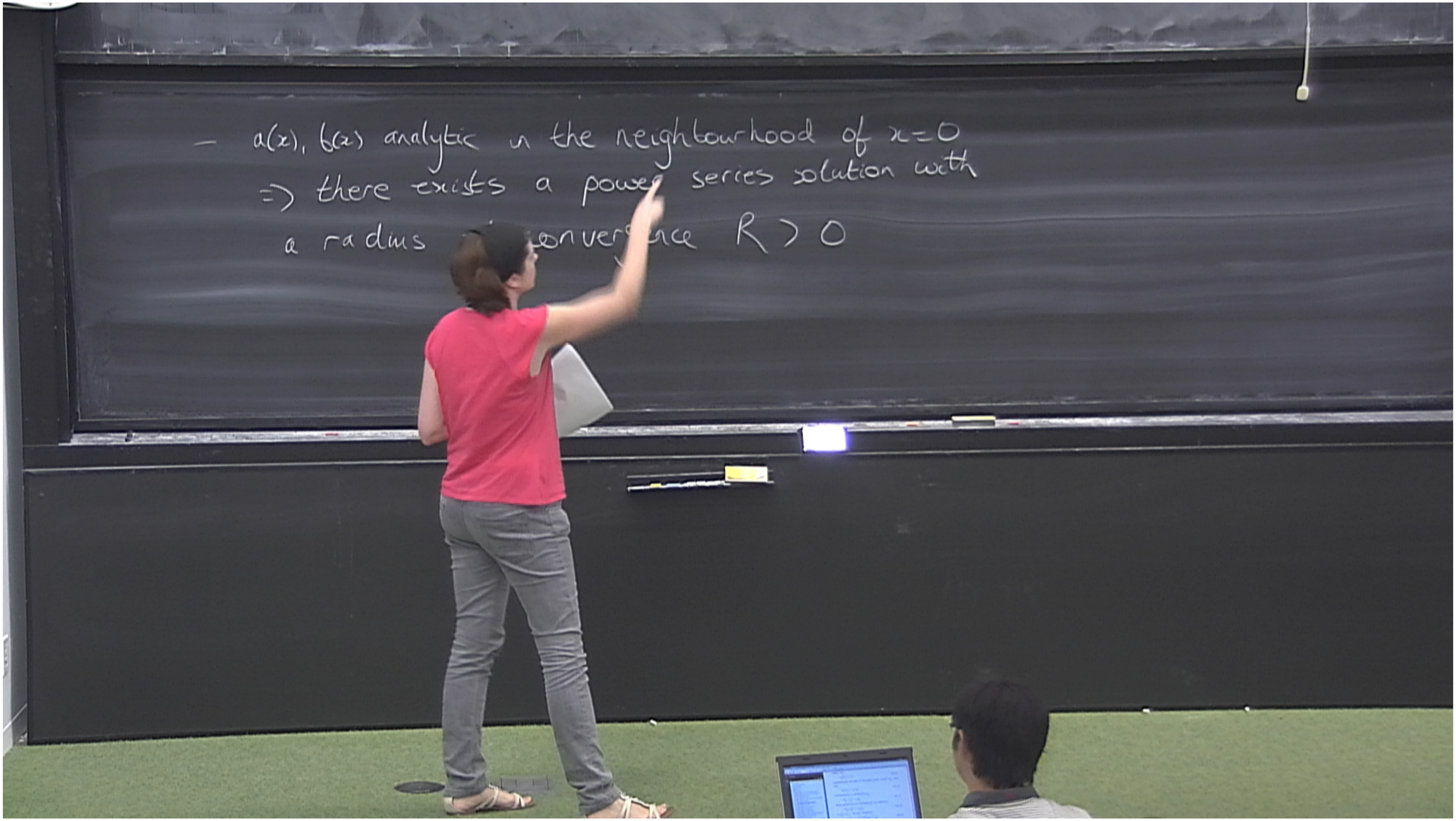

Extended power series method
$y''(x) + \frac{a(x)}{x} y'(x) + \frac{b(x)}{x^2} y(x) = 0$
$a(x), b(x)$ analytic
ok for a solution of the form

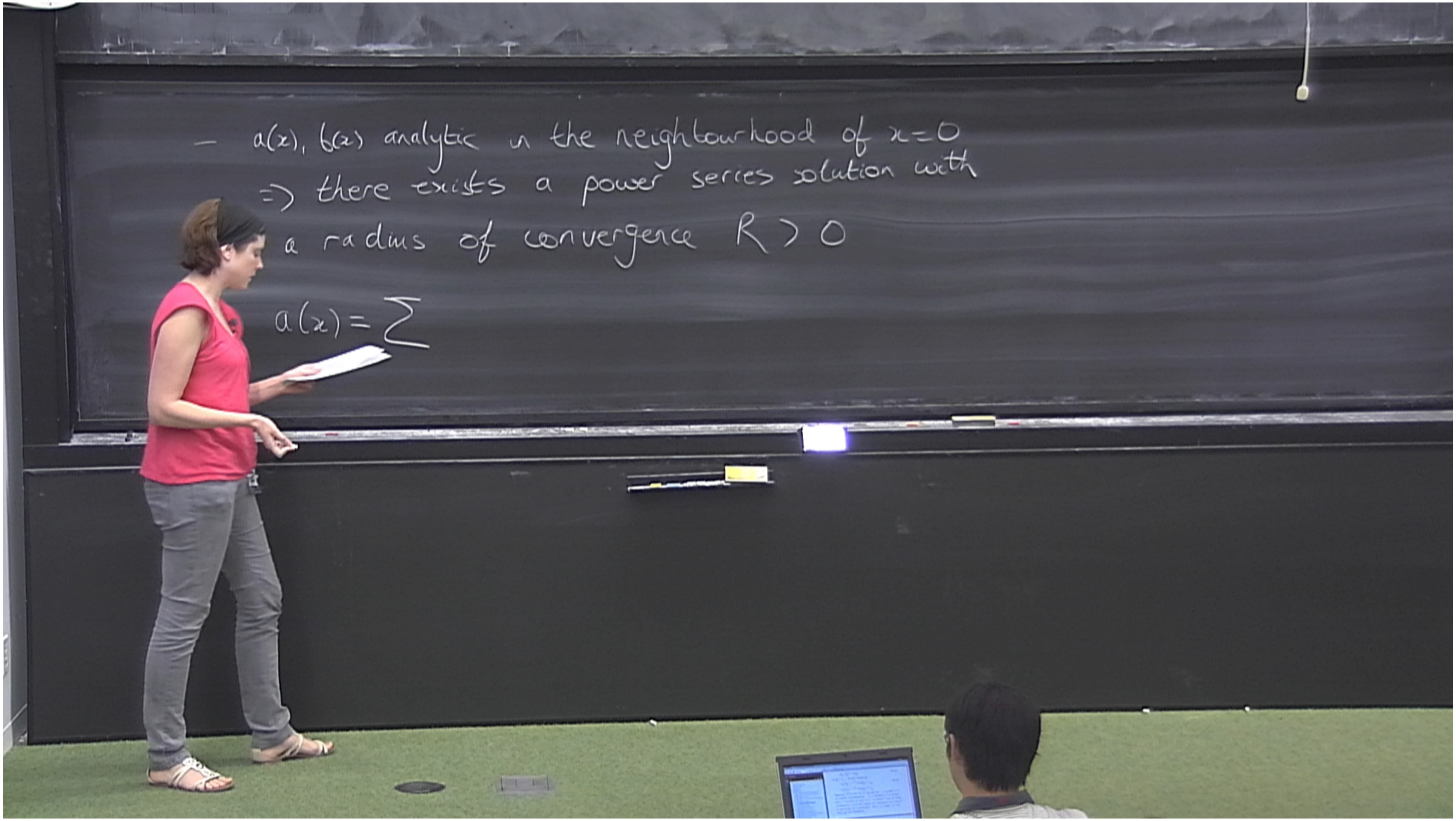

Extended power series method
$a(x), b(x)$ analytic
$y''(x) + \frac{a(x)}{x} y'(x) + \frac{b(x)}{x^2} y(x) = 0$
Look for a solution of the form $y(x) = x^r \sum_{k=0}^{\infty} c_k x^k$

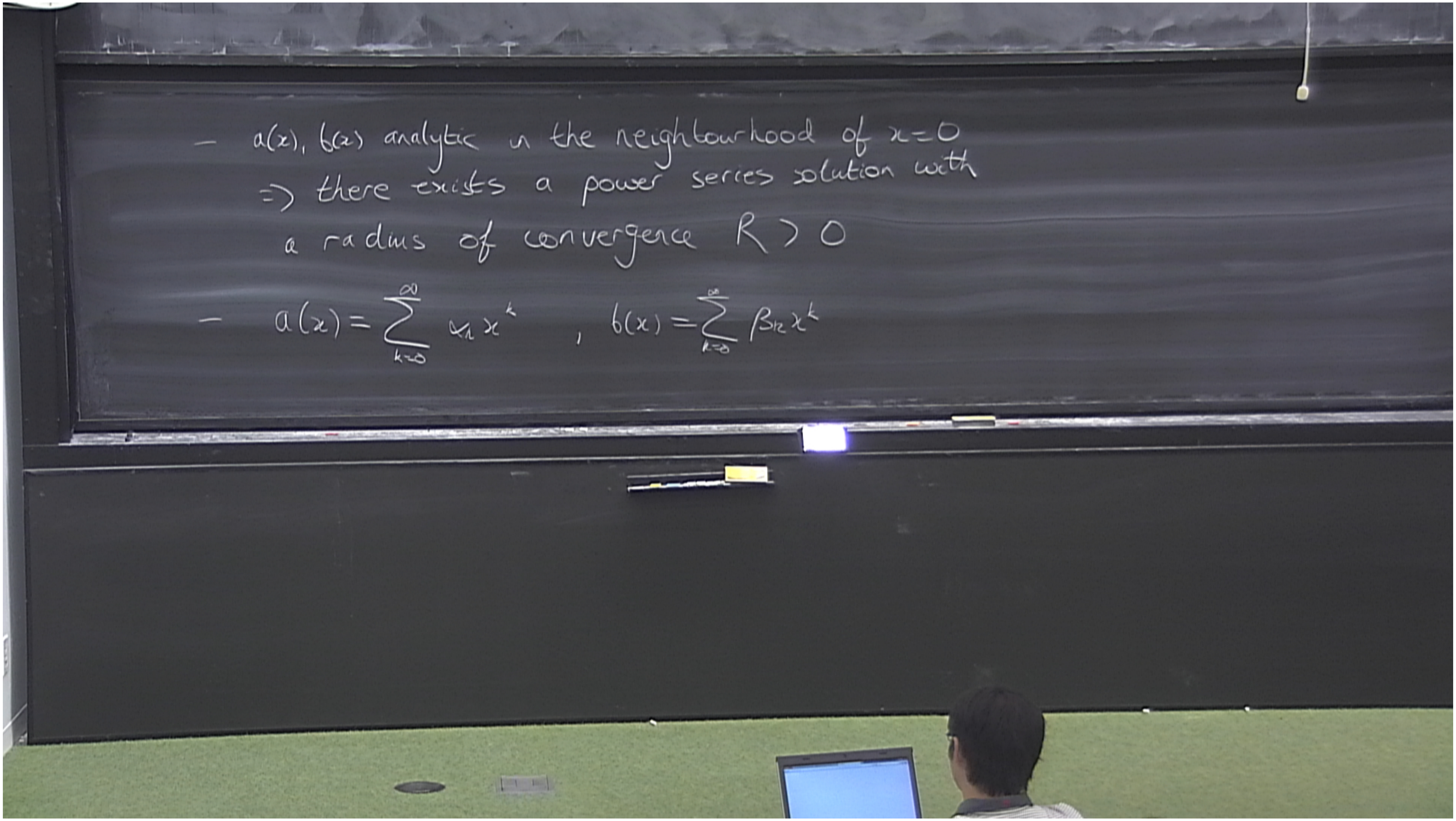

– a(x), b(x) analytic

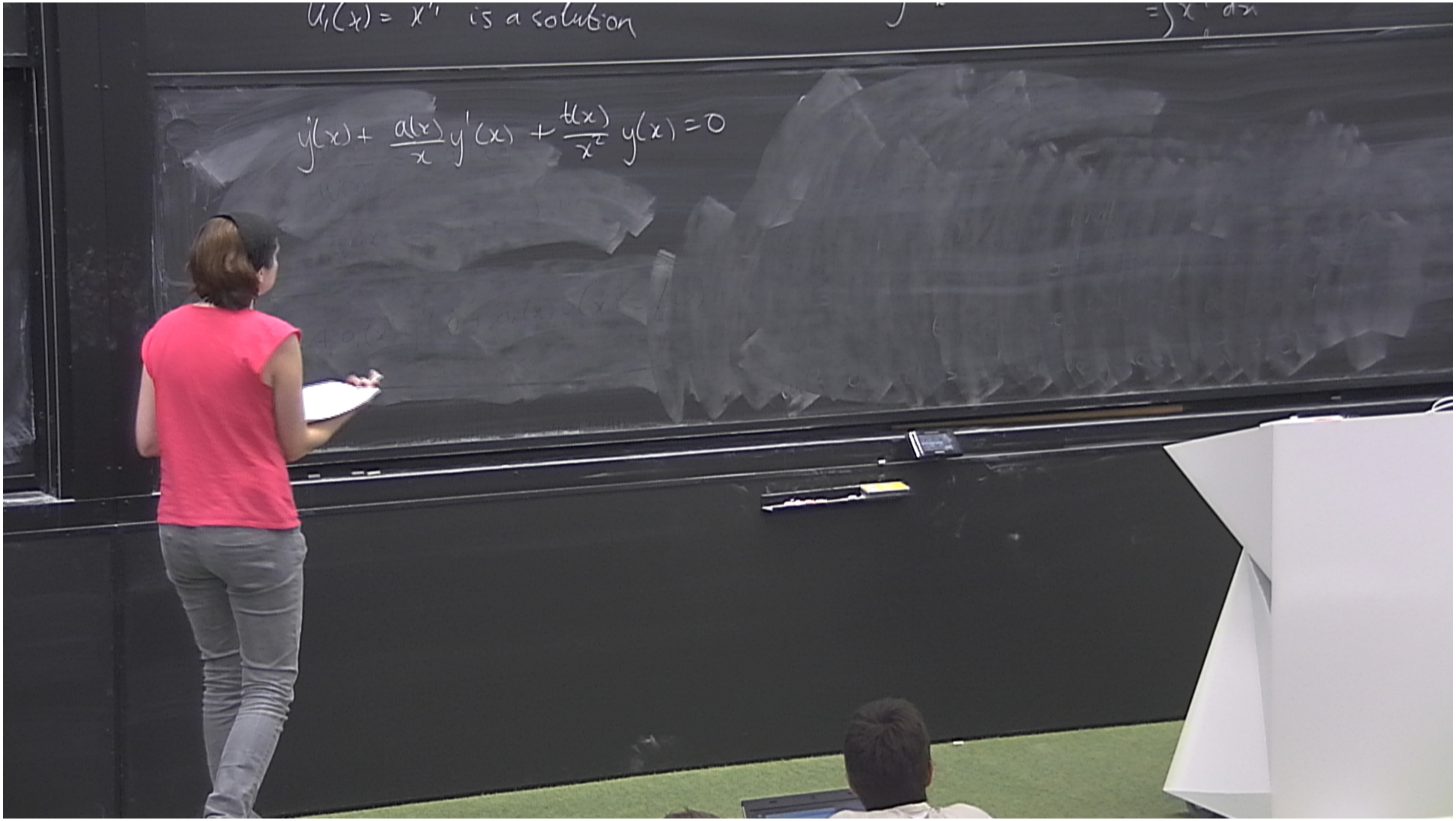

– a(x), b(x) analytic in the neighbourhood of x = 0
⇒ there exists a

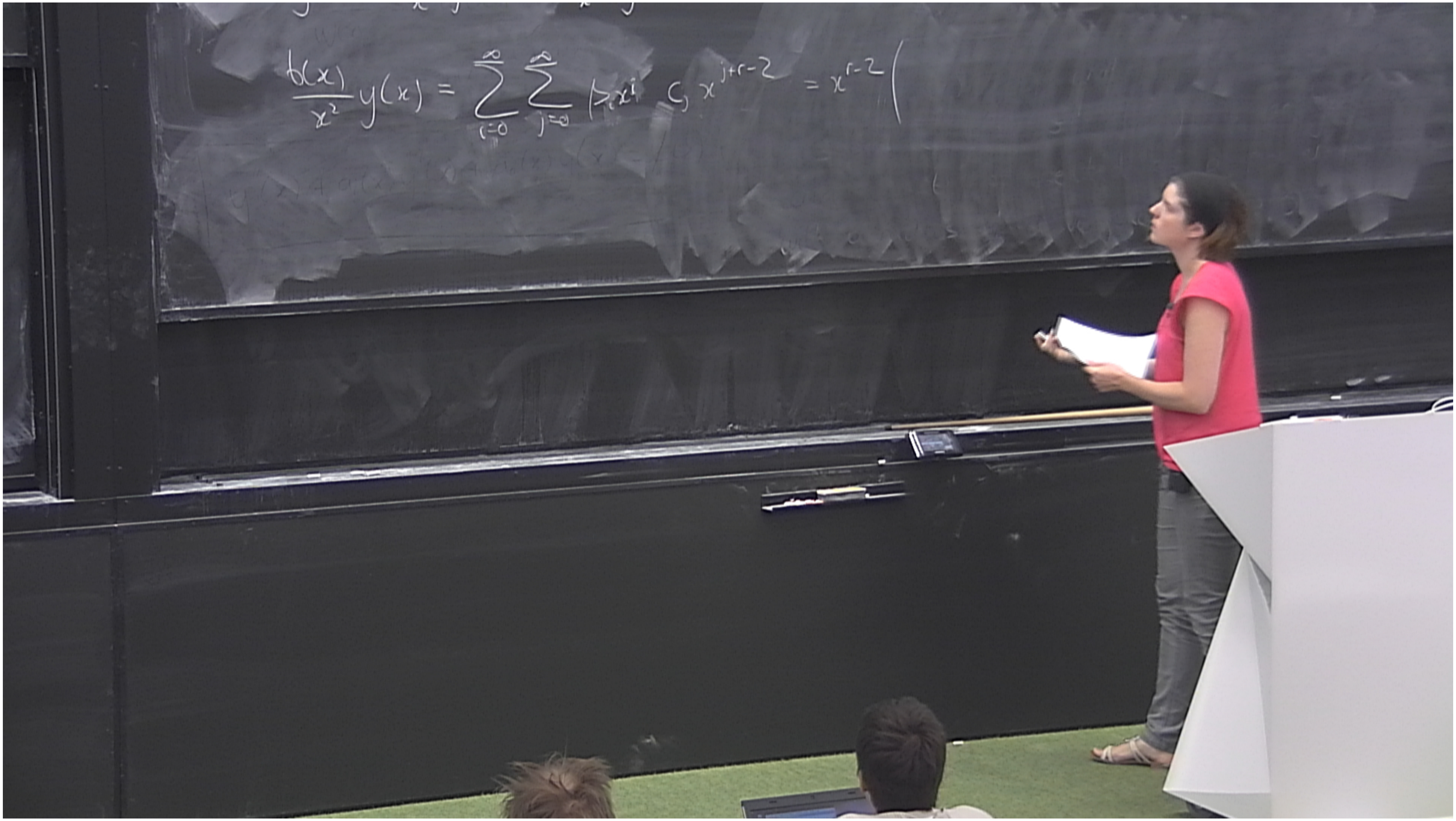

– a(x), b(x) analytic in the neighbourhood of x=0
⇒ there exists a power series solution with
a radius of convergence R > 0

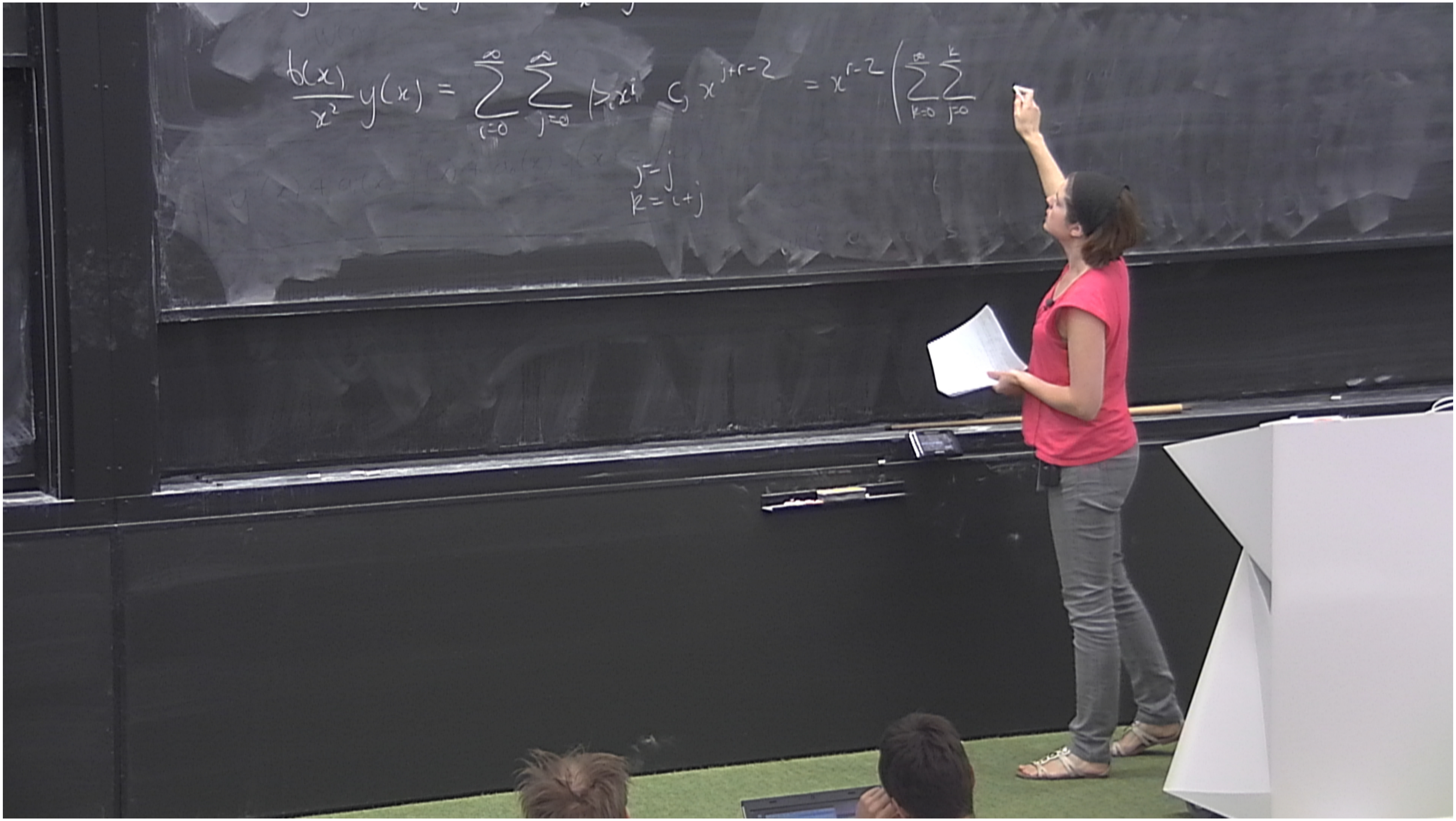

– a(x), b(x) analytic in the neighbourhood of x=0
=> there exists a power series solution with
a radius
R > 0


– $a(x)$, $b(x)$ analytic in the neighbourhood of $x=0$
$\Rightarrow$ there exists a power series solution with
a radius of convergence $R > 0$
$a(x) = \sum$

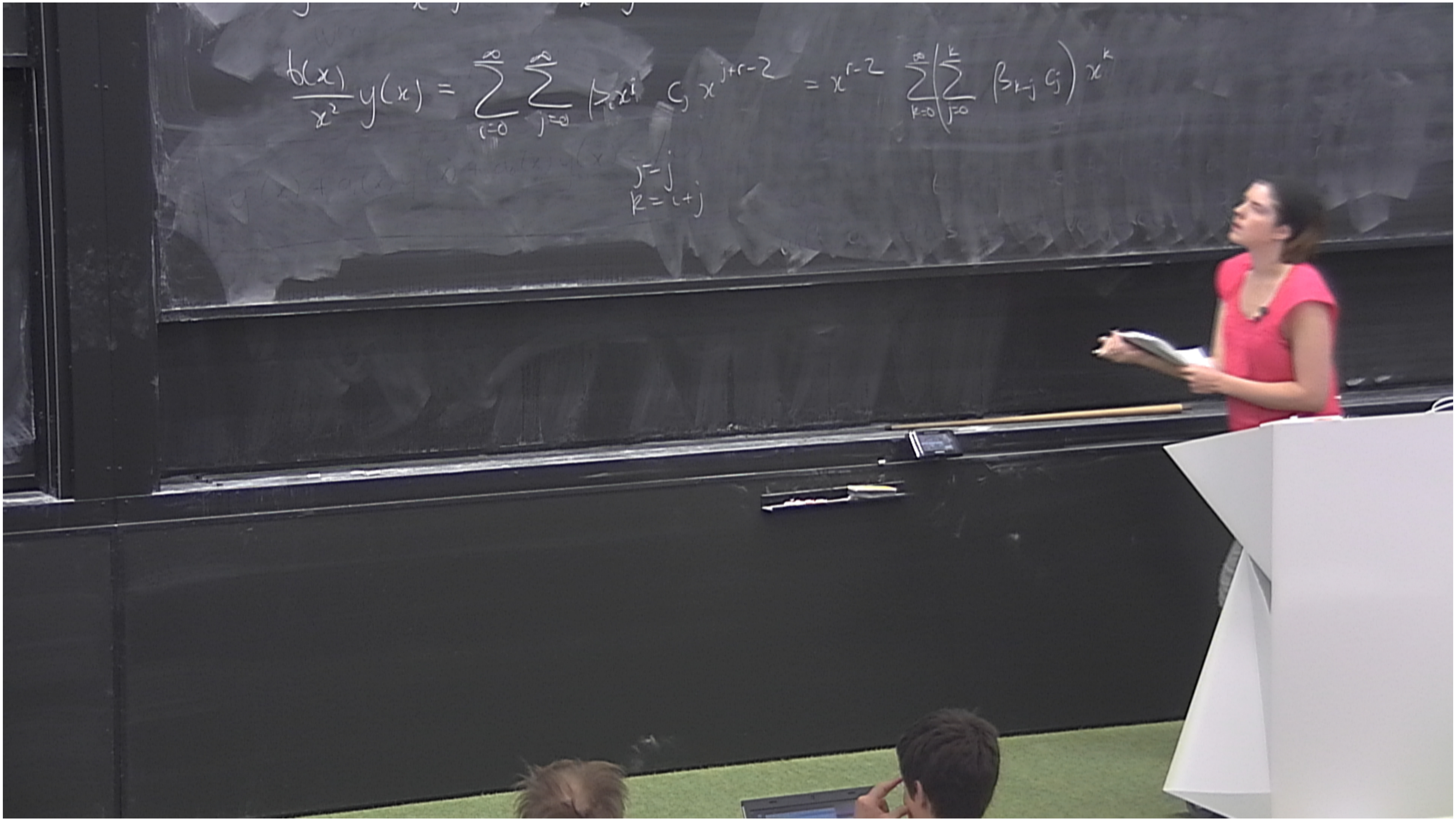

- a(x), b(x) analytic in the neighbourhood of x=0
=> there exists a power series solution with
a radius of convergence R > 0
- $a(x) = \sum_{k=0}^{\infty} \alpha_k x^k$, $b(x) = \sum_{k=0}^{\infty} \beta_k x^k$

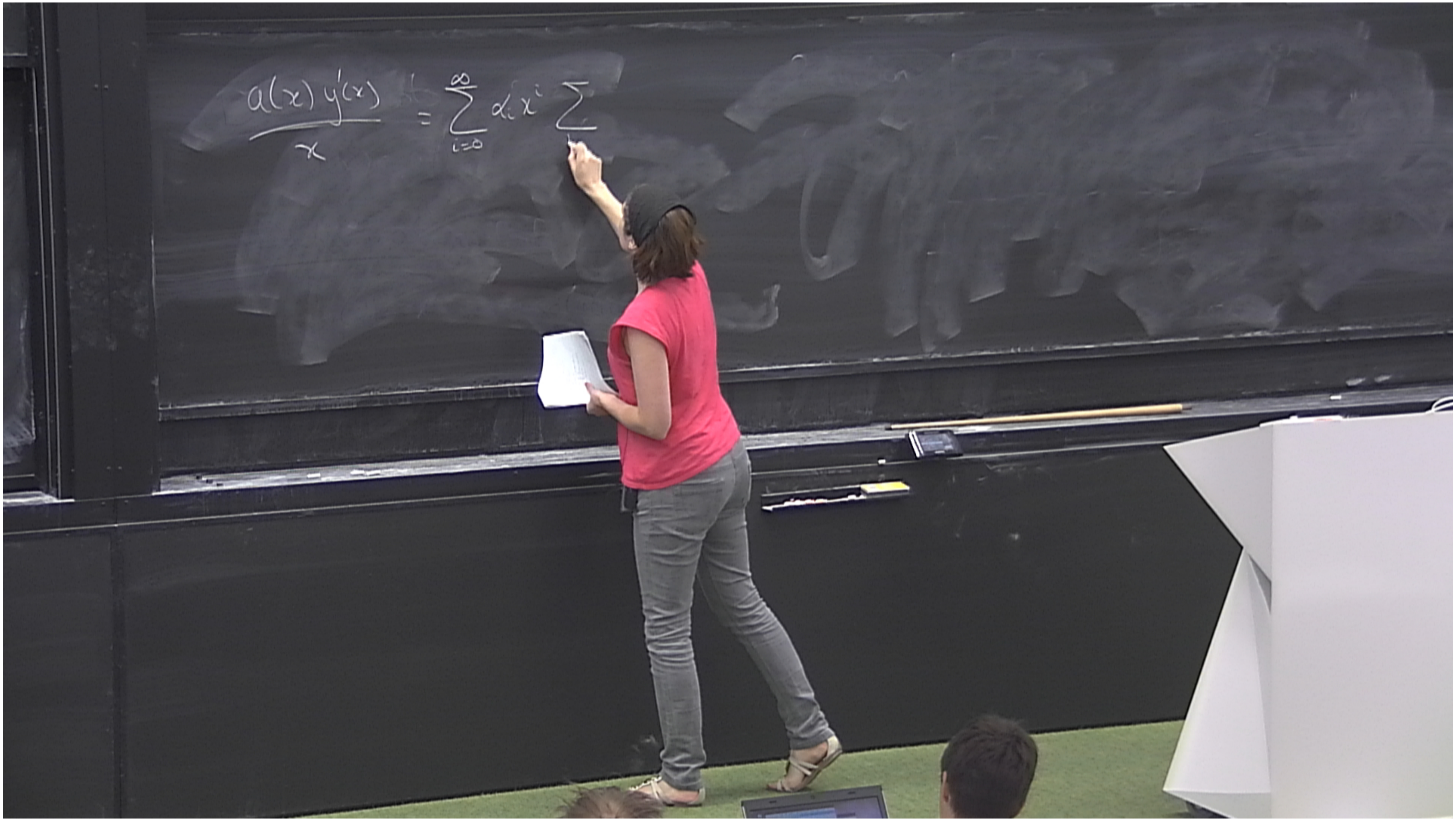

is a solution
y''(x) + a(x)/x y'(x) + b(x)/x^2 y(x) = 0

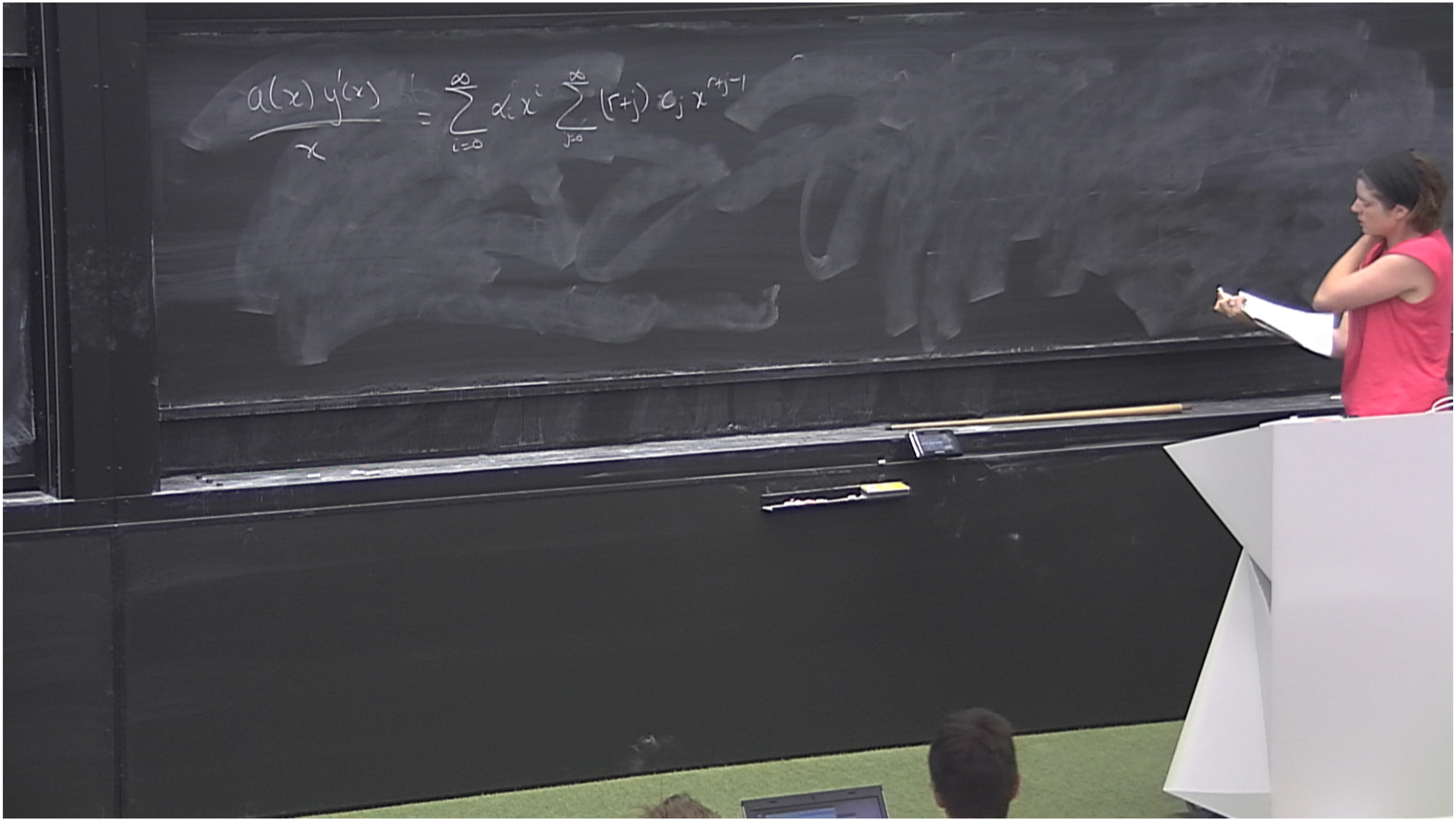

$\frac{b(x)}{x^2} y(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_i x^i c_j x^{j+r-2} = x^{r-2}($

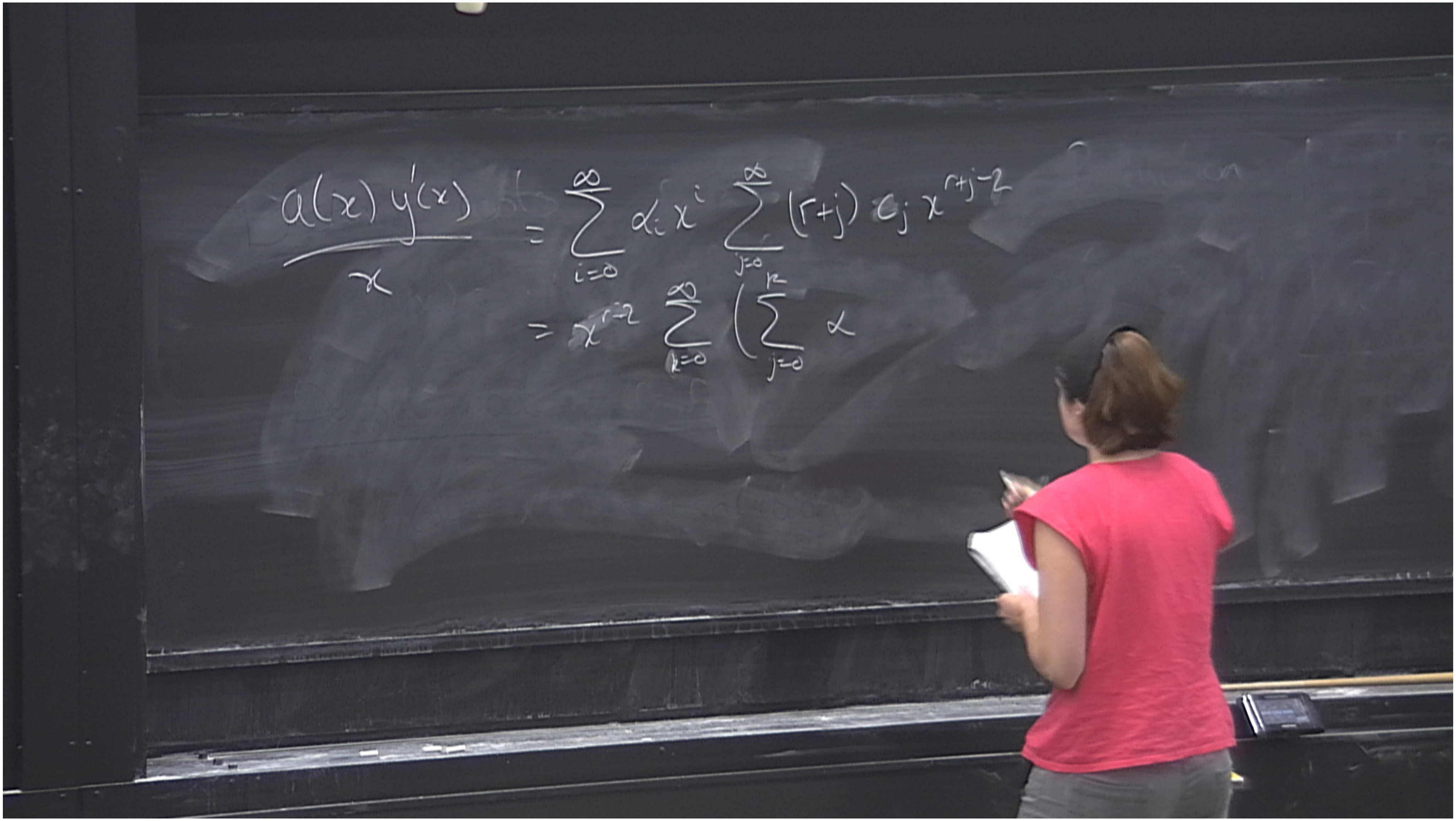


$\frac{b(x)}{x^2}y(x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_i x^i c_j x^{j+r-2} = x^{r-2}\left(\sum_{k=0}^{\infty}\sum_{j=0}^{k}\right.$
$j=j$
$k=i+j$


$\frac{b(x)}{x^2} y(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_i x^i \, c_j \, x^{j+r-2} = x^{r-2} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \beta_{k-j} c_j \right) x^k$
$j' = j$
$k = i + j$

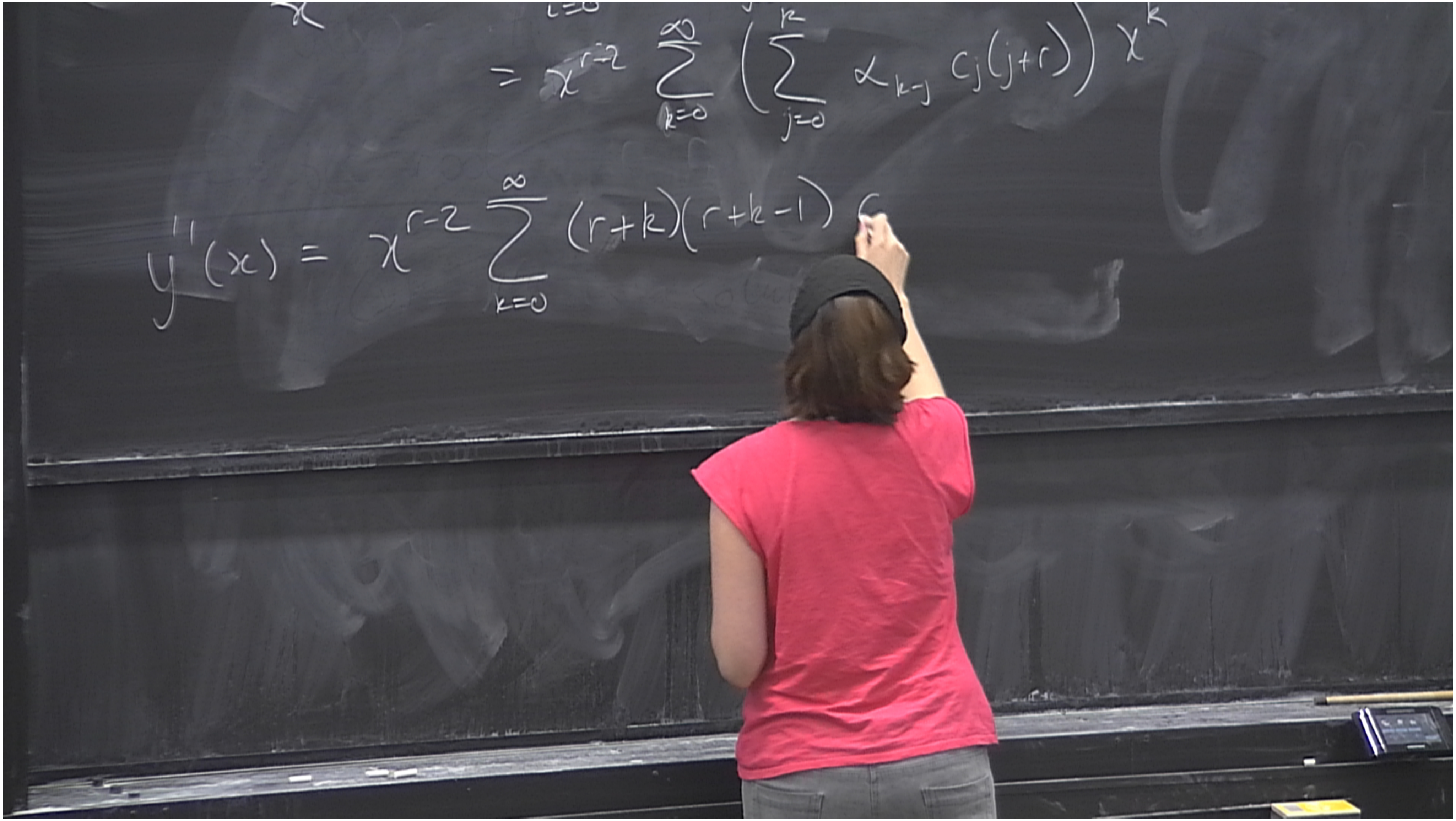

$\frac{a(x)\,y'(x)}{x} = \sum_{i=0}^{\infty} \alpha_i x^i \sum$


$\frac{a(x)\,y'(x)}{x} = \sum_{i=0}^{\infty} \alpha_i x^i \sum_{j=0}^{\infty} (r+j)\, c_j\, x^{r+j-1}$


$\frac{a(x)\,y'(x)}{x} = \sum_{i=0}^{\infty} \alpha_i x^i \sum_{j=0}^{\infty} (r+j)\, c_j\, x^{r+j-2}$
$= x^{r-2} \sum_{k=0}^{\infty} \Big(\sum_{j=0}^{k} \alpha$

$$\frac{a(x)\,y'(x)}{x} = \sum_{i=0}^{\infty} \alpha_i x^i \sum_{j=0}^{\infty} (r+j)\, c_j\, x^{r+j-2}$$

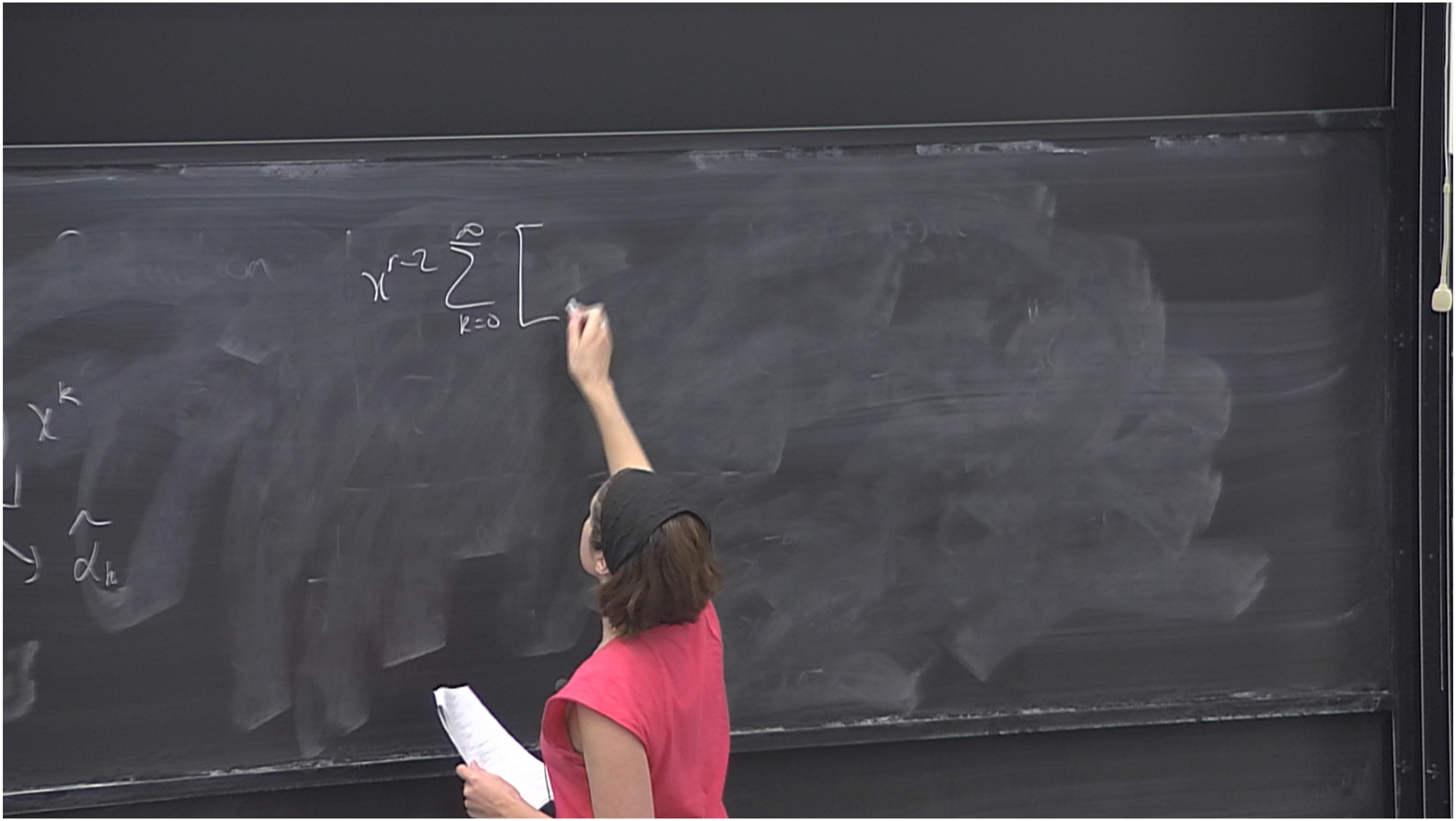
$$= x^{r-2} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \alpha_{k-j}\, c_j (j+r) \right) x^k$$


$= x^{r-2} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \alpha_{k-j} c_j (j+r) \right) x^k$
$y''(x) = x^{r-2} \sum_{k=0}^{\infty} (r+k)(r+k-1) c$

$$\frac{a(x)\,y'(x)}{x} = \sum_{i=0}^{\infty} \alpha_i x^i \sum_{j=0}^{\infty} (r+j)\, c_j\, x^{r+j-2}$$

$$= x^{r-2} \sum_{k=0}^{\infty} \underbrace{\left(\sum_{j=0}^{k} \alpha_{k-j}\, c_j (j+r) \right)} x^k$$

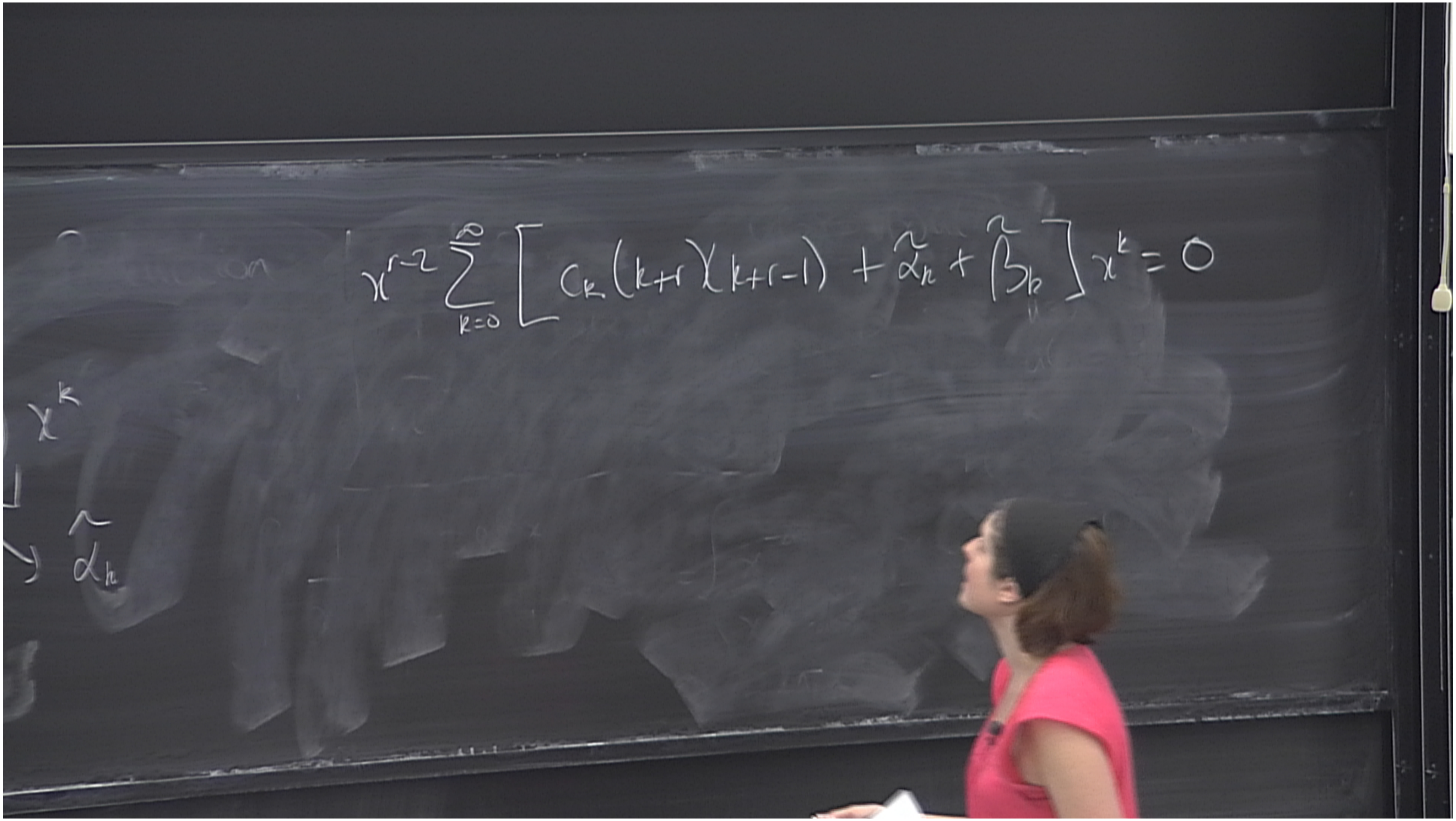
$$y''(x) = x^{r-2} \sum_{k=0}^{\infty} (r+k)(r+k-1)\, c_k\, x^k$$

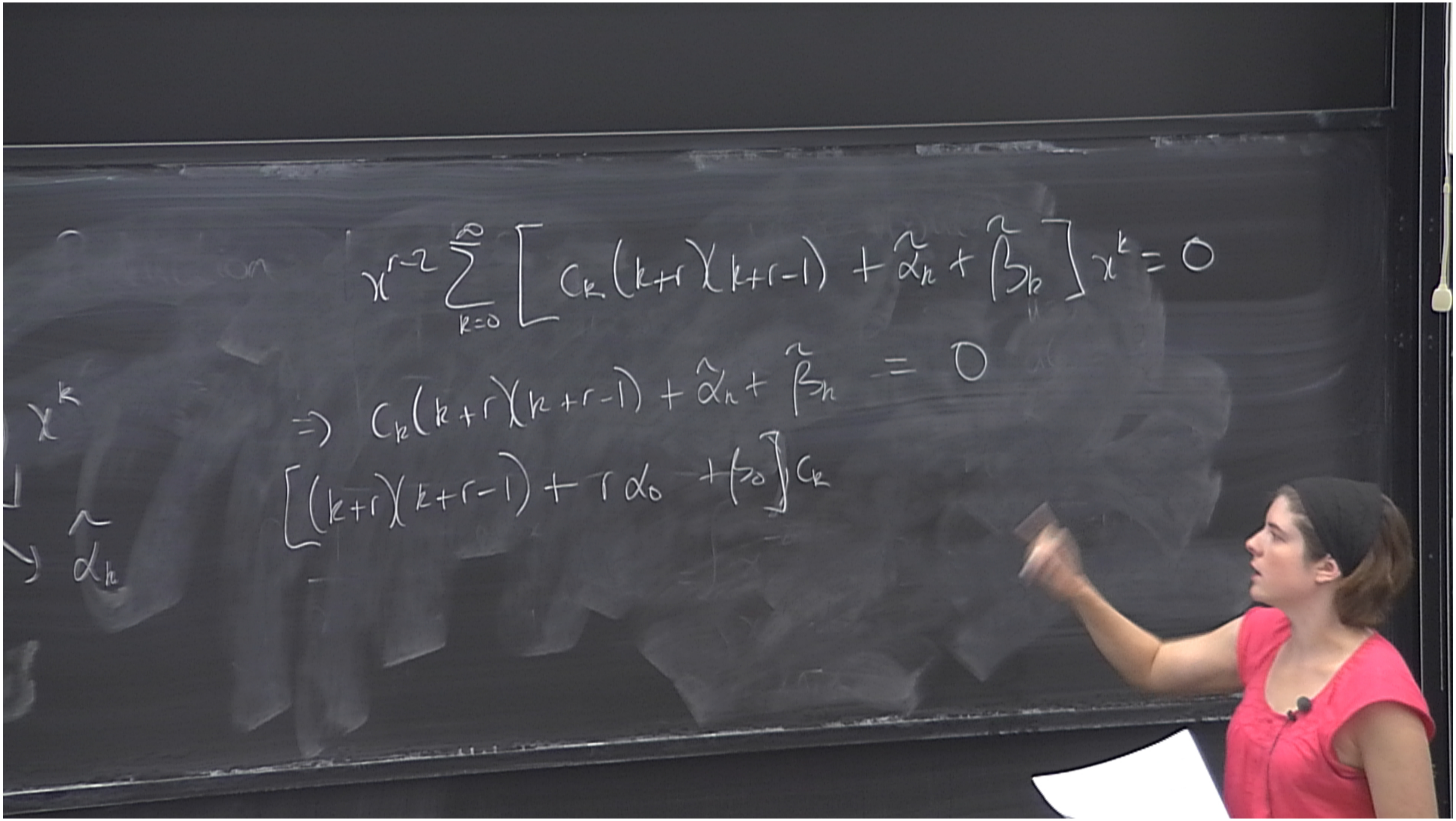

$x^{r-2}\sum_{k=0}^{\infty}\Big[$
x^k
$\hat{\alpha}_k$

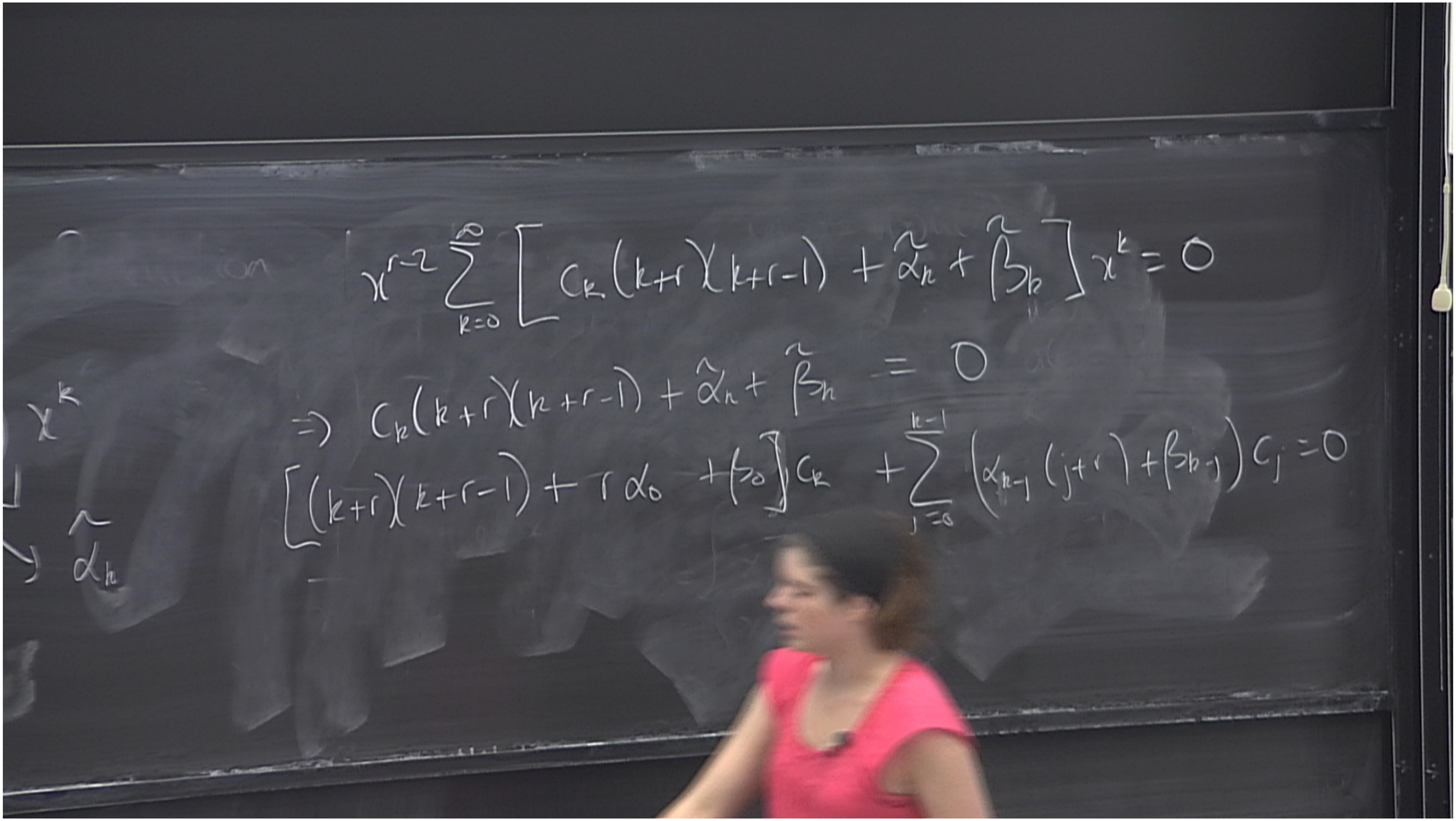
$$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$$

x^k

$\tilde{\alpha}_k$


$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$
x^k
$\tilde{\alpha}_k$


$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$
$\Rightarrow c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k=0$
$\left[(k+r)(k+r-1)+r\alpha_0+\beta_0\right]c_k$
x^k
$\tilde{\alpha}_k$


$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$
$\Rightarrow c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k=0$
$\left[(k+r)(k+r-1)+r\alpha_0+\beta_0\right]c_k+\sum_{j=0}^{k-1}\left(\alpha_{k-j}(j+r)+\beta_{k-j}\right)c_j=0$
x^k
$\tilde{\alpha}_k$

x^k

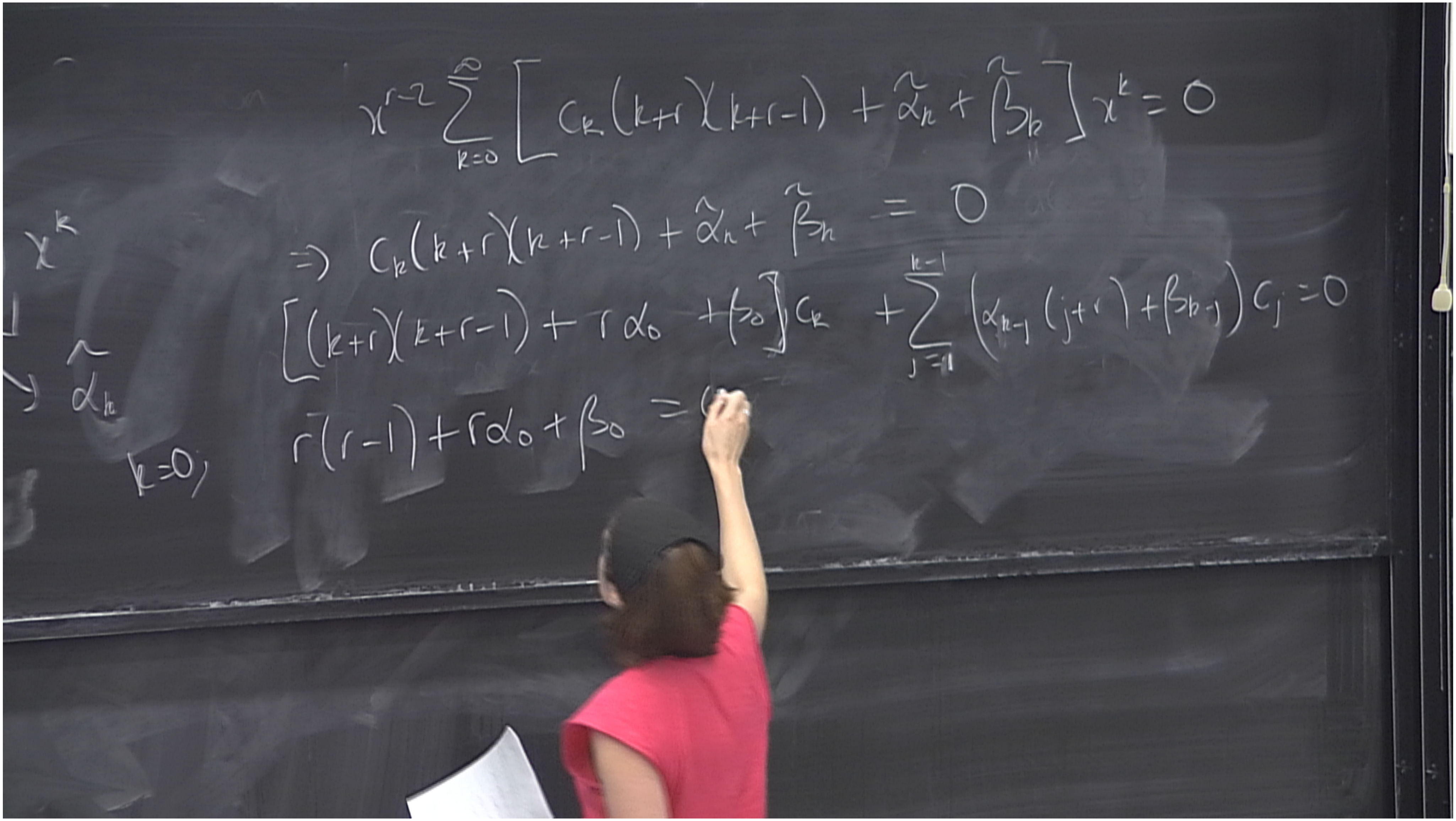
$\tilde{\alpha}_k$

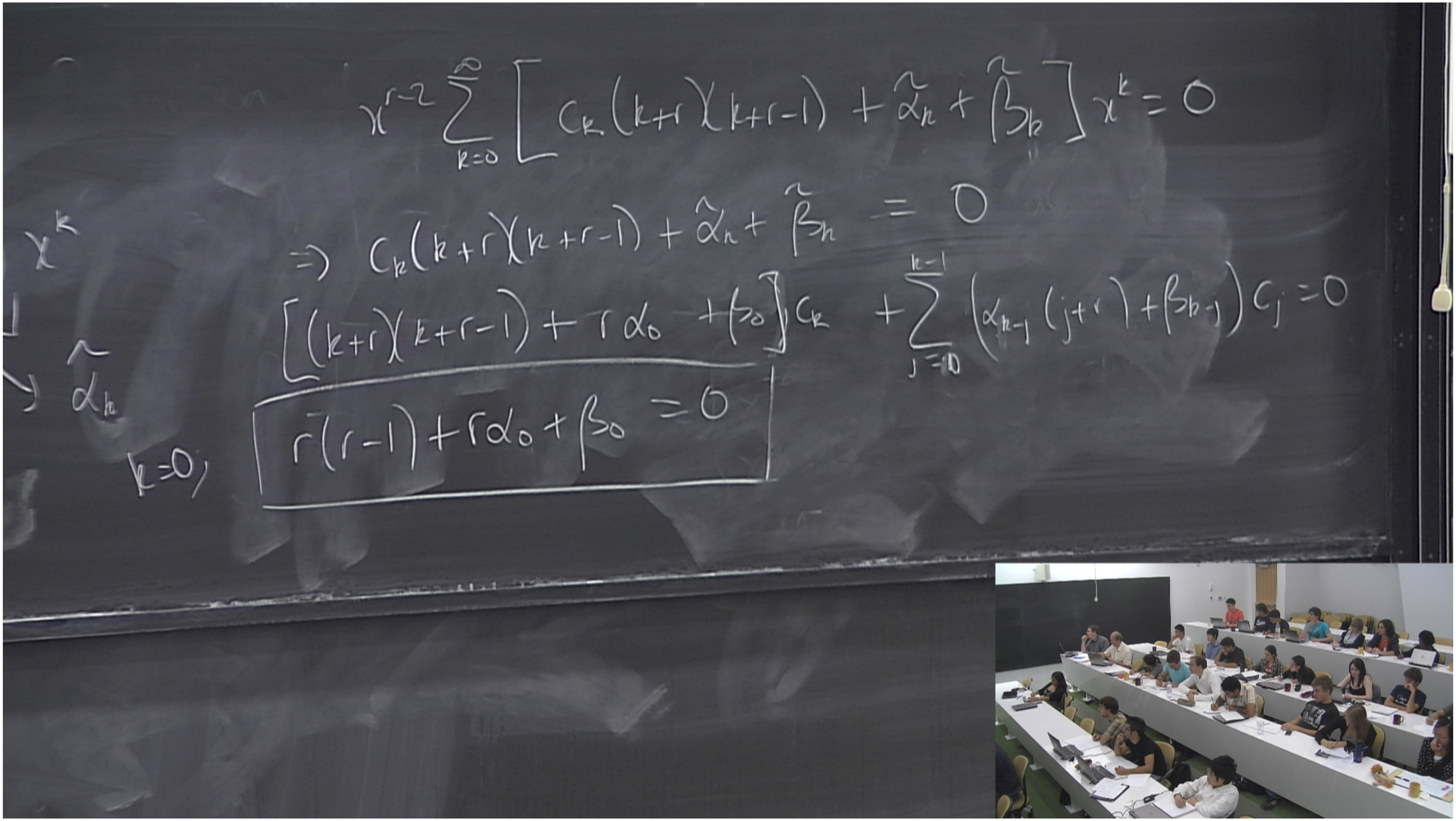
$k=0,$

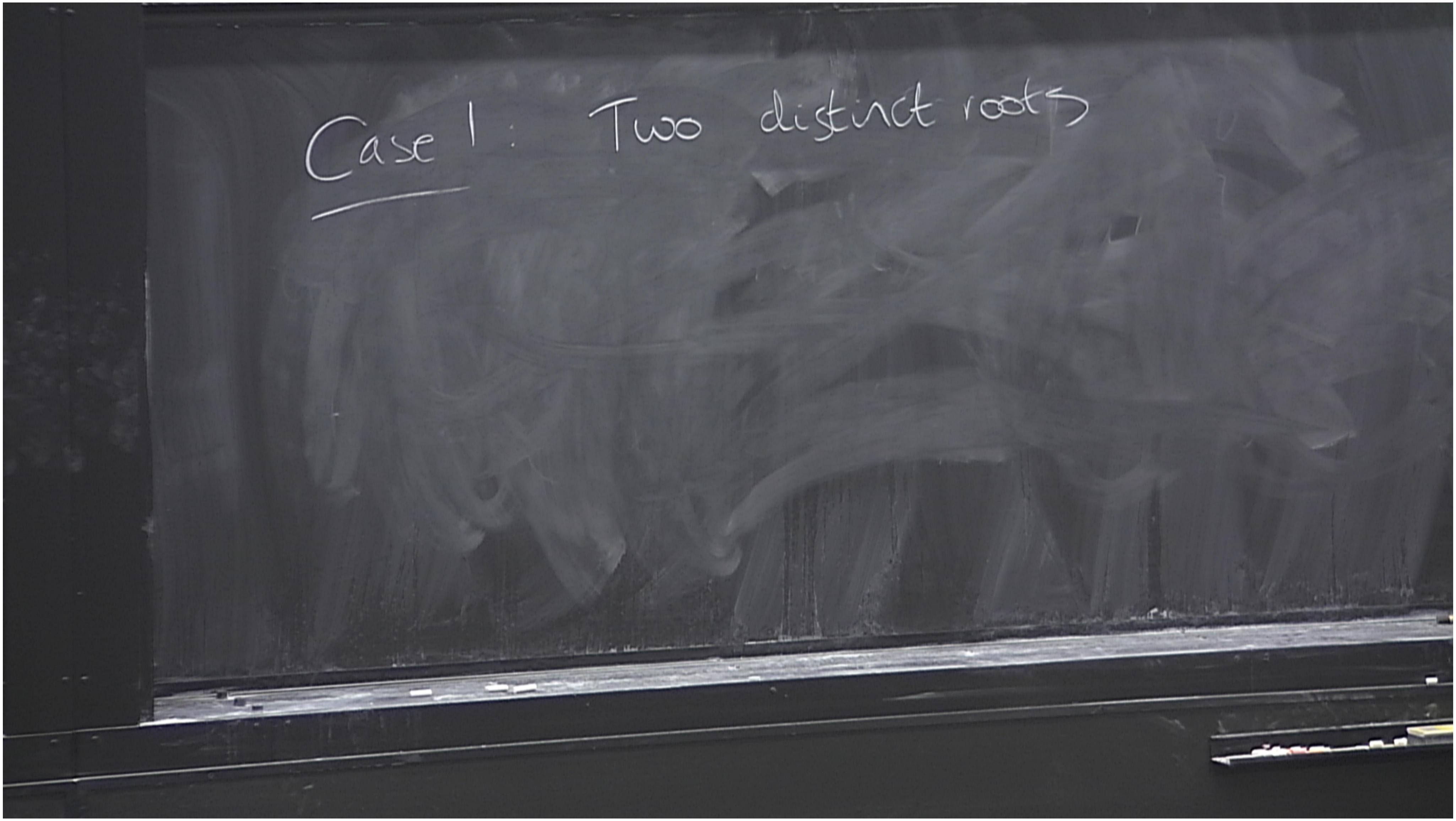
$$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$$

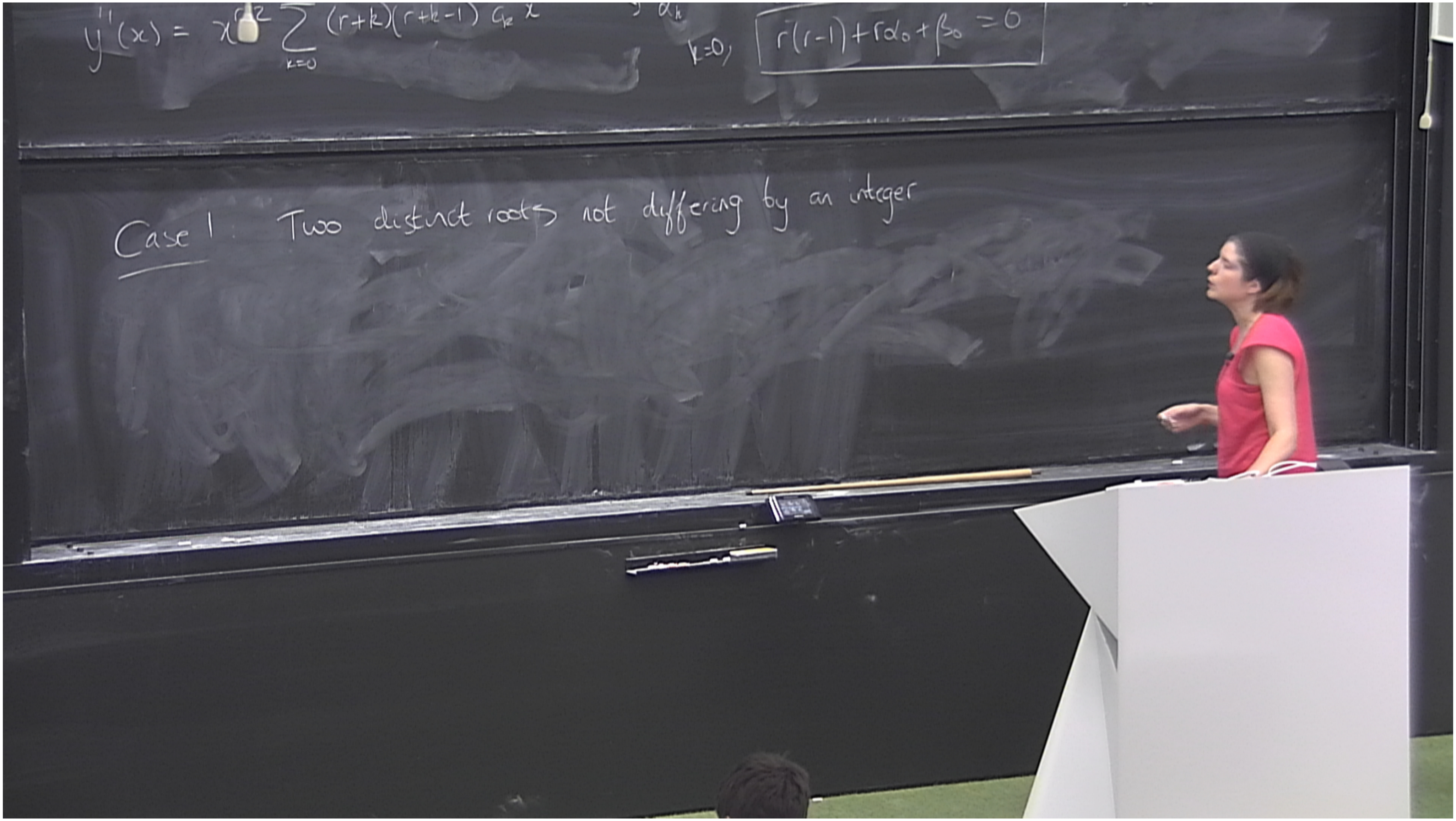
$$\Rightarrow c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k=0$$

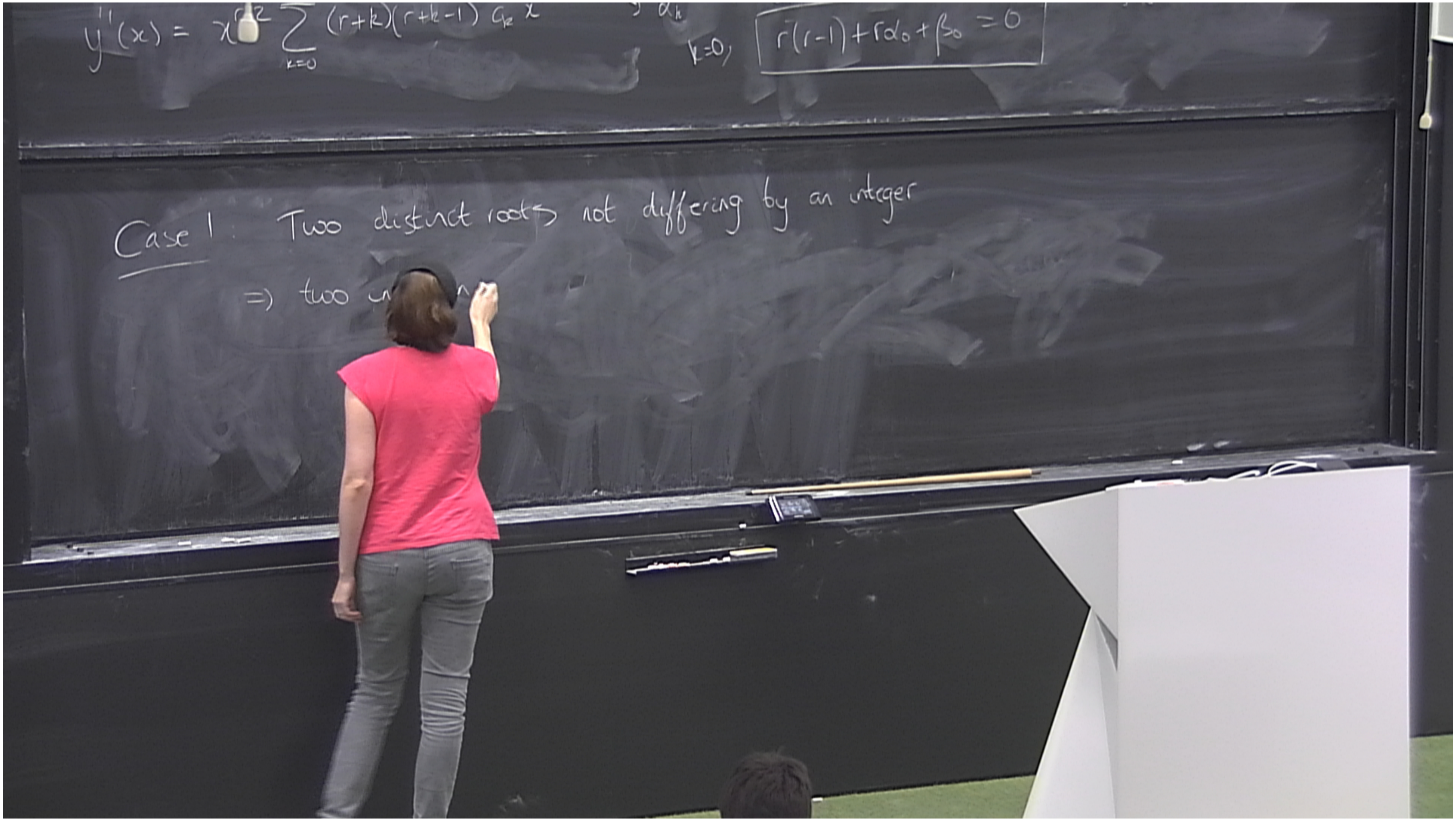
$$\left[(k+r)(k+r-1)+r\alpha_0+\beta_0\right]c_k+\sum_{j=0}^{k-1}\left(\alpha_{k-j}(j+r)+\beta_{k-j}\right)c_j=0$$


$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$
x^k
$\Rightarrow c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k=0$
$\left[(k+r)(k+r-1)+r\alpha_0+\beta_0\right]c_k+\sum_{j=1}^{k-1}\left(\alpha_{k-j}(j+r)+\beta_{k-j}\right)c_j=0$
$\tilde{\alpha}_k$
$k=0,$
$r(r-1)+r\alpha_0+\beta_0=0$

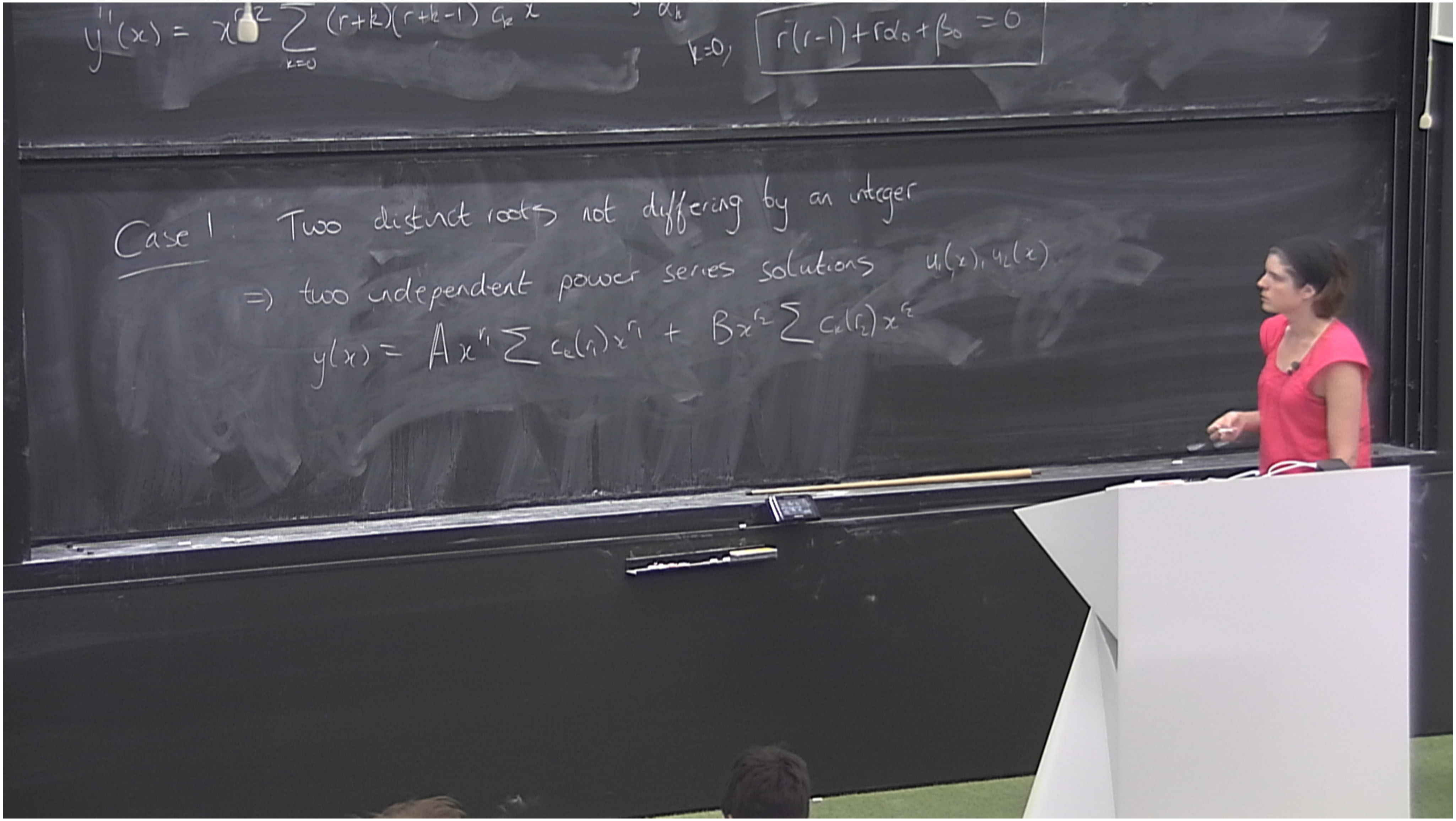

$x^{r-2}\sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k\right]x^k=0$
x^k
$\Rightarrow c_k(k+r)(k+r-1)+\tilde{\alpha}_k+\tilde{\beta}_k=0$
$[(k+r)(k+r-1)+r\alpha_0+\beta_0]c_k+\sum_{j=0}^{k-1}\left(\alpha_{k-j}(j+r)+\beta_{k-j}\right)c_j=0$
$\tilde{\alpha}_k$
$k=0;\ \boxed{r(r-1)+r\alpha_0+\beta_0=0}$

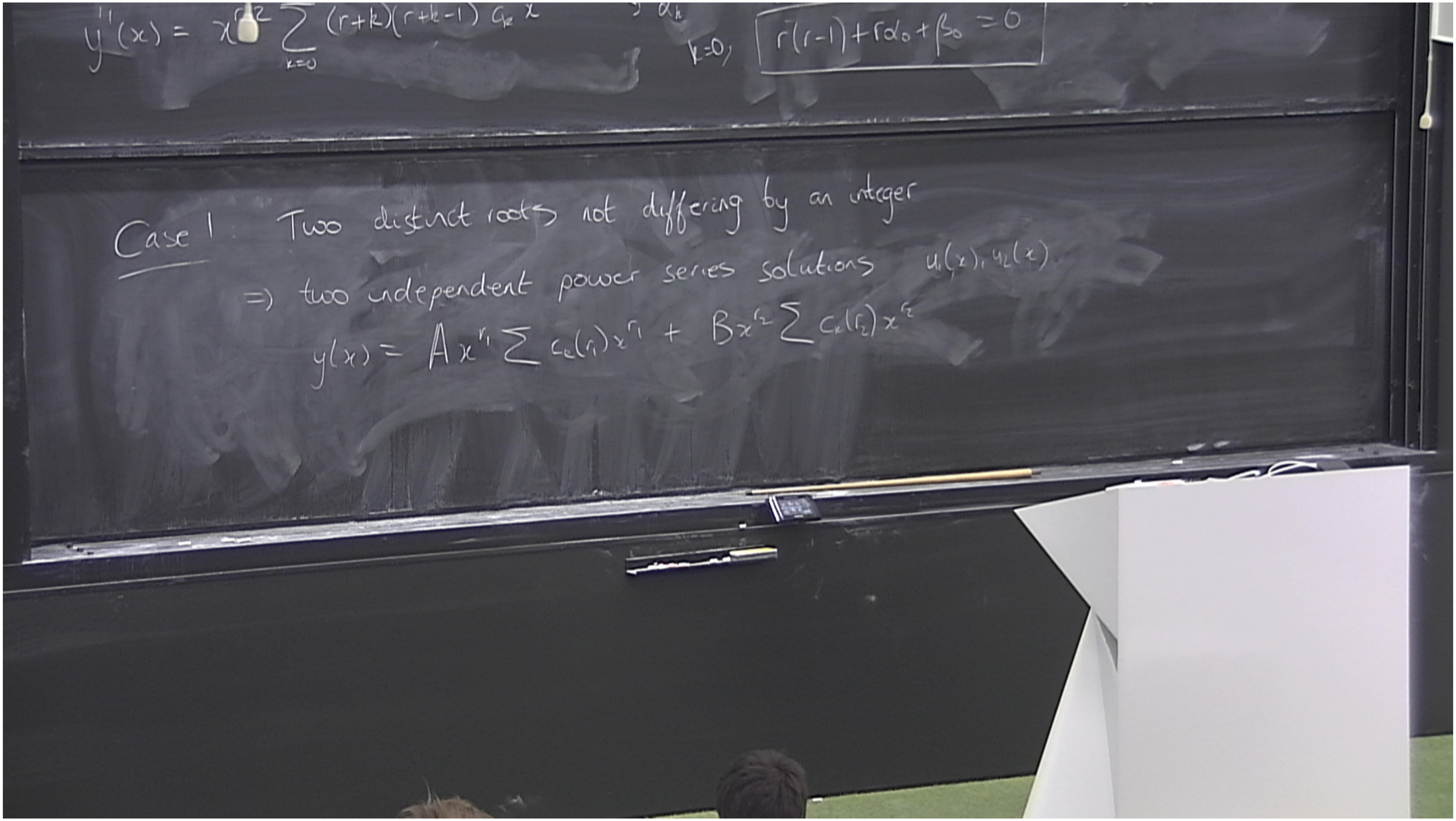

Case 1: Two distinct roots

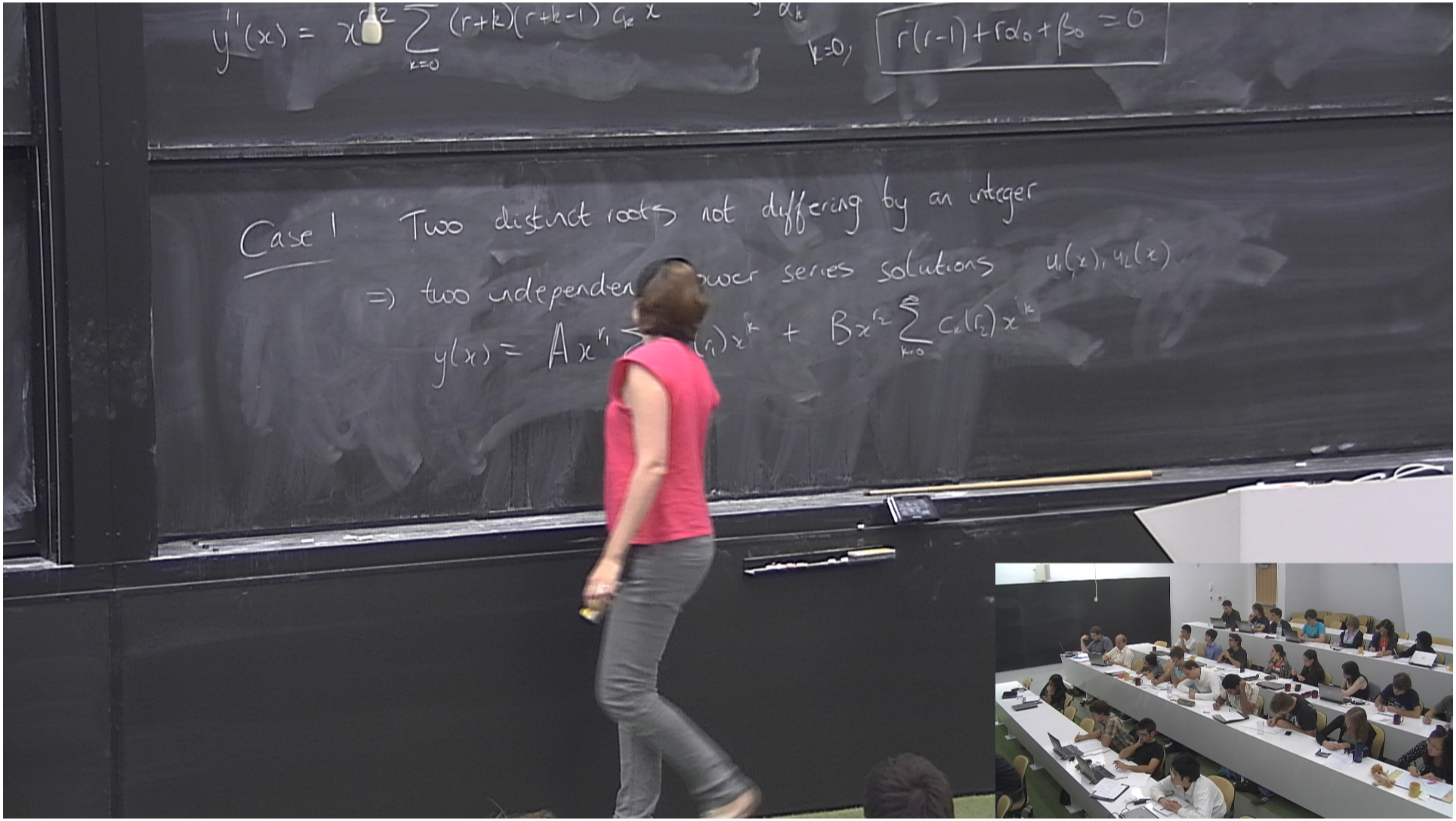

k=0, r(r-1)+rα₀+β₀=0
Case 1: Two distinct roots not differing by an integer

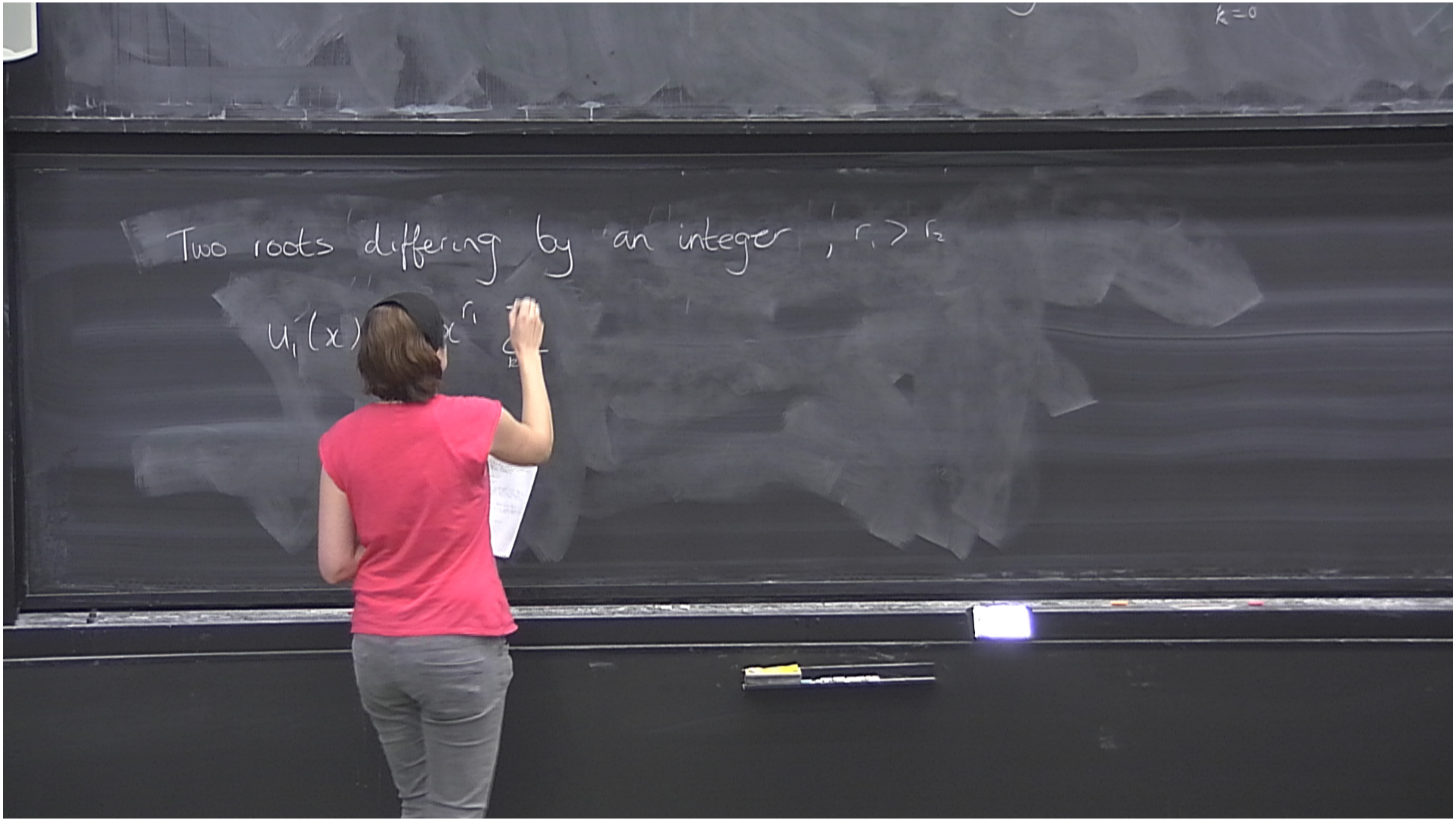


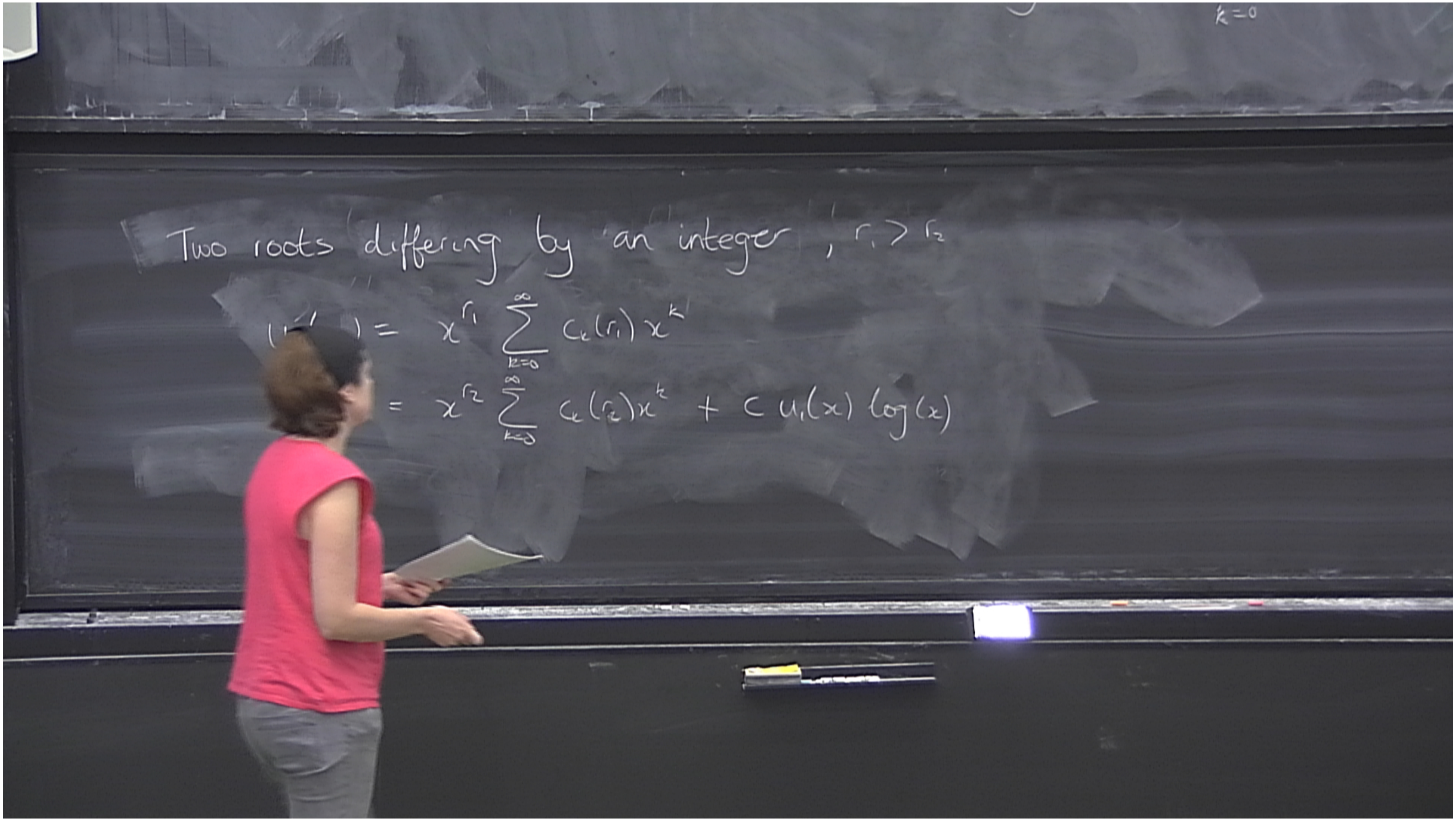
y''(x) = x^{r-2} Σ_{k=0} (r+k)(r+k-1) a_k x
k=0, r(r-1)+rα_0+β_0 = 0
Case 1: Two distinct roots not differing by an integer
⇒ two

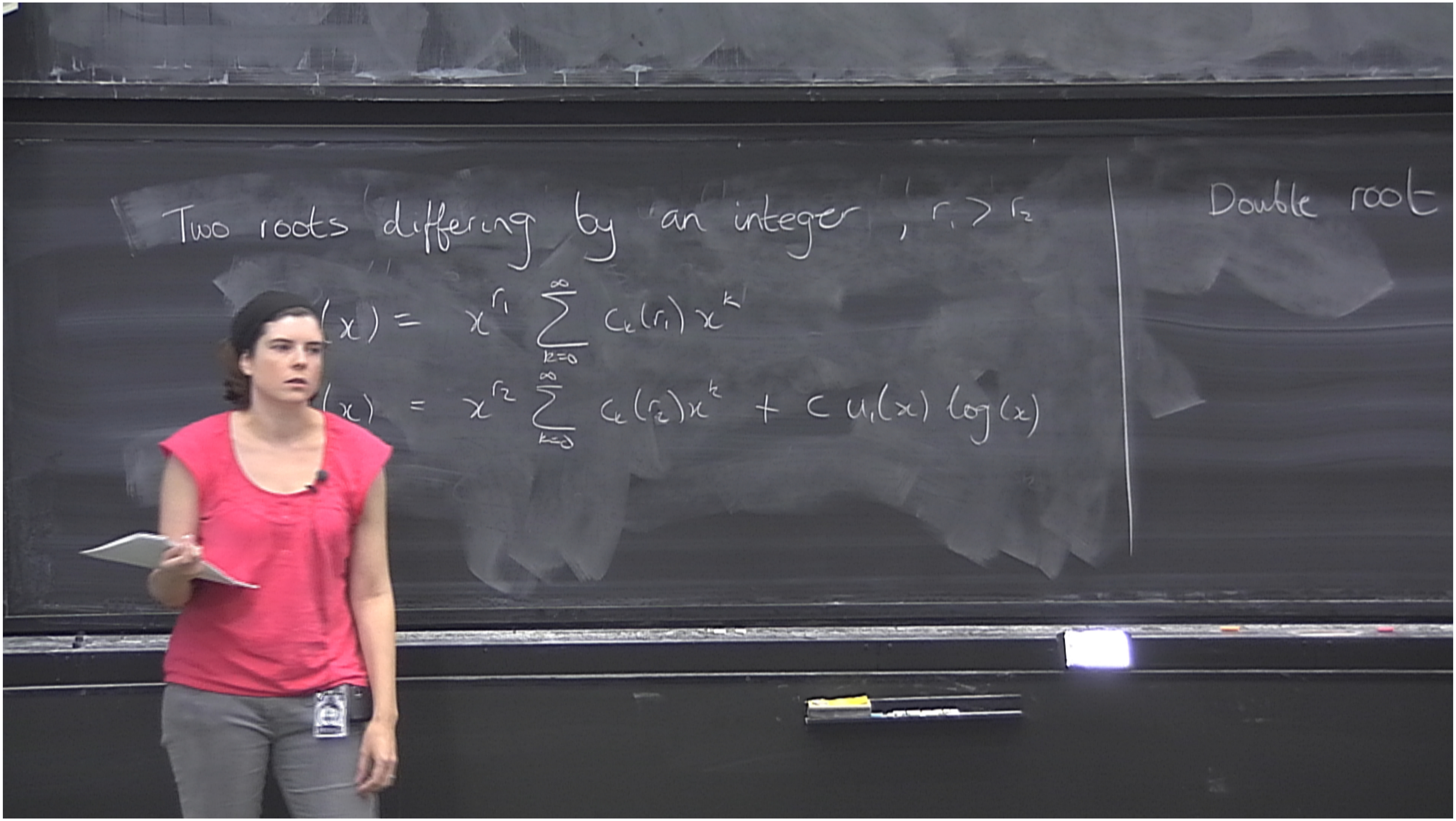

$y''(x) = x^{r-2} \sum_{k=0} (r+k)(r+k-1) c_k x$
$k=0$, $\boxed{r(r-1) + r\alpha_0 + \beta_0 = 0}$
Case 1 : Two distinct roots not differing by an integer
⇒ two independent power series solutions $u_1(x), u_2(x)$
$y(x) = A x^{r_1} \sum c_k(r_1) x^{r_1} + B x^{r_2} \sum c_k(r_2) x^{r_2}$

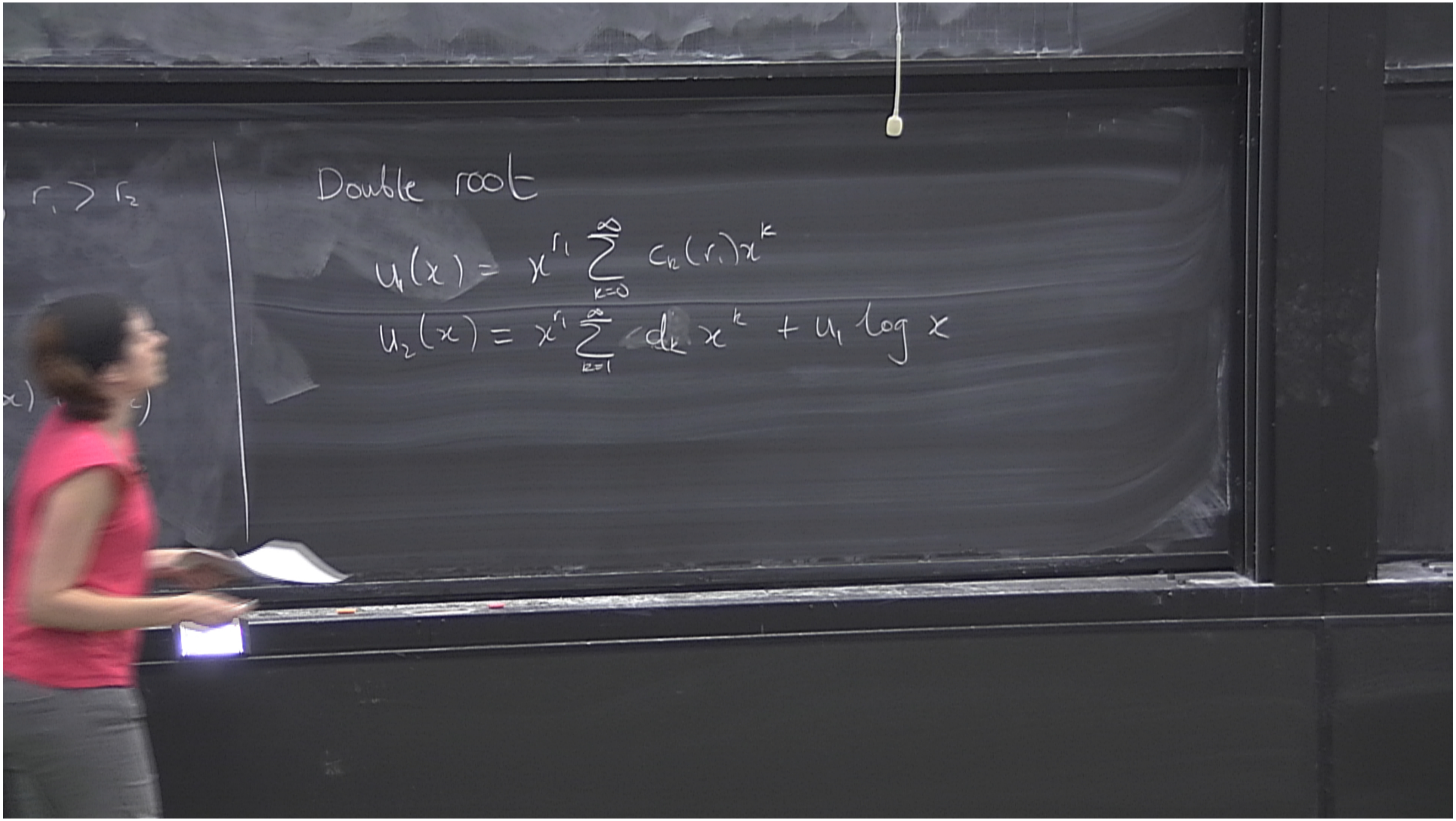

$y''(x) = x^{r-2}\sum_{k=0}(r+k)(r+k-1)c_k x$
α_k
$k=0,\ \boxed{r(r-1)+r\alpha_0+\beta_0=0}$
Case 1: Two distinct roots not differing by an integer
$\Rightarrow$ two independent power series solutions $u_1(x), u_2(x)$
$y(x) = A x^{r_1}\sum c_k(r_1)x^{r_1} + B x^{r_2}\sum c_k(r_2)x^{r_2}$

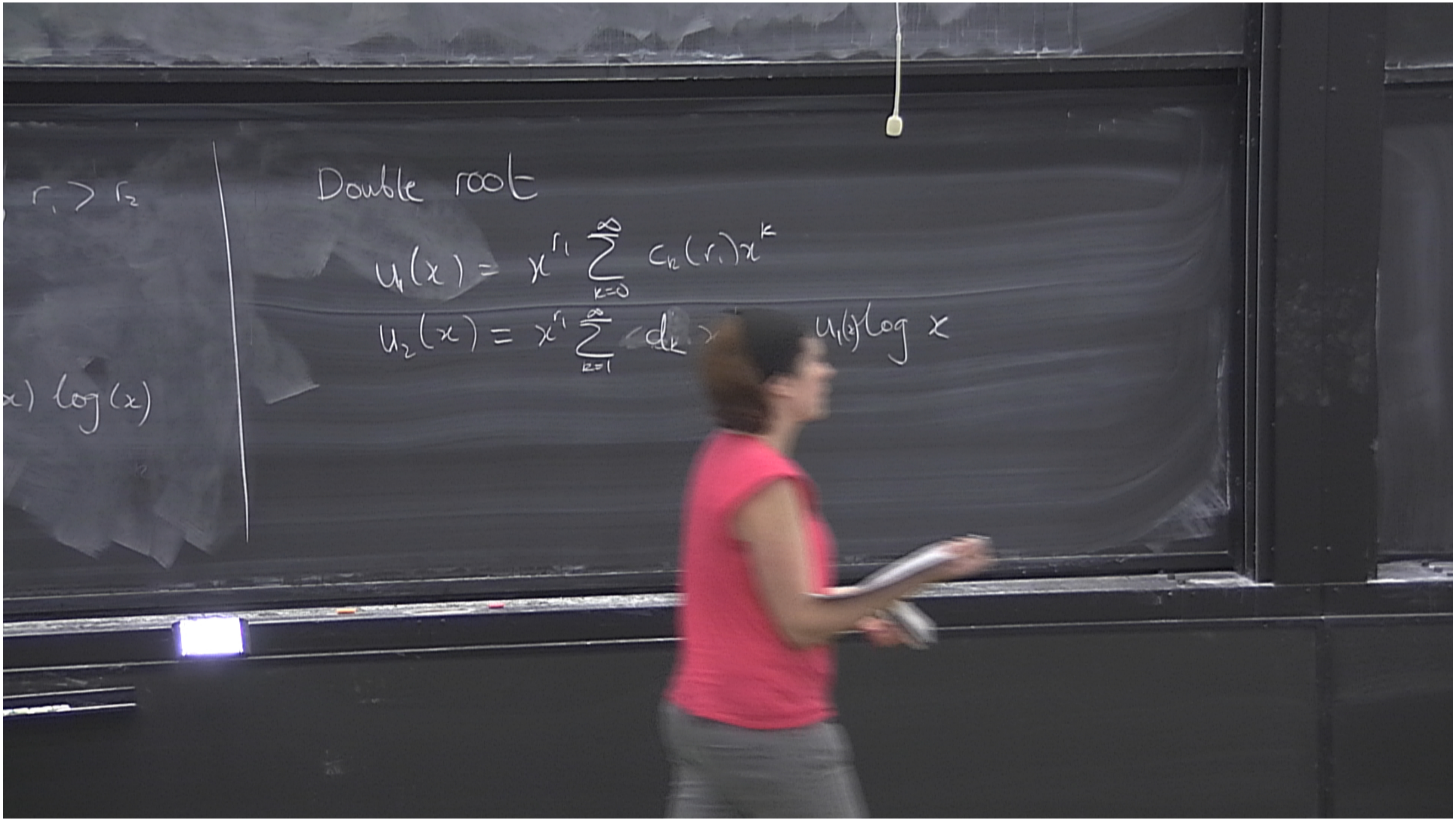

k=0, r(r-1)+rα0+β0 = 0
Case 1 Two distinct roots not differing by an integer
=) two independen ower series solutions u1(x), u2(x)

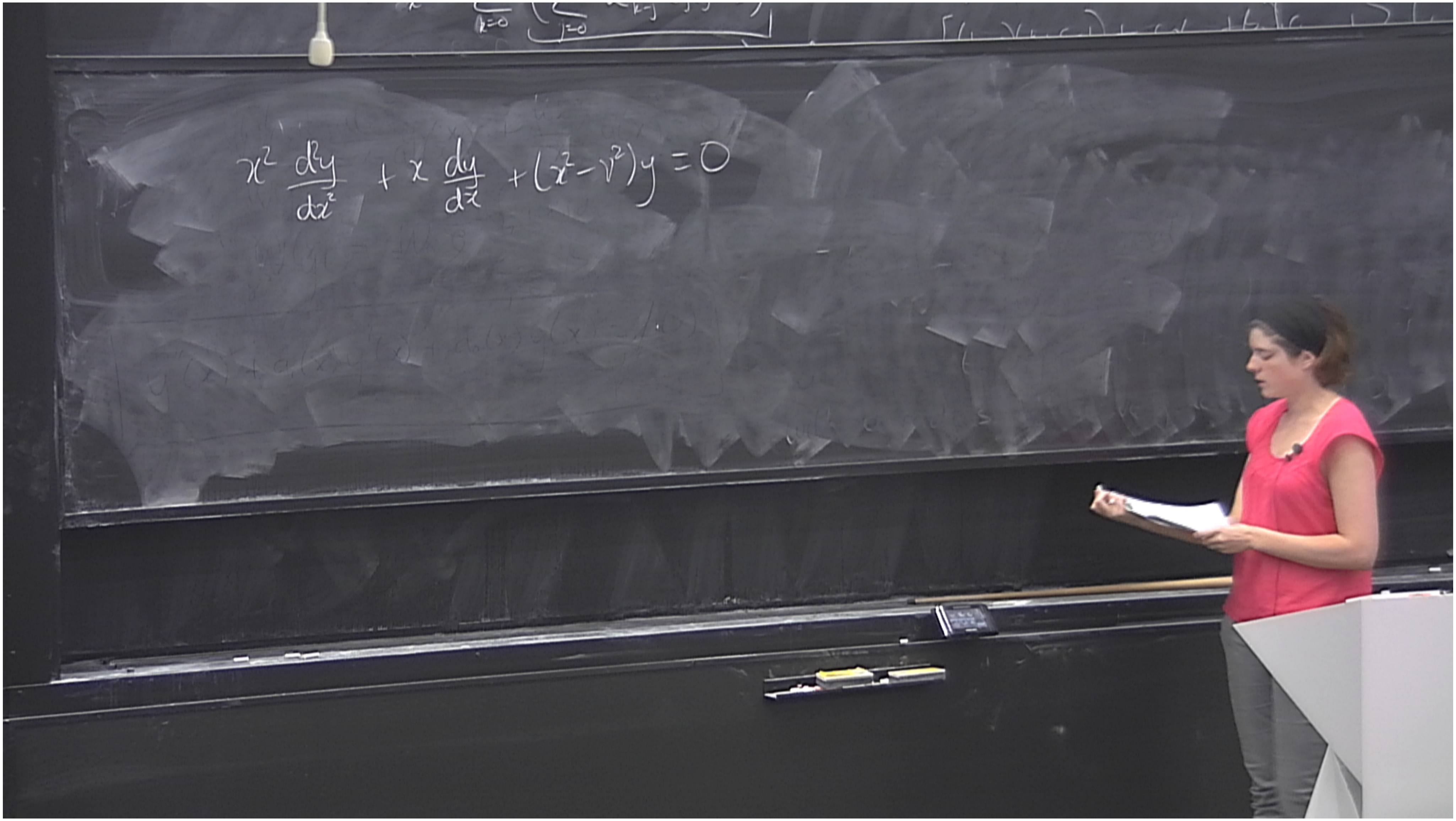

Two roots differing by an integer , $r_1 > r_2$
$u_1(x)$
x^{r_1}

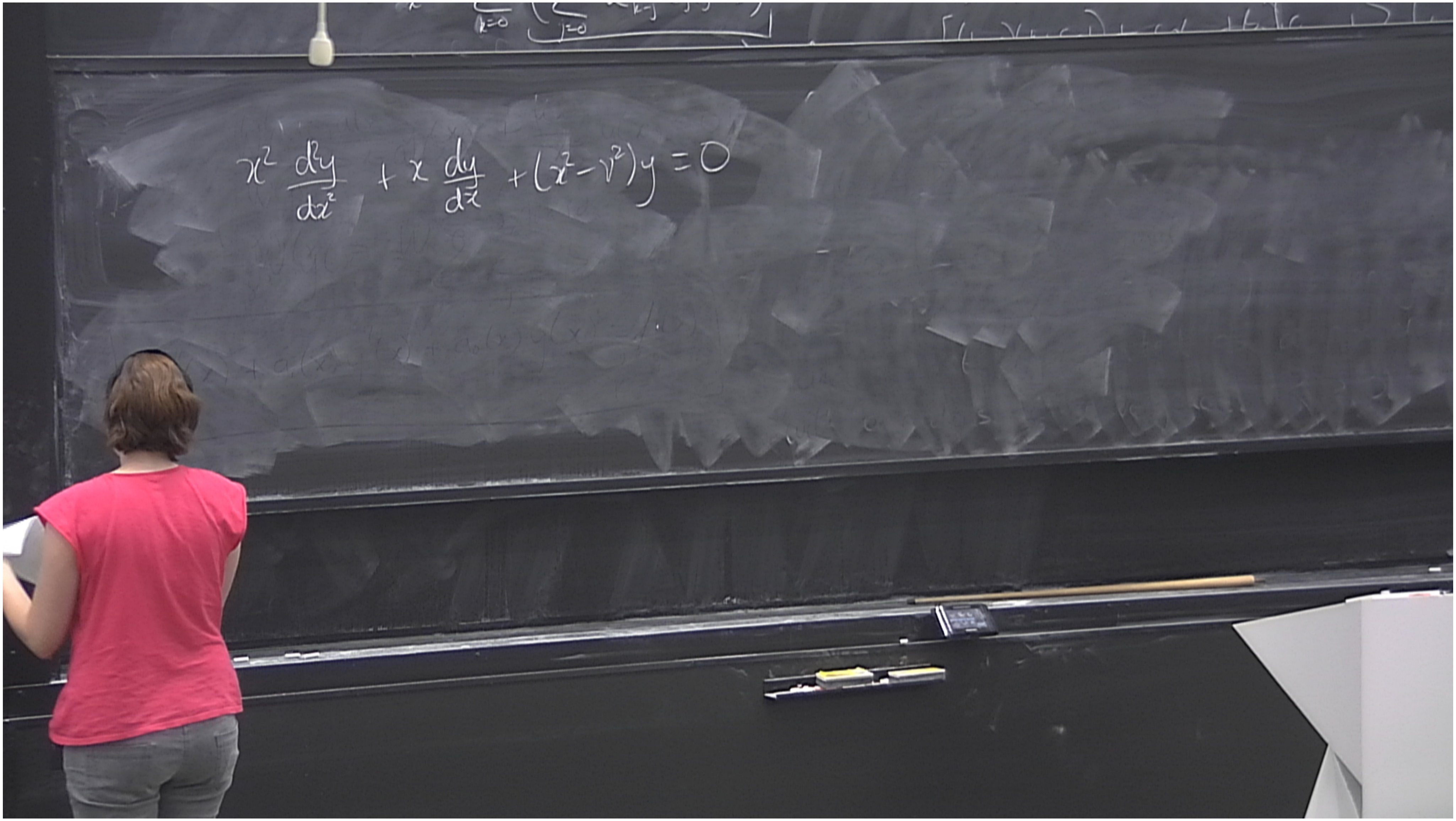

Two roots differing by an integer ; $r_1 > r_2$
$= x^{r_1} \sum_{k=0}^{\infty} c_k(r_1) x^k$
$= x^{r_2} \sum_{k=0}^{\infty} c_k(r_2) x^k + c\, u_1(x) \log(x)$

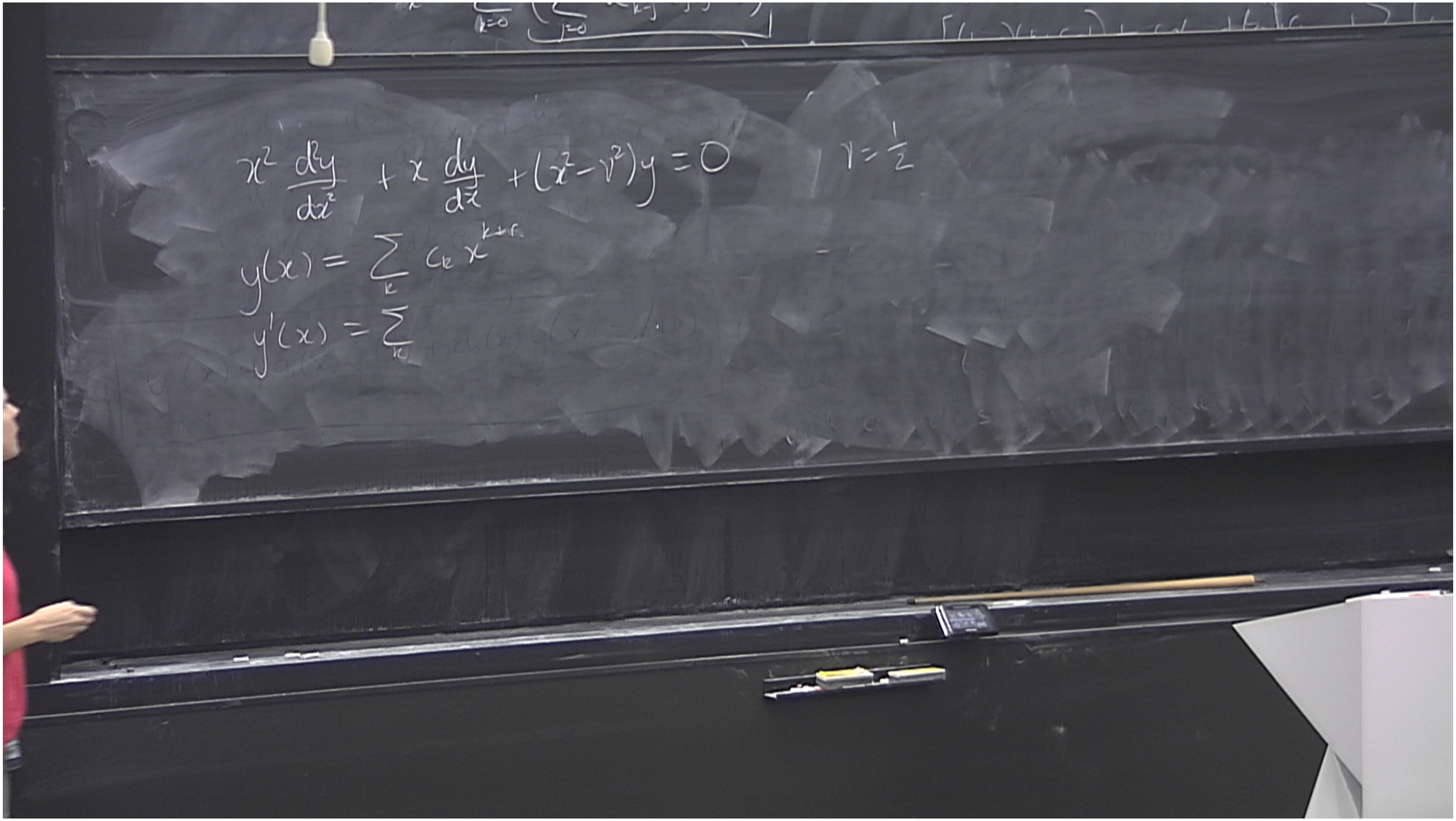

Two roots differing by an integer, $r_1 > r_2$
$(x) = x^{r_1} \sum_{k=0}^{\infty} c_k(r_1) x^k$
$(x) = x^{r_2} \sum_{k=0}^{\infty} c_k(r_2) x^k + c\, u_1(x) \log(x)$
Double root

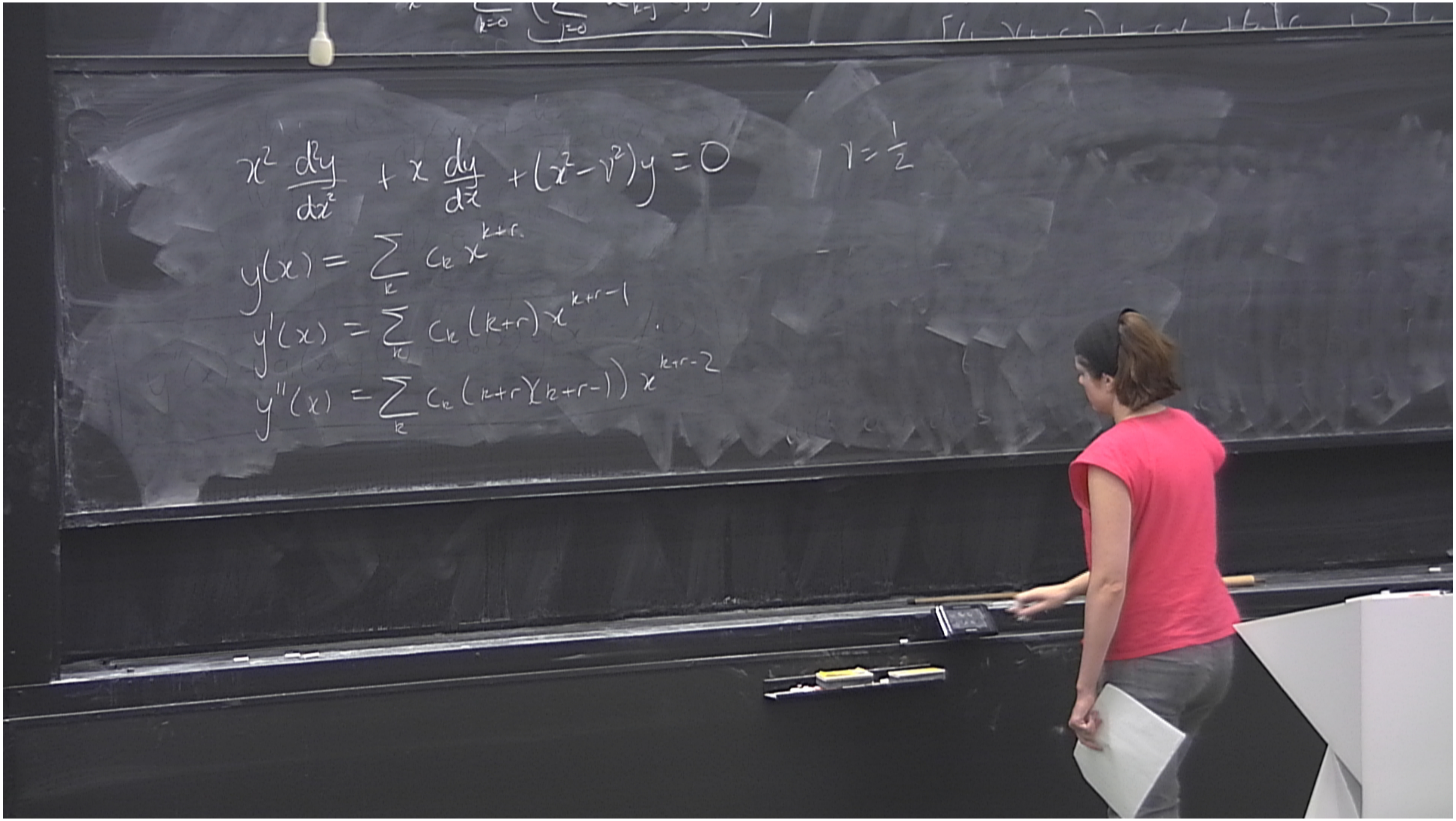

$r_1 > r_2$
Double root
$u_1(x) = x^{r_1} \sum_{k=0}^{\infty} c_k(r_1) x^k$
$u_2(x) = x^{r_1} \sum_{k=1}^{\infty} d_k x^k + u_1 \log x$


$r_1 > r_2$
$x)\log(x)$
Double root
$u_1(x) = x^{r_1}\sum_{k=0}^{\infty} c_k(r_1)x^k$
$u_2(x) = x^{r_1}\sum_{k=1}^{\infty} d_k$
$u_1(x)\log x$

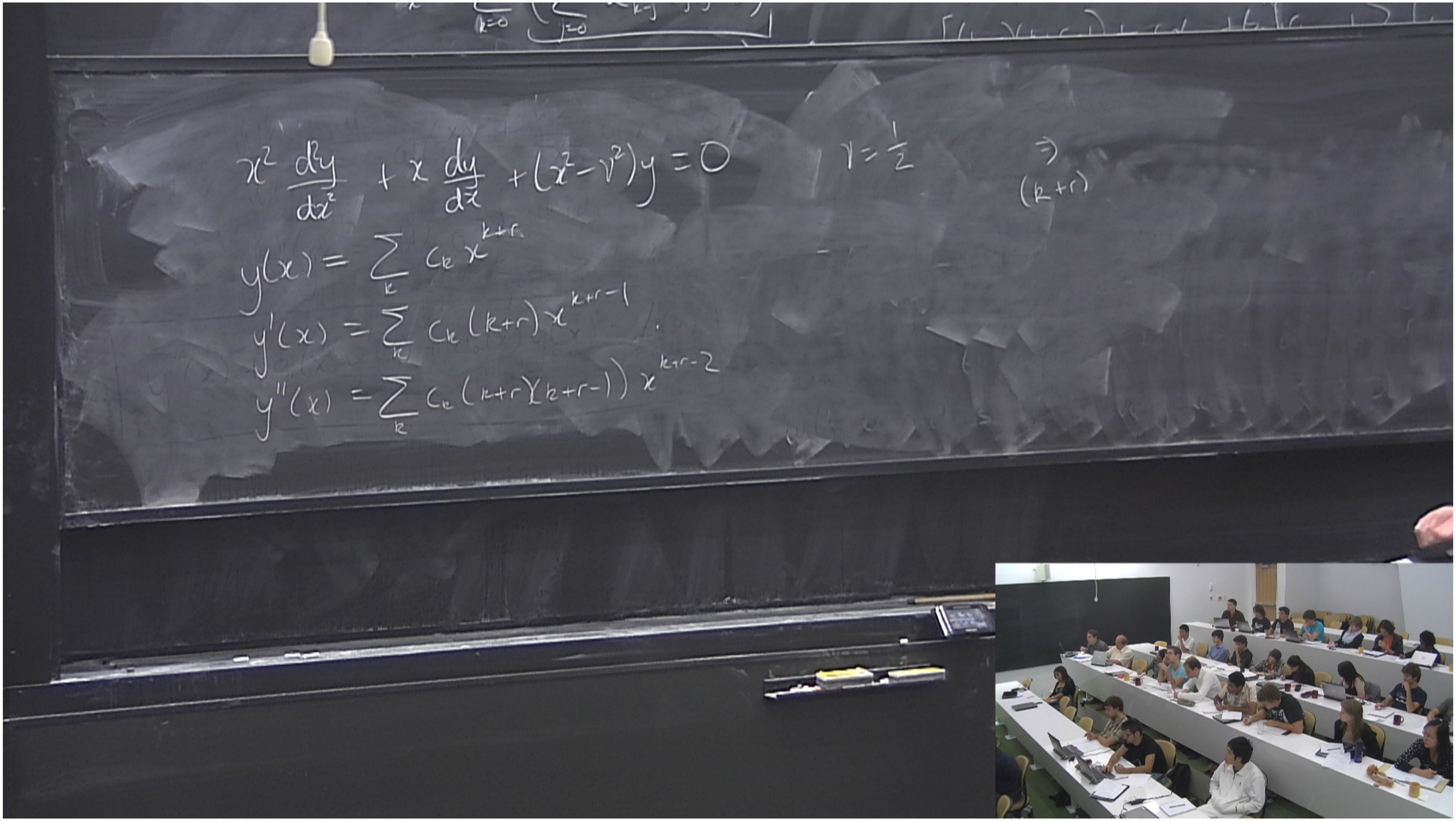

$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$


$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$


$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0$
$\nu = \frac{1}{2}$
$y(x) = \sum_k c_k x^{k+r}$
$y'(x) = \sum_k$


$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$
$\nu = \frac{1}{2}$
$y(x) = \sum_k c_k x^{k+r}$
$y'(x) = \sum_k c_k (k+r) x^{k+r-1}$
$y''(x) = \sum_k c_k (k+r)(k+r-1) x^{k+r-2}$


$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$
$\nu = \frac{1}{2}$
$\Rightarrow$
$(k+r)$
$y(x) = \sum_k c_k x^{k+r}$
$y'(x) = \sum_k c_k (k+r) x^{k+r-1}$
$y''(x) = \sum_k c_k (k+r)(k+r-1) x^{k+r-2}$


$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$
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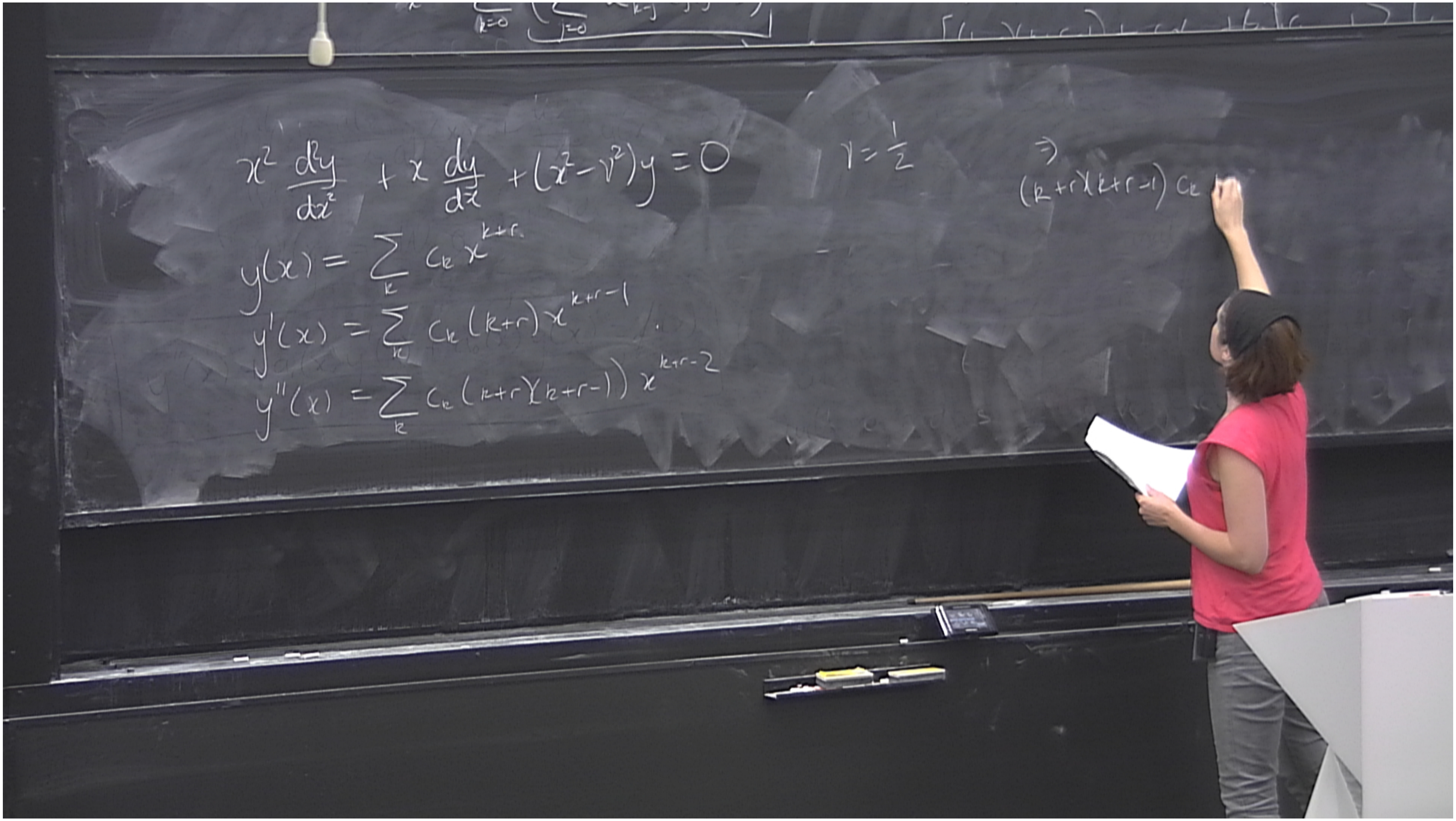
$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2) y = 0$
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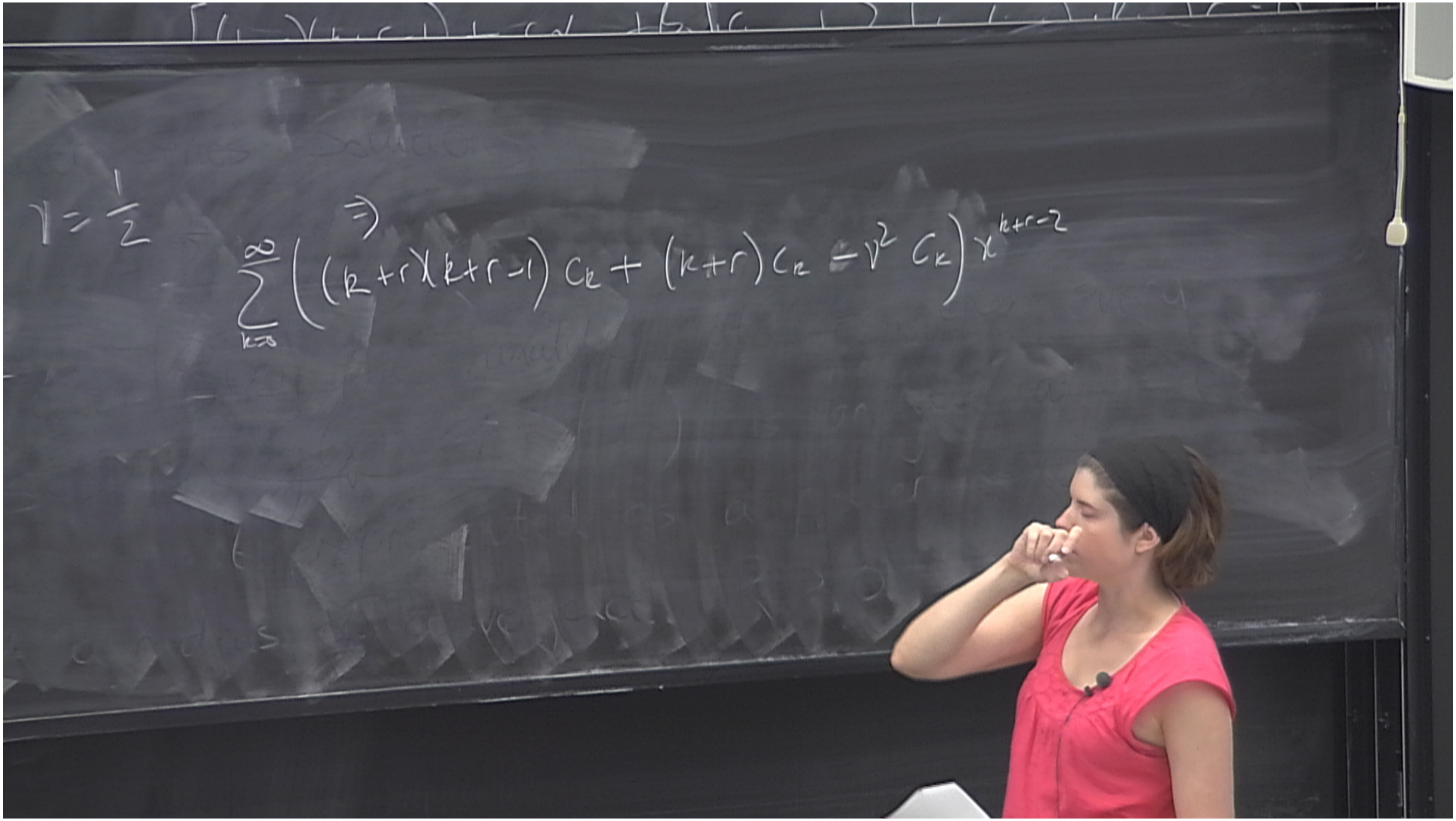
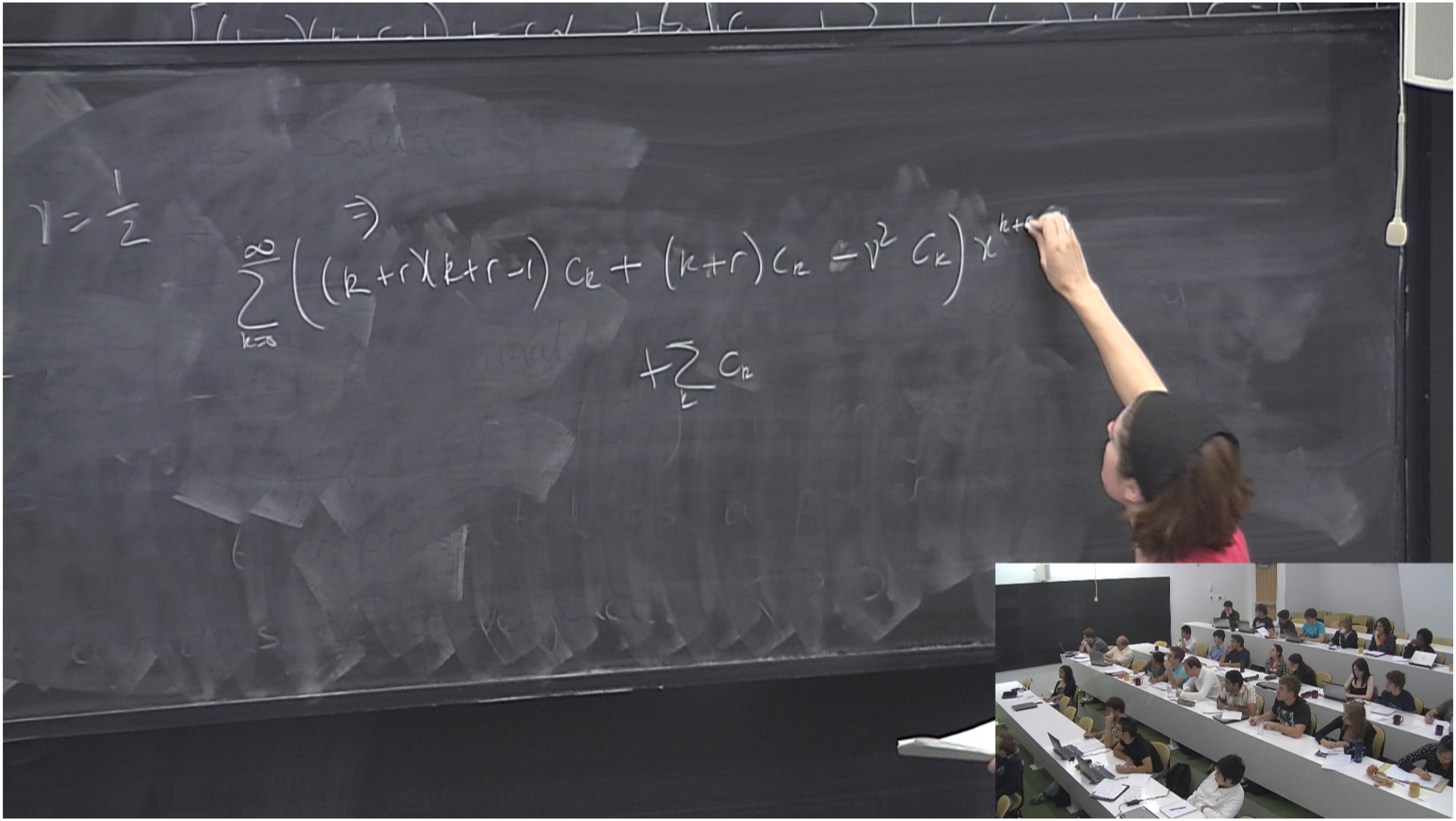
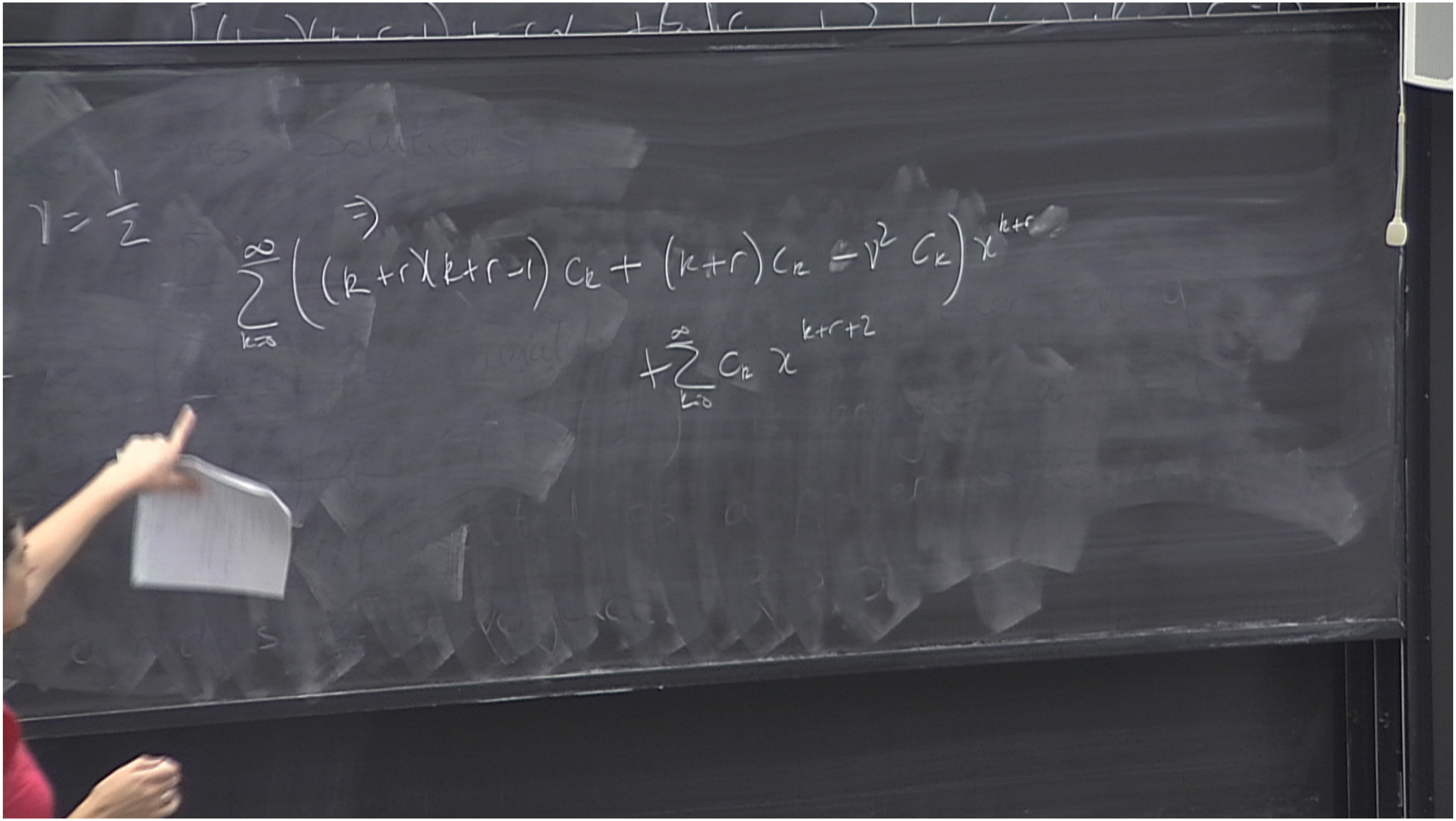
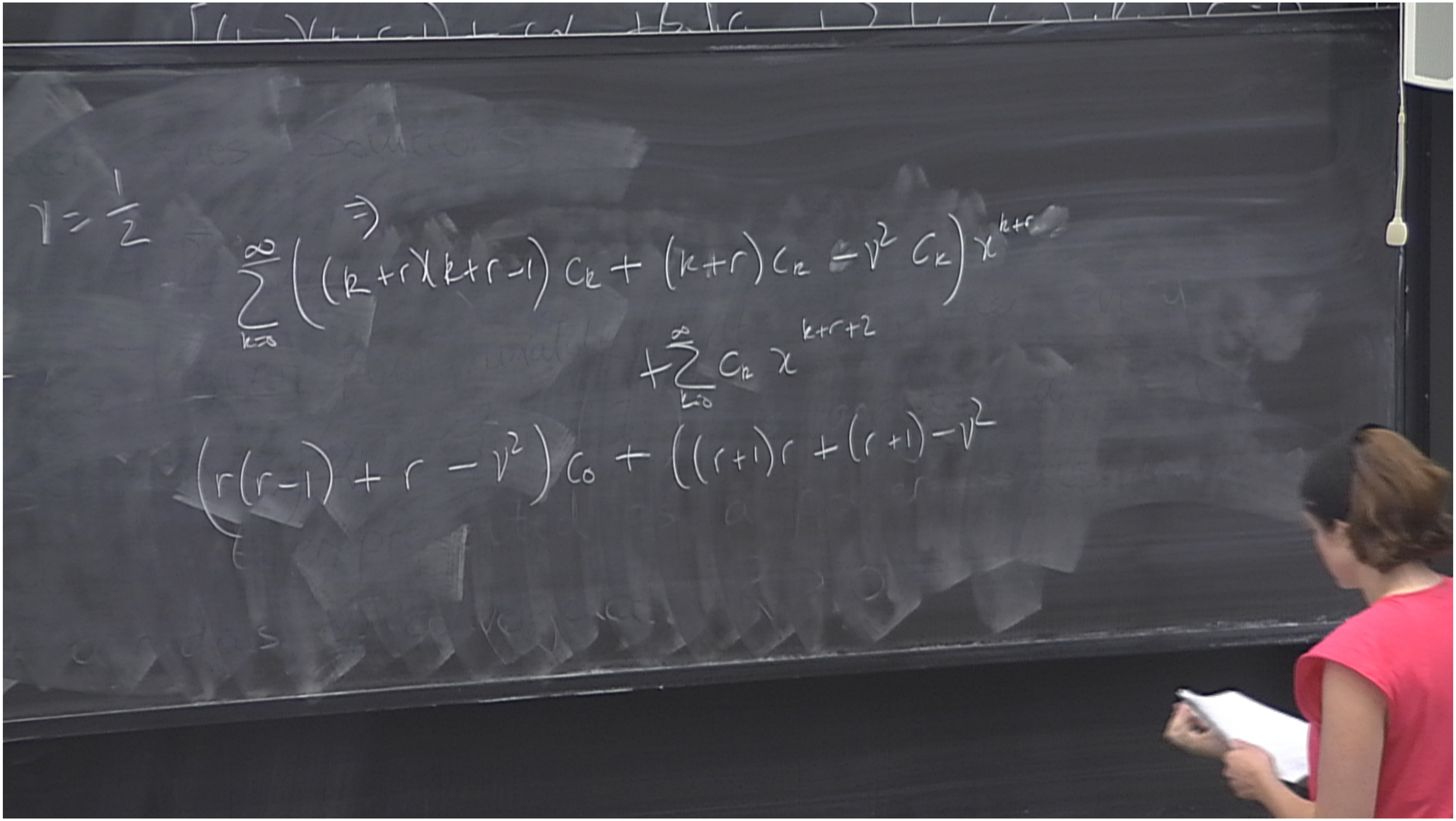
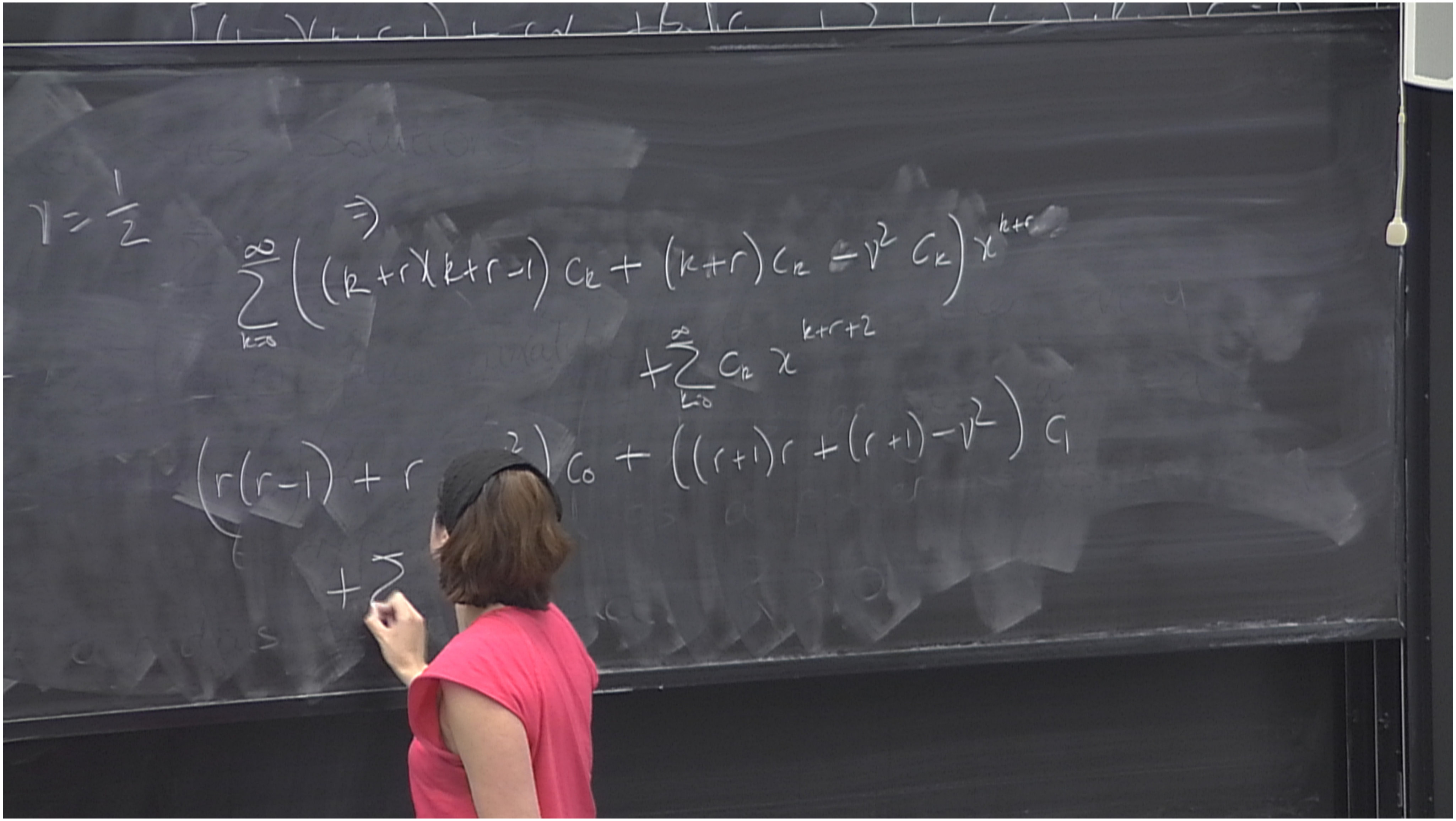
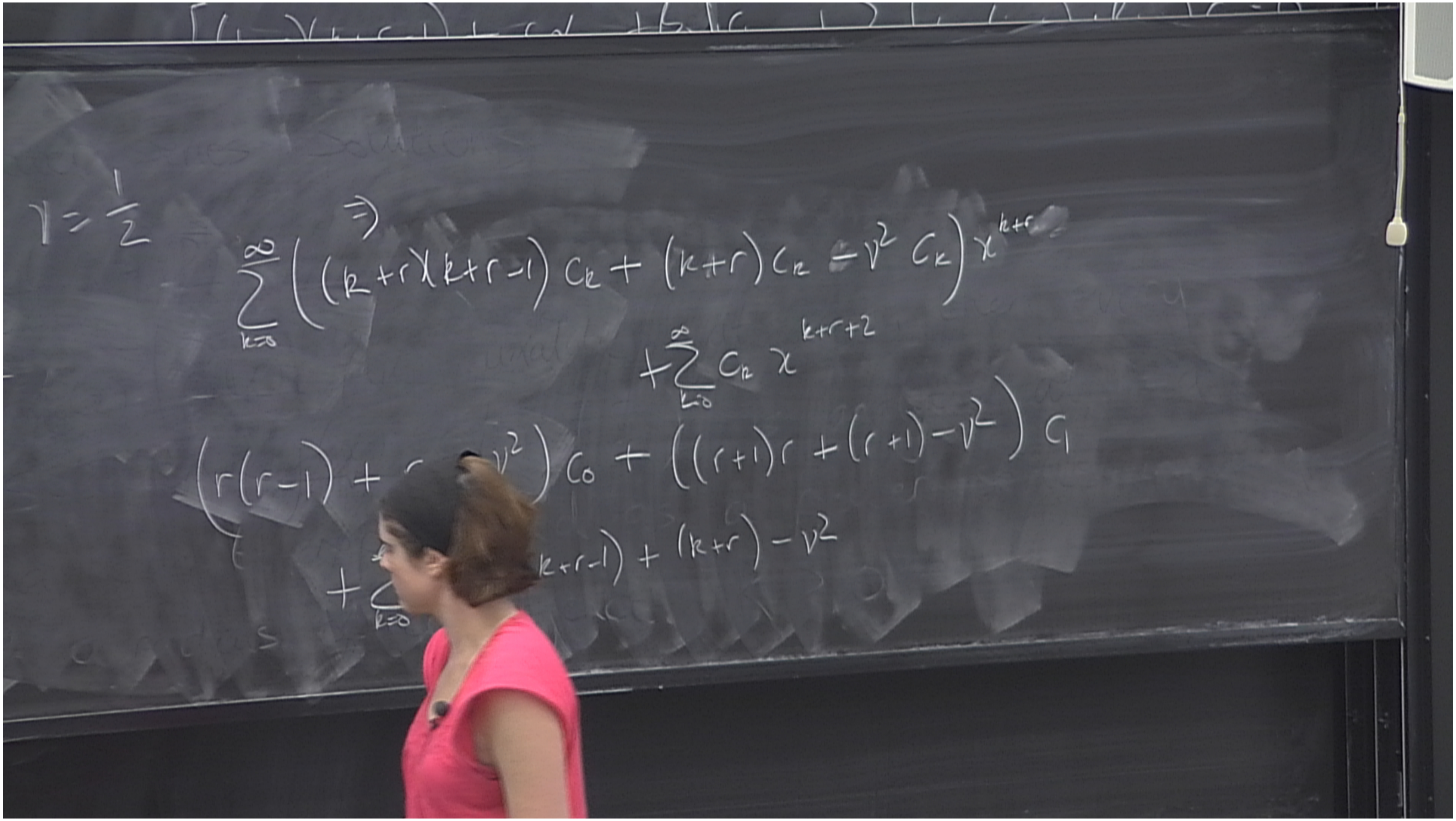

$r = \frac{1}{2} \Rightarrow$
$\sum_{k=0}^{\infty} \left((k+r)(k+r-1) c_k + (k+r) c_k - \nu^2 c_k \right) x^{k+r-2}$


r = 1/2 ⇒
∑_{k=0}^∞ ((k+r)(k+r−1) c_k + (k+r) c_k − ν² c_k) x^{k+r}
+ ∑_k c_k


$r = \frac{1}{2}$ ⇒
$\sum_{k=0}^{\infty}\left((k+r)(k+r-1)c_k + (k+r)c_k - \nu^2 c_k\right)x^{k+r}$
$+\sum_{k=0}^{\infty} c_k x^{k+r+2}$


$\nu = \frac{1}{2} \Rightarrow$
$\sum_{k=0}^{\infty}\Big((k+r)(k+r-1)c_k + (k+r)c_k - \nu^2 c_k\Big)x^{k+r}$
$+\sum_{k=0}^{\infty} c_k x^{k+r+2}$
$\big(r(r-1) + r - \nu^2\big)c_0 + \big((r+1)r + (r+1) - \nu^2$


$\nu = \frac{1}{2}$
$\Rightarrow$
$\sum_{k=0}^{\infty}\left((k+r)(k+r-1)c_k + (k+r)c_k - \nu^2 c_k\right)x^{k+r}$
$+\sum_{k=0}^{\infty} c_k x^{k+r+2}$
$c_0 + \left((r+1)r + (r+1) - \nu^2\right)c_1$


$r = \frac{1}{2}$ $\Rightarrow$
$\sum_{k=0}^{\infty} \left((k+r)(k+r-1) c_k + (k+r) c_k - \nu^2 c_k \right) x^{k+r}$
$+ \sum_{k=0}^{\infty} c_k x^{k+r+2}$
$\left(r(r-1) + \quad \nu^2 \right) c_0 + \left((r+1)r + (r+1) - \nu^2 \right) c_1$
$+ \sum_{k=0} \quad k+r-1) + (k+r) - \nu^2$

$$\sum_{k=0}^{\infty}\Big((k+r)(k+r-1)c_k + (k+r)c_k - \nu^2 c_k\Big)x^{k+r}$$

$$+\sum_{k=0}^{\infty} c_k\, x^{k+r+2}$$

$$\big(r(r-1) + r - \nu^2\big)c_0 + \big((r+1)r + (r+1) - \nu^2\big)c_1$$

$$+\sum_{k=0}^{\infty}\Big(\big((k+r)(k+r-1) + (k+r) - \nu^2\big)c_k + c_{k-2}\Big)$$

$$B x^{r_2}\sum_{k=0}^{\infty} c_k(r_2)\, x^k$$

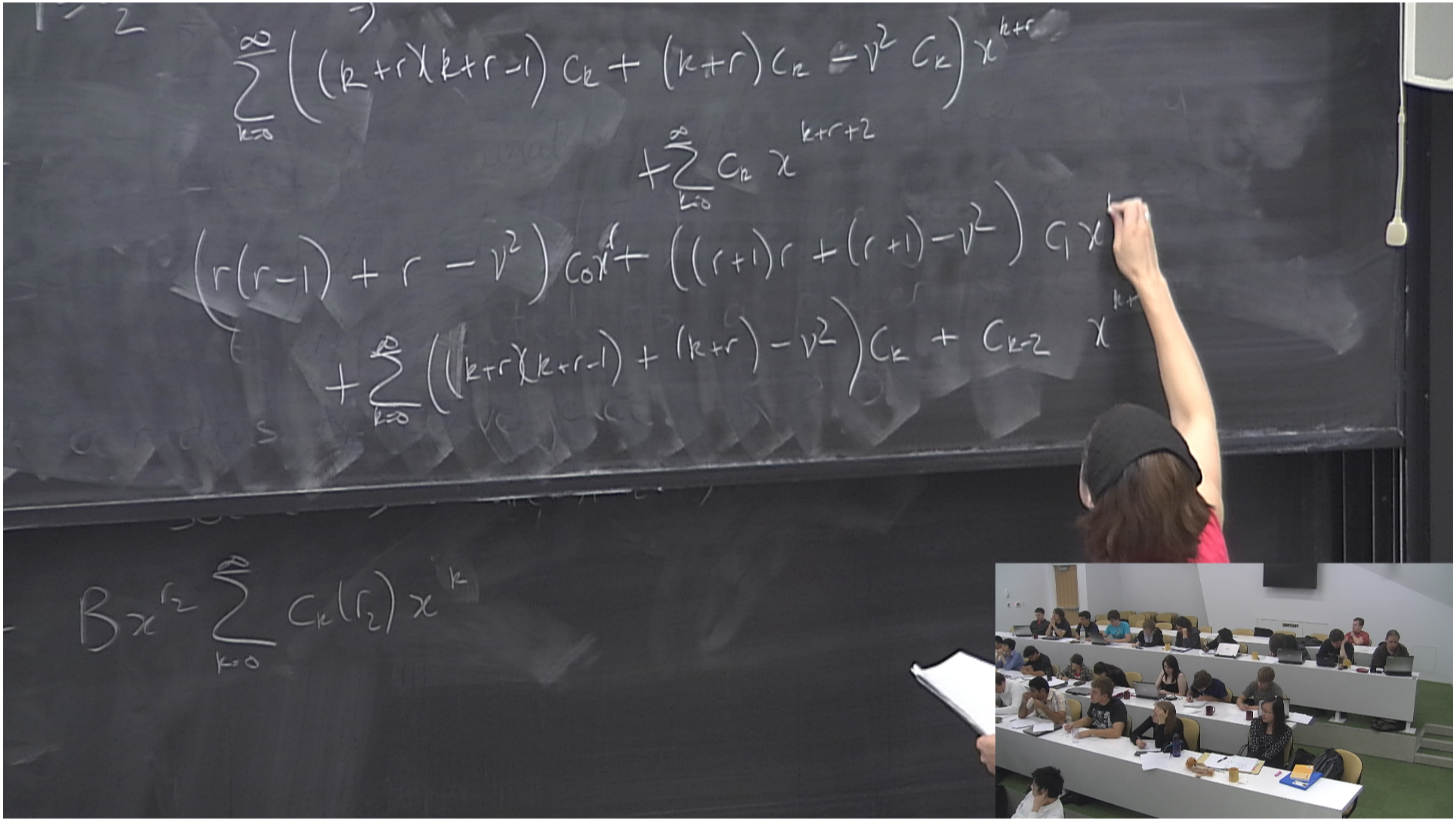
$$\sum_{k=0}^{\infty} \left((k+r)(k+r-1) c_k + (k+r) c_k - \nu^2 c_k \right) x^{k+r}$$

$$+ \sum_{k=0}^{\infty} c_k \, x^{k+r+2}$$

$$\left(r(r-1) + r - \nu^2 \right) c_0 x^0 + \left((r+1)r + (r+1) - \nu^2 \right) c_1 x^1$$

$$+ \sum_{k=0}^{\infty} \left(\left((k+r)(k+r-1) + (k+r) - \nu^2 \right) c_k + c_{k-2} \right) x^{k+r}$$

$$- B x^{r_2} \sum_{k=0}^{\infty} c_k(r_2) \, x^k$$


$\sum_{k=0}^{\infty}\left((k+r)(k+r-1)c_k + (k+r)c_k - \nu^2 c_k\right)x^{k+r}$
$+\sum_{k=0}^{\infty} c_k x^{k+r+2}$
$\left(r(r-1)+r-\nu^2\right)c_0 x^r + \left((r+1)r+(r+1)-\nu^2\right)c_1 x$
$+\sum_{k=0}^{\infty}\left((k+r)(k+r-1)+(k+r)-\nu^2\right)c_k + c_{k-2}\ x^{k+r}$
$Bx^{r_2}\sum_{k=0}^{\infty} c_k(r_2)x^k$

$$(r^2-\nu^2)c_0 = 0$$

$$r^2 = \nu^2$$

$$r = \pm\nu$$


$(r^2 - v^2)c_0 = 0$
$r^2 = v^2$
$r = \pm v$

$$(r^2 - \nu^2) c_0 = 0$$

$$r^2 = \nu^2$$

$$r = \pm \nu$$

ν integer $\Rightarrow$ $y(x) = A J_\nu(x) + B K_\nu(x)$

$$(r^2 - \nu^2) c_0 = 0$$

$$r^2 = \nu^2$$

$$r = \pm \nu$$

ν integer $\Rightarrow$ $y(x) = A J_\nu(x) + B K_\nu(x)$

ν $\frac{1}{2}$ integer $\Rightarrow$

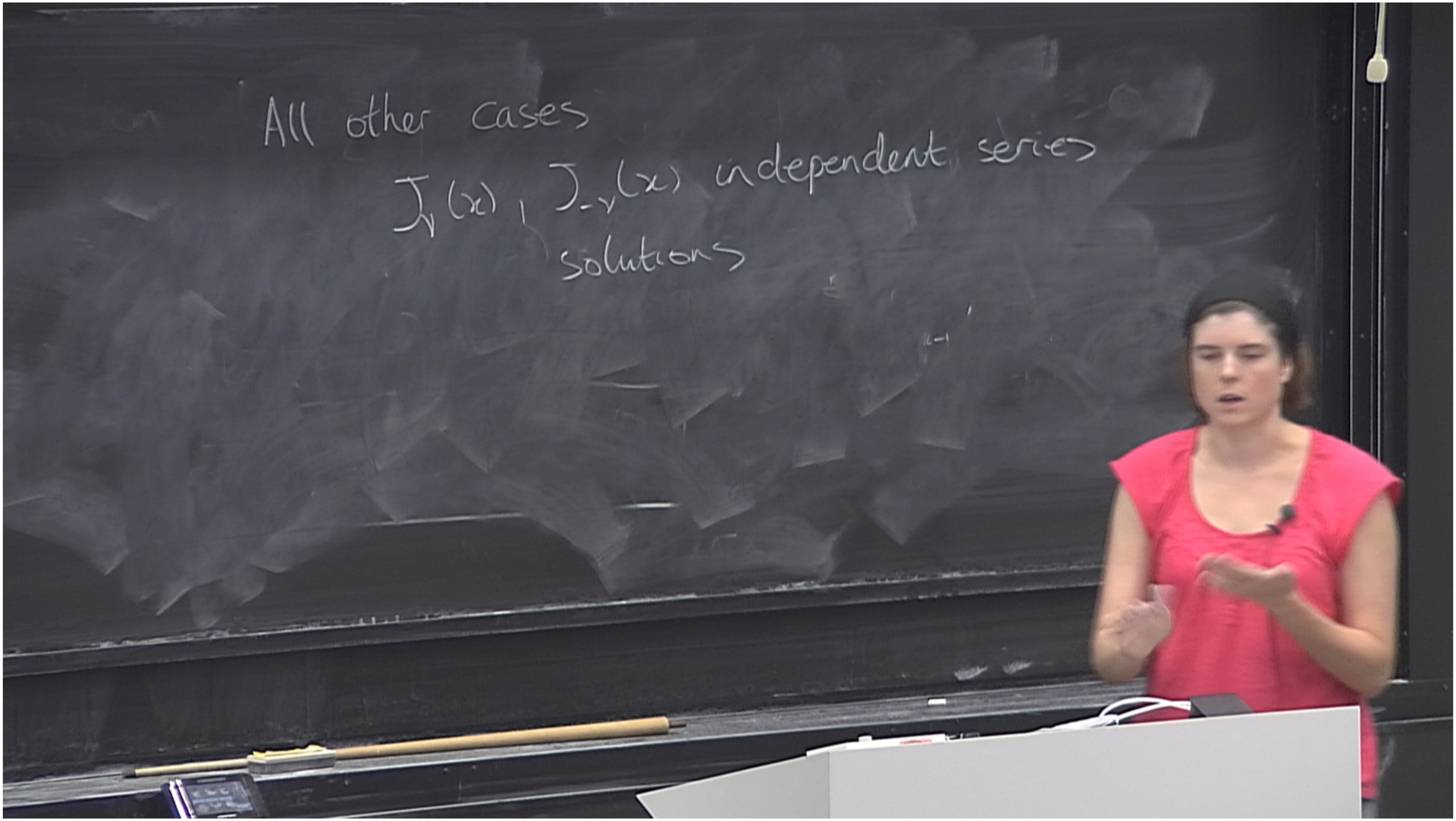
$$(r^2 - \nu^2) c_0 = 0$$

$$r^2 = \nu^2$$

$$r = \pm \nu$$

ν integer $\Rightarrow$ $y(x) = A J_\nu(x) + B K_\nu(x)$

ν $\frac{1}{2}$ integer $\Rightarrow$ $c = 0$ $\Rightarrow$ no $\log(x)$ term


All other cases
$J_\nu(x)$, $J_{-\nu}(x)$ independent series
solutions