Title: Complex Analysis - Lecture 4

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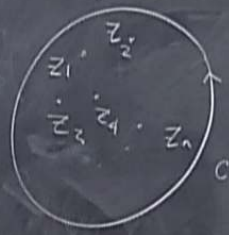
URL: http://pirsa.org/11080126

Abstract:

Recall The Residue theorem

$f(z)$ singularities $z_1, \dots, z_n$



$\cdot z_{n+1}$
$\cdot z_{n+2}$

$$\oint_C f(z)dz = 2\pi i \sum_{i=1}^{n} \text{Res } f(z) \text{ at } z = z_i$$

1. $I = \int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta$, $a \neq b \in \mathbb{R}$ $a > b > 0$

$z = e^{i\theta}$

$d\theta = -i\frac{dz}{z}$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$= \frac{z + 1/z}{2}$

$I = \oint_C \frac{-i\,dz}{z} \frac{1}{a + b\frac{1+z^2}{2z}}$

$= -2i \oint_C \frac{dz}{2az + b + bz^2}$

$z_\pm = -\frac{a}{b} \pm \sqrt{\frac{a^2 - b^2}{b}}$

$z_- \times z_+$

$= \frac{a^2}{b^2} - \frac{a^2 - b^2}{b^2}$

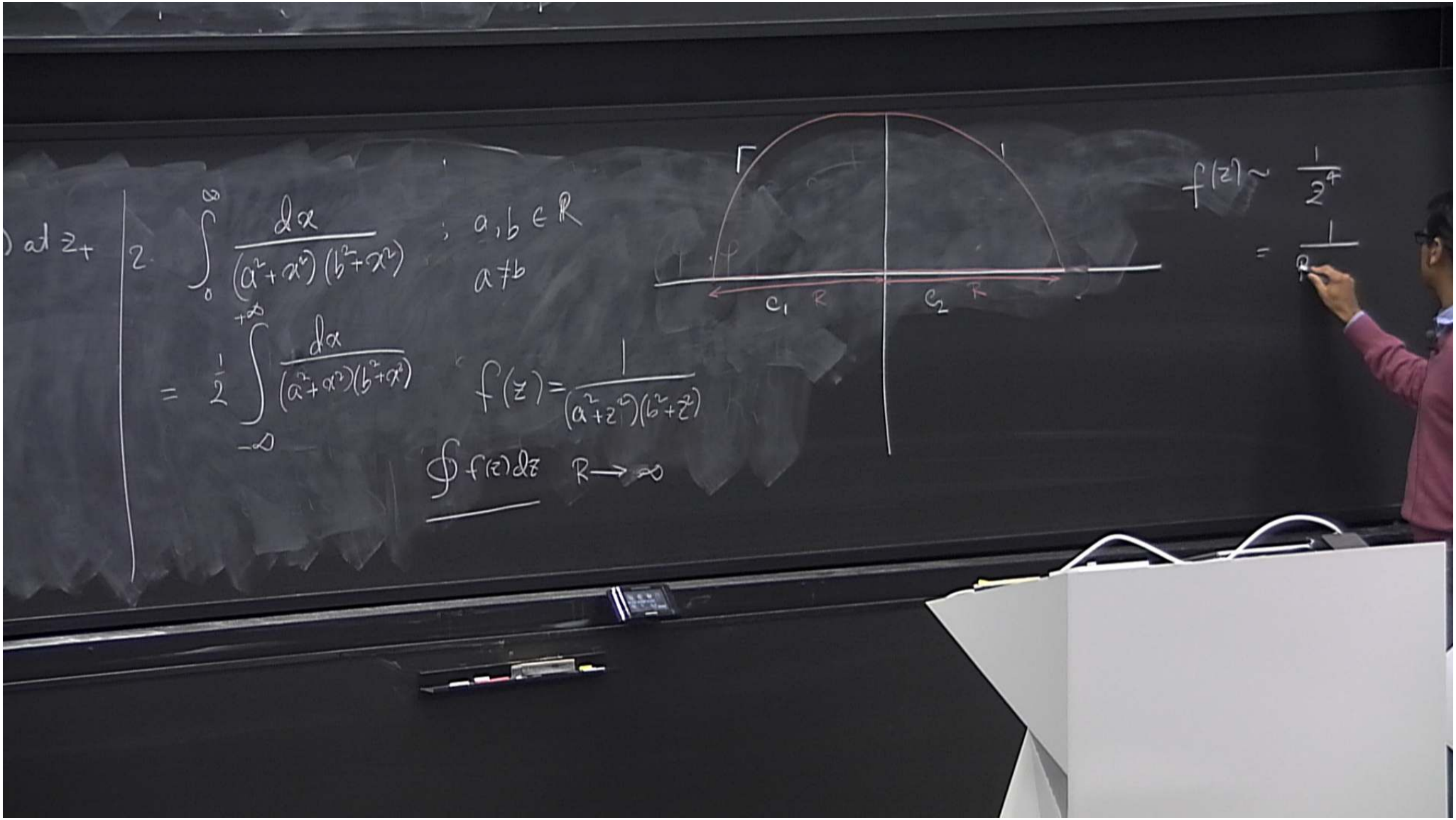
$= 1$

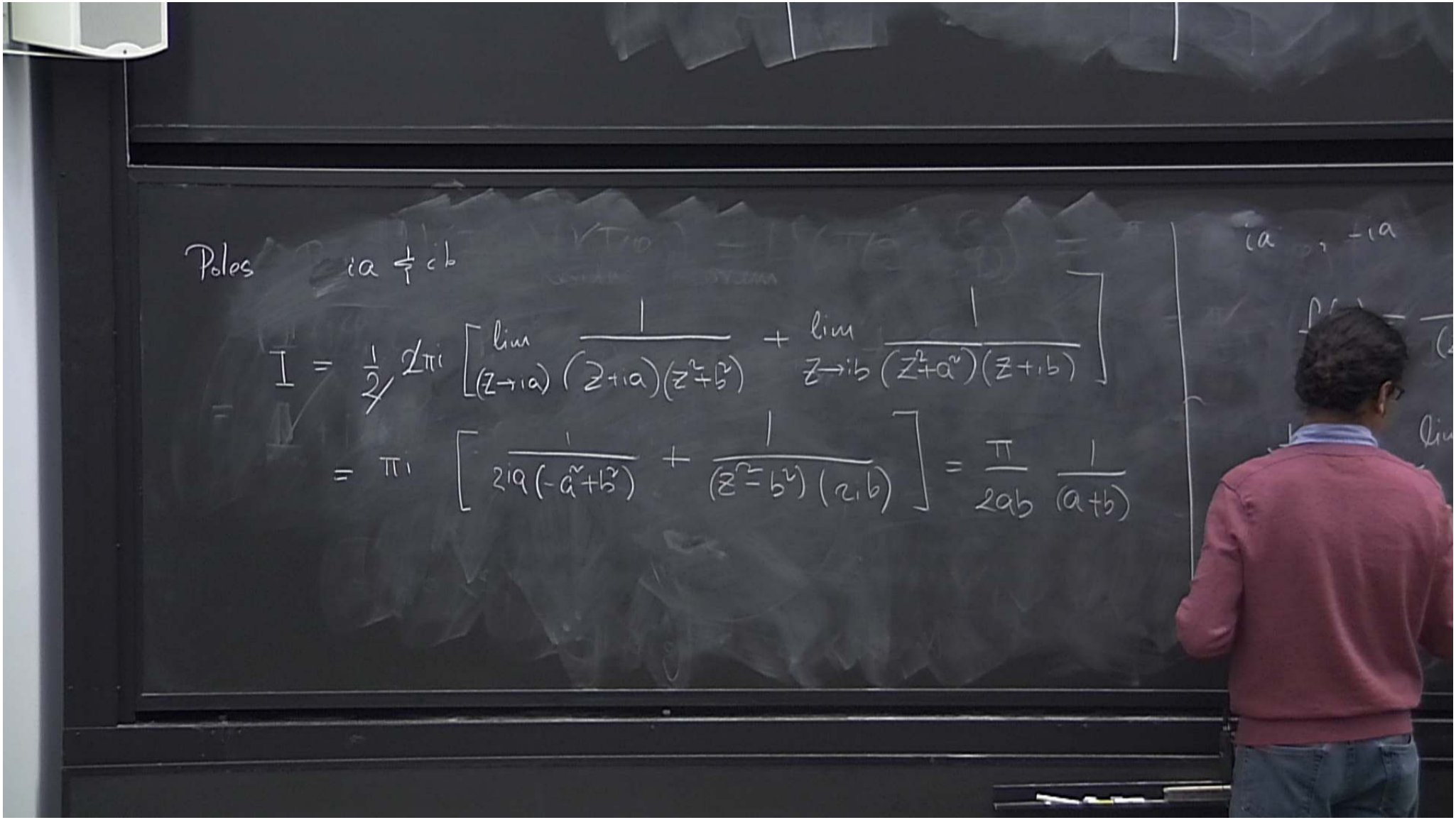
$$I = -2i \times 2\pi i \, \text{Res}\, f(z) \text{ at } z_+$$

$$= \frac{2\pi b}{\sqrt{a^2 - b^2}}$$

$$2 \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} \quad ; \; a, b \in \mathbb{R}, \; a \neq b$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$$


at z+
2 ∫₀^∞ dx / (a²+x²)(b²+x²) ; a, b ∈ R, a ≠ b
= 1/2 ∫_{-∞}^{+∞} dx / (a²+x²)(b²+x²)
f(z) = 1 / (a²+z²)(b²+z²)
∮ f(z) dz R → ∞
Γ
C₁ R
C₂ R
f(z) ~ 1/z⁴
= 1/R⁴


Poles ia ≠ ib
$I = \frac{1}{2} 2\pi i \left[\lim_{(z \to ia)} \frac{1}{(z+ia)(z^2+b^2)} + \lim_{z \to ib} \frac{1}{(z^2+a^2)(z+ib)} \right]$
$= \pi i \left[\frac{1}{2ia(-a^2+b^2)} + \frac{1}{(z^2-b^2)(2ib)} \right] = \frac{\pi}{2ab} \frac{1}{(a+b)}$
ia , -ia

$$\frac{1}{(z+ia)(z^2+b^2)} + \lim_{z\to ib}\frac{1}{(z^2+a^2)(z+ib)}\Bigg]$$

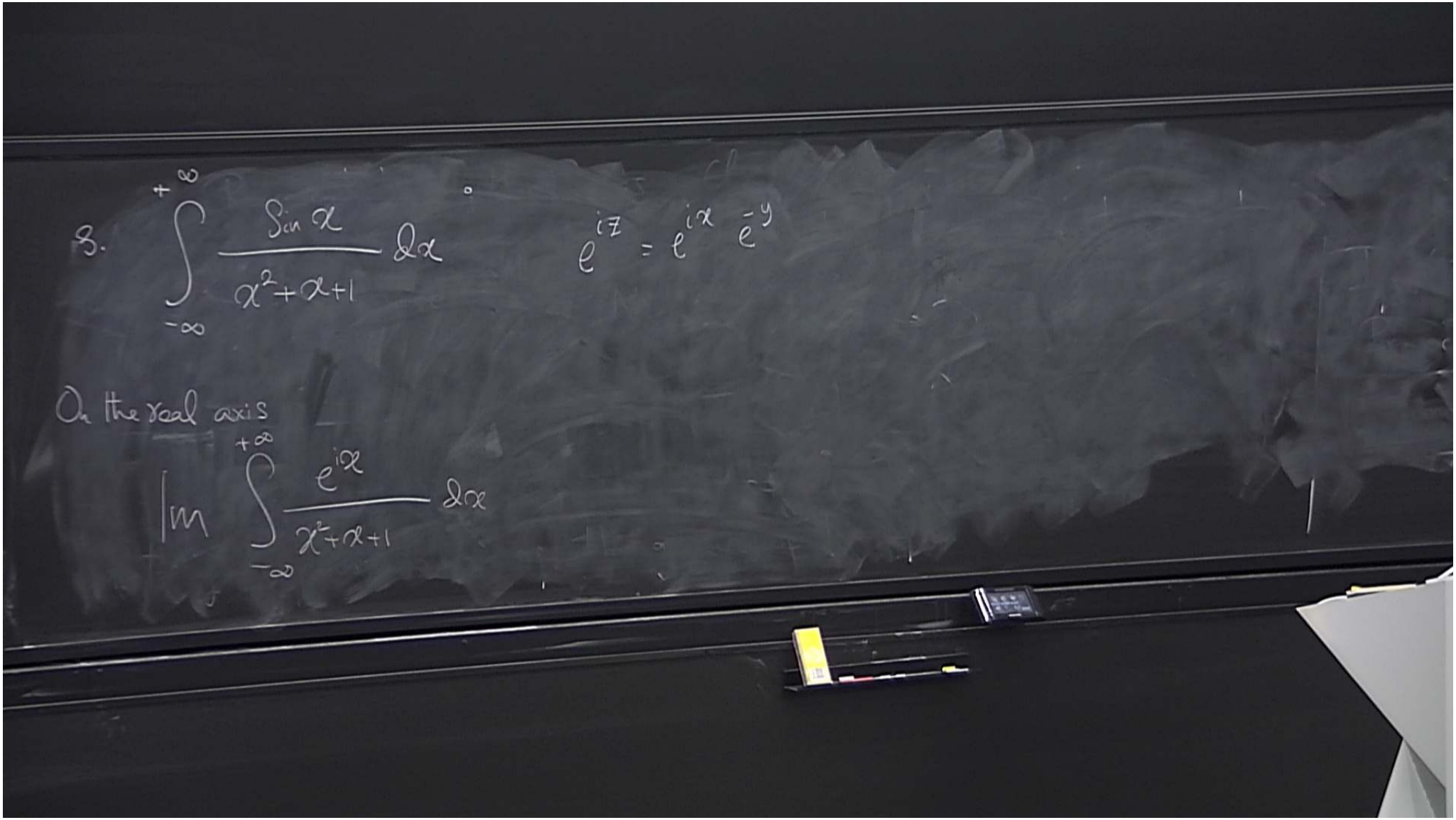
$$\frac{1}{-a^2+b^2} + \frac{1}{(z^2-b^2)(z+ib)}\Bigg] = \frac{\pi}{2ab}\,\frac{1}{(a+b)}$$

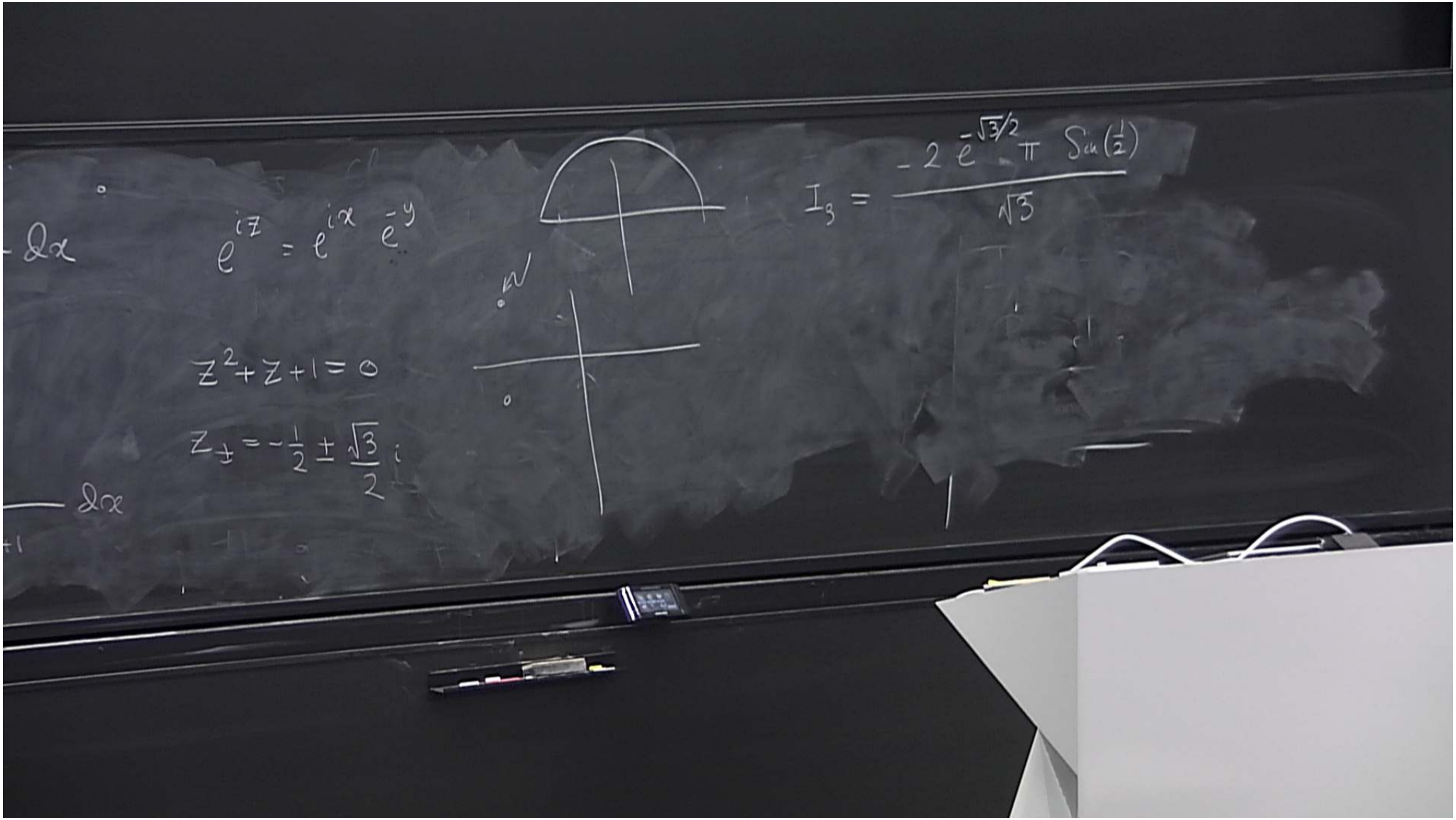
ia , $-ia$

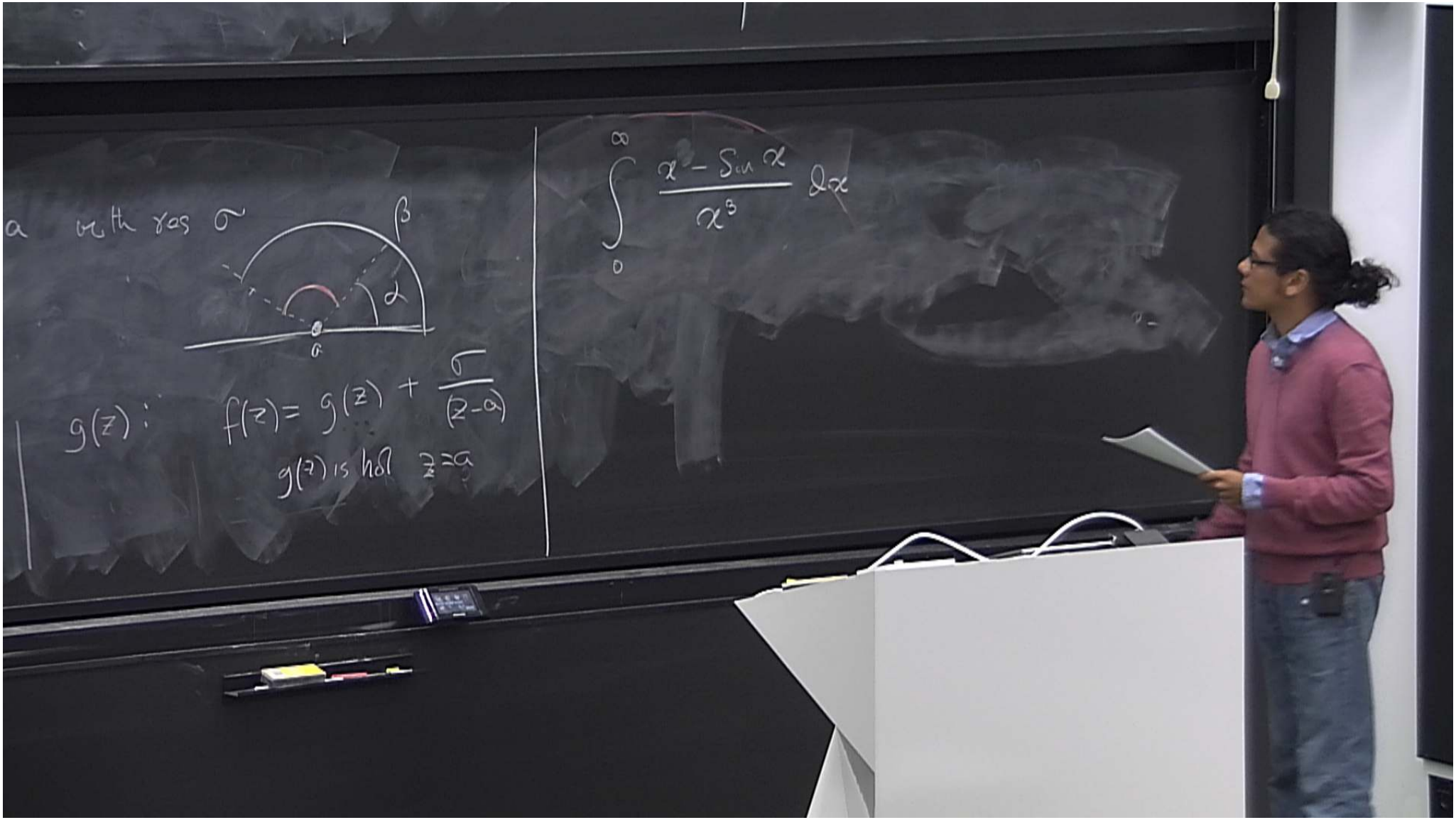
$$f(z) = \frac{1}{(z^2+a^2)^2}$$

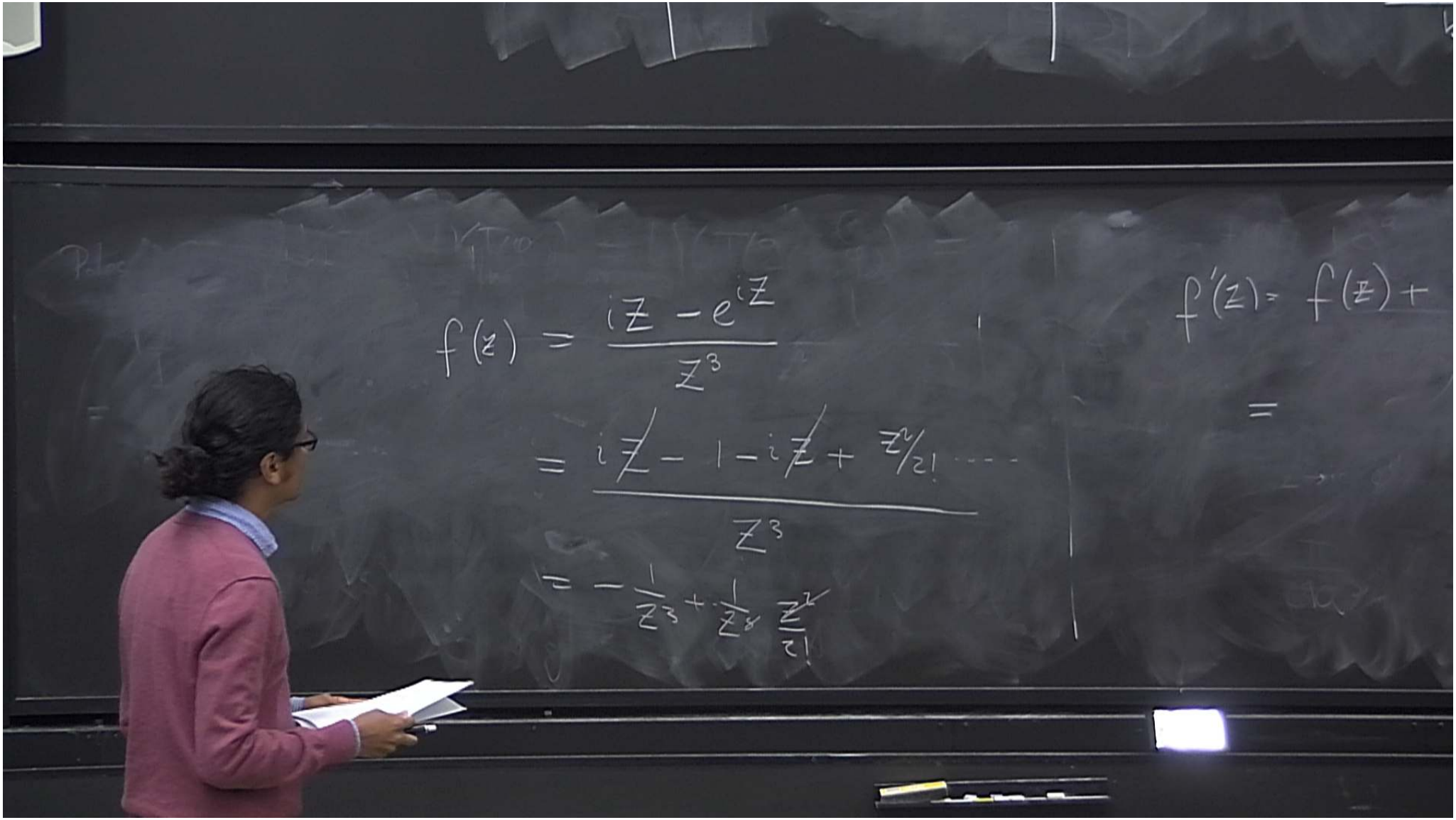
$$\frac{1}{2}\,2\pi i \lim_{z\to ia}\frac{d}{dz}(z-ia)^2\,\frac{1}{(z-ia)^2(z+ia)^2}$$

$$= \frac{\pi}{4a^3}$$


3. $\int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\,dx$
$e^{iz} = e^{ix}\,e^{-y}$
On the real axis
$\mathrm{Im} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\,dx$


$e^{iz} = e^{ix} e^{-y}$
$z^2 + z + 1 = 0$
$z_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
$I_3 = \frac{-2 e^{-\sqrt{3}/2} \pi \sin(\frac{1}{2})}{\sqrt{3}}$


a with rad σ
β
α
0
g(z):
f(z) = g(z) + σ/(z-a)
g(z) is hol z=a
∫ from 0 to ∞ (x - sin x)/x³ dx


$f(z) = \frac{iz - e^{iz}}{z^3}$
$= \frac{iz - 1 - iz + z^2/2! \cdots}{z^3}$
$= -\frac{1}{z^3} + \frac{1}{z^2} \frac{z^2}{2!}$
$f'(z) = f(z) +$
$=$

$$f(z) = \frac{iz - e^{iz}}{z^3}$$

$$= \frac{iz - 1 - iz + z^2/2! \cdots}{z^3}$$

$$= -\frac{1}{z^3} + \frac{1}{z^2} \frac{z^2}{2!}$$

$$f'(z) = f(z) + \frac{1}{z^3}$$

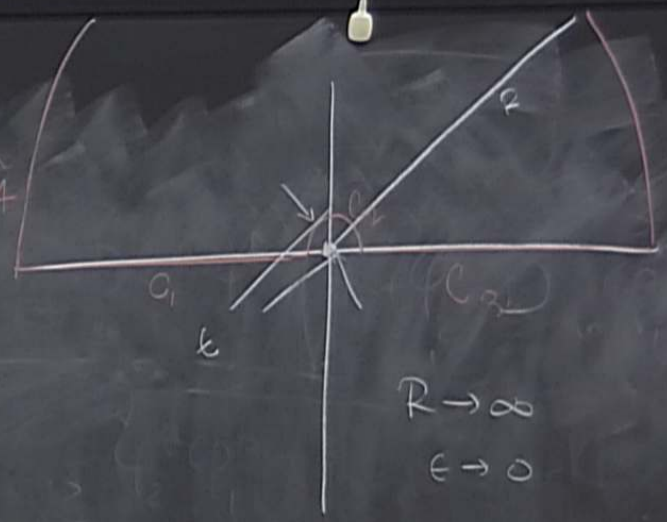
$$= \frac{iz + 1 - e^{iz}}{z^3}$$

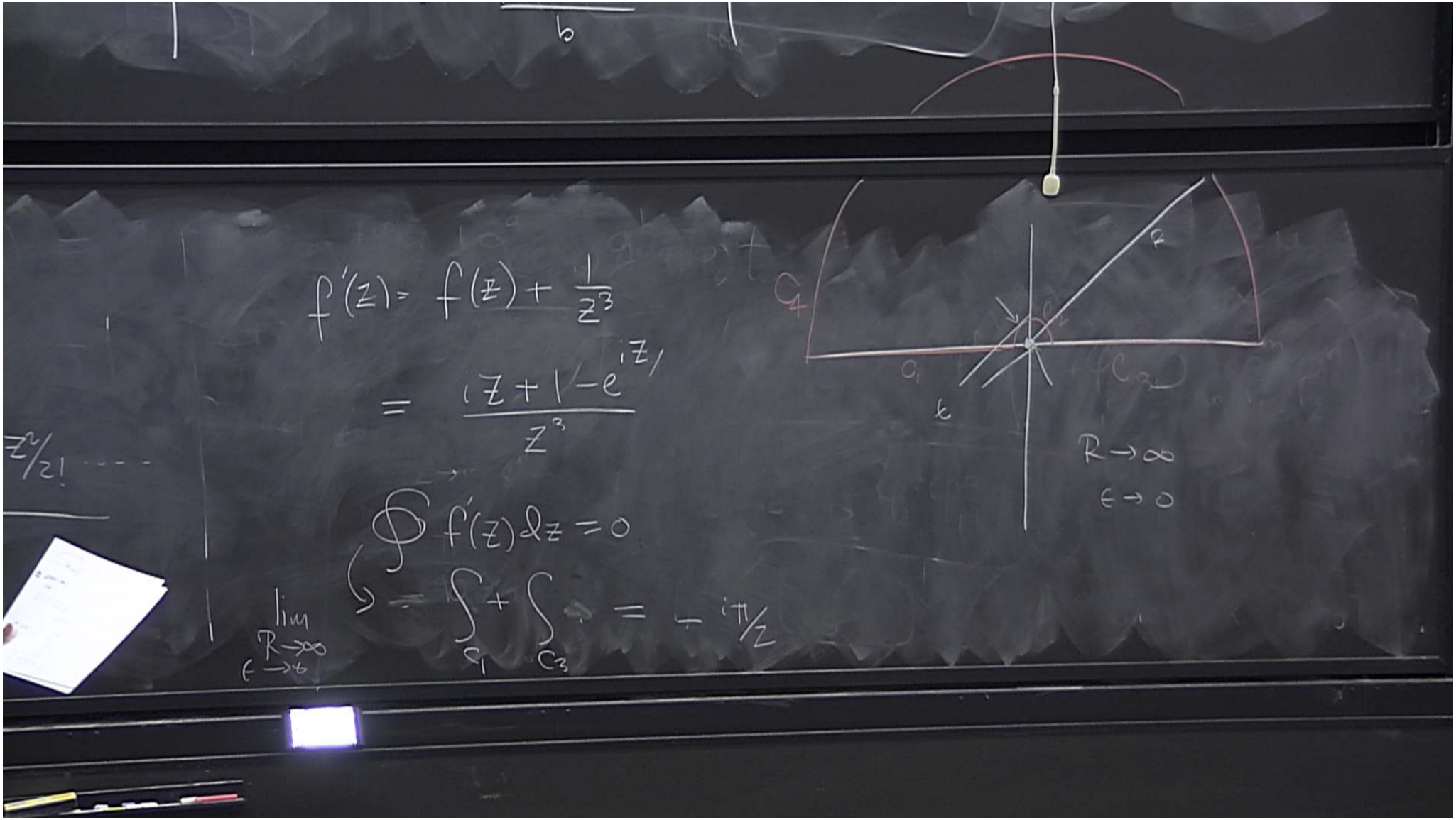
$$f'(z) = f(z) + \frac{1}{z^3}$$

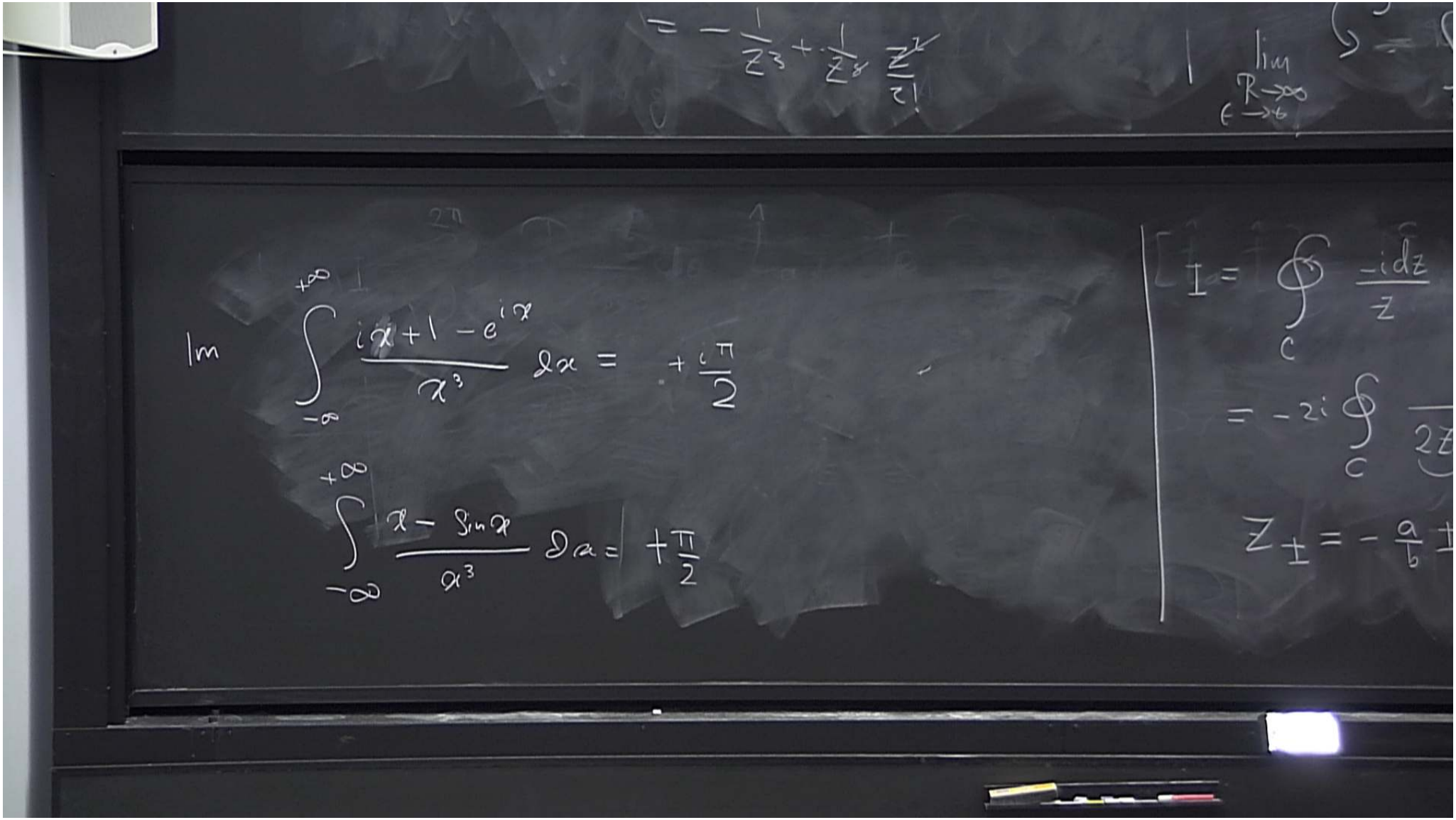
$$= \frac{iz + 1 - e^{iz}}{z^3}$$

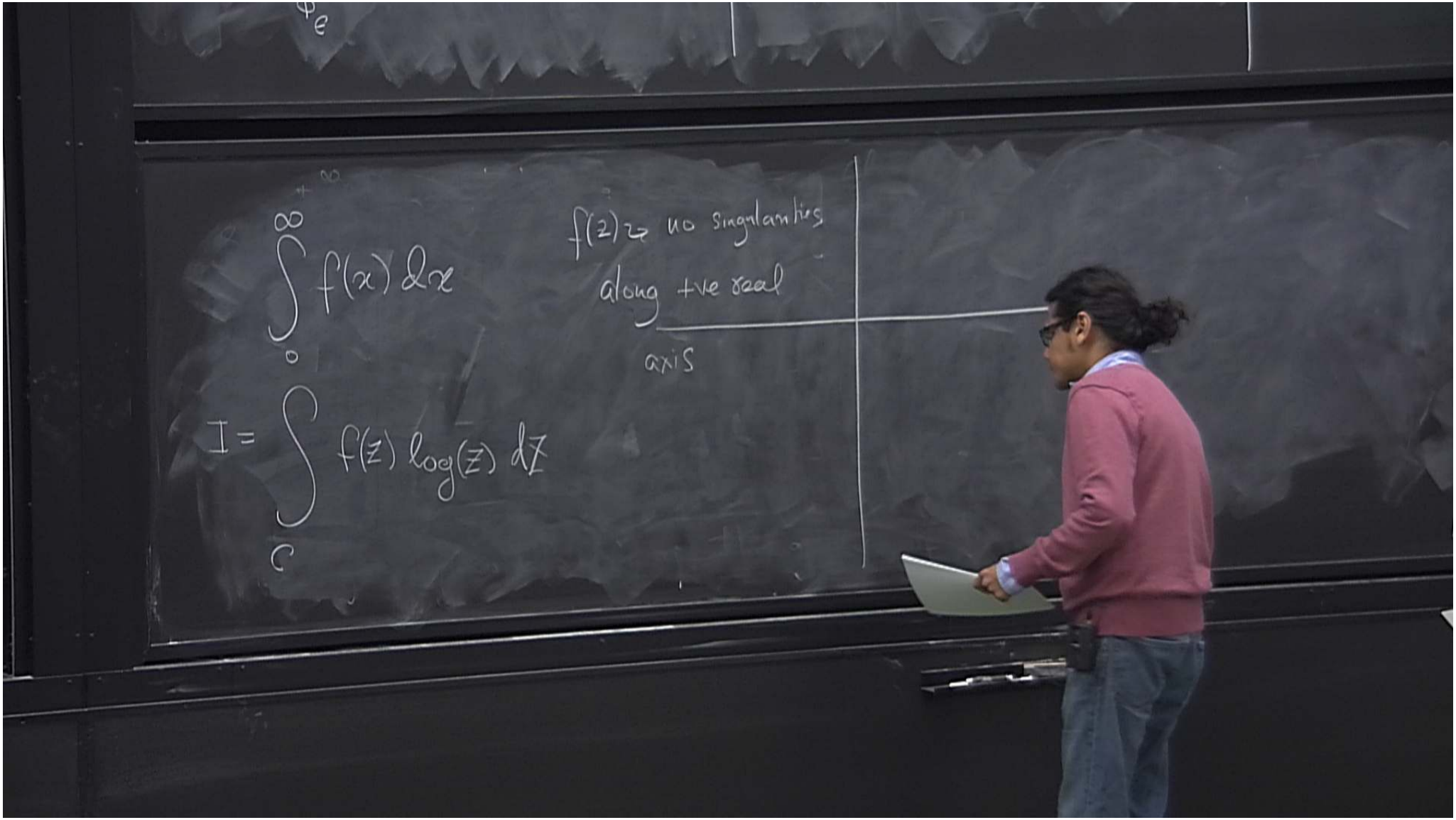
$$\oint f'(z)\,dz$$

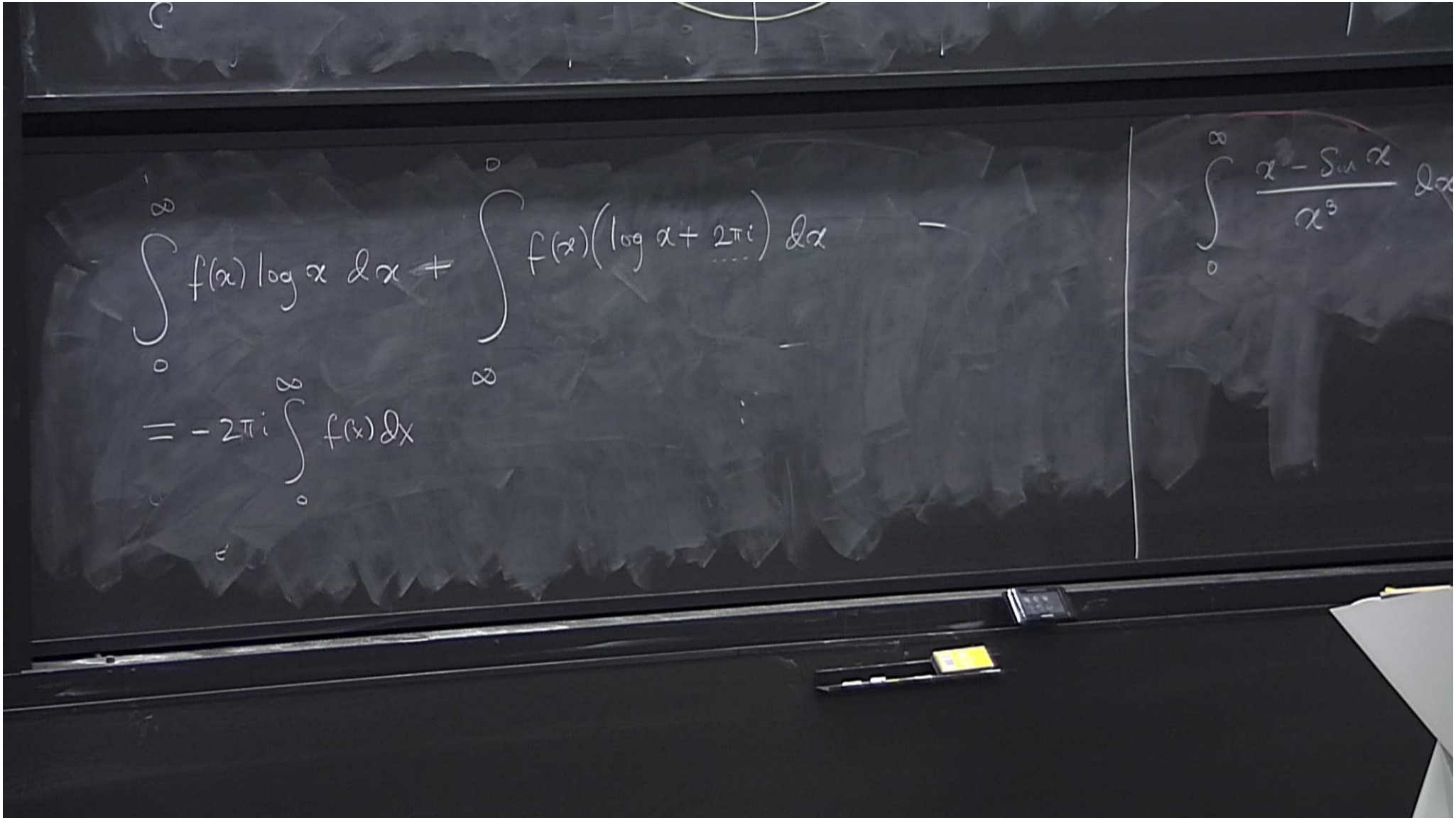
$$= \int_{C_1} + \int_{C_3}$$

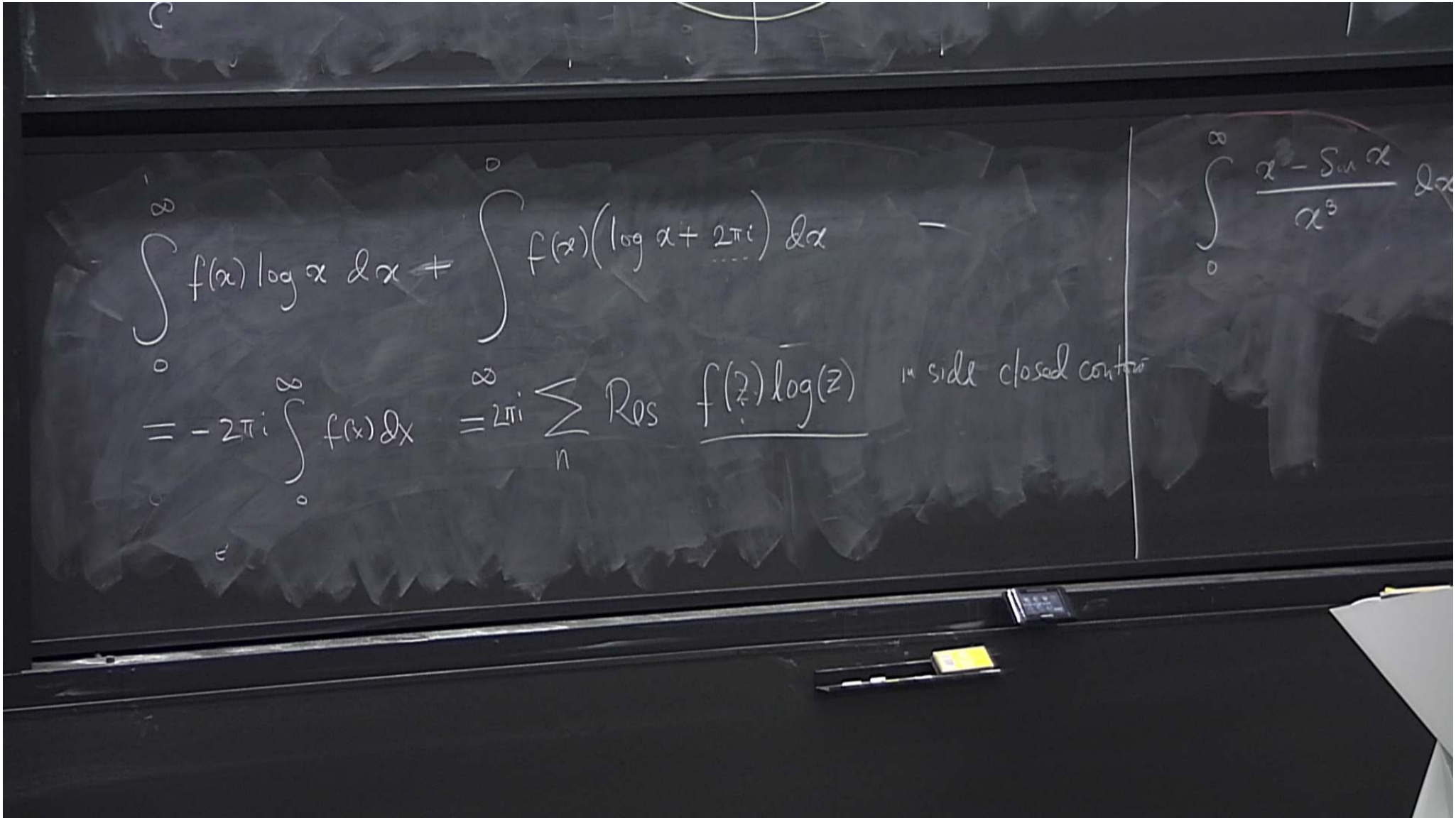



b
$f'(z) = f(z) + \frac{1}{z^3}$
$= \frac{iz + 1 - e^{iz}}{z^3}$
$z^2/2!$ - - -
$\oint f'(z)\,dz = 0$
$\lim_{\substack{R\to\infty \\ \epsilon\to 0}}$
$\int_{C_1} + \int_{C_3} = -i\pi/2$
C_4
C_1
C_2
C_3
R
ϵ
$R \to \infty$
$\epsilon \to 0$


$= -\frac{1}{z^3} + \frac{1}{z^5}\frac{z^2}{2!}$
$\lim_{\substack{R\to\infty \\ \epsilon\to 0}}$
$\mathrm{Im} \int_{-\infty}^{+\infty} \frac{ix + 1 - e^{ix}}{x^3}\,dx = +\frac{i\pi}{2}$
$\int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3}\,dx = +\frac{\pi}{2}$
$I = \oint_C \frac{-i\,dz}{z}$
$= -2i \oint_C$
$z_\pm = -\frac{a}{b} \pm$


$\int_0^\infty f(x)\,dx$
$f(z) \rightarrow$ no singularities along +ve real axis
$I = \int_C f(z) \log(z)\,dz$


$\int_0^\infty f(x) \log x \, dx + \int_\infty^0 f(x)(\log x + 2\pi i) \, dx$
$= -2\pi i \int_0^\infty f(x) \, dx$
$\int_0^\infty \frac{x - \sin x}{x^3} \, dx$


$\int_0^\infty f(x)\log x\,dx + \int_0^\infty f(x)(\log x + 2\pi i)\,dx$ −
$\int_0^\infty \frac{x - \sin x}{x^3}\,dx$
$= -2\pi i \int_0^\infty f(x)\,dx$
$= 2\pi i \sum_n \text{Res}\ f(z)\log(z)$
in side closed contour