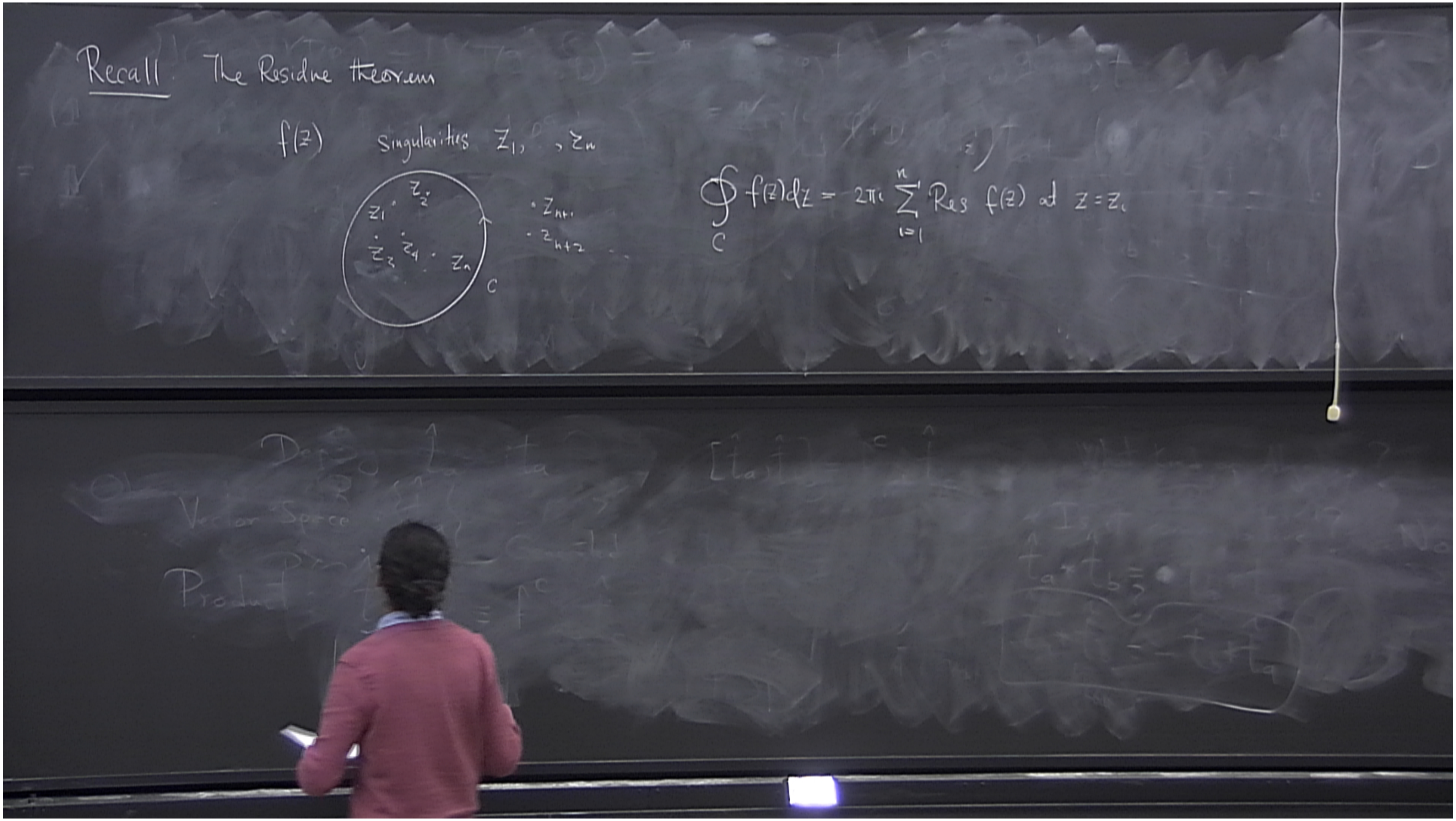
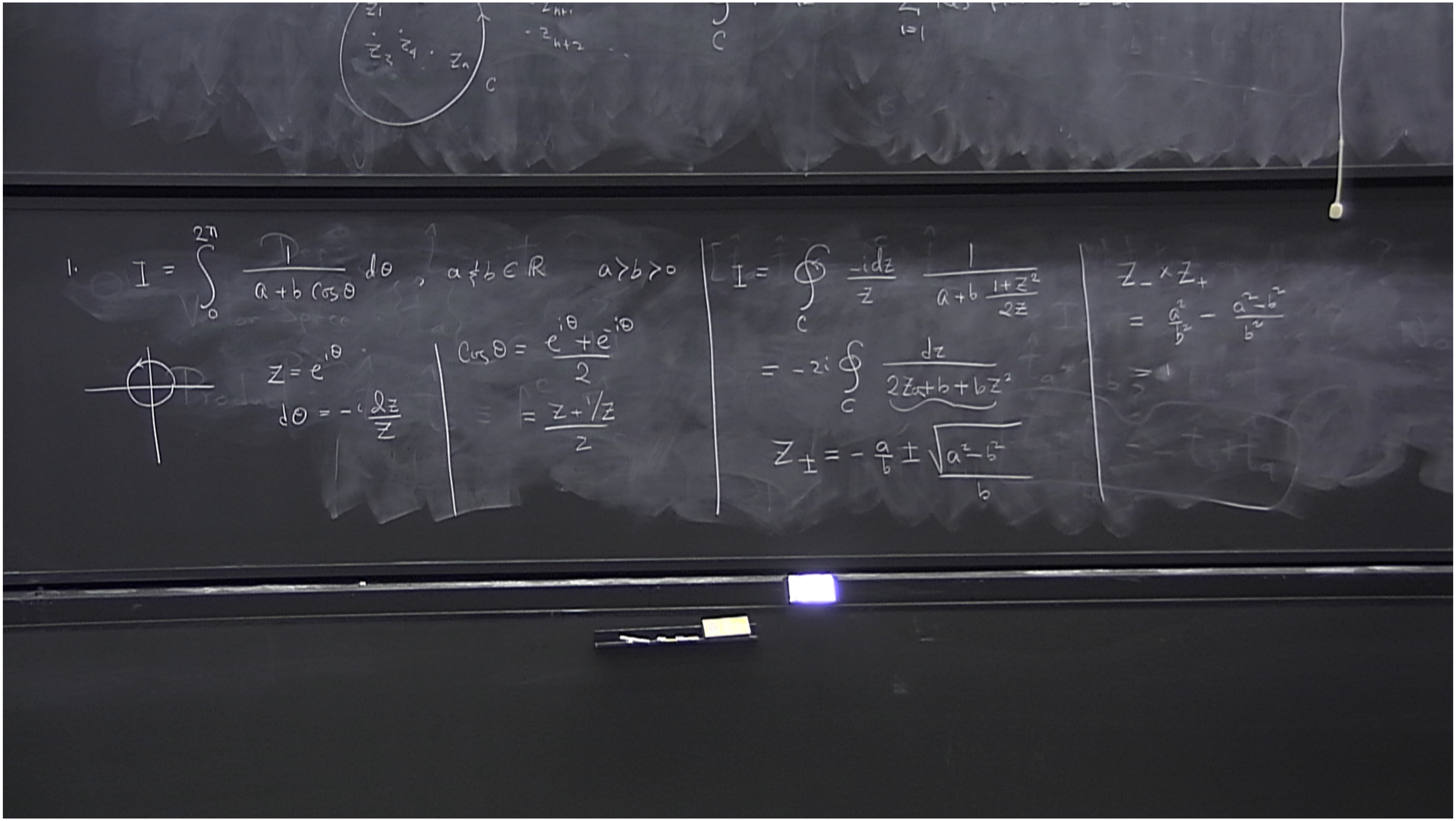
Title: Complex Analysis - Lecture 4

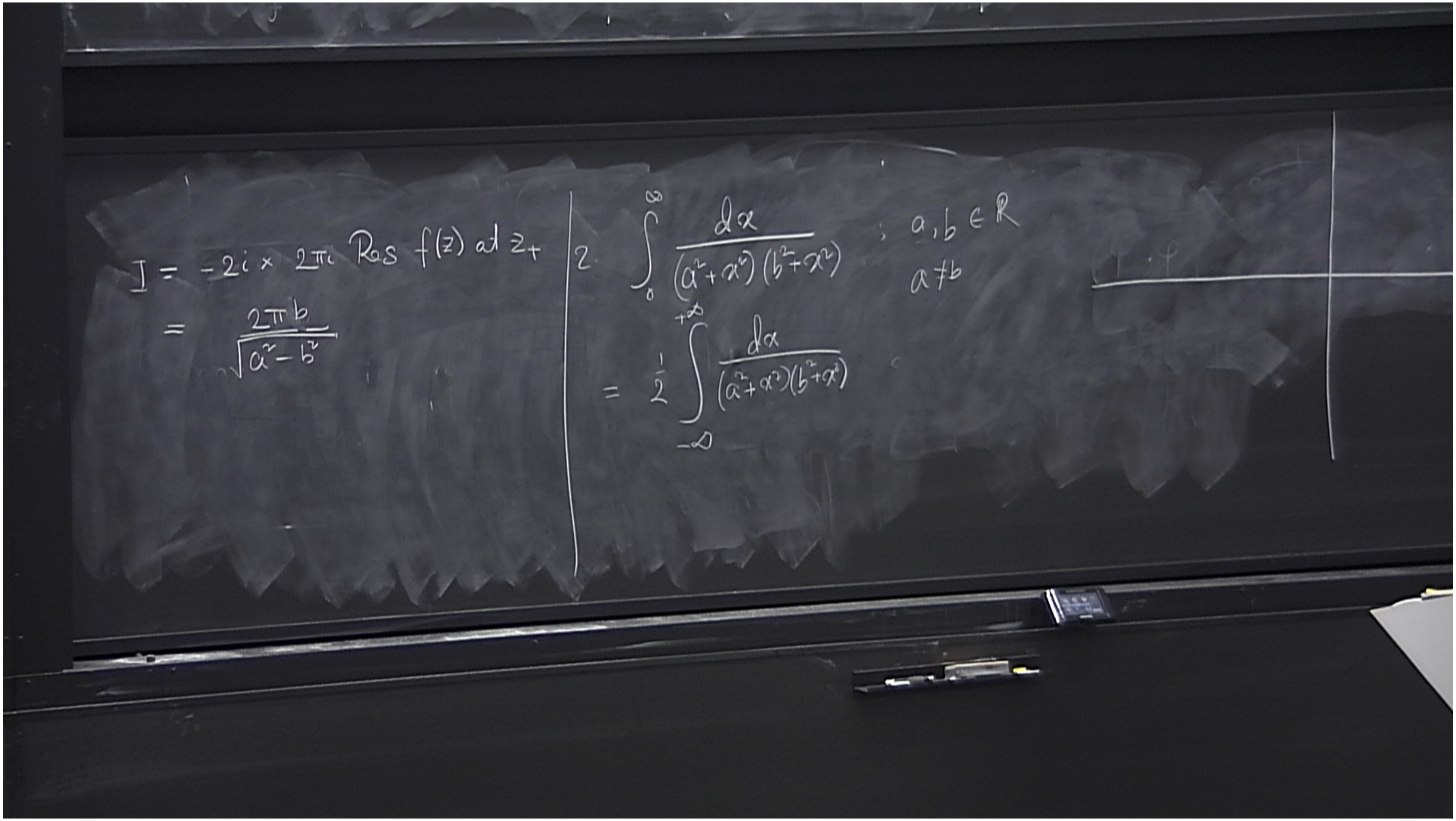
Date: Aug 22, 2011 10:45 AM

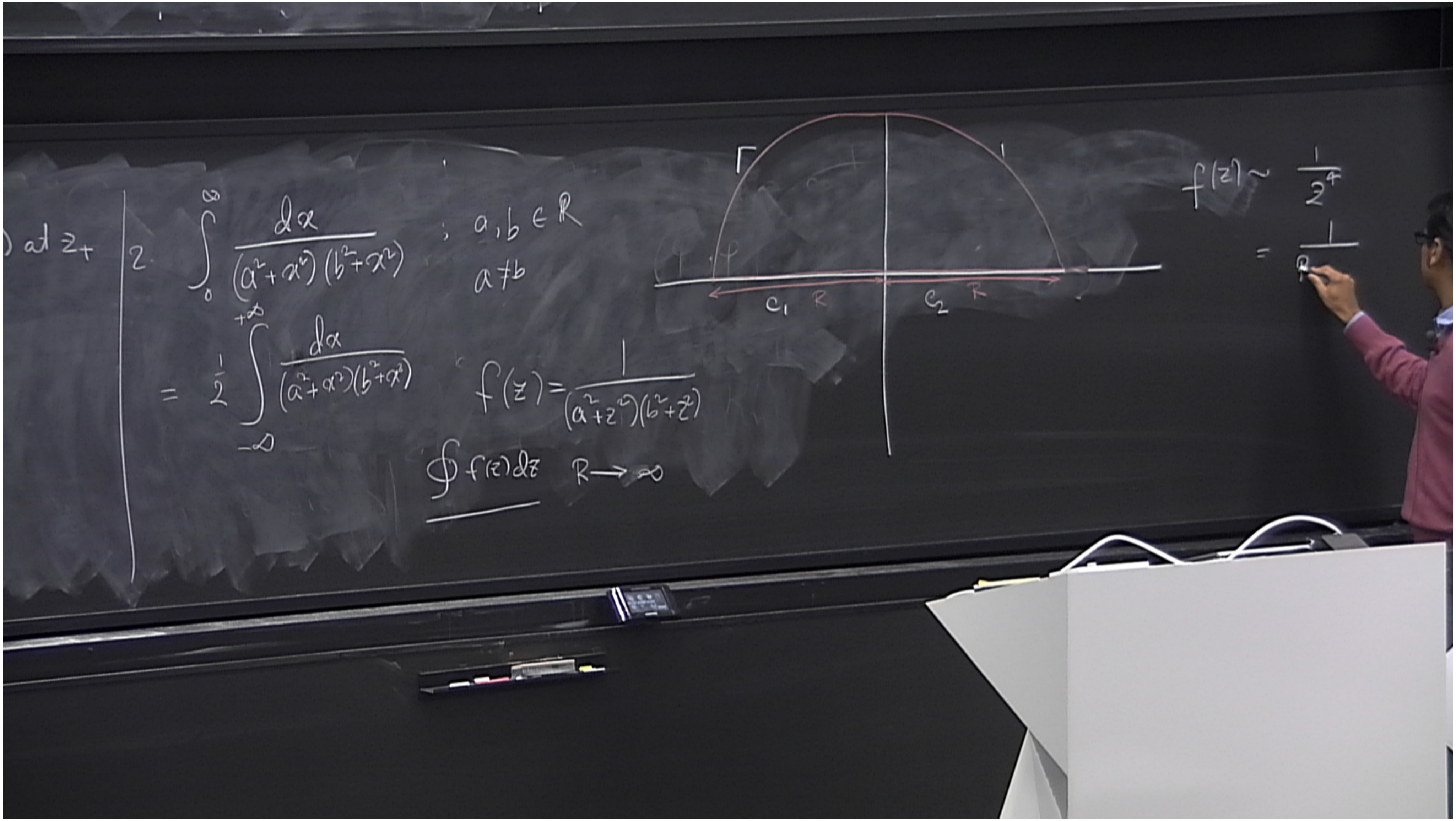
URL: http://pirsa.org/11080126

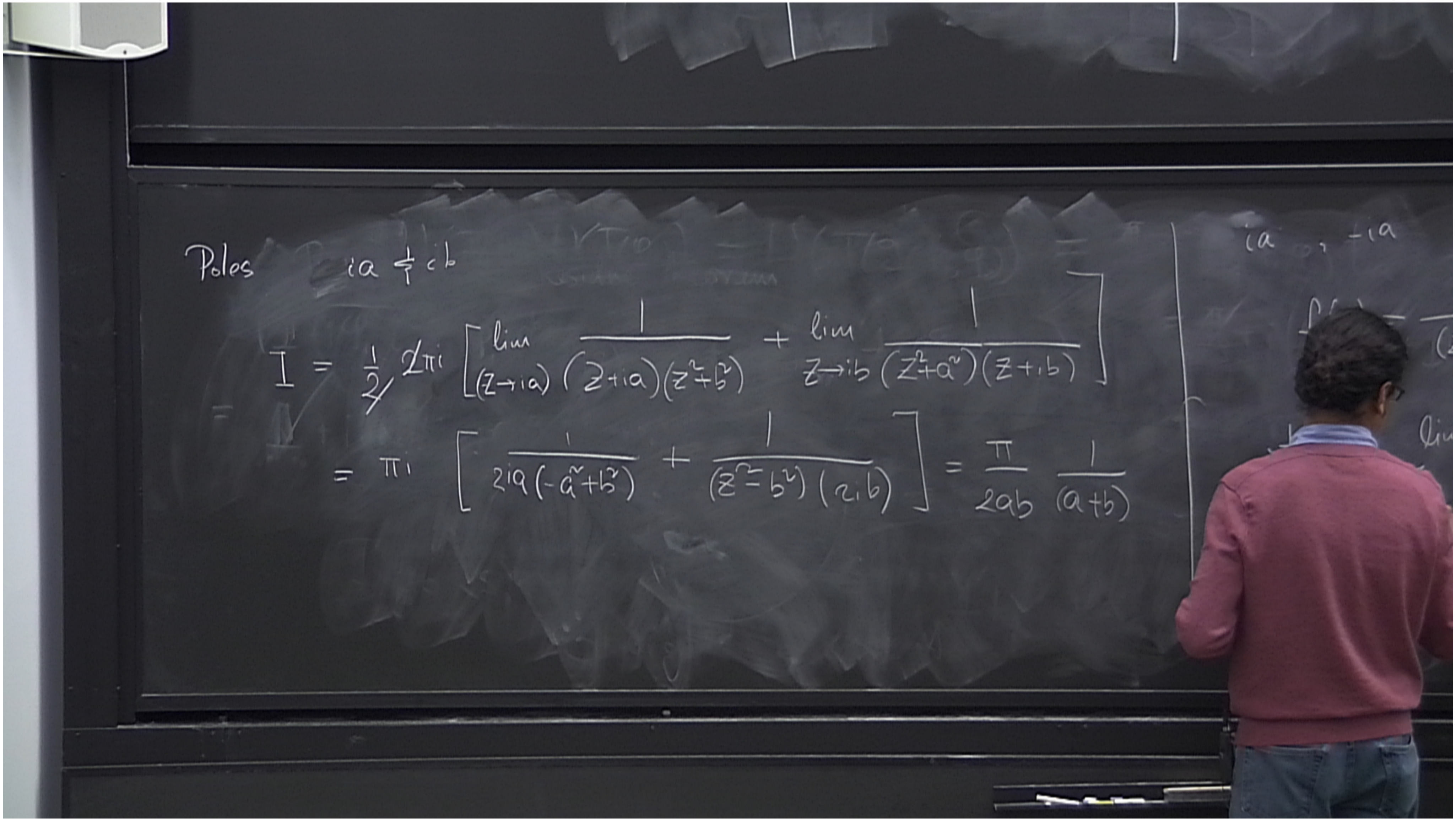
Abstract:

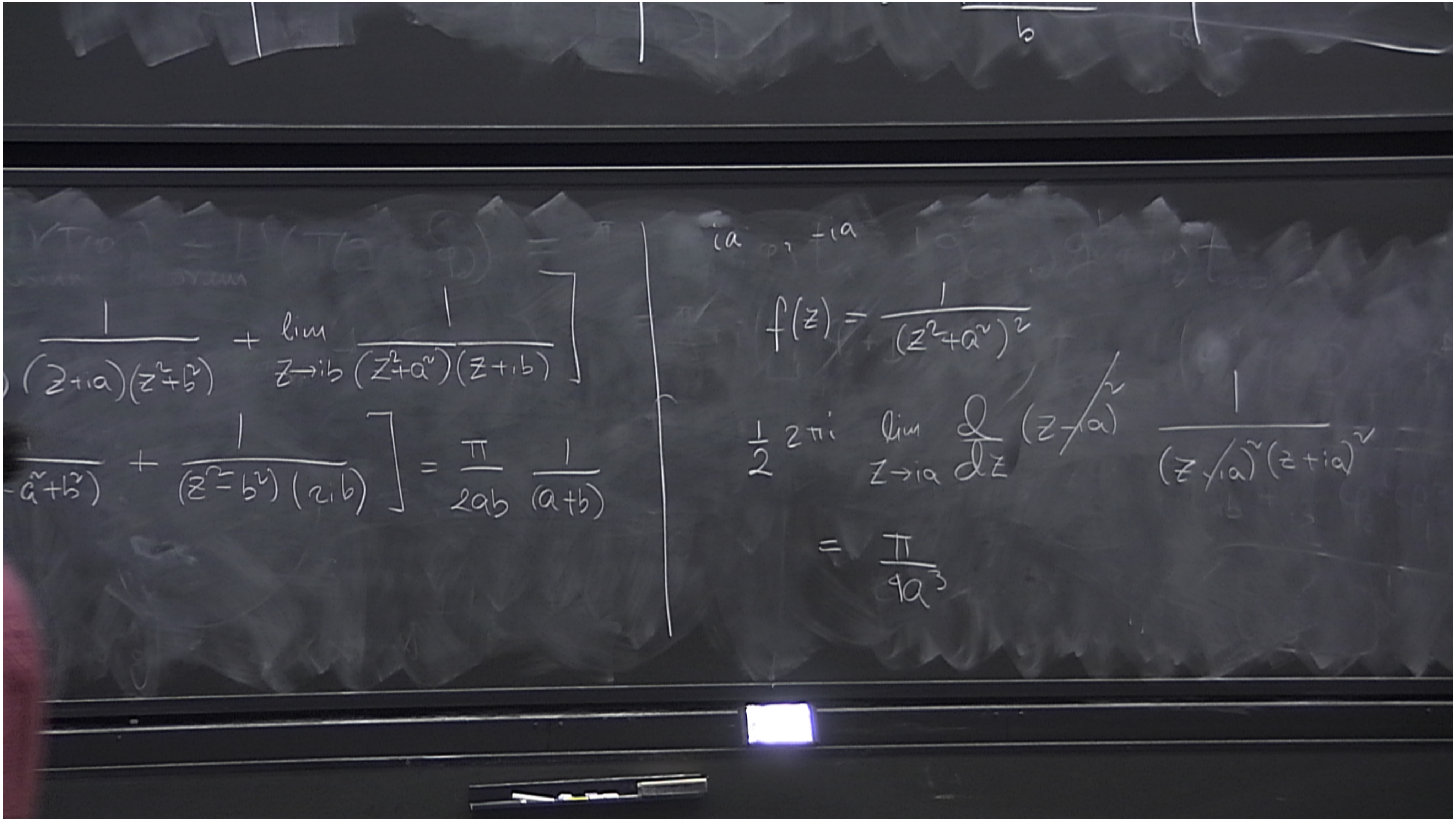

Recall: The Residue theorem
f(z) singularities z1, , zn
z1 z2 z3 z4 zn C
zn+1 zn+2
∮C f(z)dz = 2πi Σ(i=1 to n) Res f(z) at z = zi

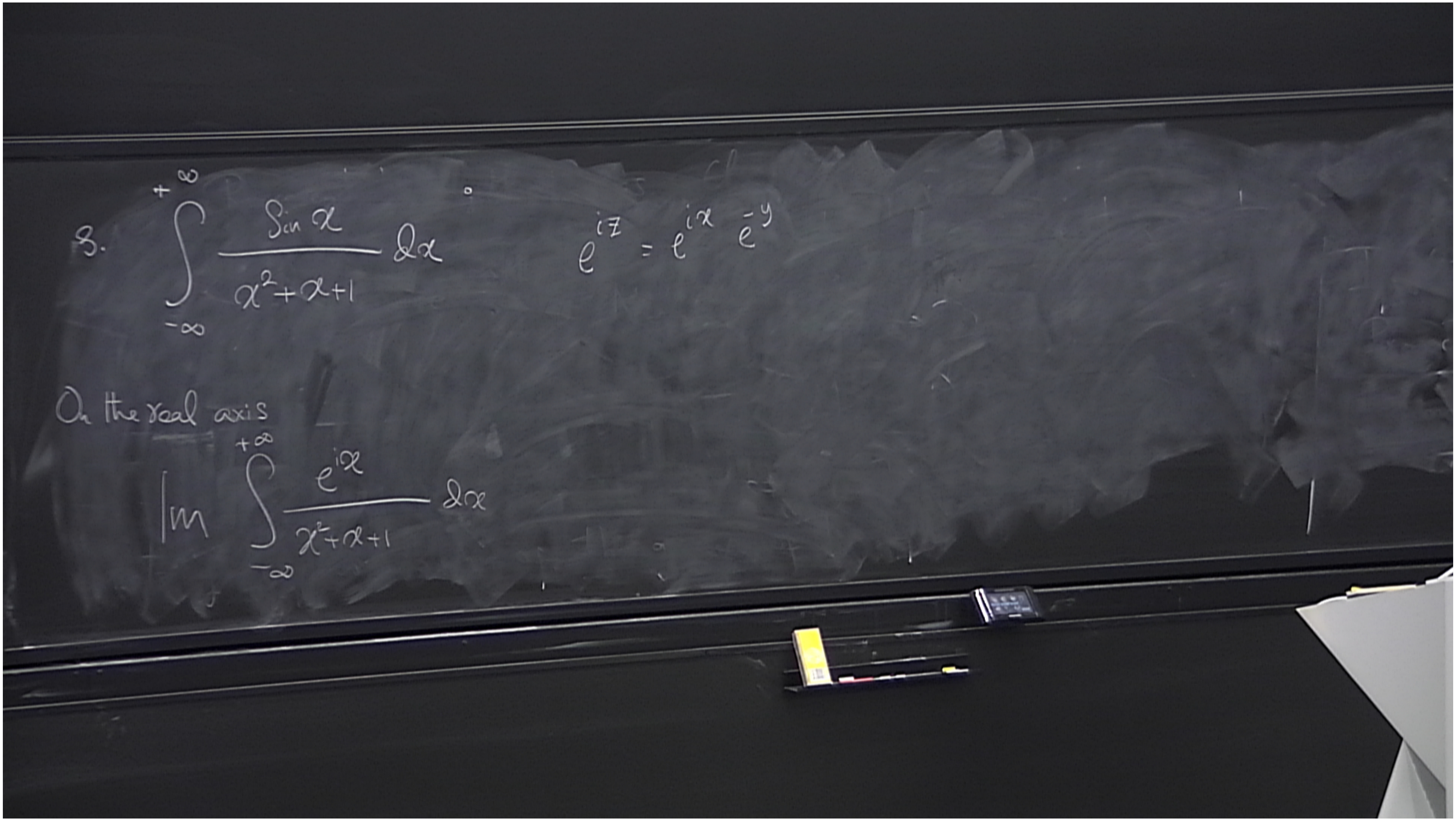

1. $I = \int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta$, $a \,\&\, b \in \mathbb{R}$ $a > b > 0$
$z = e^{i\theta}$
$d\theta = -i \frac{dz}{z}$
$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$
$I = \oint_C \frac{-i\,dz}{z} \frac{1}{a + b\frac{1+z^2}{2z}}$
$= -2i \oint_C \frac{dz}{2za + b + bz^2}$
$z_\pm = -\frac{a}{b} \pm \frac{\sqrt{a^2 - b^2}}{b}$
$z_- \times z_+$
$= \frac{a^2}{b^2} - \frac{a^2 - b^2}{b^2}$

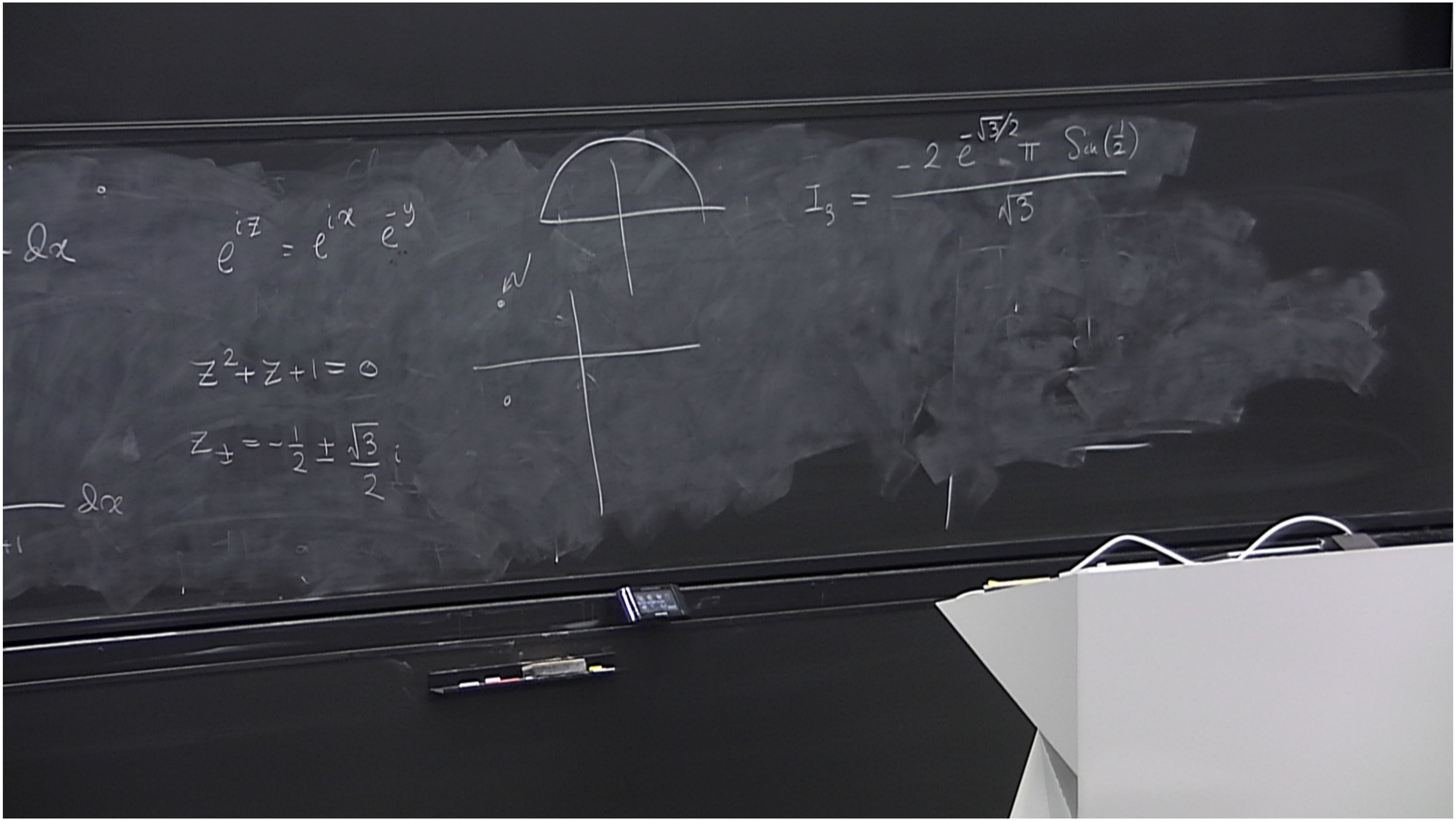

$I = -2i \times 2\pi i \operatorname{Res} f(z)$ at z_+
$= \frac{2\pi b}{\sqrt{a^2-b^2}}$
2. $\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$; $a, b \in \mathbb{R}$
$a \neq b$
$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$

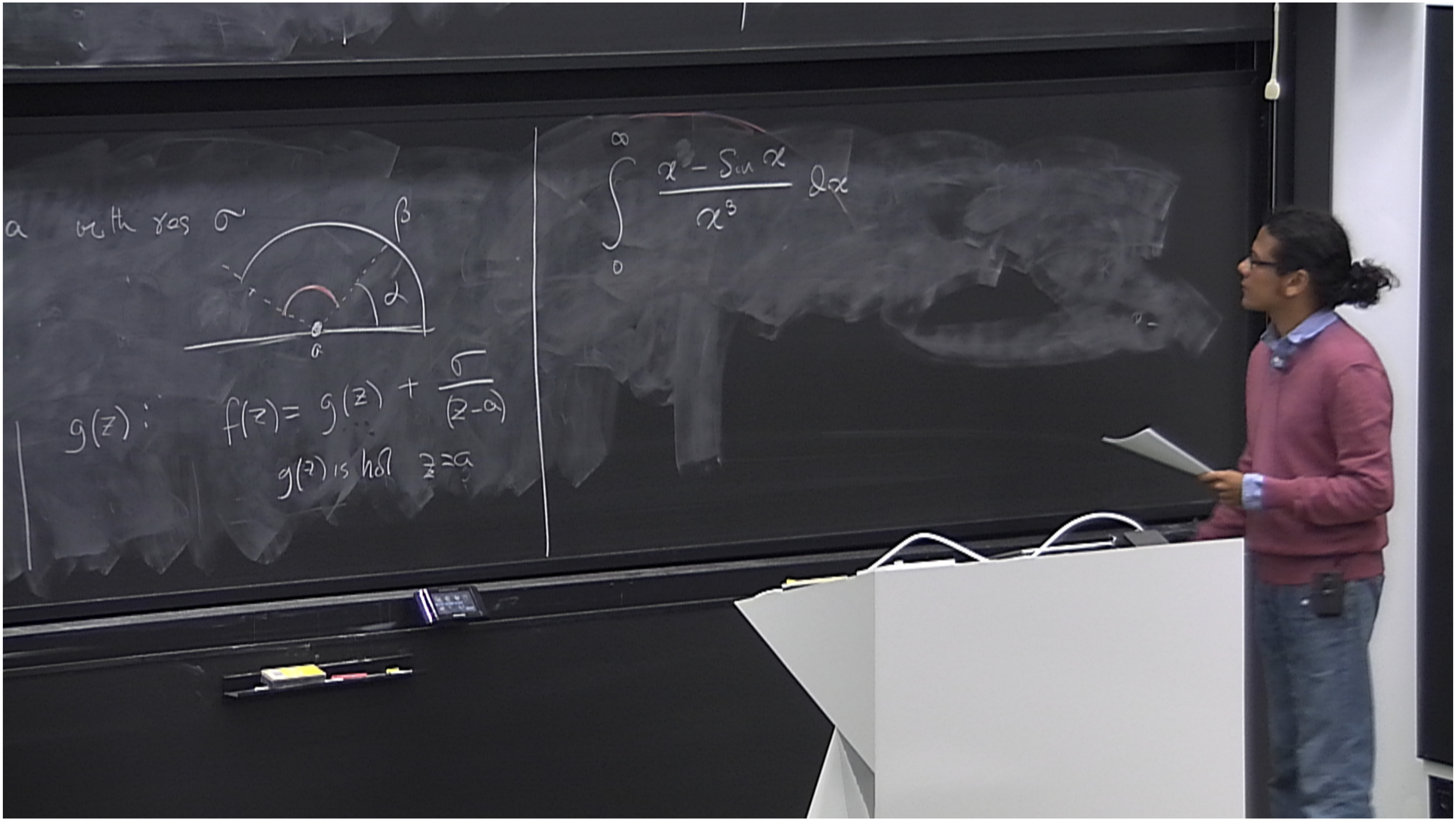

at z_+
2.
$\int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)}$; $a, b \in \mathbb{R}$
$a \neq b$
$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{(a^2+x^2)(b^2+x^2)}$
$f(z) = \frac{1}{(a^2+z^2)(b^2+z^2)}$
$\oint f(z)\,dz \quad R \to \infty$
Γ
C_1 R
C_2 R
$f(z) \sim \frac{1}{z^4}$
$= \frac{1}{R}$


Poles ia & ib
$I = \frac{1}{2} 2\pi i \left[\lim_{z \to ia} \frac{1}{(z+ia)(z^2+b^2)} + \lim_{z \to ib} \frac{1}{(z^2+a^2)(z+ib)} \right]$
$= \pi i \left[\frac{1}{2ia(-a^2+b^2)} + \frac{1}{(z^2-b^2)(2ib)} \right] = \frac{\pi}{2ab} \frac{1}{(a+b)}$
ia , -ia

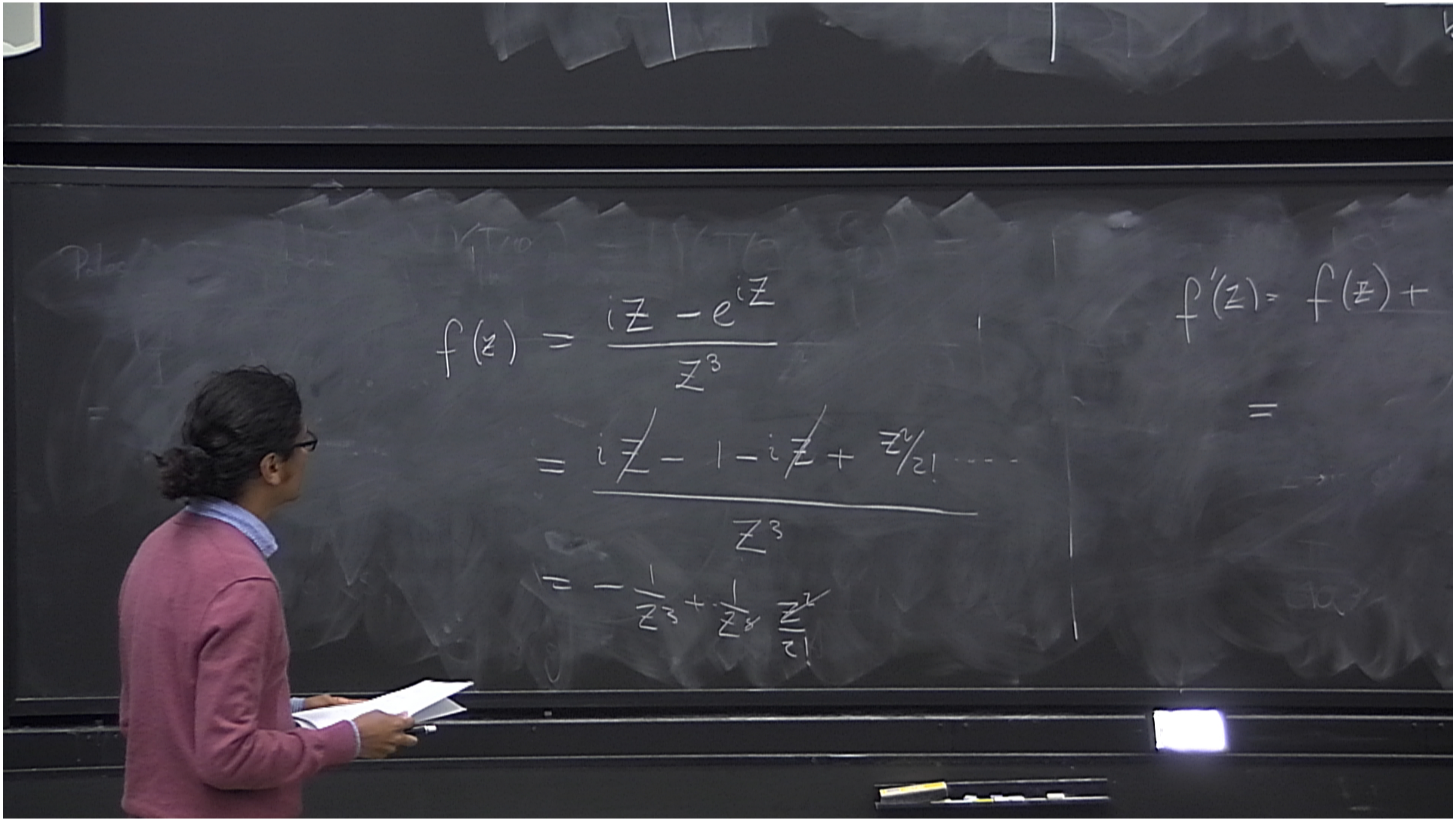

ia, $-ia$
$f(z) = \frac{1}{(z^2+a^2)^2}$
$\frac{1}{(z+ia)(z^2+b^2)} + \lim_{z\to ib} \frac{1}{(z^2+a^2)(z+ib)}$
$\frac{1}{-a^2+b^2} + \frac{1}{(z^2-b^2)(2ib)} = \frac{\pi}{2ab}\frac{1}{(a+b)}$
$\frac{1}{2} 2\pi i \lim_{z\to ia} \frac{d}{dz} (z-ia)^2 \frac{1}{(z-ia)^2(z+ia)^2}$
$= \frac{\pi}{4a^3}$


8. $\int_{-\infty}^{+\infty} \frac{\sin x}{x^2+x+1}\, dx$
$e^{iz} = e^{ix}\, e^{-y}$
On the real axis
$\mathrm{Im} \int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+x+1}\, dx$


$e^{iz} = e^{ix} e^{-y}$
$I_3 = \frac{-2 e^{-\sqrt{3}/2 \pi} \sin(\frac{1}{2})}{\sqrt{3}}$
$z^2 + z + 1 = 0$
$z_\pm = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$



a with ras σ
β
α
a
$\int_0^\infty \frac{x^3 - \sin x}{x^3}\,dx$
g(z) :
$f(z) = g(z) + \frac{\sigma}{(z-a)}$
g(z) is hol z=a



$f(z) = \frac{iz - e^{iz}}{z^3}$
$= \frac{i\cancel{z} - 1 - i\cancel{z} + z^2/2! \cdots}{z^3}$
$= -\frac{1}{z^3} + \frac{1}{z^3}\frac{z^2}{2!}$
$f'(z) = f(z) +$
$=$

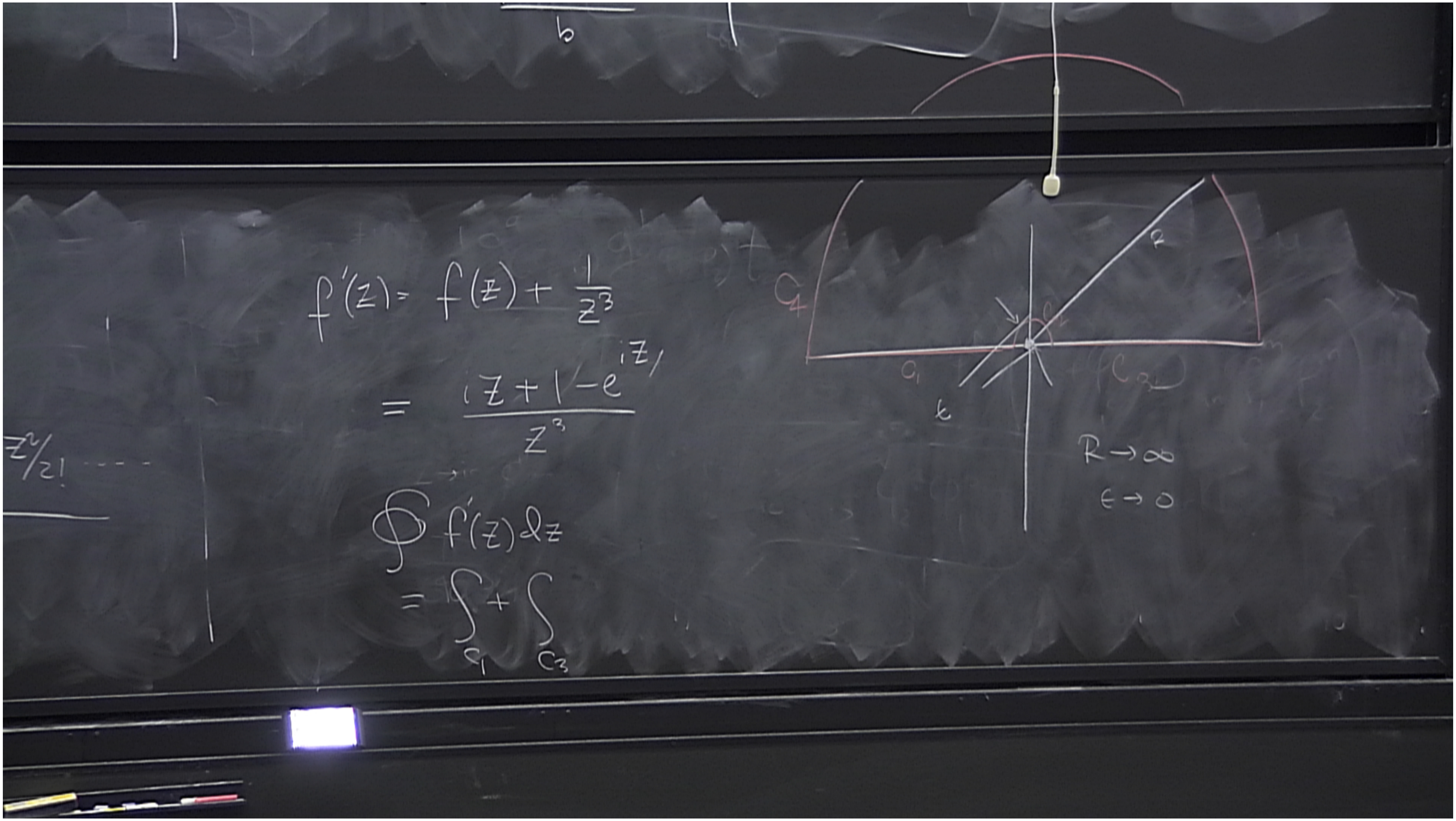
$$f(z) = \frac{iz - e^{iz}}{z^3}$$

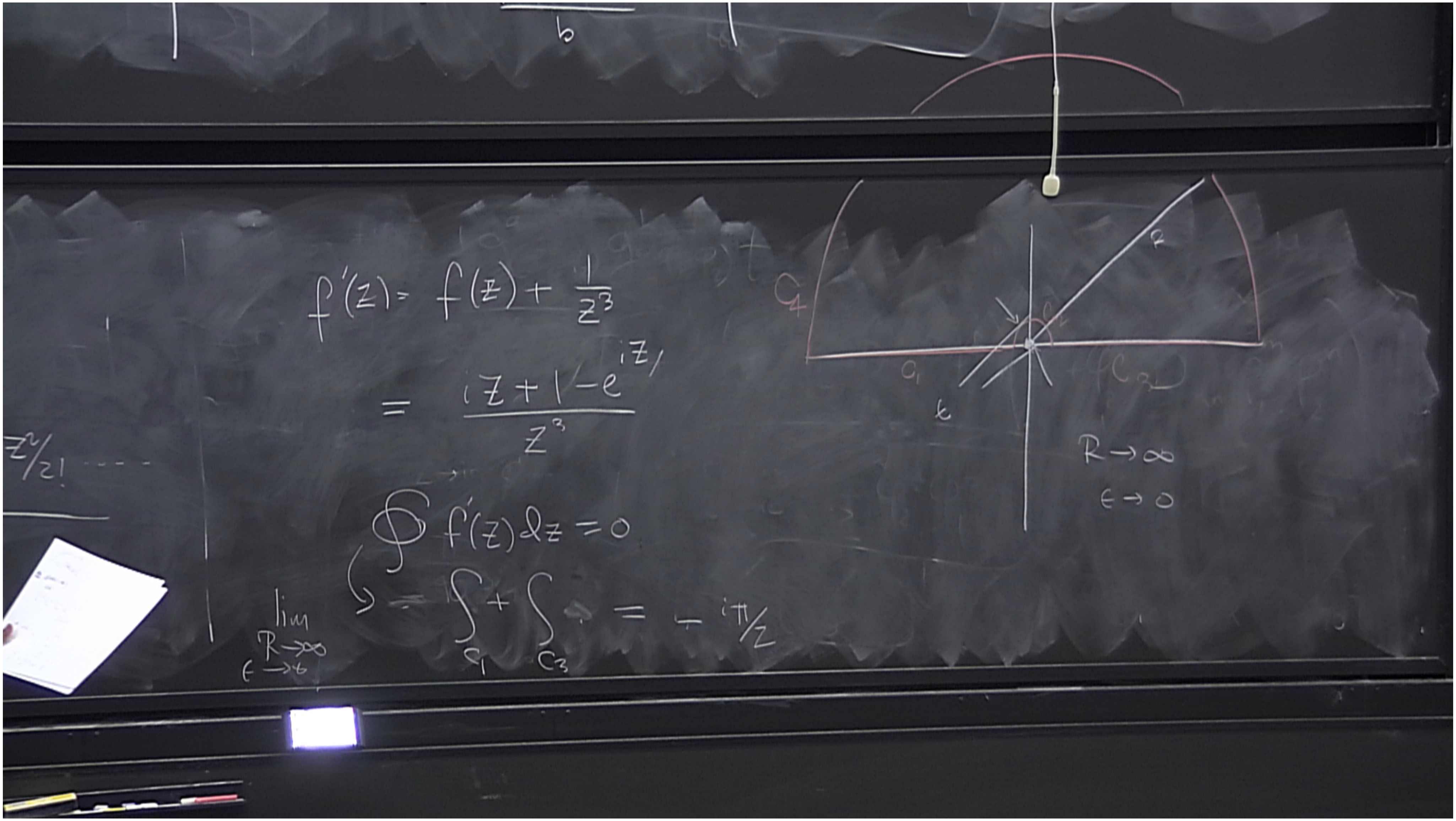
$$= \frac{i\cancel{z} - 1 - i\cancel{z} + z^2/2! \cdots}{z^3}$$

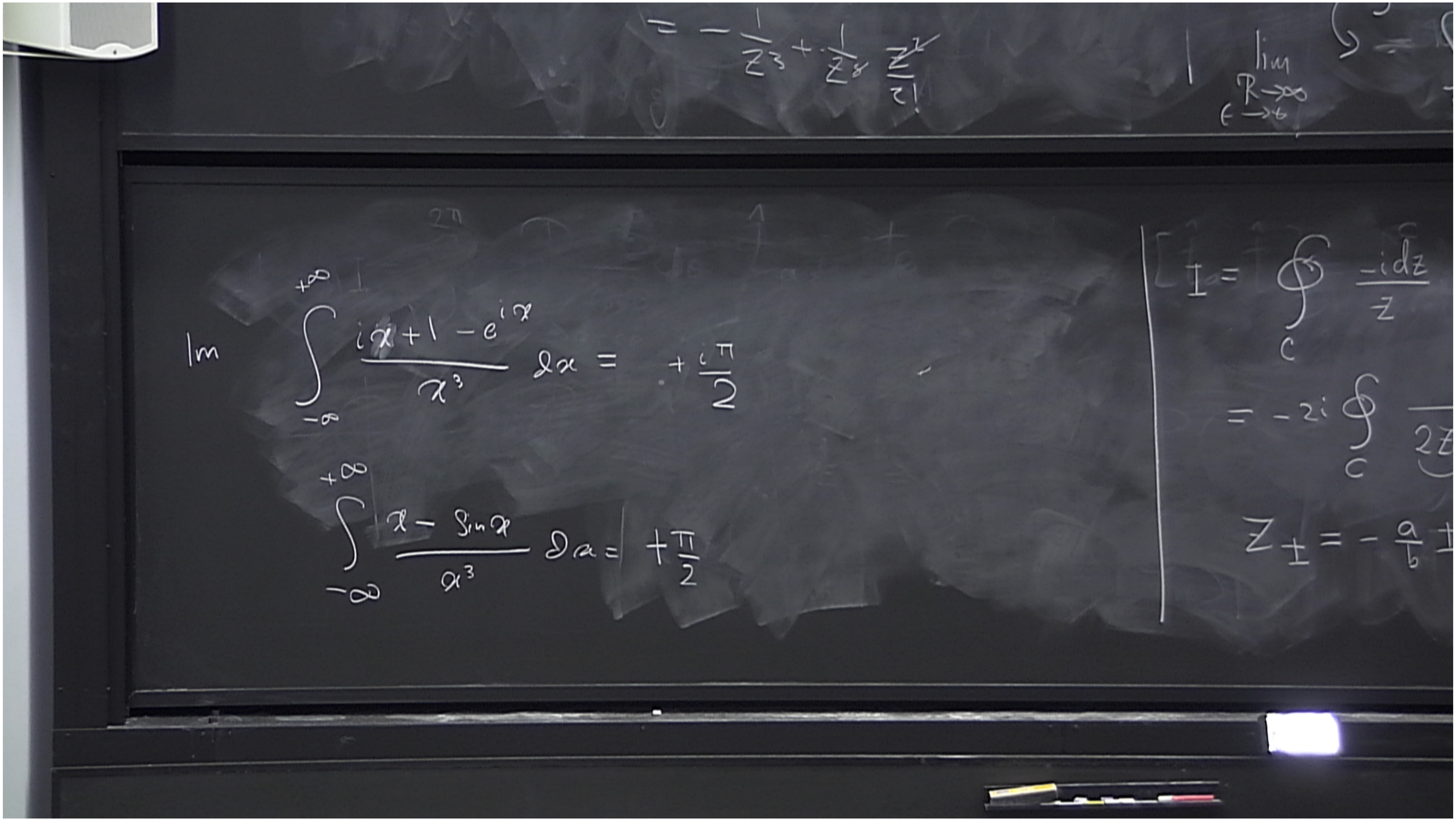
$$= -\frac{1}{z^3} + \frac{1}{z^3}\frac{z^2}{2!}$$

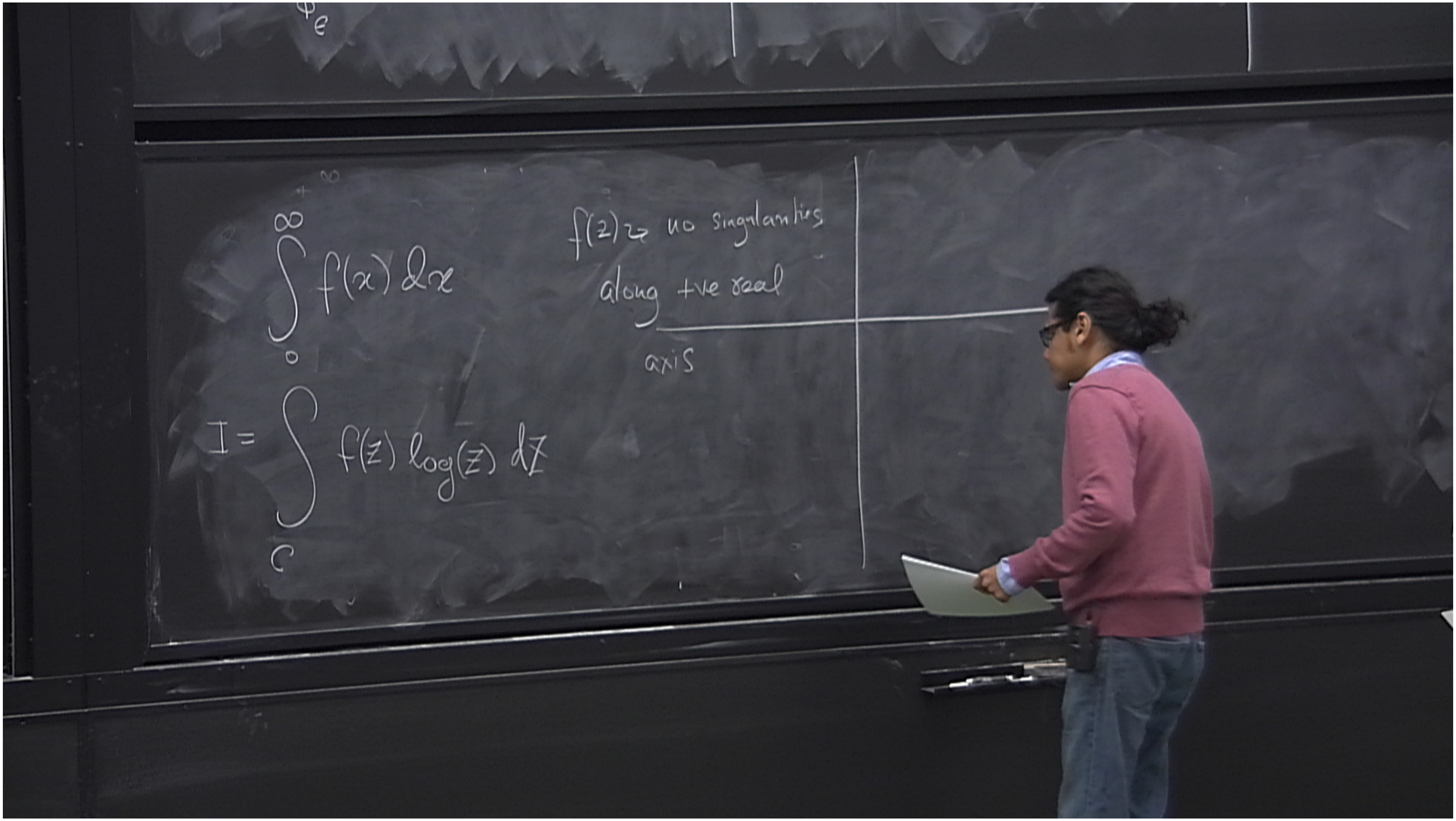
$$f'(z) = f(z) + \frac{1}{z^3}$$

$$= \frac{iz + 1 - e^{iz}}{z^3}$$

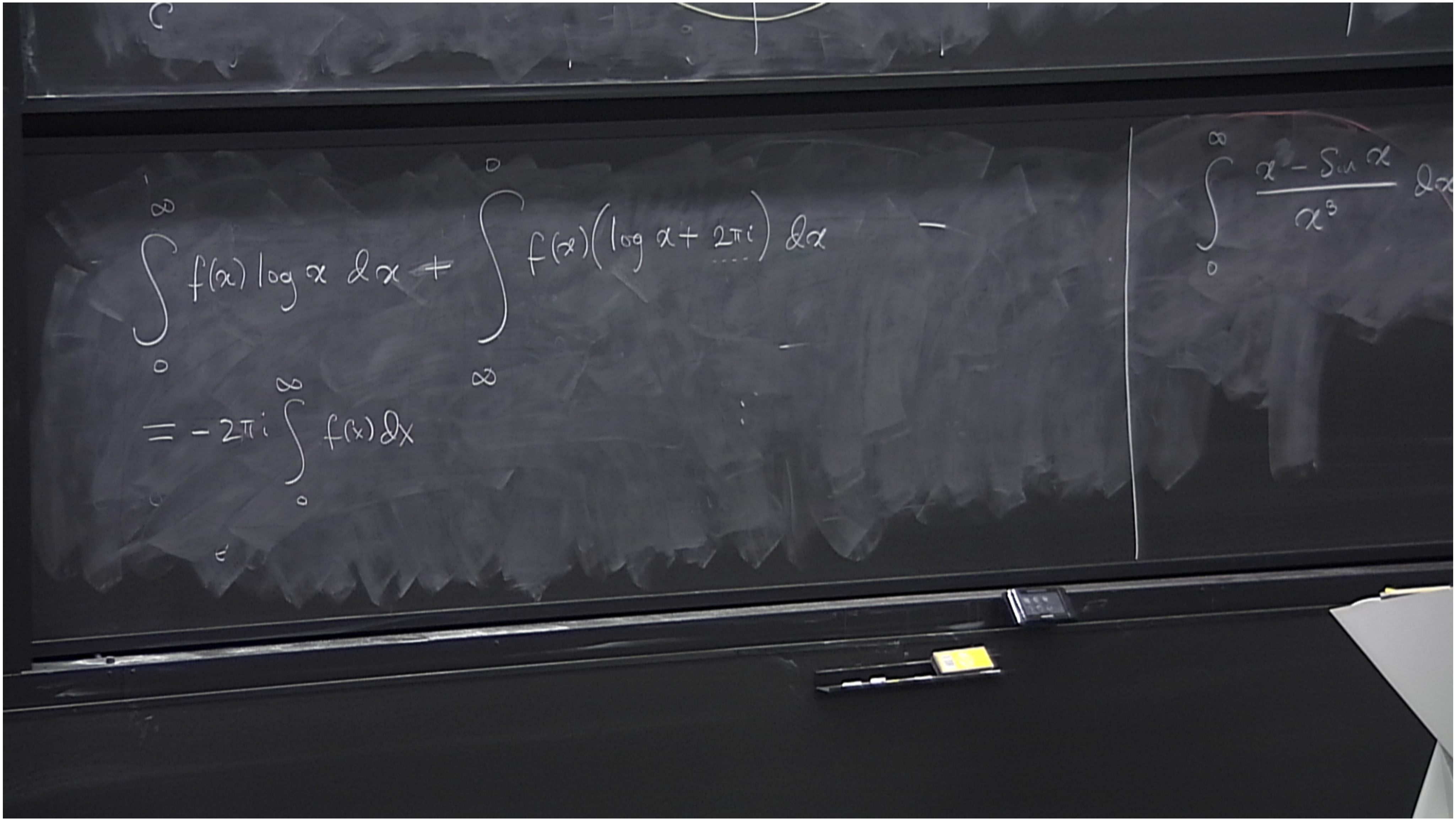

f'(z) = f(z) + 1/z^3
= (iz + 1 - e^{iz})/z^3
∮ f'(z) dz
= ∫_{C1} + ∫_{C3}
C4
C1
C2
ε
R
R → ∞
ε → 0


$f'(z) = f(z) + \frac{1}{z^3}$
$= \frac{iz + 1 - e^{iz}}{z^3}$
$\oint f'(z)\,dz = 0$
$\lim_{R\to\infty, \epsilon\to 0} \int_{C_1} + \int_{C_3} = -i\pi/2$
C_1
C_2
C_3
C_4
$R \to \infty$
$\epsilon \to 0$
$z^2/2!$


$= -\frac{1}{z^3} + \frac{1}{z} \frac{z^2}{2!}$
$\lim_{R\to\infty, \epsilon\to 0}$
$\text{Im} \int_{-\infty}^{+\infty} \frac{ix + 1 - e^{ix}}{x^3} dx = + \frac{i\pi}{2}$
$\int_{-\infty}^{+\infty} \frac{x - \sin x}{x^3} dx = + \frac{\pi}{2}$
$I = \oint_C \frac{-i dz}{z}$
$= -2i \oint_C$
$z_\pm = -\frac{a}{b} \pm$



$\int_0^\infty f(x)\,dx$
f(z) ⇒ no singularities
along +ve real
axis
$I = \int_C f(z) \log(z)\, dz$



$\int_0^\infty f(x)\log x\,dx + \int_\infty^0 f(x)(\log x + 2\pi i)\,dx$
$= -2\pi i \int_0^\infty f(x)\,dx$
$\int_0^\infty \frac{x - \sin x}{x^3}\,dx$

$$\int_0^\infty f(x)\log x \, dx + \int_\infty^0 f(x)\left(\log x + 2\pi i\right) dx \; -$$

$$= -2\pi i \int_0^\infty f(x)\, dx = 2\pi i \sum_n \mathrm{Res}\; f(z)\log(z) \quad \text{inside closed contour}$$

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx$$