Title: Algebra - Lecture 4

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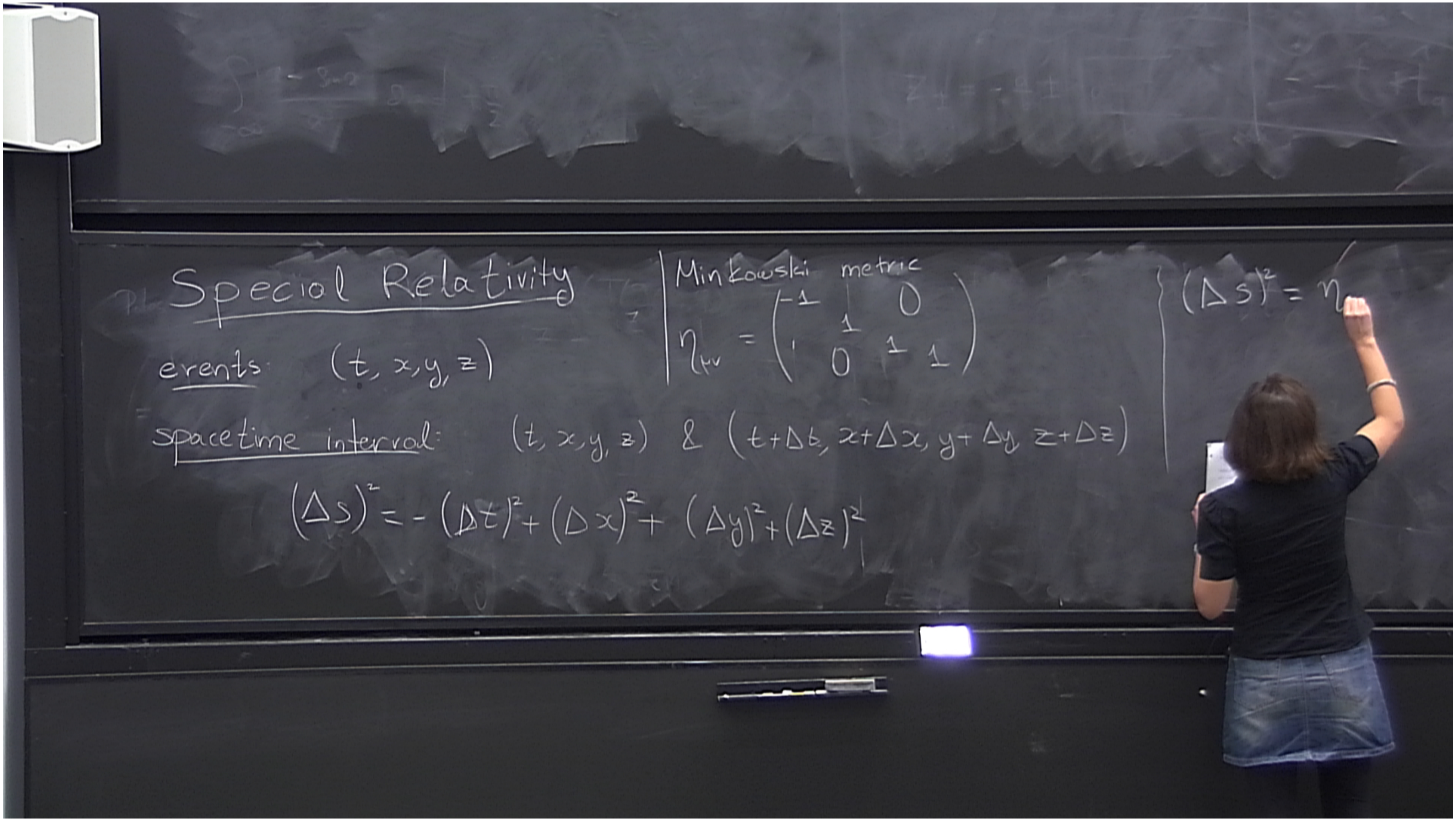
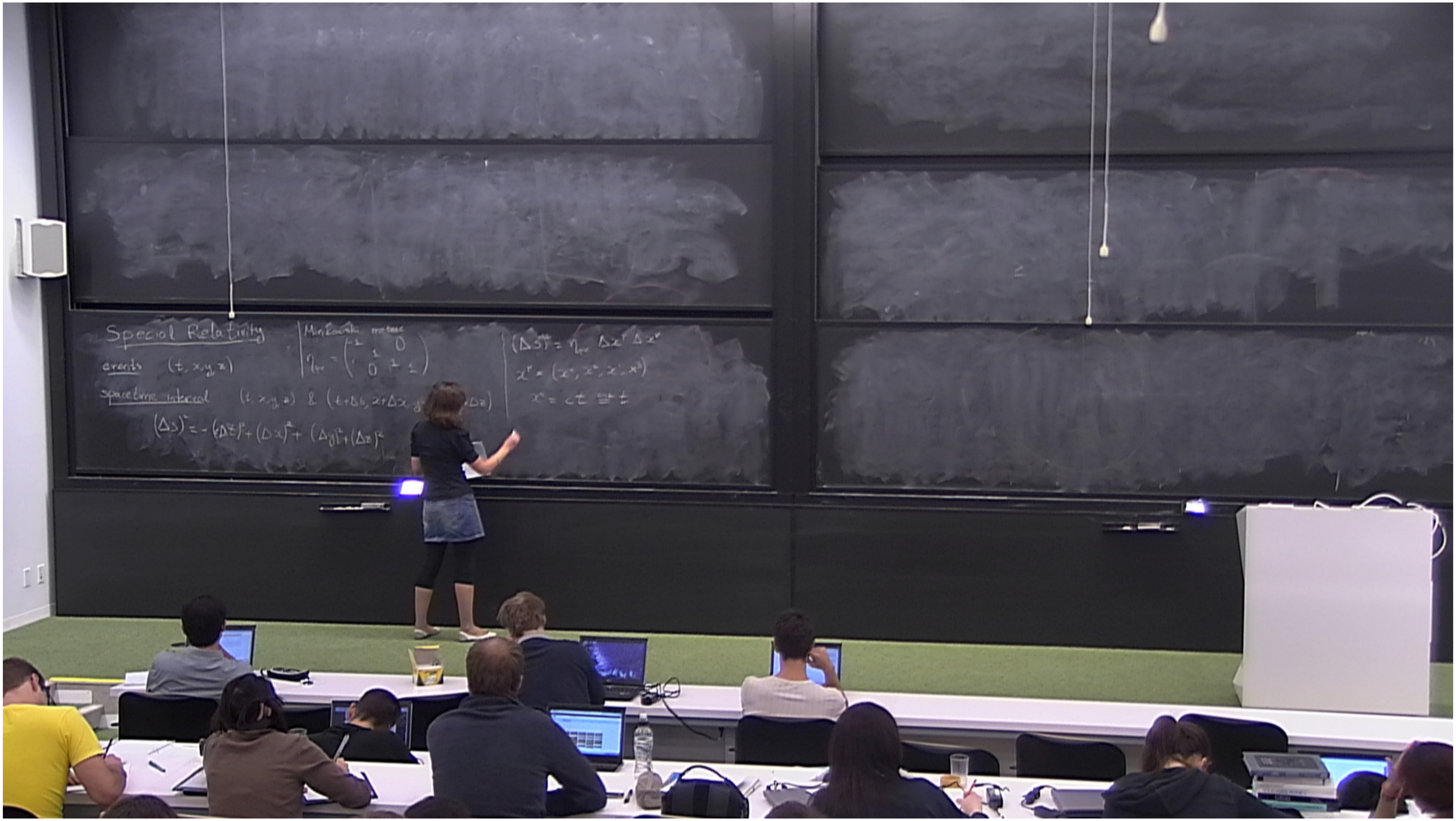
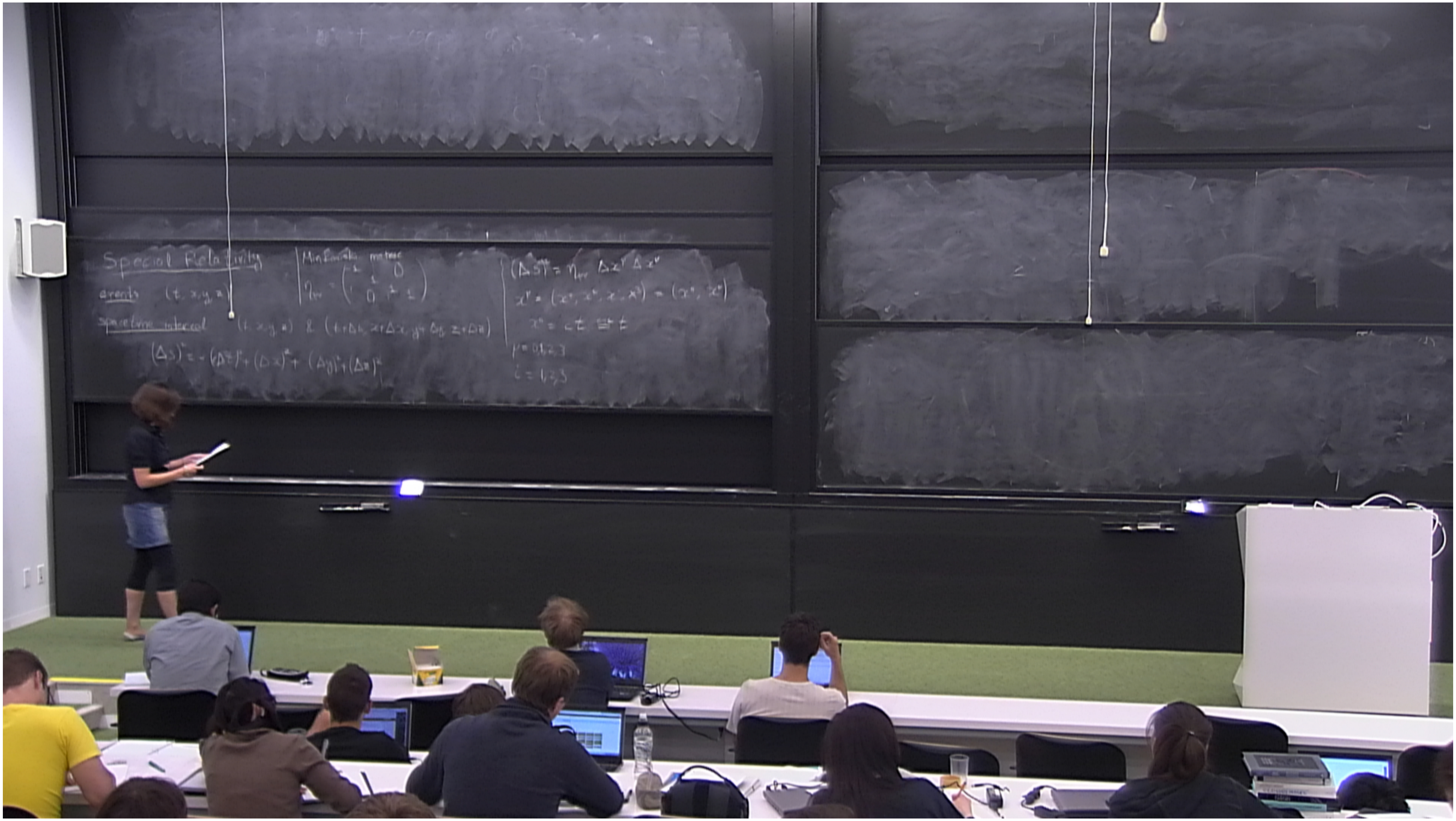
URL: http://pirsa.org/11080127

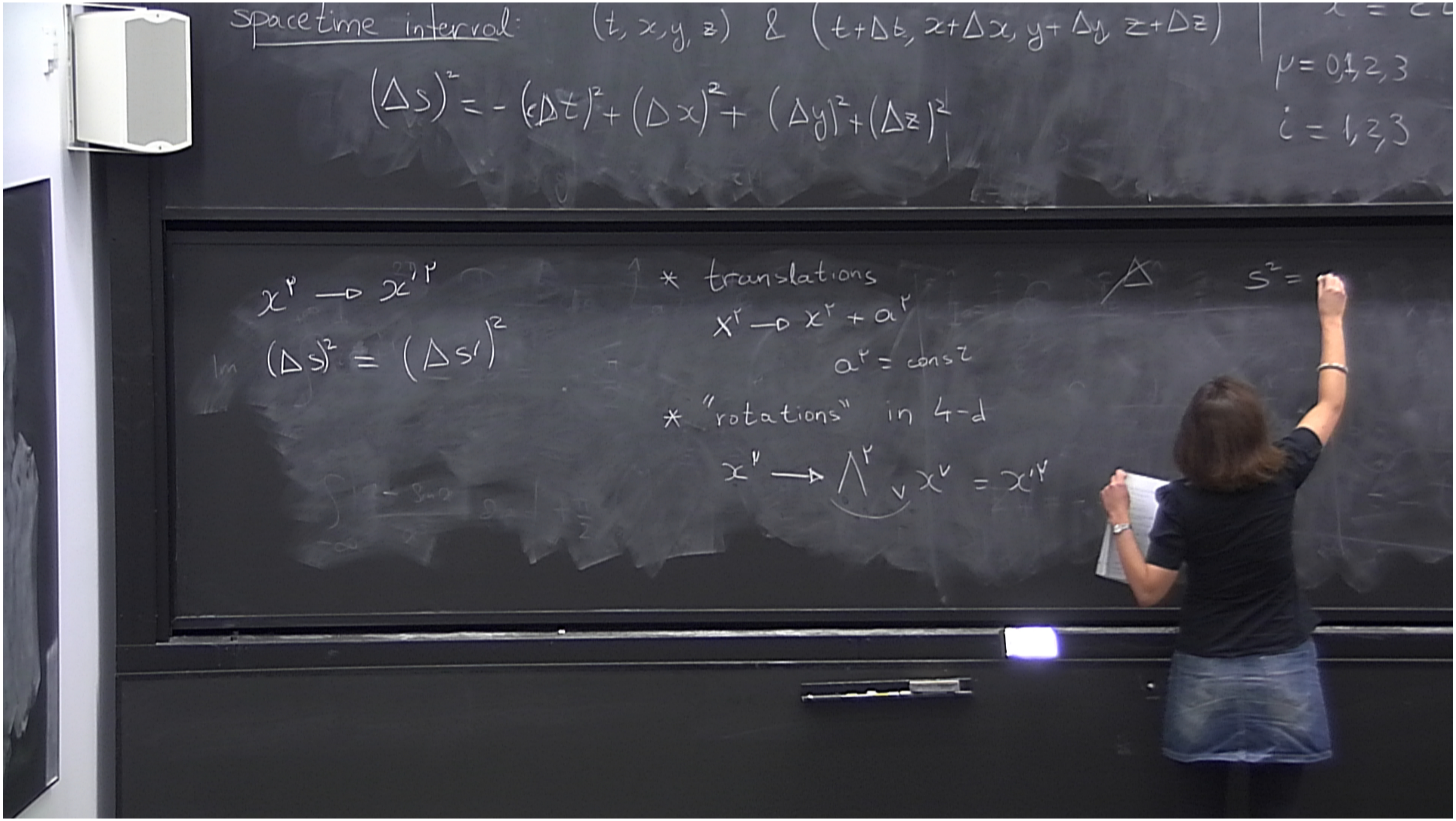
Abstract:

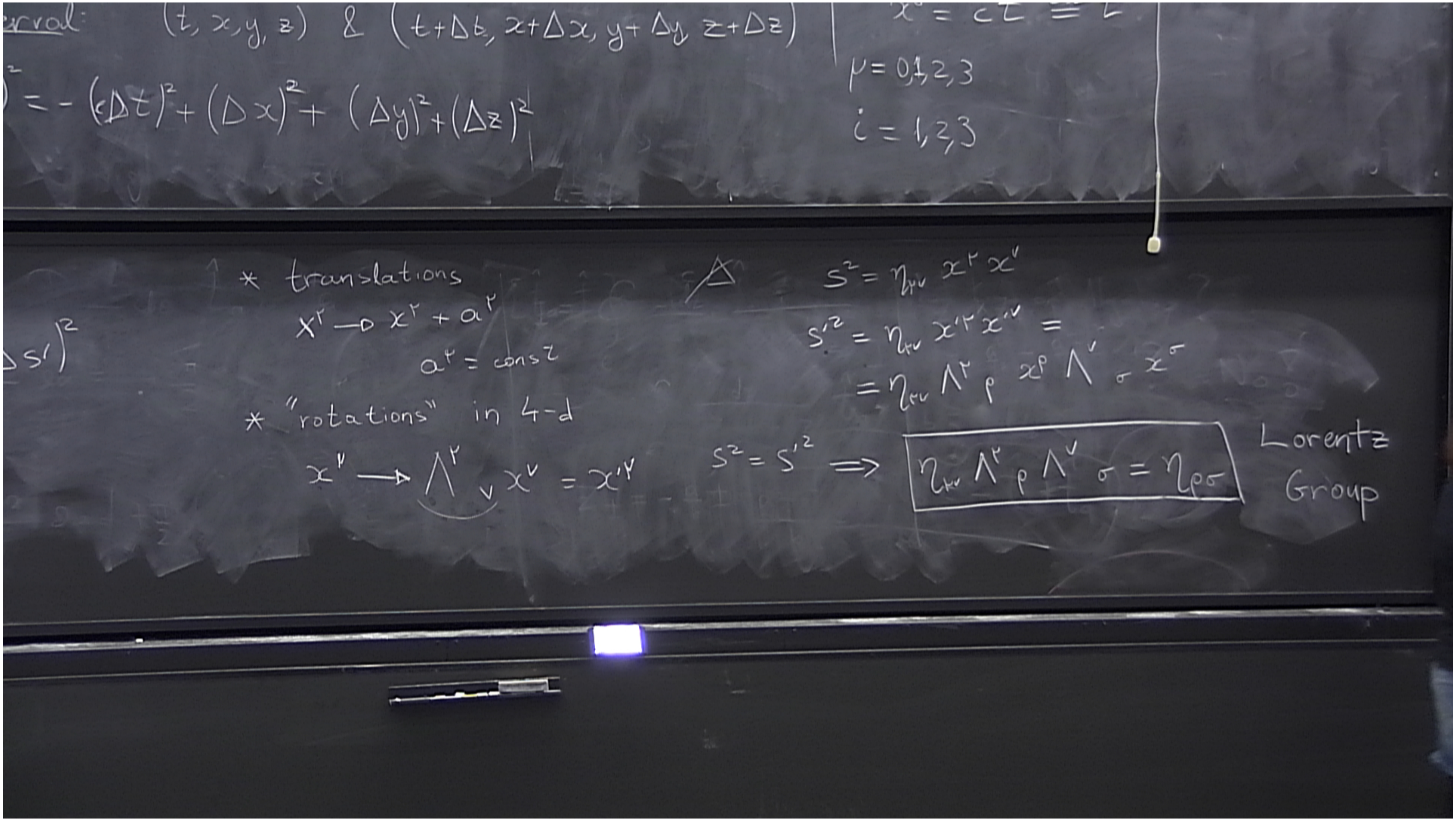

Special Relativity
events (t, x, y, z)
spacetime interval: (t, x, y, z) & (t+Δt, x+Δx, y+Δy, z+Δz)
(Δs)² = -(Δt)² + (Δx)² + (Δy)² + (Δz)²
Minkowski metric
ημν = (-1 0 / 1 / 0 1 1)
(Δs)² = η


Special Relativity
events (t, x, y, z)
spacetime interval (t, x, y, z) & (t+Δt, x+Δx, y
(Δs)² = −(Δt)² + (Δx)² + (Δy)² + (Δz)²
Minkowski metric
(Δs)² = η_μν Δx^μ Δx^ν
x^μ = (x⁰, x¹, x², x³)
x⁰ = ct = t

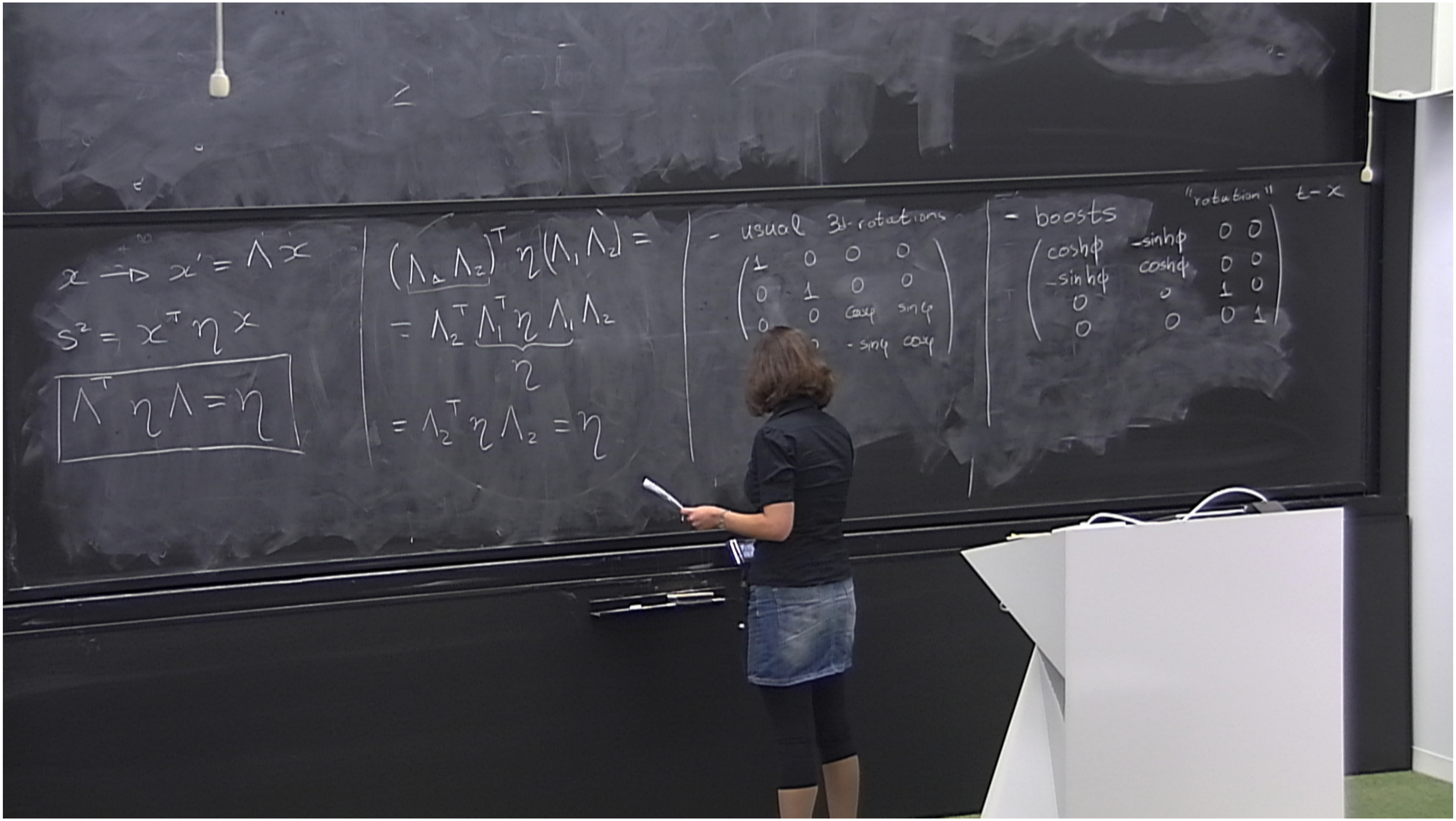


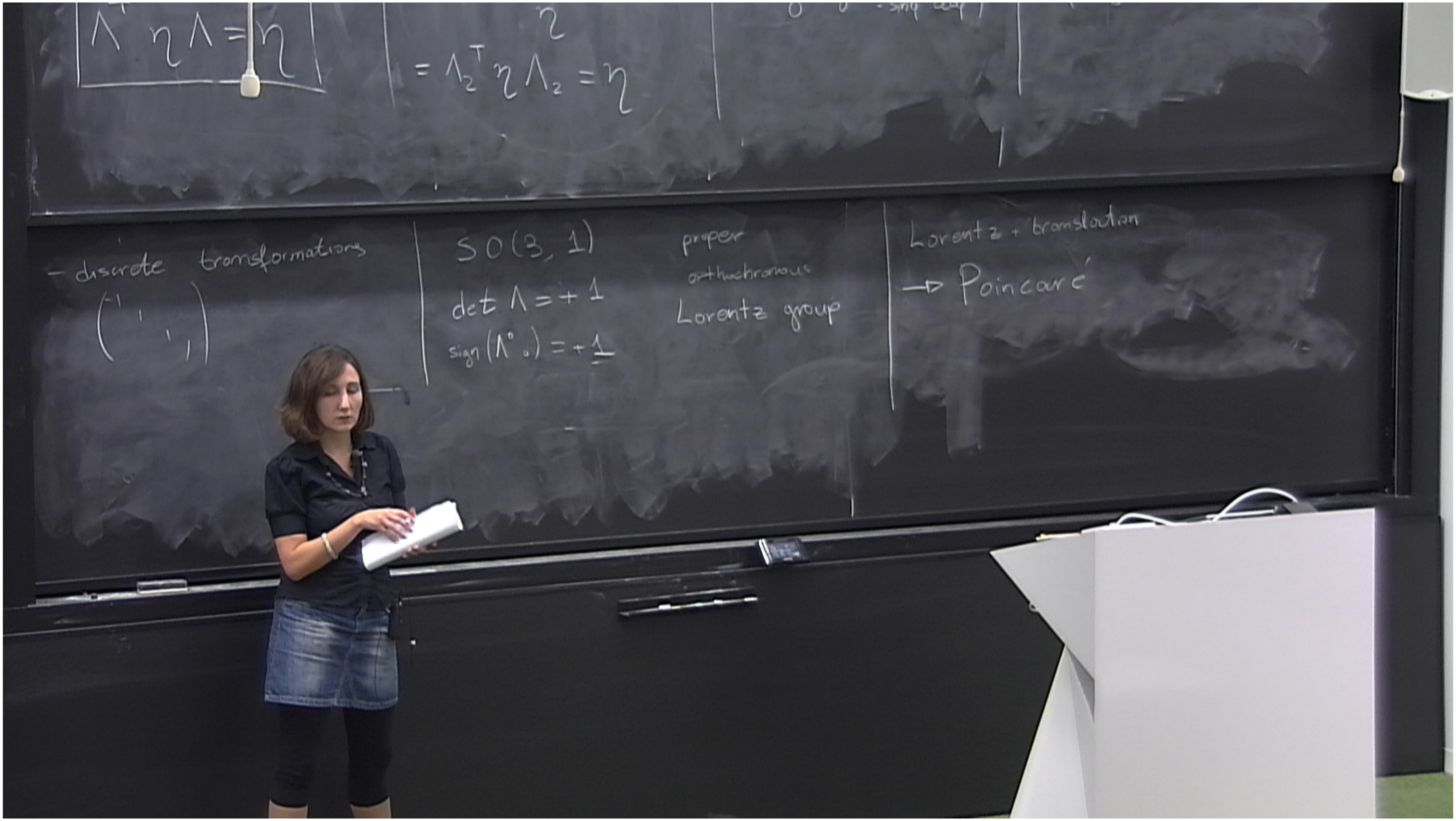
Special Relativity
events (t, x, y, z)
spacetime interval
μ = 0,1,2,3
i = 1,2,3

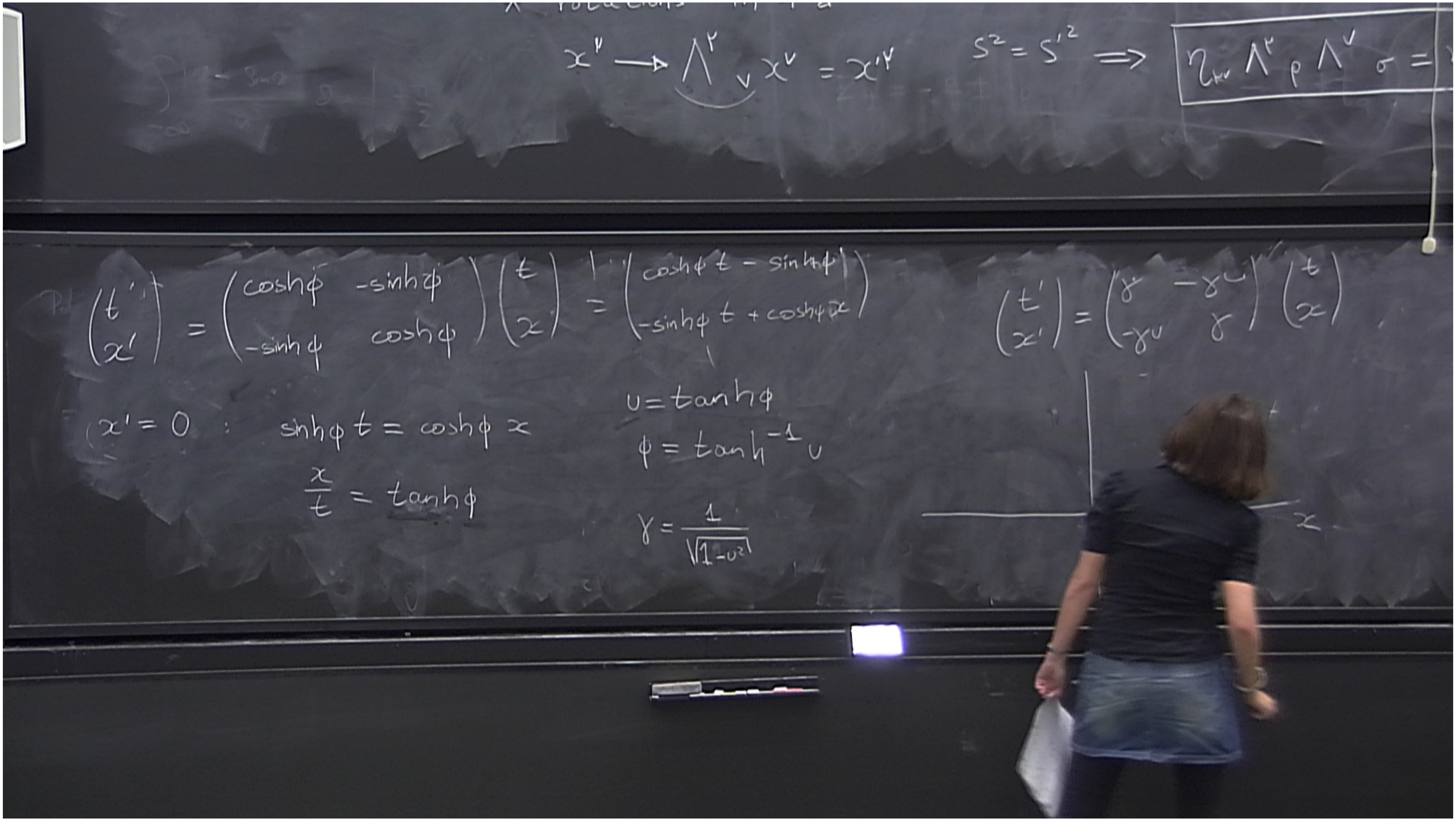

spacetime interval: (t, x, y, z) & (t+Δt, x+Δx, y+Δy, z+Δz)
(Δs)² = -(cΔt)² + (Δx)² + (Δy)² + (Δz)²
μ = 0,1,2,3
i = 1,2,3
x^μ → x'^μ
(Δs)² = (Δs')²
* translations
x^μ → x^μ + a^μ
a^μ = const
* "rotations" in 4-d
x^μ → Λ^μ_ν x^ν = x'^μ
s² =

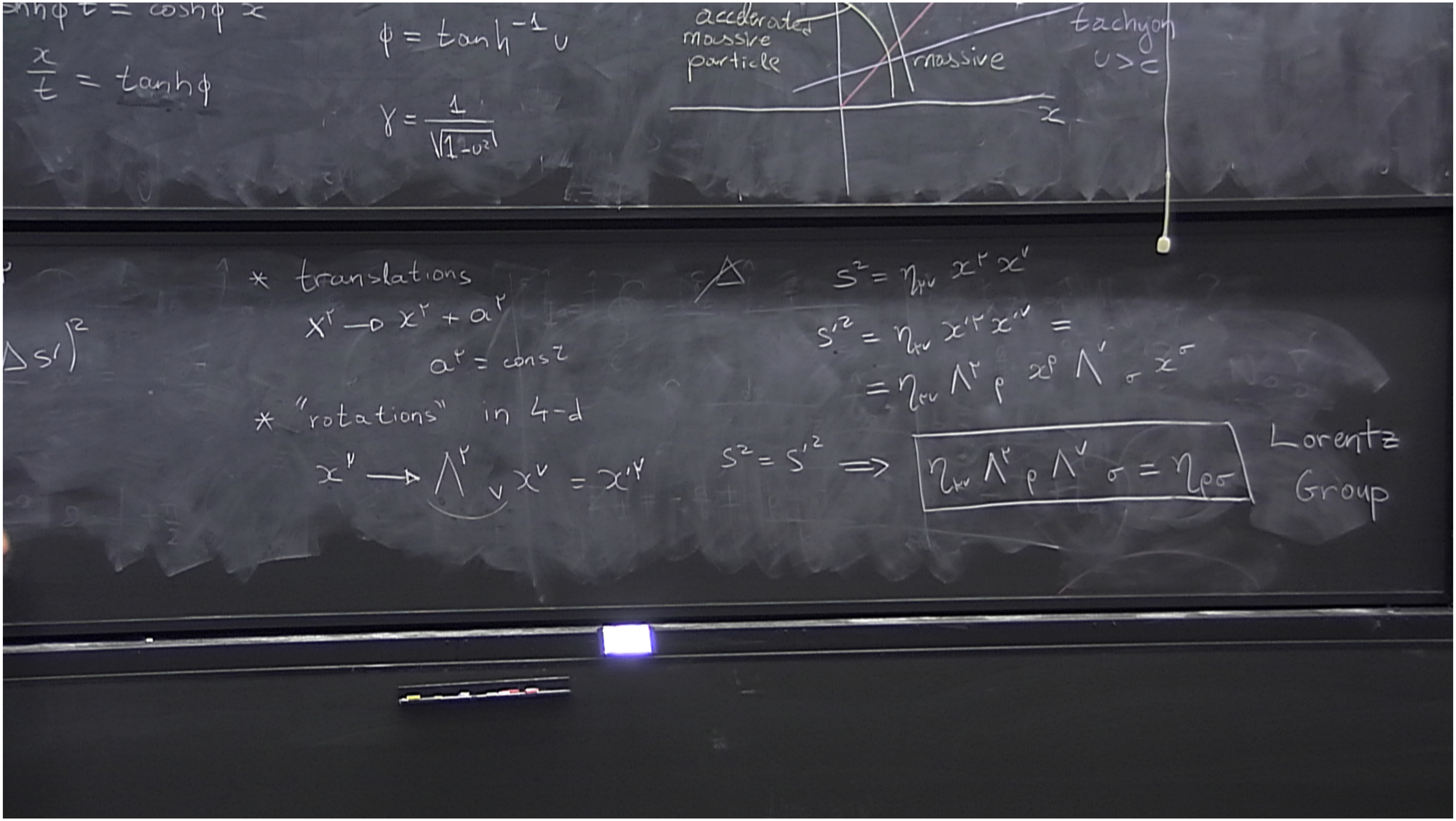


ᴇrval: (t, x, y, z) & $(t+\Delta t, x+\Delta x, y+\Delta y, z+\Delta z)$
$= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$
$\mu = 0,1,2,3$
$i = 1,2,3$
* translations
$x^\mu \to x^\mu + a^\mu$
$a^\mu = const$
* "rotations" in 4-d
$x^\mu \to \Lambda^\mu{}_\nu x^\nu = x'^\mu$
$s^2 = \eta_{\mu\nu} x^\mu x^\nu$
$s'^2 = \eta_{\mu\nu} x'^\mu x'^\nu =$
$= \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma$
$s^2 = s'^2 \Rightarrow$ $\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$
Lorentz Group

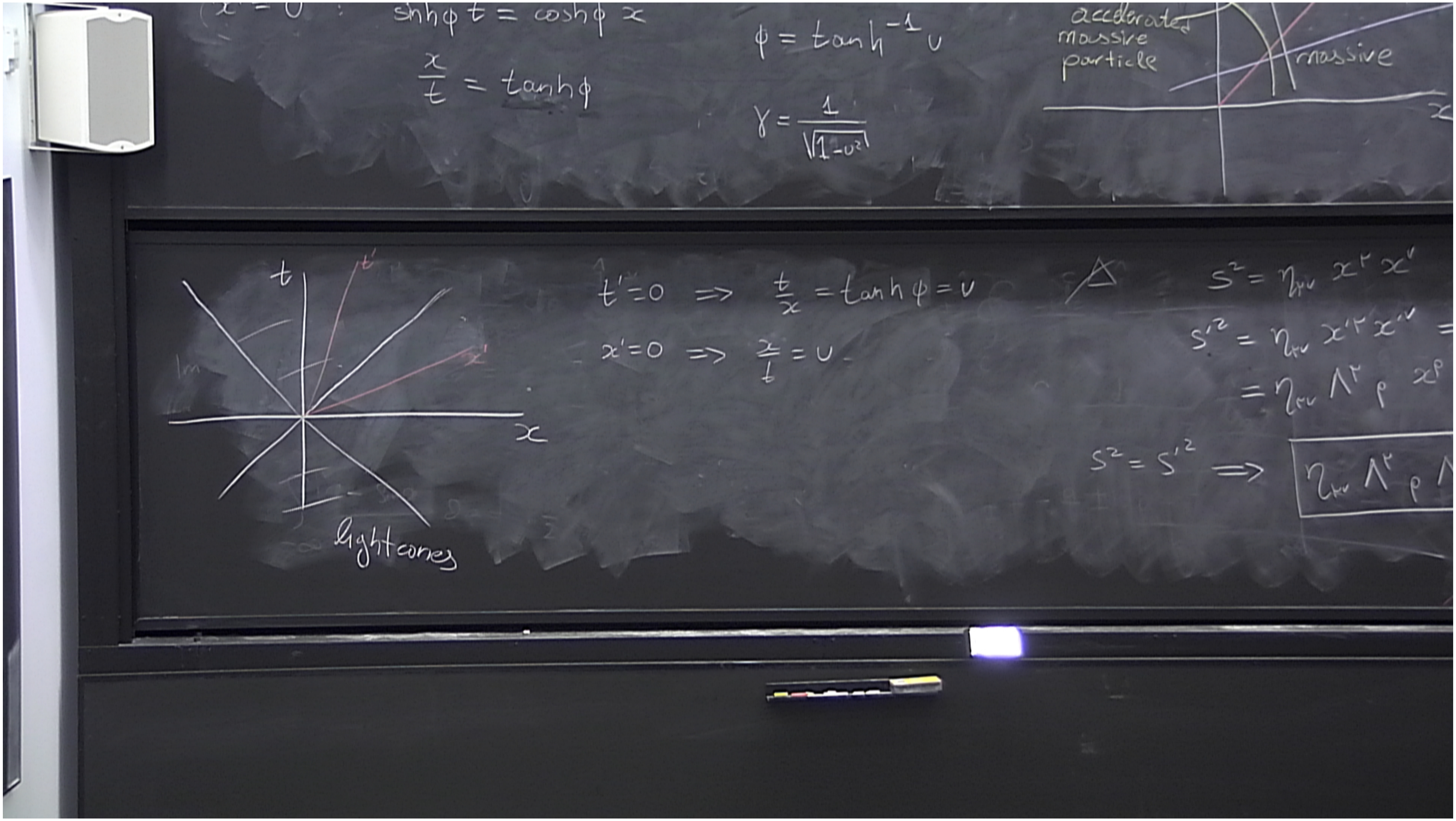

x → x' = Λx
s² = xᵀ η x
Λᵀ η Λ = η
(Λ₁Λ₂)ᵀ η (Λ₁Λ₂) =
= Λ₂ᵀ Λ₁ᵀ η Λ₁ Λ₂
η
= Λ₂ᵀ η Λ₂ = η
- usual 3d-rotations
- boosts
"rotation" t-x
cosh φ -sinh φ 0 0
-sinh φ cosh φ 0 0

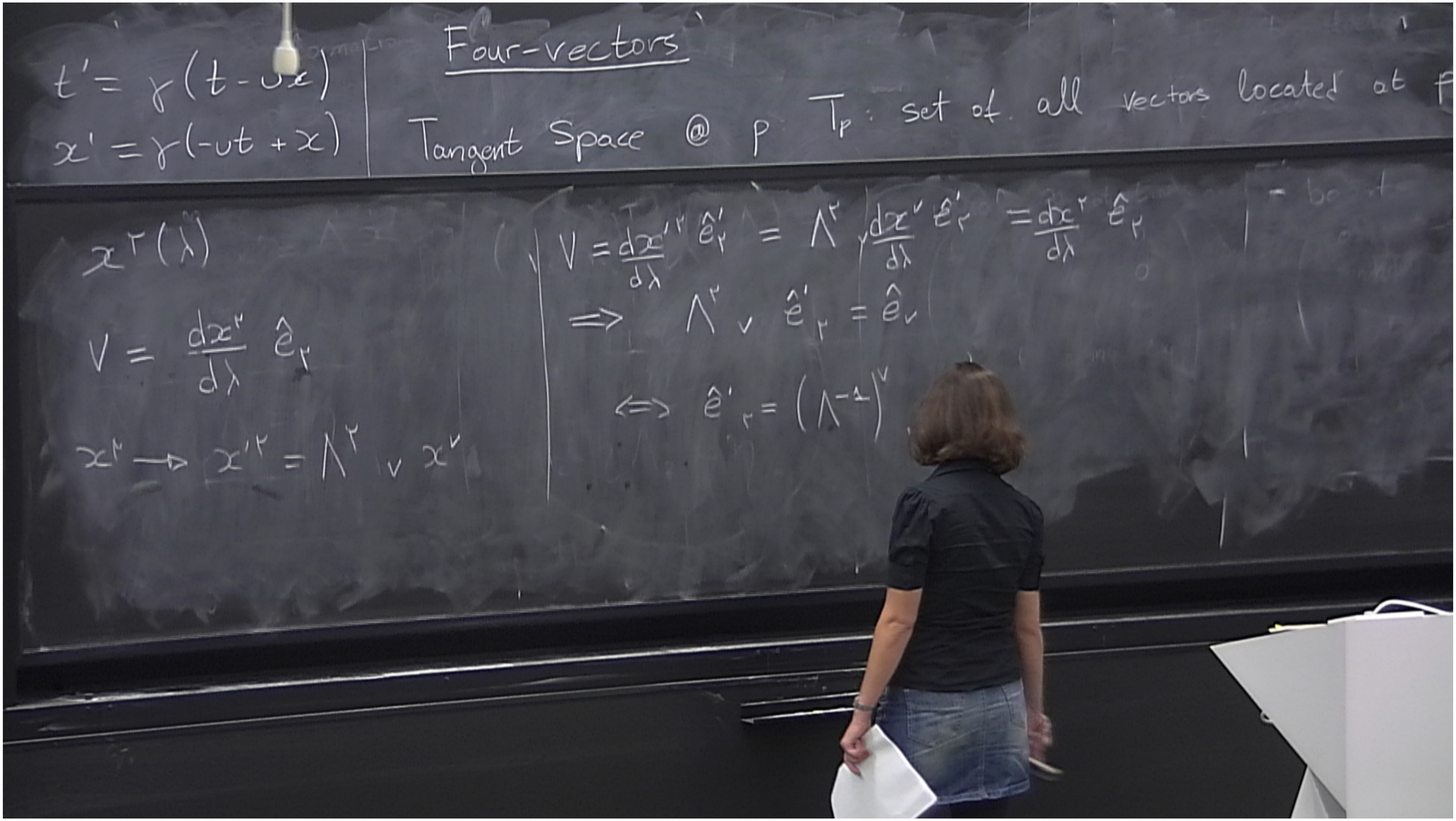

ΛT η Λ = η
= Λ2T η Λ2 = η
- discrete transformations
SO(3, 1)
det Λ = +1
sign(Λ0 0) = +1
proper
orthochronous
Lorentz group
Lorentz + translation
→ Poincaré

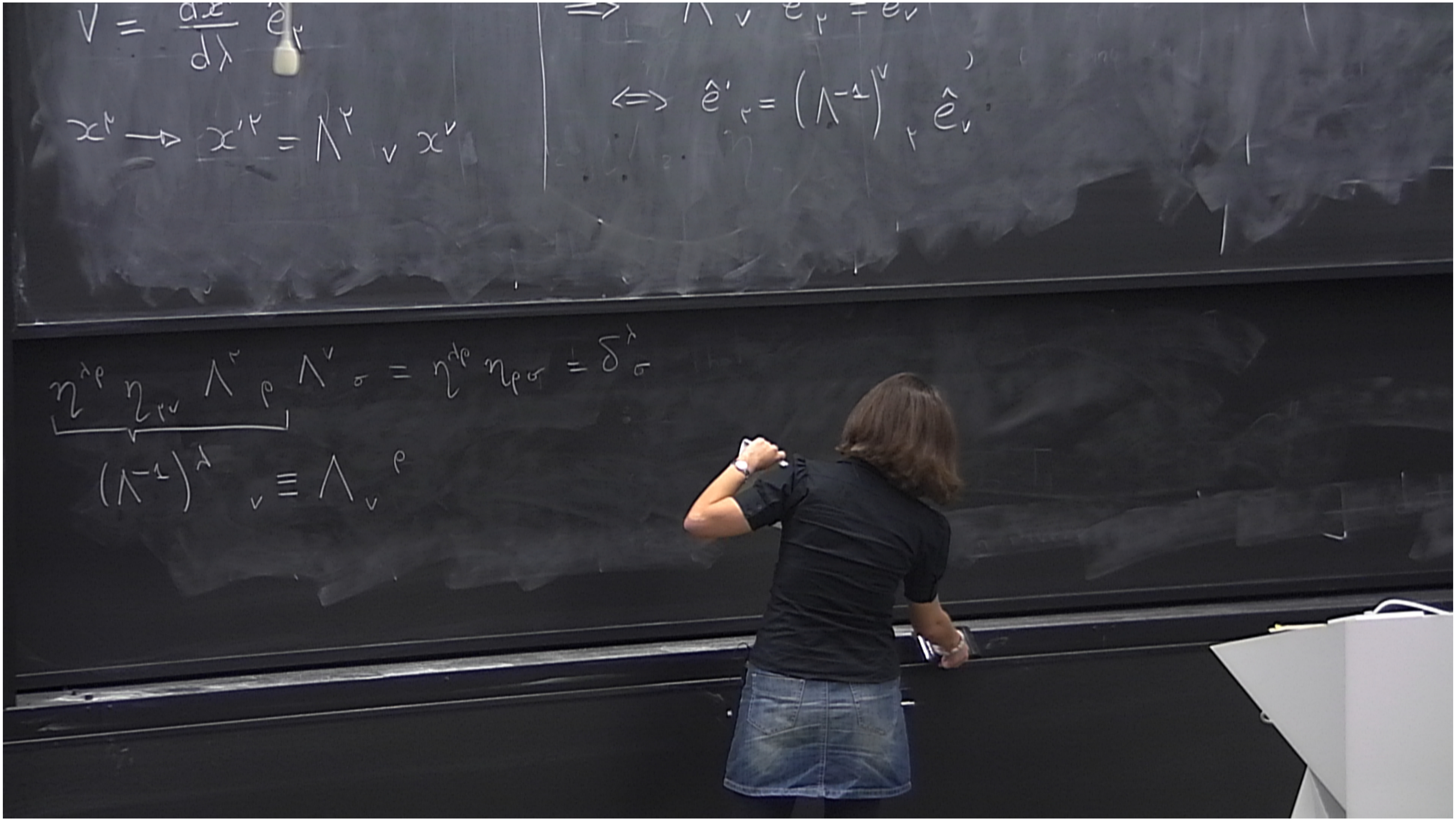

x^μ → Λ^μ_ν x^ν = x'^μ
s² = s'² ⟹ η_μν Λ^μ_ρ Λ^ν_σ =
(t' x') = (coshφ −sinhφ; −sinhφ coshφ)(t x) = (coshφ t − sinhφ x; −sinhφ t + coshφ x)
(t' x') = (γ −γv; −γv γ)(t x)
x' = 0 : sinhφ t = coshφ x
x/t = tanhφ
v = tanhφ
φ = tanh⁻¹ v
γ = 1/√(1−v²)
x

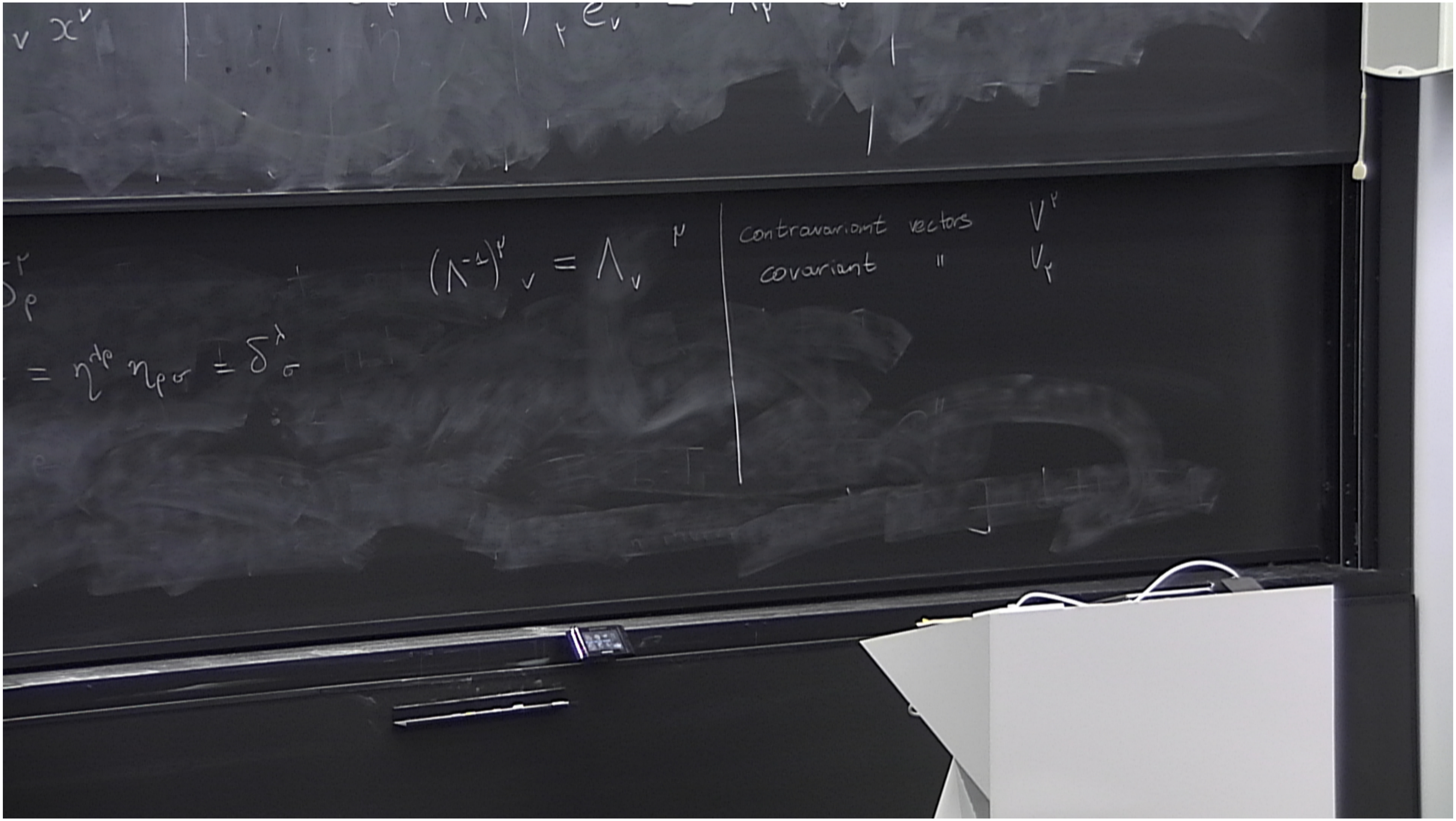


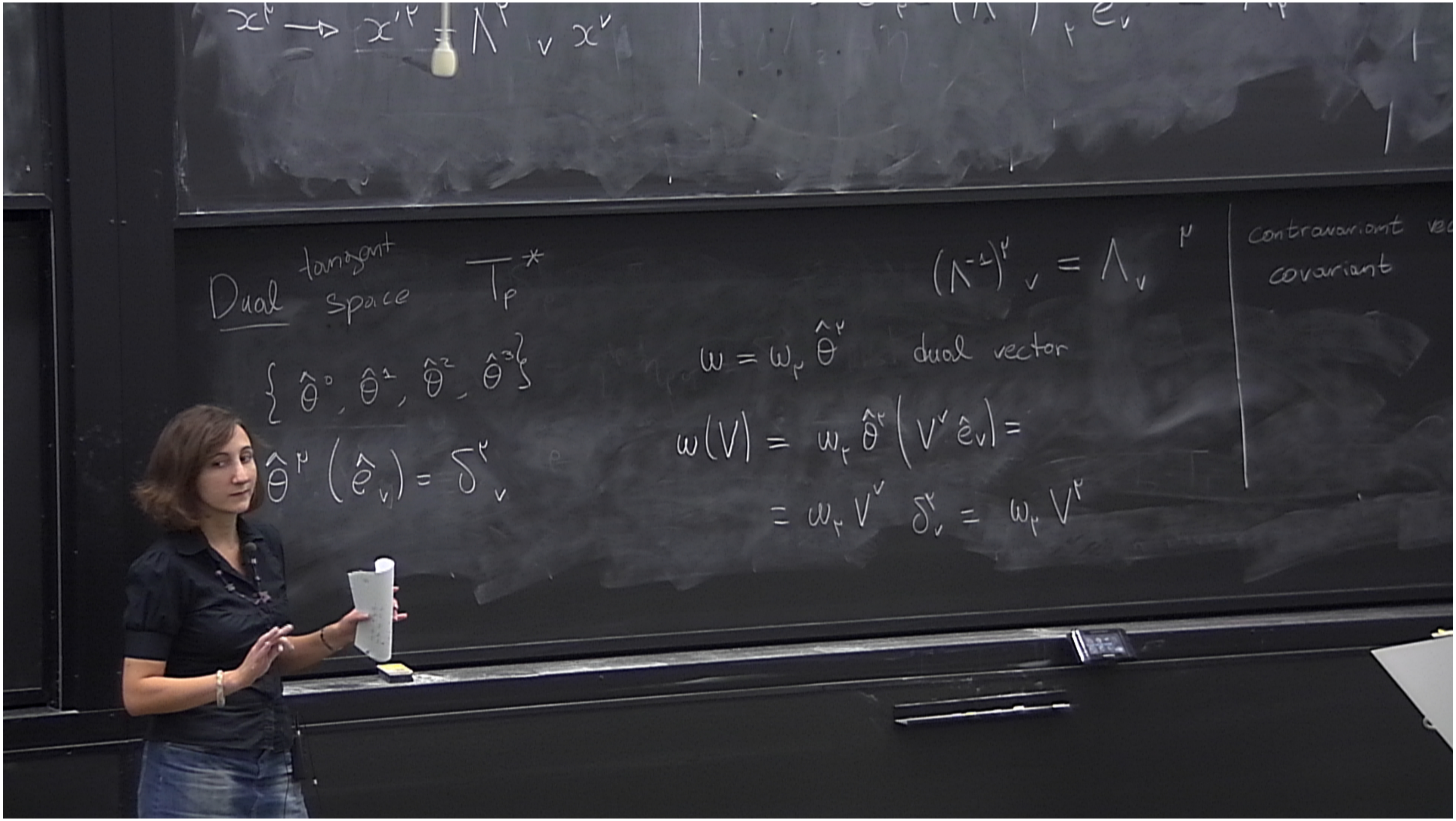
$\phi = \tanh^{-1} v$
$\frac{x}{t} = \tanh\phi$
$\gamma = \frac{1}{\sqrt{1-v^2}}$
accelerated massive particle
massive
tachyon
$v > c$
x
* translations
$x^\mu \to x^\mu + a^\mu$
$a^\mu = const$
* "rotations" in 4-d
$x^\mu \to \Lambda^\mu{}_\nu x^\nu = x'^\mu$
$s^2 = \eta_{\mu\nu} x^\mu x^\nu$
$s'^2 = \eta_{\mu\nu} x'^\mu x'^\nu =$
$= \eta_{\mu\nu} \Lambda^\mu{}_\rho x^\rho \Lambda^\nu{}_\sigma x^\sigma$
$s^2 = s'^2 \Rightarrow \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$
Lorentz Group

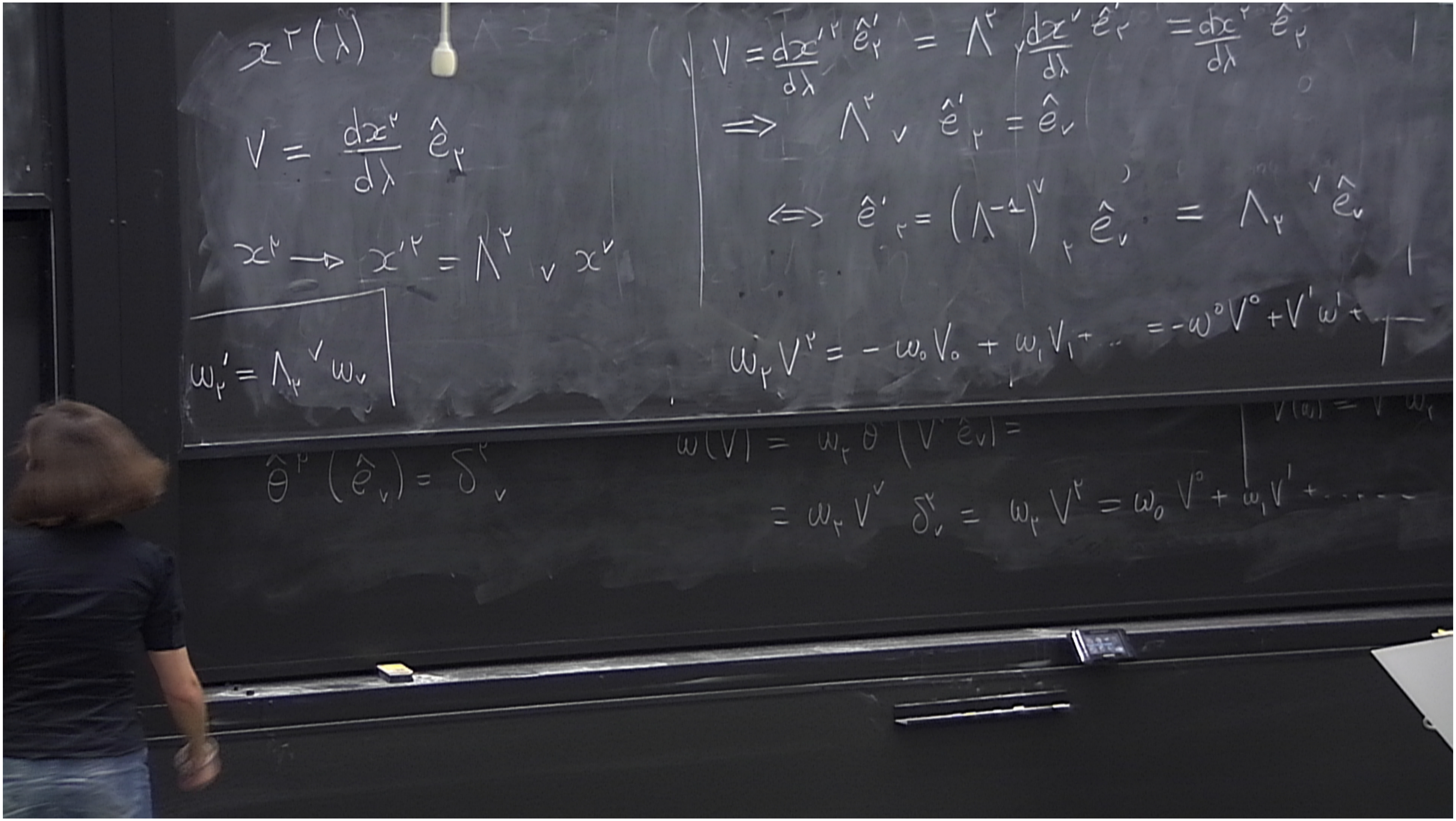

sinh φ t = cosh φ x
x/t = tanh φ
φ = tanh^-1 v
γ = 1/√(1-v²)
accelerated massive particle
massive
x
t
t'
x'
x
lightcones
t'=0 => t/x = tanh φ = v
x'=0 => x/t = v
s² = η_μν x^μ x^ν
s'² = η_μν x'^μ x'^ν =
= η_μν Λ^μ_ρ x^ρ
s² = s'² => η_μν Λ^μ_ρ Λ

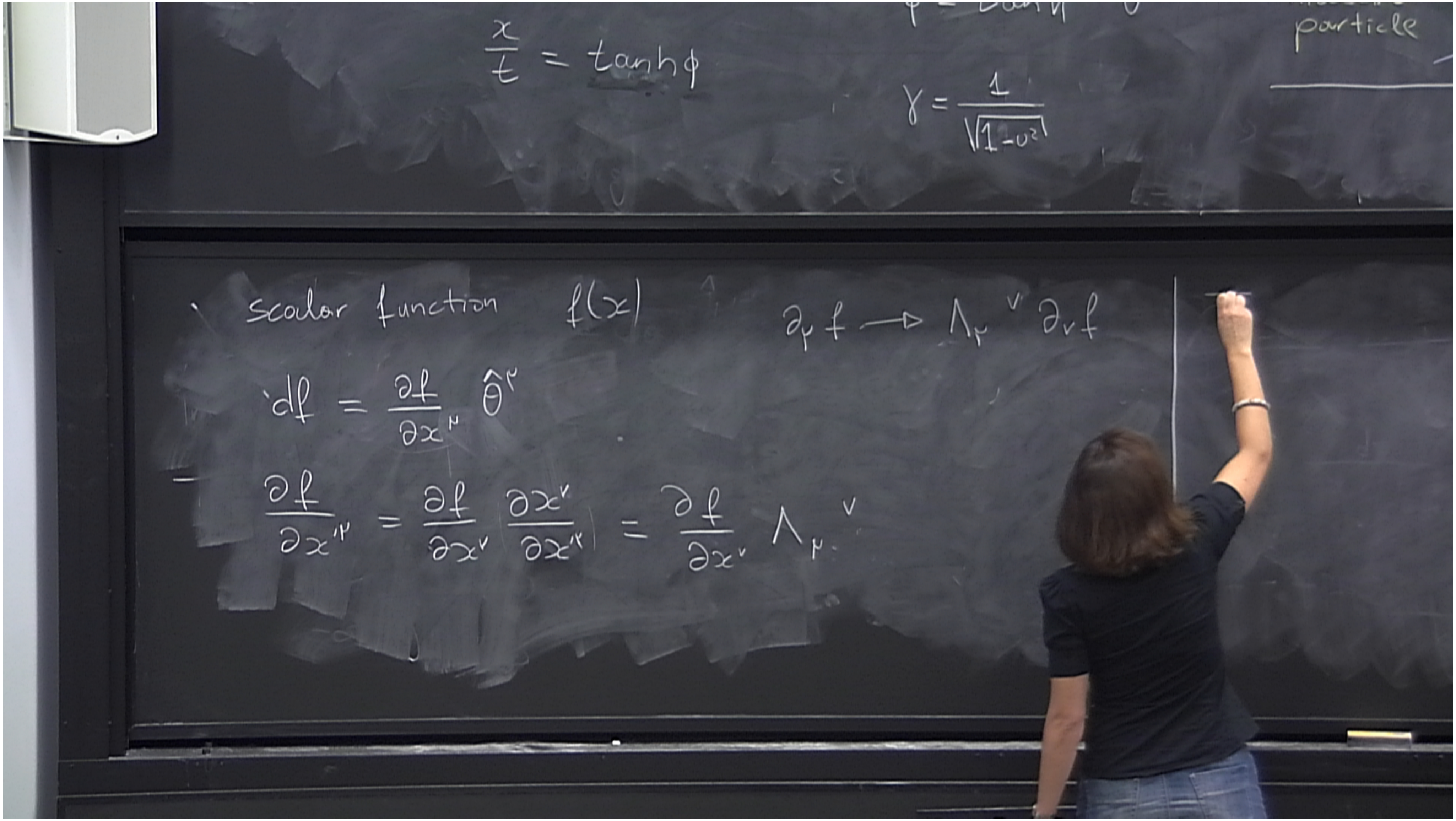

Four-vectors
t' = γ(t - vx)
x' = γ(-vt + x)
Tangent Space @ p   T_p : set of all vectors located at p
x^μ(λ)
V = dx^μ/dλ ê_μ
x^μ → x'^μ = Λ^μ_ν x^ν
V = dx'^μ/dλ ê'_μ = Λ^μ_ν dx^ν/dλ ê'_μ = dx^μ/dλ ê_μ
⟹ Λ^μ_ν ê'_μ = ê_ν
⟺ ê'_μ = (Λ^{-1})^ν

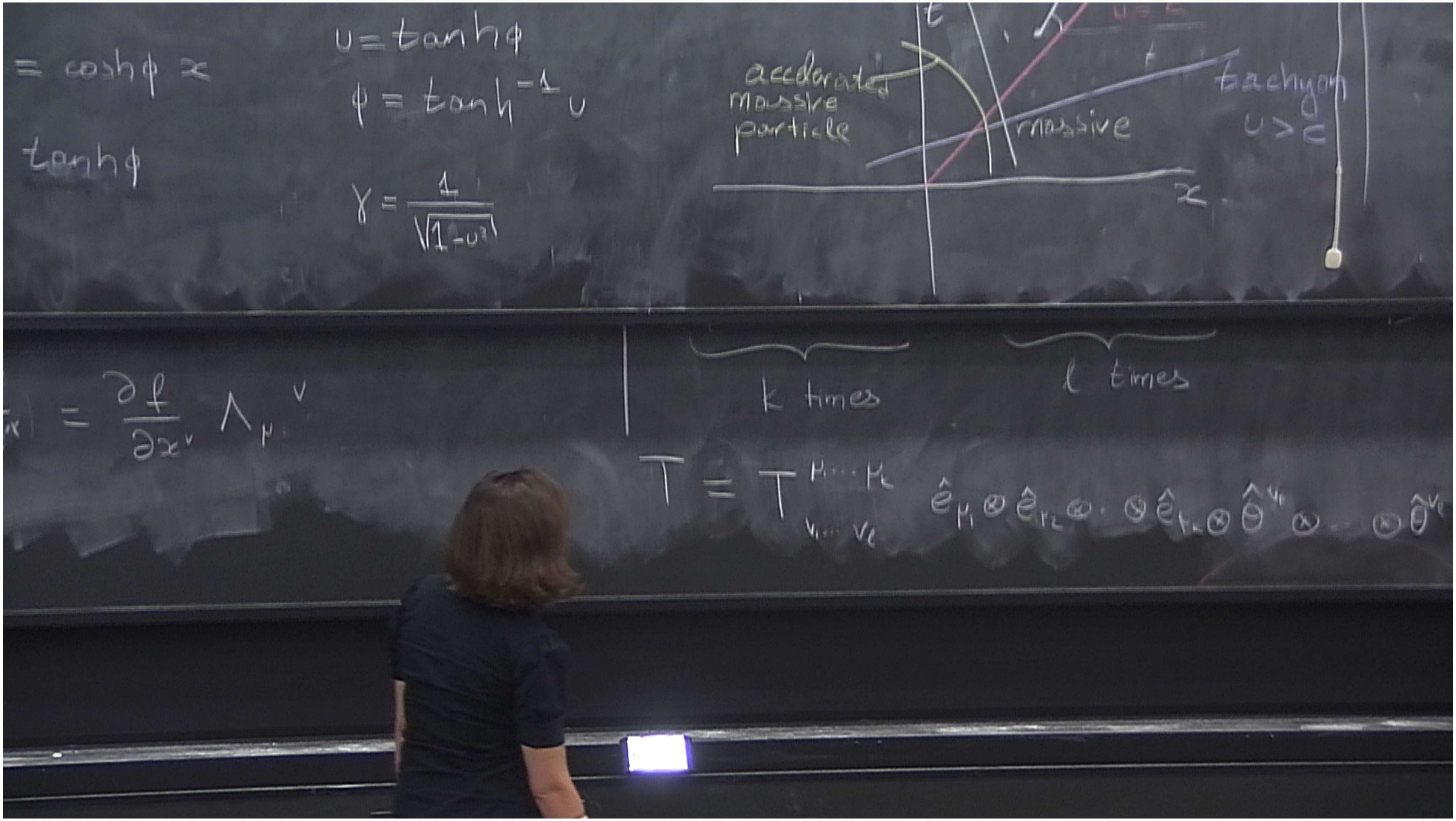

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$
$\Leftrightarrow \hat{e}'_\mu = (\Lambda^{-1})^\nu{}_\mu \hat{e}_\nu$
$\eta^{\lambda\rho} \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta^{\lambda\rho} \eta_{\rho\sigma} = \delta^\lambda{}_\sigma$
$(\Lambda^{-1})^\lambda{}_\nu \equiv \Lambda_\nu{}^\rho$

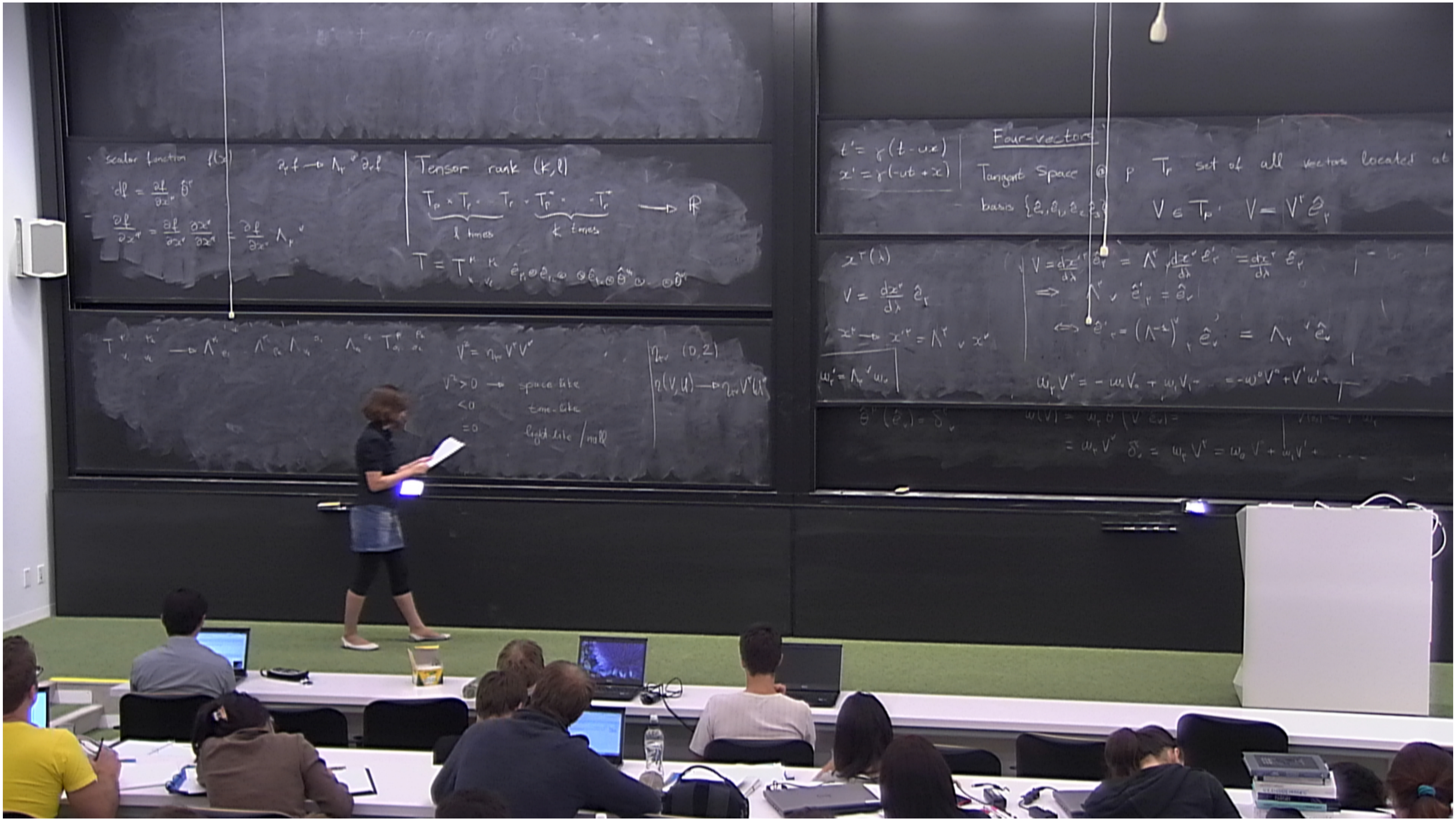

$(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}$
contravariant vectors V^{μ}
covariant " V_{μ}
$= \eta^{\lambda\rho}\eta_{\rho\sigma} = \delta^{\lambda}{}_{\sigma}$

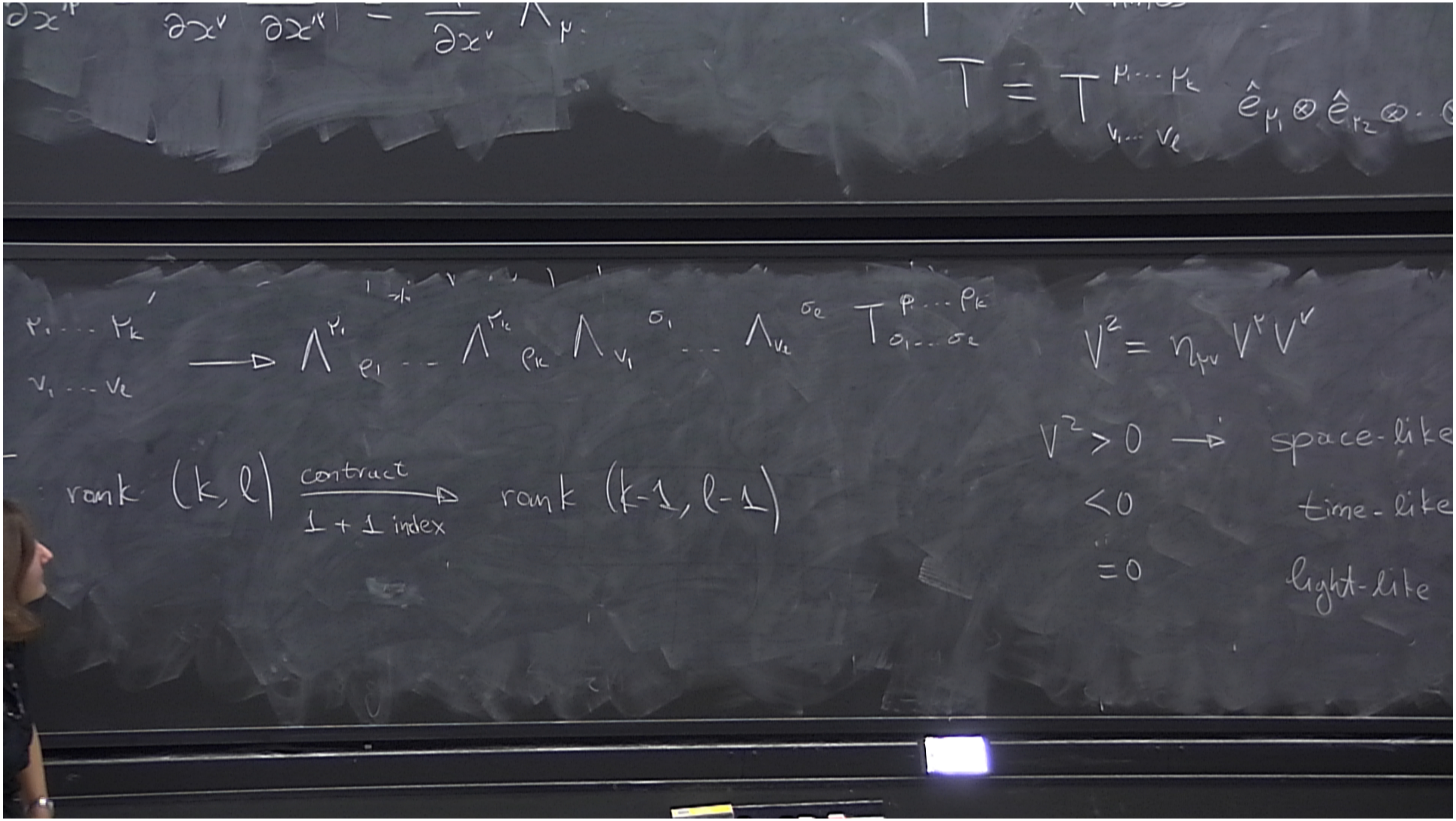

x^μ → x'^μ = Λ^μ_ν x^ν
Dual tangent space T_p^*
{θ̂^0, θ̂^1, θ̂^2, θ̂^3}
θ̂^μ (ê_ν) = δ^μ_ν
(Λ^{-1})^μ_ν = Λ_ν^μ
ω = ω_μ θ̂^μ dual vector
ω(V) = ω_μ θ̂^μ (V^ν ê_ν) =
= ω_μ V^ν δ^μ_ν = ω_μ V^μ
contravariant ve
covariant


$x^\mu(\lambda)$
$V = \frac{dx^\mu}{d\lambda} \hat{e}_\mu$
$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$
$\omega'_\mu = \Lambda_\mu{}^\nu \omega_\nu$
$V = \frac{dx'^\mu}{d\lambda} \hat{e}'_\mu = \Lambda^\mu{}_\nu \frac{dx^\nu}{d\lambda} \hat{e}'_\mu = \frac{dx^\mu}{d\lambda} \hat{e}_\mu$
$\Longrightarrow \Lambda^\mu{}_\nu \hat{e}'_\mu = \hat{e}_\nu$
$\Longleftrightarrow \hat{e}'_\mu = (\Lambda^{-1})^\nu{}_\mu \hat{e}_\nu = \Lambda_\mu{}^\nu \hat{e}_\nu$
$\omega_\mu V^\mu = -\omega_0 V_0 + \omega_1 V_1 + \dots = -\omega^0 V^0 + V^1 \omega^1 + \dots$
$\hat{\theta}^\mu(\hat{e}_\nu) = \delta^\mu{}_\nu$
$\omega(V) = \omega_\mu \theta^\mu(V^\nu \hat{e}_\nu) =$
$= \omega_\mu V^\nu \delta^\mu_\nu = \omega_\mu V^\mu = \omega_0 V^0 + \omega_1 V^1 + \dots$


x/t = tanhφ
particle
γ = 1/√(1-v²)
scalar function f(x)
∂μ f → Λμ ν ∂ν f
df = ∂f/∂x^μ θ̂^μ
∂f/∂x'^μ = ∂f/∂x^ν ∂x^ν/∂x'^μ = ∂f/∂x^ν Λμ ν


= cosh φ x
v = tanh φ
φ = tanh⁻¹ v
tanh φ
γ = 1/√(1-v²)
accelerated
massive
particle
massive
tachyon
v > c
t
x
k times
l times
T = T^{μ1…μk}_{ν1…νl} ê_{μ1} ⊗ ê_{μ2} ⊗ … ⊗ ê_{μk} ⊗ θ̂^{ν1} ⊗ … ⊗ θ̂^{νl}


scalar function f(x)
Tensor rank (k,l)
k times
Four-vectors
Tangent Space @ p
set of all vectors located at
basis
space-like
time-like
light-like / null


T = T^{μ1…μk}_{ν1…νℓ} ê_{μ1} ⊗ ê_{μ2} ⊗ …
V² = η_{μν} V^μ V^ν
V² > 0 → space-like
< 0 time-like
= 0 light-like
rank (k, ℓ) contract 1 + 1 index → rank (k-1, ℓ-1)