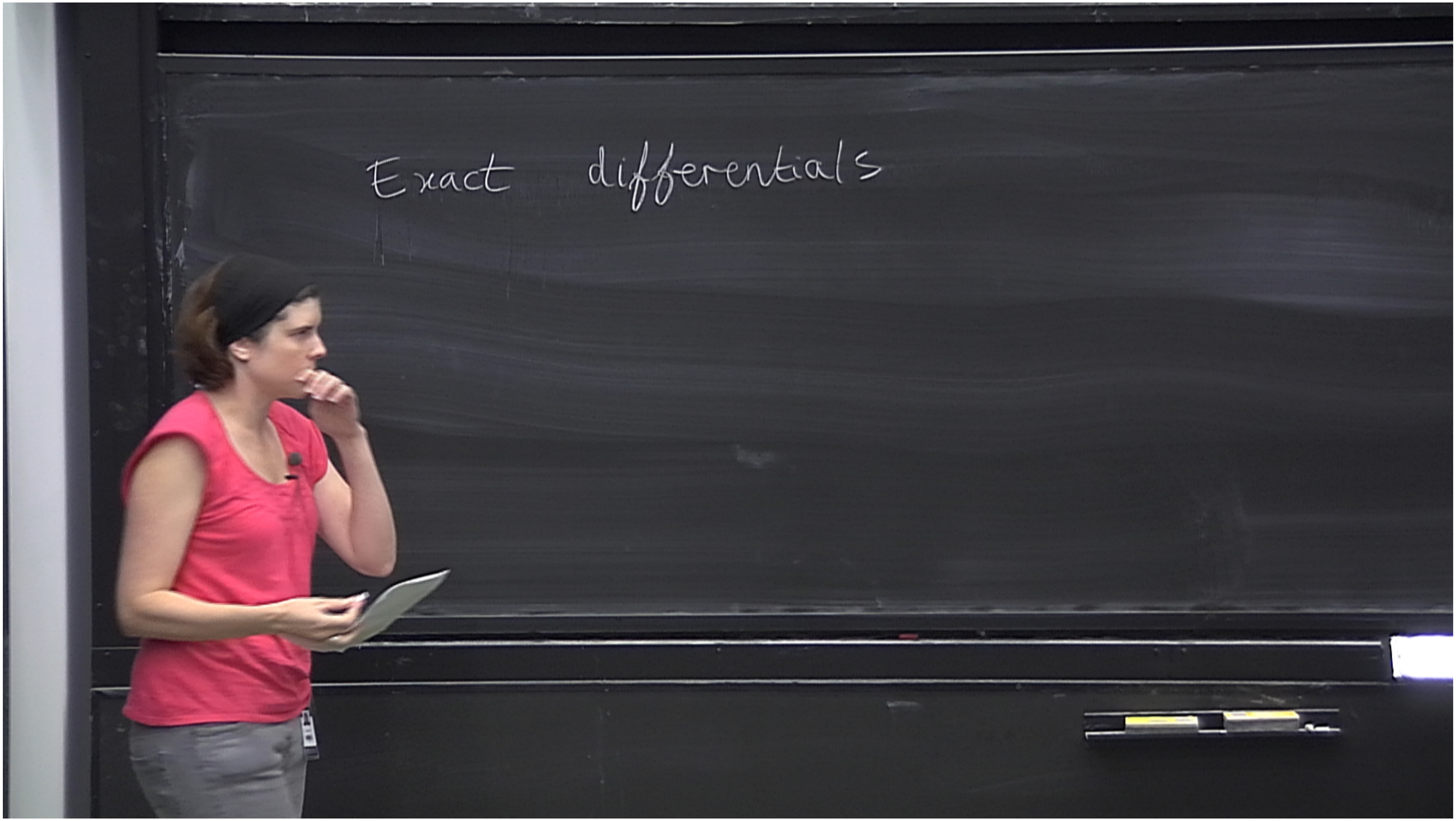
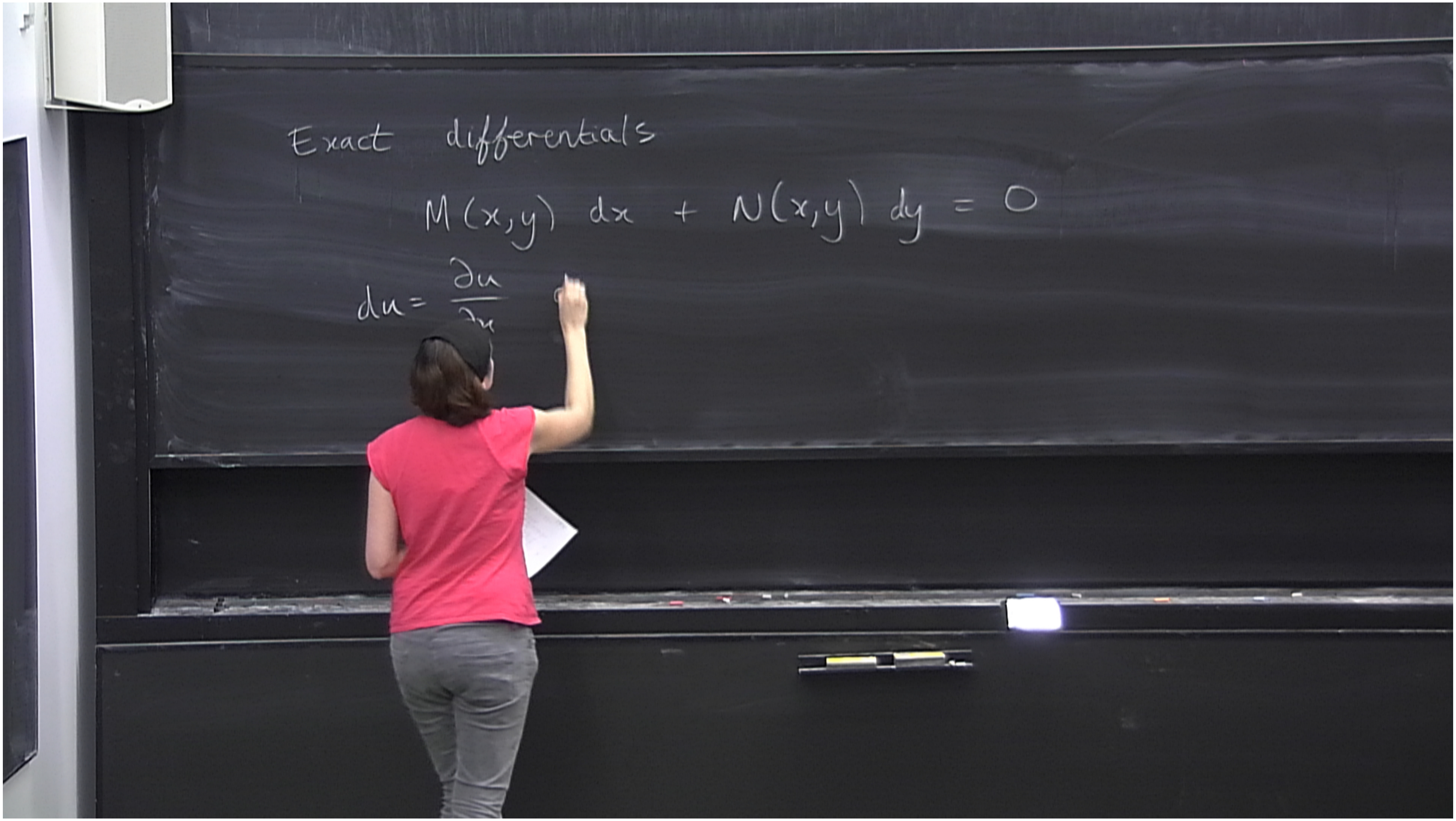
Title: Differential Equations - Lecture 2

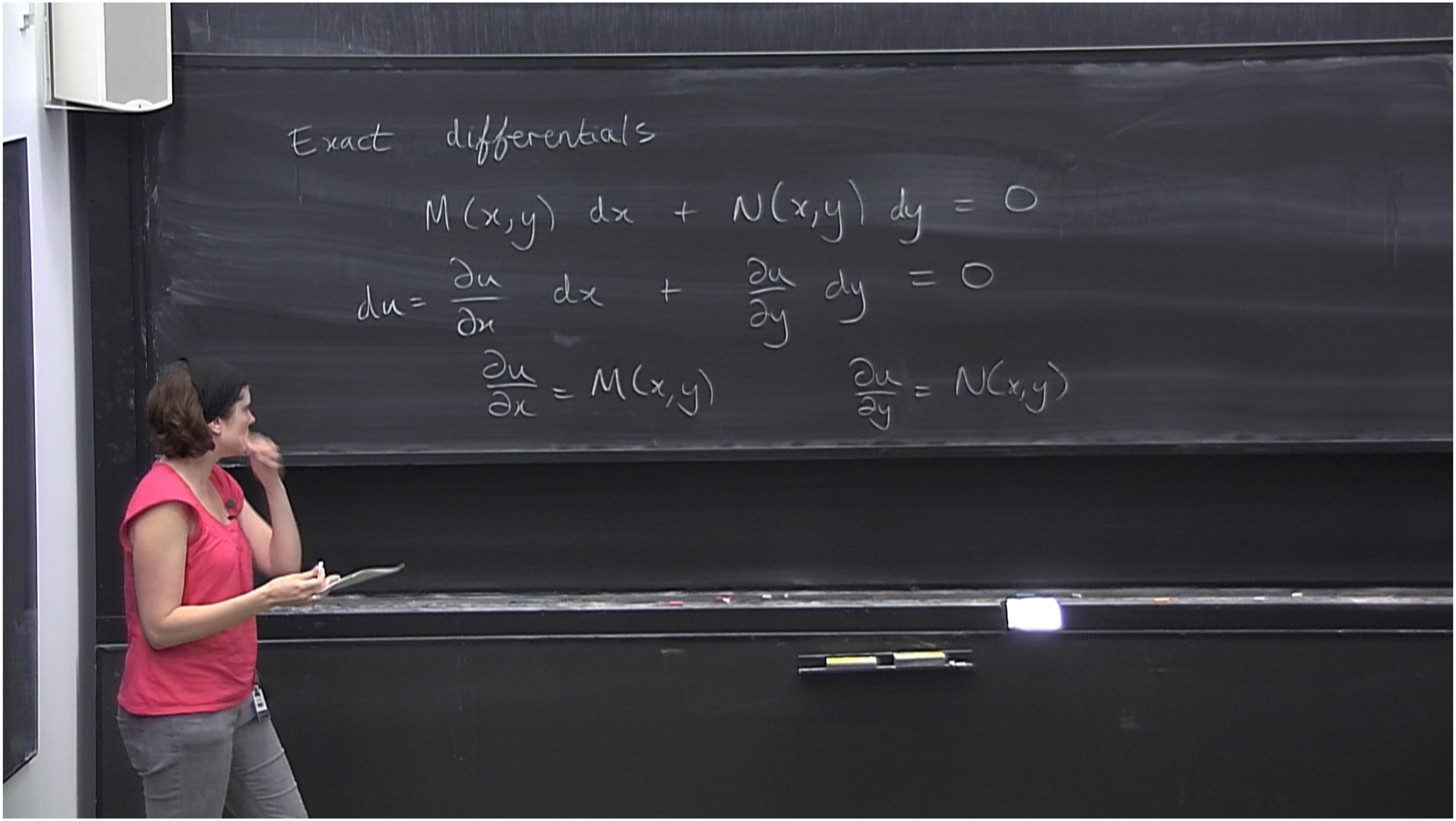
Date: Aug 29, 2011 09:00 AM

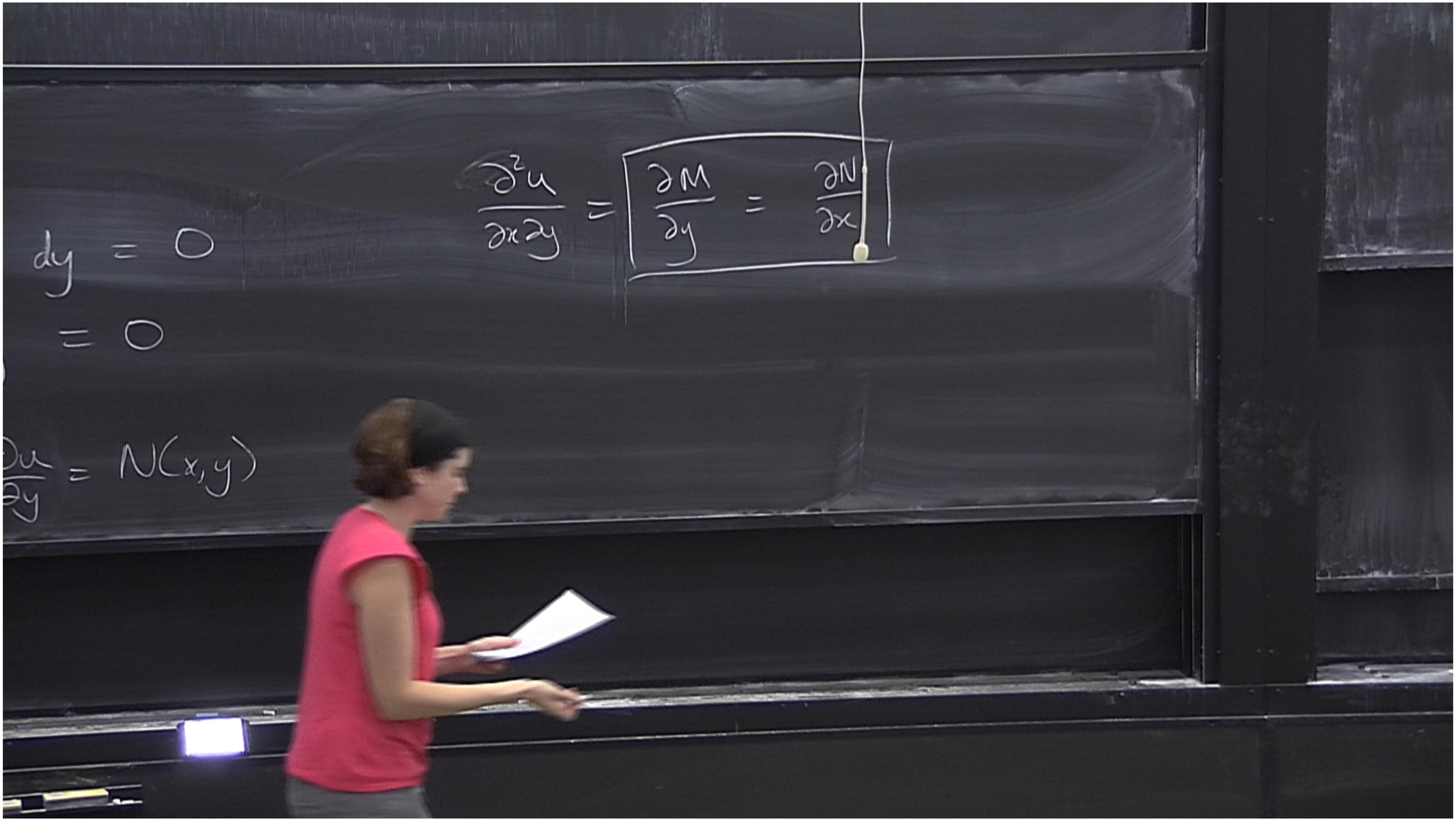
URL: http://pirsa.pi.local/11080141

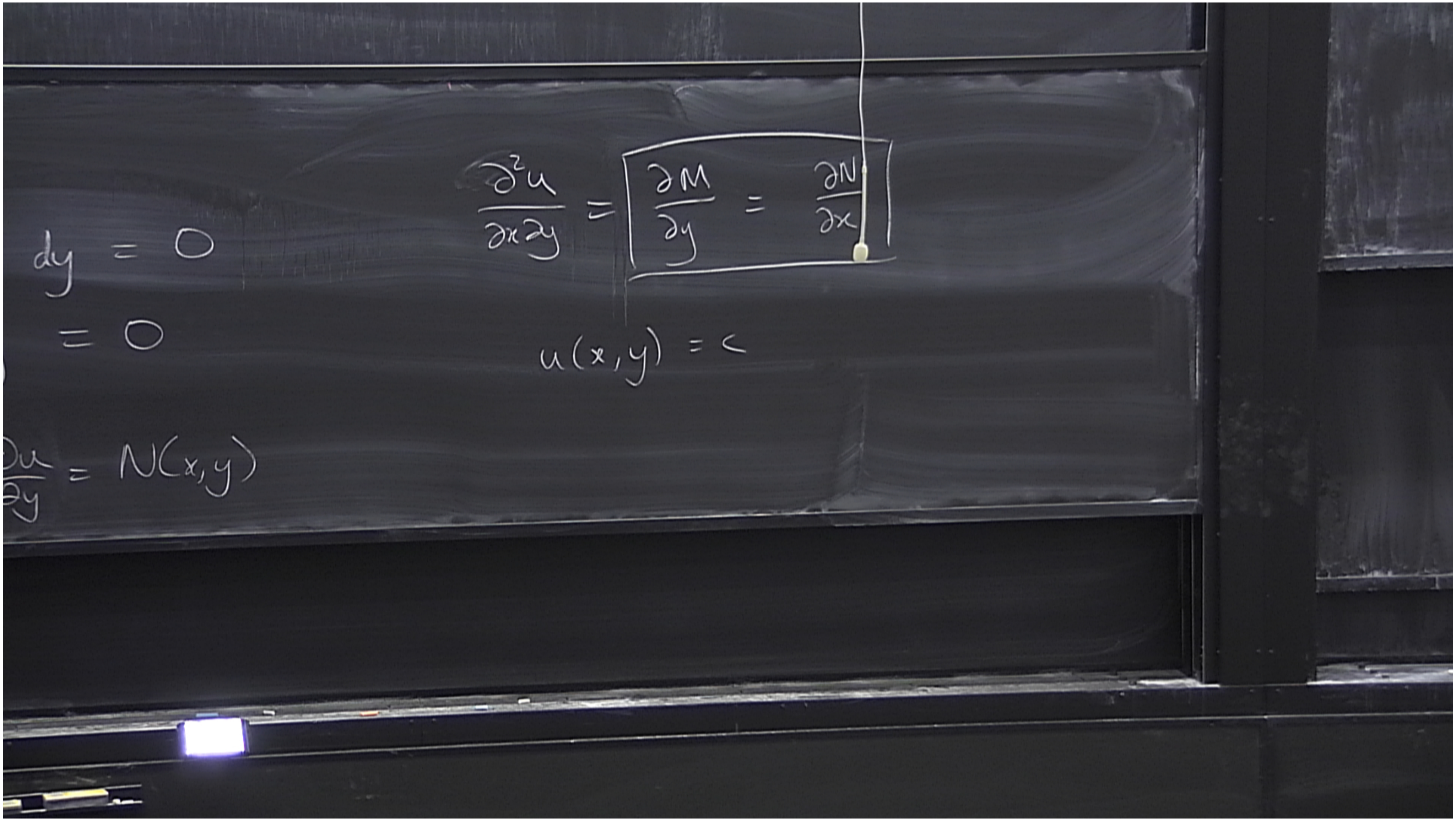
Abstract:


Exact differentials


Exact differentials
M(x,y) dx + N(x,y) dy = 0
du = ∂u/∂x


Exact differentials
$M(x,y)\,dx + N(x,y)\,dy = 0$
$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$
$\frac{\partial u}{\partial x} = M(x,y)$
$\frac{\partial u}{\partial y} = N(x,y)$


$\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$
dy = 0
= 0
= N(x,y)


$dy = 0$
$= 0$
$\frac{\partial u}{\partial y} = N(x,y)$
$\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$
$u(x,y) = c$

$$\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$dy = 0$

$= 0$

$$u(x,y) = c$$

$$u(x,y) = \int M(x,y)\, dx + f(y)$$

$$\frac{\partial u}{\partial y} = N(x,y)$$

$$dy = 0$$

$$= 0$$

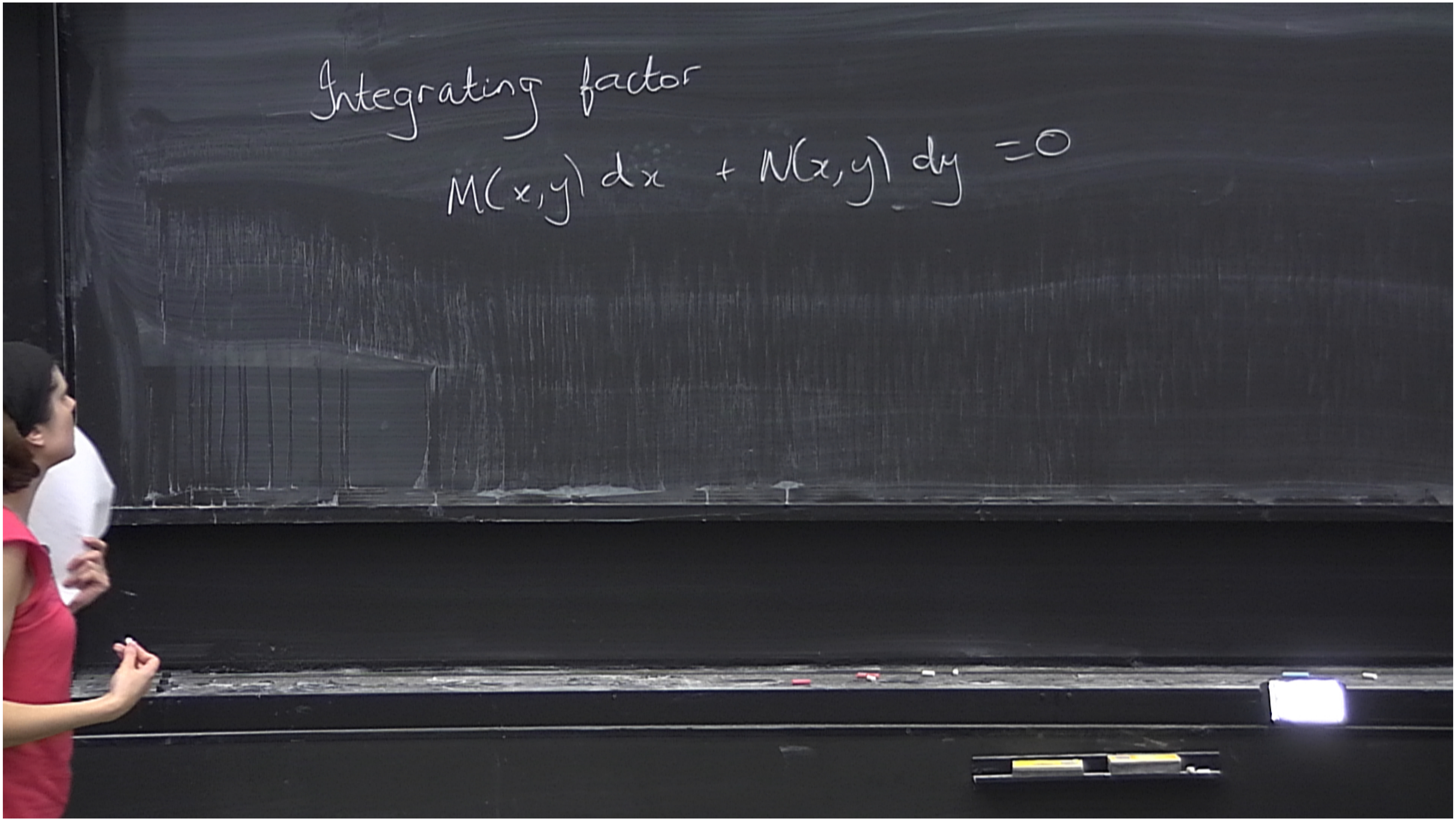
$$\frac{\partial u}{\partial y} = N(x,y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$u(x,y) = c$$

$$u(x,y) = \int M(x,y)\,dx + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\,dx\right) + f'(y) = N(x,y)$$


Integrating factor
$M(x,y)\,dx + N(x,y)\,dy = 0$

Integrating factor

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

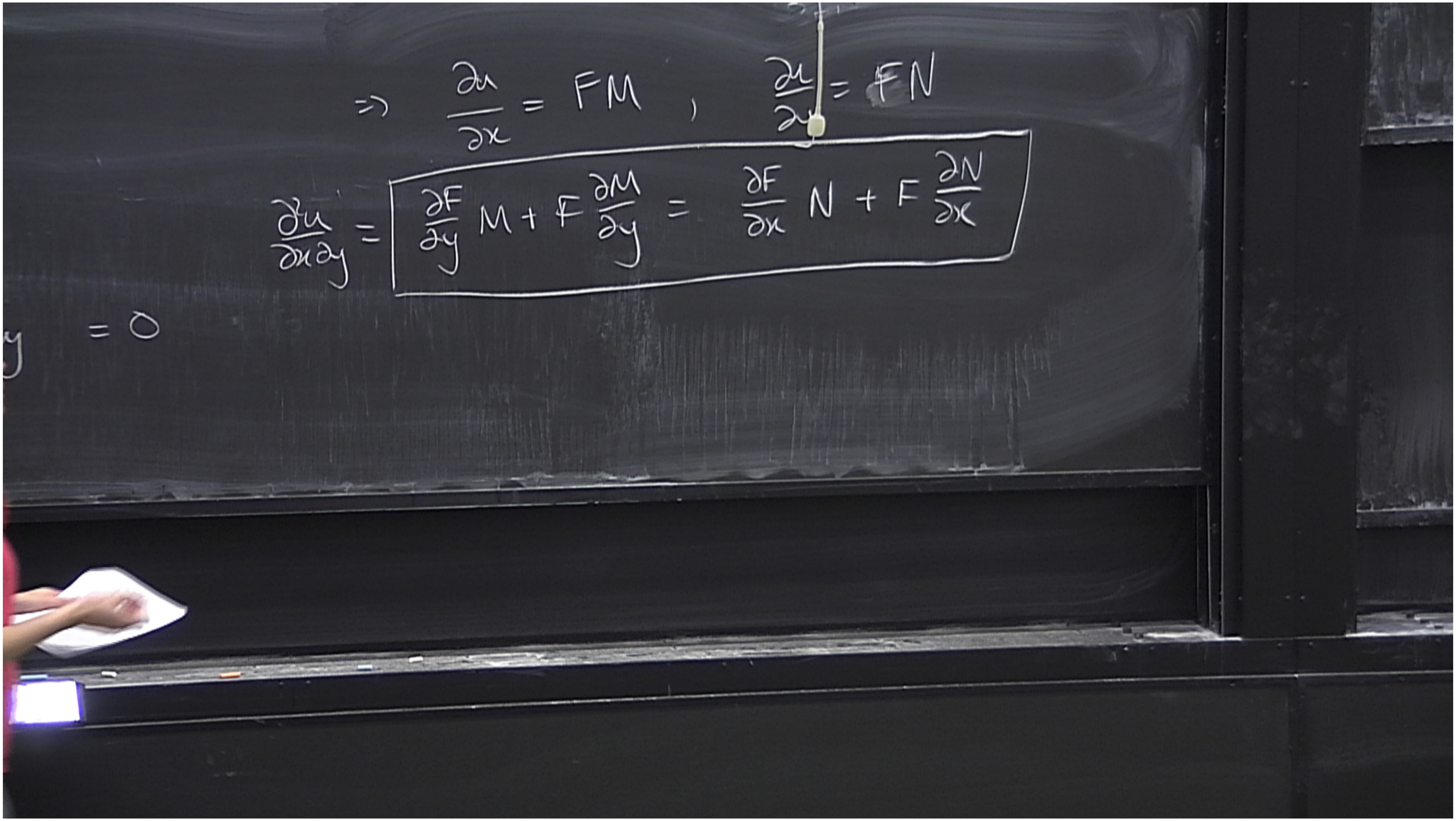
maybe $\exists\, F$ s.t

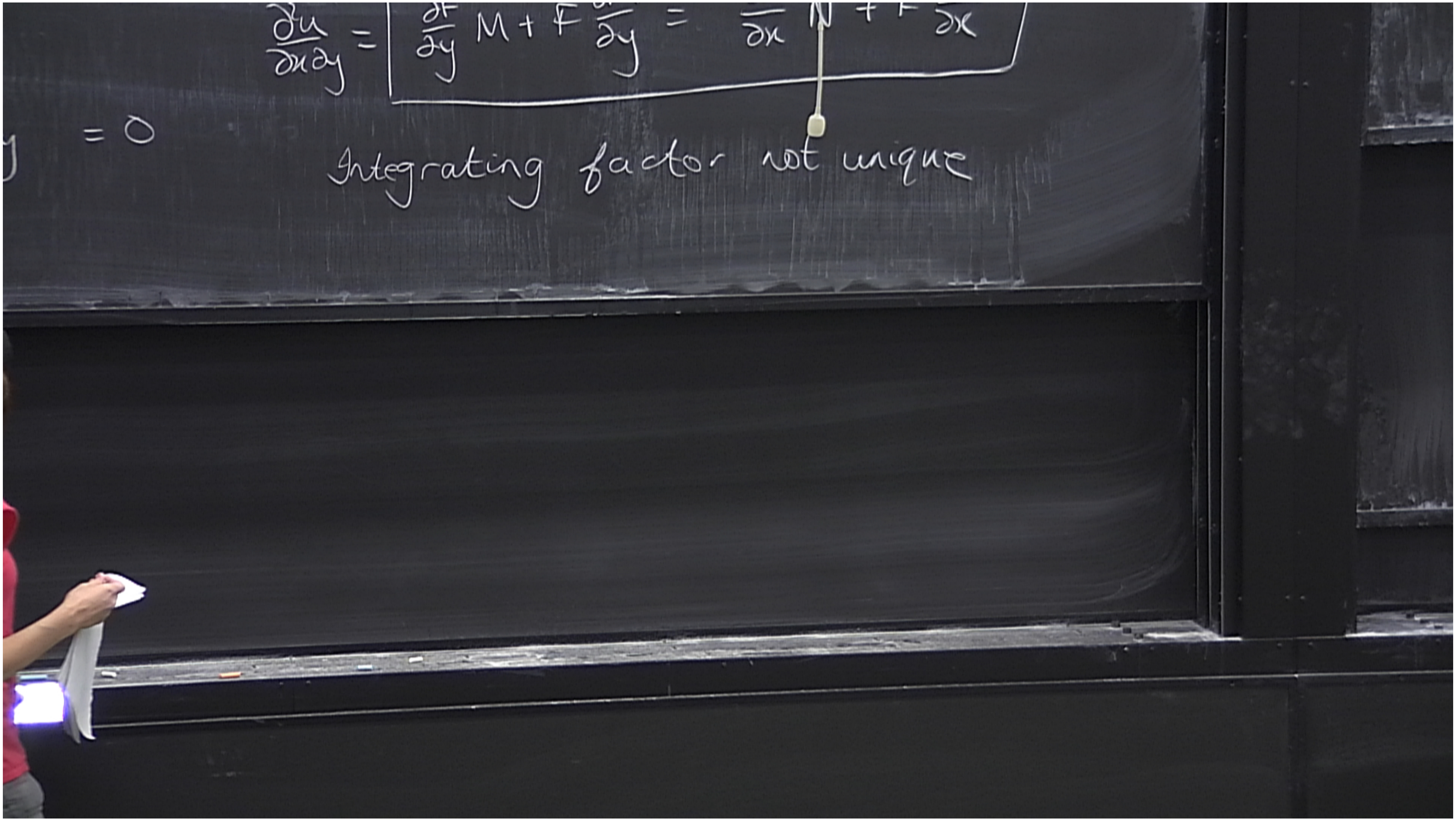
$$F(x,y)M(x,y)\,dx + F(x,y)N(x,y)\,dy = 0$$

is exact

$$\Rightarrow \quad \frac{\partial u}{\partial x} = FM \quad , \quad \frac{\partial u}{\partial y} = FN$$

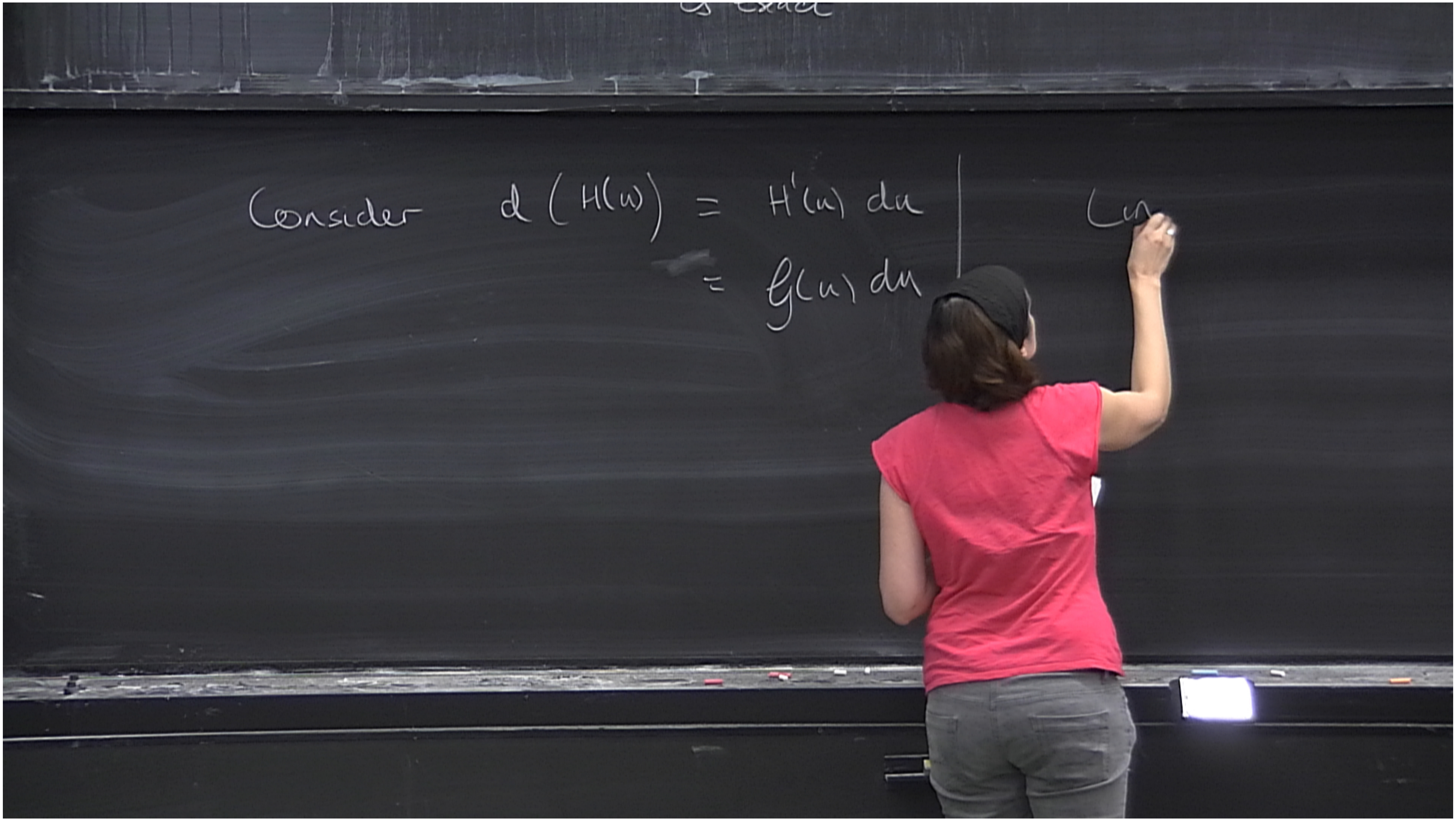
$$y \quad = 0$$

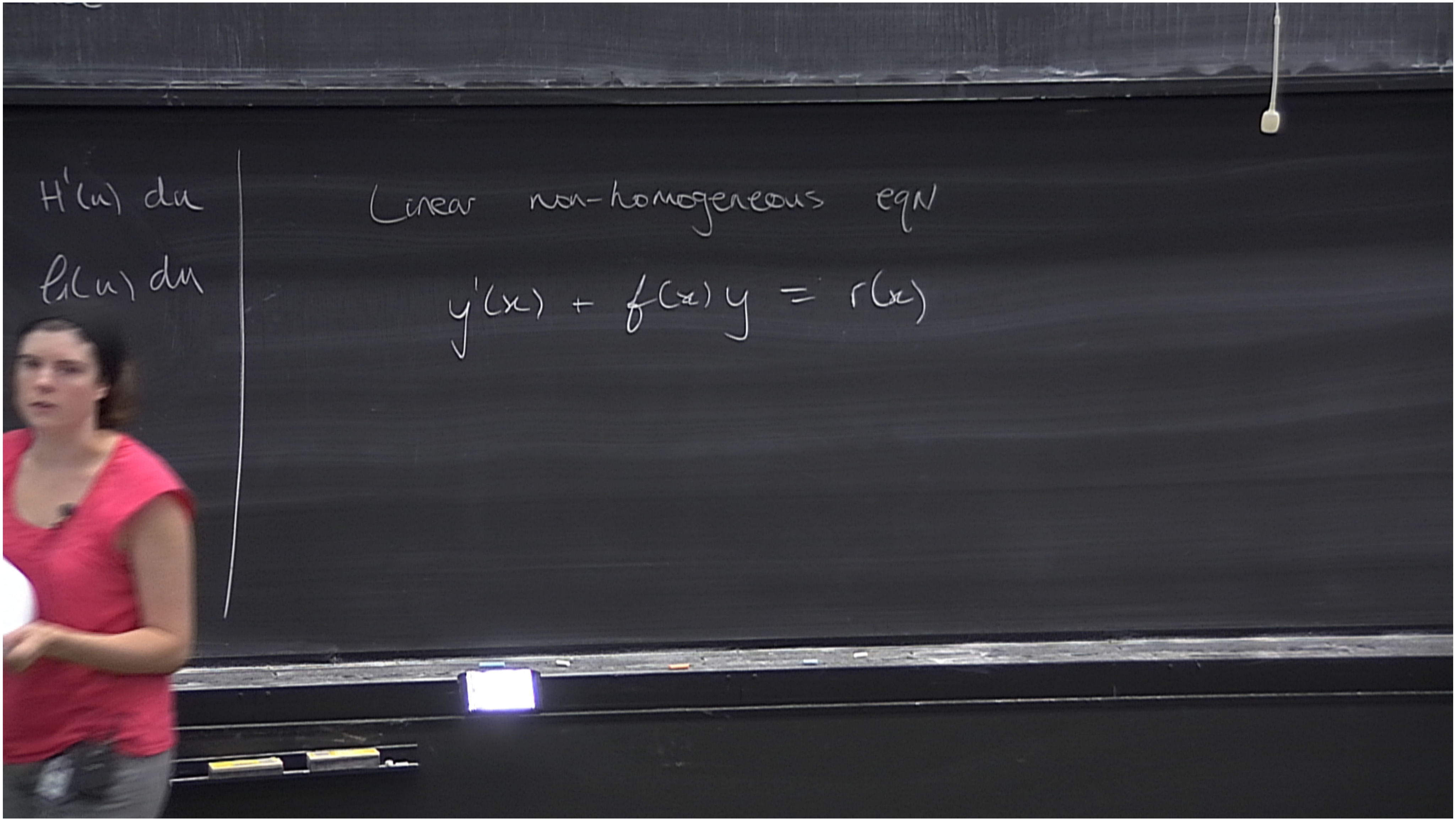

$\Rightarrow \quad \frac{\partial u}{\partial x} = FM \ , \quad \frac{\partial u}{\partial y} = FN$
$\frac{\partial u}{\partial x \partial y} = \boxed{\frac{\partial F}{\partial y} M + F \frac{\partial M}{\partial y} = \frac{\partial F}{\partial x} N + F \frac{\partial N}{\partial x}}$
$y \quad = 0$

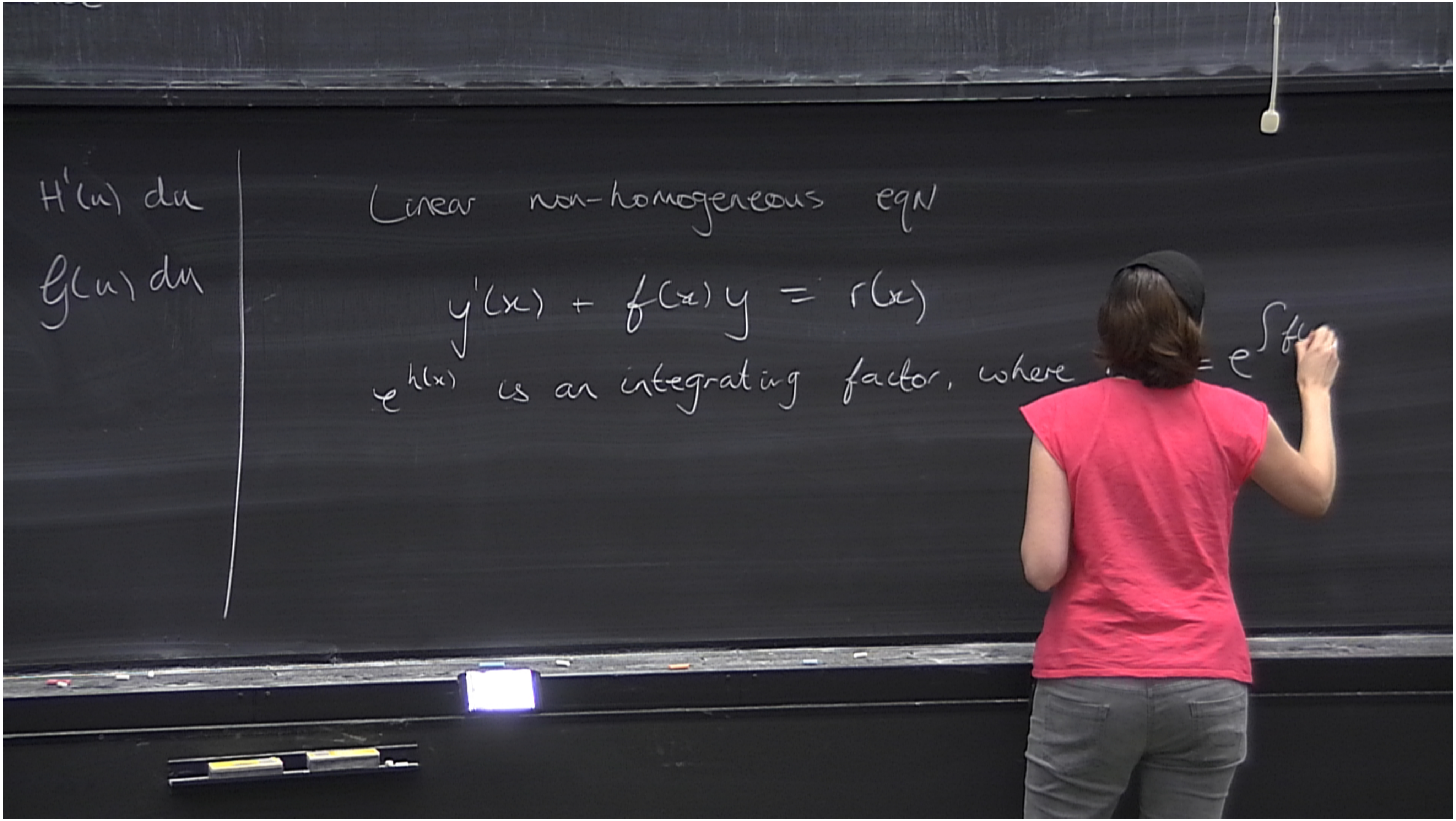

= 0
Integrating factor not unique

Consider $d\left(H(u)\right) = H'(u)\,du$

$$= f(u)\,du$$


Consider $d\left(H(u)\right) = H'(u)\,du$
$= f(u)\,du$


$H'(u)\,du$
$h(u)\,du$
Linear non-homogeneous eqN
$y'(x) + f(x)\,y = r(x)$


$H'(u)\,du$
$g(u)\,du$
Linear non-homogeneous eqN
$y'(x) + f(x)\,y = r(x)$
$e^{h(x)}$ is an integrating factor, where
$= e^{\int f}$

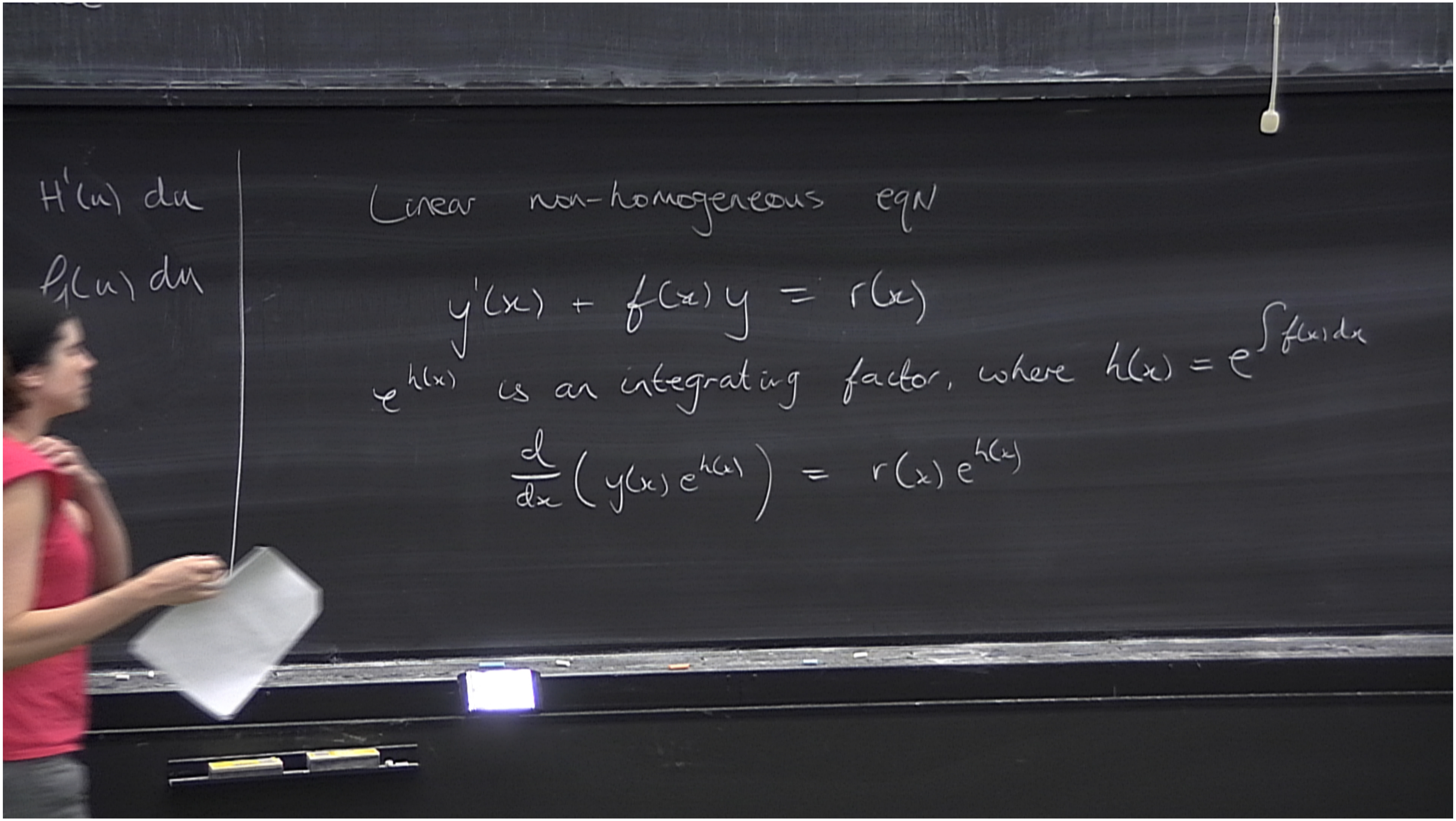
$H'(u)\,du$

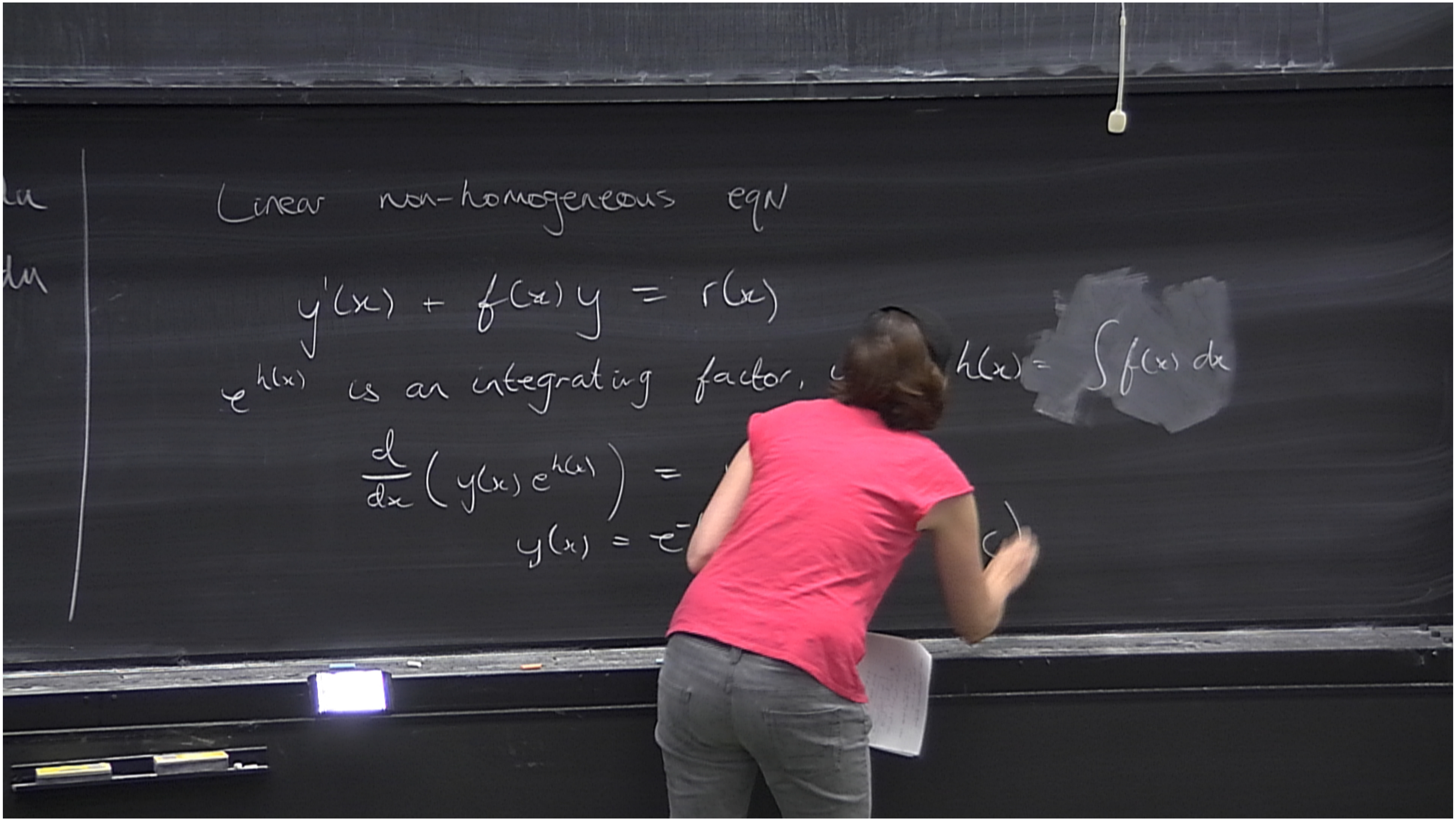
$g(u)\,du$

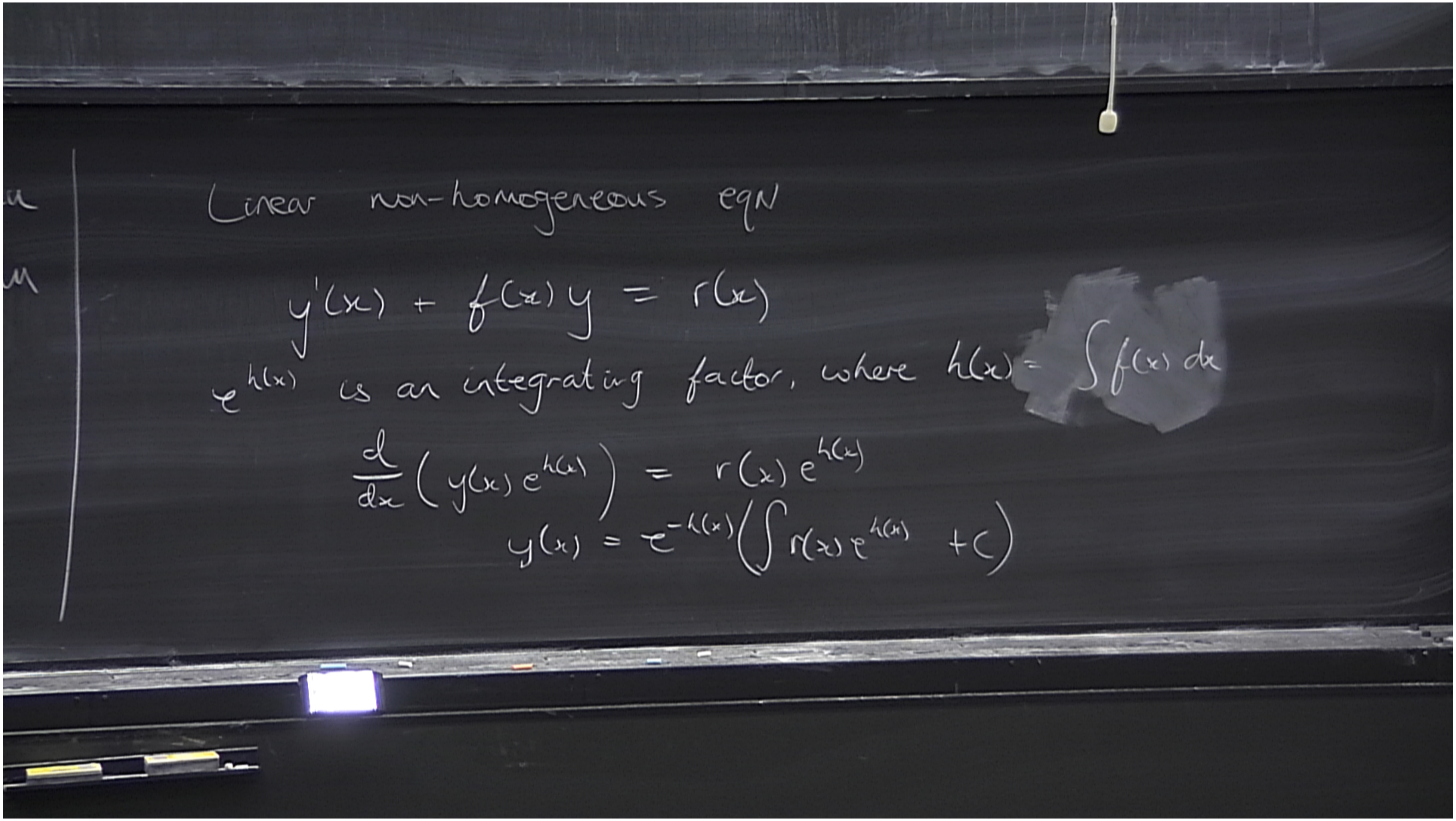
Linear non-homogeneous eqN

$$y'(x) + f(x)\,y = r(x)$$

$e^{h(x)}$ is an integrating factor, where $h(x) = e^{\int f(x)\,dx}$

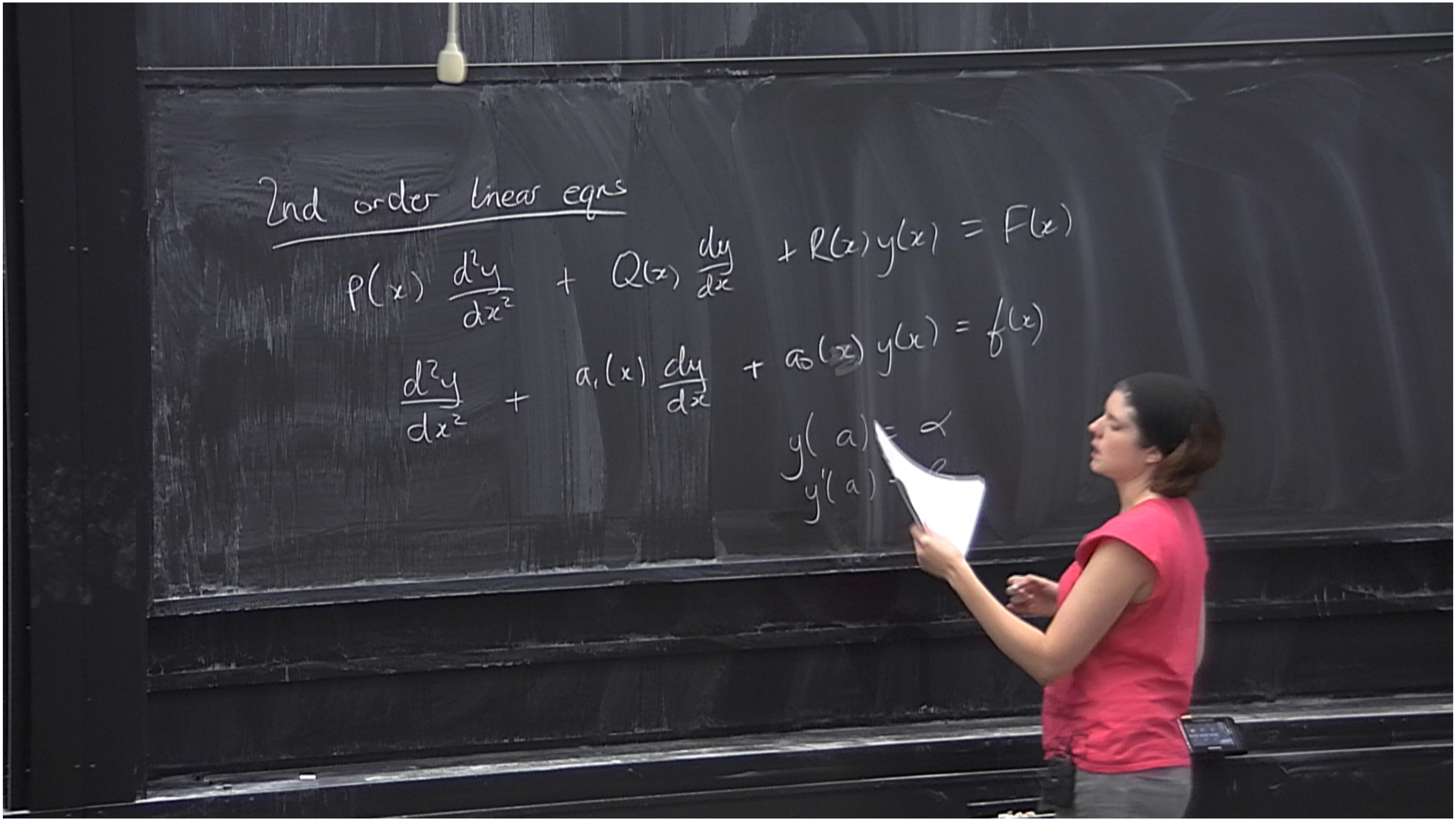

$H'(u)\,du$
$f_1(u)\,du$
Linear non-homogeneous eqN
$y'(x) + f(x)\,y = r(x)$
$e^{h(x)}$ is an integrating factor, where $h(x) = e^{\int f(x)\,dx}$
$\frac{d}{dx}\left(y(x)\,e^{h(x)}\right) = r(x)\,e^{h(x)}$

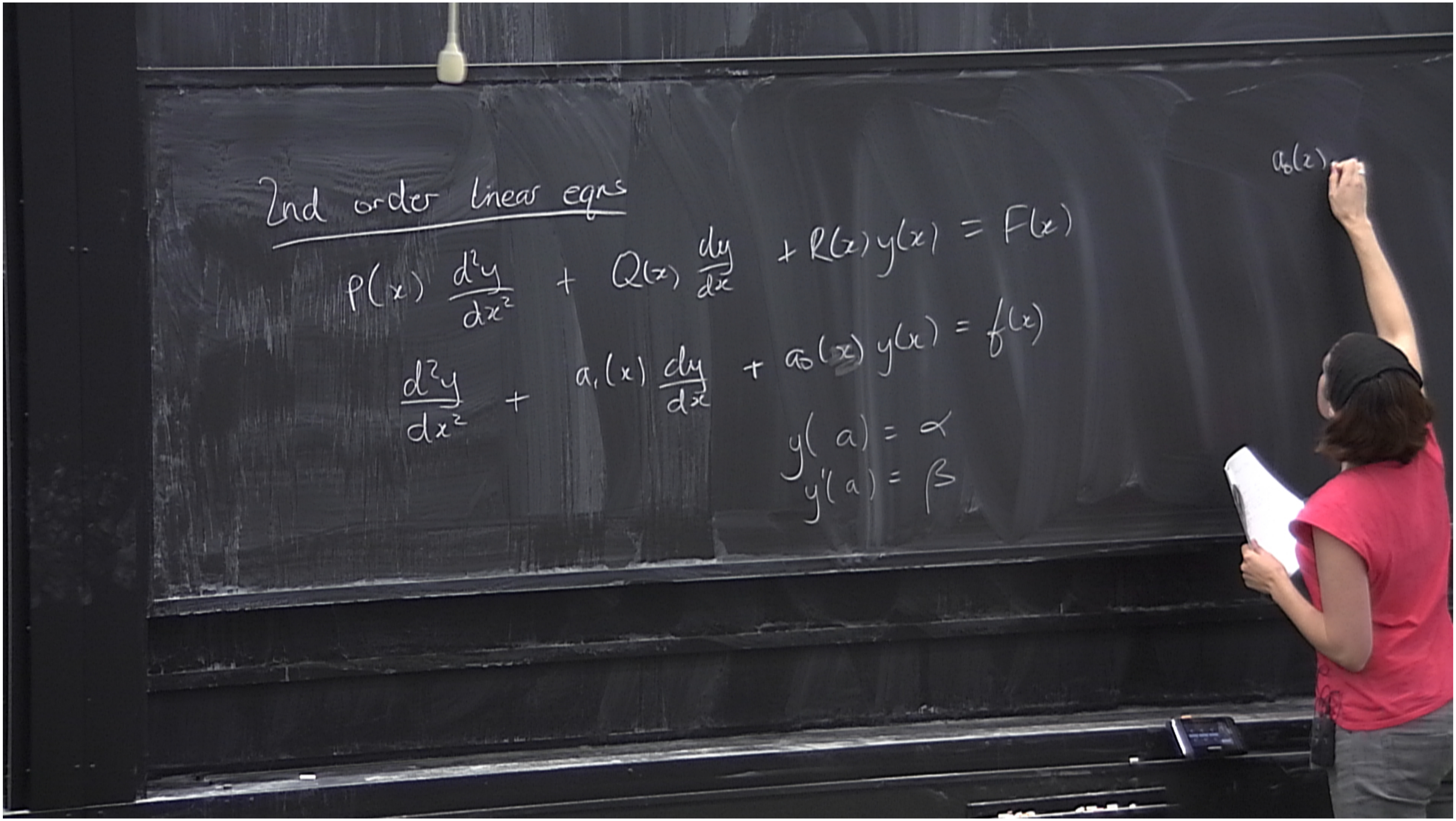

Linear non-homogeneous eqN
y'(x) + f(x) y = r(x)
e^h(x) is an integrating factor,
h(x) = ∫ f(x) dx
d/dx (y(x) e^h(x)) =
y(x) =

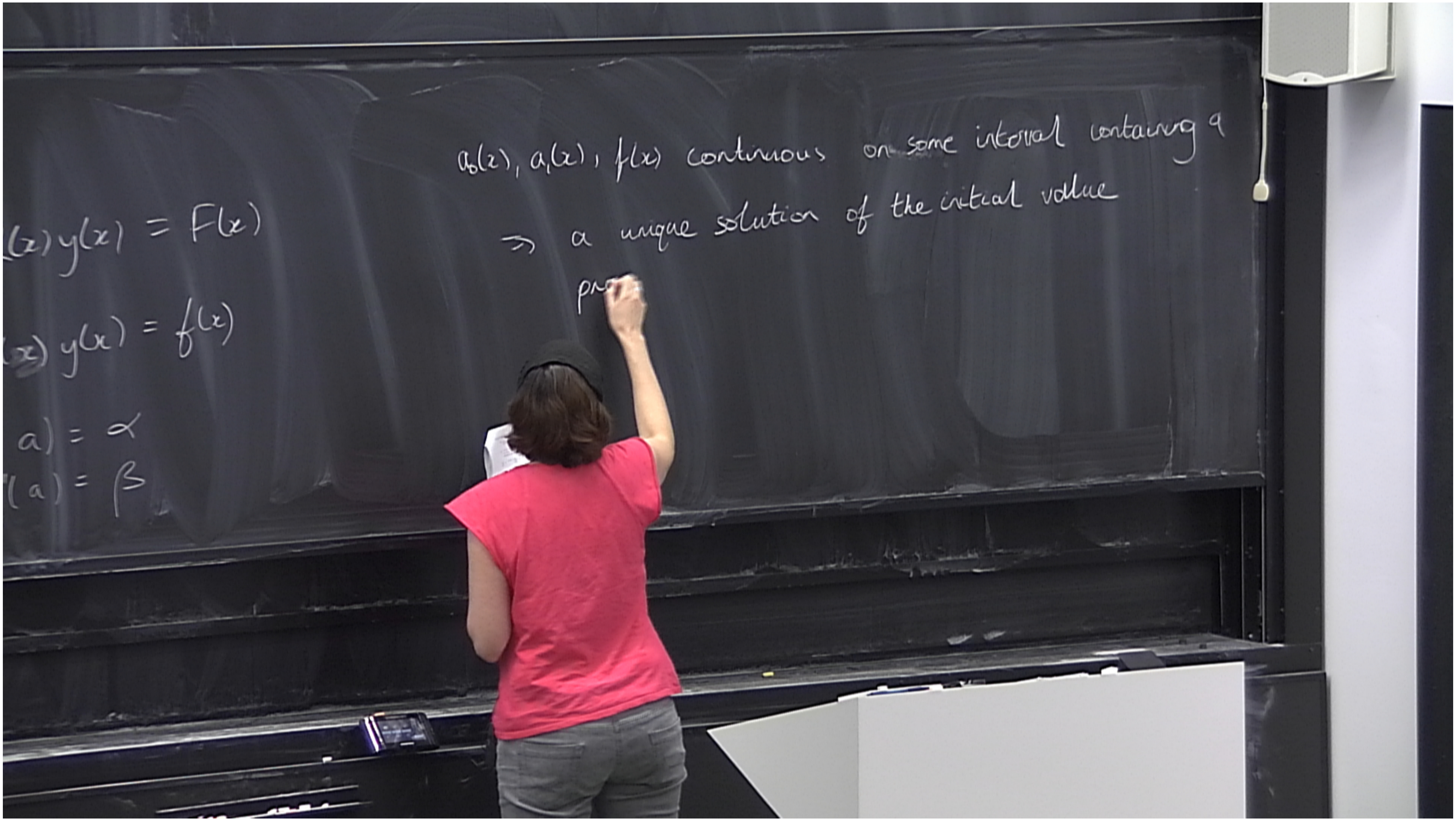

Linear non-homogeneous eqN
$y'(x) + f(x)\,y = r(x)$
$e^{h(x)}$ is an integrating factor, where $h(x) = \int f(x)\,dx$
$\frac{d}{dx}\left(y(x)\,e^{h(x)}\right) = r(x)\,e^{h(x)}$
$y(x) = e^{-h(x)}\left(\int r(x)\,e^{h(x)} + C\right)$

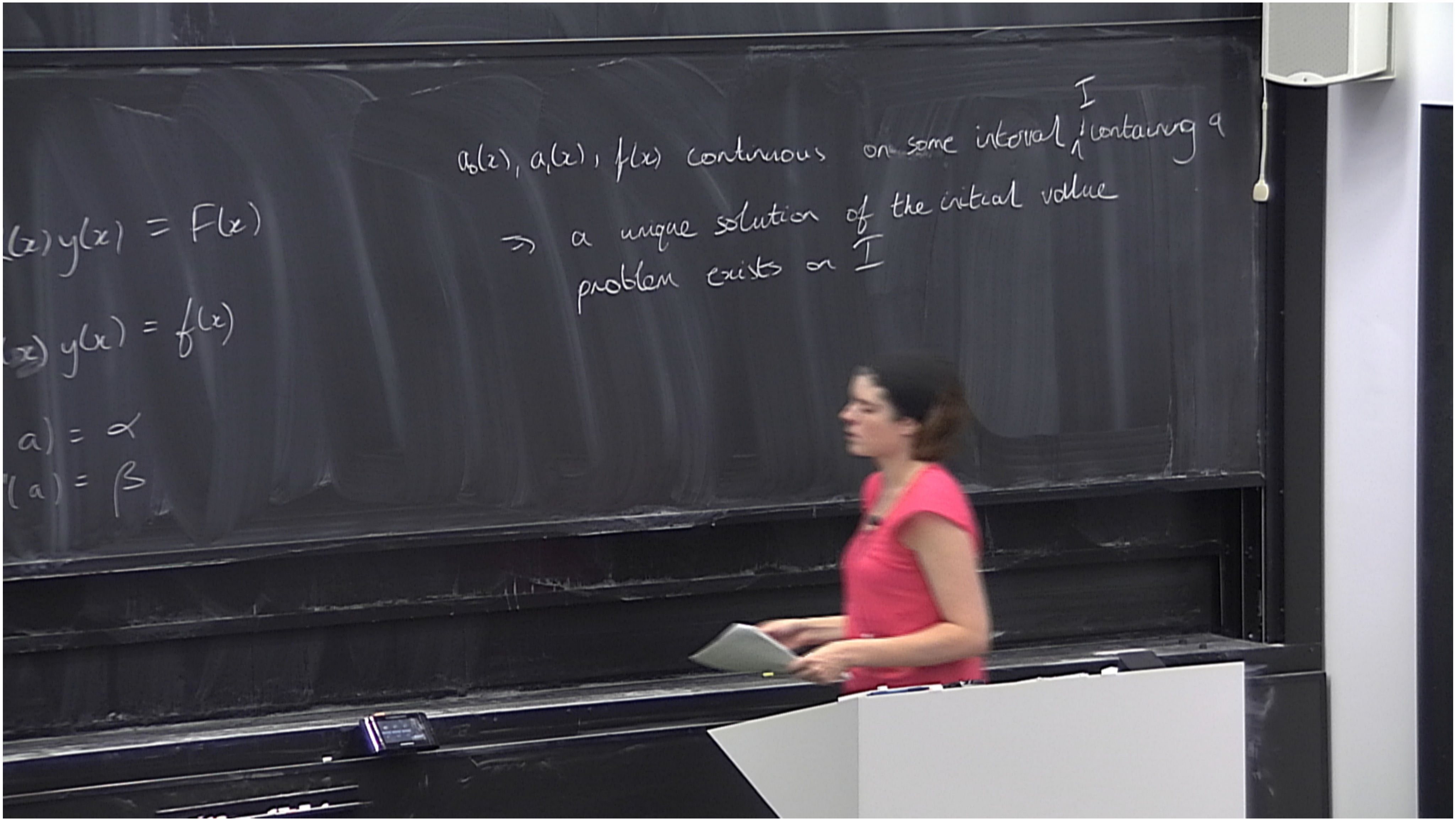
2nd order linear eqns

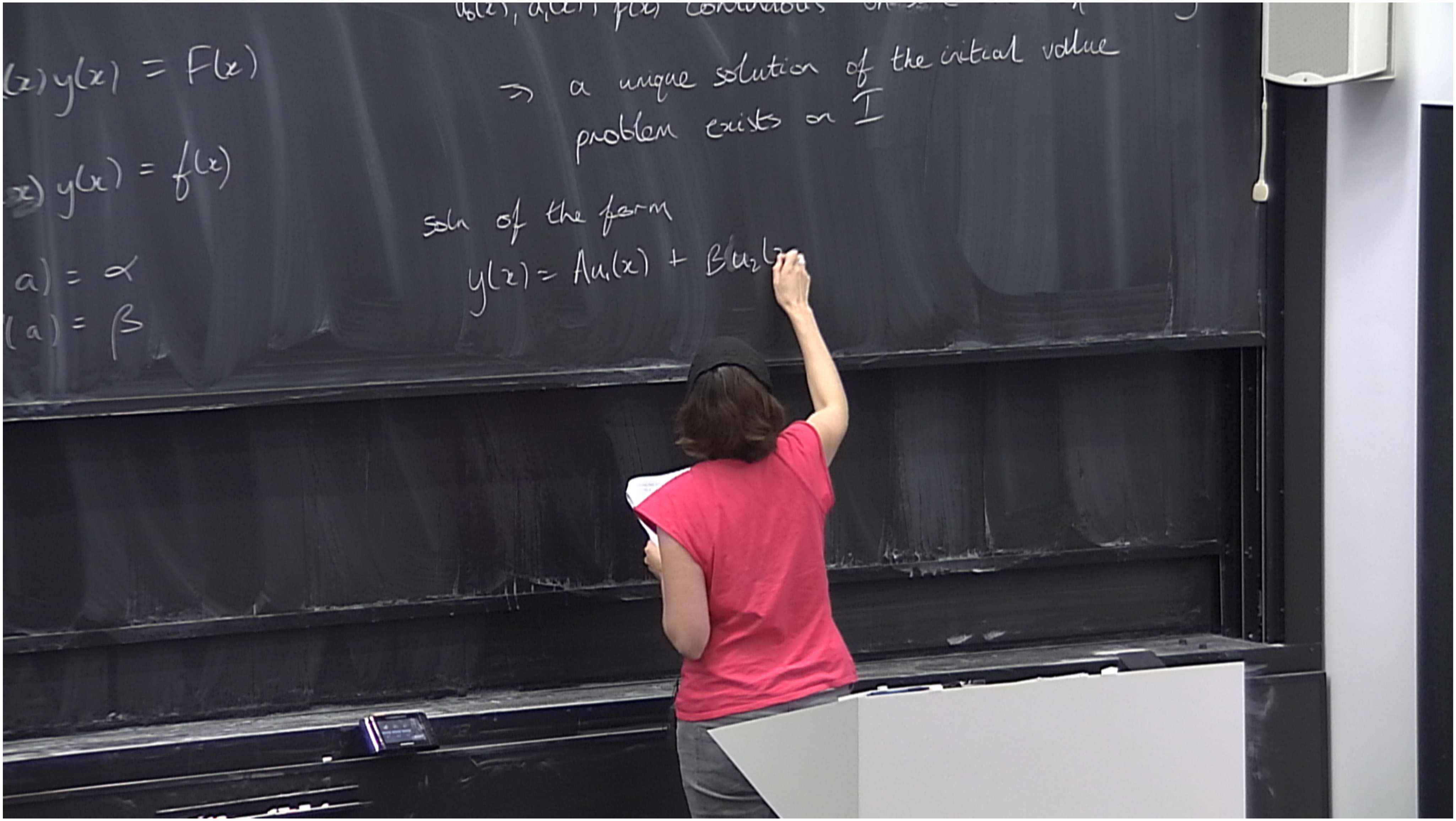
$$P(x)\,\frac{d^2y}{dx^2} + Q(x)\,\frac{dy}{dx} + R(x)\,y(x) = F(x)$$

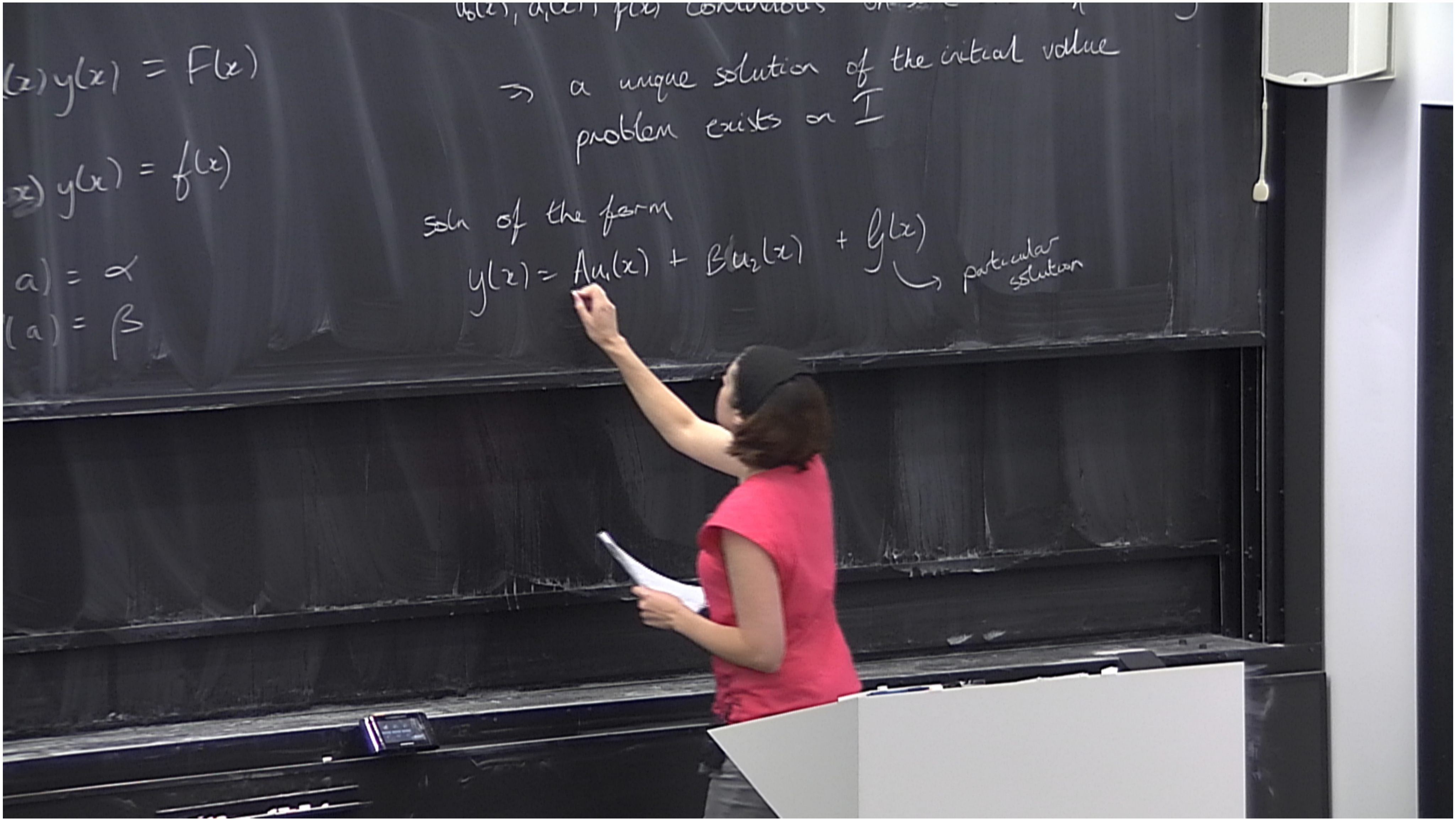

2nd order linear eqns
P(x) d²y/dx² + Q(x) dy/dx + R(x) y(x) = F(x)
d²y/dx² + a₁(x) dy/dx + a₀(x) y(x) = f(x)
y(a) = α
y'(a)


2nd order linear eqns
$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x) y(x) = F(x)$
$\frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x) y(x) = f(x)$
$y(a) = \alpha$
$y'(a) = \beta$
$a_0(x)$

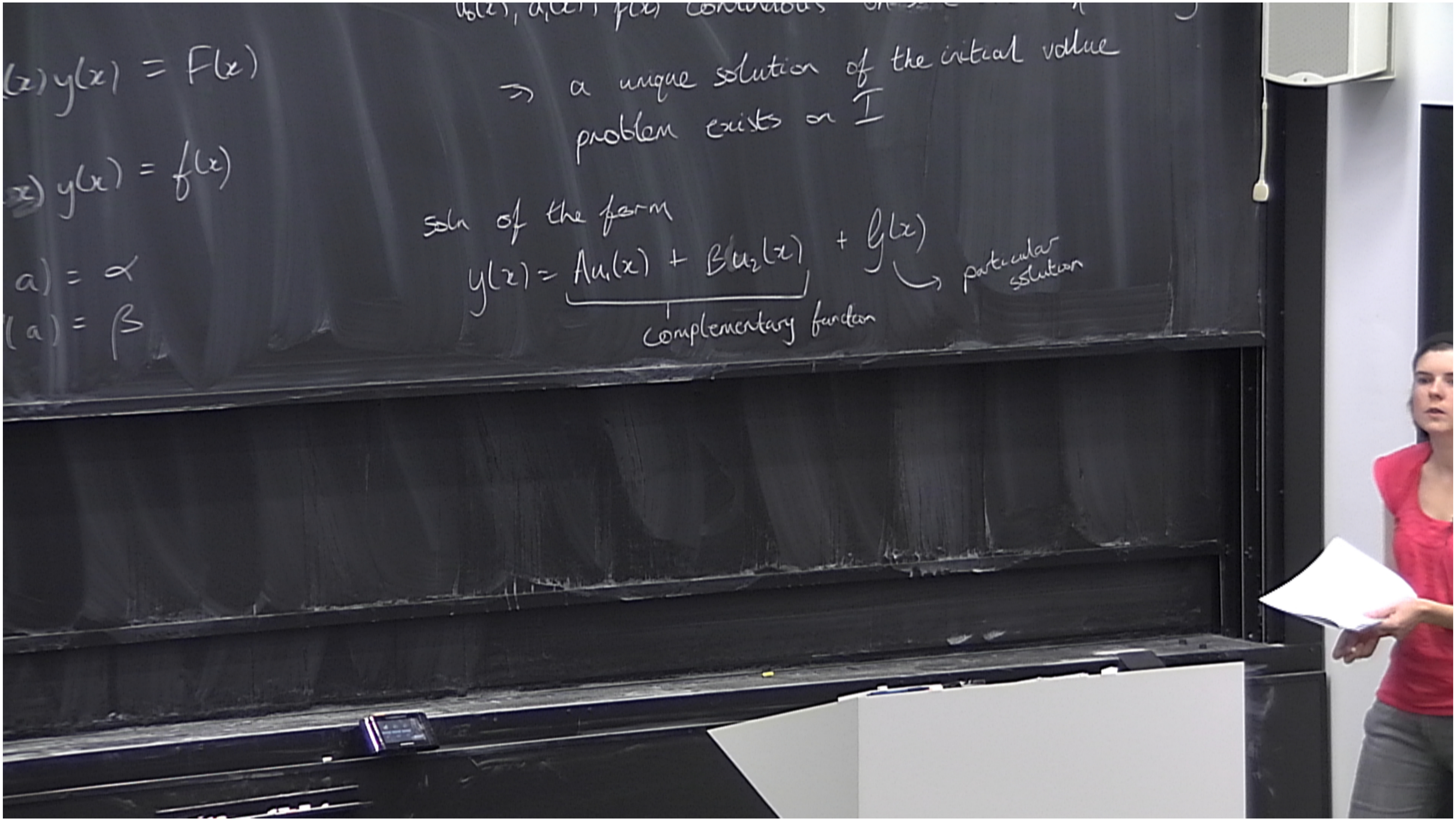

a0(x), a1(x), f(x) continuous on some interval containing a
⇒ a unique solution of the initial value
y(x) = F(x)
y(x) = f(x)
a) = α
(a) = β


$a_0(x)$, $a_1(x)$, $f(x)$ continuous on some interval I containing a
$\Rightarrow$ a unique solution of the initial value problem exists on I
$(x)\, y(x) = F(x)$
$(x)\, y(x) = f(x)$
$a) = \alpha$
$(a) = \beta$

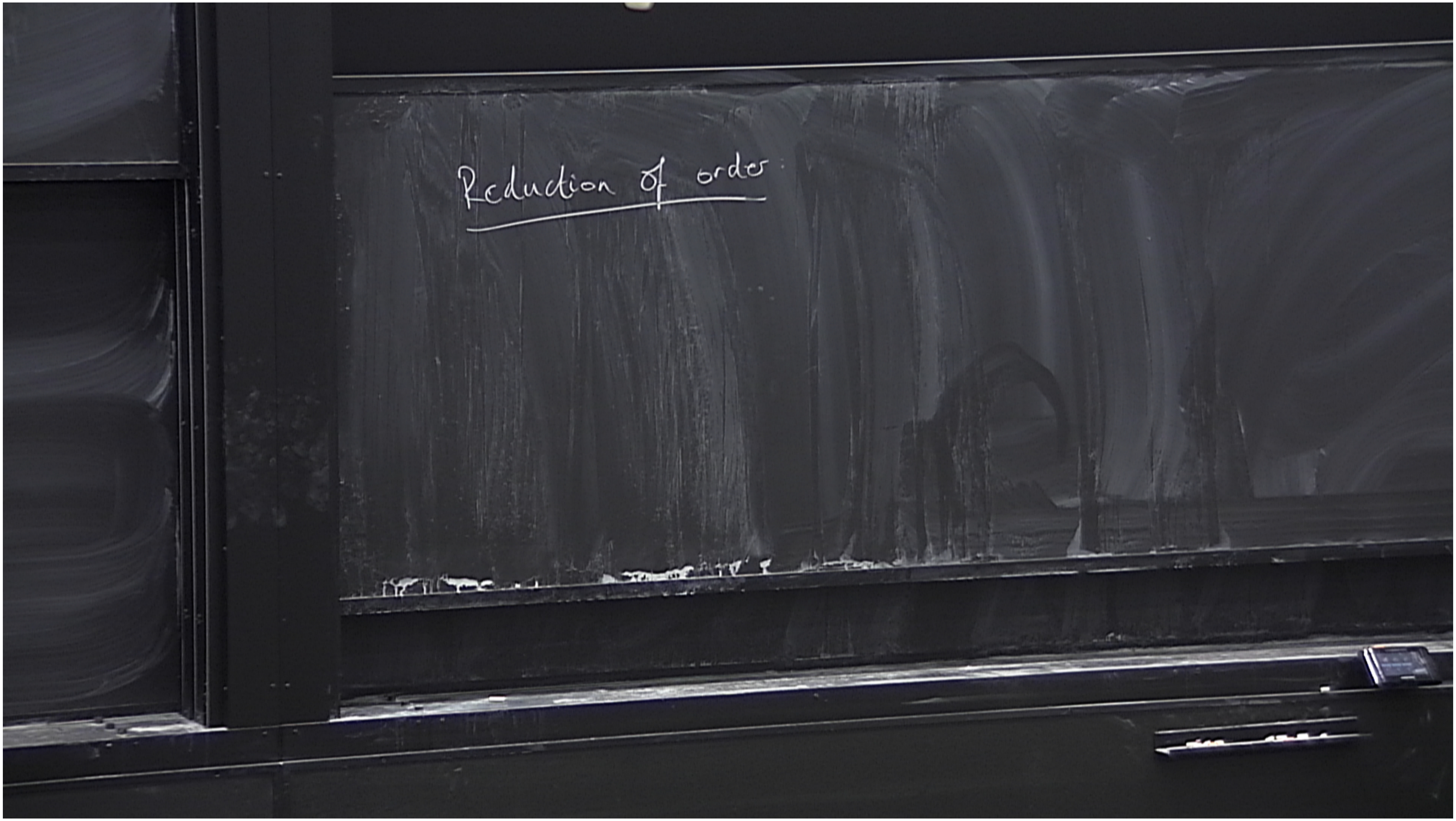

y(x) = F(x)
y(x) = f(x)
a) = α
(a) = β
⇒ a unique solution of the initial value
problem exists on I
soln of the form
y(x) = Au1(x) + Bu2(x)

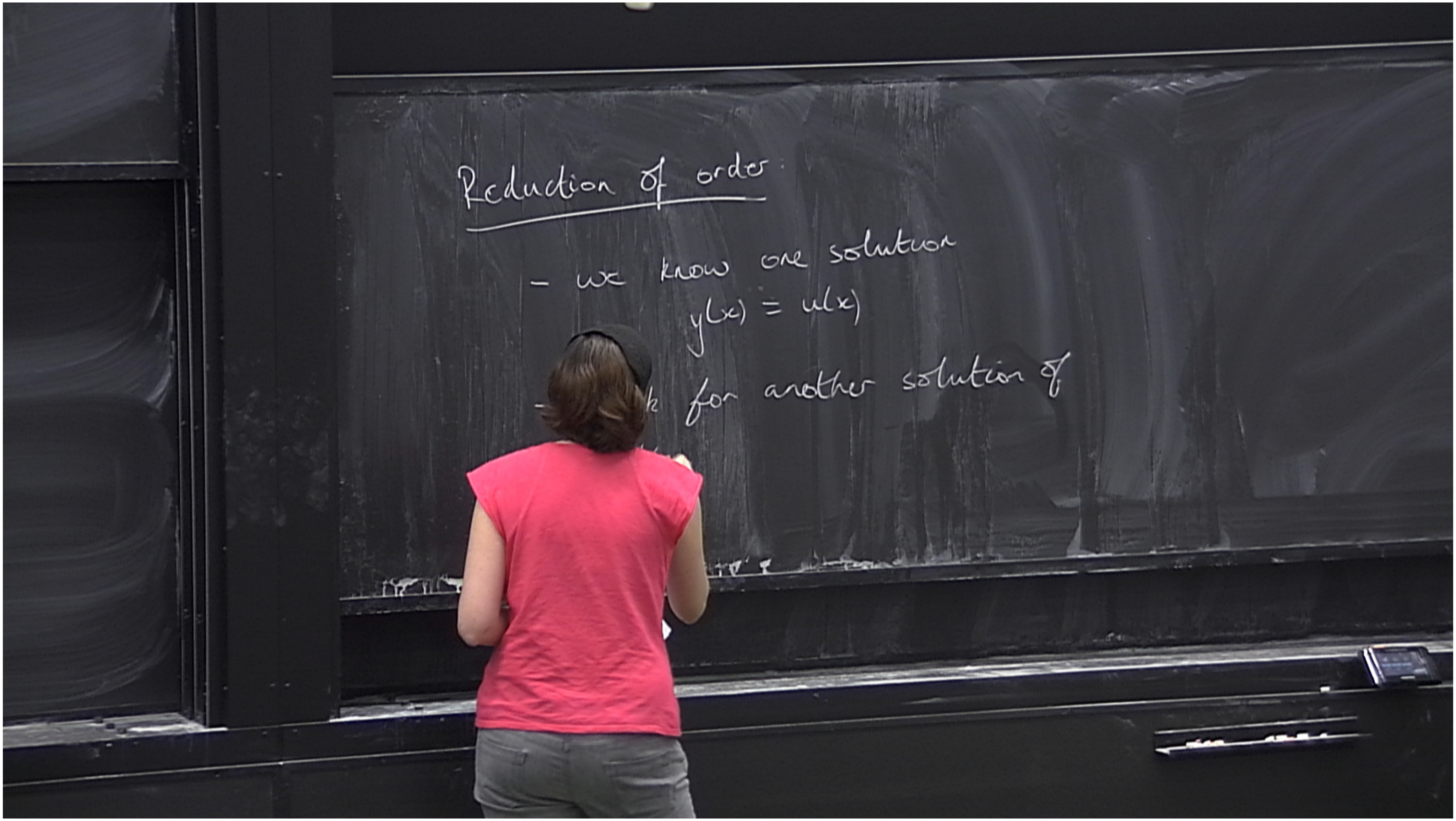


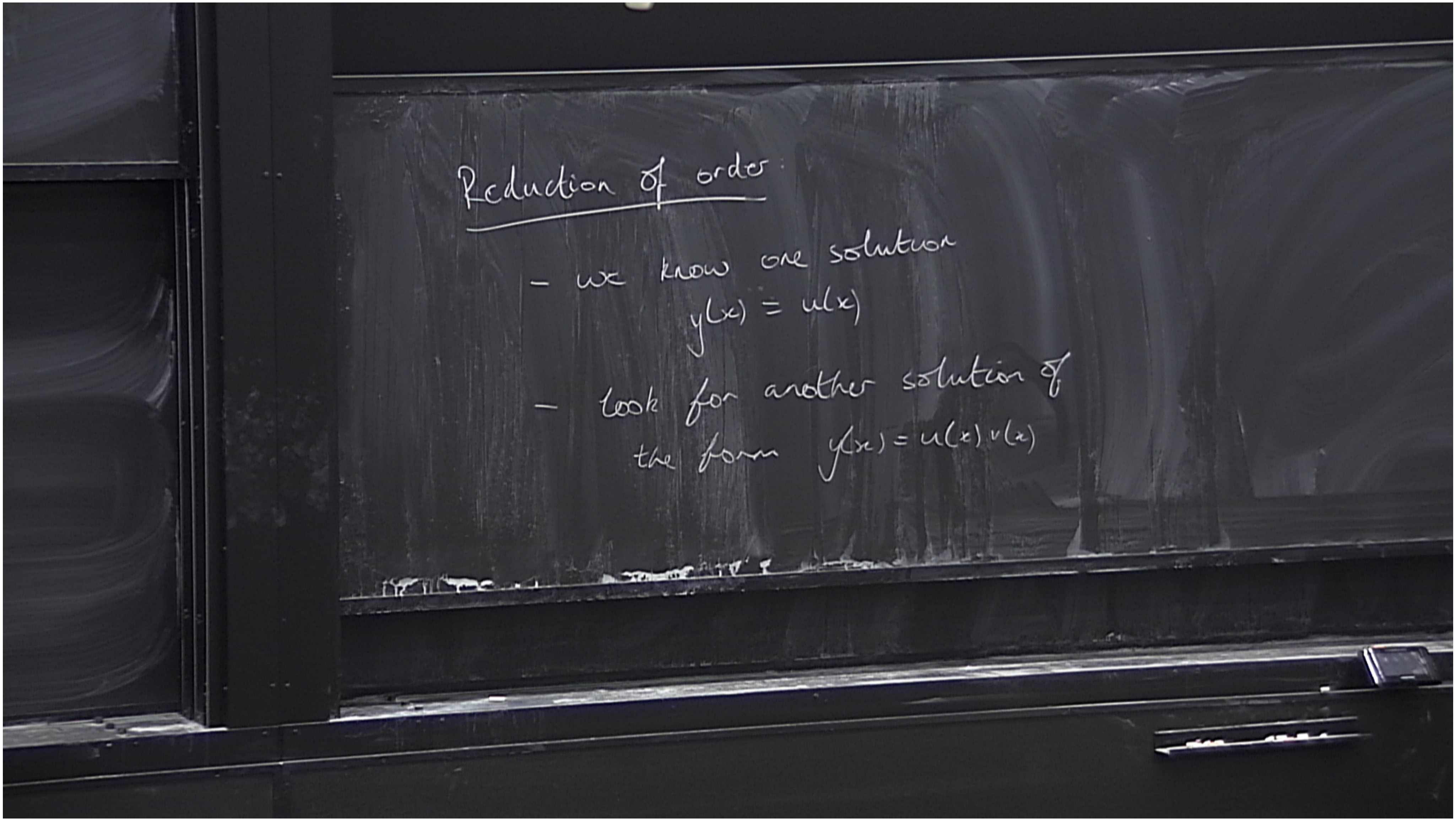
y(x) = F(x)
⇒ a unique solution of the initial value
problem exists on I
y(x) = f(x)
soln of the form
y(x) = Au_1(x) + Bu_2(x) + g(x)
particular solution
a) = α
(a) = β

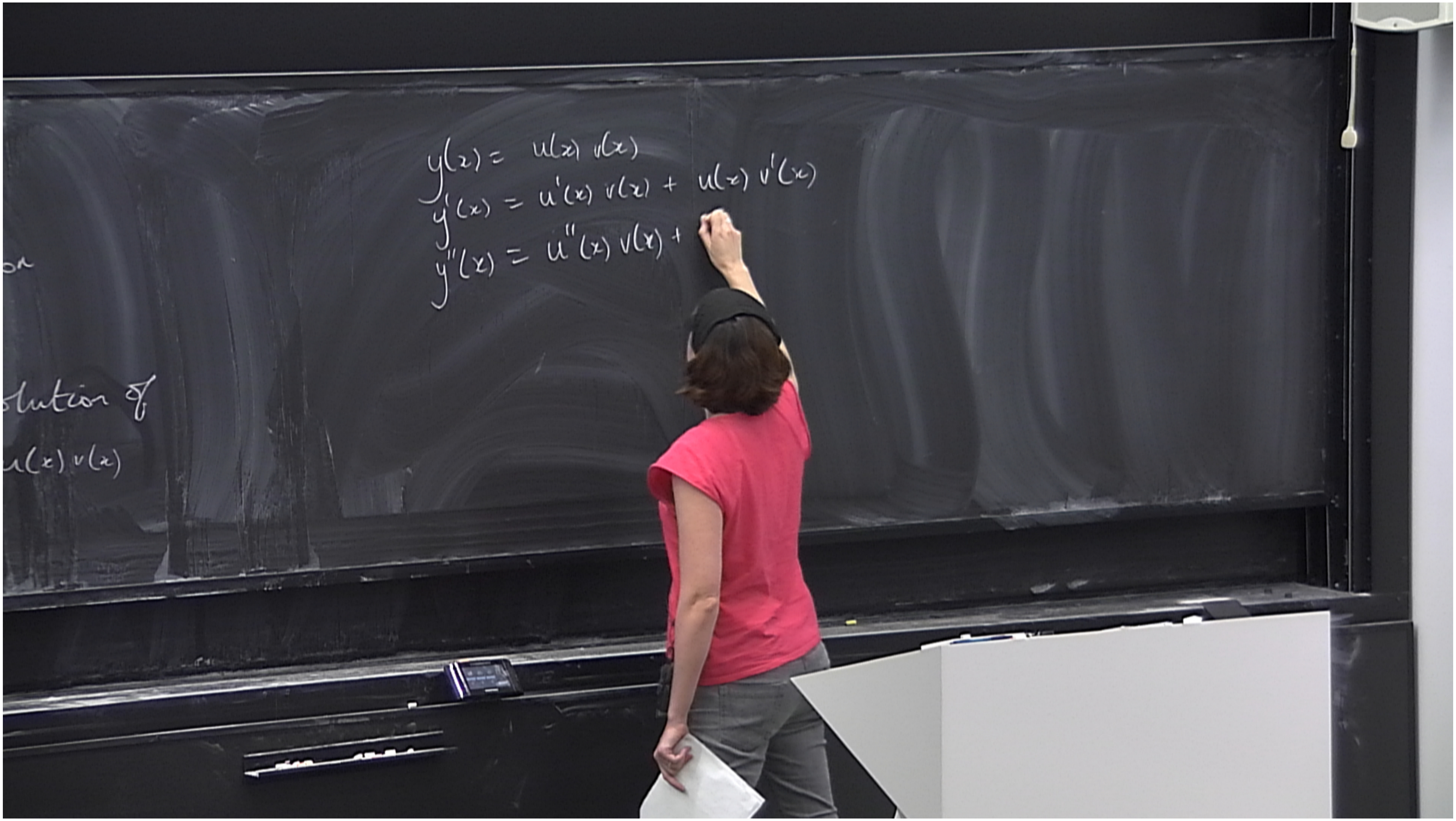


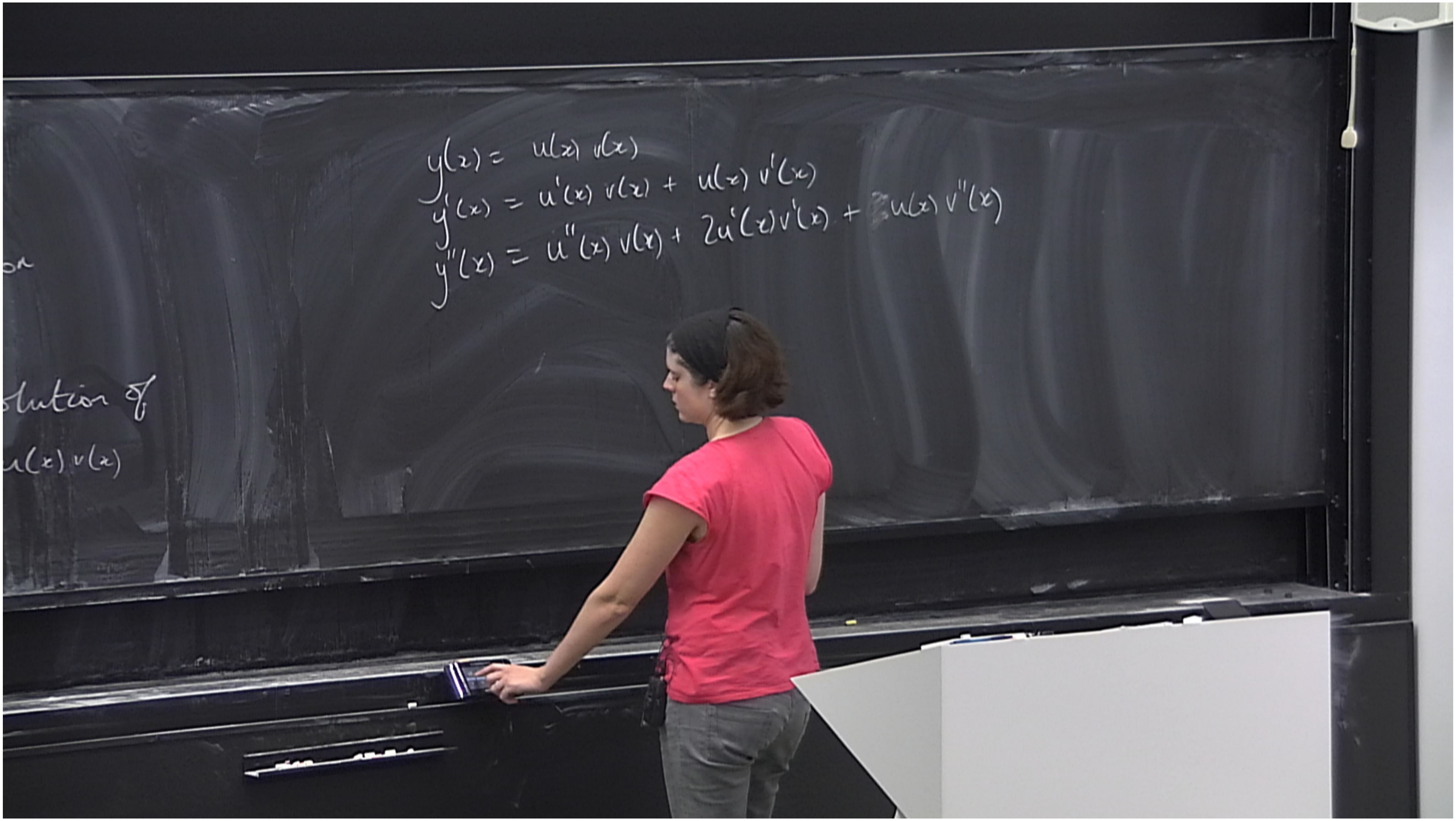
⇒ a unique solution of the initial value problem exists on I
soln of the form
y(x) = Au1(x) + Bu2(x) + g(x)
complementary function
particular solution

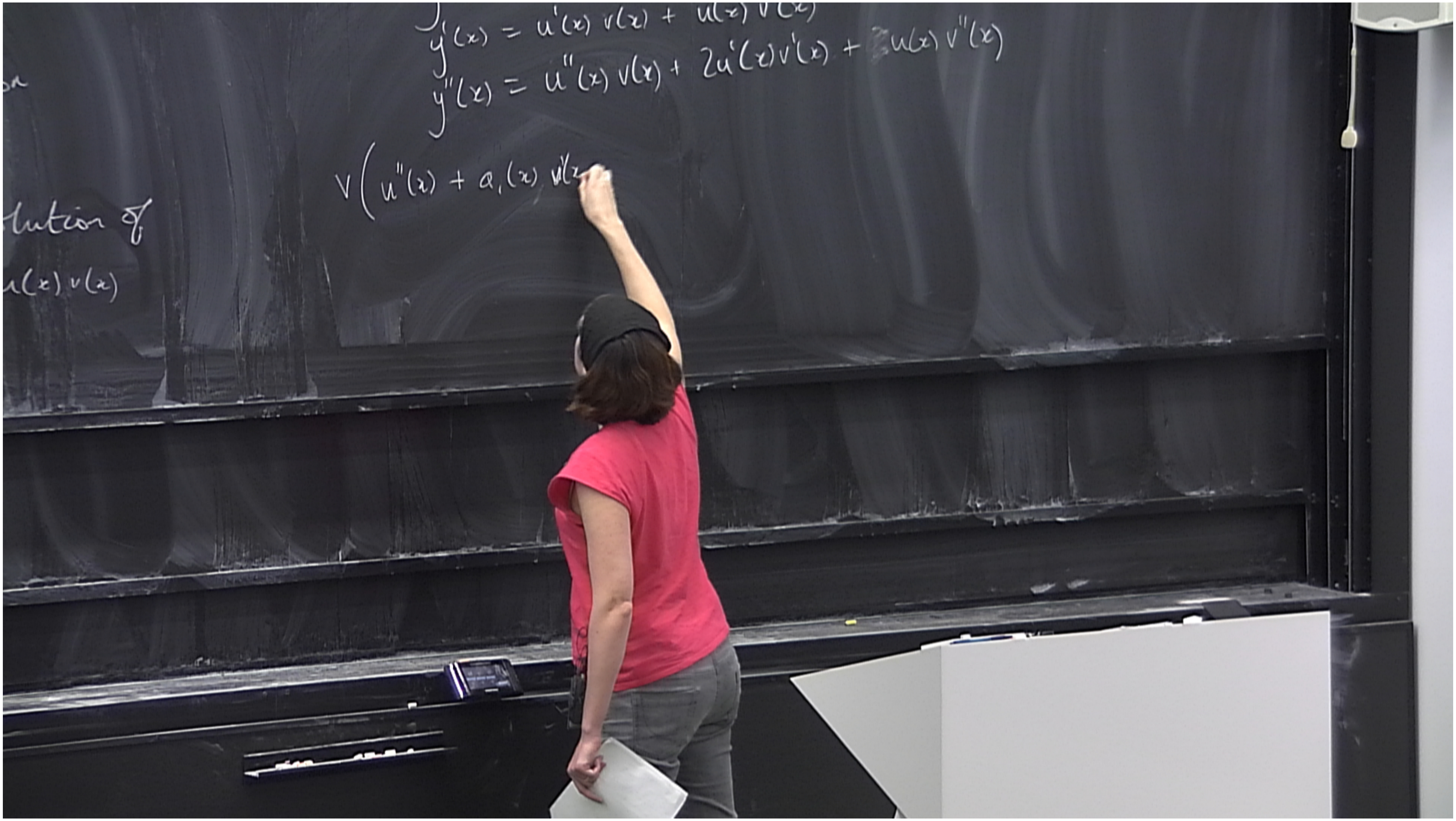

Reduction of order

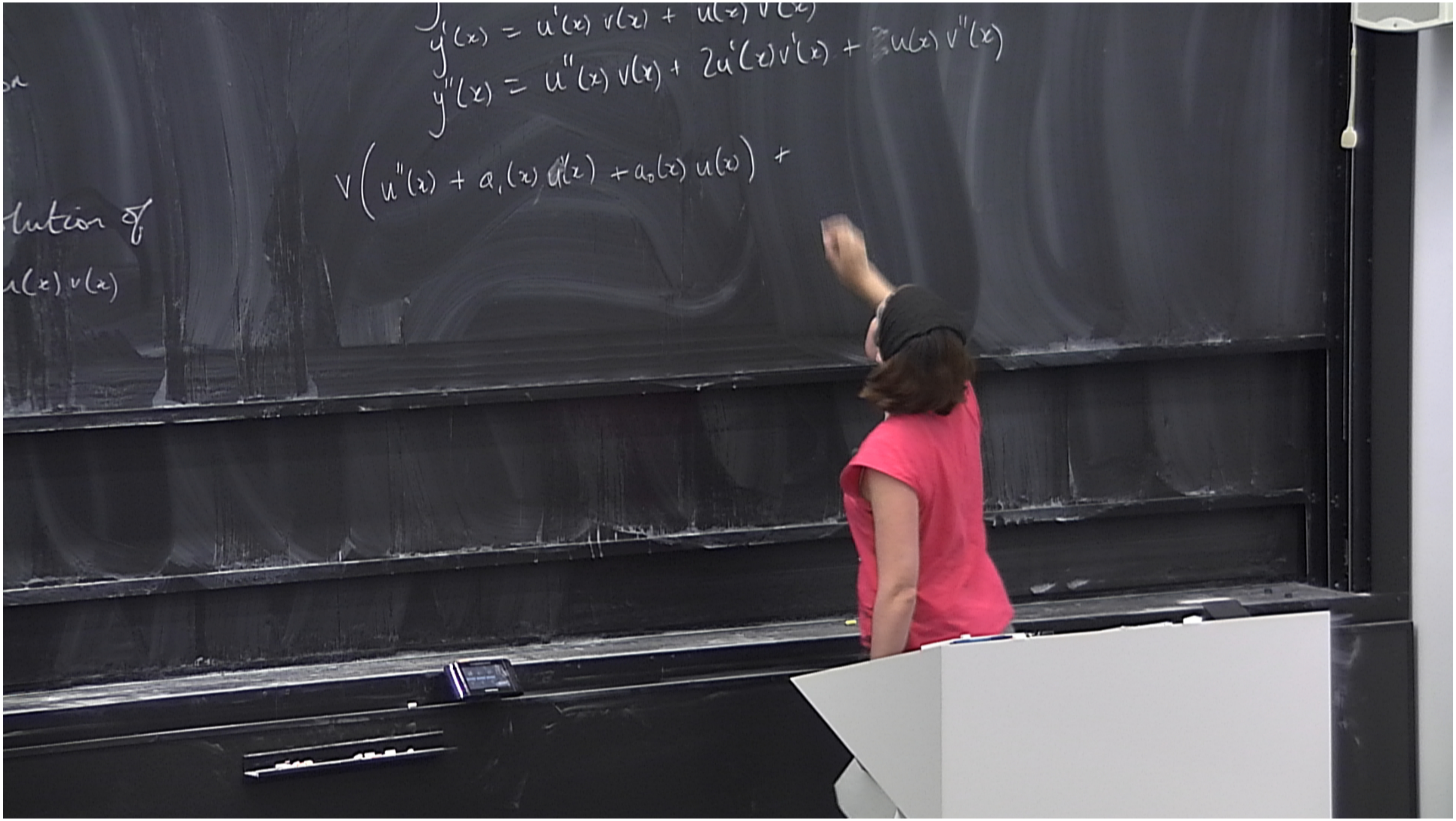

Reduction of order
- we know one solution
$y(x) = u(x)$
for another solution of


Reduction of order
– we know one solution
$y(x) = u(x)$
– look for another solution of
the form $y(x) = u(x)\,v(x)$

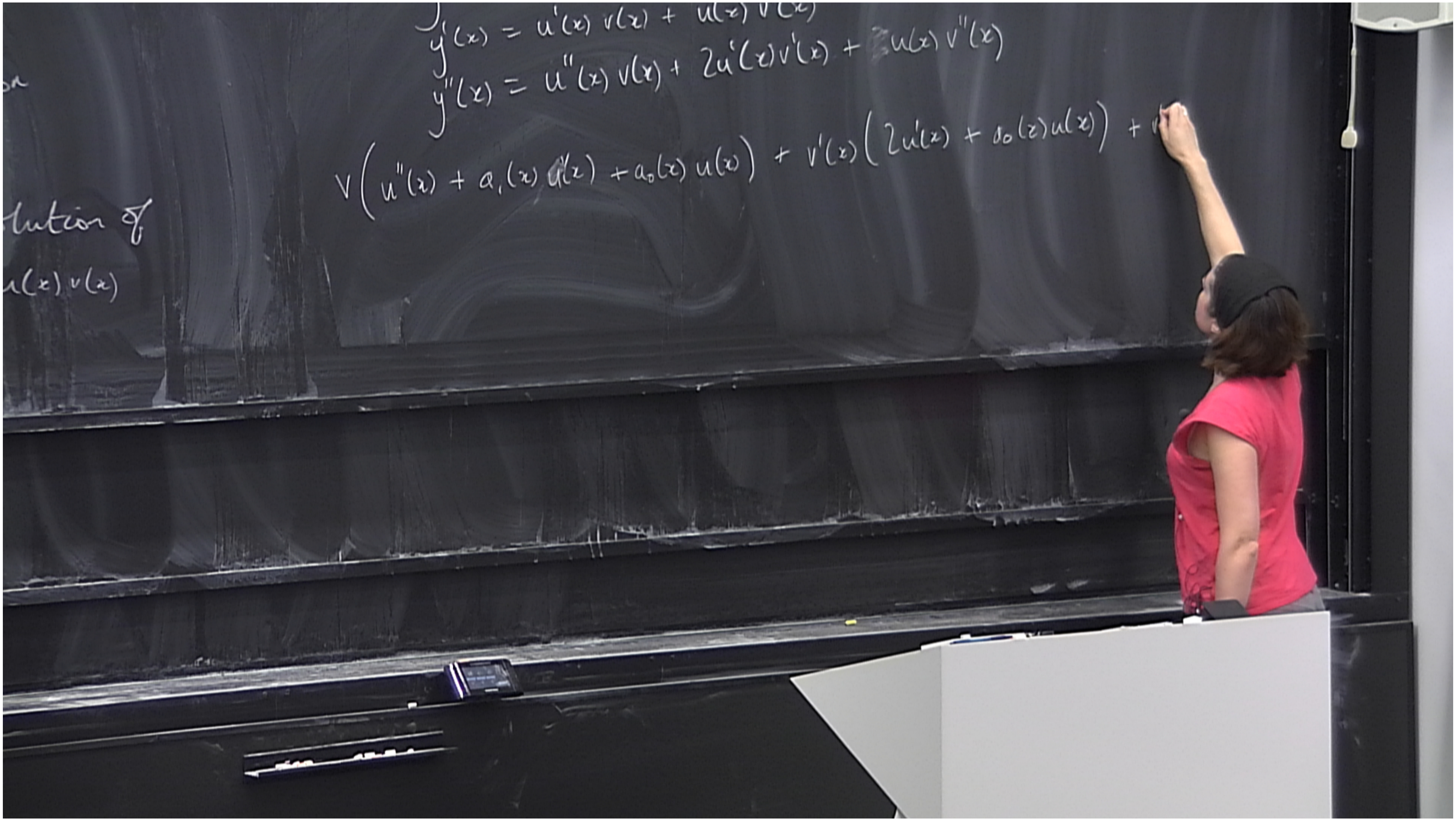

y(x) = u(x) v(x)
y'(x) = u'(x) v(x) + u(x) v'(x)
y''(x) = u''(x) v(x) +
olution of
u(x) v(x)


y(x) = u(x) v(x)
y'(x) = u'(x) v(x) + u(x) v'(x)
y''(x) = u''(x) v(x) + 2u'(x)v'(x) + u(x) v''(x)
on
olution of
u(x) v(x)

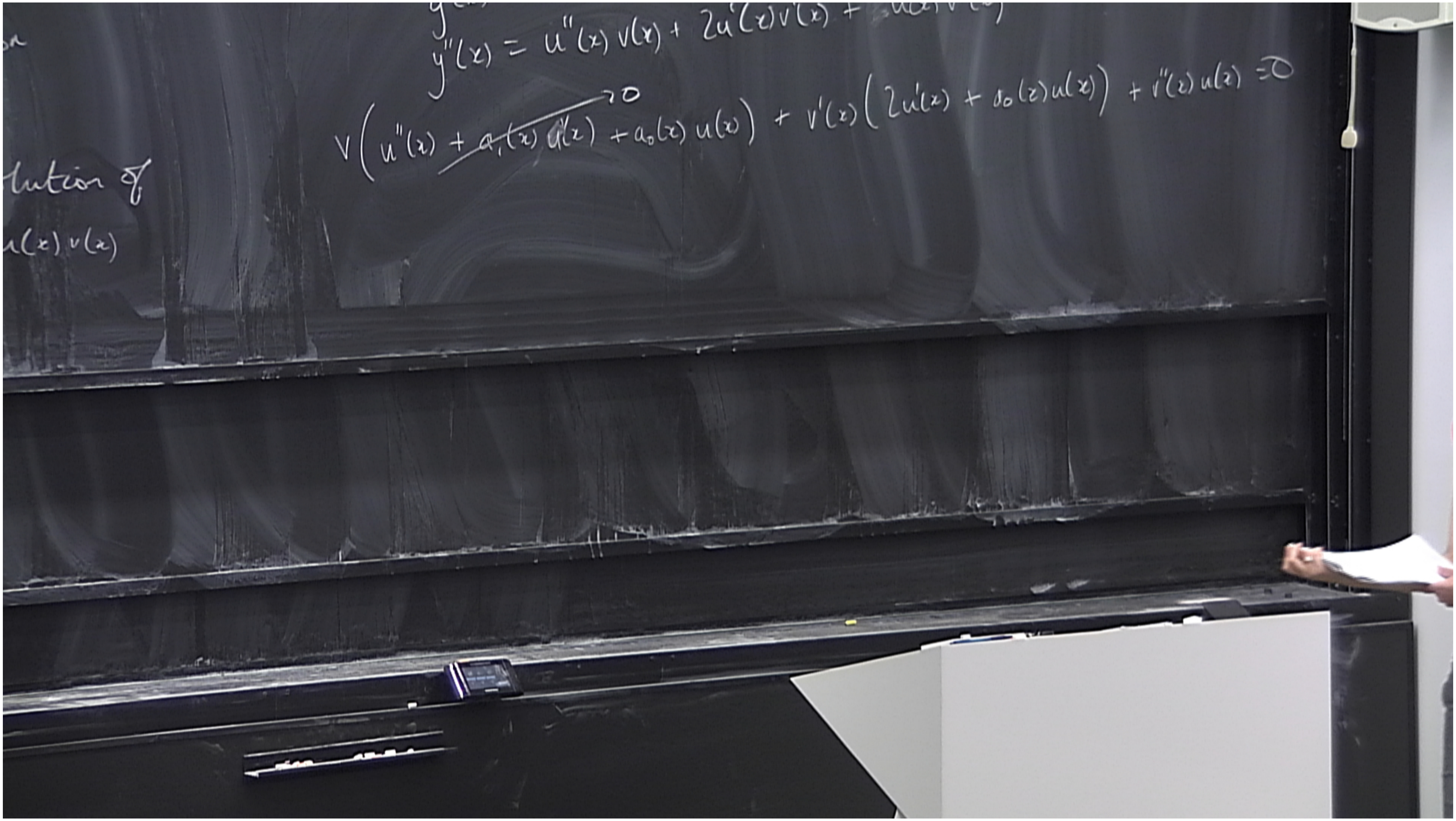

y''(x) = u''(x) v(x) + 2u'(x)v'(x) + u(x) v''(x)
lution of
u(x) v(x)



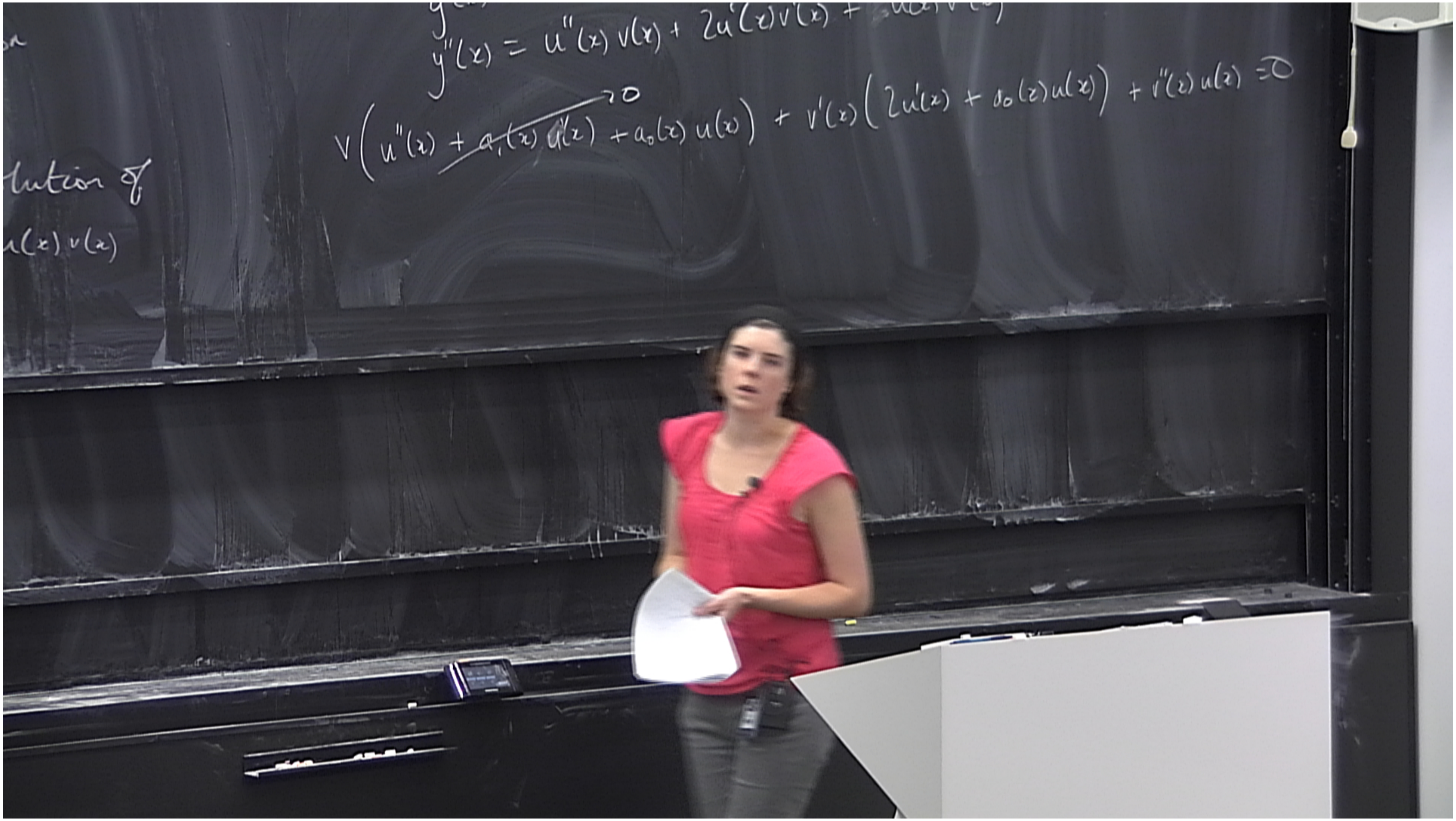
$y'(x) = u'(x)v(x) + u(x)v'(x)$
$y''(x) = u''(x)v(x) + 2u'(x)v'(x) + u(x)v''(x)$
$v\left(u''(x) + a_1(x)u'(x) + a_0(x)u(x)\right) +$
lution of
$u(x)v(x)$



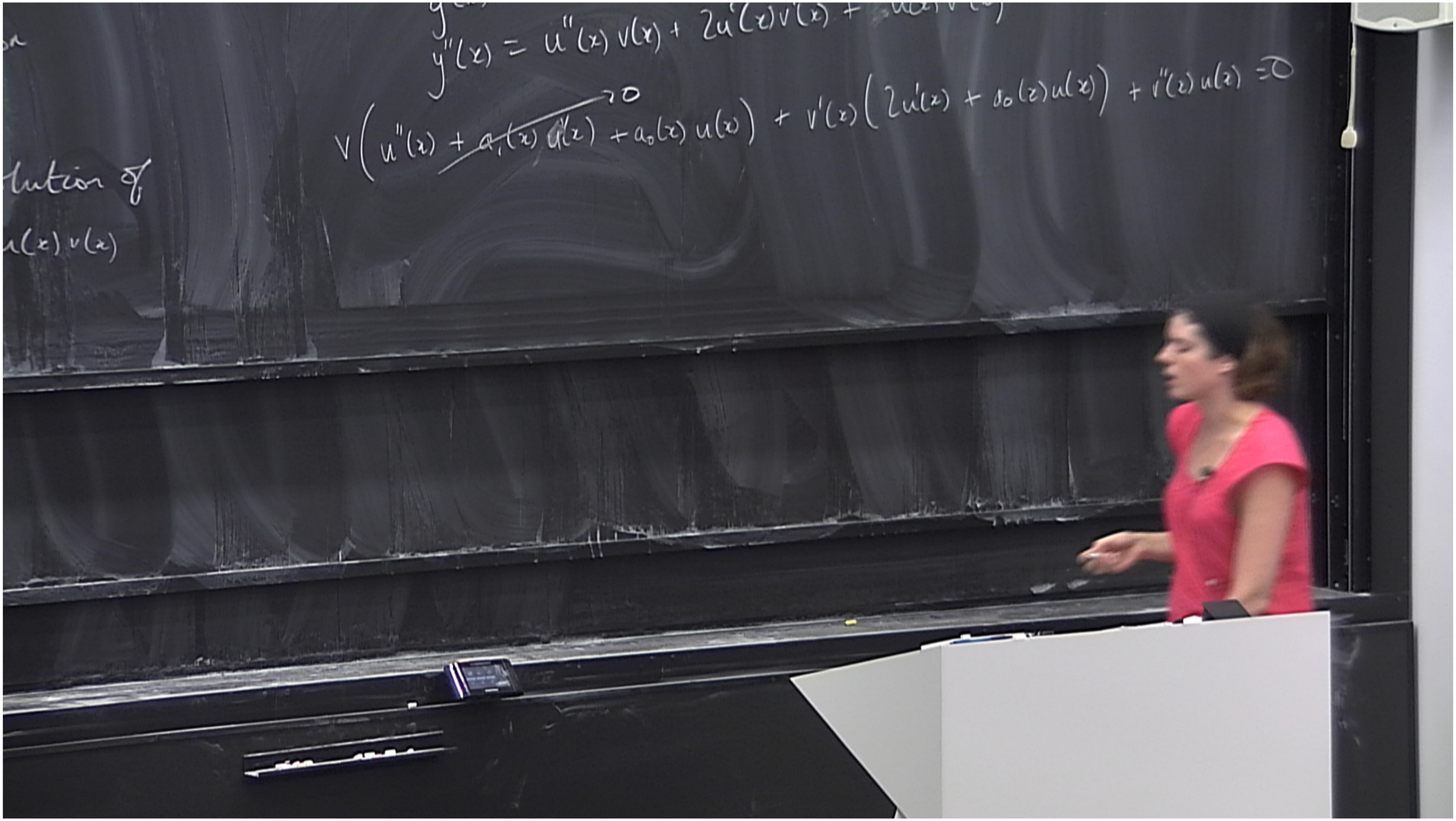
y''(x) = u''(x) v(x) + 2u'(x)v'(x) + u(x) v''(x)
v(u''(x) + a₁(x) u'(x) + a₀(x) u(x)) + v'(x)(2u'(x) + a₀(x)u(x)) +
lution of
u(x) v(x)

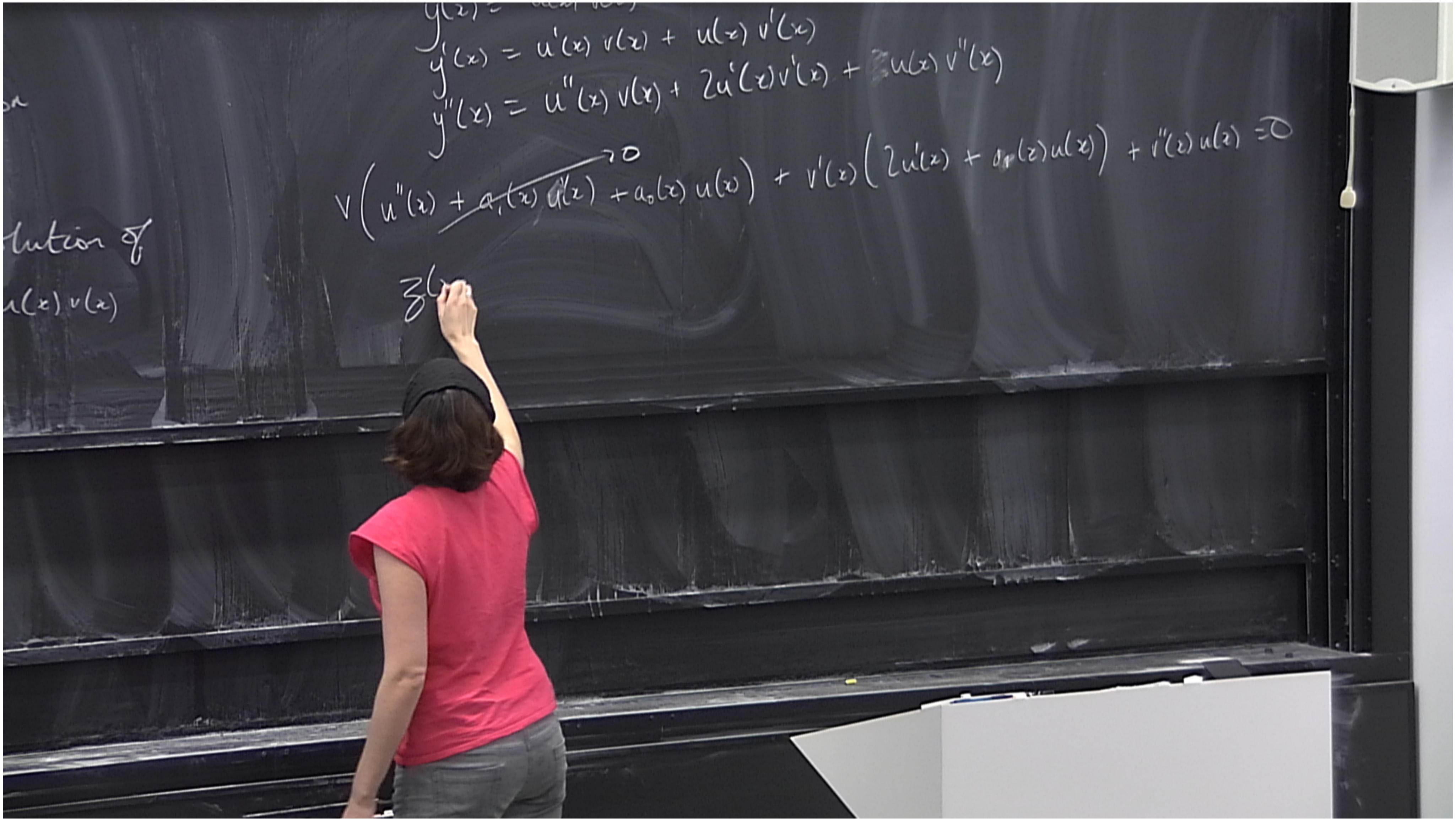


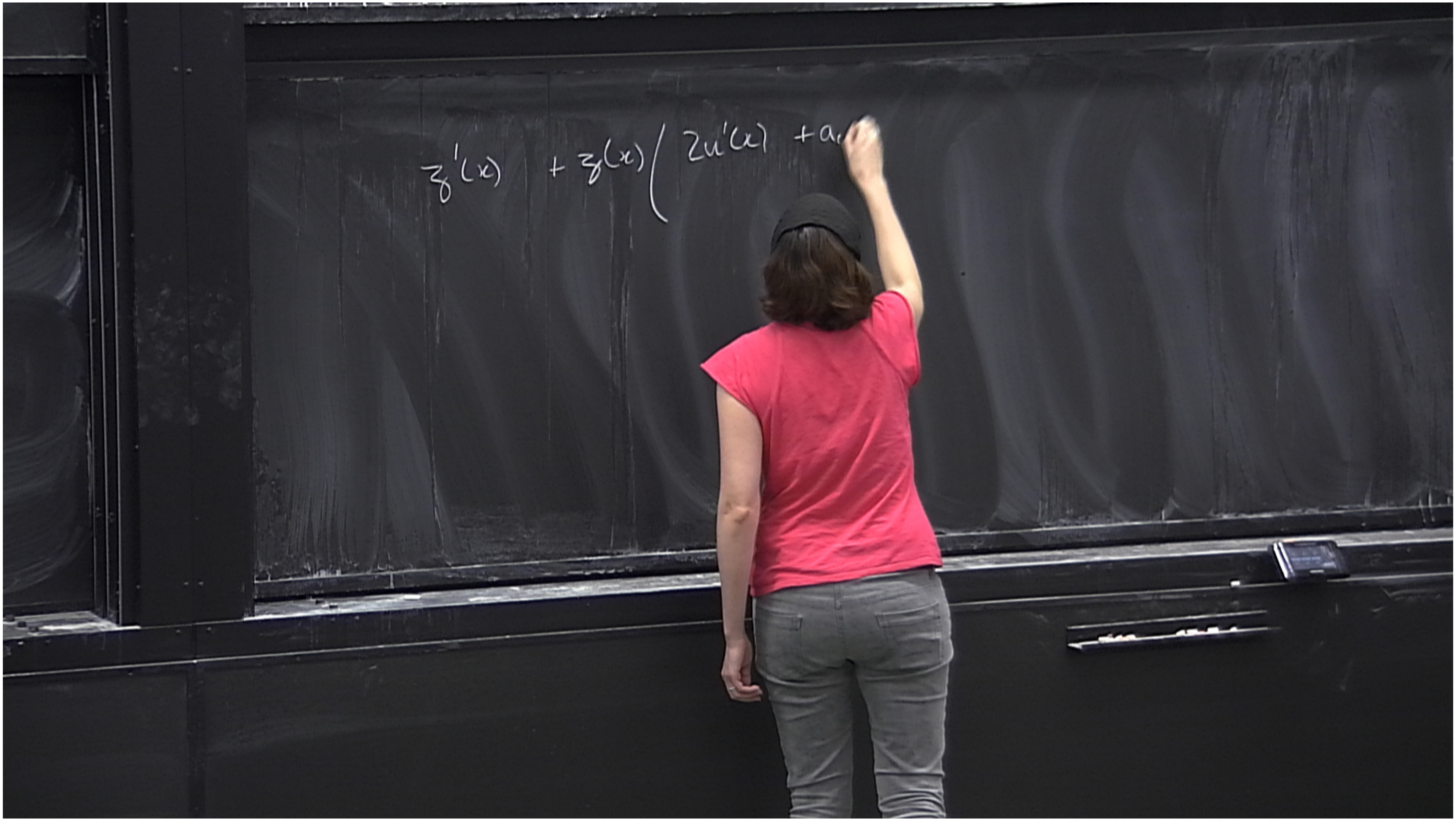
y''(x) = u''(x) v(x) + 2u'(x)v'(x) +
v(u''(x) + a1(x) u'(x) + a0(x) u(x)) + v'(x)(2u'(x) + a0(x)u(x)) + v''(x)u(x) = 0
lution of
u(x) v(x)



y''(x) = u''(x) v(x) + 2u'(x)v'(x) +
v(u''(x) + a1(x) u'(x) + a0(x) u(x)) + v'(x)(2u'(x) + a0(x)u(x)) + v''(x)u(x) = 0
lution of
u(x) v(x)

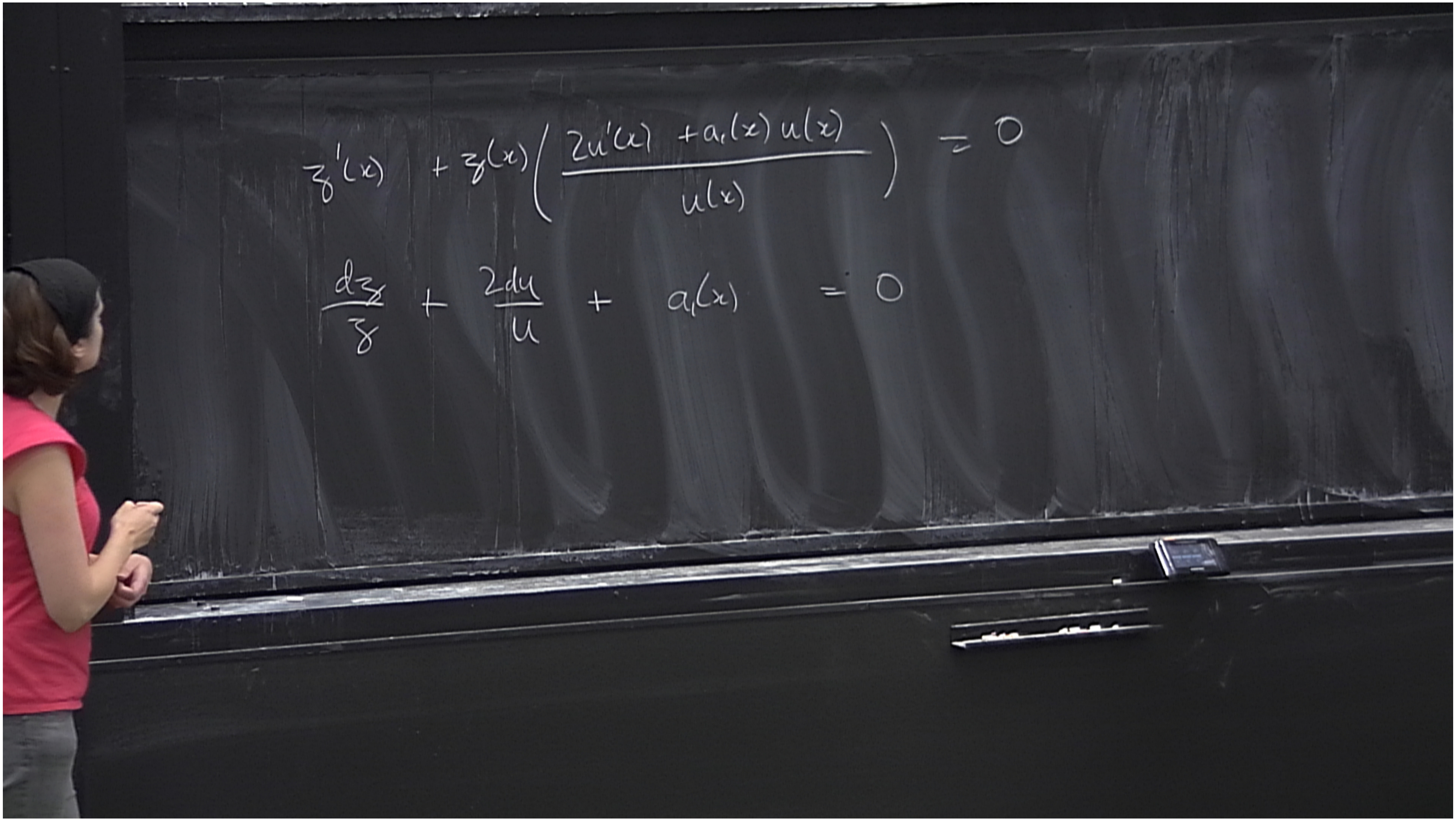

y''(x) = u''(x) v(x) + 2u'(x)v'(x) +
v (u''(x) + a_1(x) u'(x) + a_0(x) u(x)) + v'(x) (2u'(x) + a_0(x) u(x)) + v''(x) u(x) = 0
lution of
u(x) v(x)

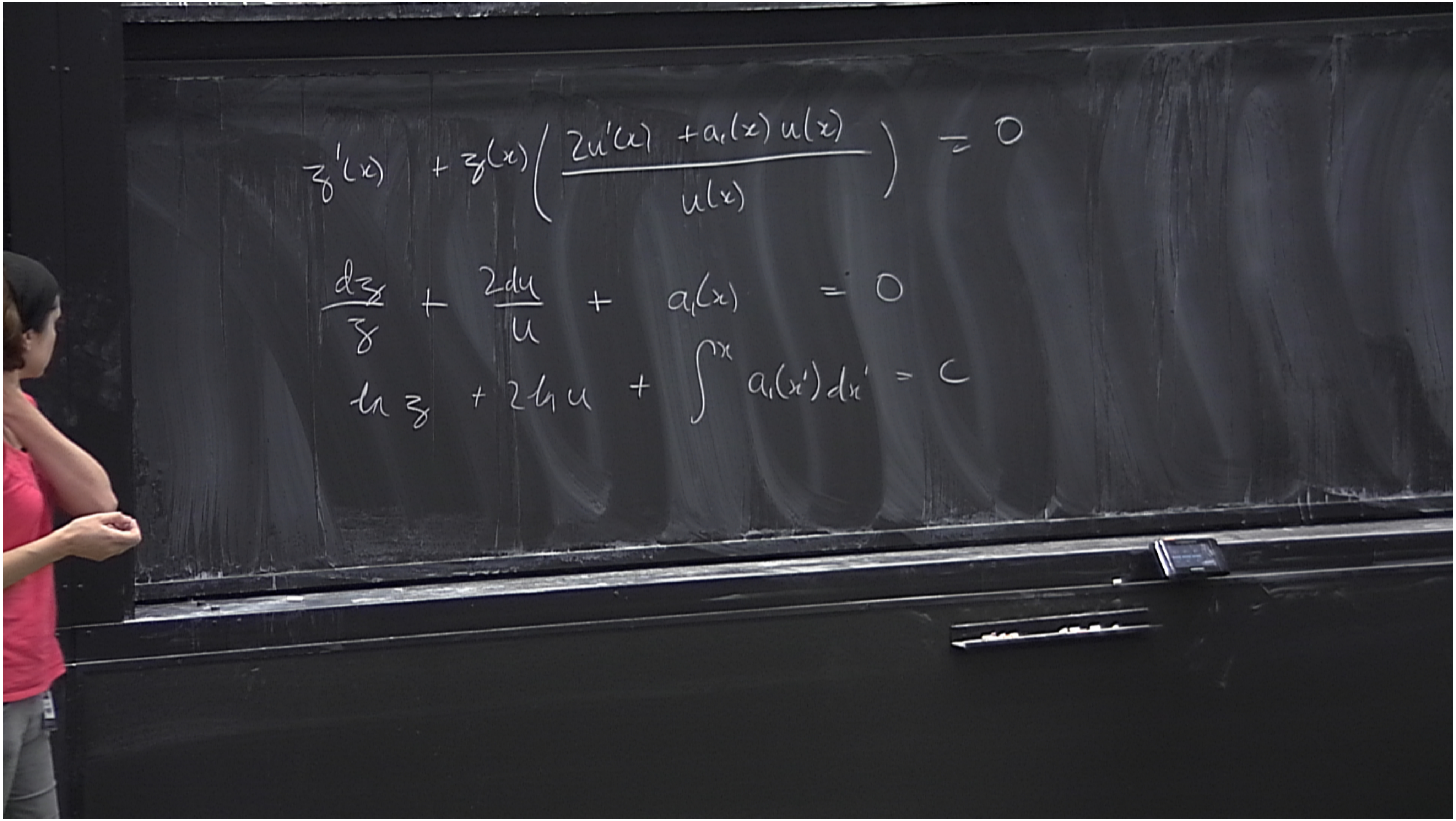

y'(x) = u'(x) v(x) + u(x) v'(x)
y''(x) = u''(x) v(x) + 2u'(x)v'(x) + u(x) v''(x)
→ 0
v(u''(x) + a1(x) u'(x) + a0(x) u(x)) + v'(x)(2u'(x) + a1(x)u(x)) + v''(x) u(x) = 0
lution of
u(x) v(x)



z'(x) + z(x)(2u'(x) + a

$$z'(x) + z(x)\left(\frac{2u'(x) + a_1(x)\,u(x)}{u(x)}\right) = 0$$


$z'(x) + z(x)\left(\frac{2u'(x) + a_1(x)\, u(x)}{u(x)}\right) = 0$
$\frac{dz}{z} + \frac{2du}{u} + a_1(x) = 0$

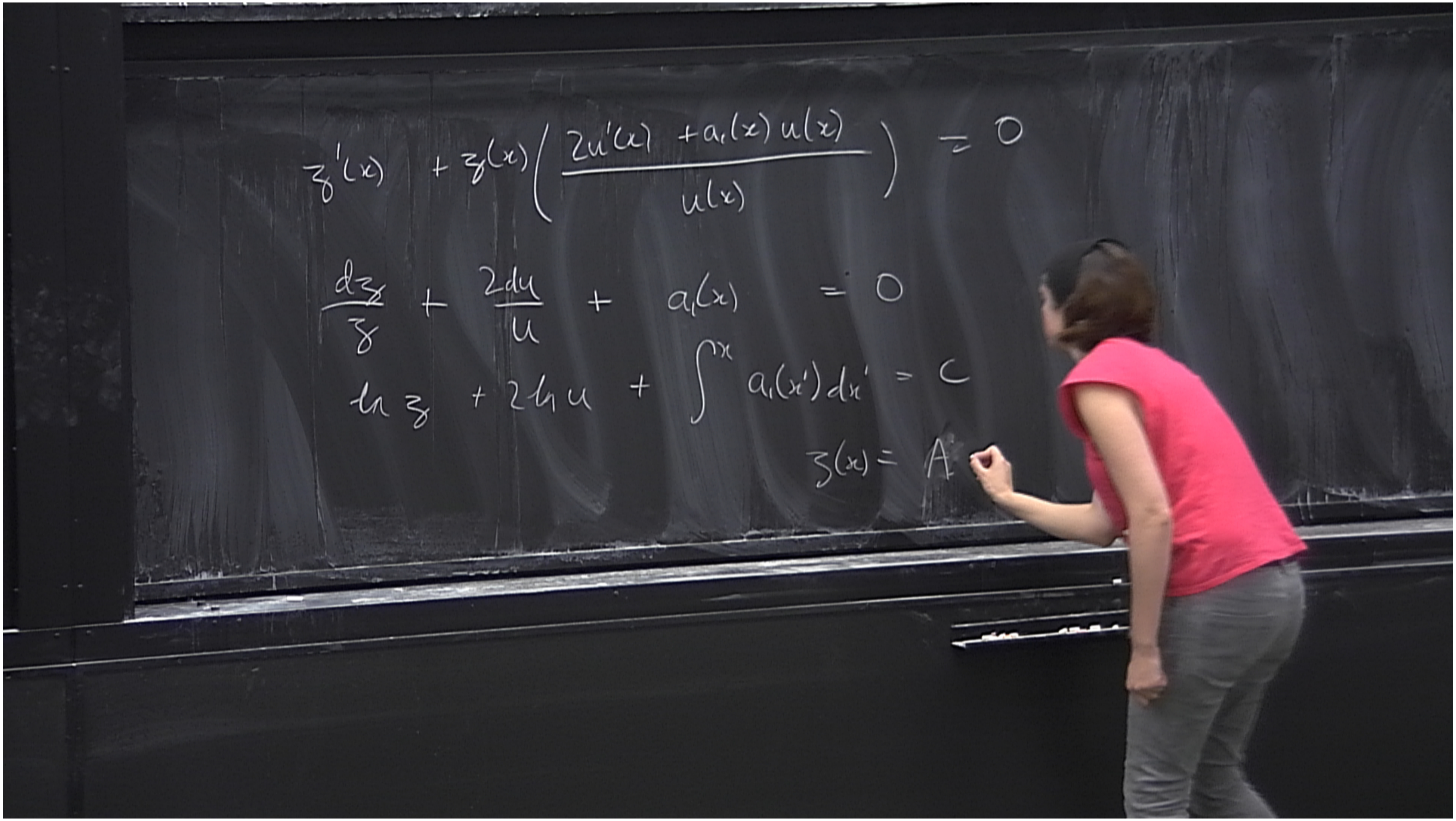

$z'(x) + z(x)\left(\frac{2u'(x) + a_1(x)\,u(x)}{u(x)}\right) = 0$
$\frac{dz}{z} + \frac{2du}{u} + a_1(x) = 0$
$\ln z + 2\ln u + \int^x a_1(x')\,dx' = c$

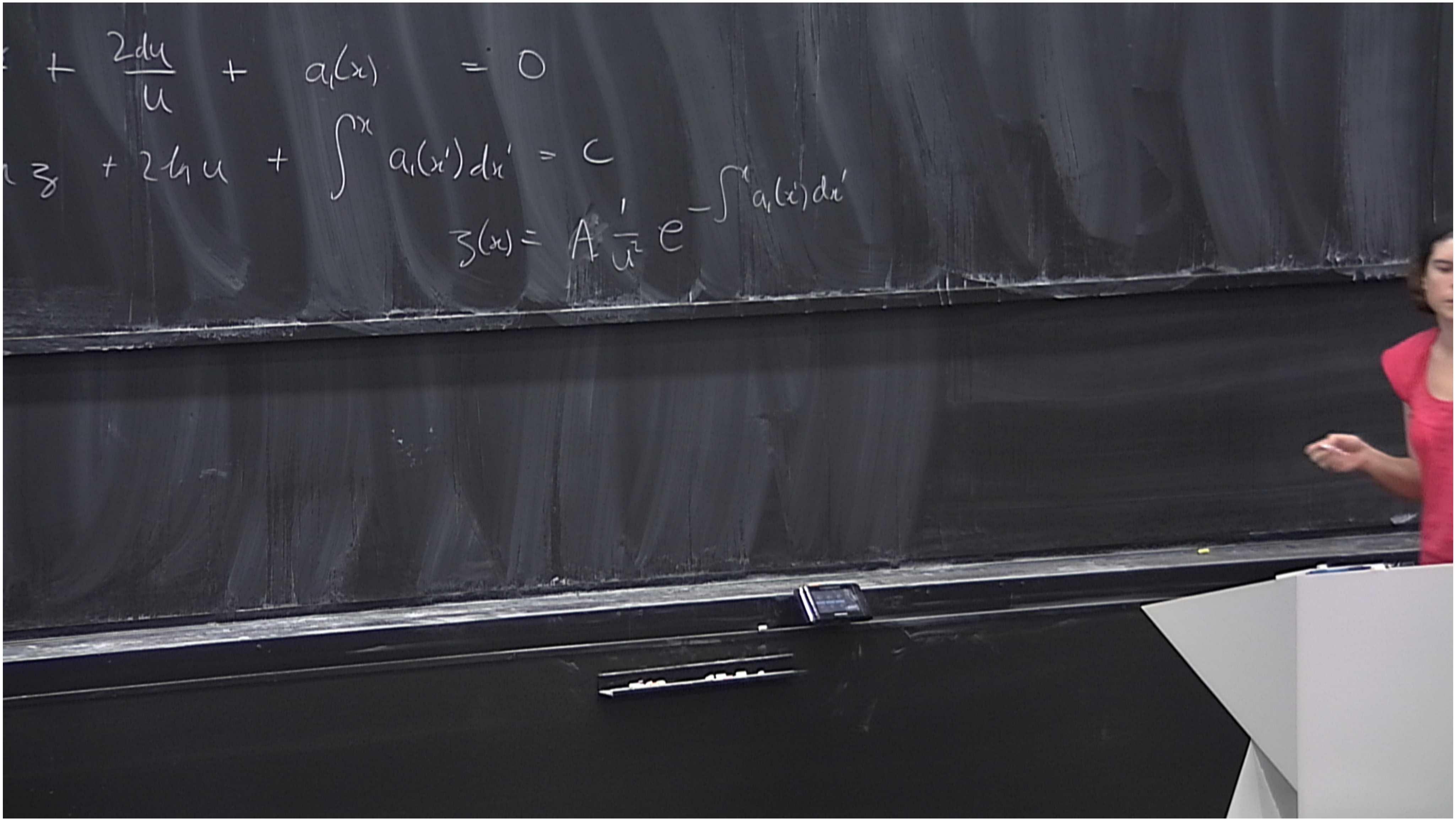
$$z'(x) + z(x)\left(\frac{2u'(x) + a_1(x)\,u(x)}{u(x)}\right) = 0$$

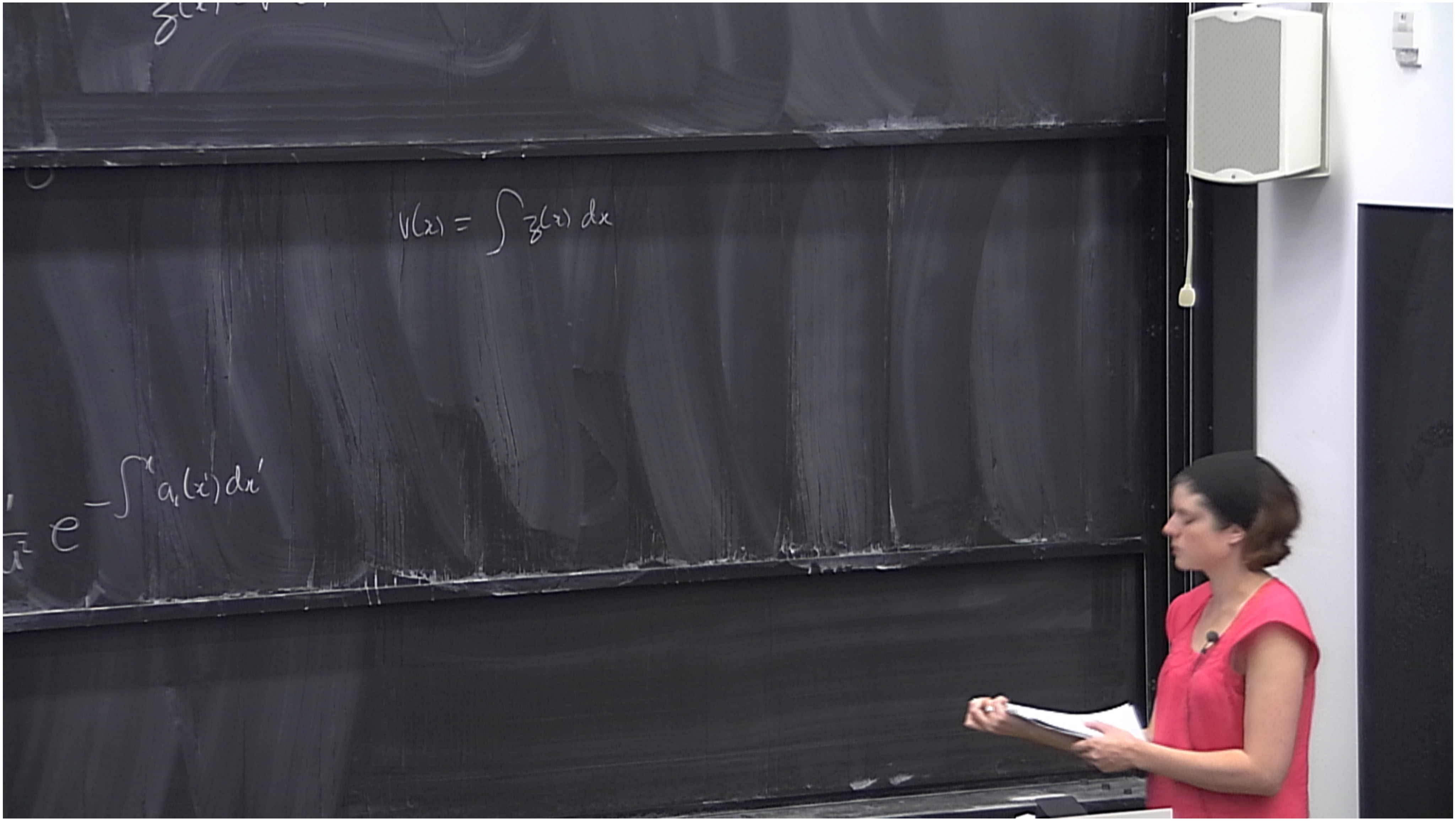
$$\frac{dz}{z} + \frac{2du}{u} + a_1(x) = 0$$

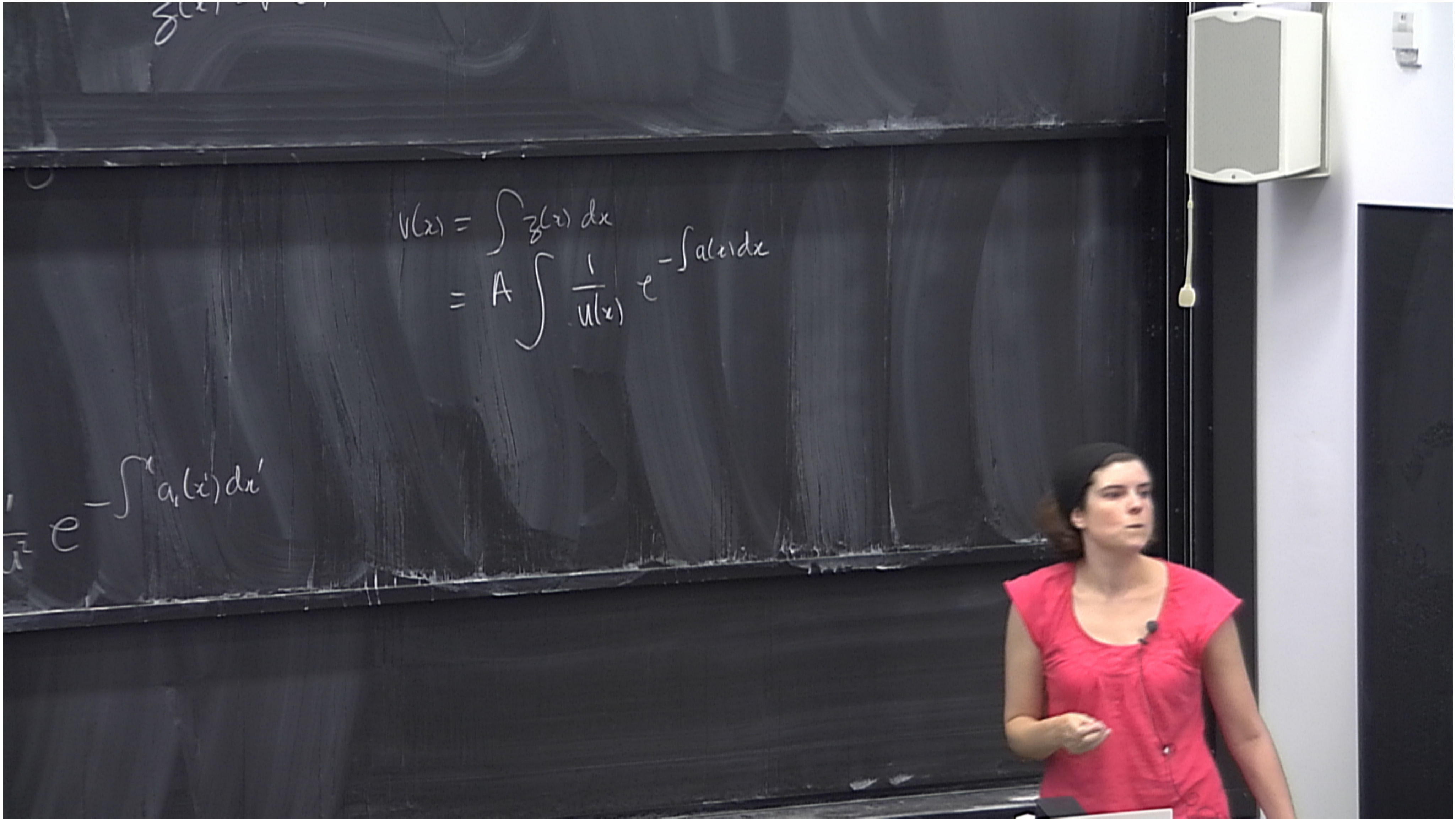
$$\ln z + 2\ln u + \int^x a_1(x')\,dx' = c$$

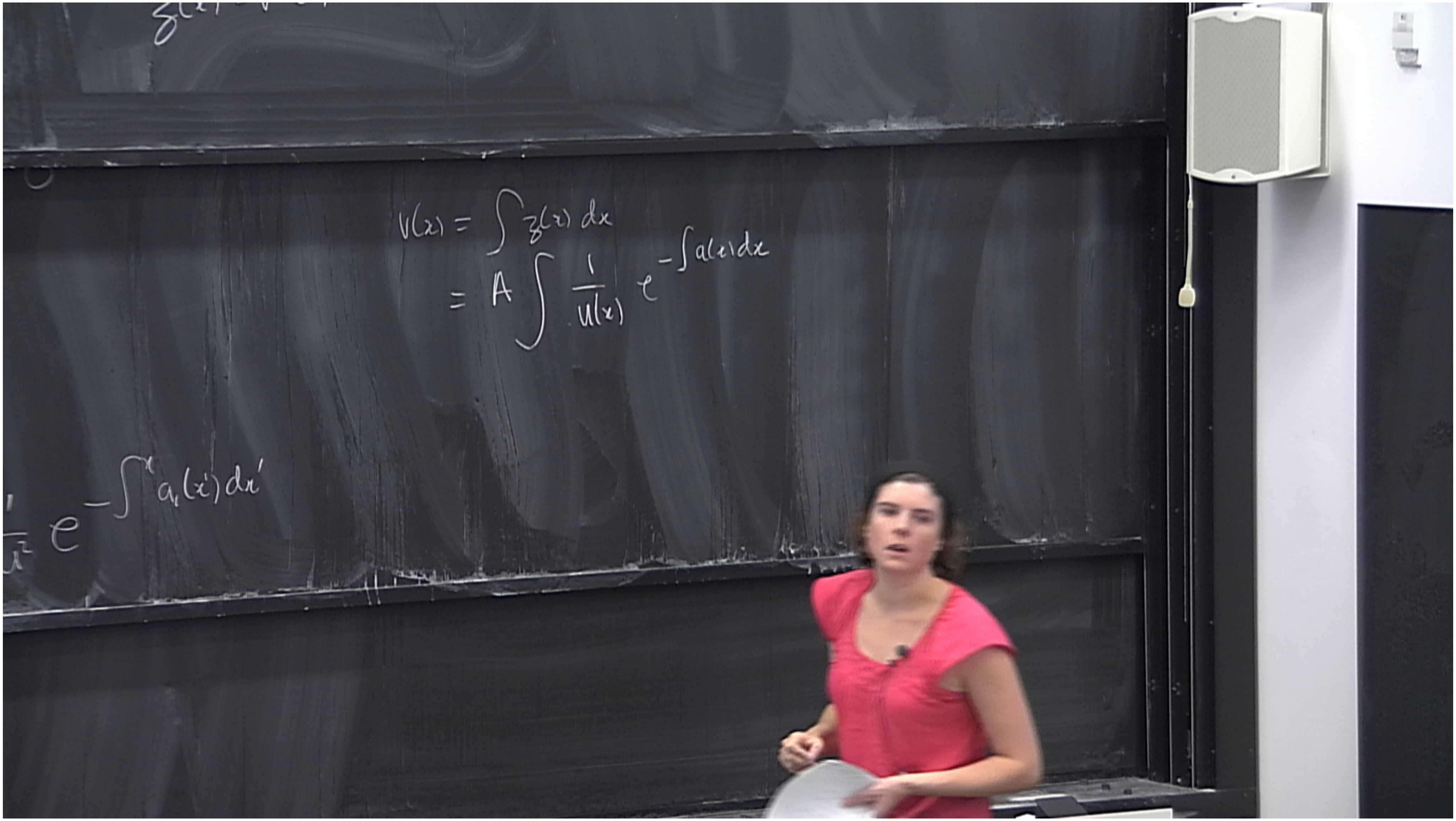
$$z(x) =$$

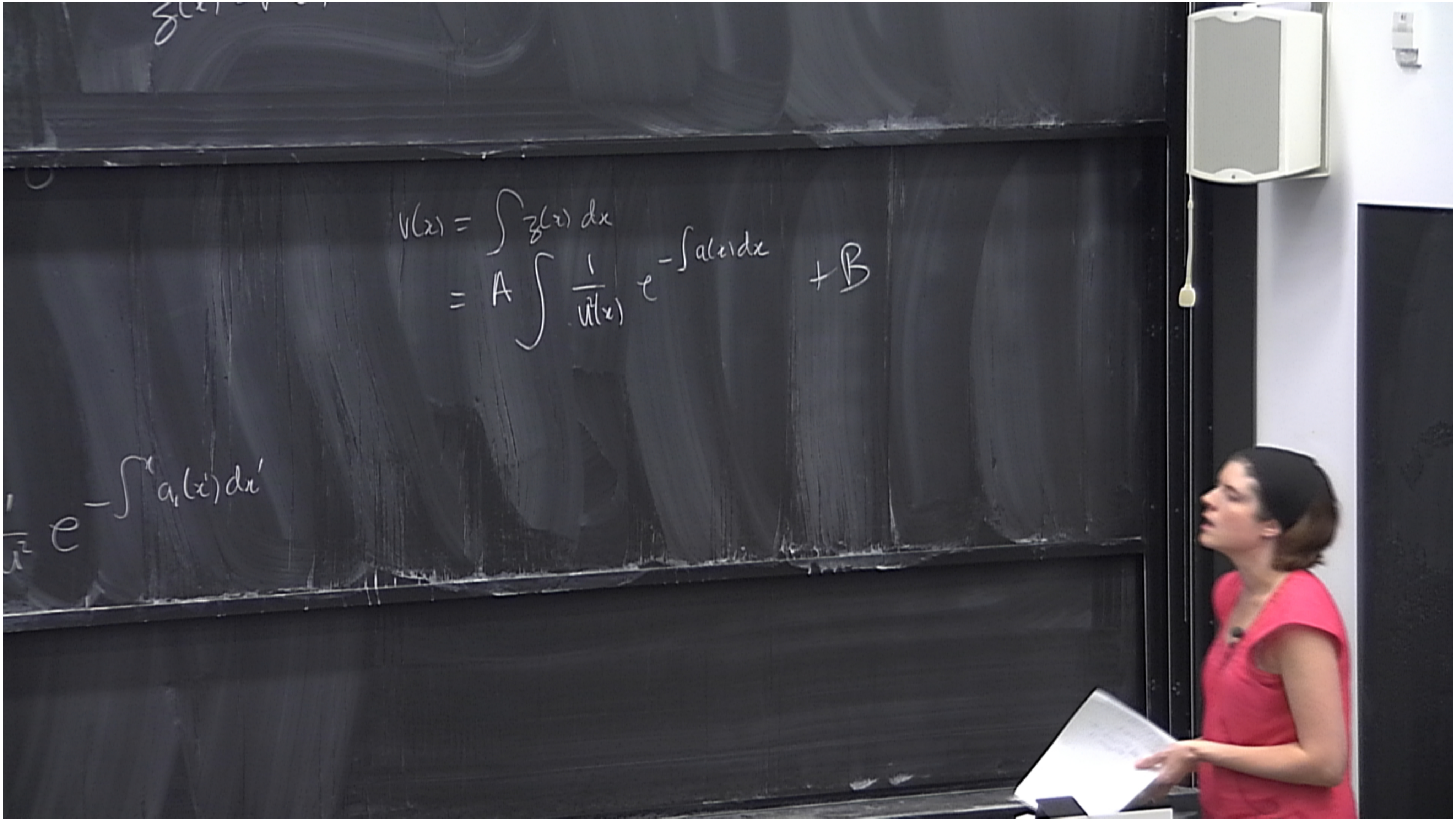

$z'(x) + z(x)\left(\frac{2u'(x) + a_1(x)\,u(x)}{u(x)}\right) = 0$
$\frac{dz}{z} + \frac{2du}{u} + a_1(x) = 0$
$\ln z + 2\ln u + \int^x a_1(x')\,dx' = C$
$z(x) = A$

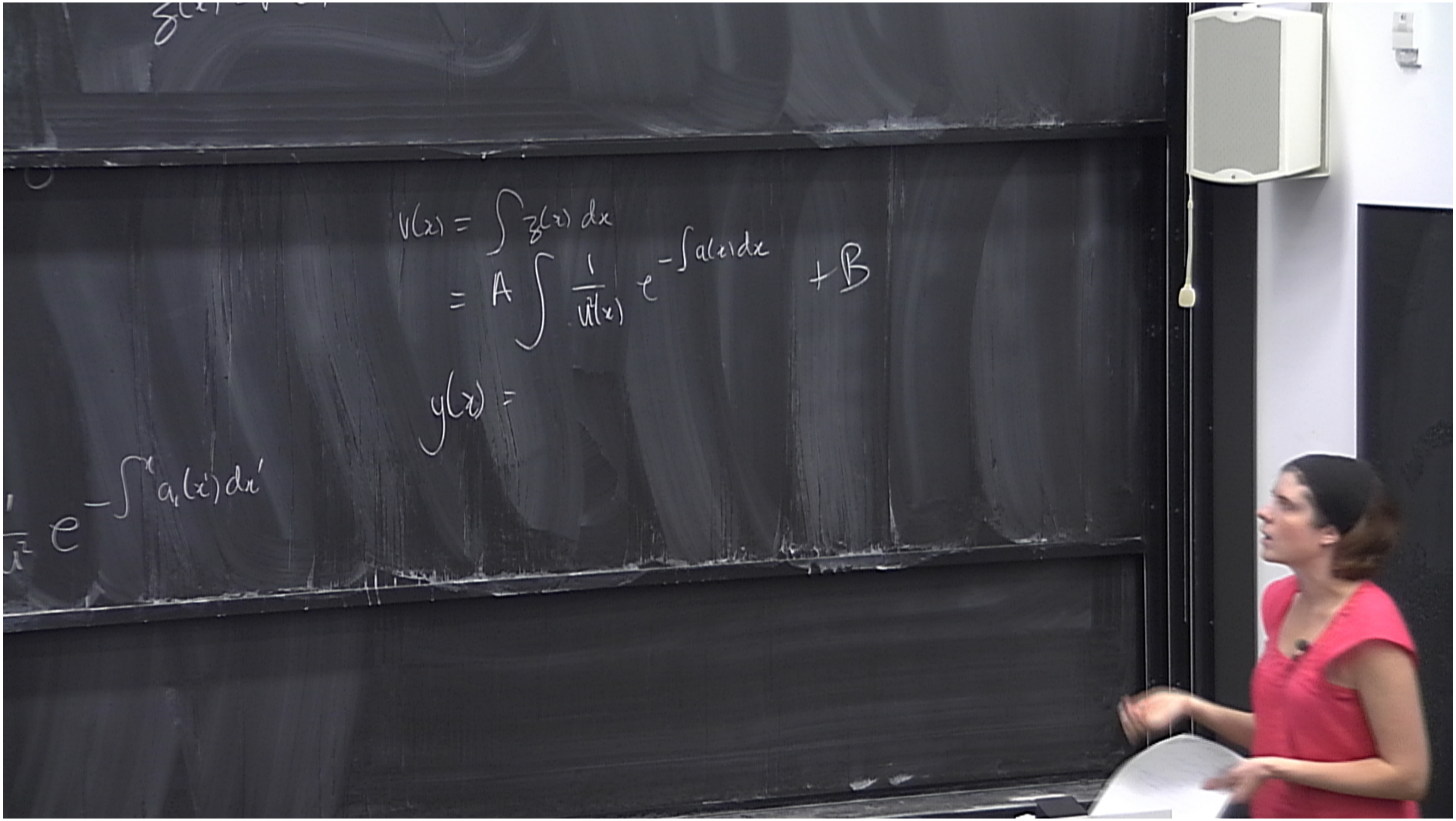

$+ \frac{2du}{u} + a_1(x) = 0$
$z + 2\ln u + \int^x a_1(x')dx' = c$
$z(x) = A \frac{1}{u^2} e^{-\int^x a_1(x')dx'}$

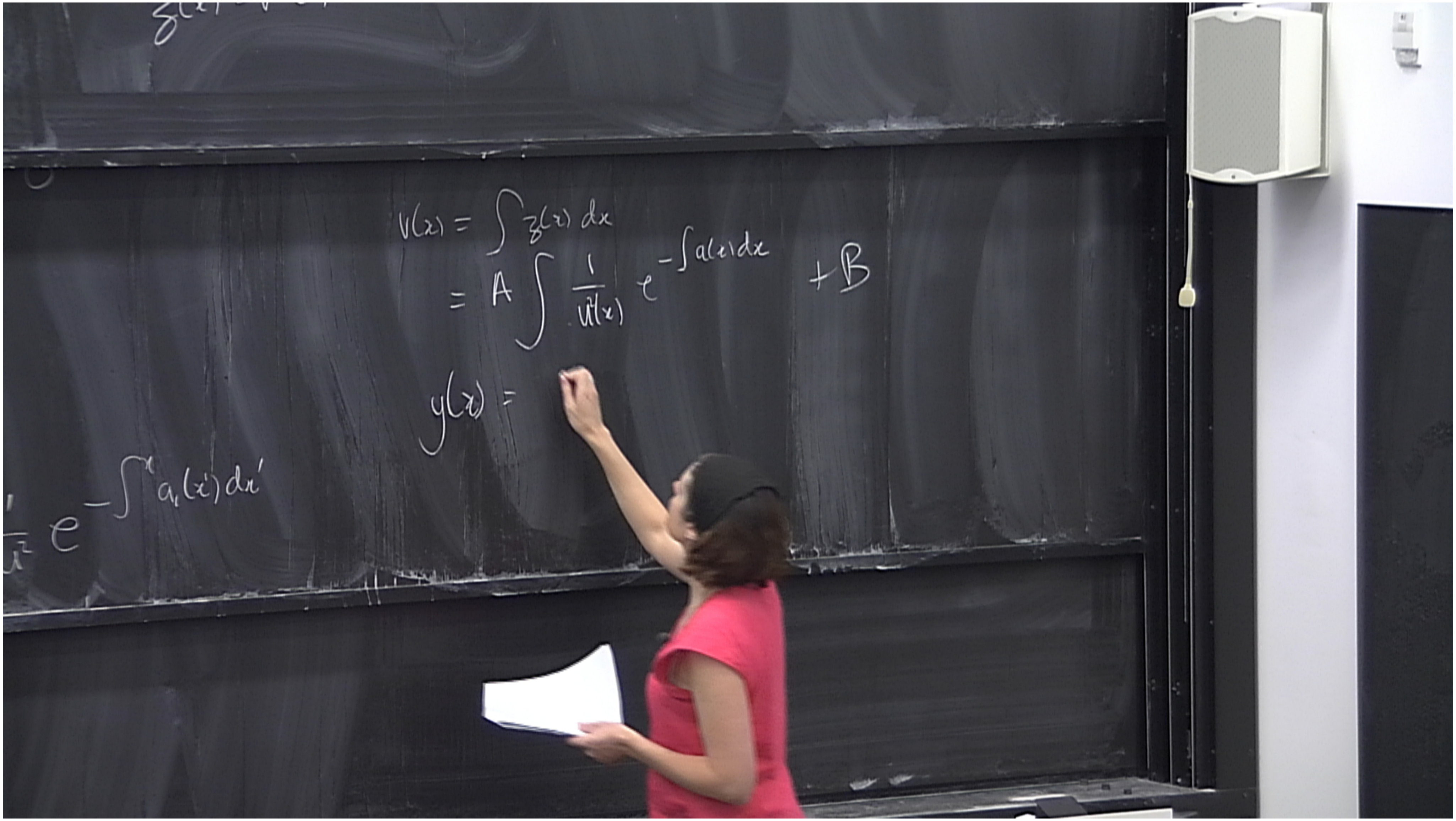

v(x) = ∫ ξ(x) dx

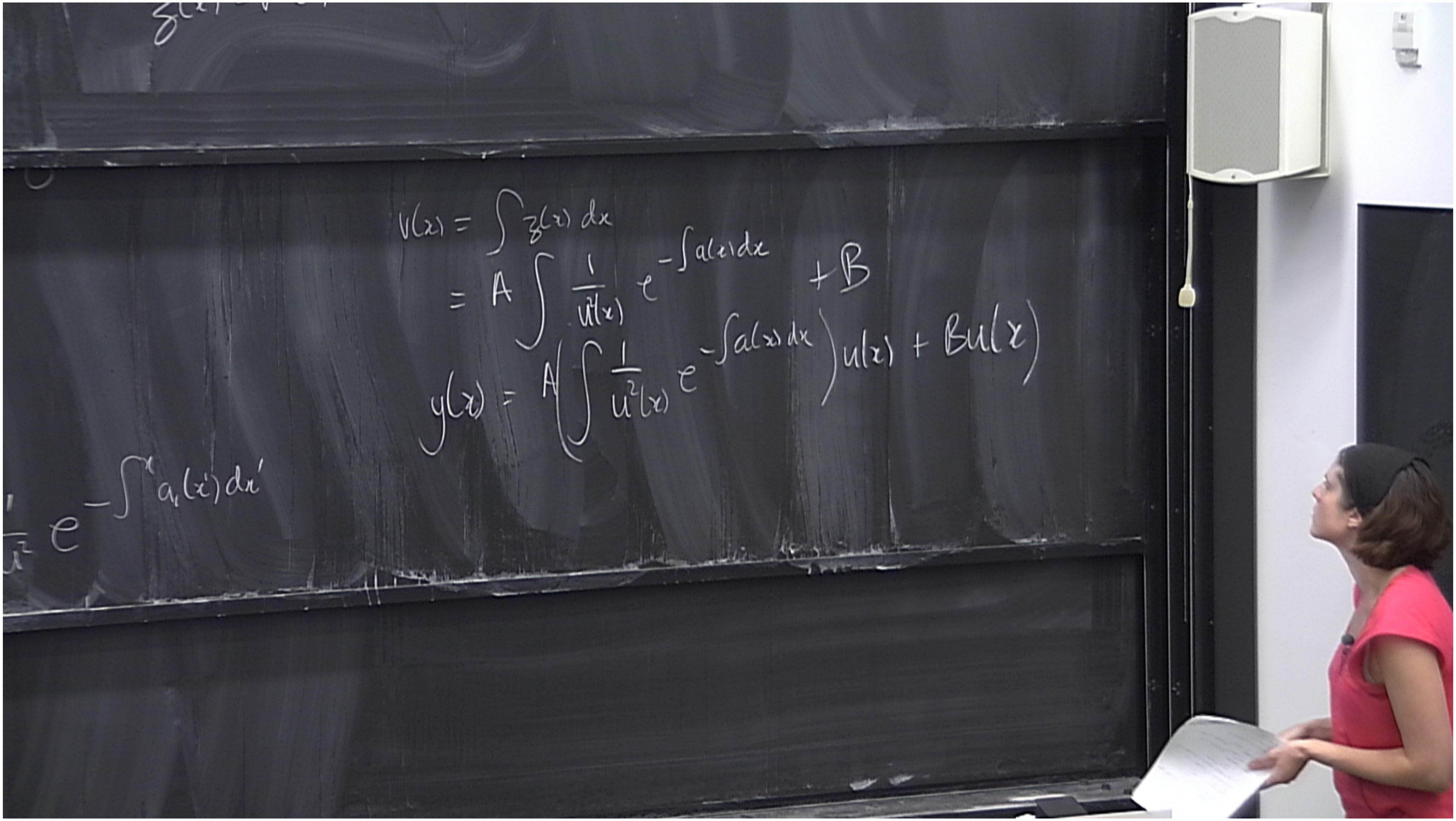

$v(x) = \int z(x)\, dx$
$= A \int \frac{1}{u(x)} e^{-\int a(x) dx}$
$e^{-\int^x a_1(x')\, dx'}$

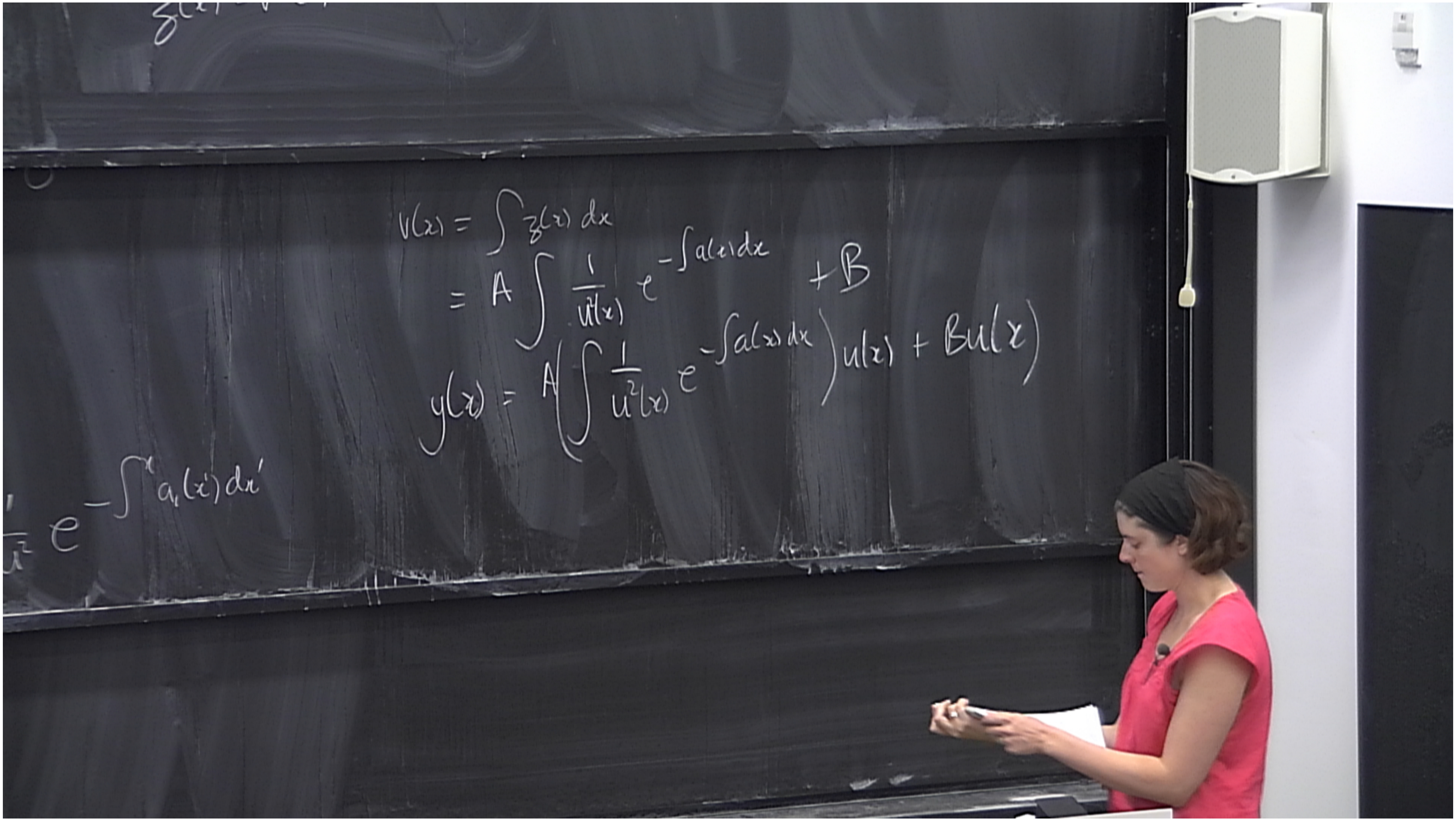

$v(x) = \int z(x)\, dx$
$= A \int \frac{1}{u(x)} e^{-\int a(x) dx}$

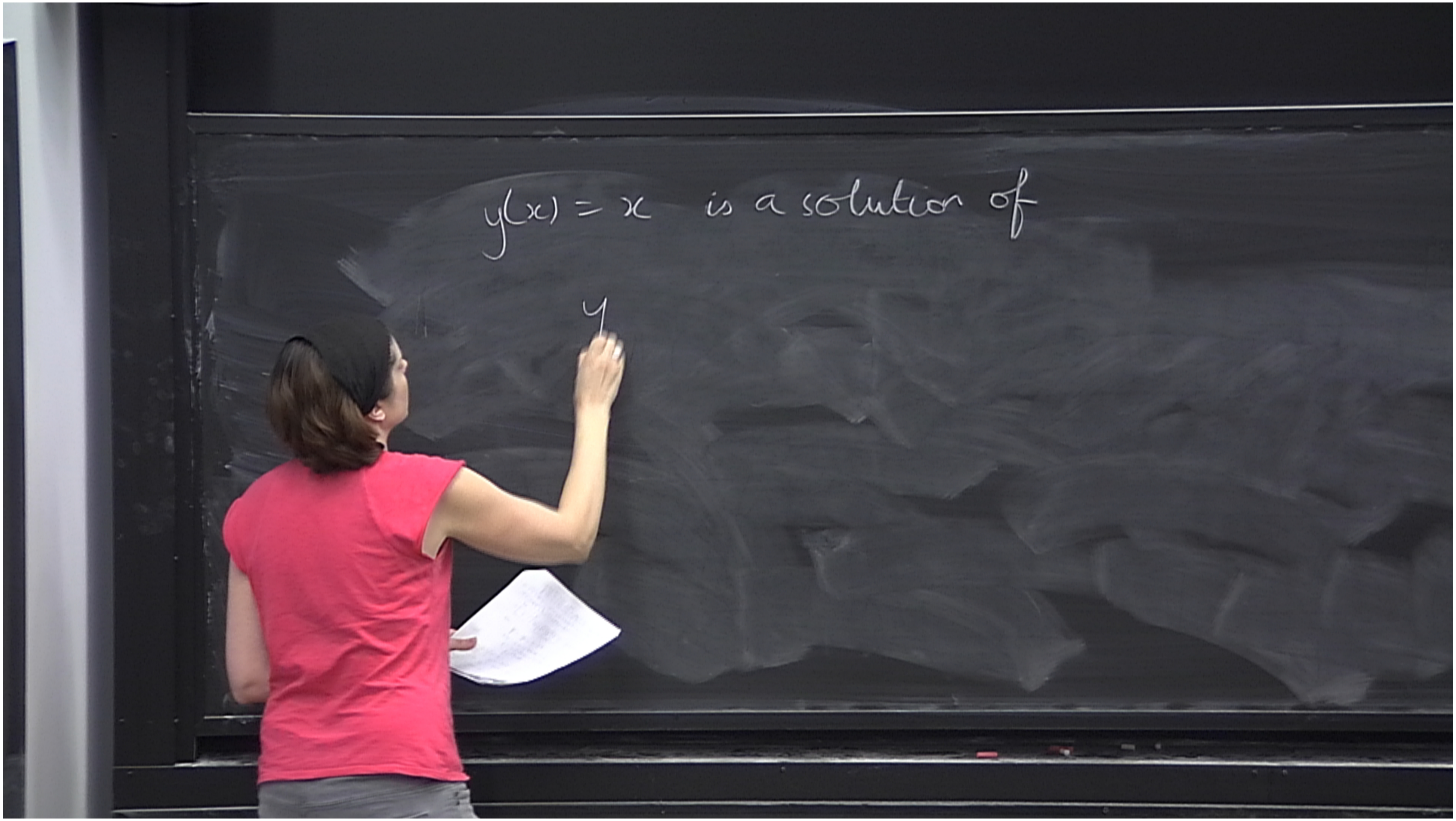

$v(x) = \int z(x)\,dx$
$= A \int \frac{1}{u^2(x)} e^{-\int a(x)\,dx} + B$
$e^{-\int^x a_1(x')\,dx'}$

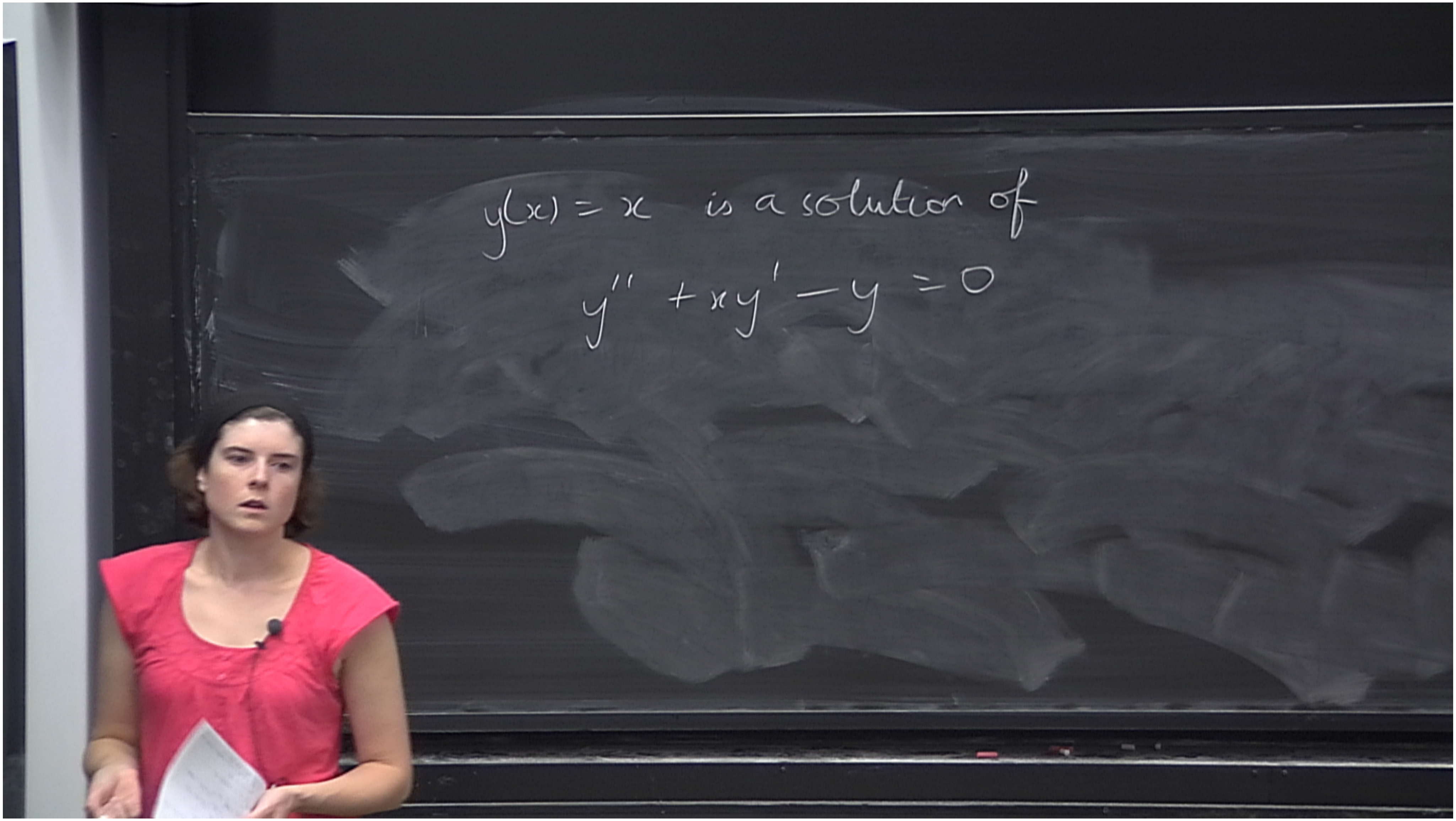

$v(x) = \int z(x)\,dx$
$= A \int \frac{1}{u^2(x)} e^{-\int a(x)\,dx} + B$
$y(x) =$
$e^{-\int^{x} a_1(x')\,dx'}$

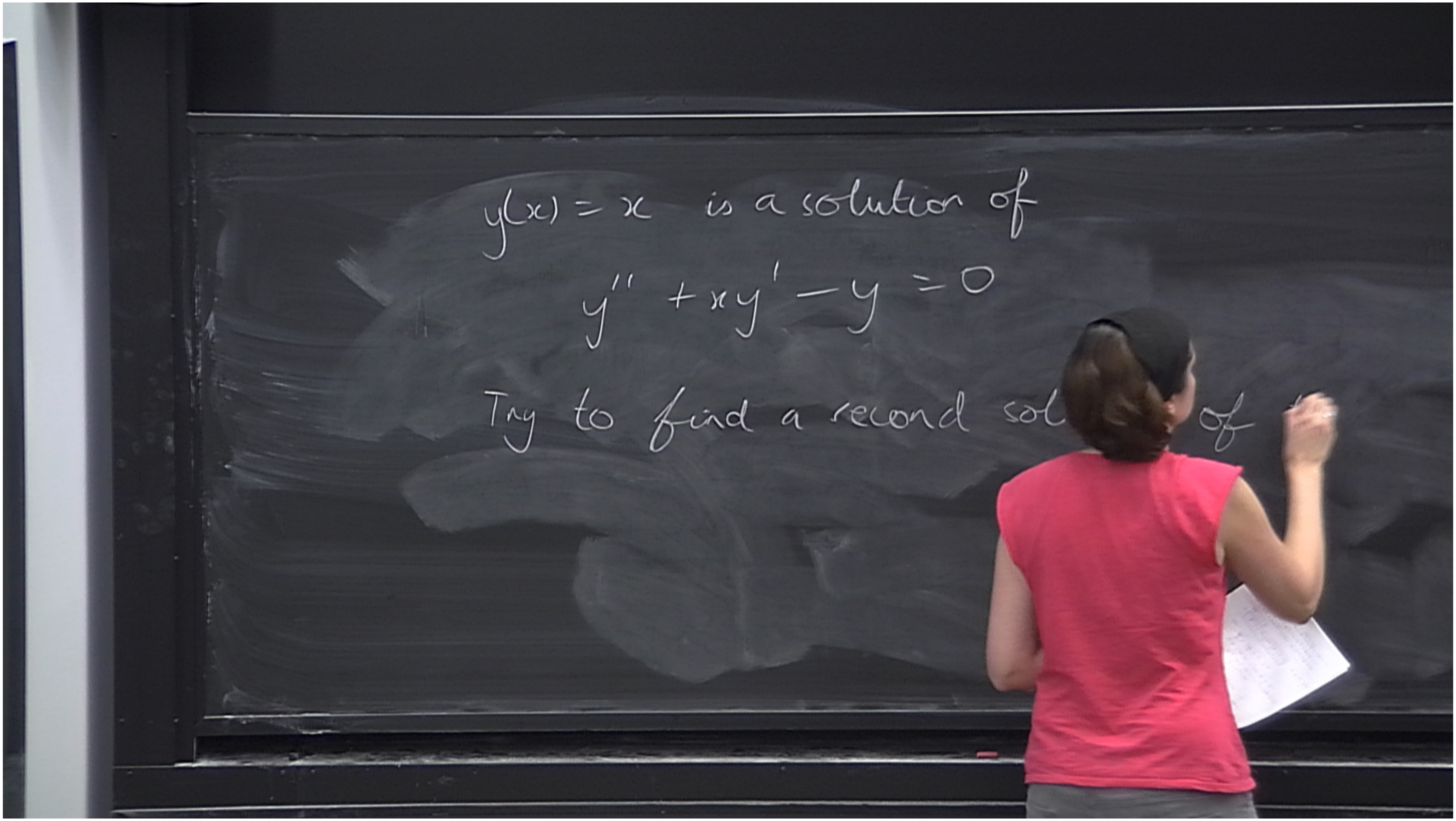

v(x) = ∫ z(x) dx
= A ∫ 1/u²(x) e^{−∫a(x)dx} + B
y(x) =
e^{−∫^x a₁(x′)dx′}

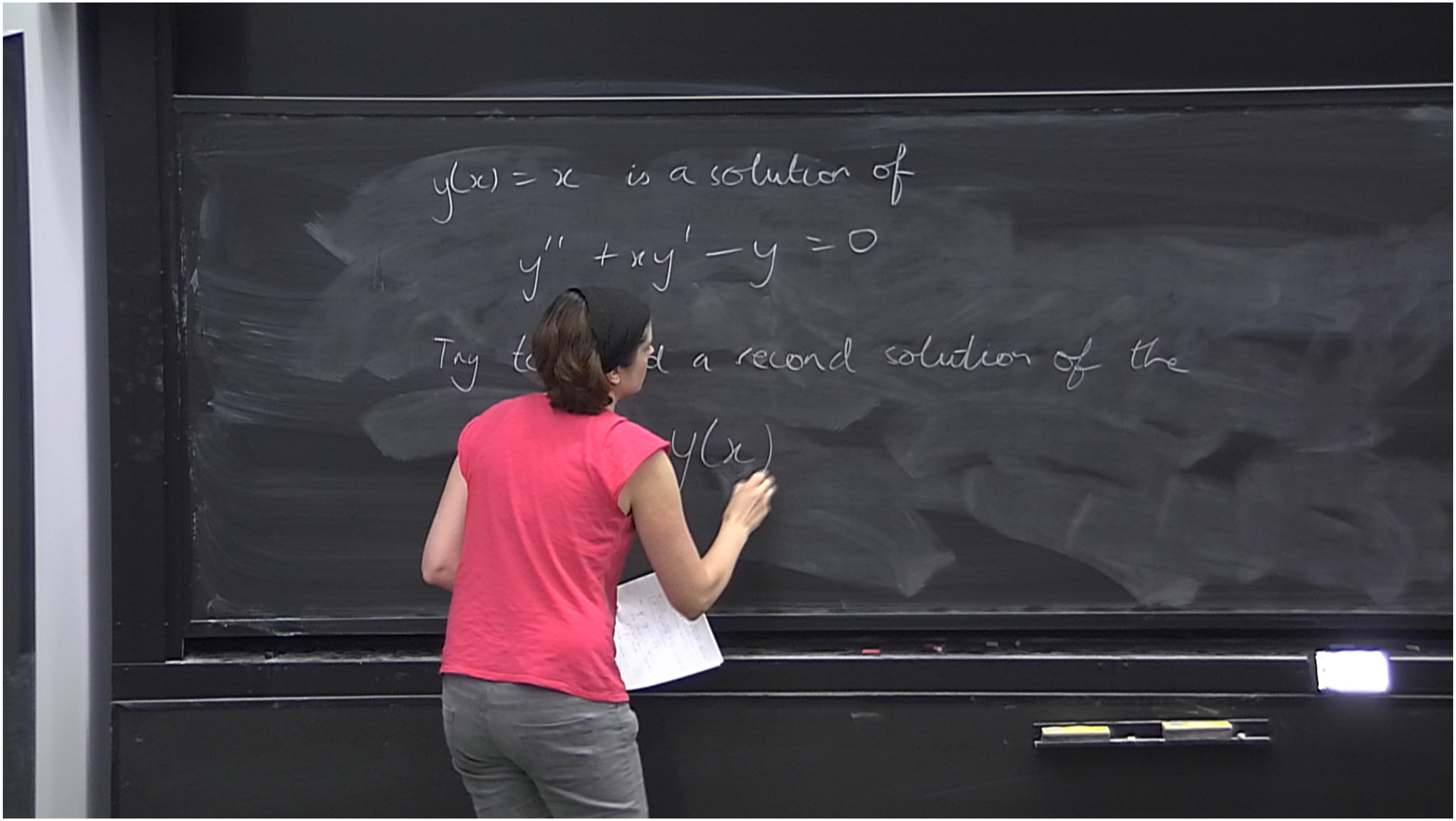

v(x) = ∫ z(x) dx
= A ∫ 1/u²(x) e^{-∫a(x)dx} + B
y(x) = A(∫ 1/u²(x) e^{-∫a(x)dx}) u(x) + Bu(x)
e^{-∫^x a_1(x') dx'}

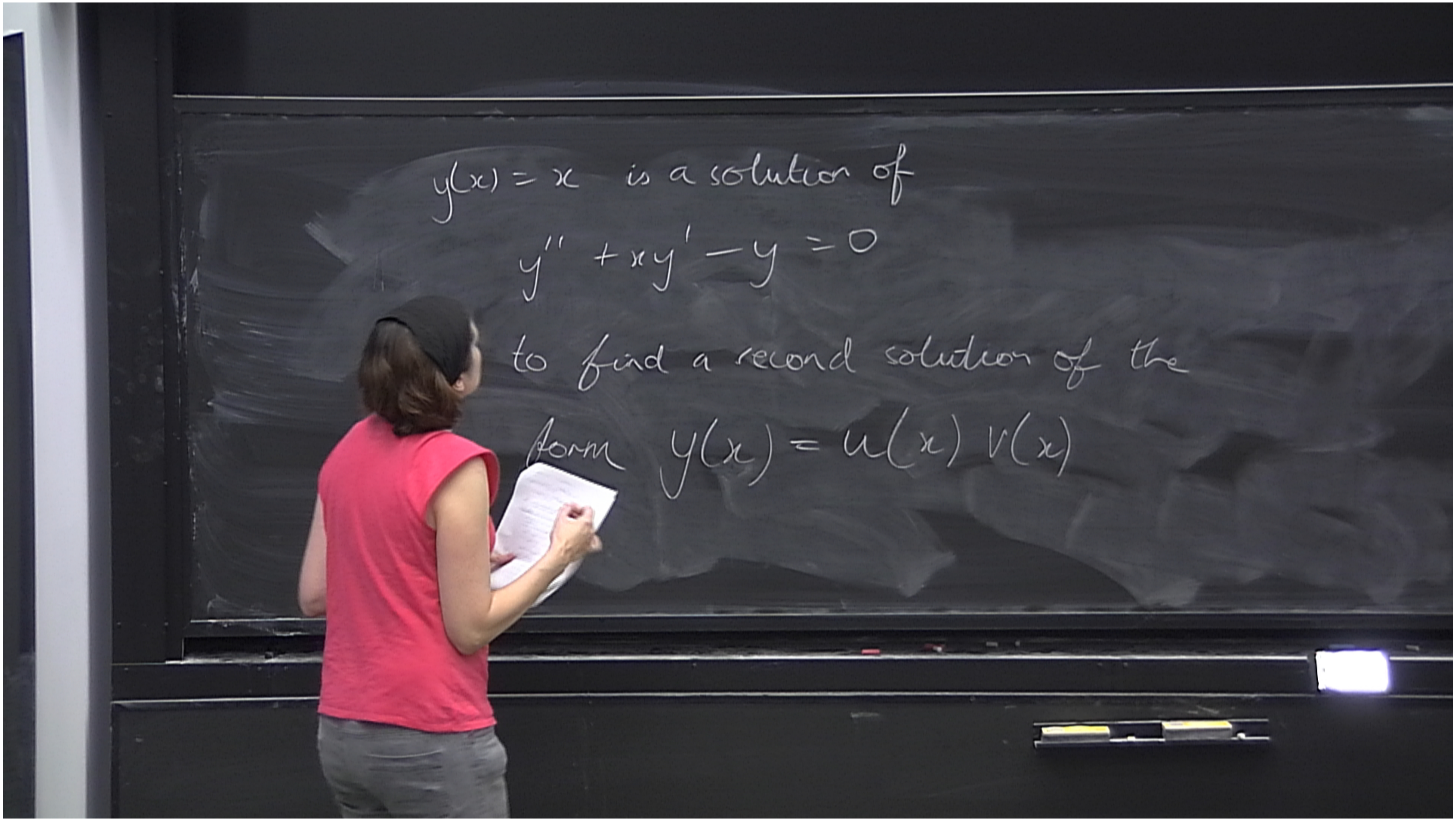

$v(x) = \int z(x)\,dx$
$= A \int \frac{1}{u^2(x)} e^{-\int a(x) dx} + B$
$y(x) = A\left(\int \frac{1}{u^2(x)} e^{-\int a(x) dx}\right) u(x) + B u(x)$
$e^{-\int^x a_1(x') dx'}$

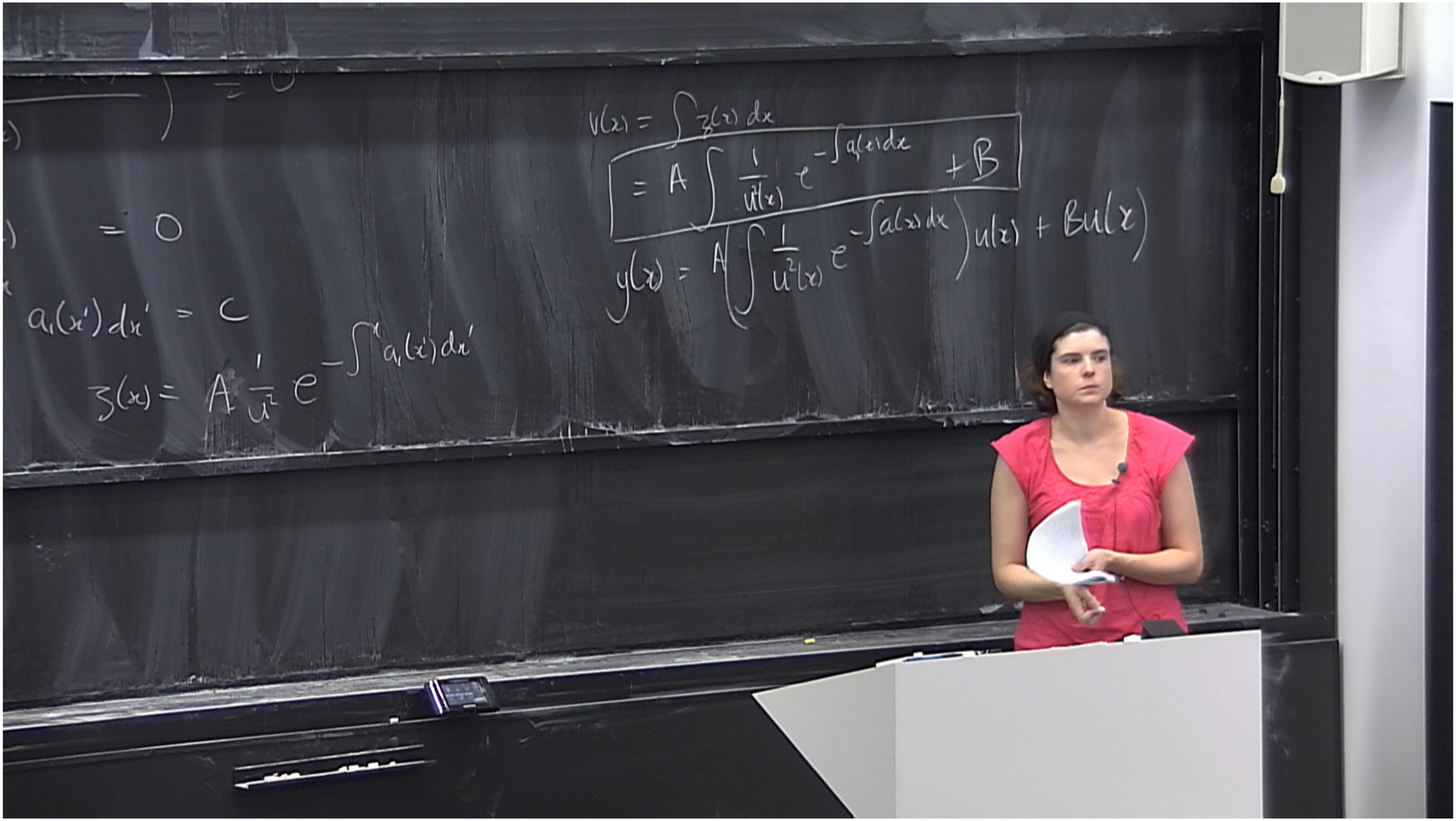

y(x) = x is a solution of
y

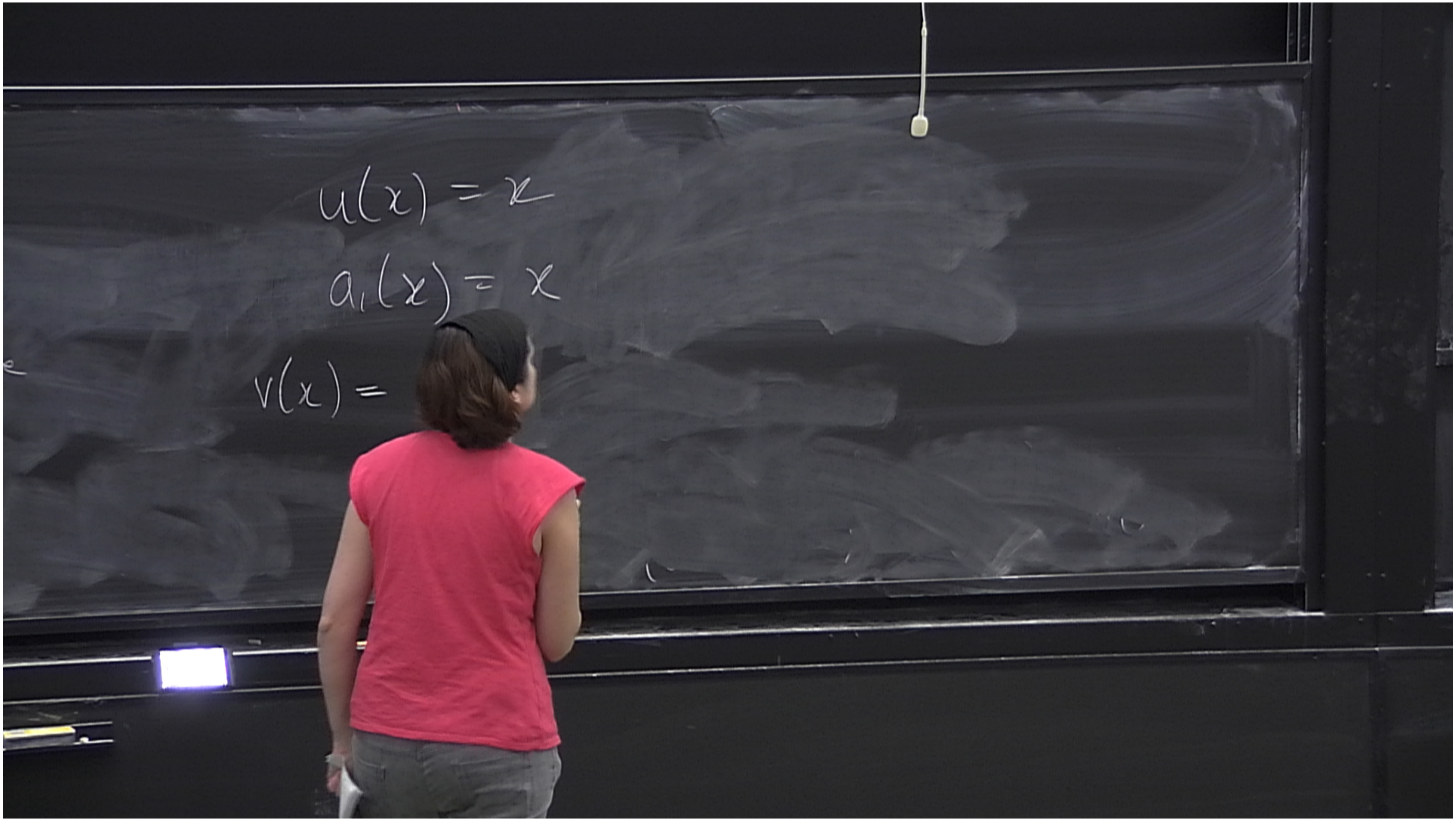

y(x) = x is a solution of
y'' + xy' − y = 0


y(x) = x is a solution of
y'' + xy' − y = 0
Try to find a second sol
of


y(x) = x is a solution of
y'' + xy' − y = 0
Try
a second solution of the
y(x)


y(x) = x is a solution of
y'' + xy' − y = 0
to find a second solution of the
form y(x) = u(x) v(x)

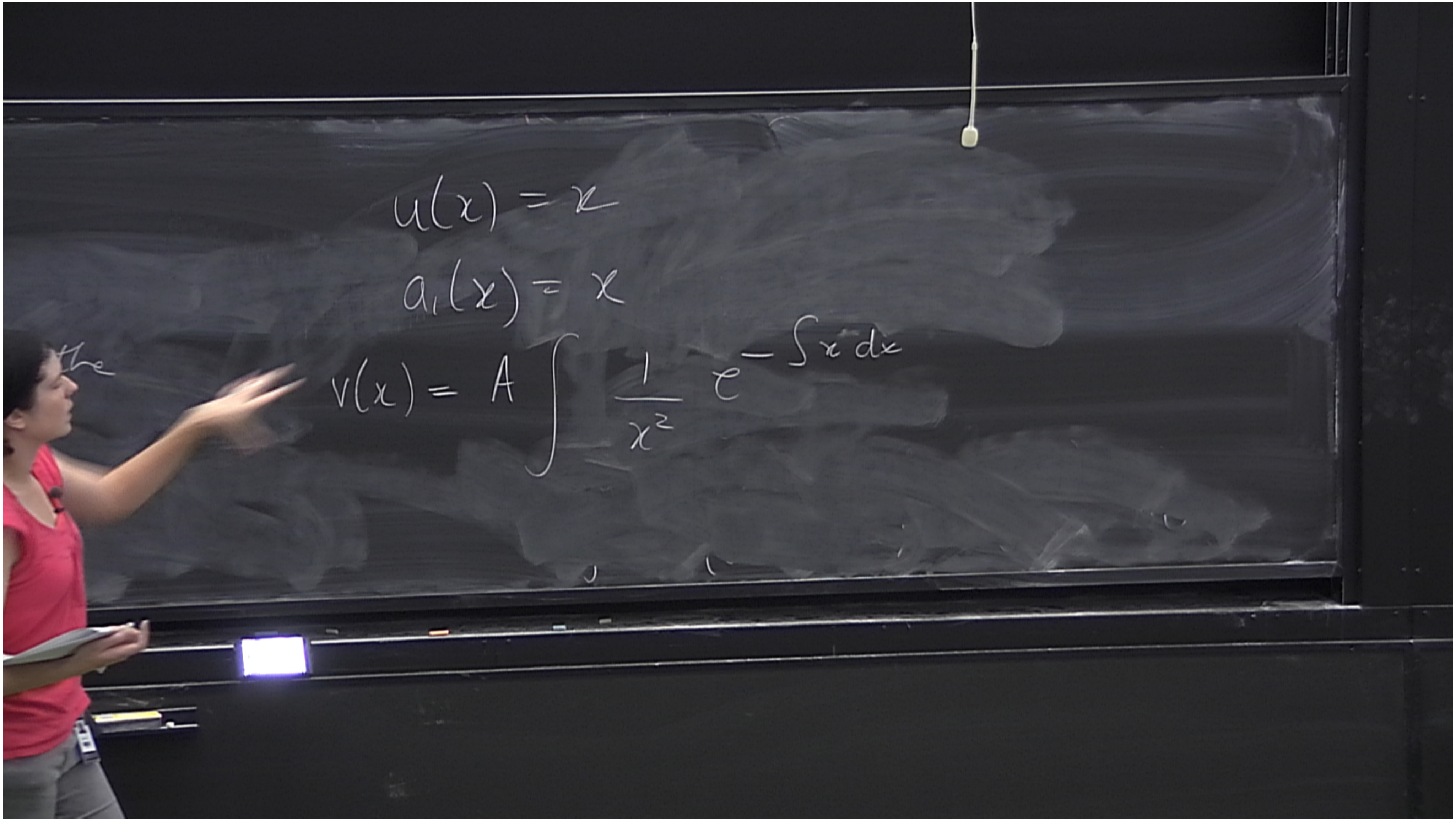

$v(x) = \int z(x)\,dx$
$= A \int \frac{1}{u^2(x)} e^{-\int a(x) dx} + B$
$y(x) = A\left(\int \frac{1}{u^2(x)} e^{-\int a(x) dx}\right) u(x) + Bu(x)$
$= 0$
$a_1(x')\,dx' = C$
$z(x) = A \frac{1}{u^2} e^{-\int^x a_1(x')dx'}$

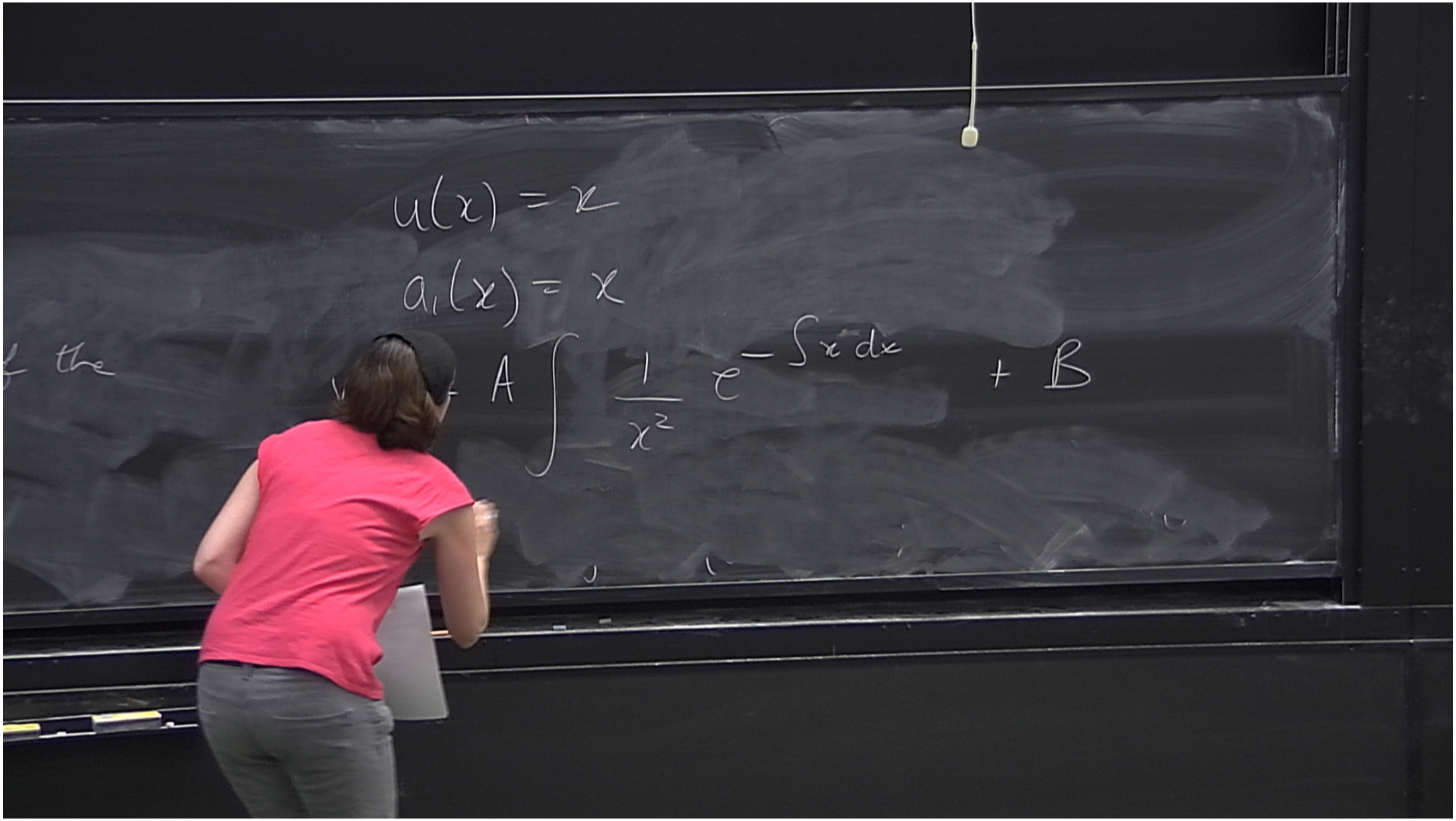

$u(x) = x$
$a_1(x) = x$
$v(x) =$

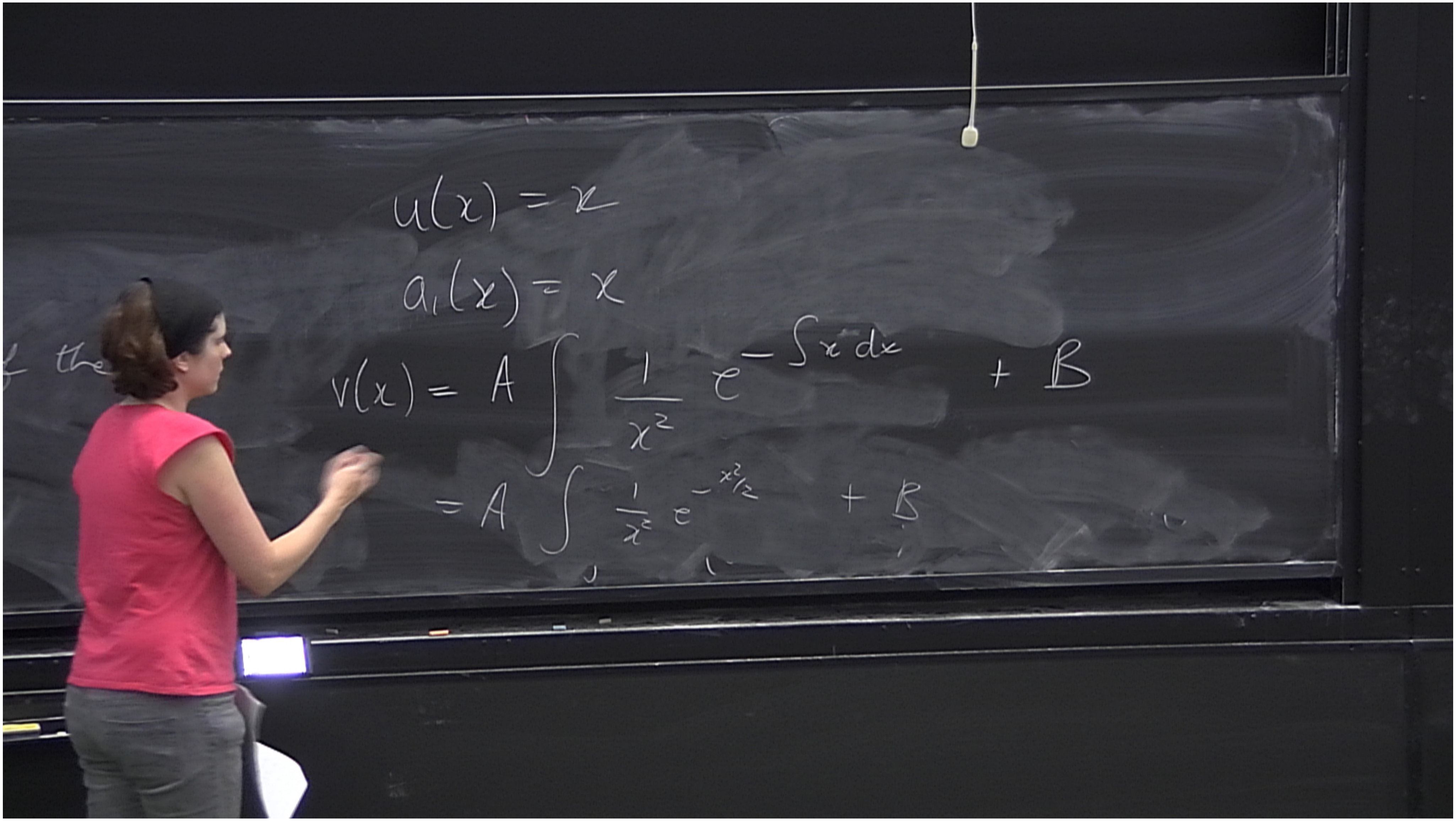
$$u(x) = x$$

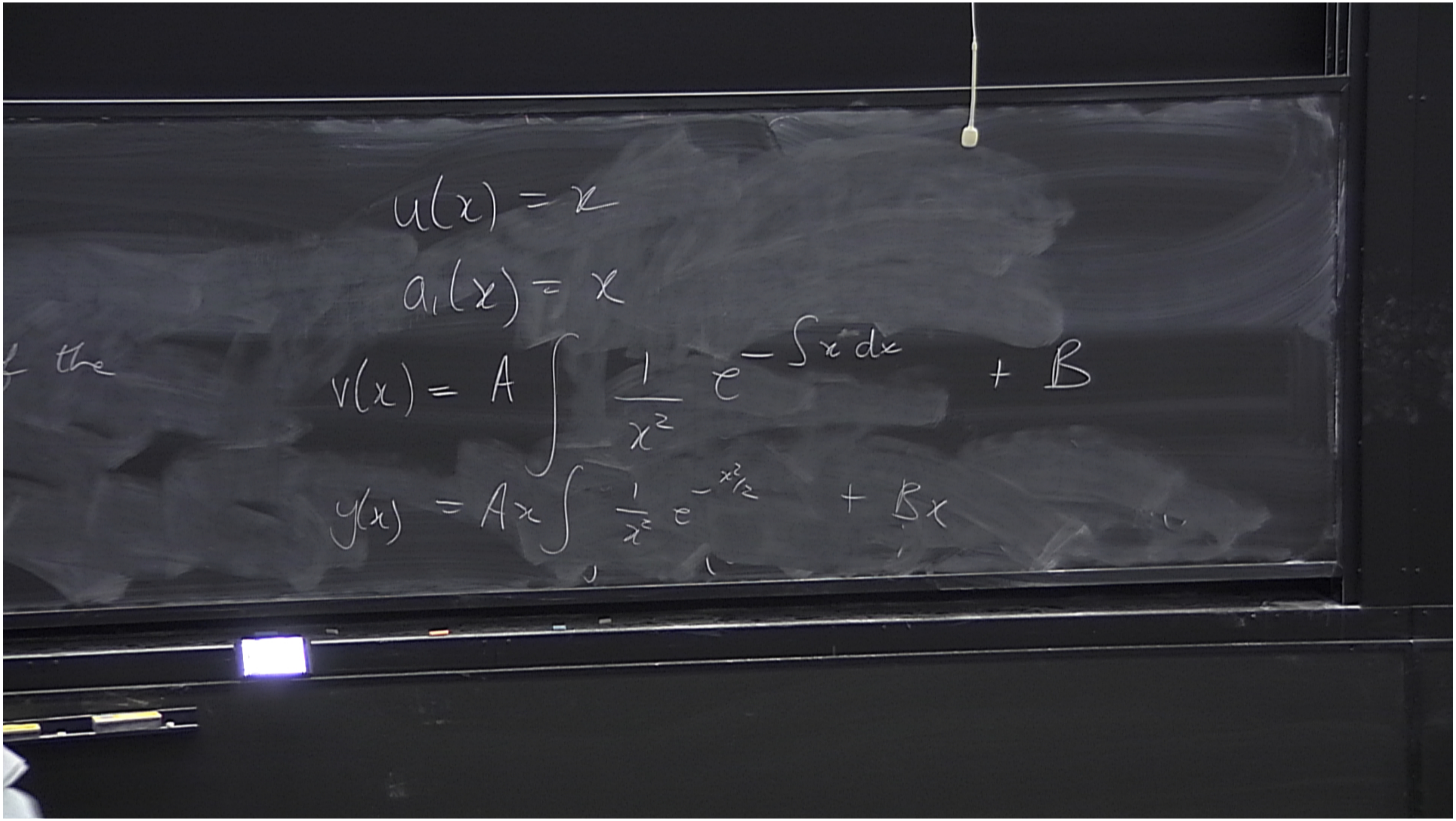
$$a_1(x) = x$$

$$v(x) = A \int \frac{1}{x^2} e^{-\int x \, dx}$$

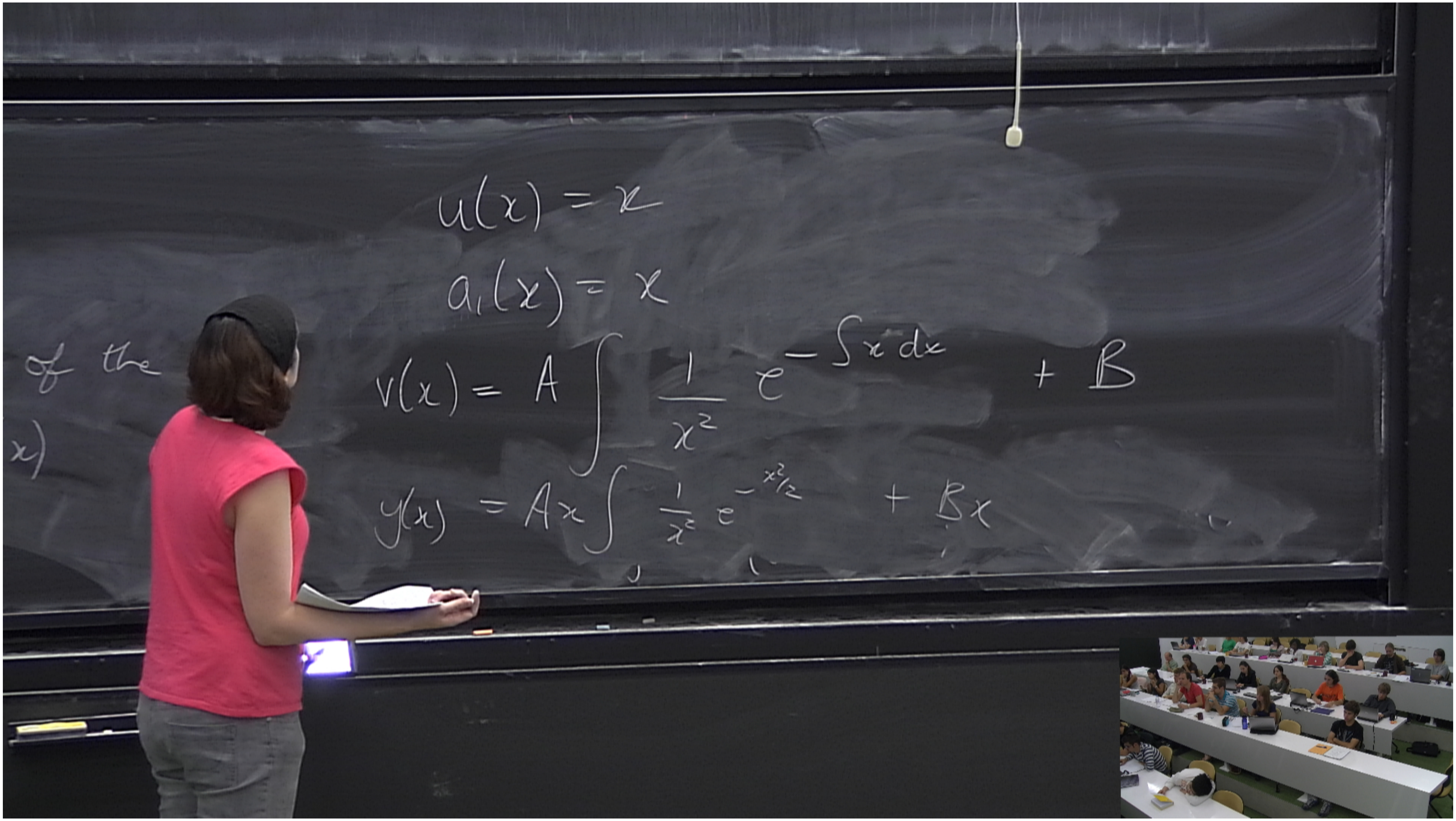

u(x) = x
a1(x) = x
the
v(x) = A ∫ 1/x² e^(−∫x dx)

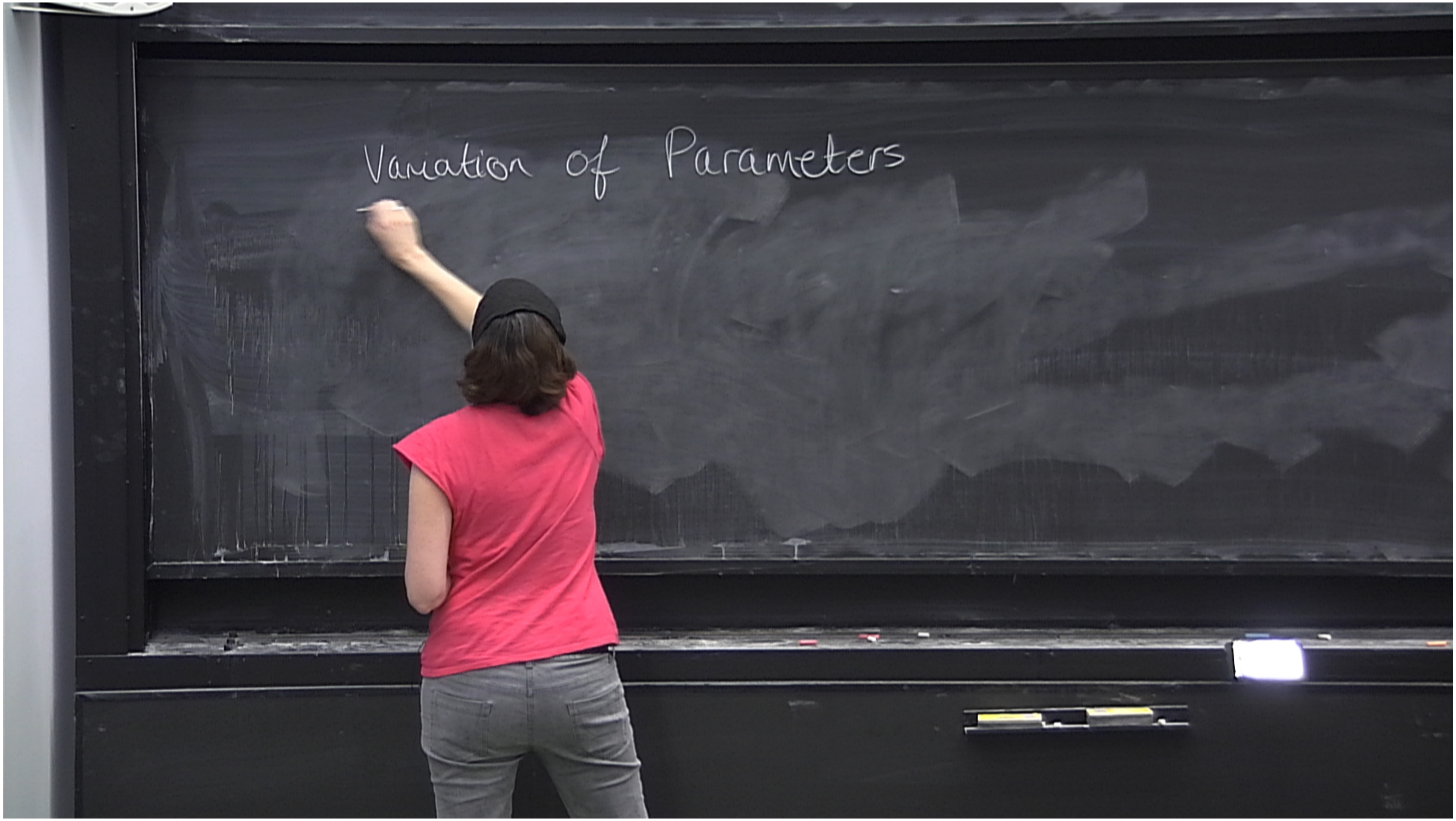

u(x) = x
a₁(x) = x
the
A ∫ 1/x² e^{−∫x dx} + B

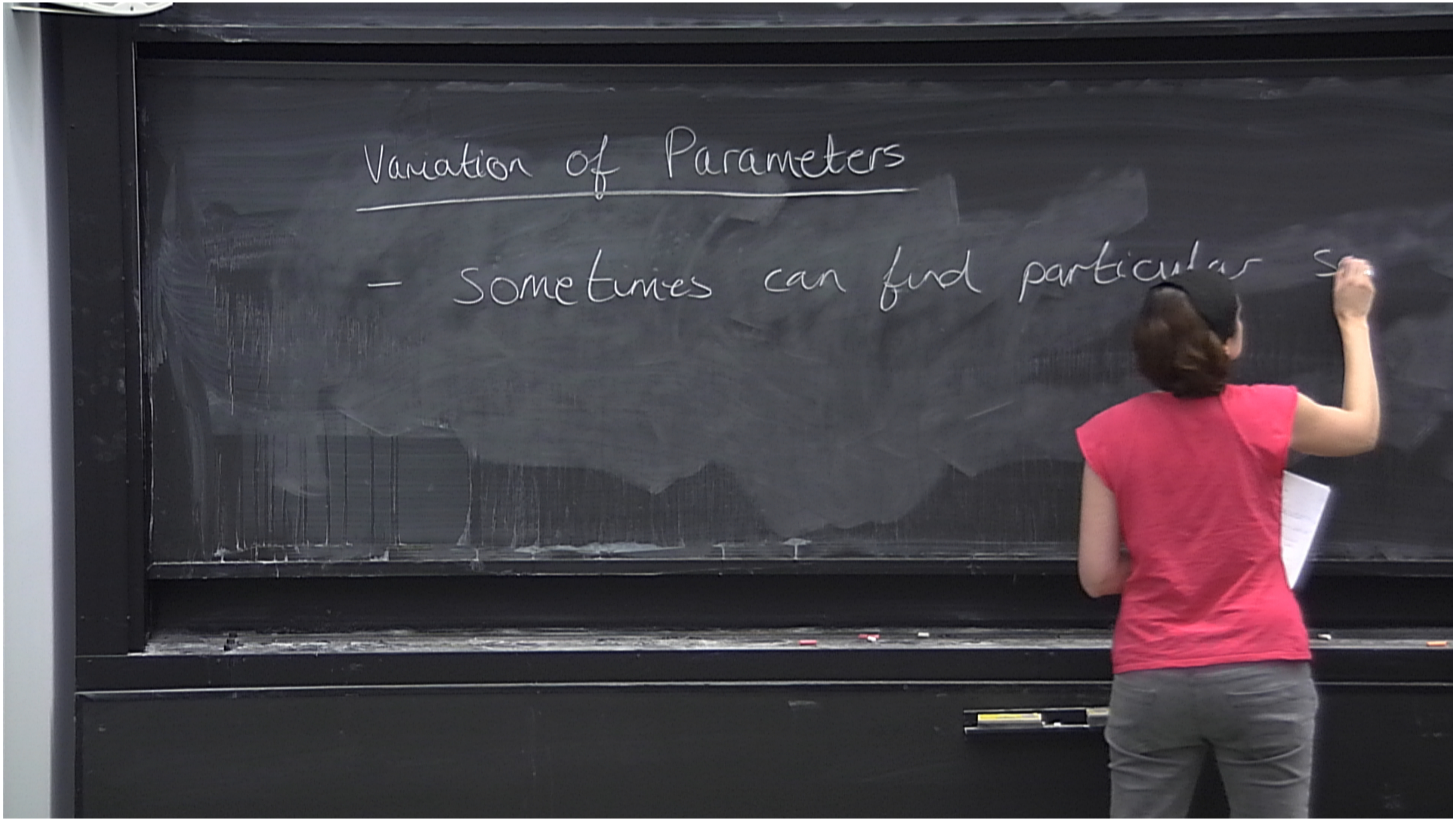

$u(x) = x$
$a_1(x) = x$
$v(x) = A \int \frac{1}{x^2} e^{-\int x \, dx} + B$
$= A \int \frac{1}{x^2} e^{-x^2/2} + B$

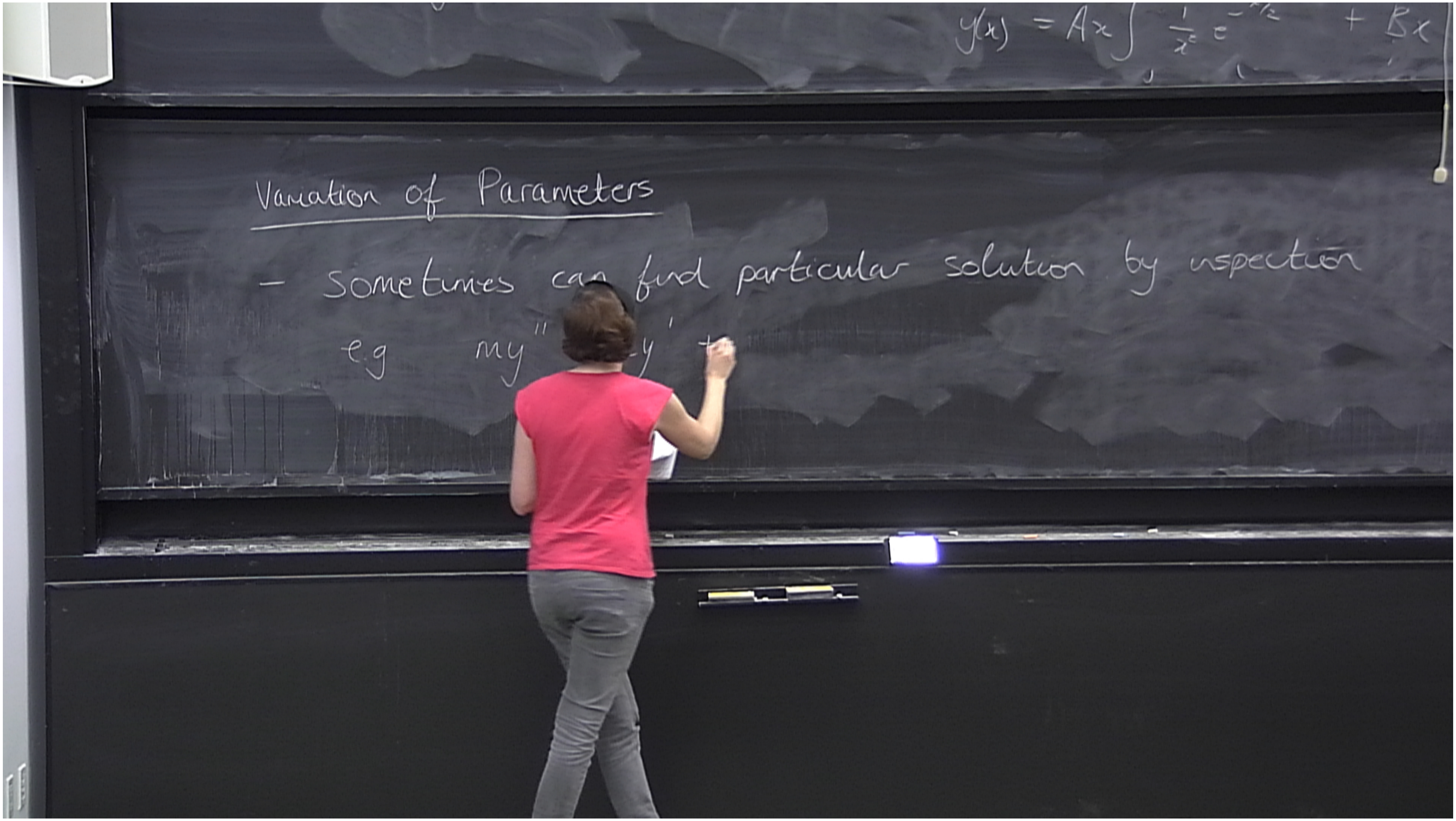


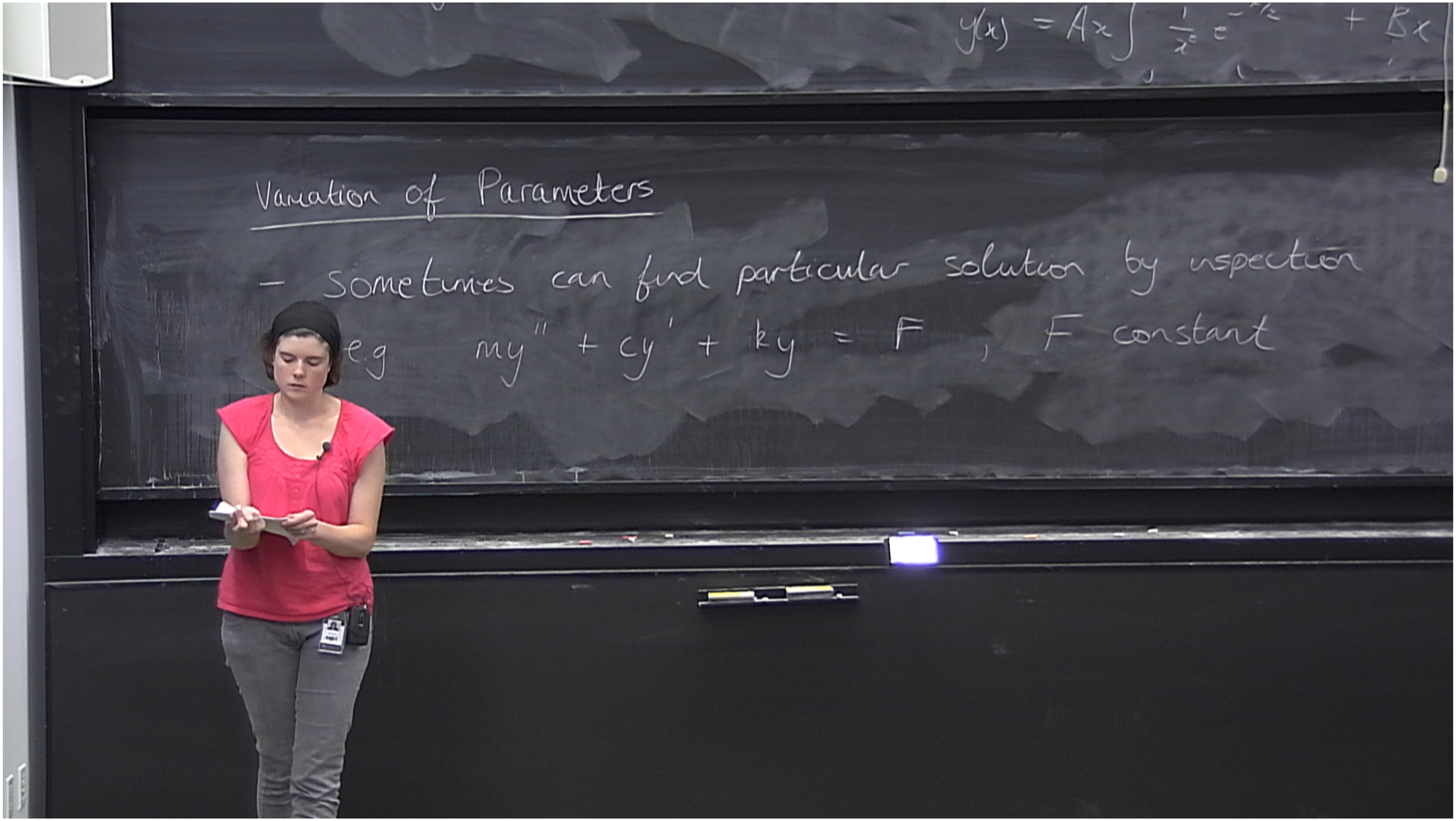
$u(x) = x$
$a_1(x) = x$
the
$v(x) = A \int \frac{1}{x^2} e^{-\int x \, dx} + B$
$y(x) = Ax \int \frac{1}{x^2} e^{-x^2/2} + Bx$

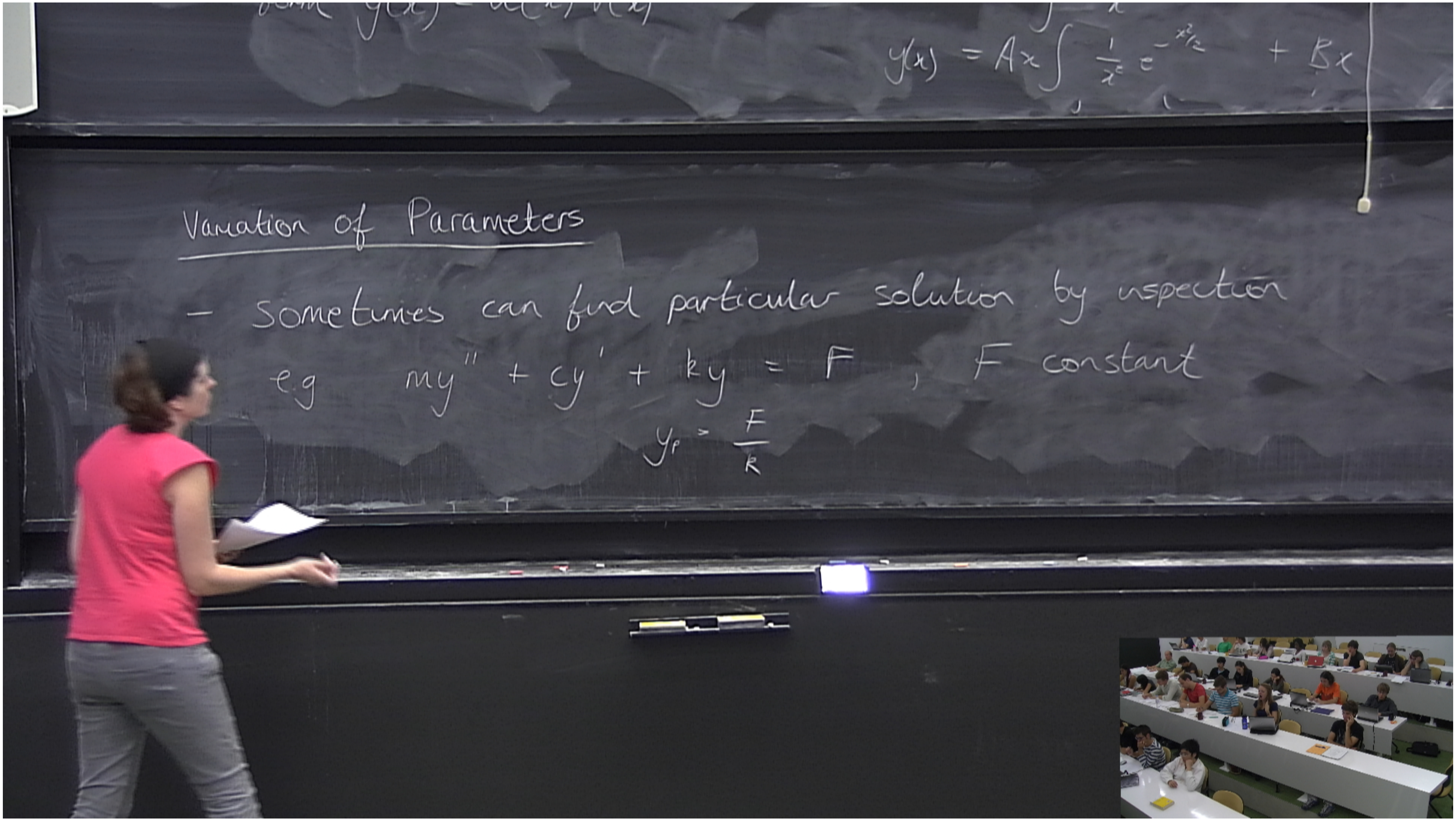

$u(x) = x$
$a_1(x) = x$
of the
$v(x) = A \int \frac{1}{x^2} e^{-\int x\, dx} + B$
$y(x) = Ax \int \frac{1}{x^2} e^{-x^2/2} + Bx$

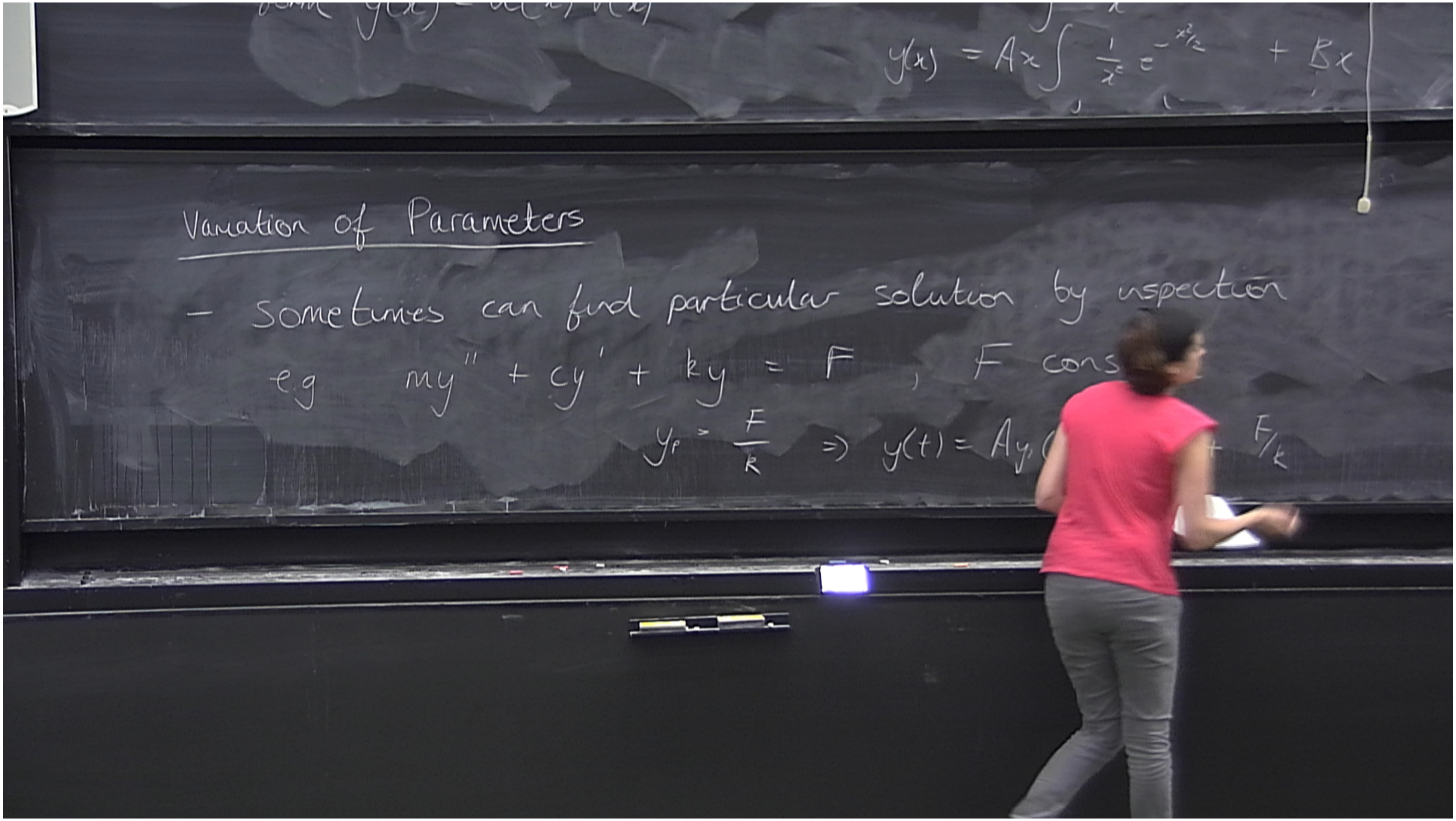

Variation of Parameters

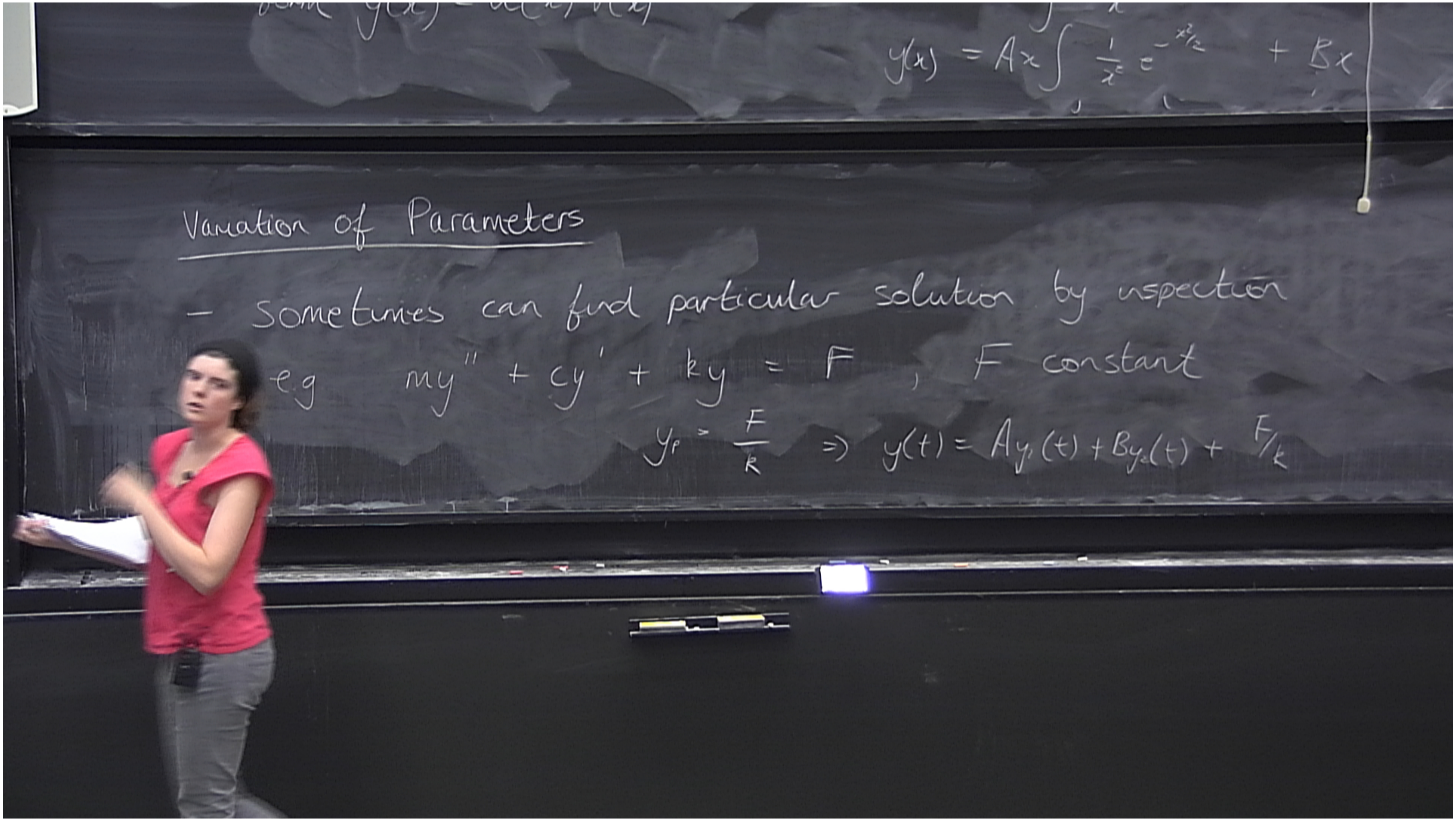

Variation of Parameters
- sometimes can find particular

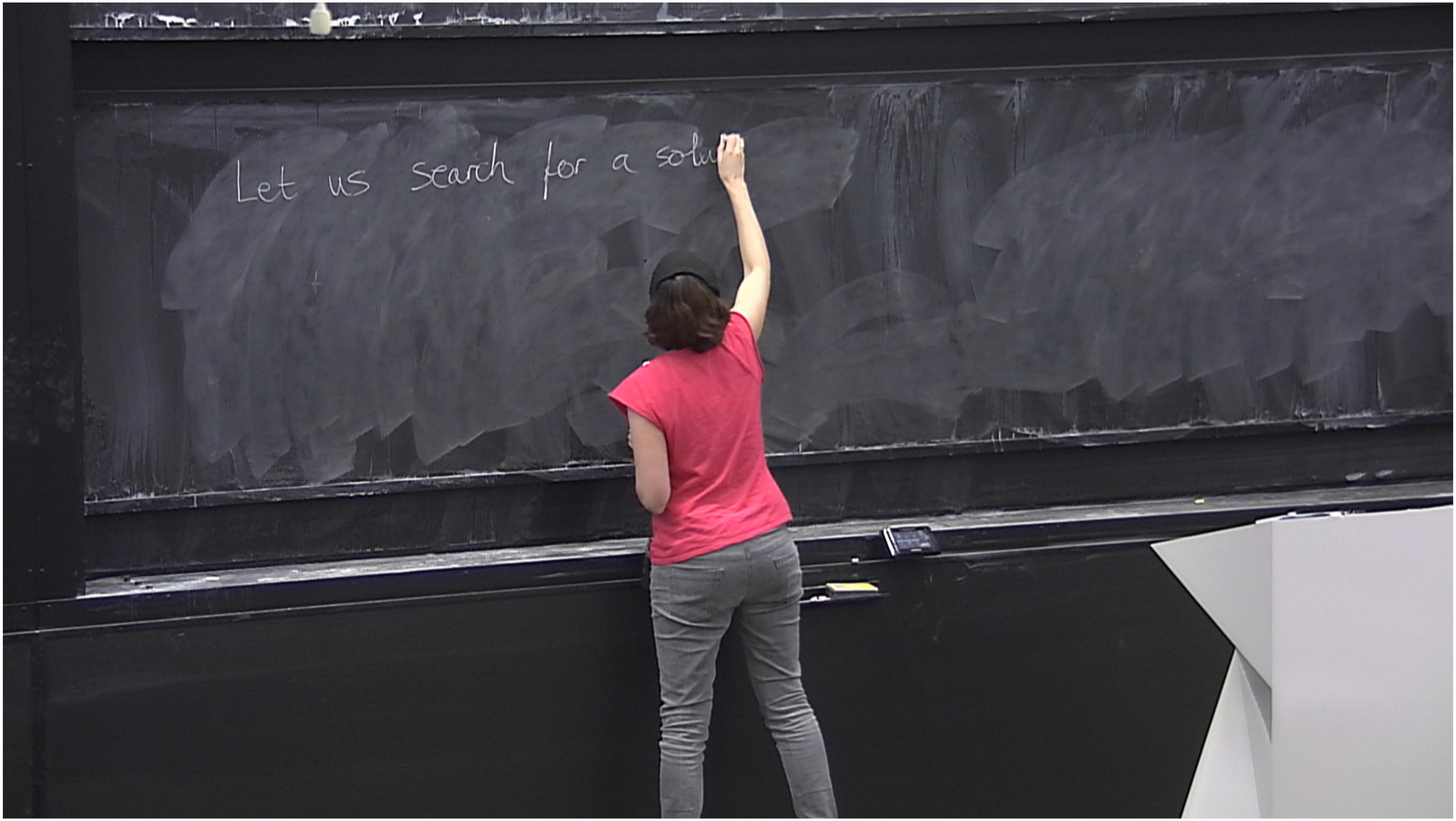

y(x) = Ax ∫ 1/x² e + Bx
Variation of Parameters
– Sometimes can find particular solution by inspection
e.g my''

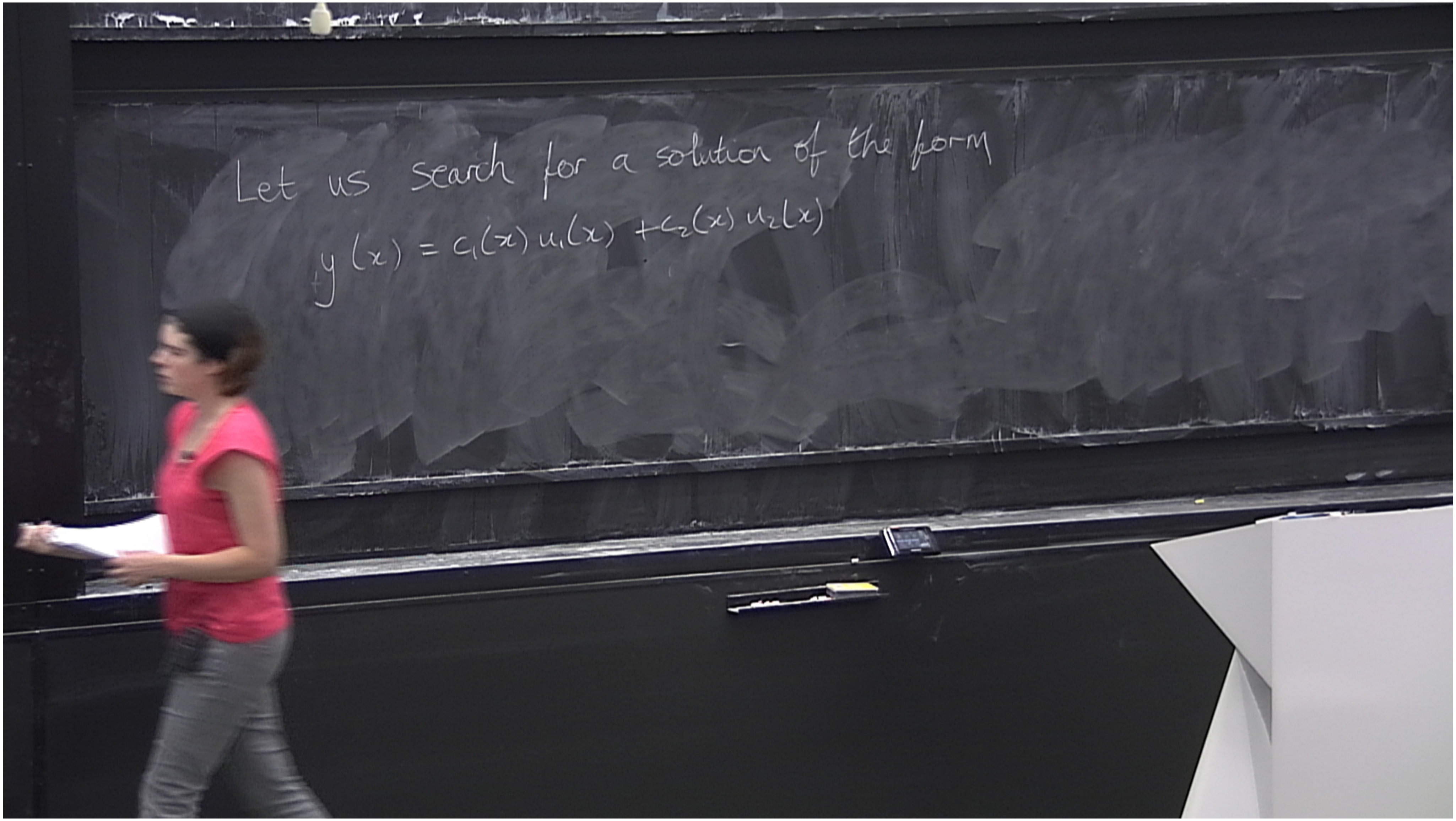

Variation of Parameters
– Sometimes can find particular solution by inspection
e.g my'' + cy' + ky = F , F constant

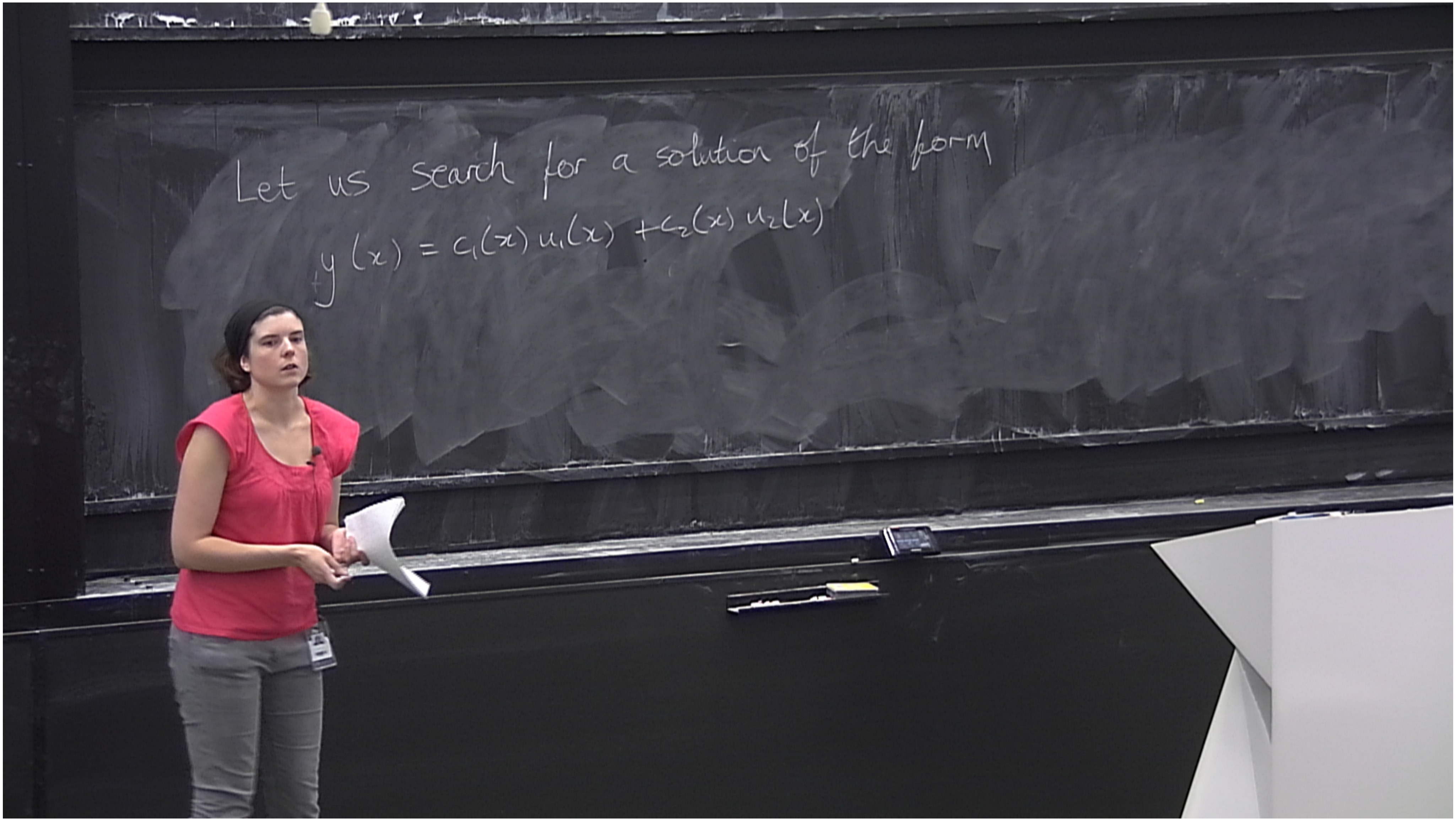

y(x) = Ax ∫ 1/x² e^{-x²/2} + Bx
Variation of Parameters
- Sometimes can find particular solution by inspection
e.g my'' + cy' + ky = F , F constant
y_p = F/k


$y(x) = Ax\int \frac{1}{x^2} e^{-x^2/2} + Bx$
Variation of Parameters
- Sometimes can find particular solution by inspection
e.g $my'' + cy' + ky = F$, F cons
$y_p = \frac{F}{k} \Rightarrow y(t) = Ay_1($
F/k


$y(x) = Ax \int \frac{1}{x^2} e^{-x^2/2} + Bx$
Variation of Parameters
- Sometimes can find particular solution by inspection
e.g $my'' + cy' + ky = F$, F constant
$y_p = \frac{F}{k} \Rightarrow y(t) = Ay_1(t) + By_2(t) + F/k$


Let us search for a solu

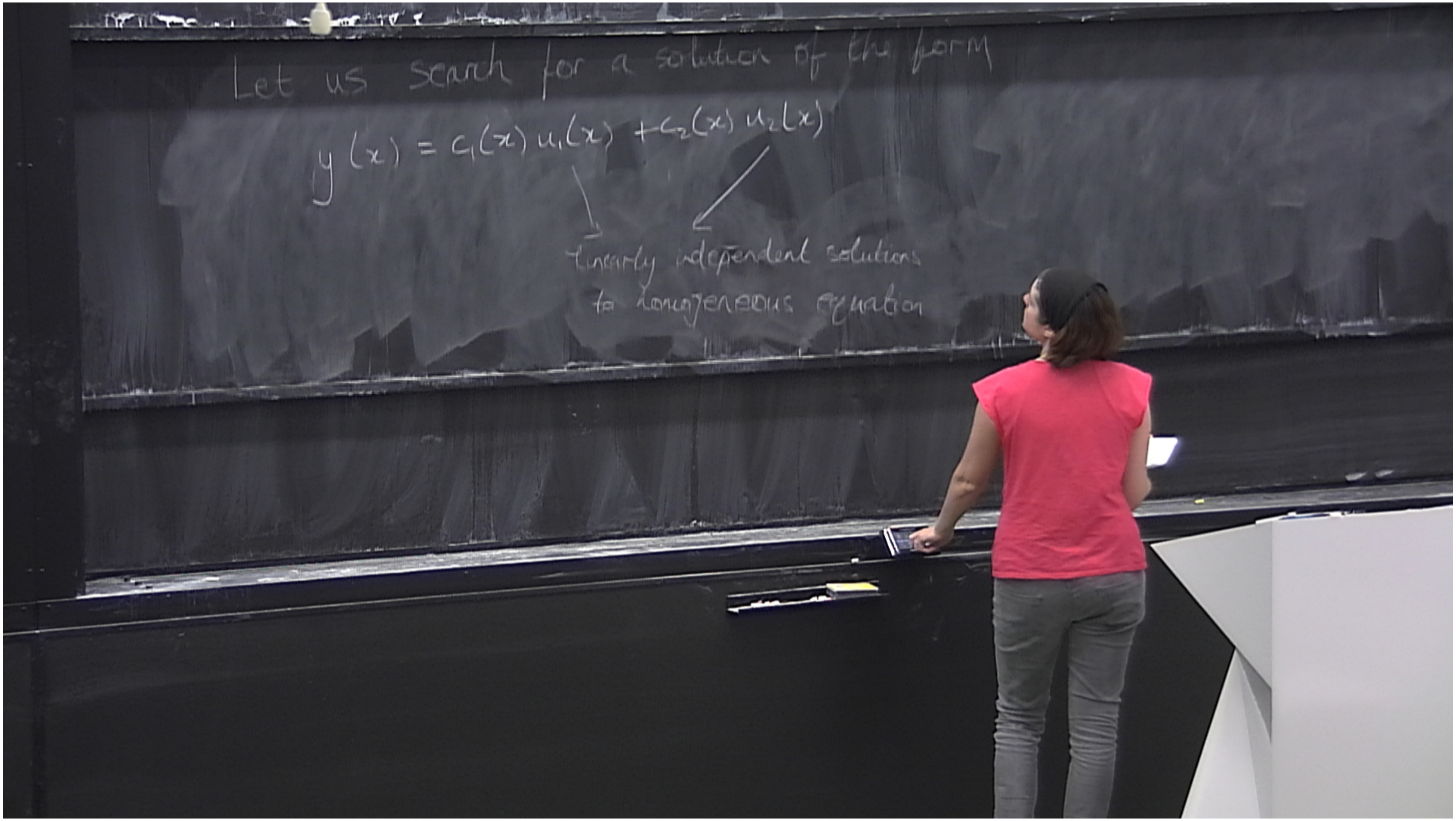

Let us search for a solution of the form
$y(x) = c_1(x)\, u_1(x) + c_2(x)\, u_2(x)$

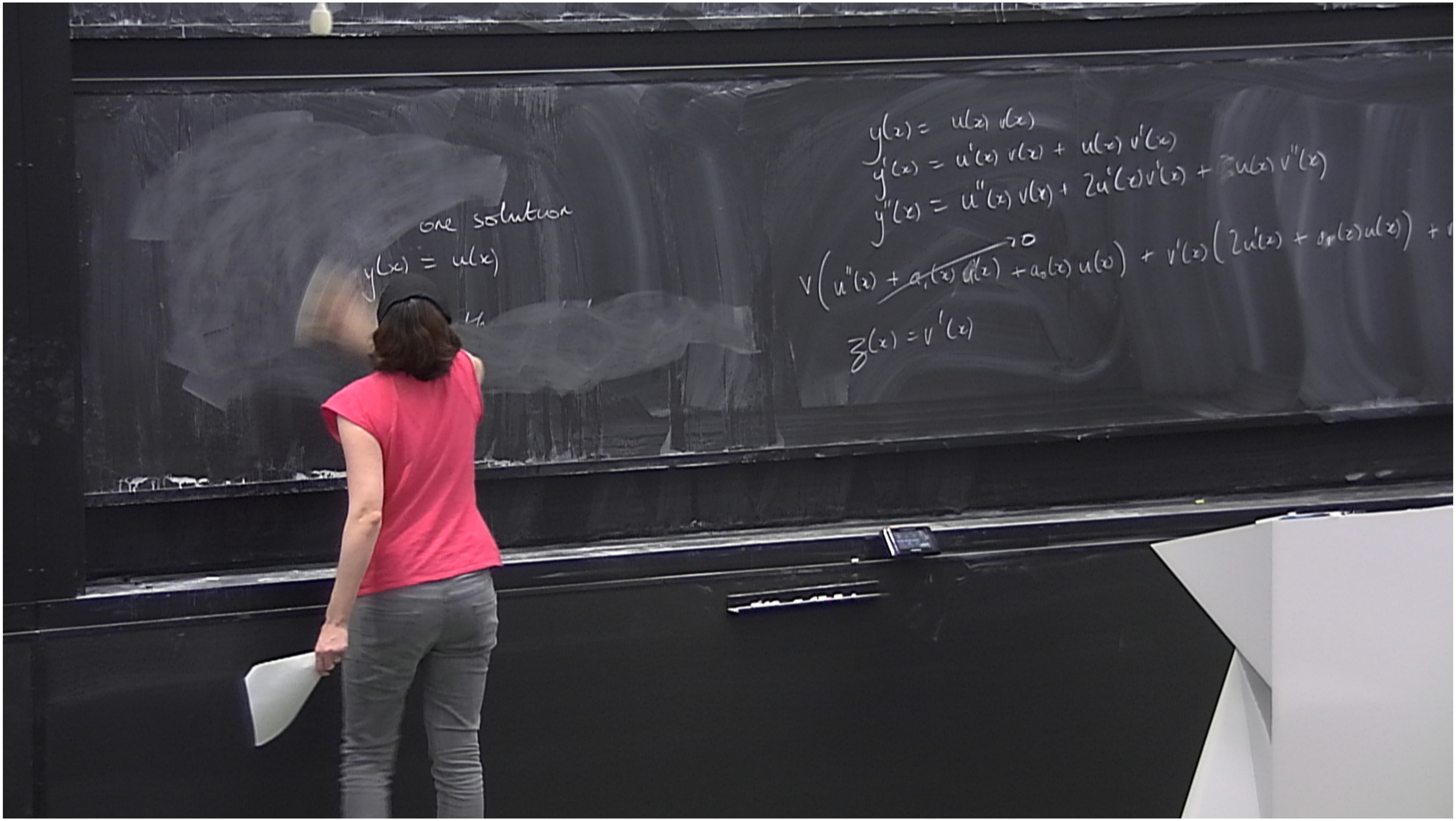

Let us search for a solution of the form
$y(x) = c_1(x)\, u_1(x) + c_2(x)\, u_2(x)$

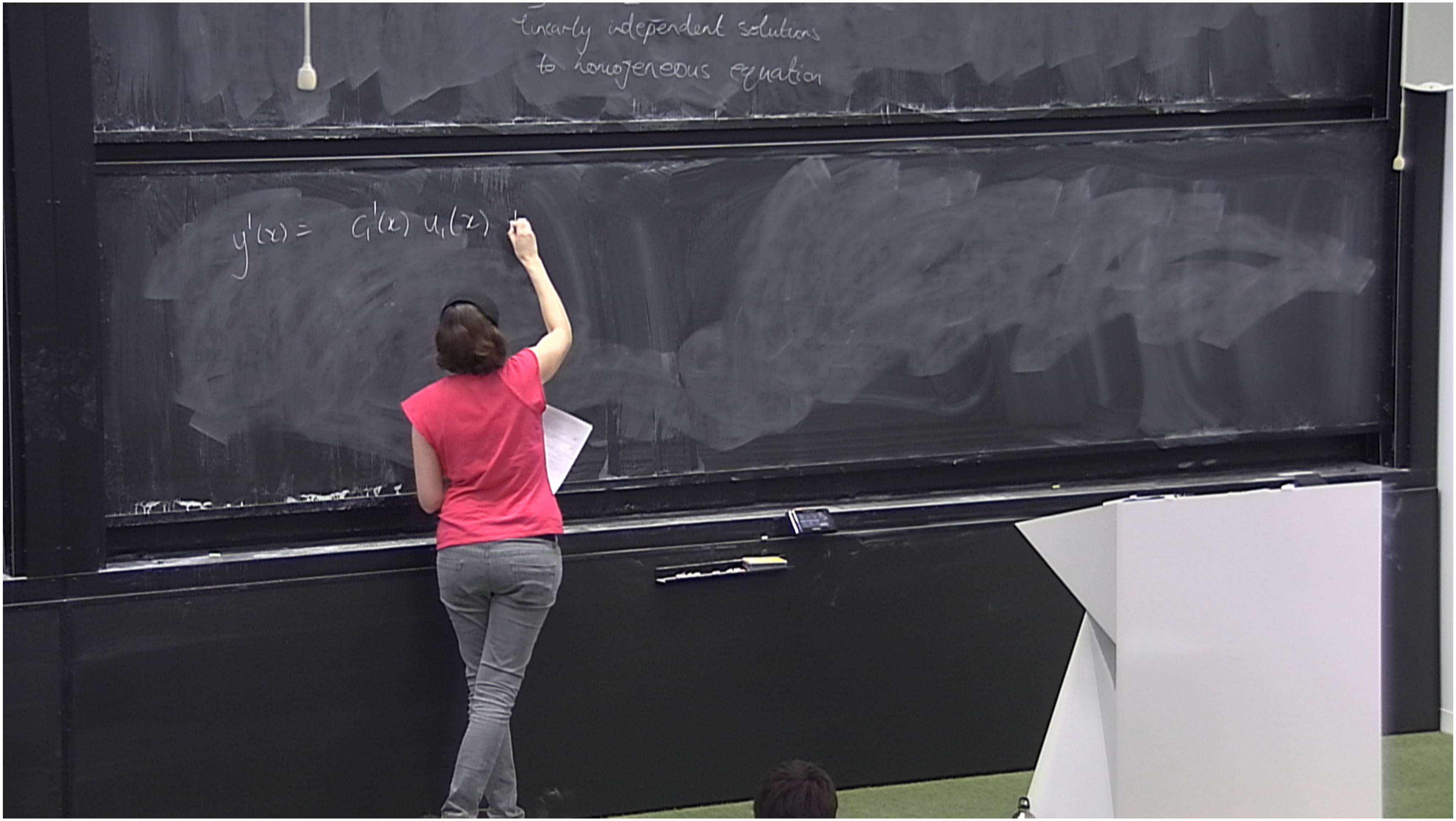
Let us search for a solution of the form

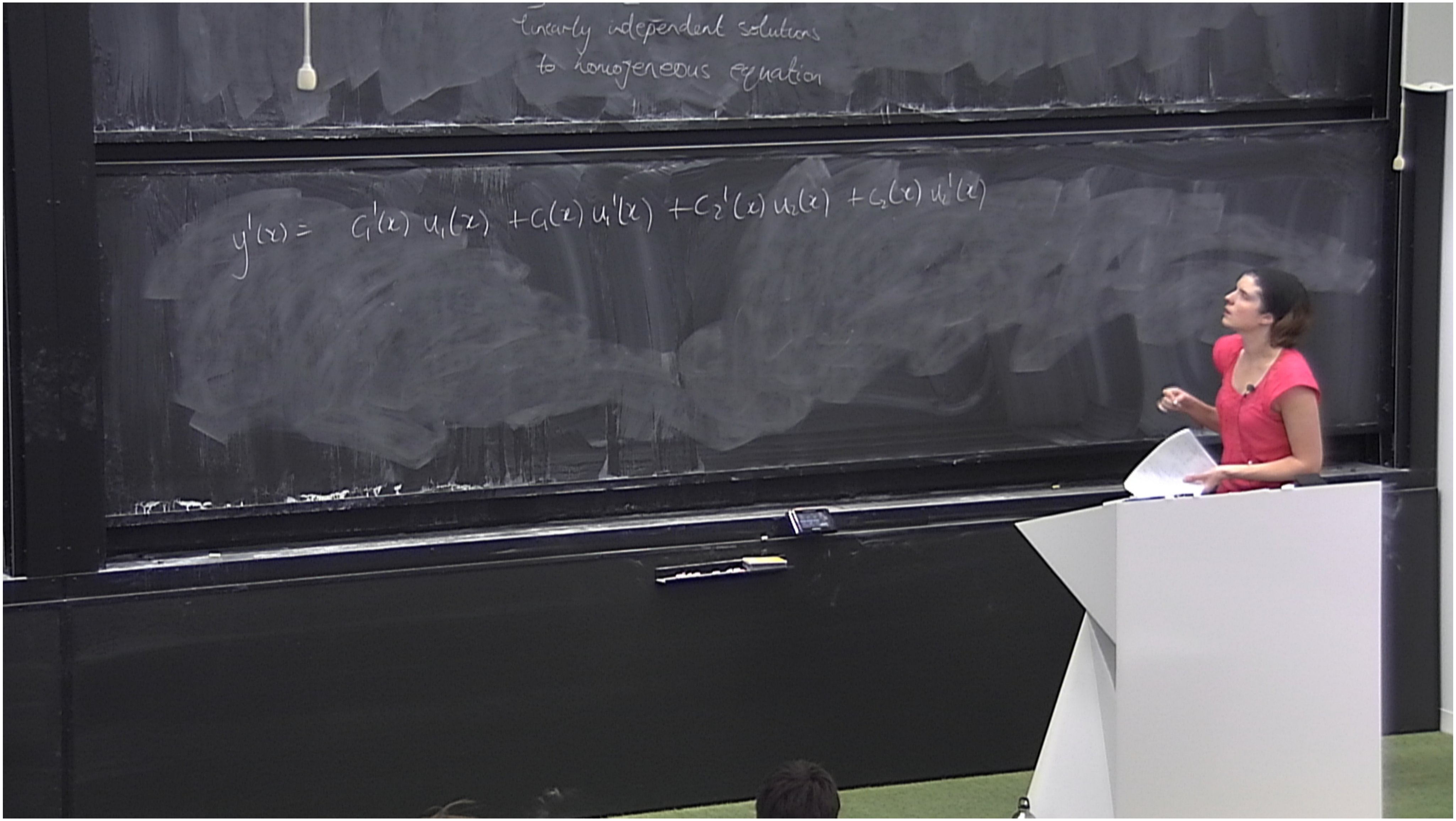
$$y(x) = c_1(x)\, u_1(x) + c_2(x)\, u_2(x)$$

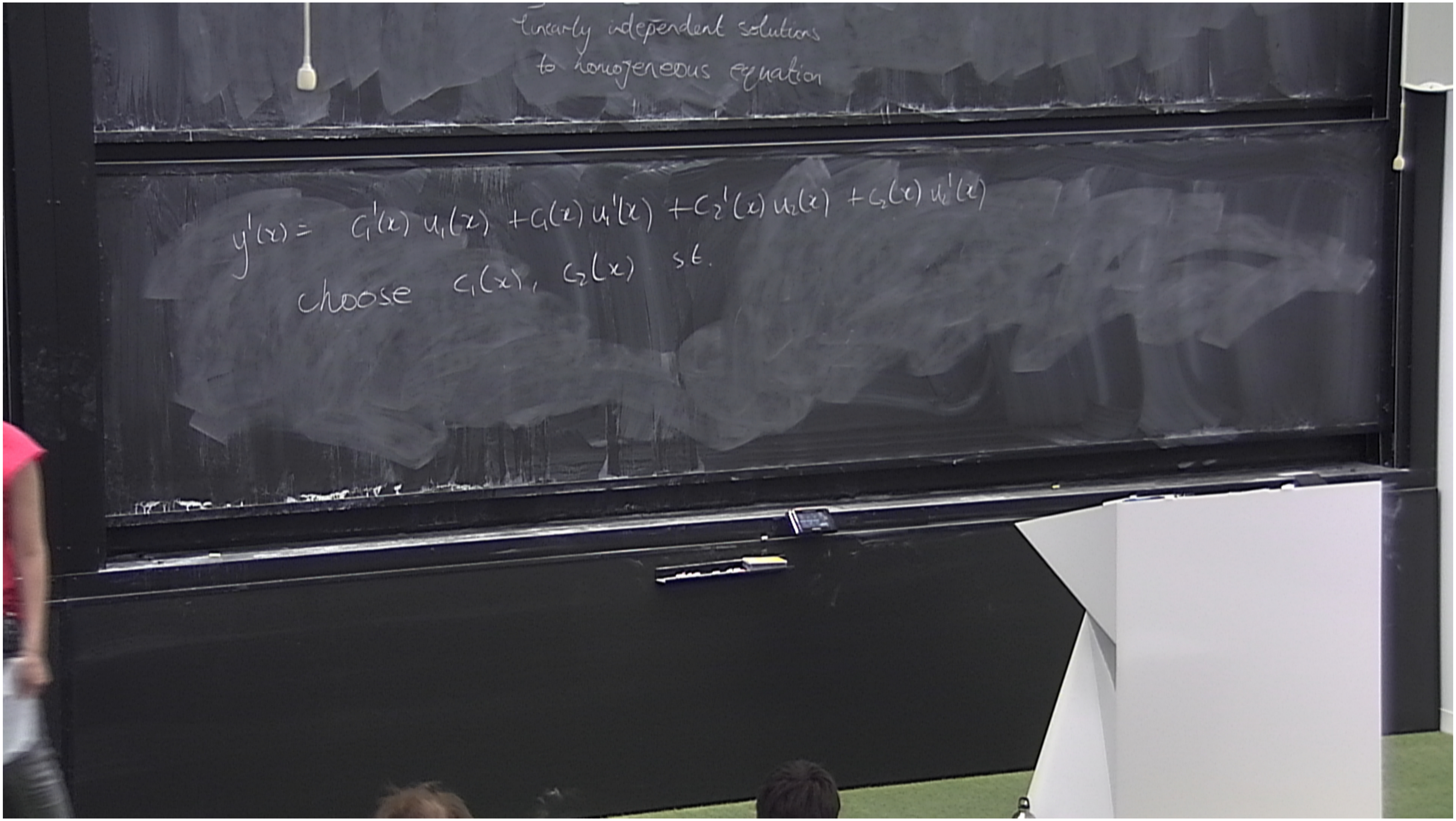
linearly independent solutions to homogeneous equation

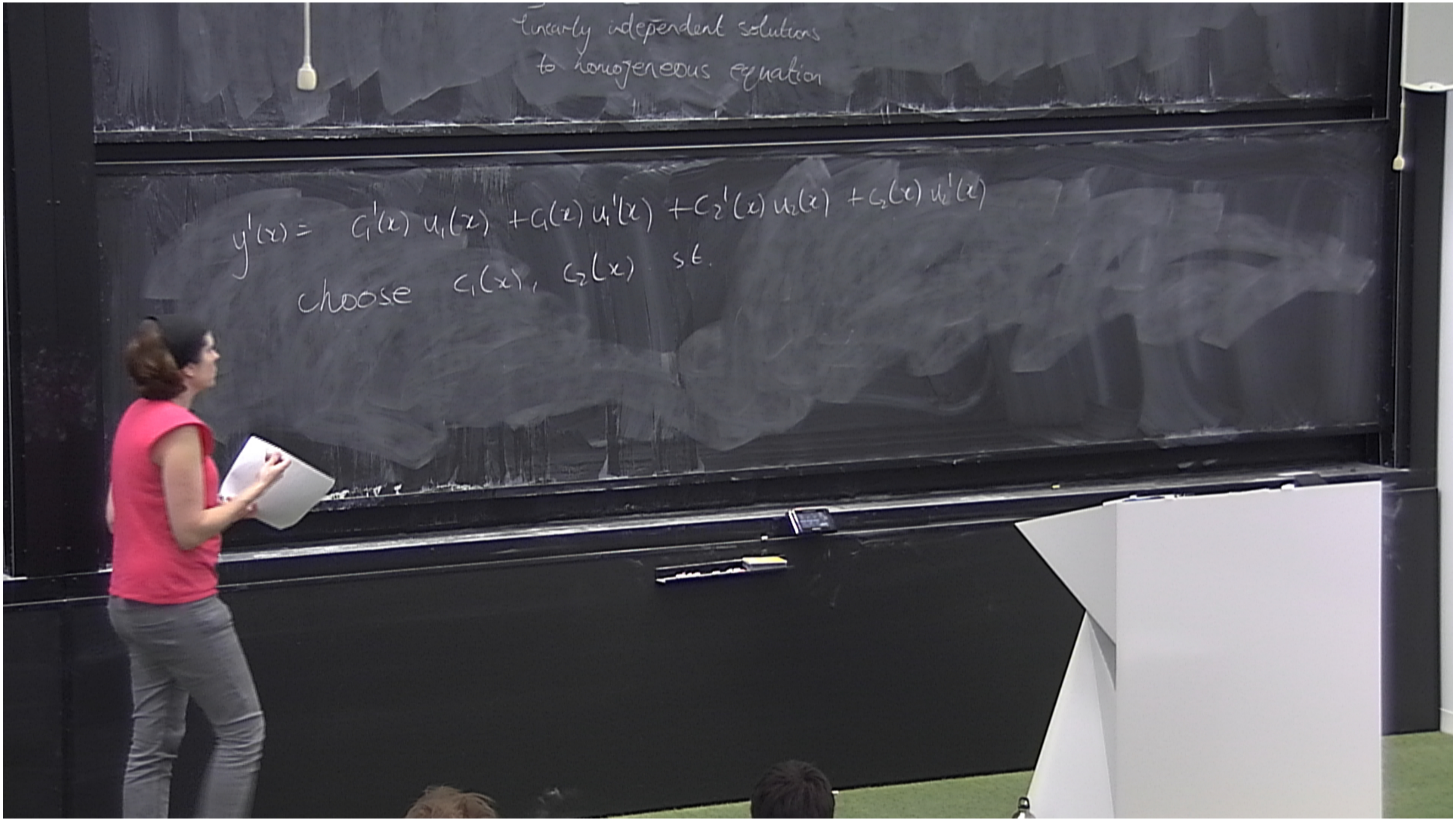

Let us search for a solution of the form
y(x) = c1(x) u1(x) + c2(x) u2(x)
linearly independent solutions
to homogeneous equation

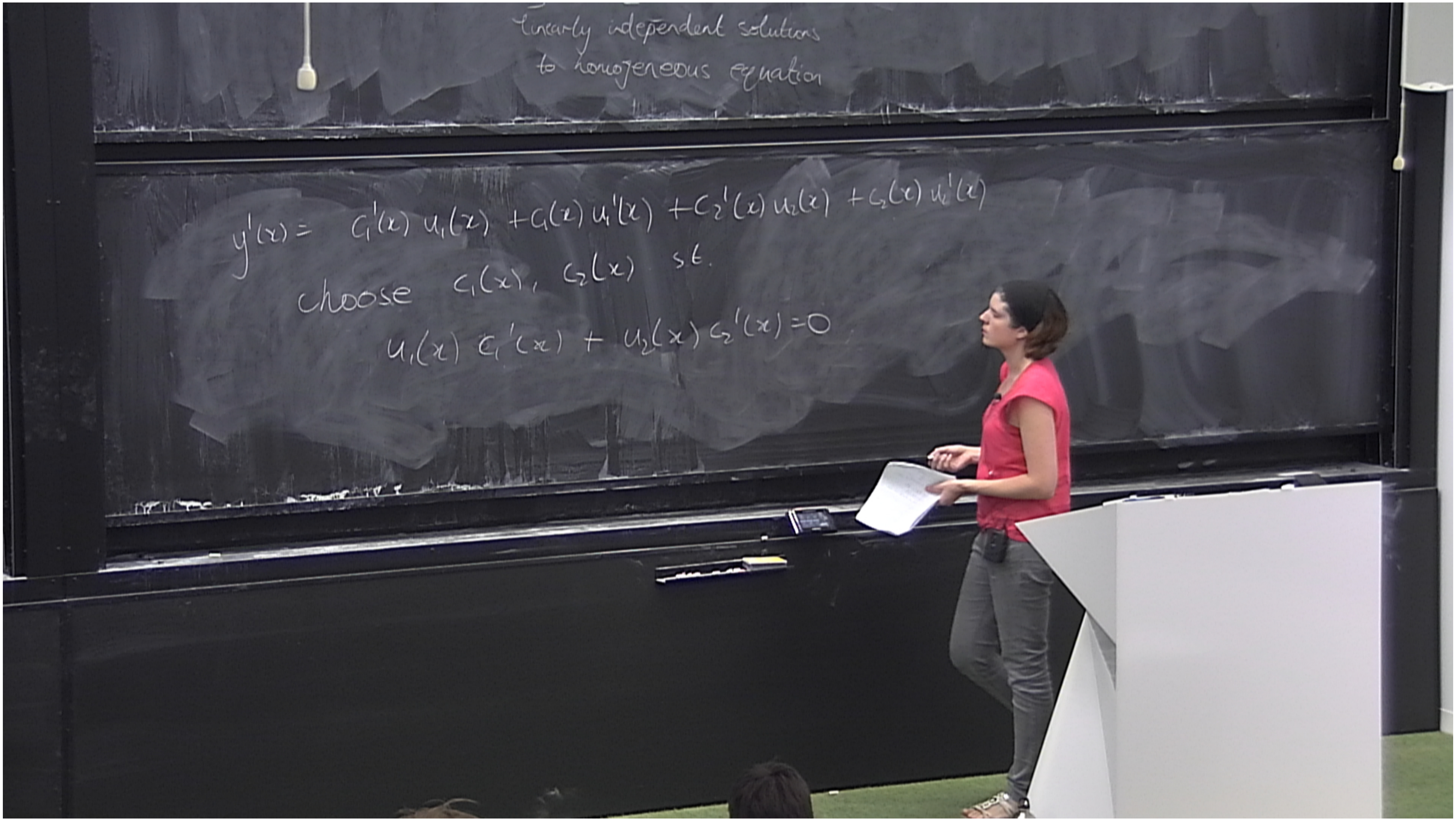

one solution
y(x) = u(x)
y(x) = u(x) v(x)
y'(x) = u'(x) v(x) + u(x) v'(x)
y''(x) = u''(x) v(x) + 2u'(x) v'(x) + u(x) v''(x)
z(x) = v'(x)

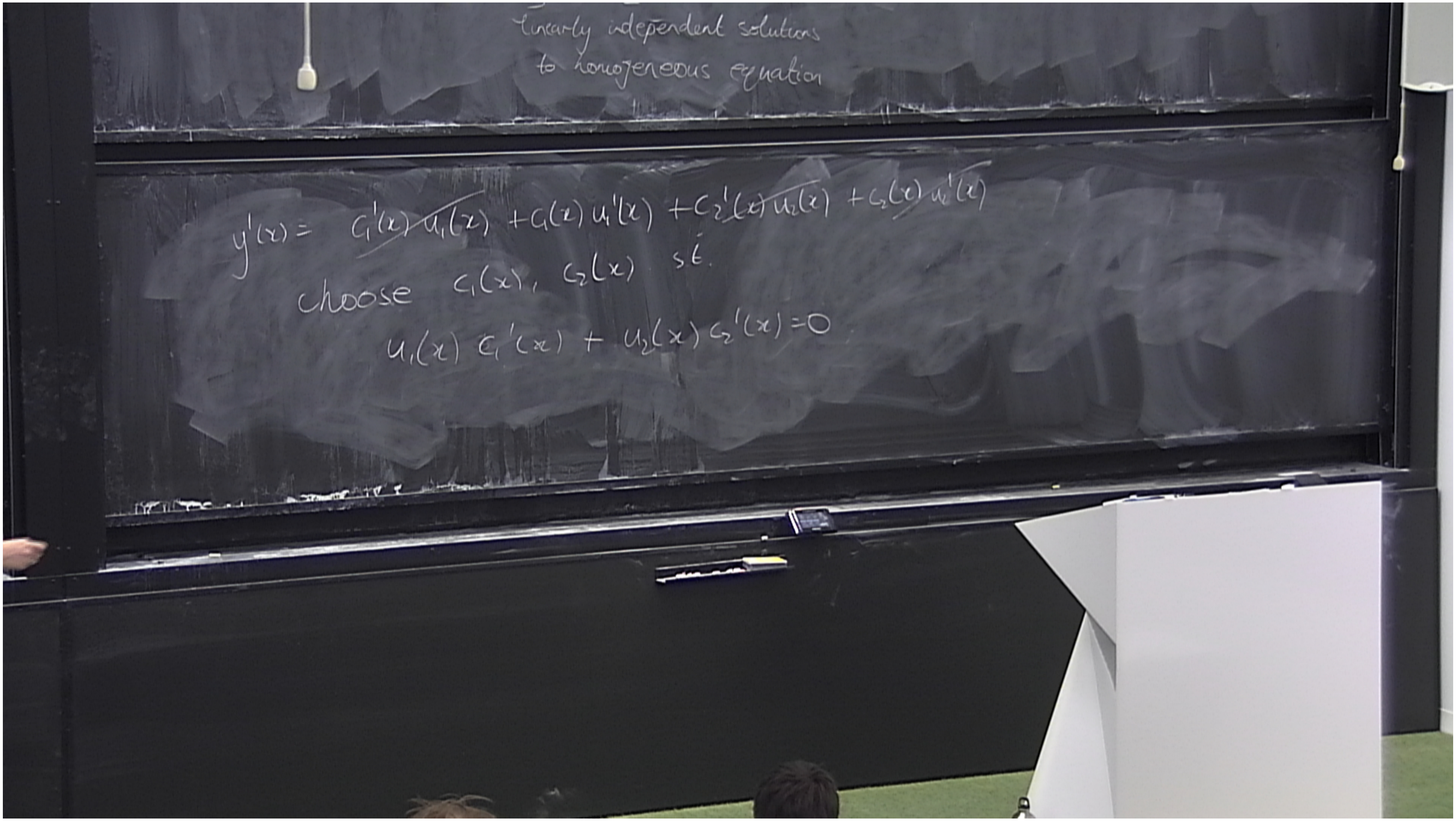

linearly independent solutions
to homogeneous equation
y'(x) = c1'(x) u1(x) +

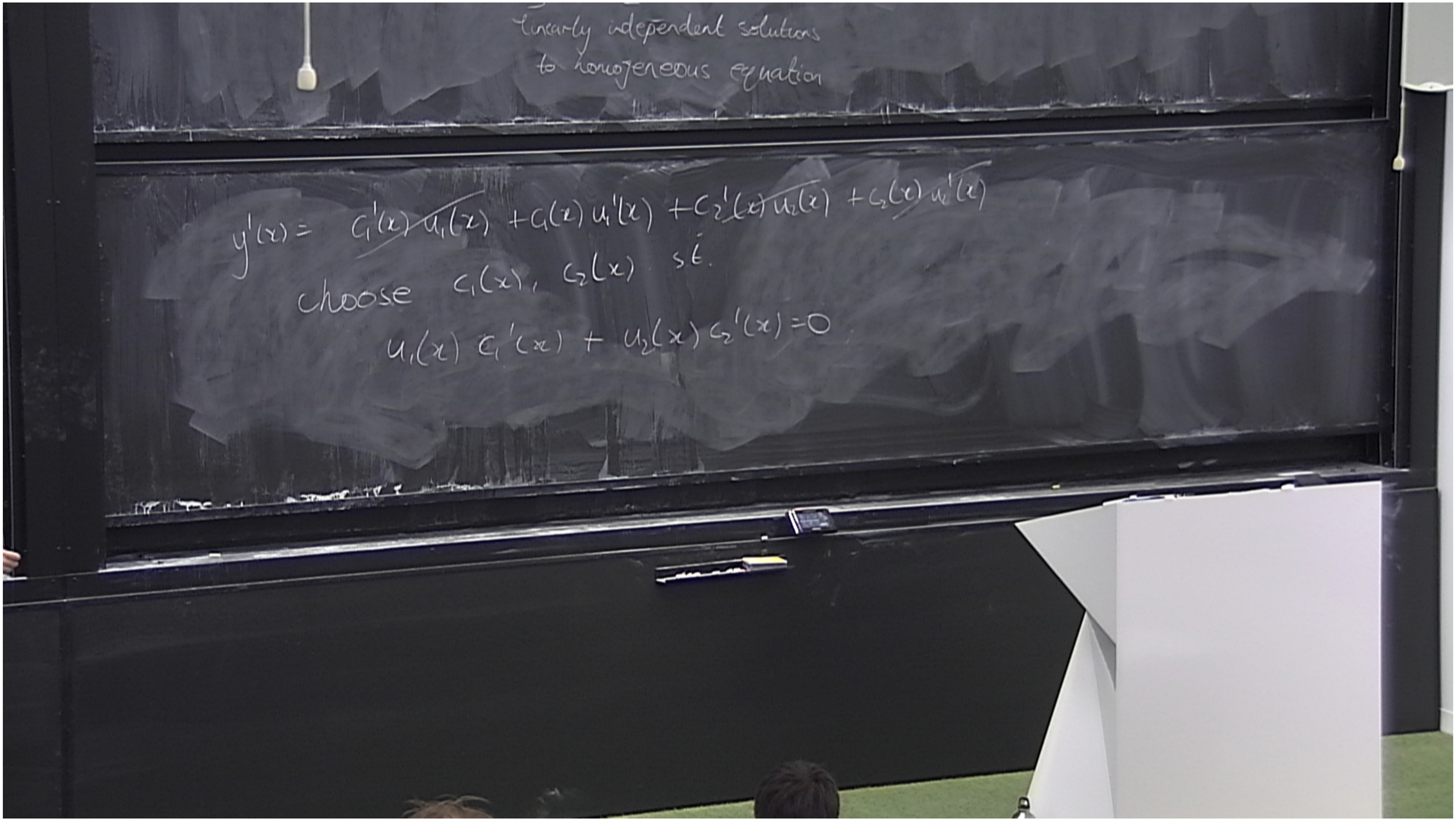

linearly independent solutions
to homogeneous equation
$y'(x) = c_1'(x)\, u_1(x) + c_1(x)\, u_1'(x) + c_2'(x)\, u_2(x) + c_2(x)\, u_2'(x)$

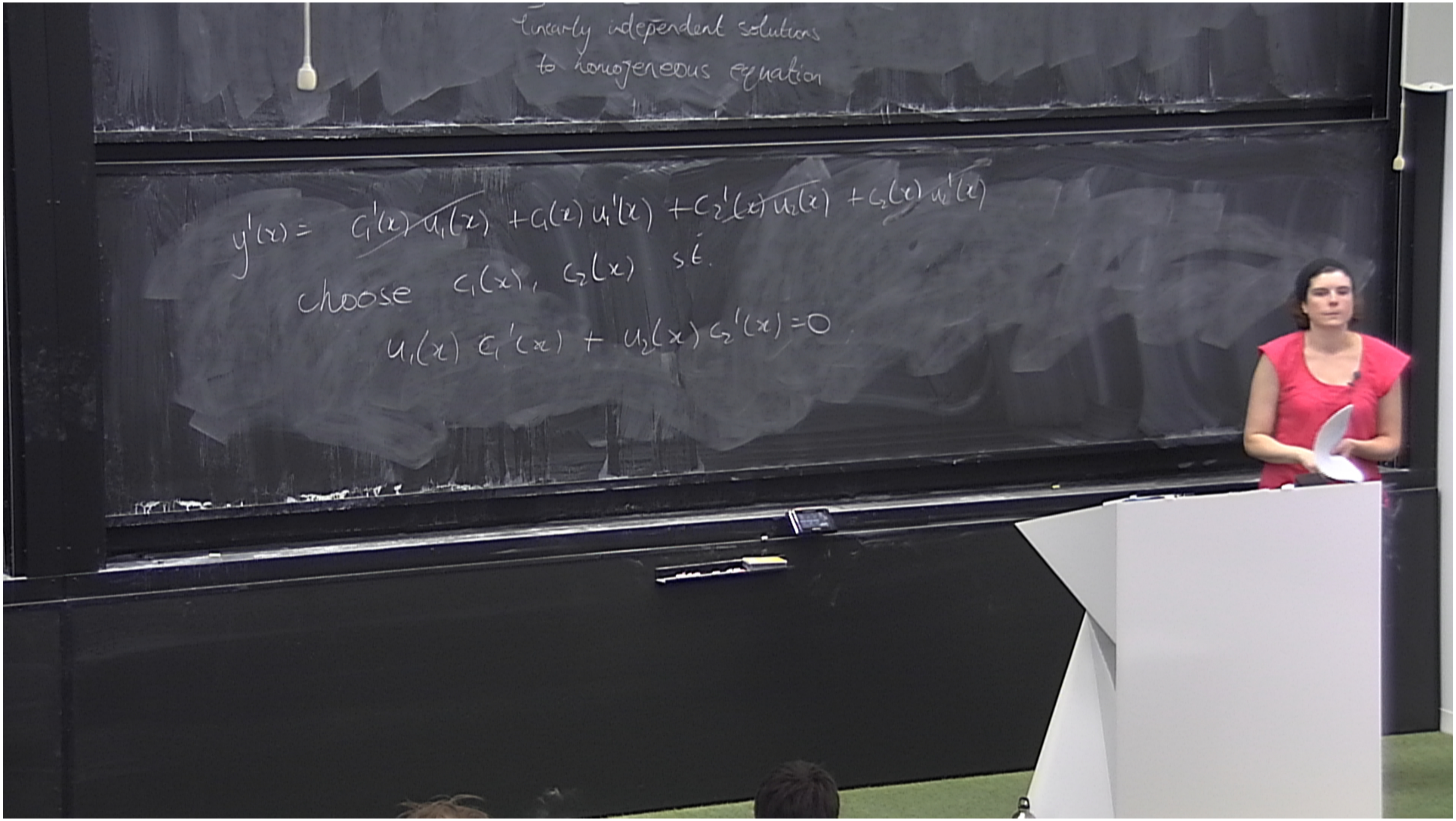

linearly independent solutions
to homogeneous equation
$y'(x) = c_1'(x)\, u_1(x) + c_1(x)\, u_1'(x) + c_2'(x)\, u_2(x) + c_2(x)\, u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.

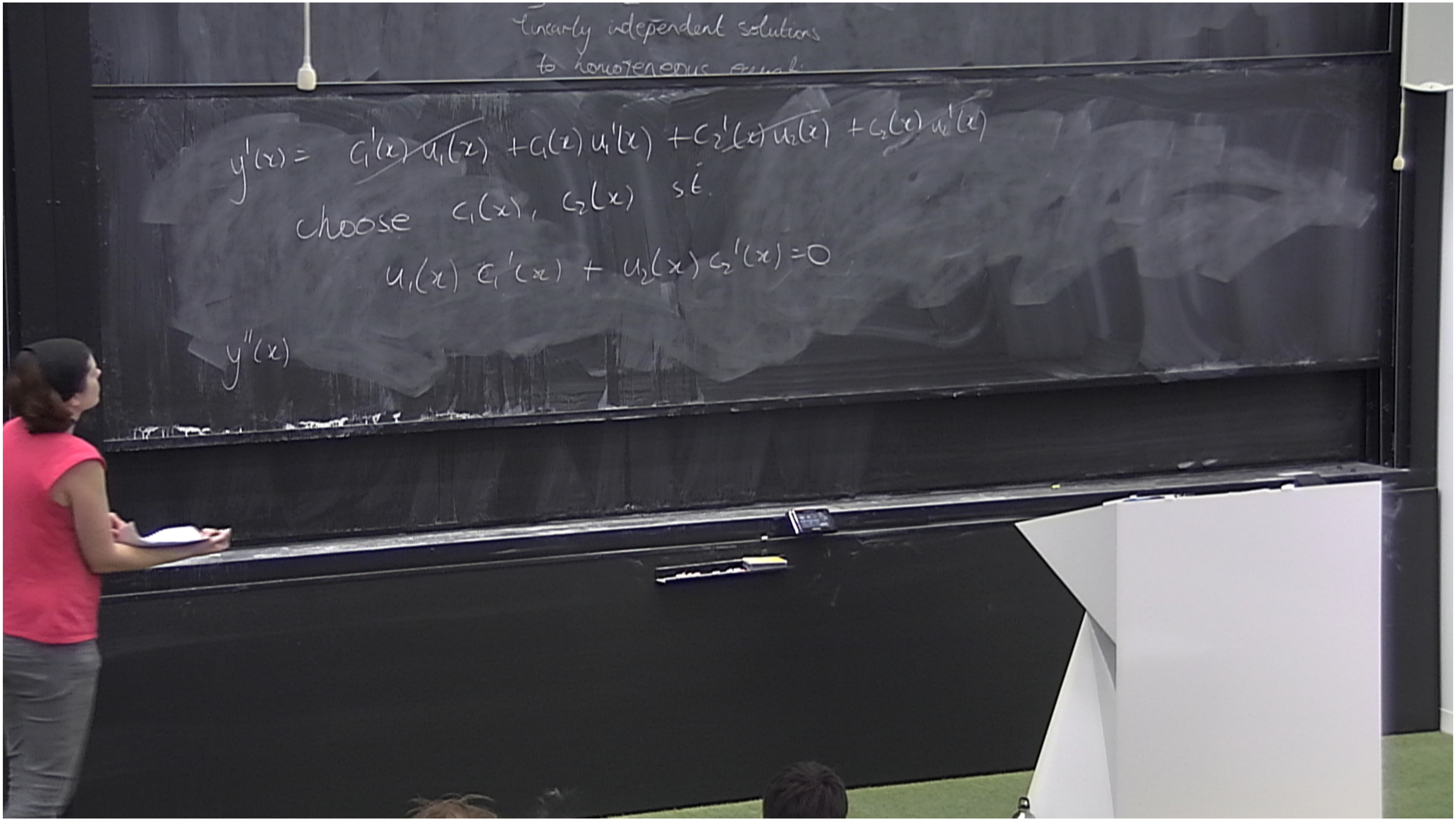

linearly independent solutions
to homogeneous equation
$y'(x) = c_1'(x)\, u_1(x) + c_1(x)\, u_1'(x) + c_2'(x)\, u_2(x) + c_2(x)\, u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.

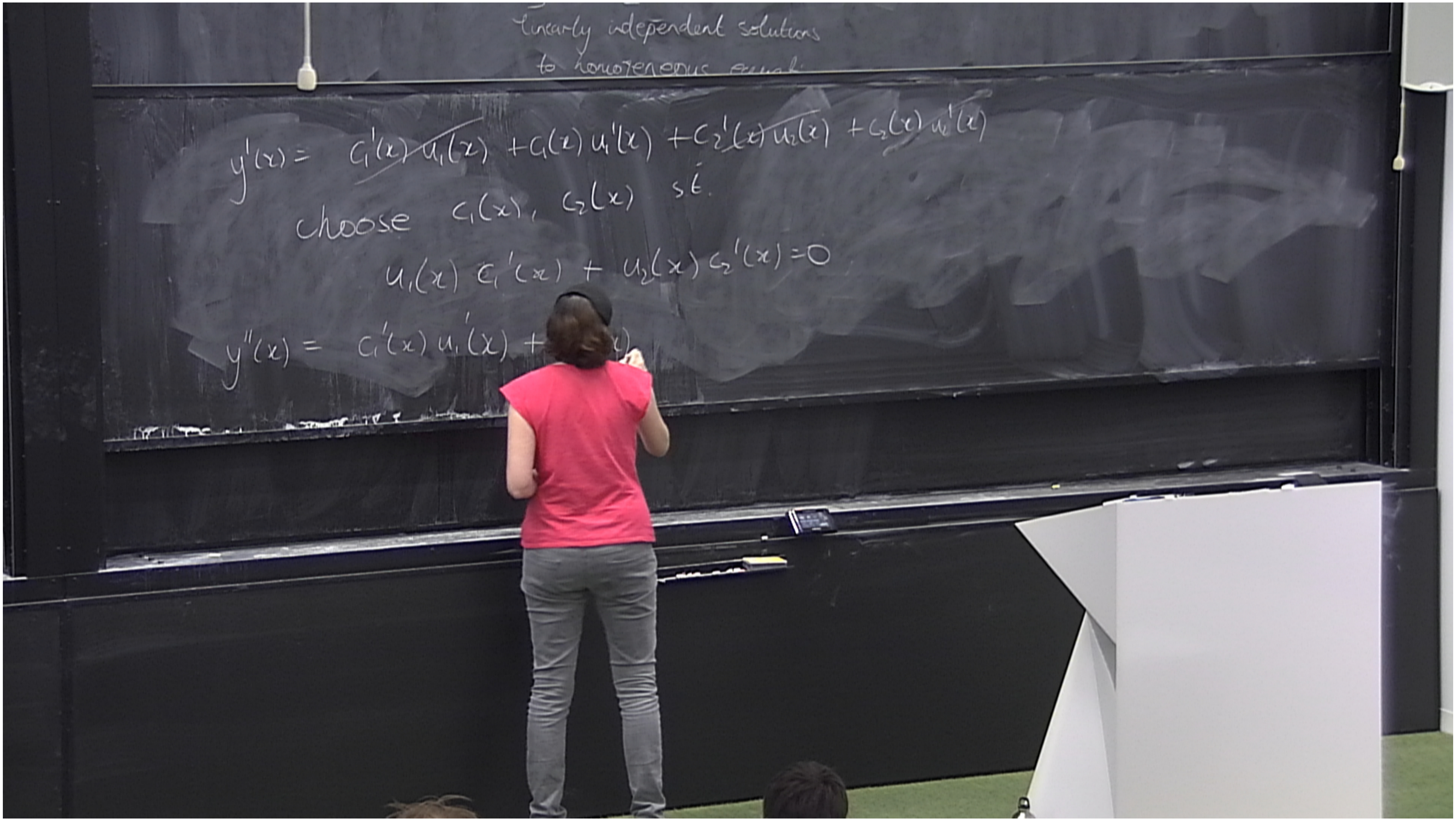

linearly independent solutions
to homogeneous equation
$y'(x) = c_1'(x)\, u_1(x) + c_1(x)\, u_1'(x) + c_2'(x)\, u_2(x) + c_2(x)\, u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.
$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$

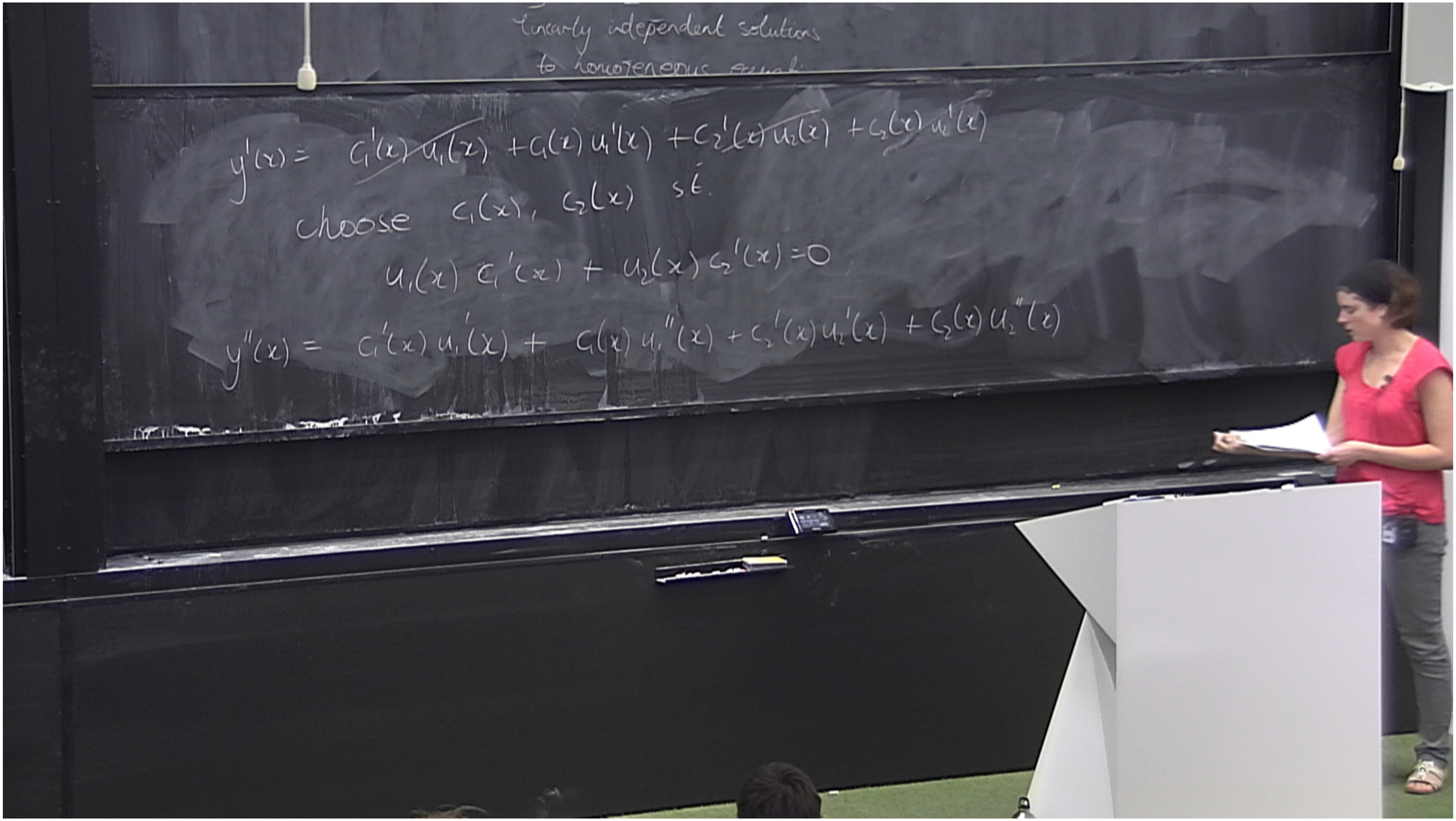

linearly independent solutions
to homogeneous equation
$y'(x) = c_1'(x)\,u_1(x) + c_1(x)\,u_1'(x) + c_2'(x)\,u_2(x) + c_2(x)\,u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.
$u_1(x)\,c_1'(x) + u_2(x)\,c_2'(x) = 0$

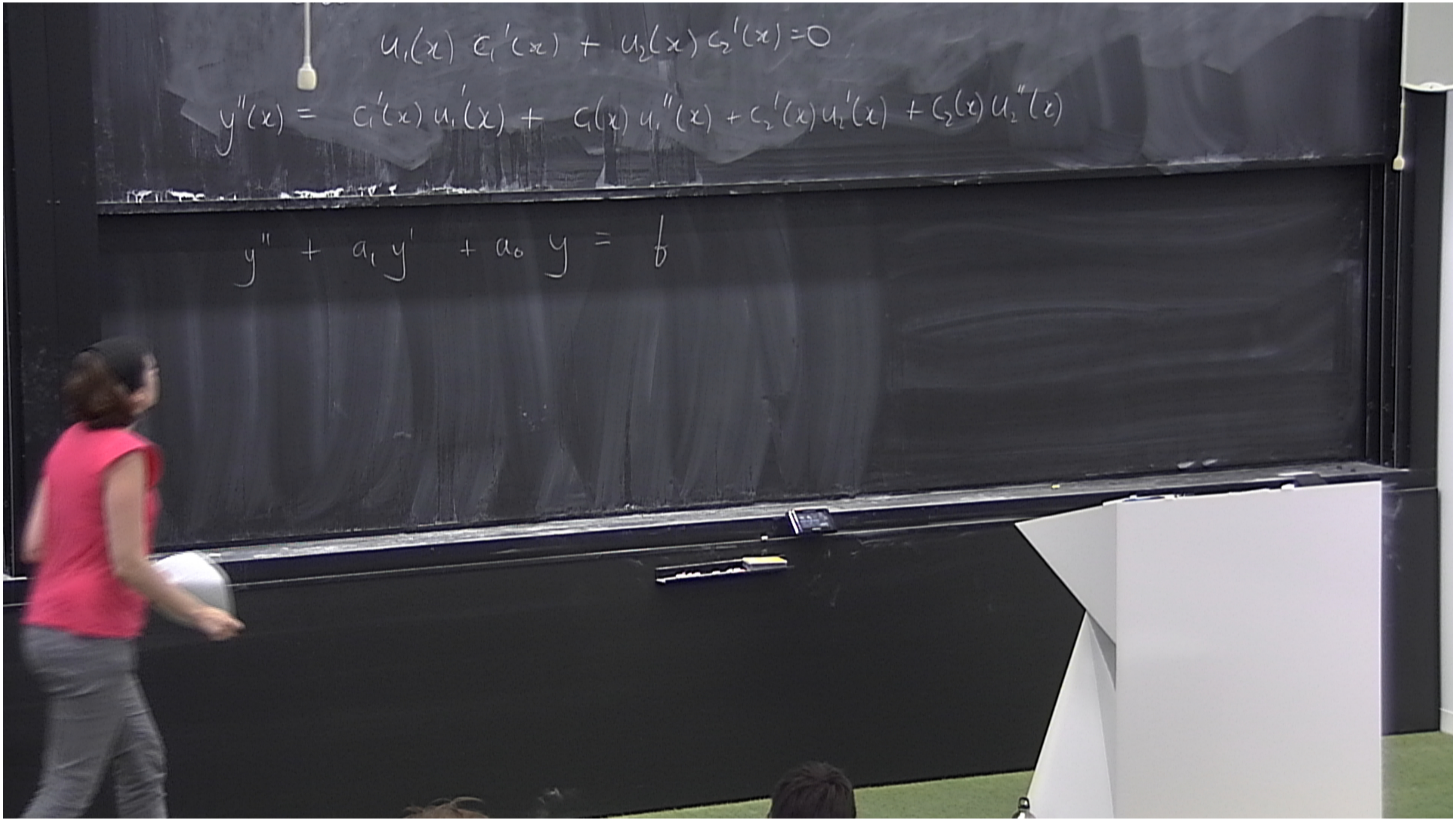

linearly independent solutions
to homogeneous equation
y'(x) = c1'(x) u1(x) + c1(x) u1'(x) + c2'(x) u2(x) + c2(x) u2'(x)
choose c1(x), c2(x) s.t.
u1(x) c1'(x) + u2(x) c2'(x) = 0

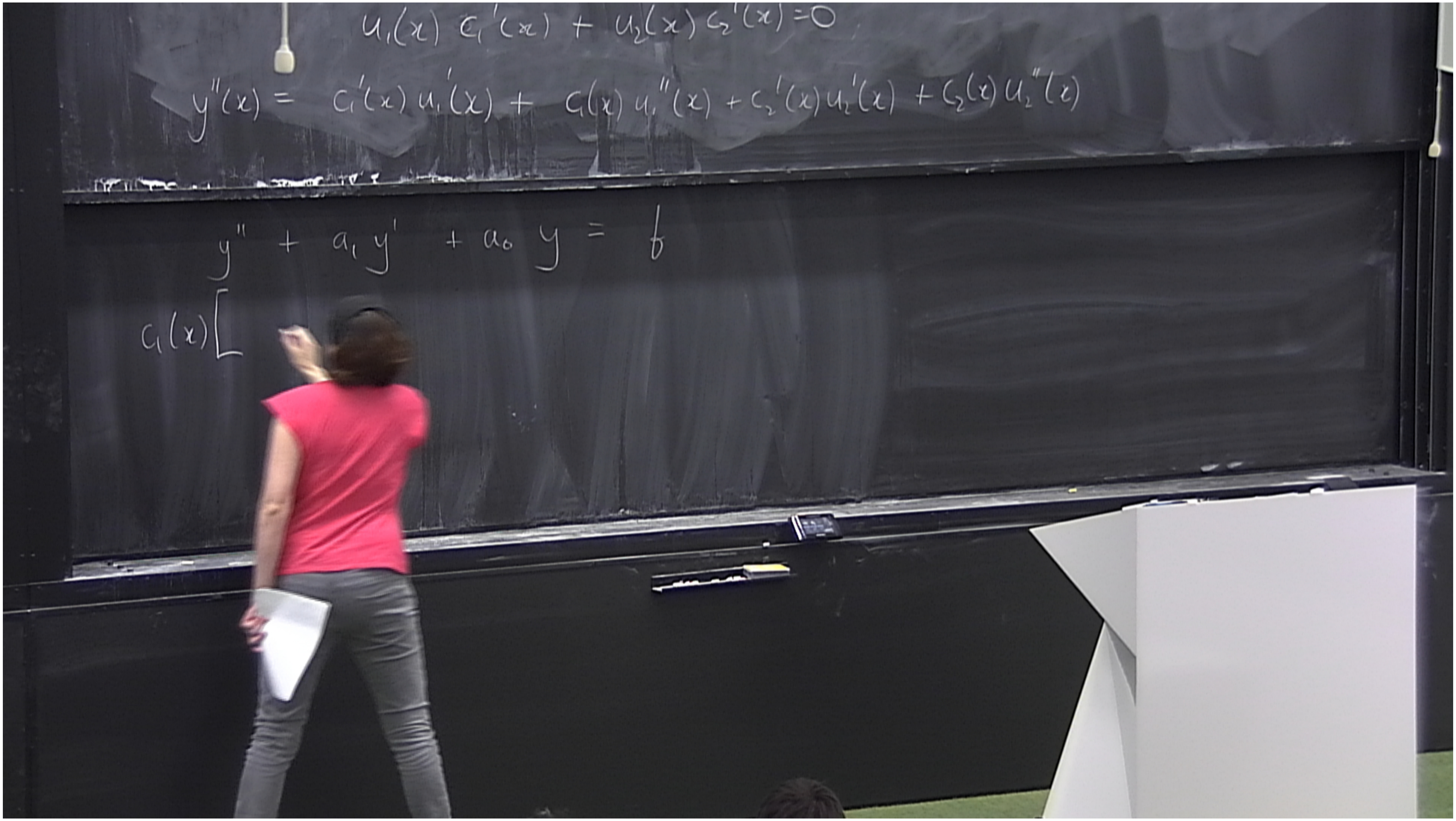

linearly independent solutions
to homogeneous equation
$y'(x) = \cancel{c_1'(x)\, u_1(x)} + c_1(x)\, u_1'(x) + \cancel{c_2'(x)\, u_2(x)} + c_2(x)\, u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.
$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$

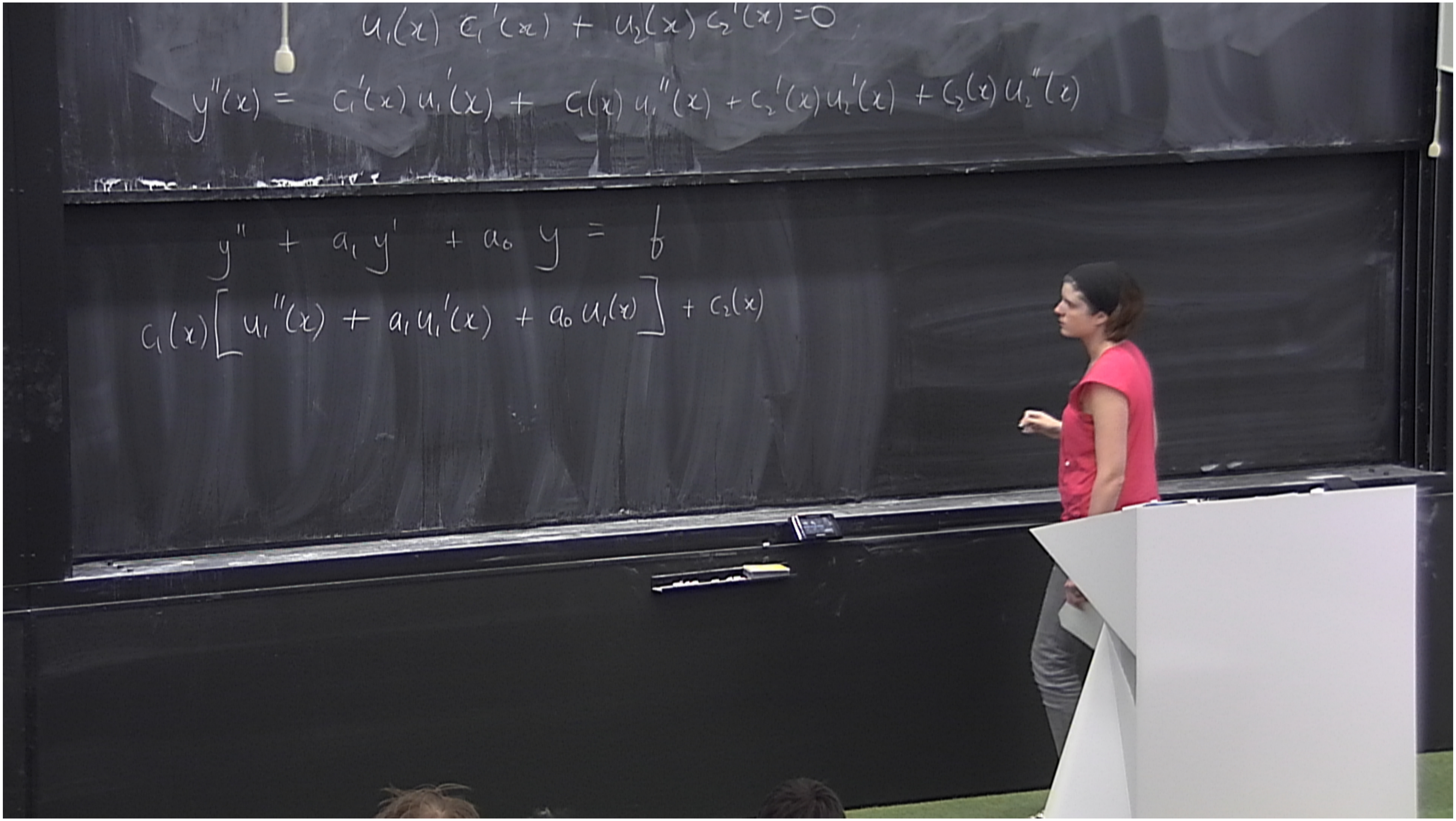

linearly independent solutions
to homogeneous
$y'(x) = \cancel{c_1'(x) u_1(x)} + c_1(x) u_1'(x) + \cancel{c_2'(x) u_2(x)} + c_2(x) u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.
$u_1(x) c_1'(x) + u_2(x) c_2'(x) = 0$
$y''(x)$

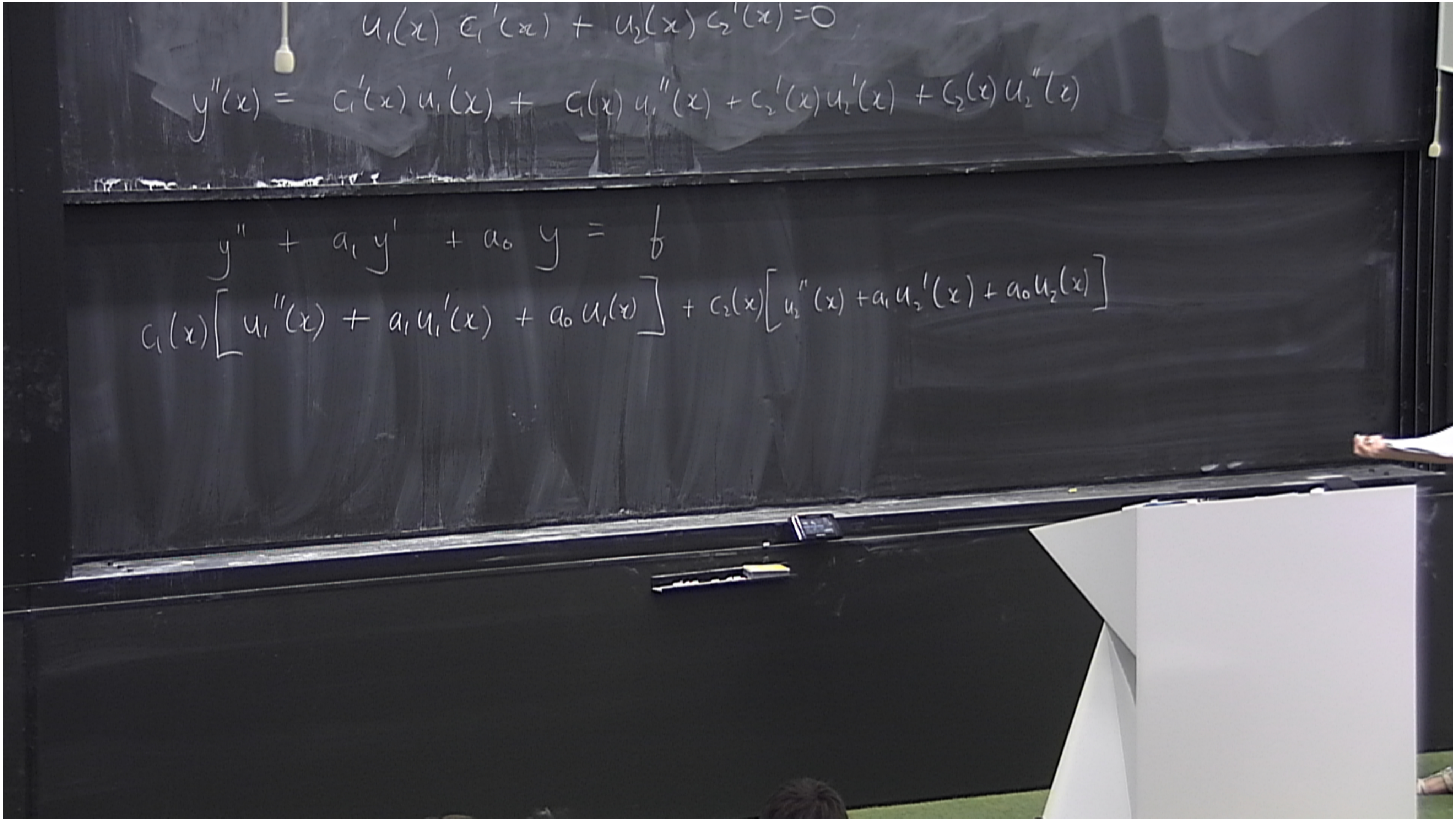

linearly independent solutions
y'(x) = c1'(x) u1(x) + c1(x) u1'(x) + c2'(x) u2(x) + c2(x) u2'(x)
choose c1(x), c2(x) s.t.
u1(x) c1'(x) + u2(x) c2'(x) = 0
y''(x) = c1'(x) u1'(x) +


linearly independent solutions
$y'(x) = c_1'(x)\,u_1(x) + c_1(x)\,u_1'(x) + c_2'(x)\,u_2(x) + c_2(x)\,u_2'(x)$
choose $c_1(x)$, $c_2(x)$ s.t.
$u_1(x)\,c_1'(x) + u_2(x)\,c_2'(x) = 0$
$y''(x) = c_1'(x)\,u_1'(x) + c_1(x)\,u_1''(x) + c_2'(x)\,u_2'(x) + c_2(x)\,u_2''(x)$

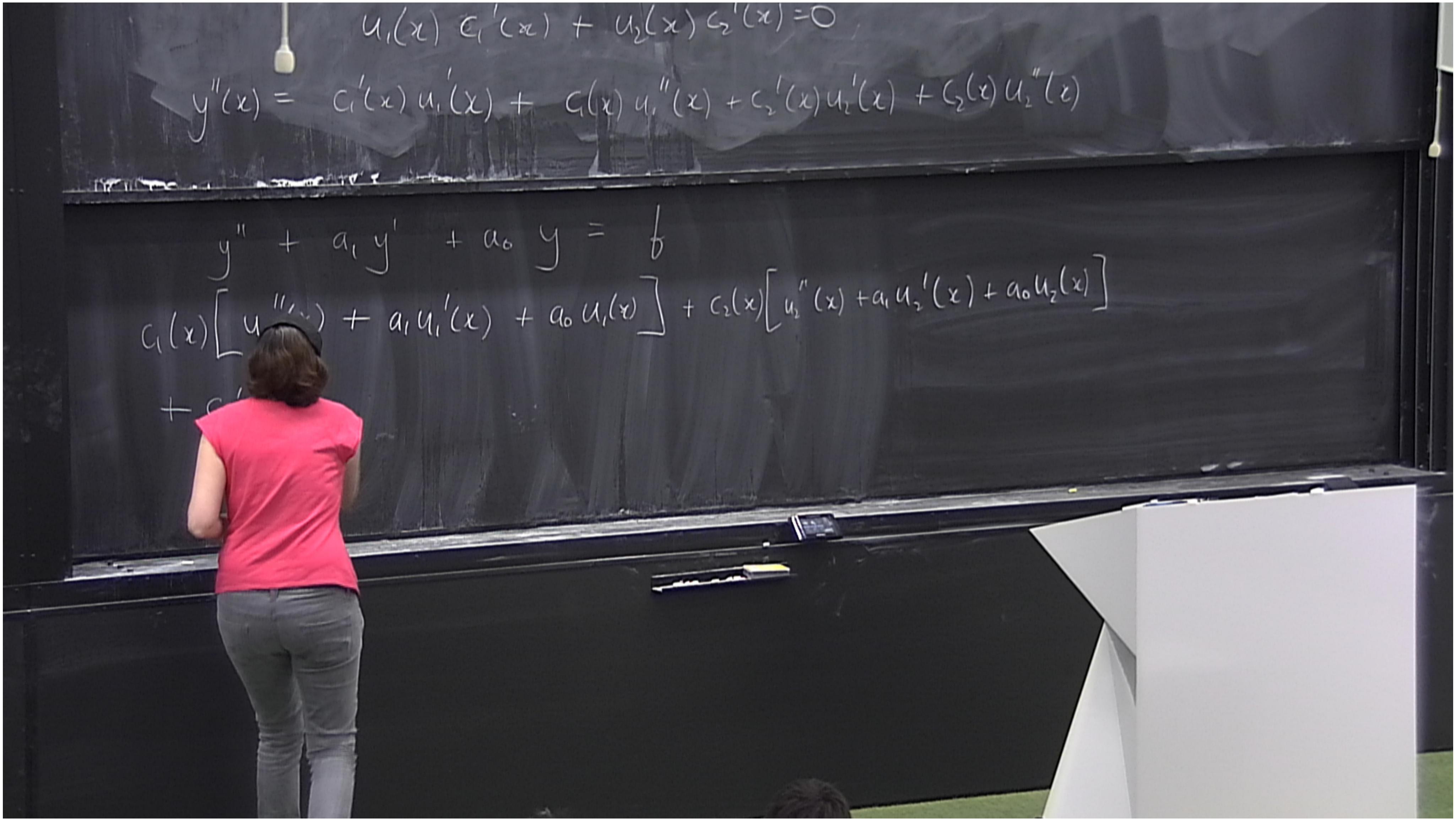

$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x)\, u_1'(x) + c_1(x)\, u_1''(x) + c_2'(x)\, u_2'(x) + c_2(x)\, u_2''(x)$
$y'' + a_1 y' + a_0 y = f$

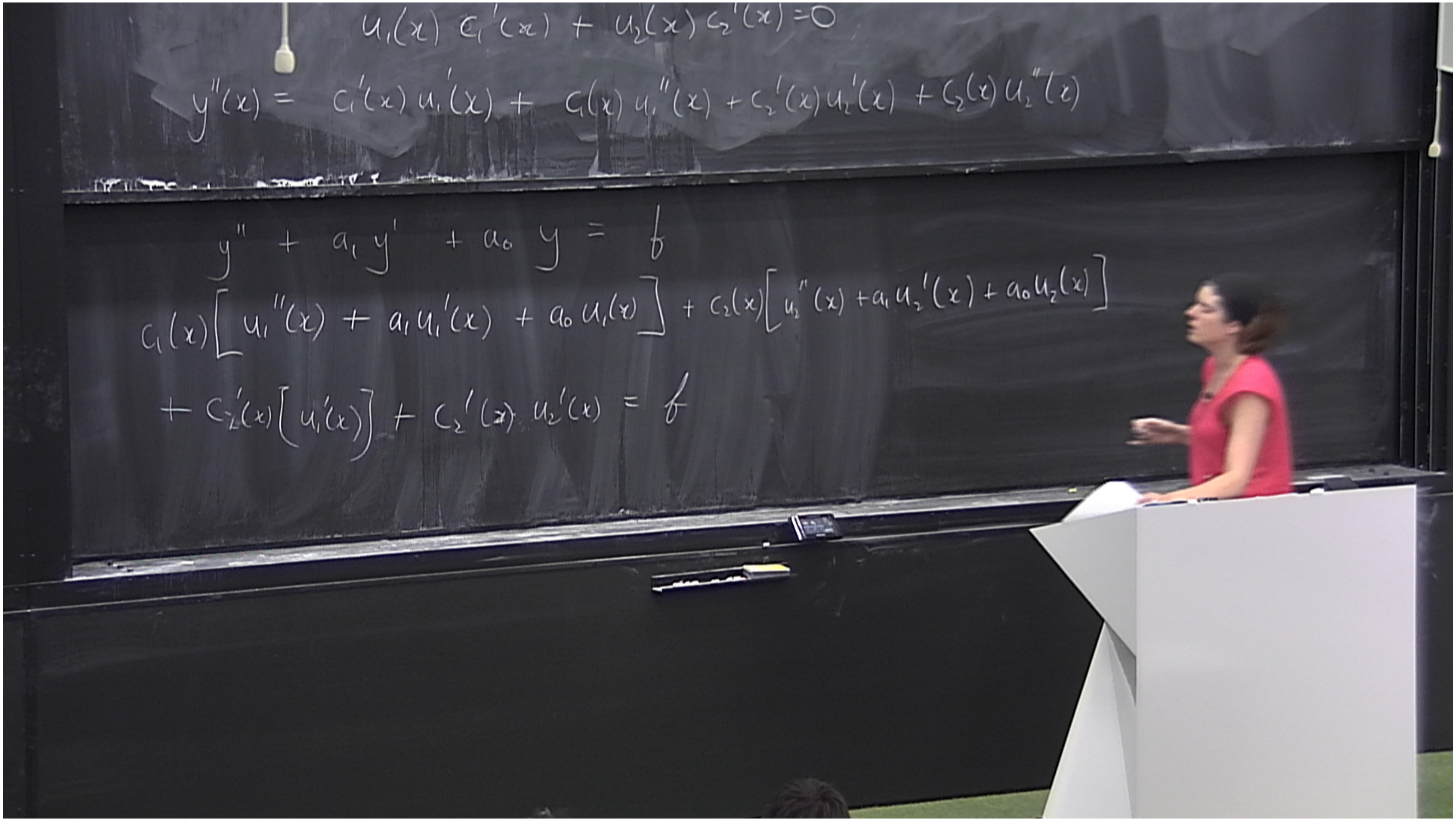

$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x)\, u_1'(x) + c_1(x)\, u_1''(x) + c_2'(x)\, u_2'(x) + c_2(x)\, u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$c_1(x)[$

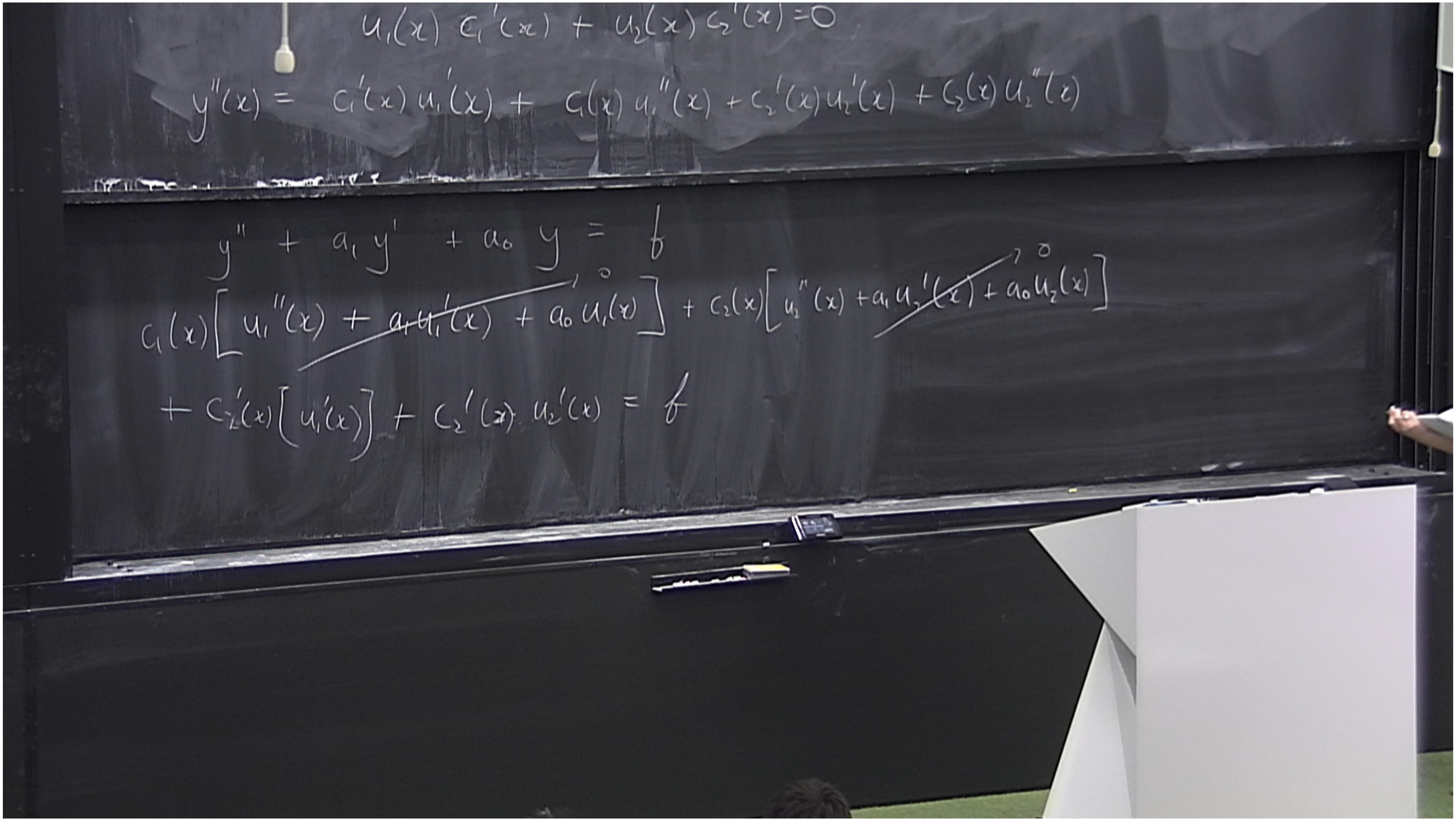

$u_1(x)\,c_1'(x) + u_2(x)\,c_2'(x) = 0$
$y''(x) = c_1'(x)\,u_1'(x) + c_1(x)\,u_1''(x) + c_2'(x)\,u_2'(x) + c_2(x)\,u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$c_1(x)\left[u_1''(x) + a_1 u_1'(x) + a_0 u_1(x)\right] + c_2(x)$

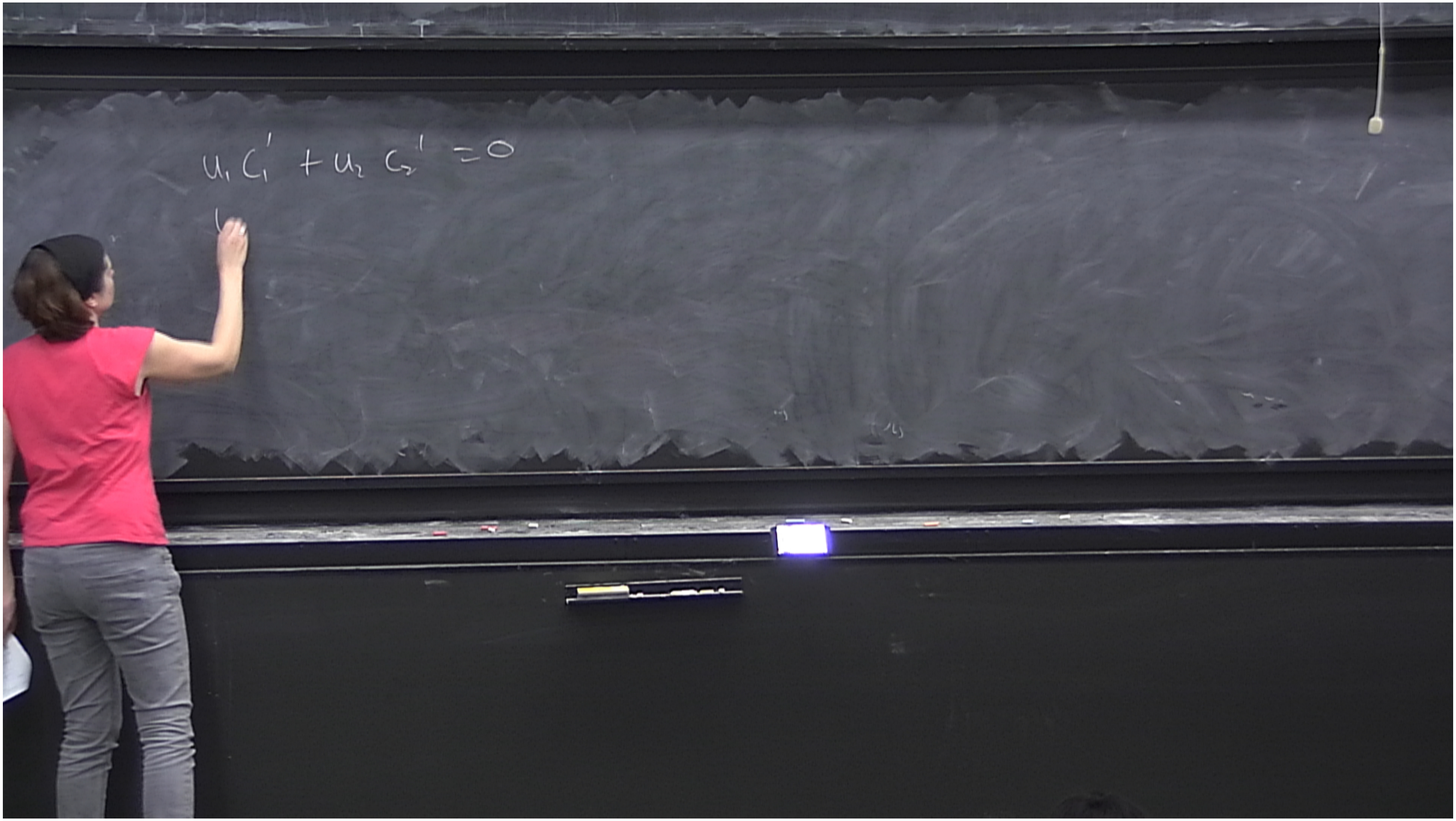


$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x) u_1'(x) + c_1(x) u_1''(x) + c_2'(x) u_2'(x) + c_2(x) u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$c_1(x)\left[u_1''(x) + a_1 u_1'(x) + a_0 u_1(x)\right] + c_2(x)\left[u_2''(x) + a_1 u_2'(x) + a_0 u_2(x)\right]$

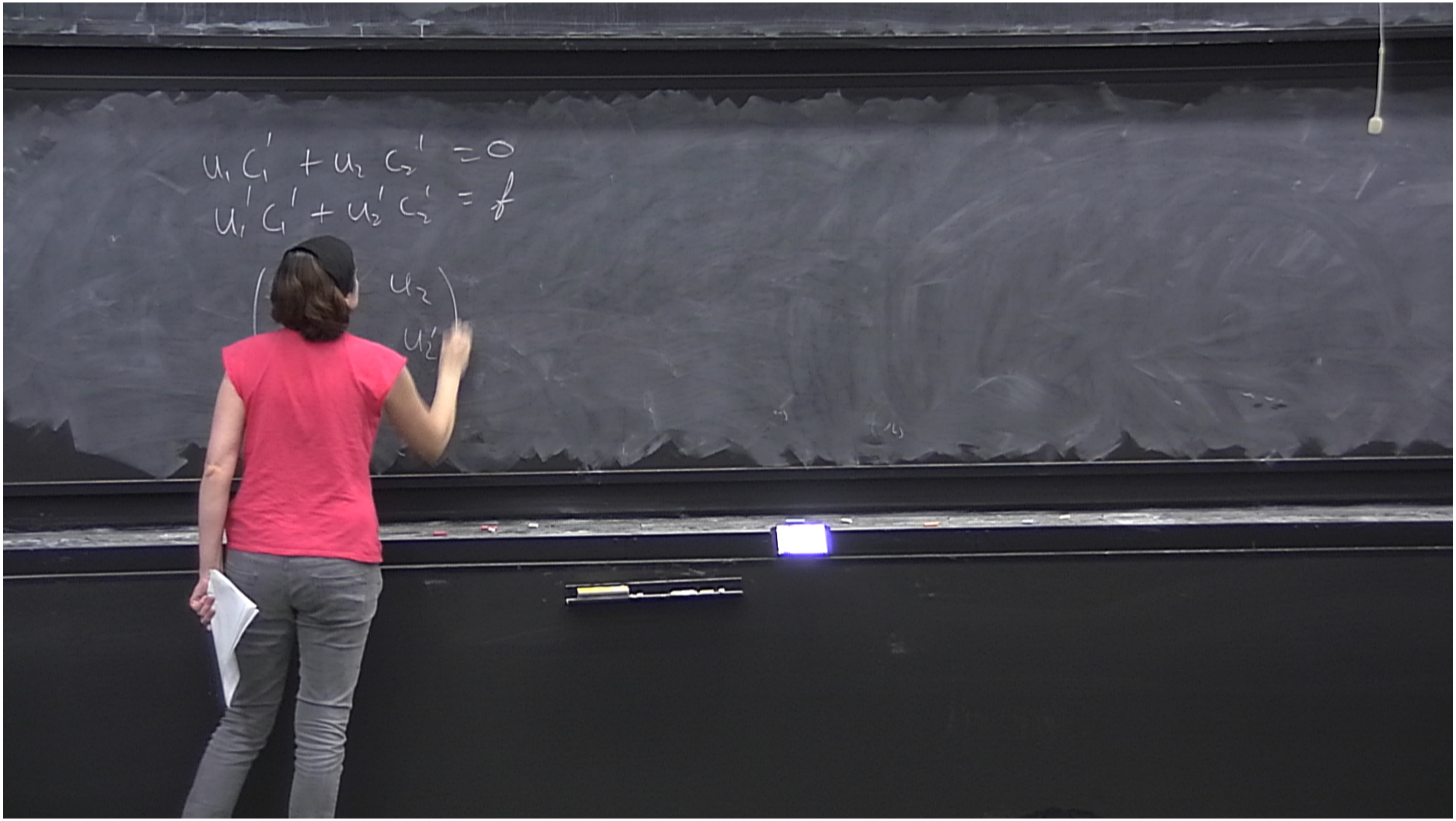

$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x)\, u_1'(x) + c_1(x)\, u_1''(x) + c_2'(x)\, u_2'(x) + c_2(x)\, u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$+ a_1 u_1'(x) + a_0 u_1(x)] + c_2(x) [u_2''(x) + a_1 u_2'(x) + a_0 u_2(x)]$

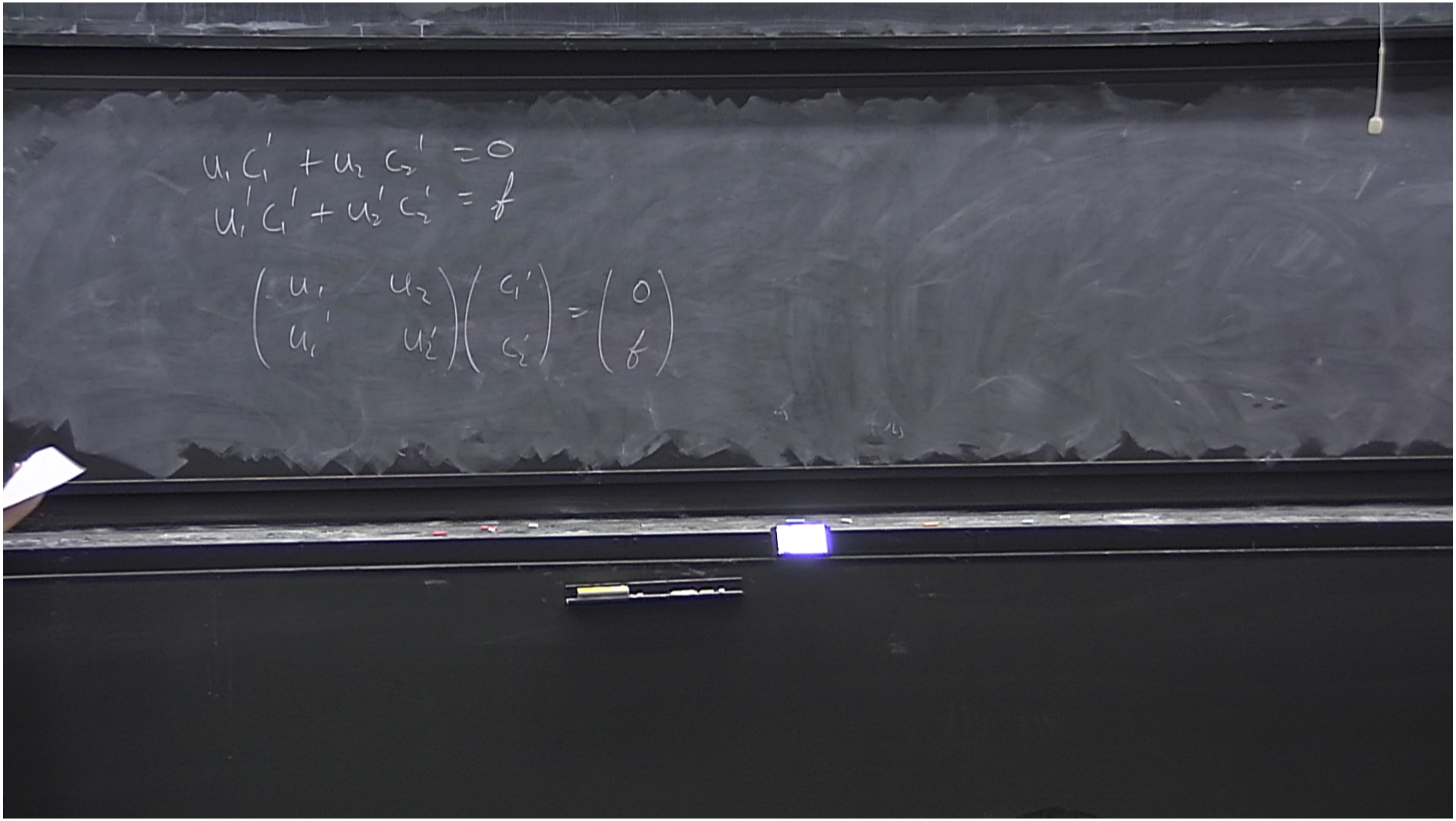

$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x)\, u_1'(x) + c_1(x)\, u_1''(x) + c_2'(x)\, u_2'(x) + c_2(x)\, u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$c_1(x)\left[u_1''(x) + a_1 u_1'(x) + a_0 u_1(x)\right] + c_2(x)\left[u_2''(x) + a_1 u_2'(x) + a_0 u_2(x)\right]$
$+ c_1'(x)\left[u_1'(x)\right] + c_2'(x)\, u_2'(x) = f$


$u_1(x)\, c_1'(x) + u_2(x)\, c_2'(x) = 0$
$y''(x) = c_1'(x)\, u_1'(x) + c_1(x)\, u_1''(x) + c_2'(x)\, u_2'(x) + c_2(x)\, u_2''(x)$
$y'' + a_1 y' + a_0 y = f$
$c_1(x)\big[\underbrace{u_1''(x) + a_1 u_1'(x) + a_0 u_1(x)}_{=0}\big] + c_2(x)\big[\underbrace{u_2''(x) + a_1 u_2'(x) + a_0 u_2(x)}_{=0}\big]$
$+ c_1'(x)\big[u_1'(x)\big] + c_2'(x)\, u_2'(x) = f$


$u_1 c_1' + u_2 c_2' = 0$


Sometimes can find particular solution by inspection
F constant
complementary function
choose

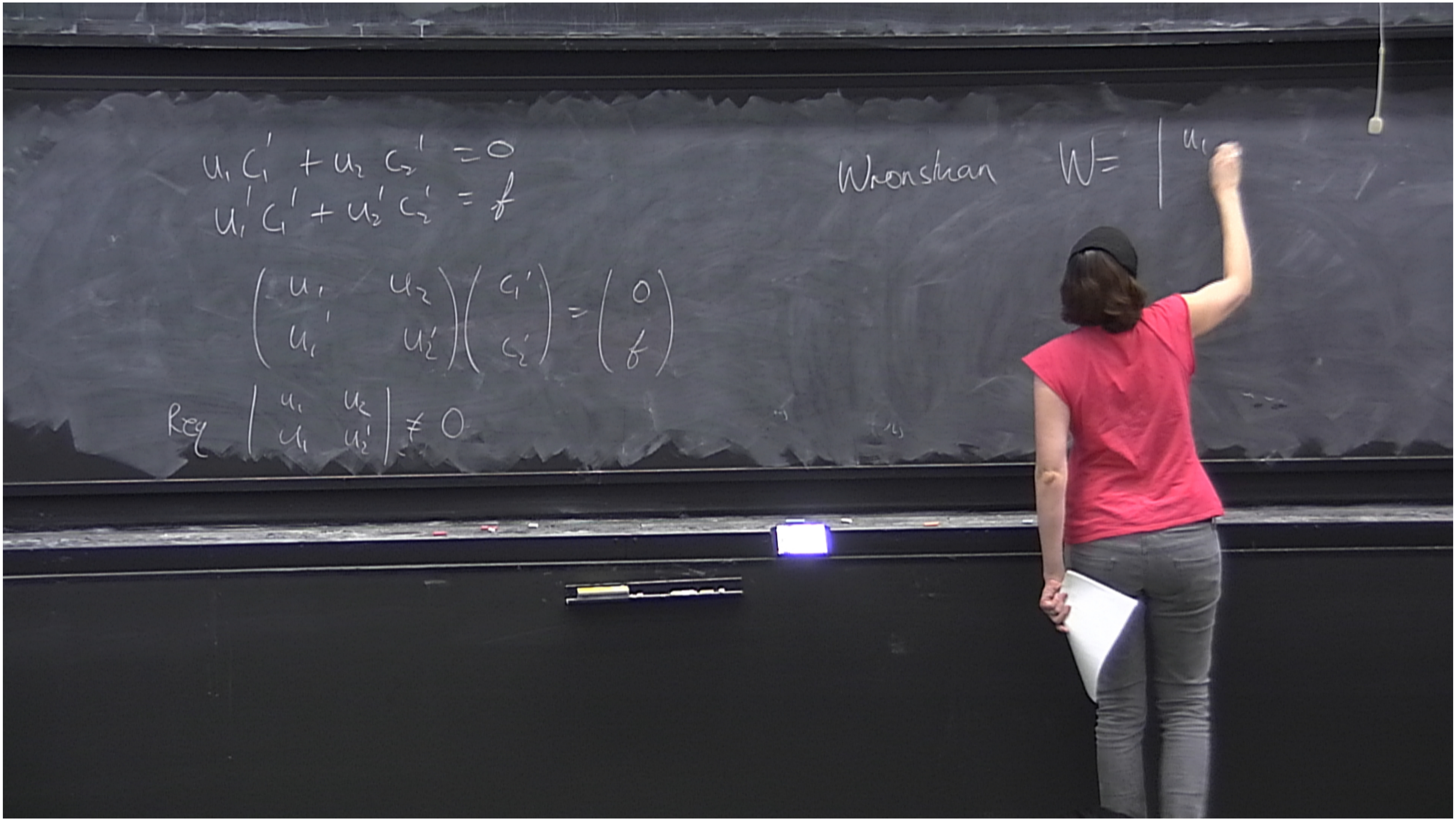

u1 c1' + u2 c2' = 0
u1' c1' + u2' c2' = f
u2
u2'

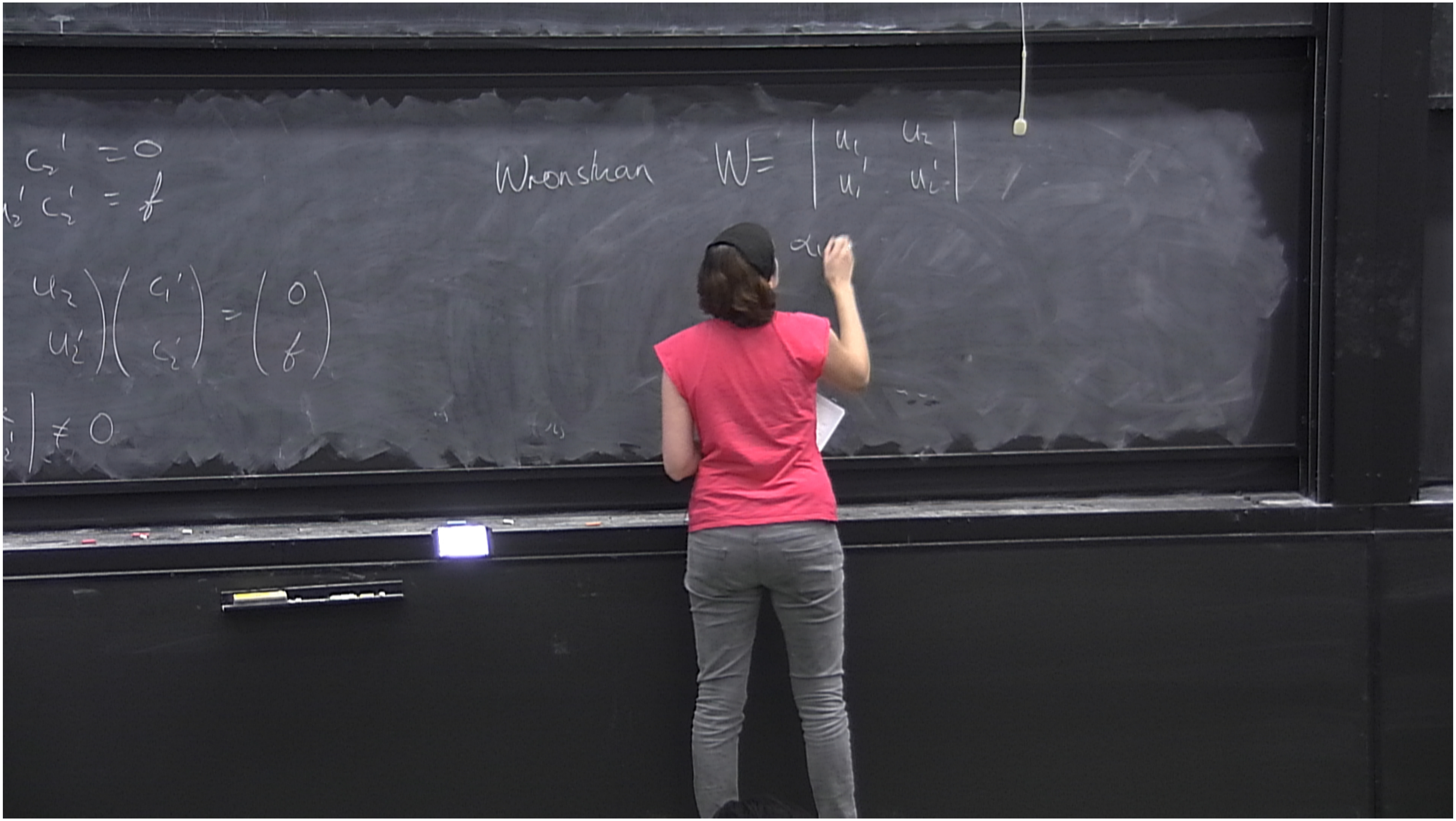

$u_1 c_1' + u_2 c_2' = 0$
$u_1' c_1' + u_2' c_2' = f$
$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$

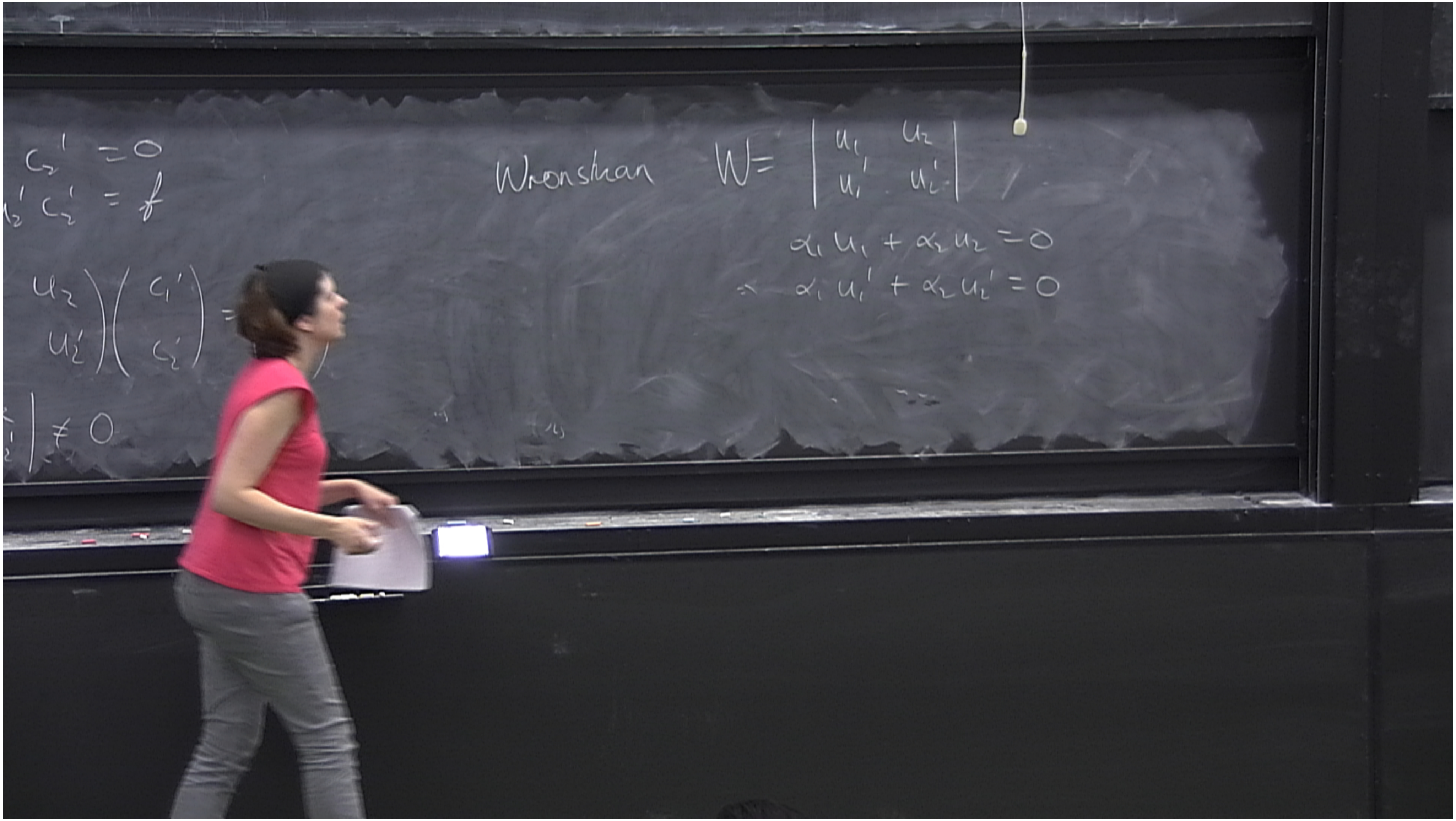
$$u_1 c_1' + u_2 c_2' = 0$$
$$u_1' c_1' + u_2' c_2' = f$$

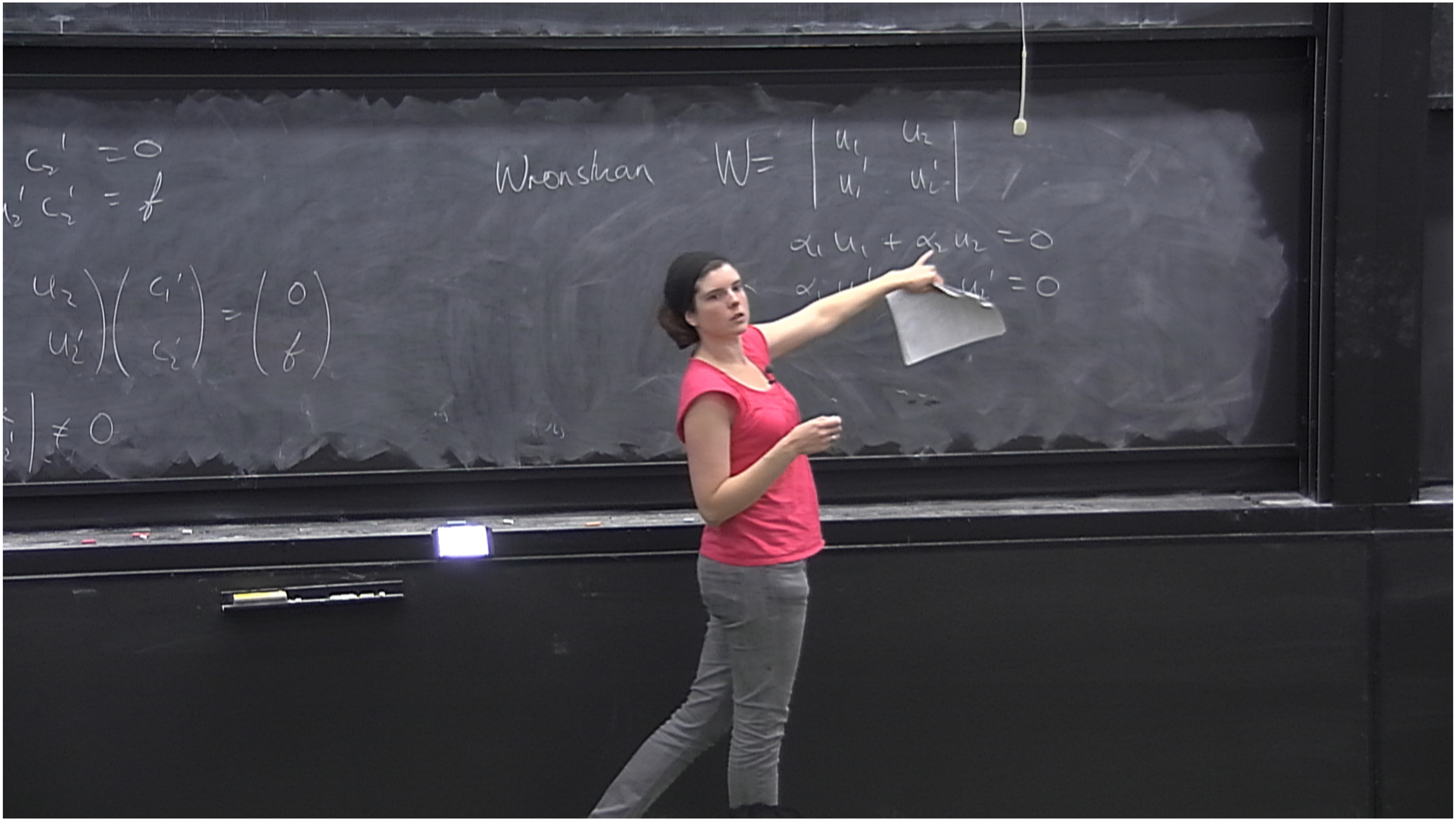
$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

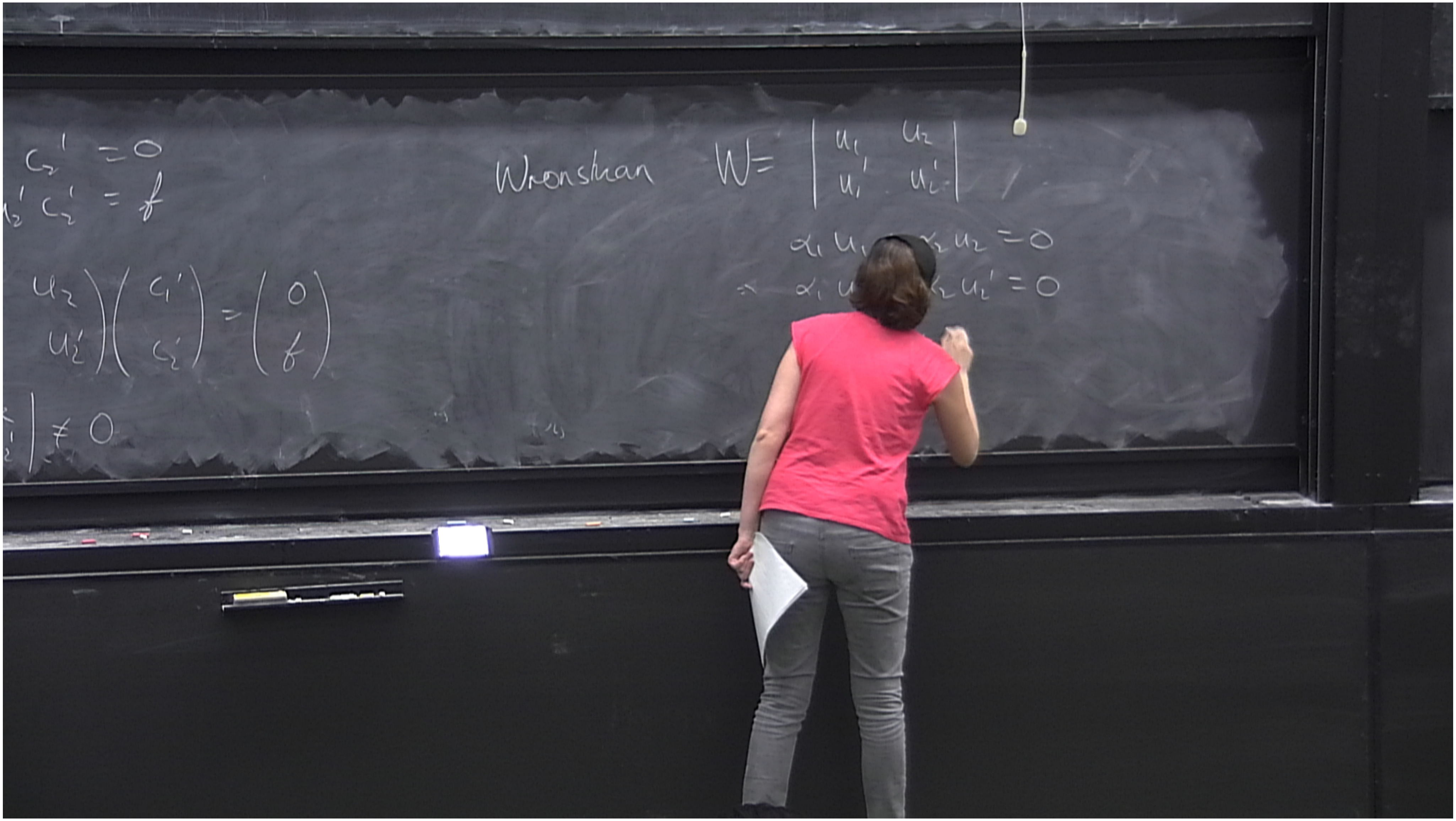
Req $\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \neq 0$

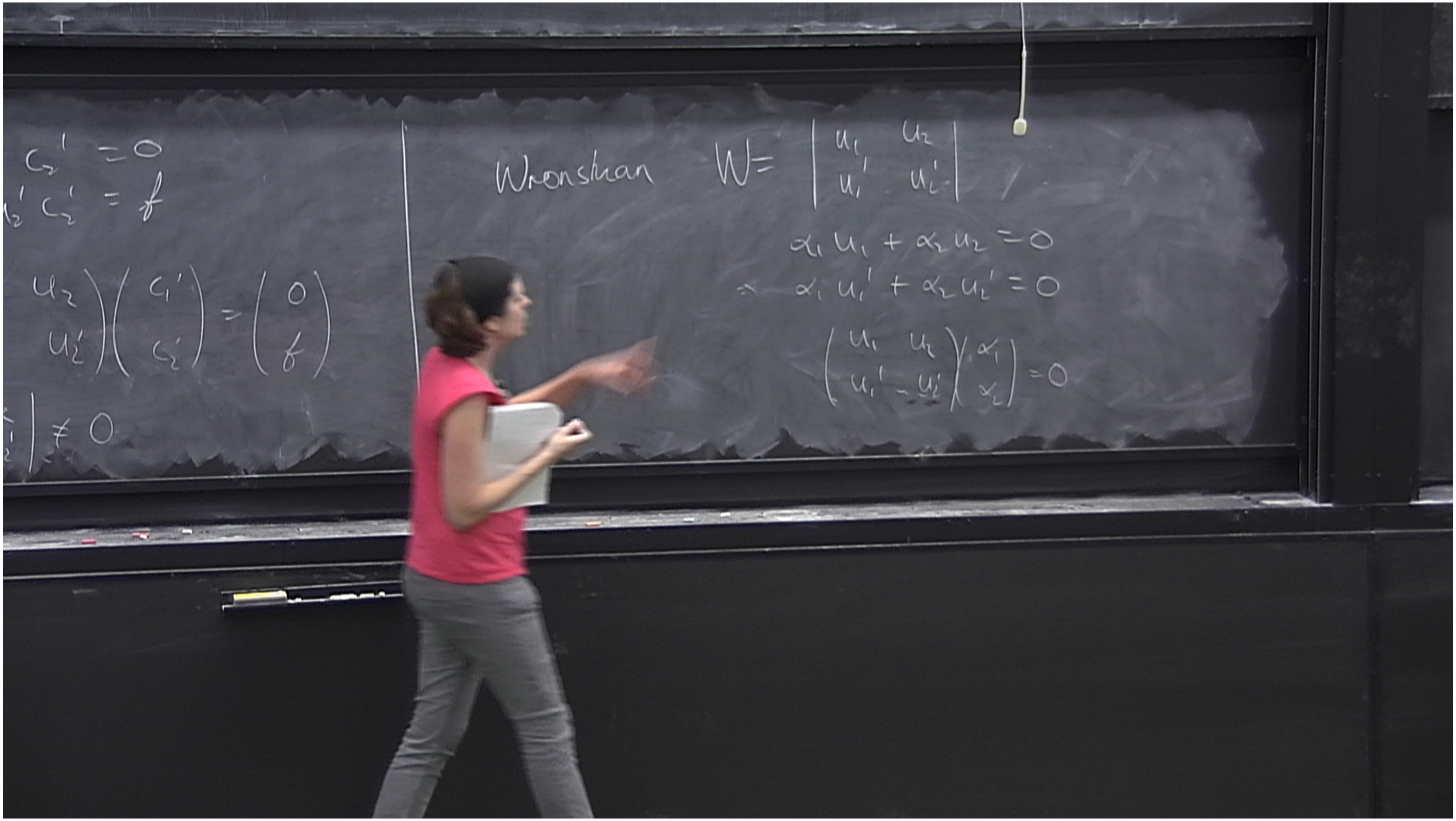

$u_1 c_1' + u_2 c_2' = 0$
$u_1' c_1' + u_2' c_2' = f$
$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix}\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$
Req $\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} \neq 0$
Wronskian $W = | u_1$


$c_2' = 0$
$u_2' c_2' = f$
$\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$
$\neq 0$
Wronskian
$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$


Wronskian W= | u1 u2 / u1' u2' |
α1 u1 + α2 u2 = 0
α1 u1' + α2 u2' = 0
c2' = 0
u2' c2' = f
≠ 0


Wronskian


Wronskian
W=
= 0
= f
≠ 0


Wronskian
W=
$\alpha_1 u_1 + \alpha_2 u_2 = 0$
$\alpha_1 u_1' + \alpha_2 u_2' = 0$
$=0$
$\neq 0$

$$c_2' = 0$$

$$u_2' c_2' = f$$

$$\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

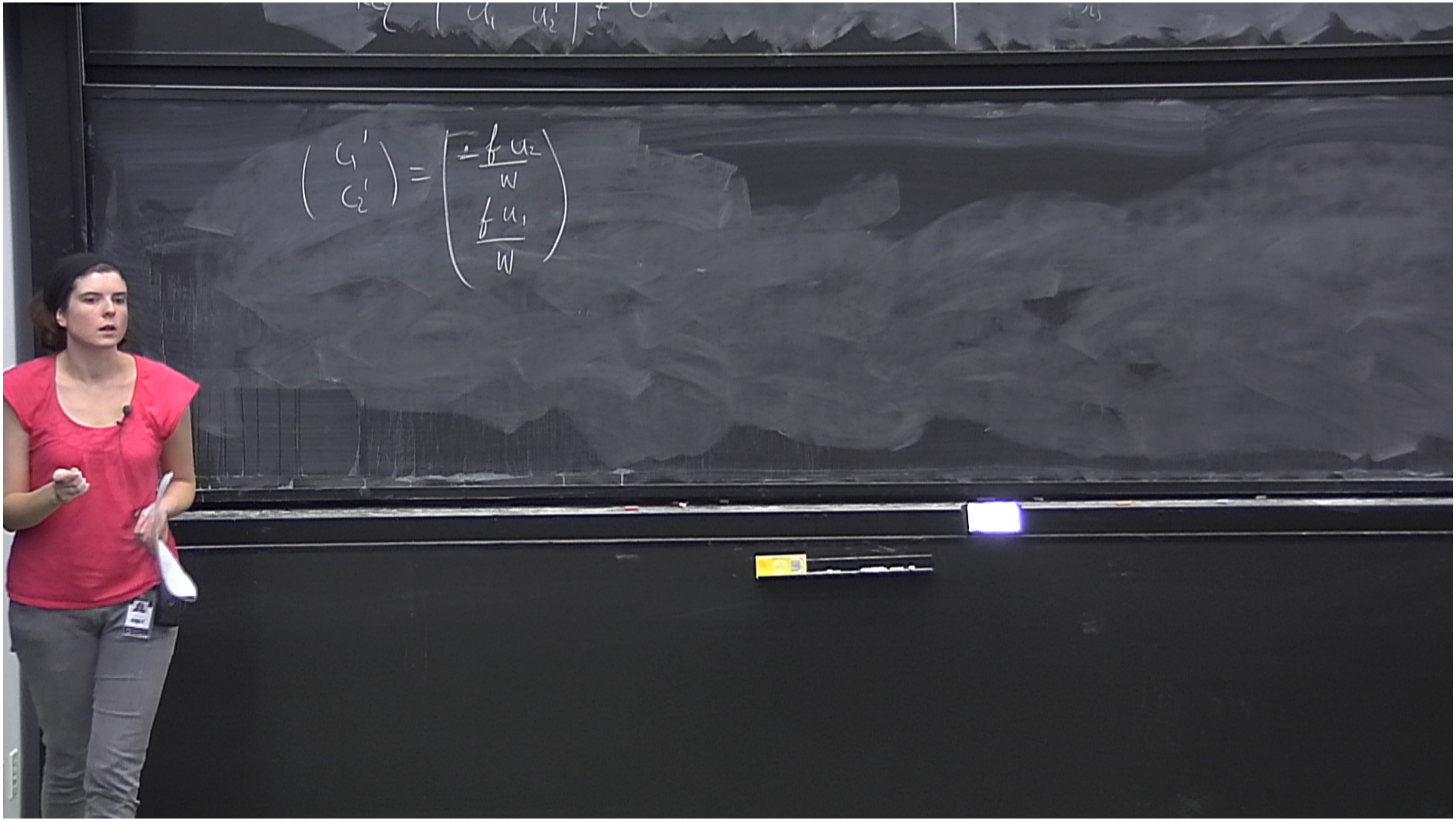
$$\Big| \ \Big| \neq 0$$

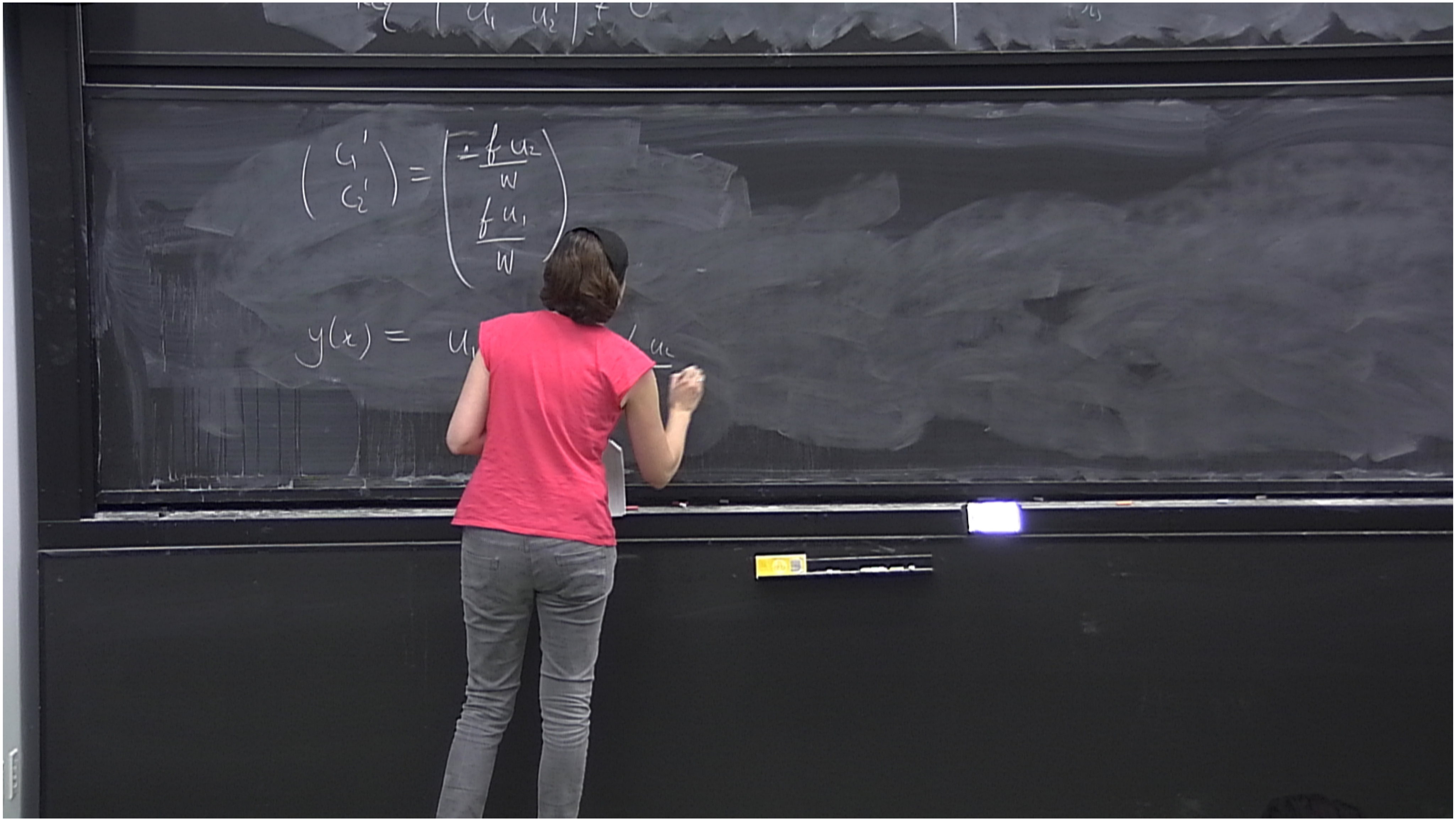
Wronskian $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$

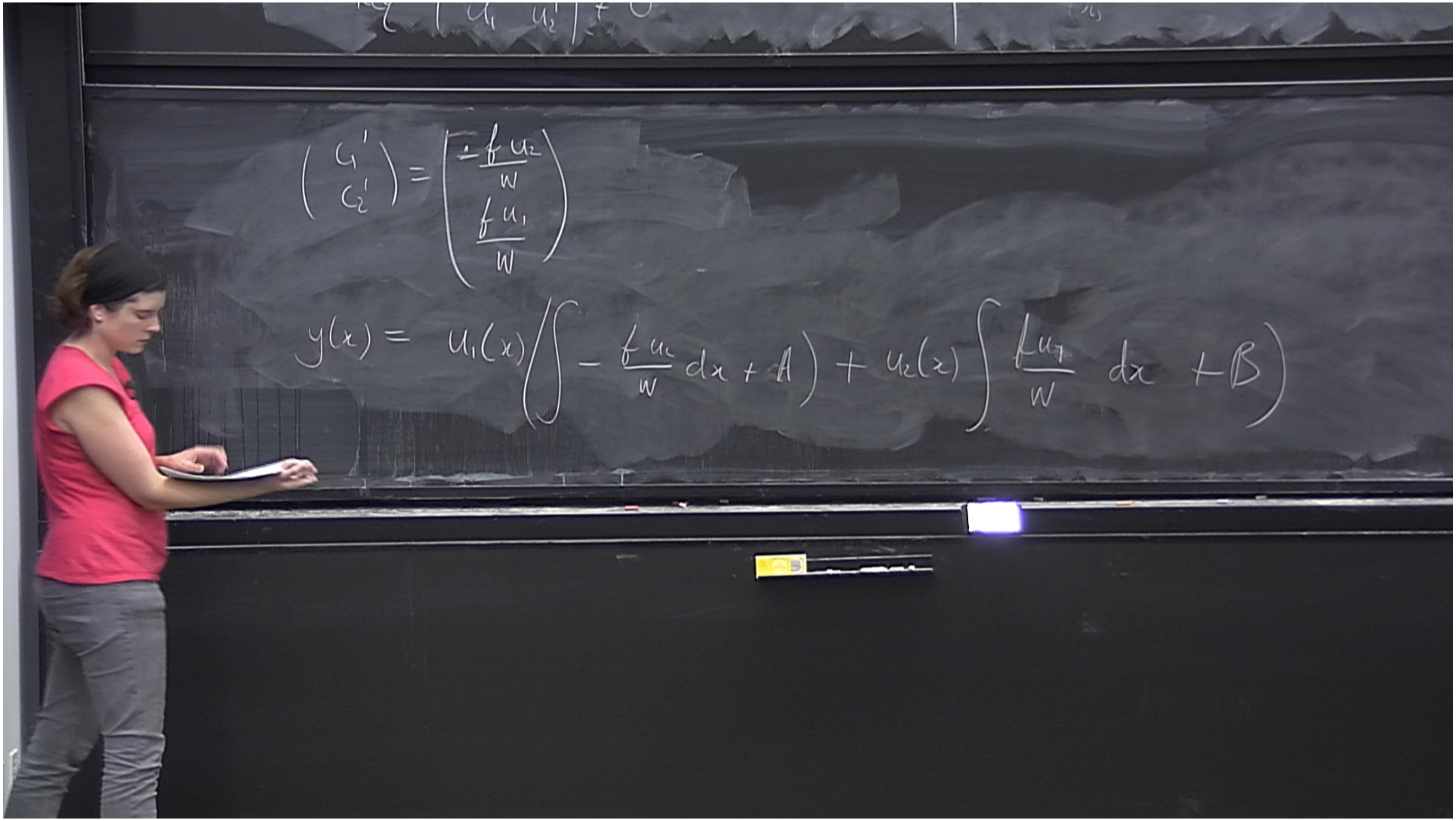
$$\alpha_1 u_1 + \alpha_2 u_2 = 0$$

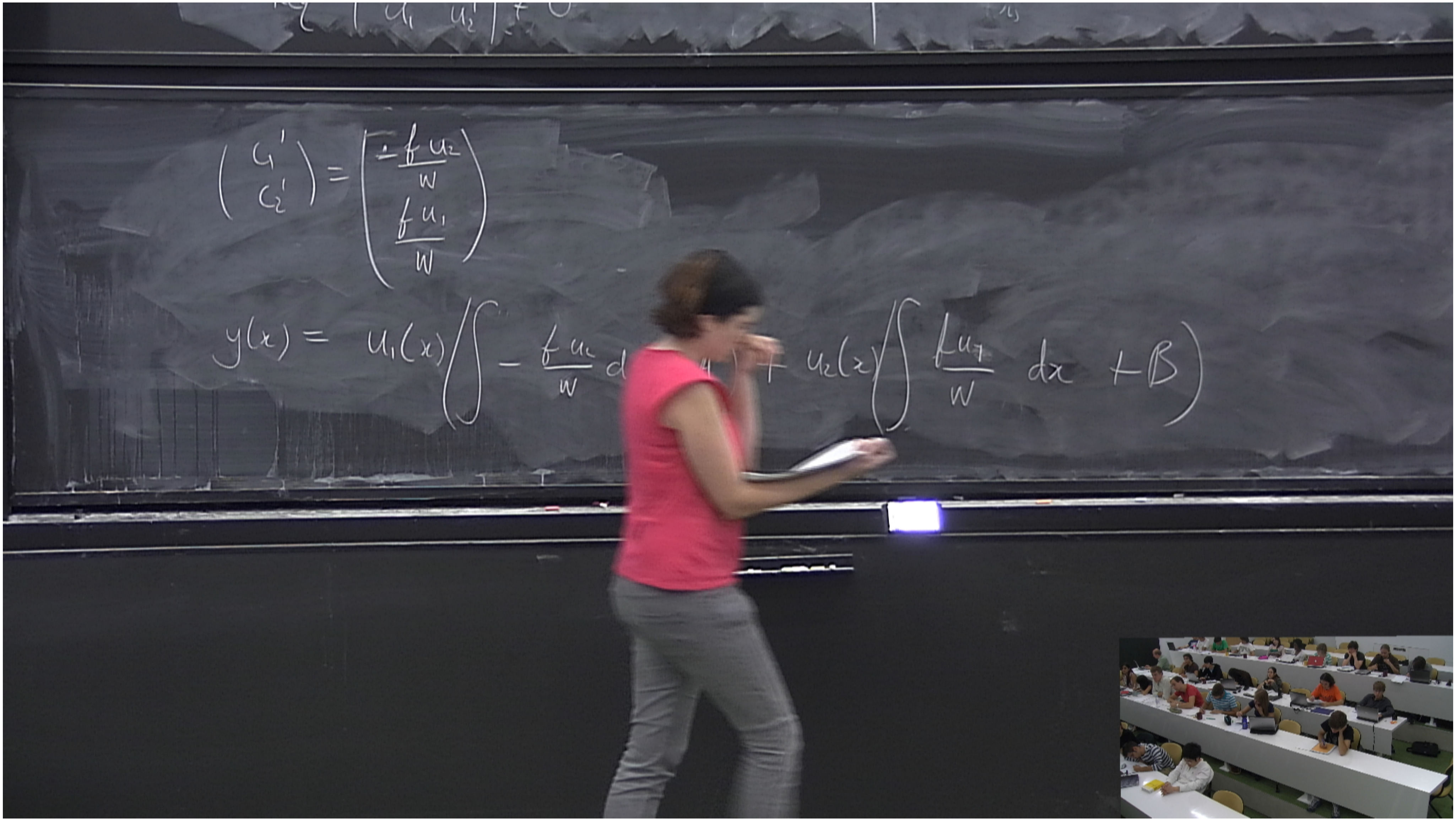
$$\Rightarrow \alpha_1 u_1' + \alpha_2 u_2' = 0$$

$$\begin{pmatrix} u_1 & u_2 \\ u_1' & u_2' \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

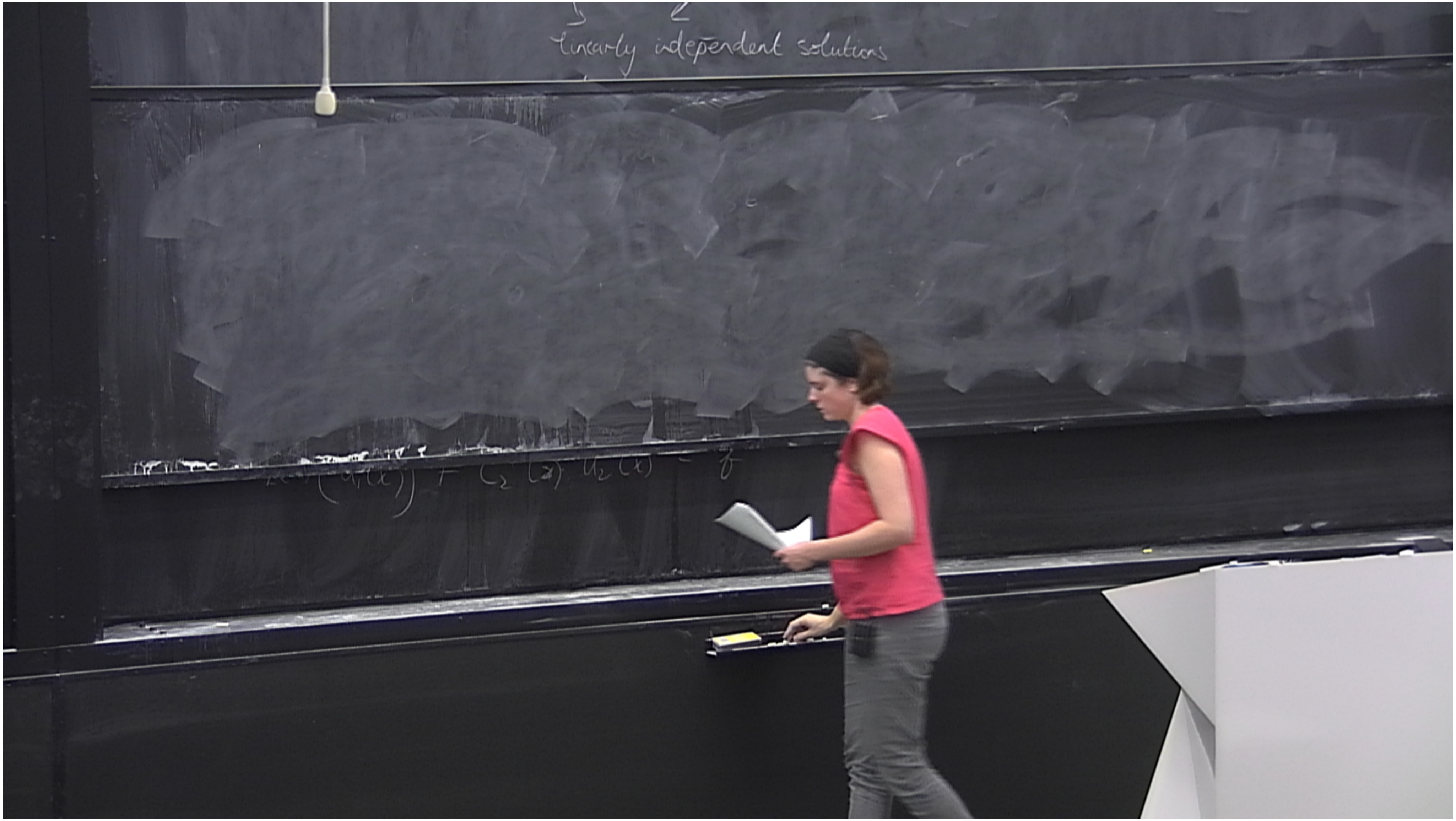

$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} -\frac{f u_2}{W} \\ \frac{f u_1}{W} \end{pmatrix}$

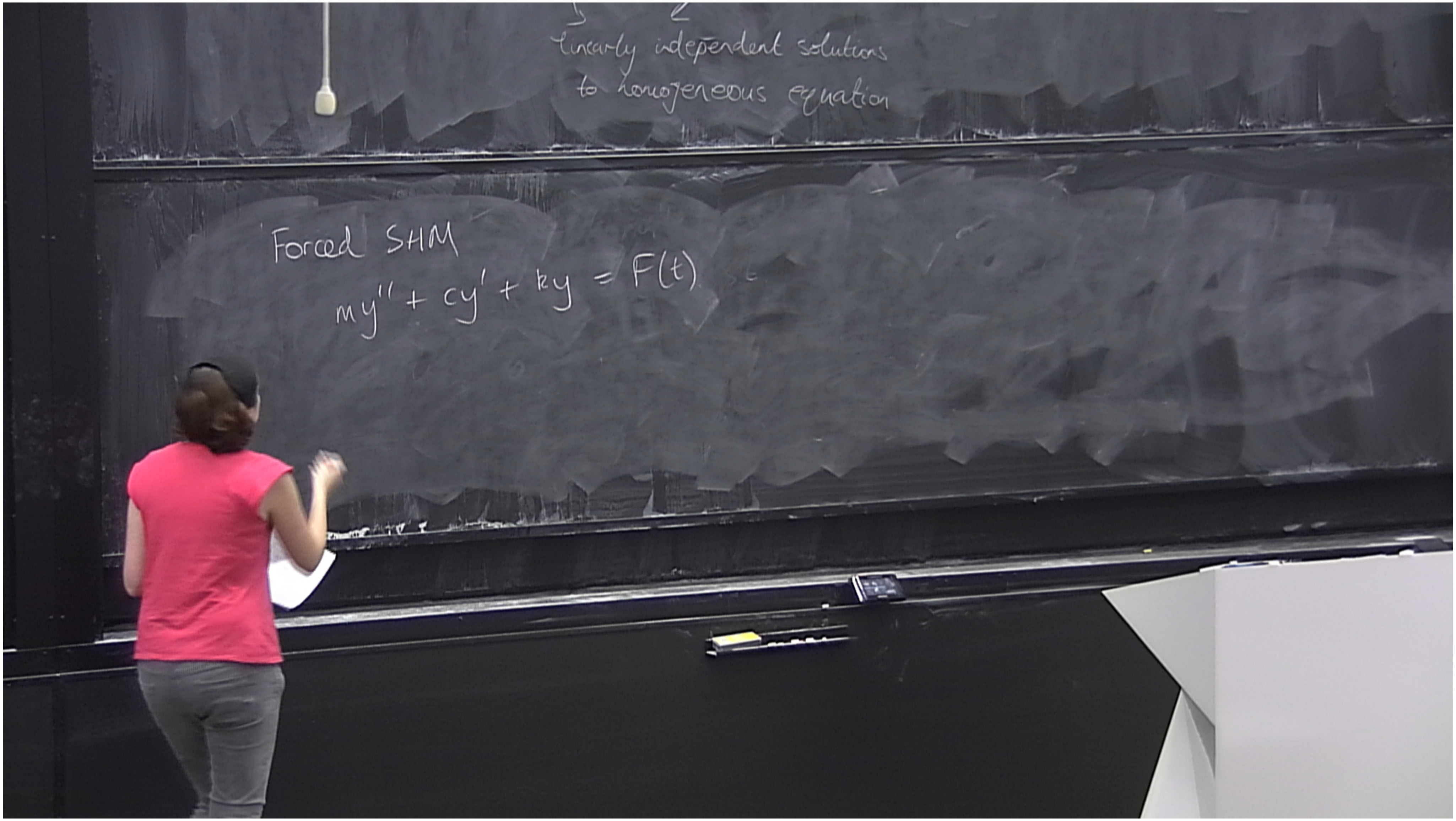

$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} -\frac{f u_2}{W} \\ \frac{f u_1}{W} \end{pmatrix}$
$y(x) =$

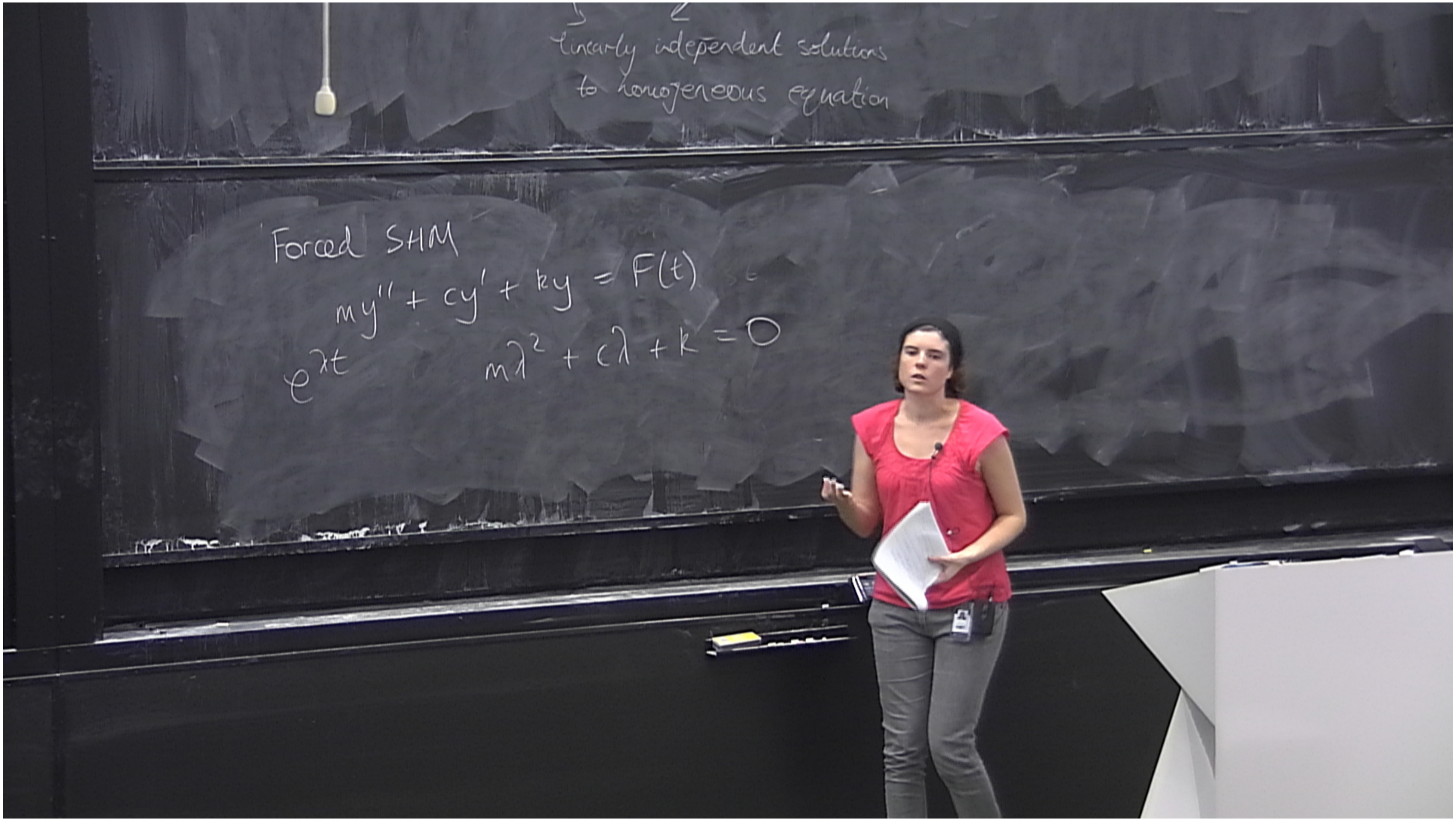

$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} -\frac{f u_2}{W} \\ \frac{f u_1}{W} \end{pmatrix}$
$y(x) = u_1(x)\left(\int -\frac{f u_2}{W}\,dx + A\right) + u_2(x)\left(\int \frac{f u_1}{W}\,dx + B\right)$

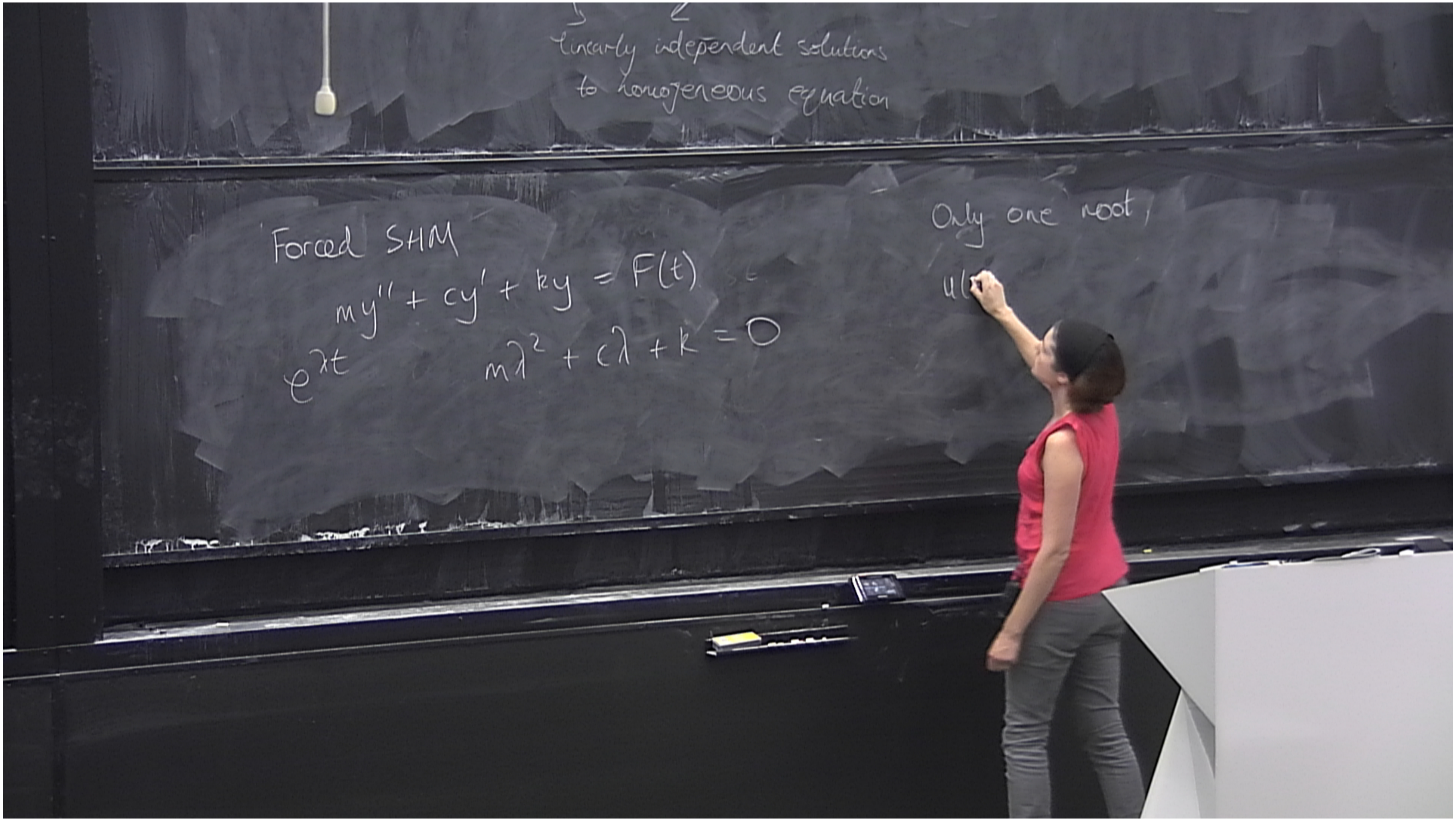


$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} -\frac{f u_2}{W} \\ \frac{f u_1}{W} \end{pmatrix}$
$y(x) = u_1(x)\left(\int -\frac{f u_2}{W} d\right.$
$+ u_2(x)\left(\int \frac{f u_1}{W} dx + B\right)$


linearly independent solutions

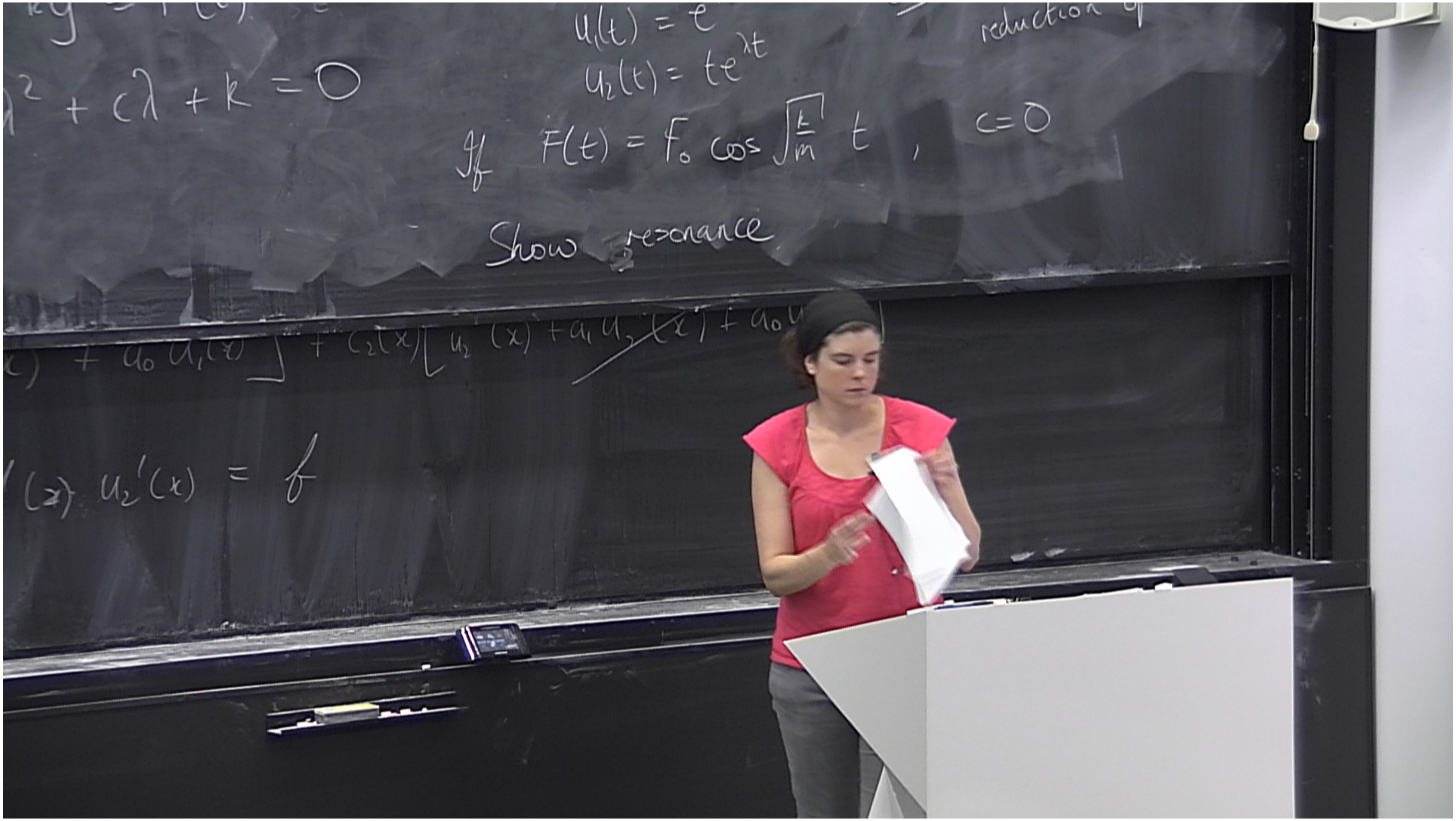

linearly independent solutions
to homogeneous equation
Forced SHM
my'' + cy' + ky = F(t)


linearly independent solutions
to homogeneous equation
Forced SHM
my'' + cy' + ky = F(t)
e^λt
mλ^2 + cλ + k = 0

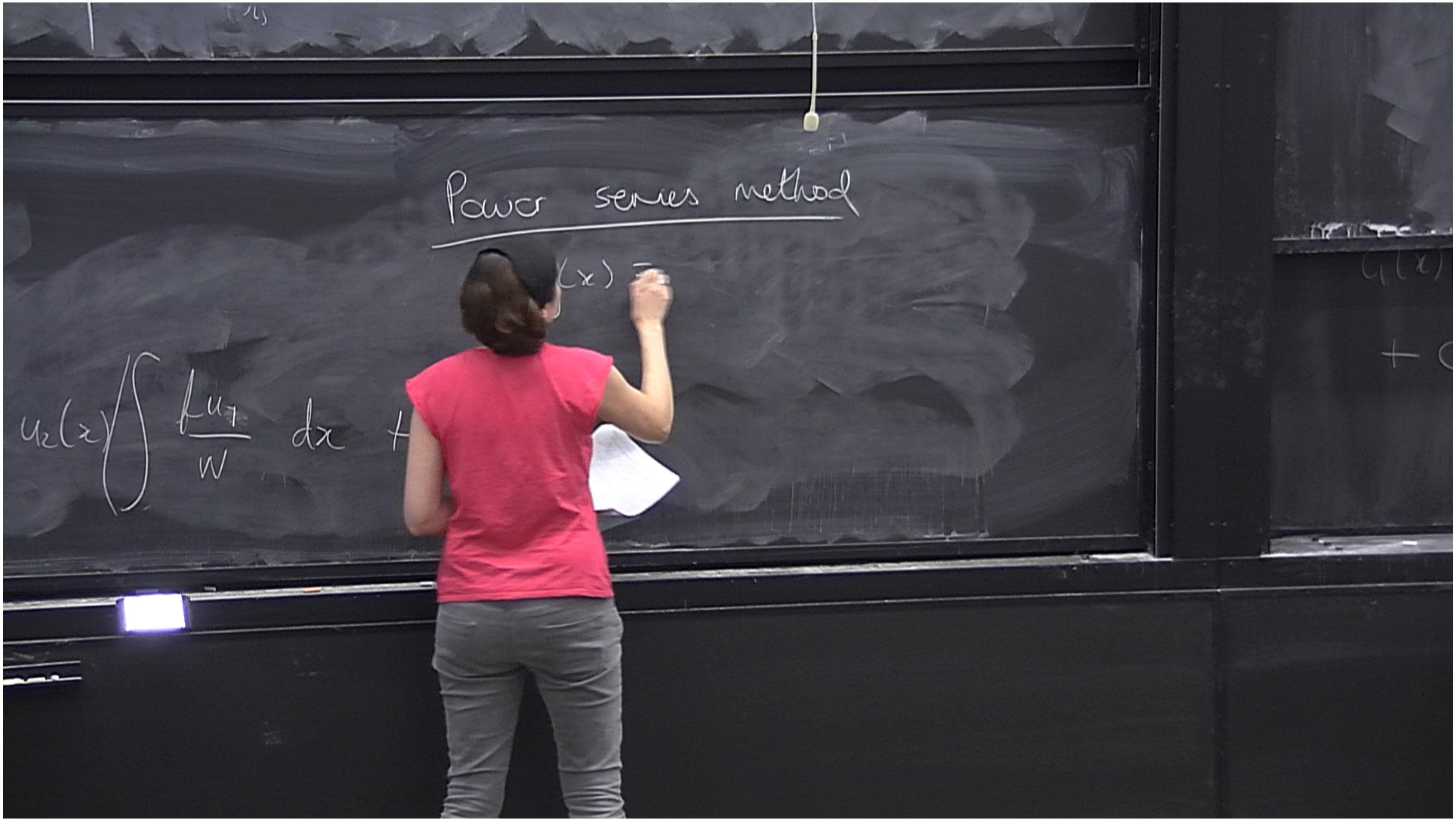

linearly independent solutions
to homogeneous equation
Forced SHM
my'' + cy' + ky = F(t)
e^λt
mλ² + cλ + k = 0
Only one root,

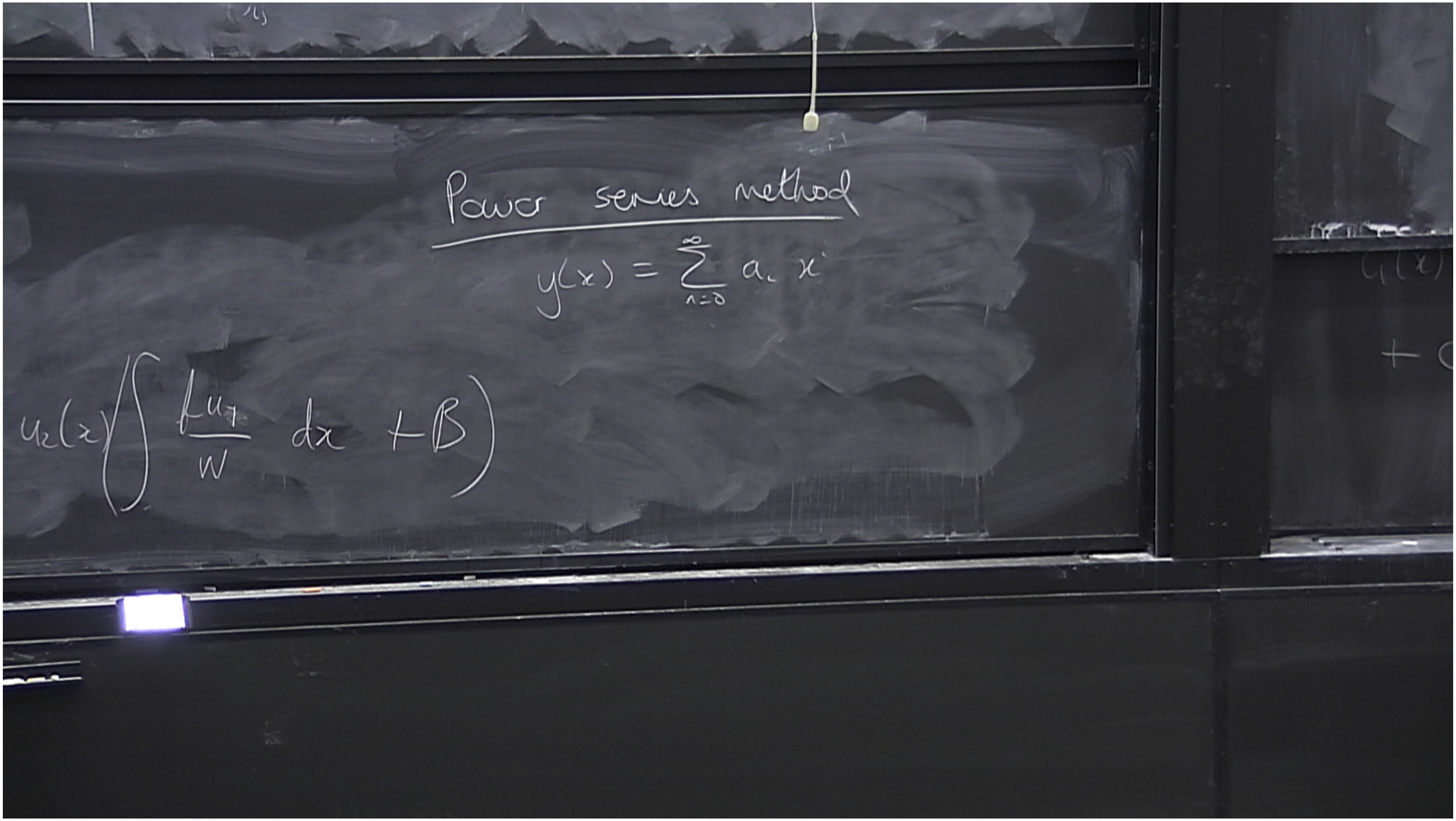

Forced SHM


Forced SHM



$u_2(t) = te^{\lambda t}$
reduction
$+ c\lambda + k = 0$
If $F(t) = F_0 \cos \sqrt{\frac{k}{m}}\, t$, $c=0$
Show resonance


Power series method
$u_2(x)\int \frac{f u_1}{W}\,dx +$


Power series method
$y(x) = \sum_{n=0}^{\infty} a_n x^n$
$u_2(x)\left(\int \frac{f u_1}{W}\, dx + B\right)$