Title: Lie Groups & Lie Algebras - Lecture 2

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Abstract:

Commutator $\leftarrow \underbrace{[\hat{t}_a, \hat{t}_b]}_{\hat{t}_a * \hat{t}_b} = f^c{}_{ab}\, \hat{t}_c$

$\hat{t}_a \hat{t}_b - \hat{t}_b \hat{t}_a$

$\hat{t}_a * \hat{t}_b = f^c{}_{ab}\, \hat{t}_c$

$\underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,b,c} + t * (t * t$

- $\hat{t}_a * \hat{t}_b = -\hat{t}_b * \hat{t}_a$
- $\hat{t}_a * (\hat{t}_b * \hat{t}_c) - (\hat{t}_a * \hat{t}_b) * \hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$

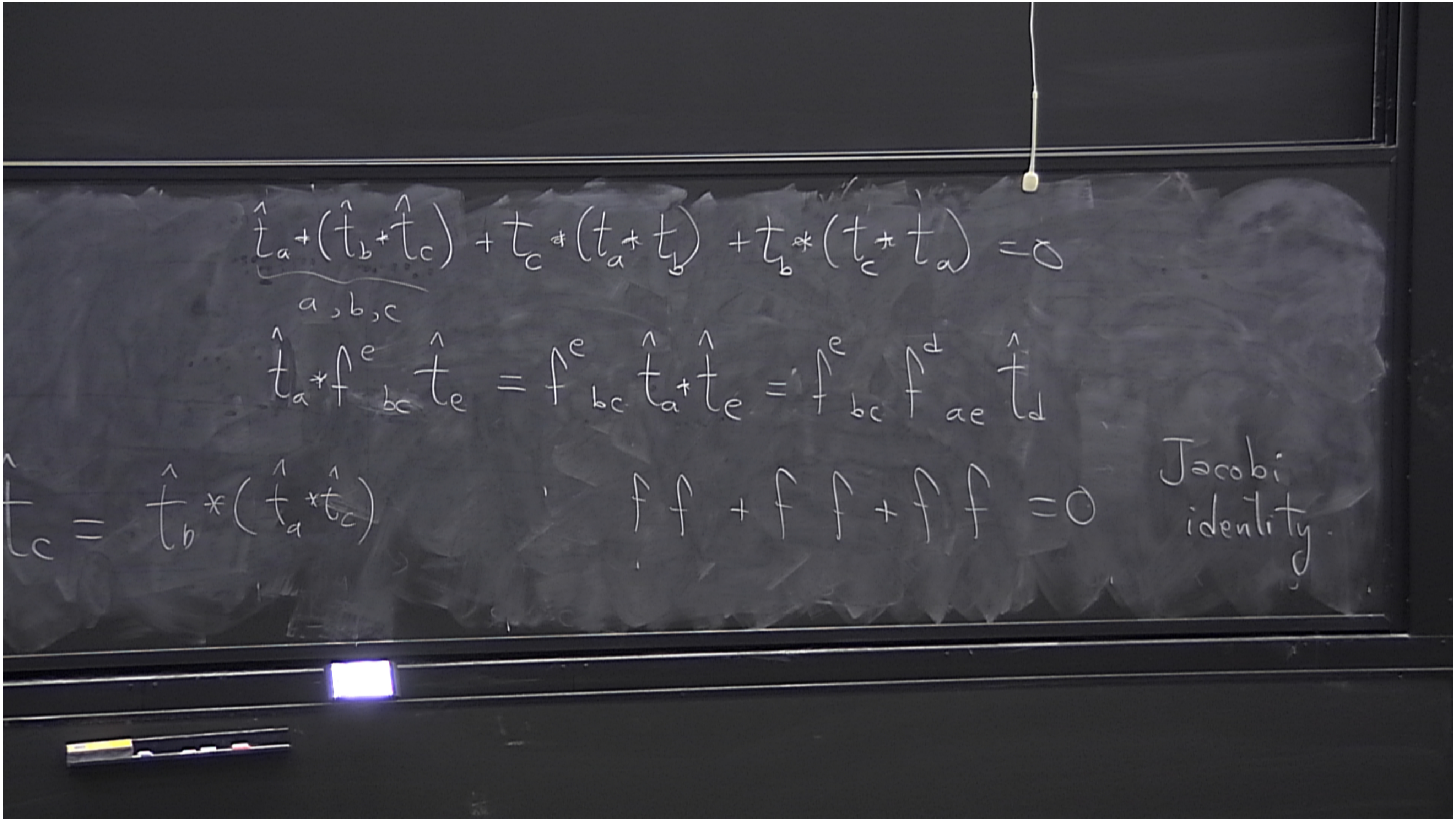
$$\ldots] = f^{c}{}_{ab}\, t_c$$

$$= f^{c}{}_{ab}\, \hat{t}_c$$

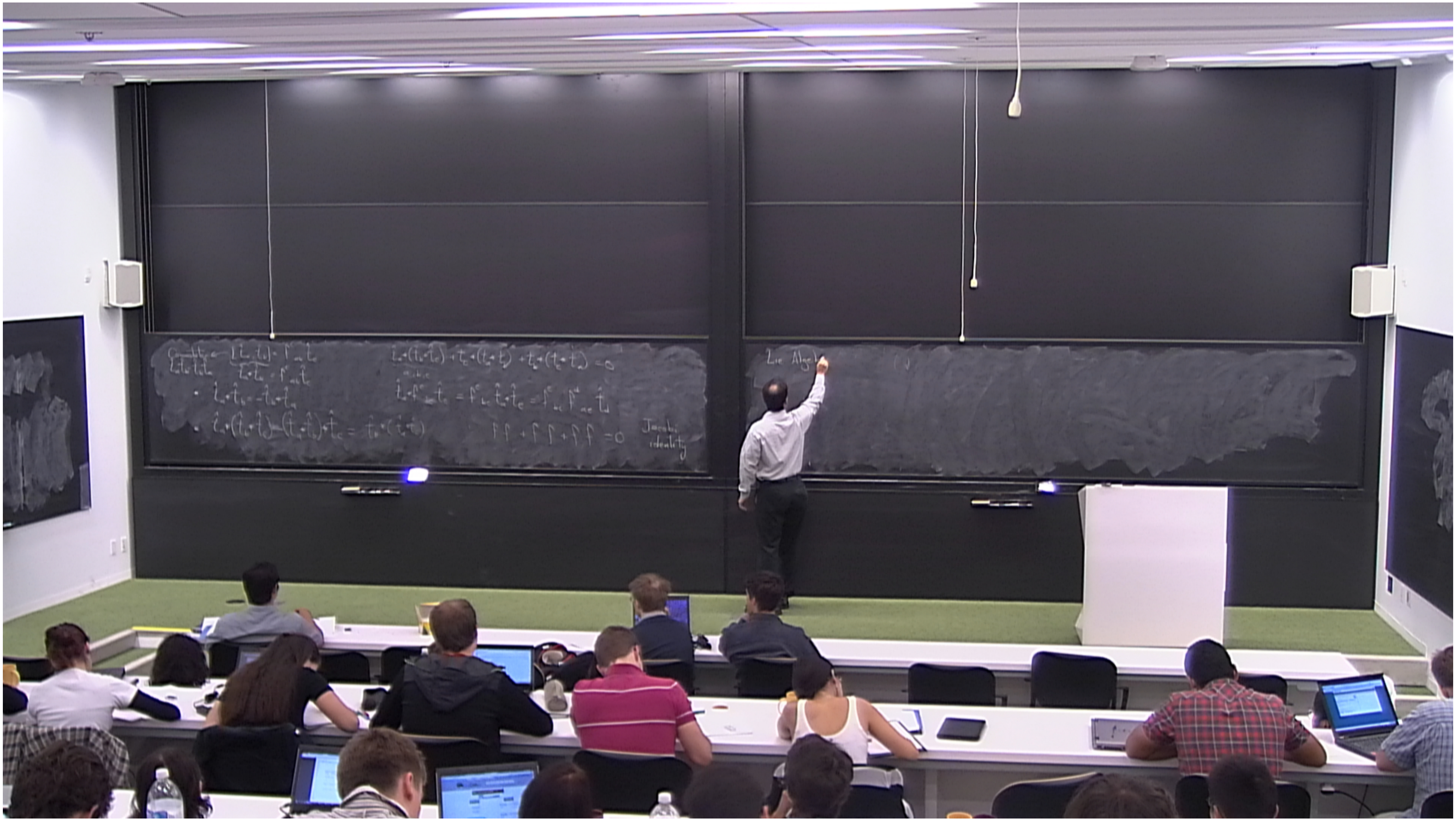
$$\hat{t}_b * \hat{t}_a$$

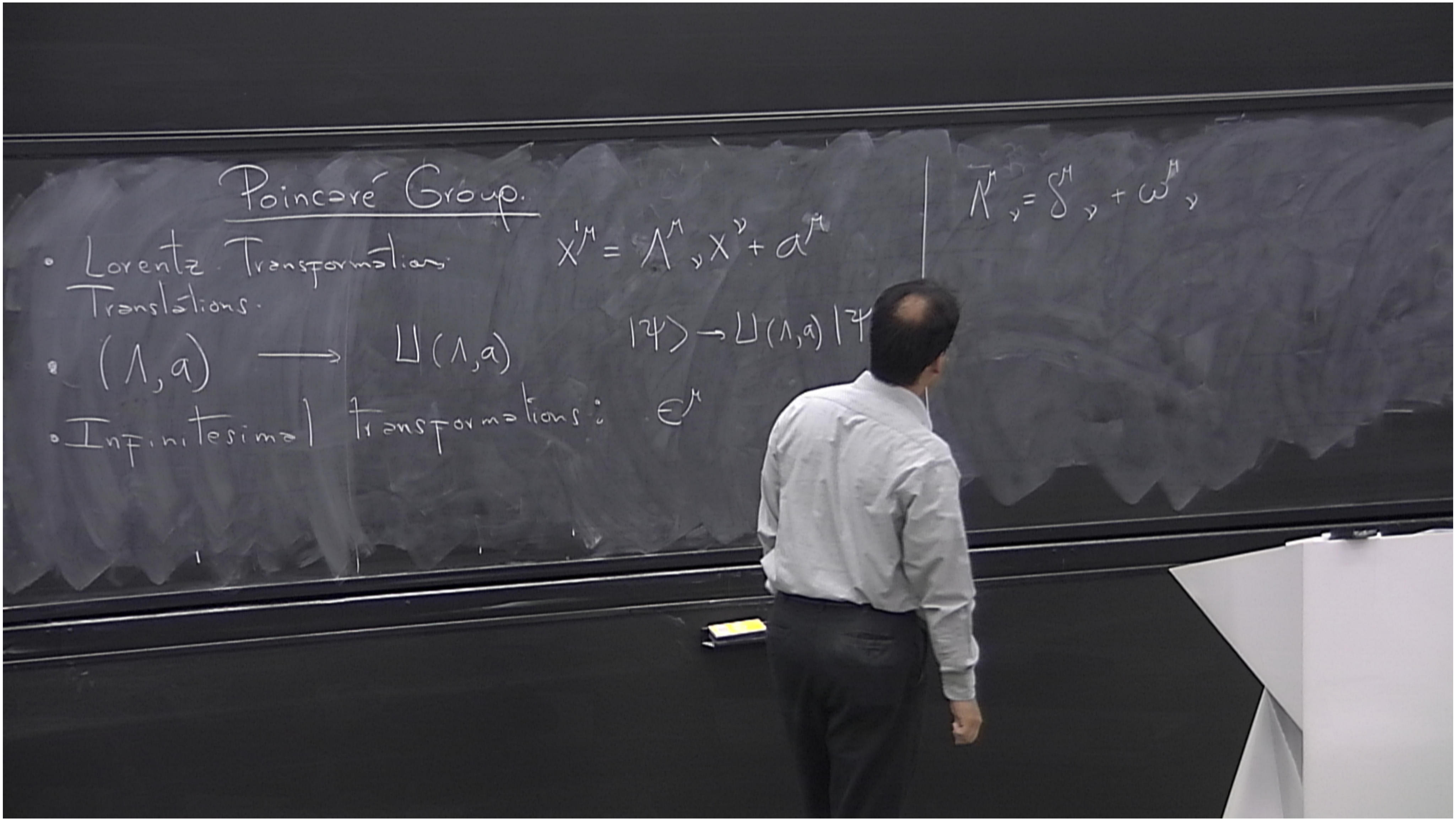
$$\ldots) - (\hat{t}_a * \hat{t}_b) * \hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$$

$$\underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,\, b,\, c} + t_c * (t_a * t_b) + t_b * (t_c * t_a) = 0$$

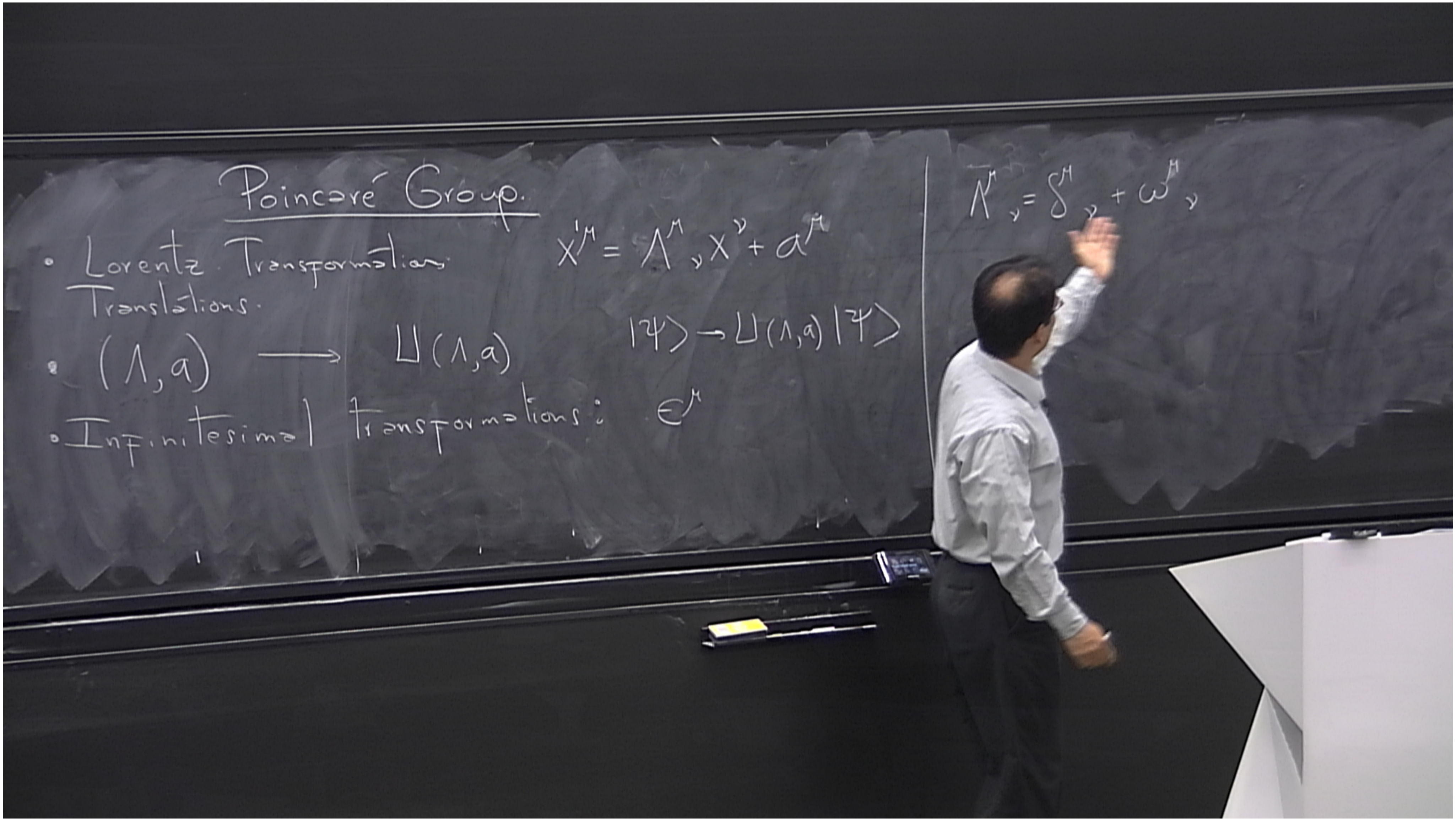


$\underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,b,c} + t_c * (t_a * t_b) + t_b * (t_c * t_a) = 0$
$\hat{t}_a * f^e{}_{bc} \hat{t}_e = f^e{}_{bc} \hat{t}_a * \hat{t}_e = f^e{}_{bc} f^d{}_{ae} \hat{t}_d$
$\hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$
$ff + ff + ff = 0$
Jacobi identity





Poincaré Group.
• Lorentz Transformations
Translations.
• $(\Lambda, a) \longrightarrow U(\Lambda, a)$
$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$
$|\psi\rangle \rightarrow U(\Lambda, a)|\psi$
• Infinitesimal transformations: ϵ^{μ}
$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$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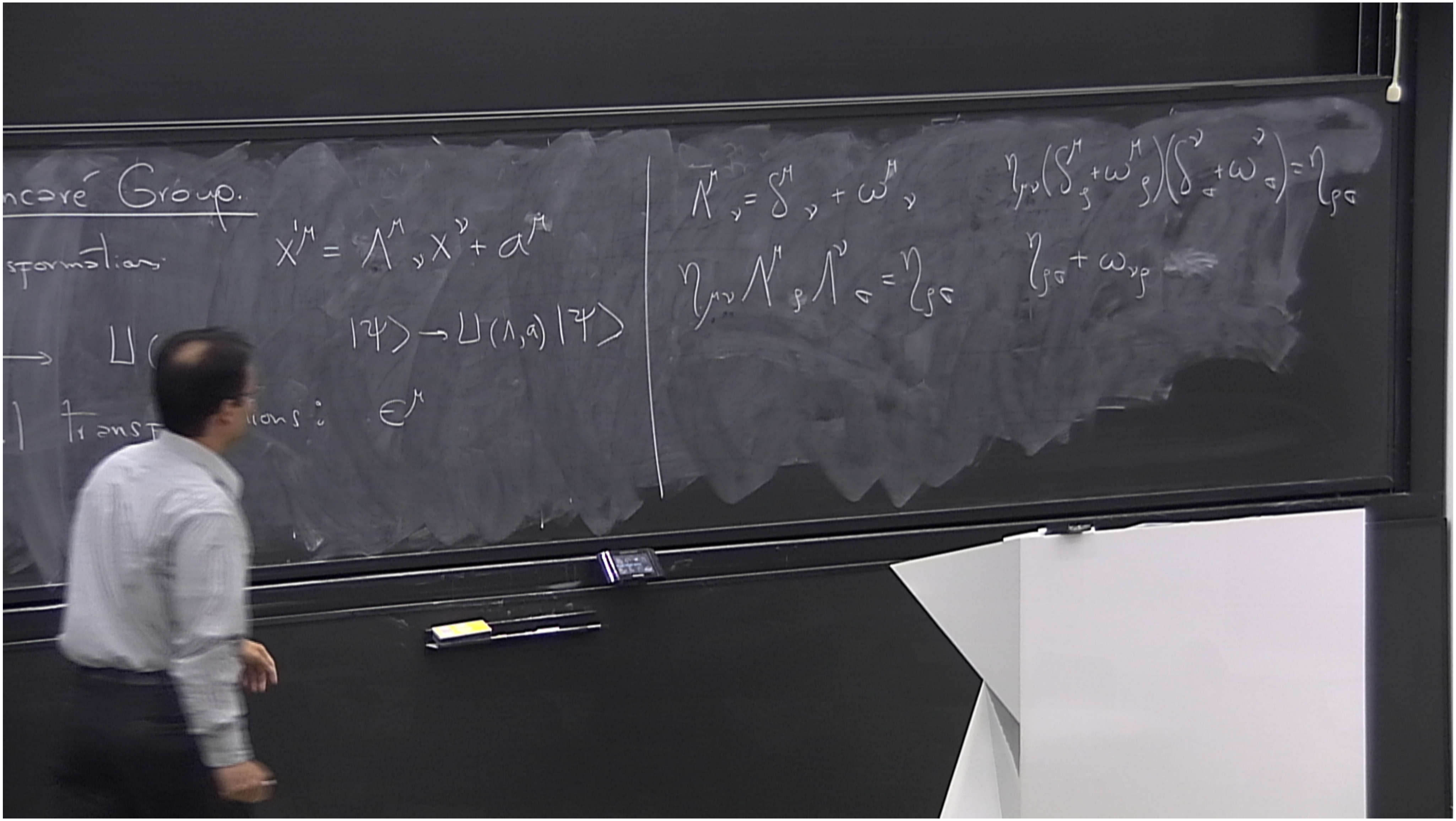
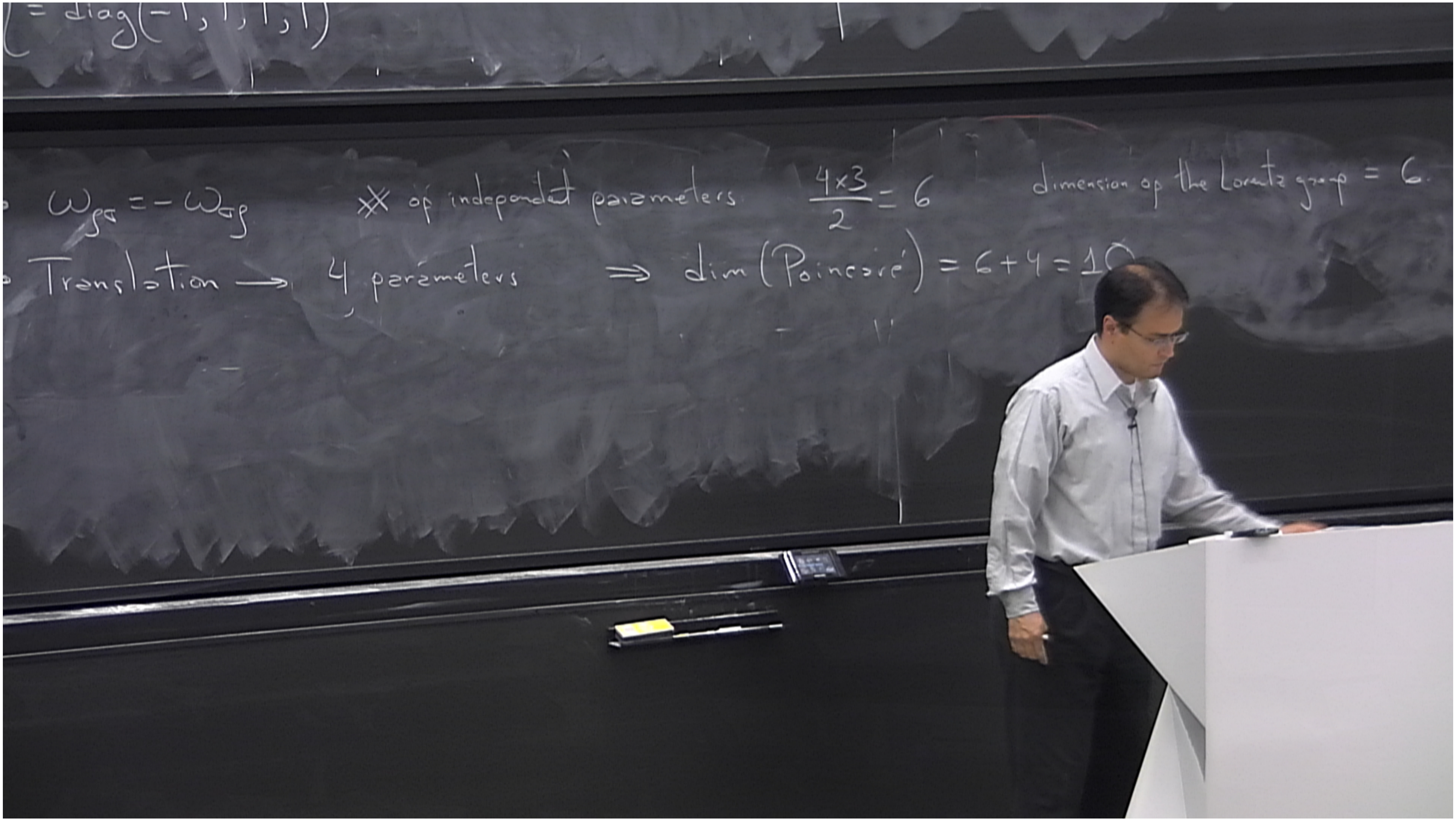

ncaré Group.
sformations
$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}$
$|\psi\rangle \to U(\Lambda, a)|\psi\rangle$
ϵ^{μ}
$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$
$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$
$\eta_{\mu\nu}(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho})(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) = \eta_{\rho\sigma}$
$\eta_{\rho\sigma} + \omega_{\nu\rho}$


= diag(−1, 1, 1, 1)
ω_ρσ = −ω_σρ
# of independent parameters
4×3/2 = 6
dimension of the Lorentz group = 6.
Translation → 4 parameters
⇒ dim(Poincaré) = 6+4 = 10

$\eta = \mathrm{diag}(-1,1,1,1)$

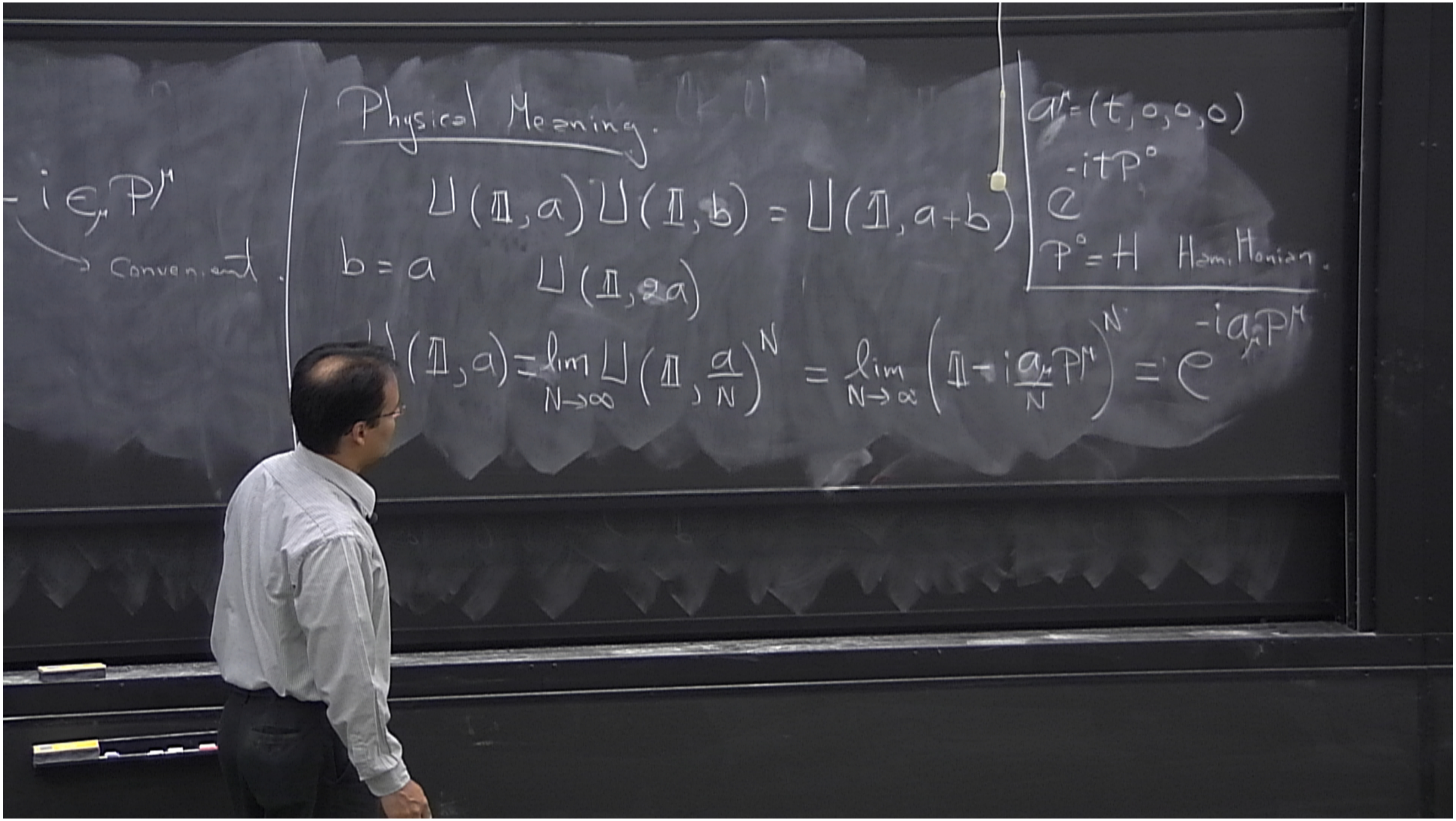
- $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ # of independent parameters $\frac{4\times 3}{2} = 6$ dimension of the Lorentz group =
- Translation $\longrightarrow$ 4 parameters $\Rightarrow$ dim(Poincaré) $= 6+4 = 10$

  ↳ Abelian $U(\mathbb{1}, a)$
- Lorentz $U(\Lambda, 0)$

Unitary Operator

$$U(\mathbb{1}+\omega, \epsilon) = \mathbb{1} + i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_{\mu} P^{\mu}$$

$J^{\mu\nu}, P^{\mu}$ Hermitian. Convenient.



Physical Meaning.
$U(\mathbb{1},a)\,U(\mathbb{1},b) = U(\mathbb{1},a+b)$
$b=a$ $\quad U(\mathbb{1},2a)$
$U(\mathbb{1},a) = \lim_{N\to\infty} U\left(\mathbb{1},\frac{a}{N}\right)^N = \lim_{N\to\infty}\left(\mathbb{1} - i\frac{a_\mu}{N}P^\mu\right)^N = e^{-ia_\mu P^\mu}$
$a^\mu = (t,0,0,0)$
e^{-itP^0}
$P^0 = H$ Hamiltonian.
$-i\epsilon_\mu P^\mu$
Convenient.

$\mathcal{D}(\mathbb{1}, a) = \lim_{N\to\infty} \mathcal{D}\left(\mathbb{1}, \frac{a}{N}\right) =$

$\vec{\mathbb{P}} = (P^1, P^2, P^3)$ $\longrightarrow$ Momentum Operators

$\vec{\mathbb{J}} = (J^{23}, J^{31}, J^{12})$ $\longrightarrow$ Angular momentum operators

$$U(\mathbb{1}, a) = \lim_{N \to \infty} U\left(\mathbb{1}, \frac{a}{N}\right)^N = \lim_{N \to \infty} \left(\mathbb{1} - i \frac{a_\mu}{N} P^\mu\right)^N = e$$

→ Momentum Operators

→ Angular momentum operators

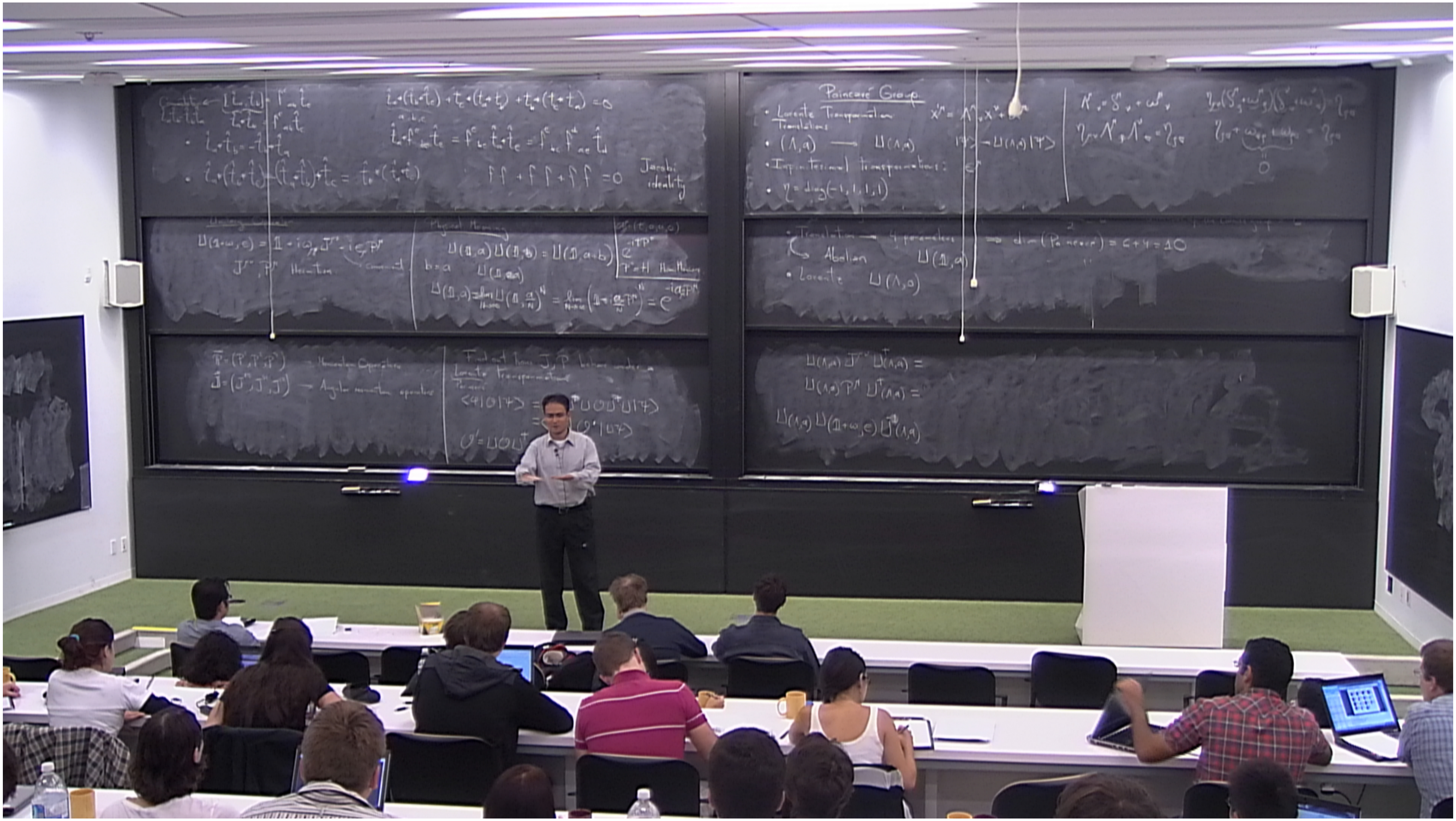
Find out how J, P behave under a Lorentz transformation

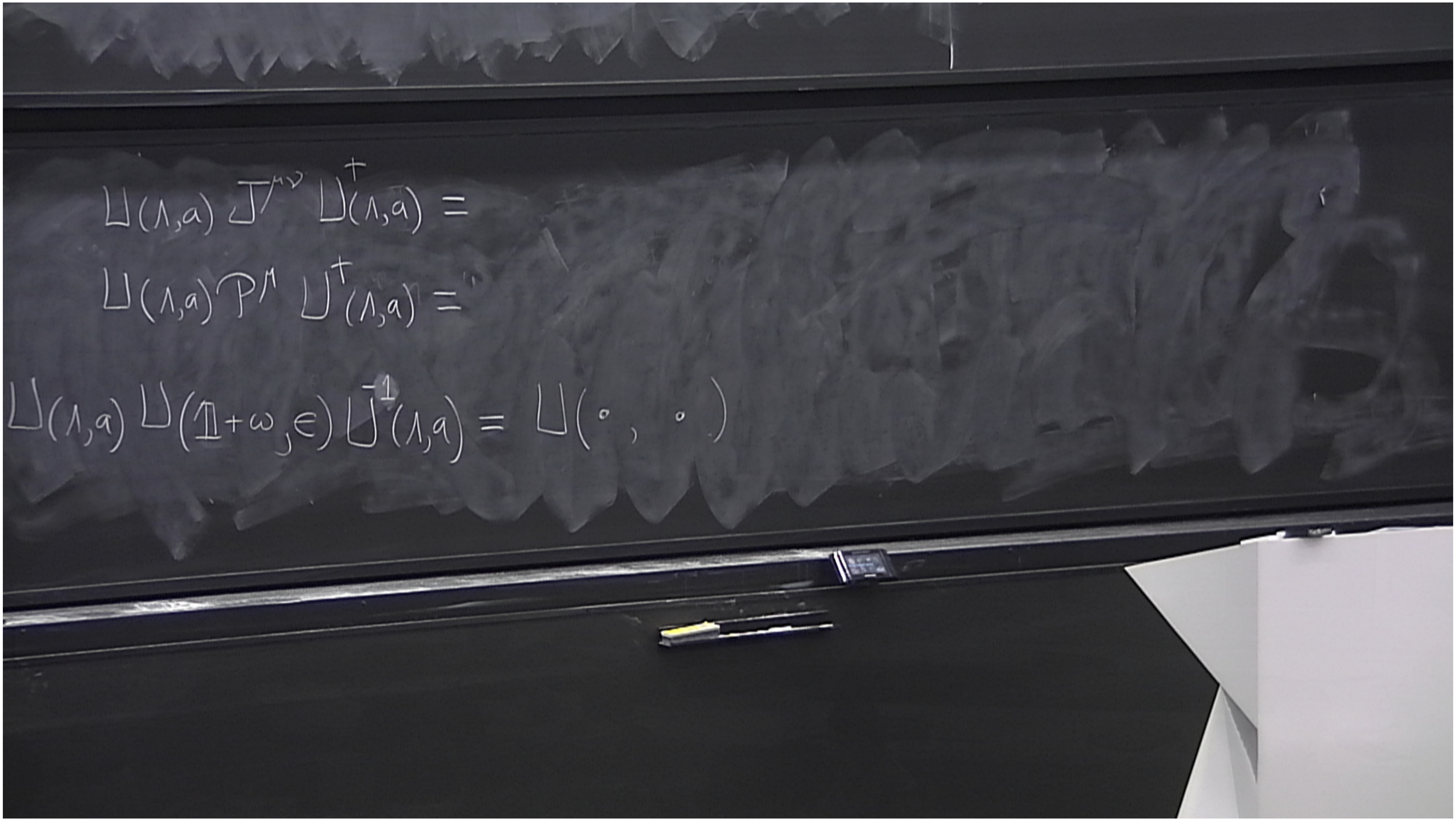
Poincaré

$$\langle \varphi | \mathcal{O} | \psi \rangle = \langle \varphi | U^\dagger U \mathcal{O} U^\dagger U | \psi \rangle$$

$$= \langle U\varphi | \mathcal{O}' | U\psi \rangle$$

$$\mathcal{O}' = U \mathcal{O} U^\dagger$$


Poincaré Group
Lorentz Transformation
Translations
Jacobi identity
Abelian
Lorentz
Hermitian



$U(\Lambda,a)\, J^{\mu\nu}\, U^\dagger(\Lambda,a) =$
$U(\Lambda,a)\, P^\mu\, U^\dagger(\Lambda,a) =$
$U(\Lambda,a)\, U(\mathbb{1}+\omega,\epsilon)\, U^{-1}(\Lambda,a) = U(\,\cdot\,,\,\cdot\,)$

$$U(\Lambda,a)\left(\mathbb{1} + i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,a) = U($$

$$U(\Lambda,a)\, J^{\mu\nu}\, U^\dagger(\Lambda,a) =$$

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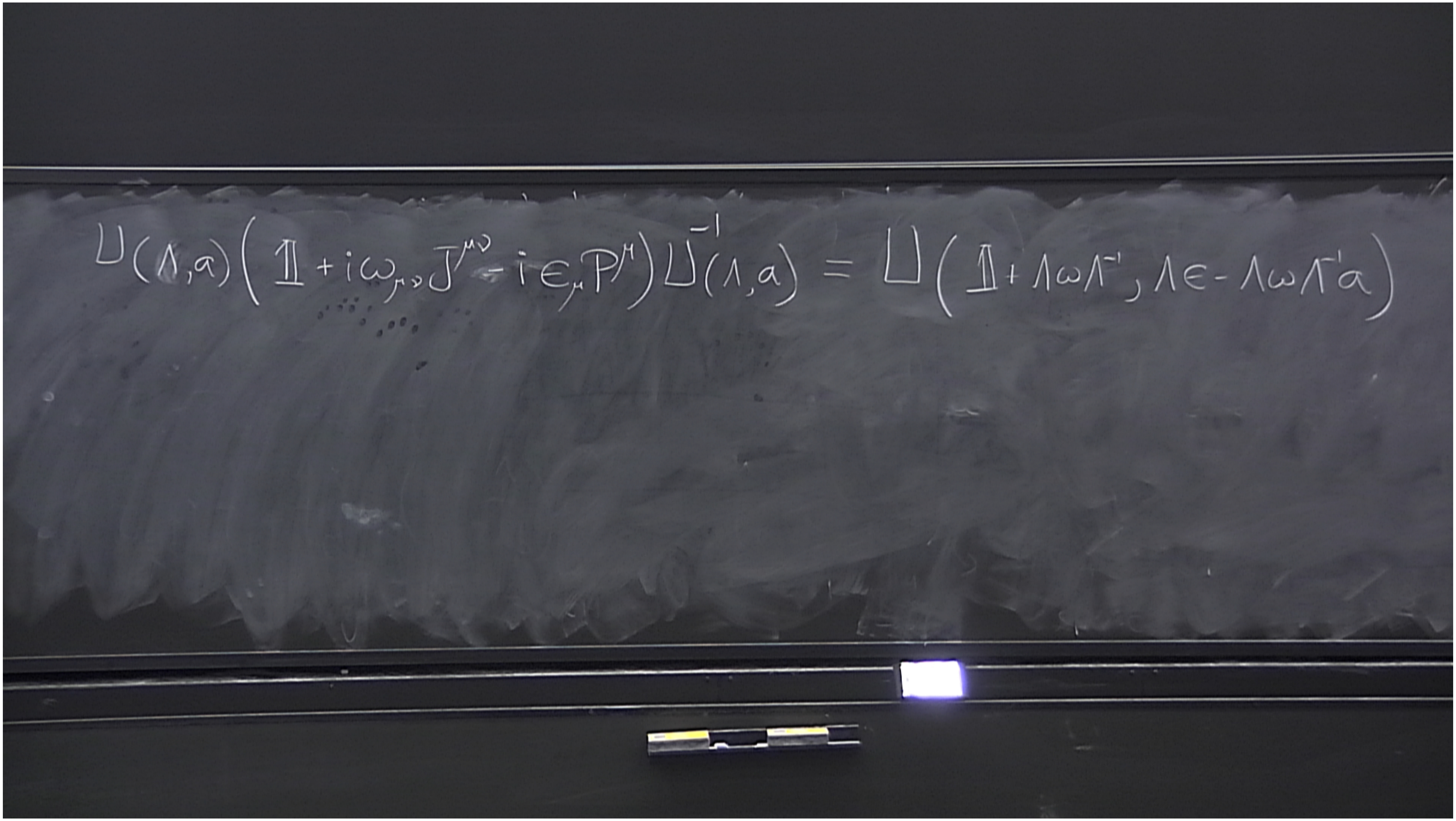


$U(\Lambda,a)\left(\mathbb{1} + i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,a) = U\left(\mathbb{1} + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\right)$

Ex:

$$U(\Lambda,a)\, J^{\mu\nu}\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right)$$

$$U(\Lambda,a)\, P^\mu\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu P^\rho$$

Find the Poincaré Algebra

$$\Lambda = \mathbb{1} + \tilde{\omega} \qquad a^\mu = \tilde{\epsilon}^\mu$$

Poincaré Algebra

$$i\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}$$

$$i\left[P^{\mu}, J^{\rho\sigma}\right] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}$$

$$\left[P^{\mu}, P^{\nu}\right] = 0$$

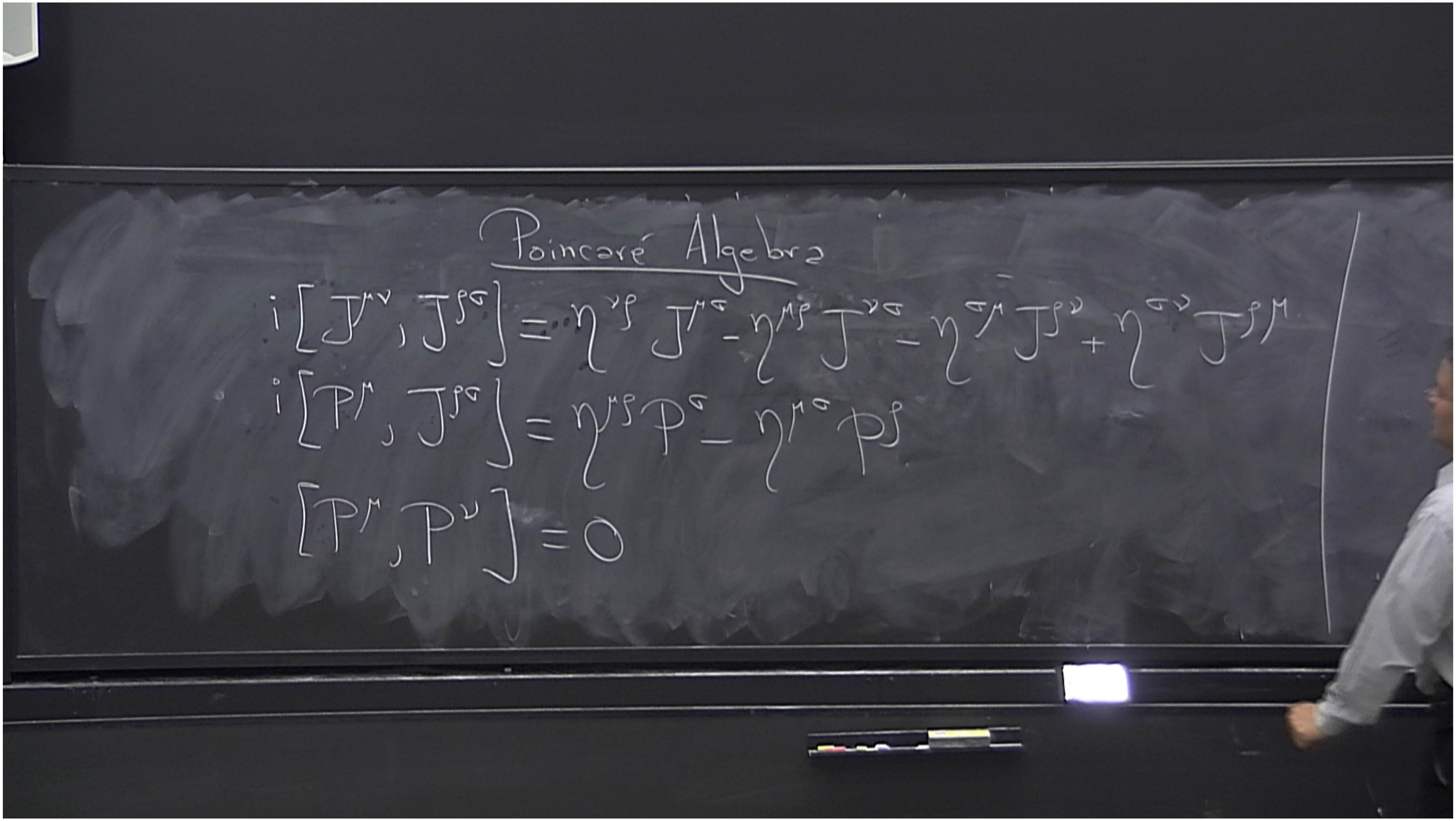
Ex:

$$U(\Lambda,a)\, J^{\mu\nu}\, U^{\dagger}(\Lambda,a) = \Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}\left(J^{\rho\sigma} - a^{\rho}P^{\sigma} + a^{\sigma}P^{\rho}\right)$$

$$U(\Lambda,a)\, P^{\mu}\, U^{\dagger}(\Lambda,a) = \Lambda_{\rho}{}^{\mu} P^{\rho}$$

Find the Poincaré Algebra

$$\Lambda = \mathbb{1} + \tilde{\omega} \qquad a^{\mu} = \tilde{\epsilon}^{\mu}$$


Poincaré Algebra
$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}$
$i[P^{\mu}, J^{\rho\sigma}] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}$
$[P^{\mu}, P^{\nu}] = 0$

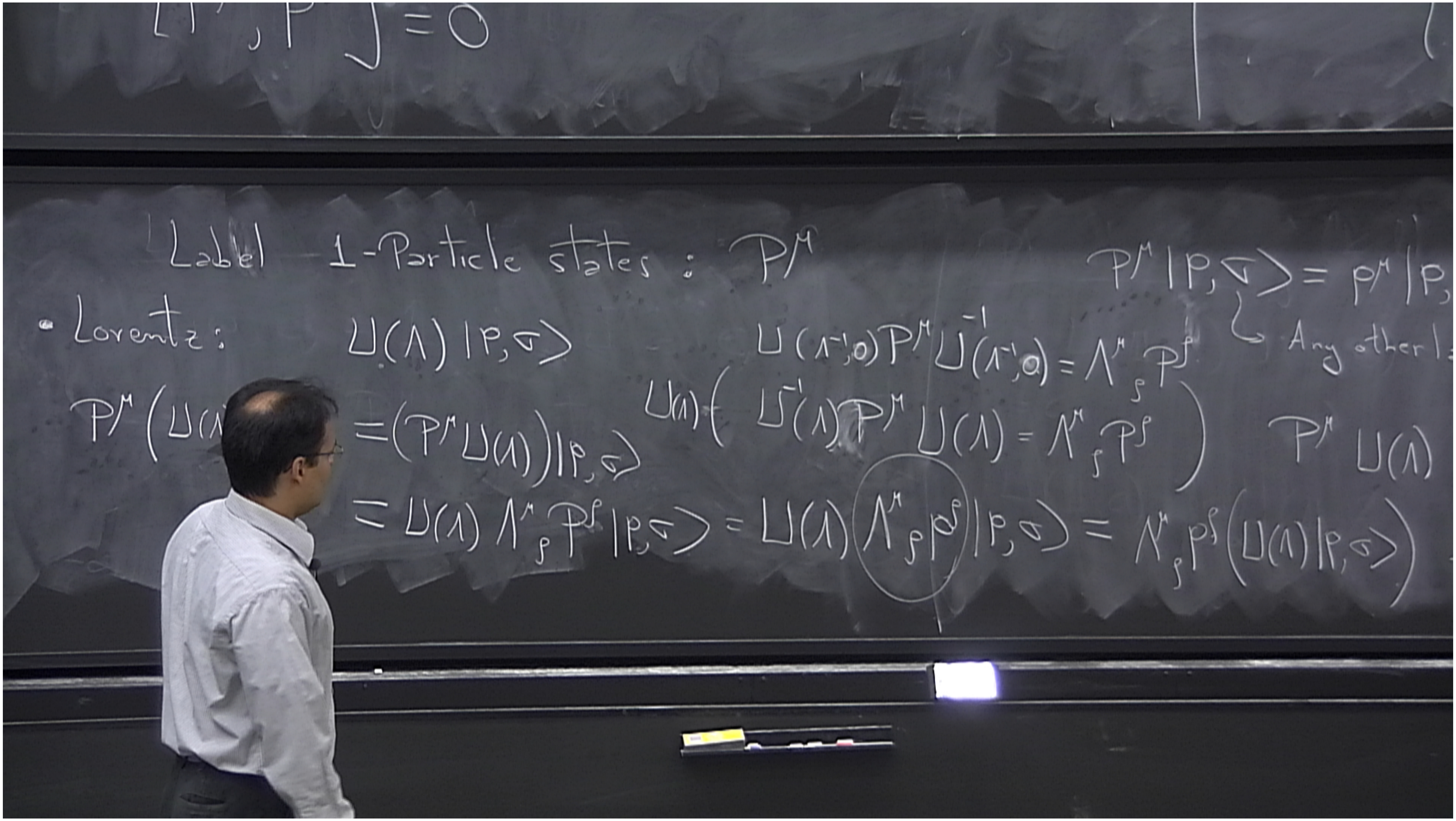
$[\,,P\,] = 0$

Label 1-Particle states : P^μ

$P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle$

Any other label nee

entz:


Label 1-Particle states : P^μ
$P^\mu |p,\sigma\rangle = p^\mu |p,$
Any other
• Lorentz: $U(\Lambda)|p,\sigma\rangle$
$U(\Lambda^{-1},0) P^\mu U^{-1}(\Lambda^{-1},0) = \Lambda^\mu{}_\rho P^\rho$
$U(\Lambda)\left(U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\rho P^\rho \right)$
$P^\mu U(\Lambda)$
$= (P^\mu U(\Lambda))|p,\sigma\rangle$
$= U(\Lambda)\Lambda^\mu{}_\rho P^\rho |p,\sigma\rangle = U(\Lambda)\Lambda^\mu{}_\rho p^\rho |p,\sigma\rangle = \Lambda^\mu{}_\rho p^\rho \left(U(\Lambda)|p,\sigma\rangle\right)$

$[\ldots, P\ ] = 0$

Label 1-Particle states : P^μ $\quad$ $P^\mu |p,\sigma\rangle = p^\mu |p,$

- Lorentz: $U(\Lambda)|p,\sigma\rangle$ $\quad$ $U(\Lambda^{-1},0)\, P^\mu\, U^{-1}(\Lambda^{-1},0) = \Lambda^\mu{}_\rho P^\rho$ $\quad$ Any other |

$$P^\mu \big(U(\Lambda)|p,\sigma\rangle\big) = \big(P^\mu U(\Lambda)\big)|p,\sigma\rangle \qquad U(\Lambda)\Big(U^{-1}(\Lambda)\, P^\mu\, U(\Lambda) = \Lambda^\mu{}_\rho P^\rho\Big) \qquad P^\mu U(\Lambda)$$

$$= U(\Lambda)\,\Lambda^\mu{}_\rho P^\rho |p,\sigma\rangle = U(\Lambda)\,\Lambda^\mu{}_\rho p^\rho |p,\sigma\rangle = \Lambda^\mu{}_\rho p^\rho \big(U(\Lambda)|p,\sigma\rangle\big)$$

Conclusion: $\boxed{U(\Lambda)\,|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle}$

<u>Trick:</u> Note that $p^\mu p_\mu = p^2 = -M^2$ First label.

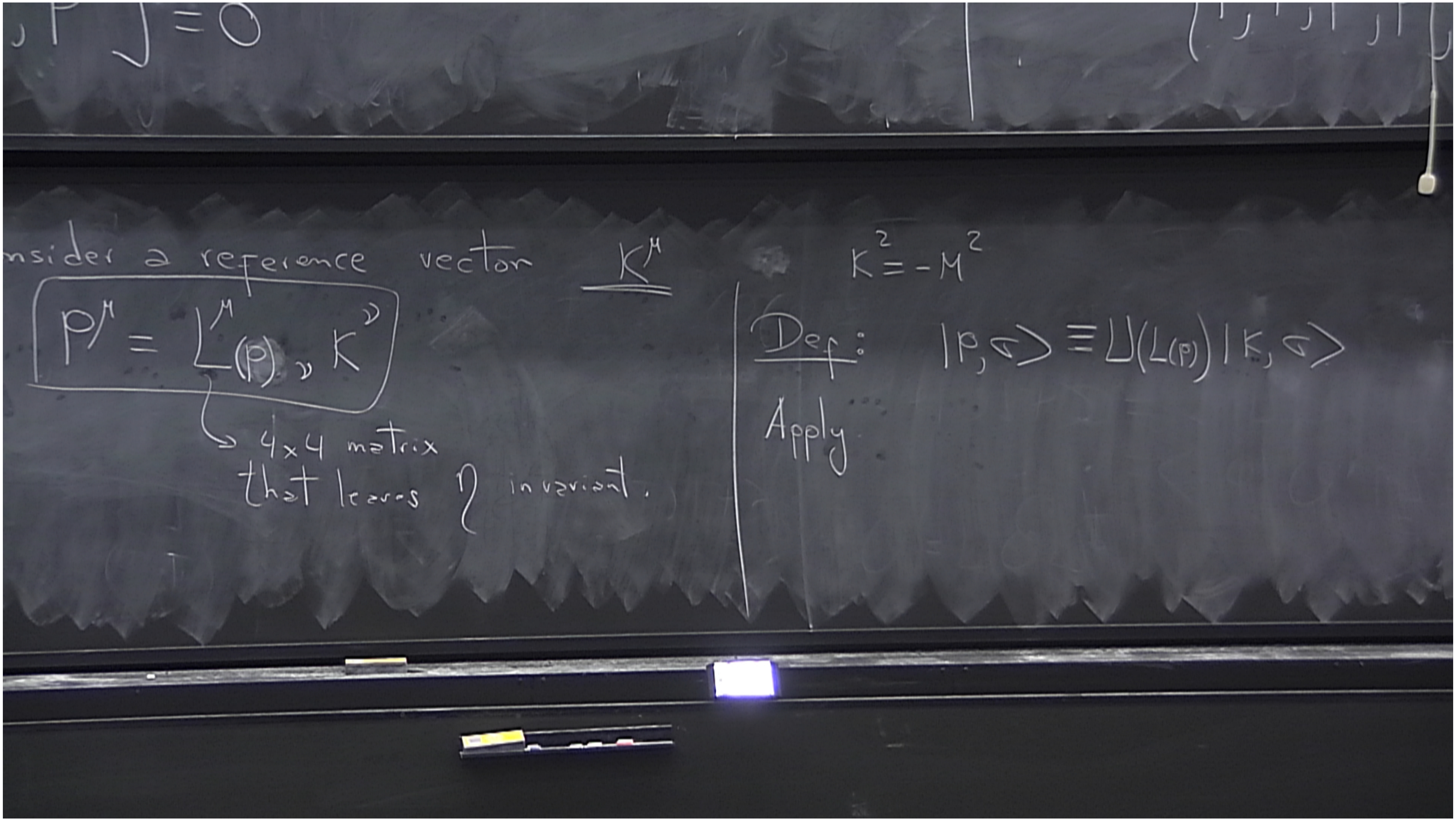
$M \to$ Mass

$p^2 < 0$ sign(p^0) is Lorentz invariant.

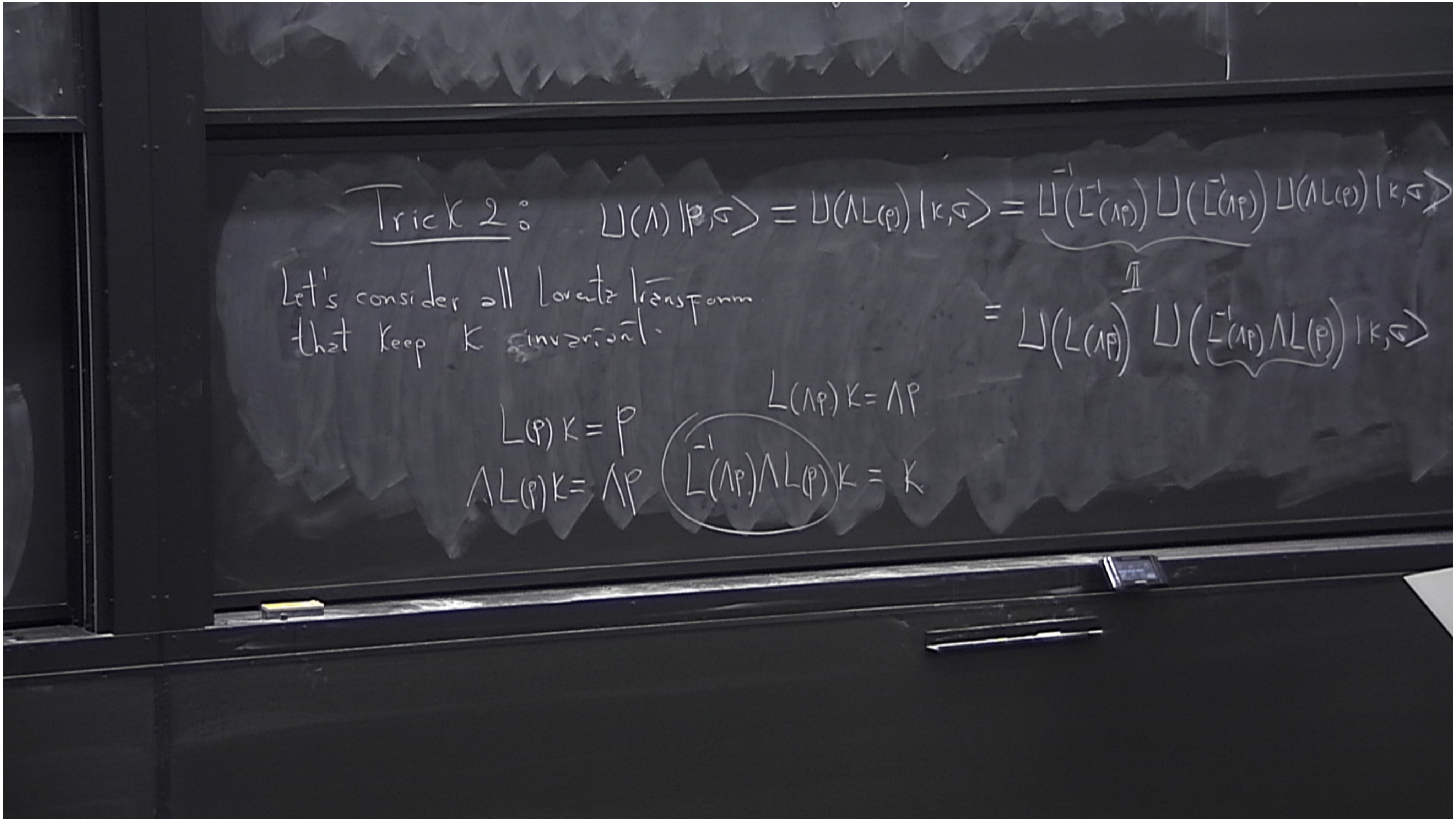
Let's consider a reference vector $\underline{\underline{K^\mu}}$ $\quad K^2 = -M^2$

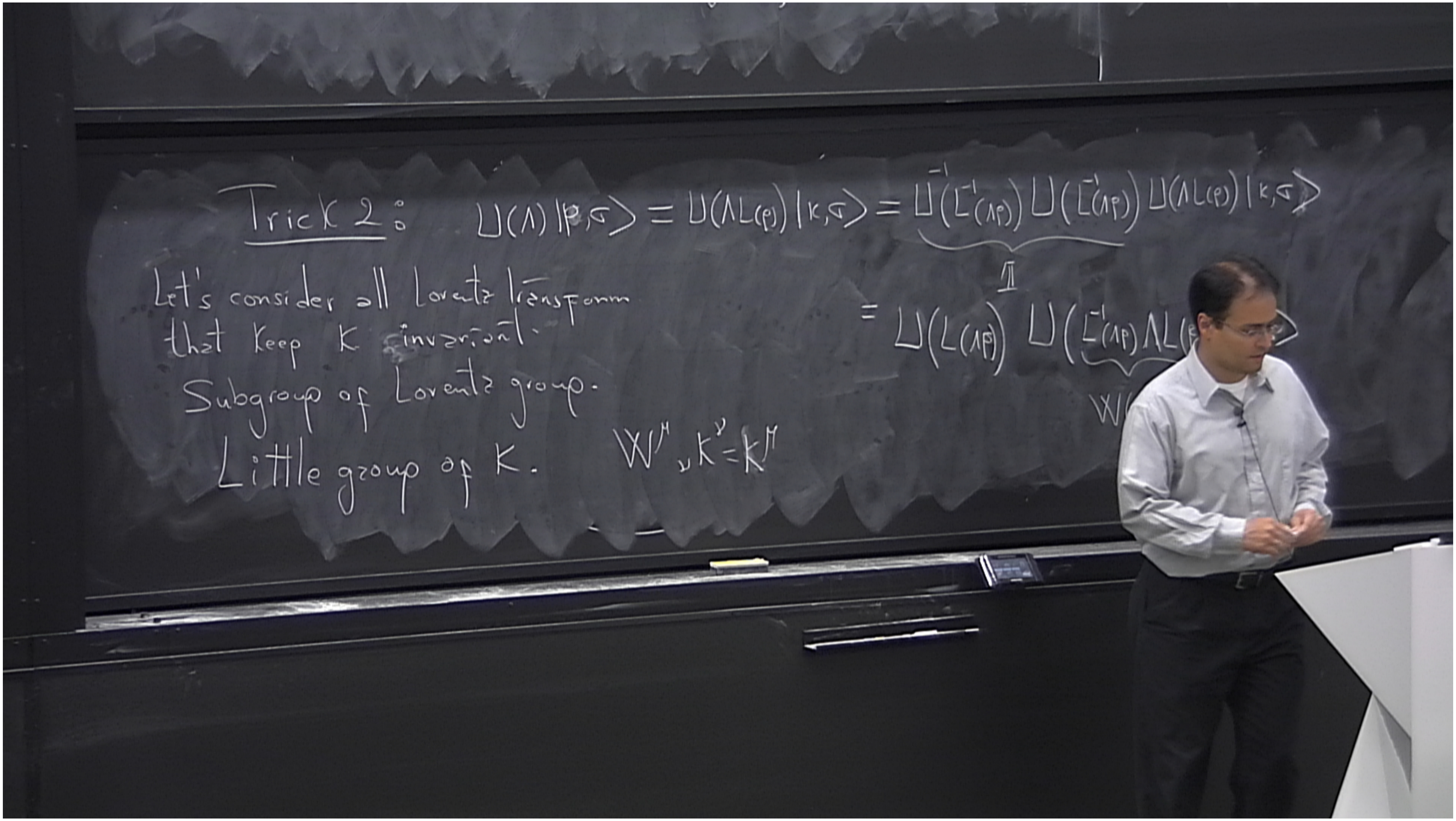
$$\boxed{p^\mu = L^\mu{}_\nu(p)\, K^\nu}$$

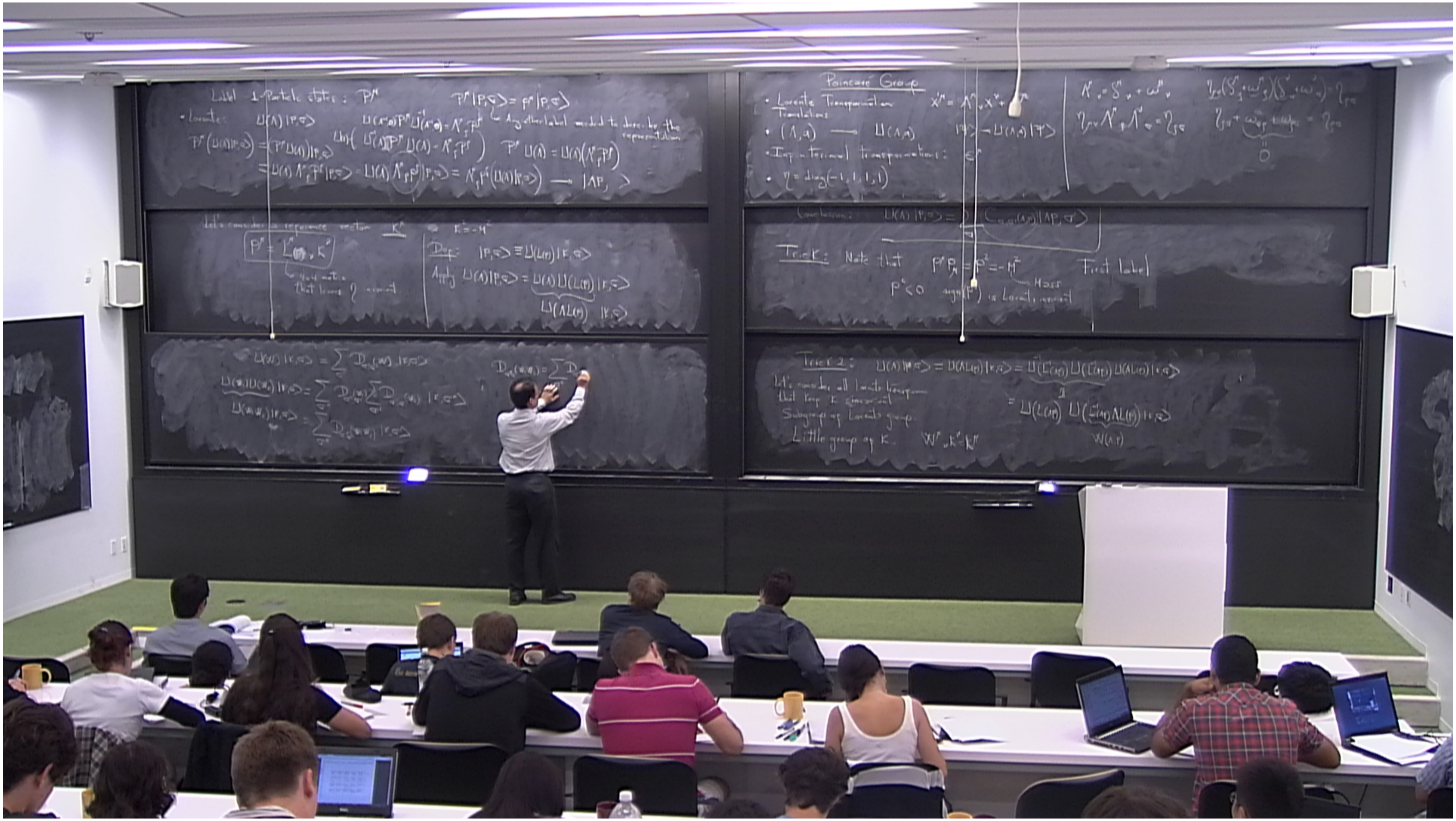
4x4 matrix that leaves η invariant.



, P] = 0
nsider a reference vector K^μ
p^μ = L^μ_ν(p) K^ν
4x4 matrix that leaves η invariant.
K^2 = -M^2
Def: |p,σ> ≡ U(L(p)) |k,σ>
Apply


Trick 2: $U(\Lambda)|p,\sigma\rangle = U(\Lambda L(p))|k,\sigma\rangle = \underbrace{U^{-1}(L^{-1}(\Lambda p))\,U(L^{-1}(\Lambda p))}_{\mathbb{1}}\,U(\Lambda L(p))|k,\sigma\rangle$
$= U(L(\Lambda p))\,U(\underbrace{L^{-1}(\Lambda p)\Lambda L(p)}_{})|k,\sigma\rangle$
Let's consider all Lorentz transform that keep k invariant.
$L(\Lambda p)k = \Lambda p$
$L(p)k = p$
$\Lambda L(p)k = \Lambda p$
$L^{-1}(\Lambda p)\Lambda L(p)k = k$


Trick 2: $U(\Lambda)|p,\sigma\rangle = U(\Lambda L(p))|k,\sigma\rangle = U^{-1}(L^{-1}(\Lambda p))\,U(L^{-1}(\Lambda p))\,U(\Lambda L(p))|k,\sigma\rangle$
$= U(L(\Lambda p))\,U(L^{-1}(\Lambda p)\Lambda L(p)$
Let's consider all Lorentz transform
that keep k invariant.
Subgroup of Lorentz group.
Little group of k.
$W^\mu{}_\nu k^\nu = k^\mu$



$U(\Lambda L(p))\ |k,\sigma\rangle$

$= \sum_{\sigma'} D_{\sigma'\sigma}(W)\ |k,\sigma'\rangle$

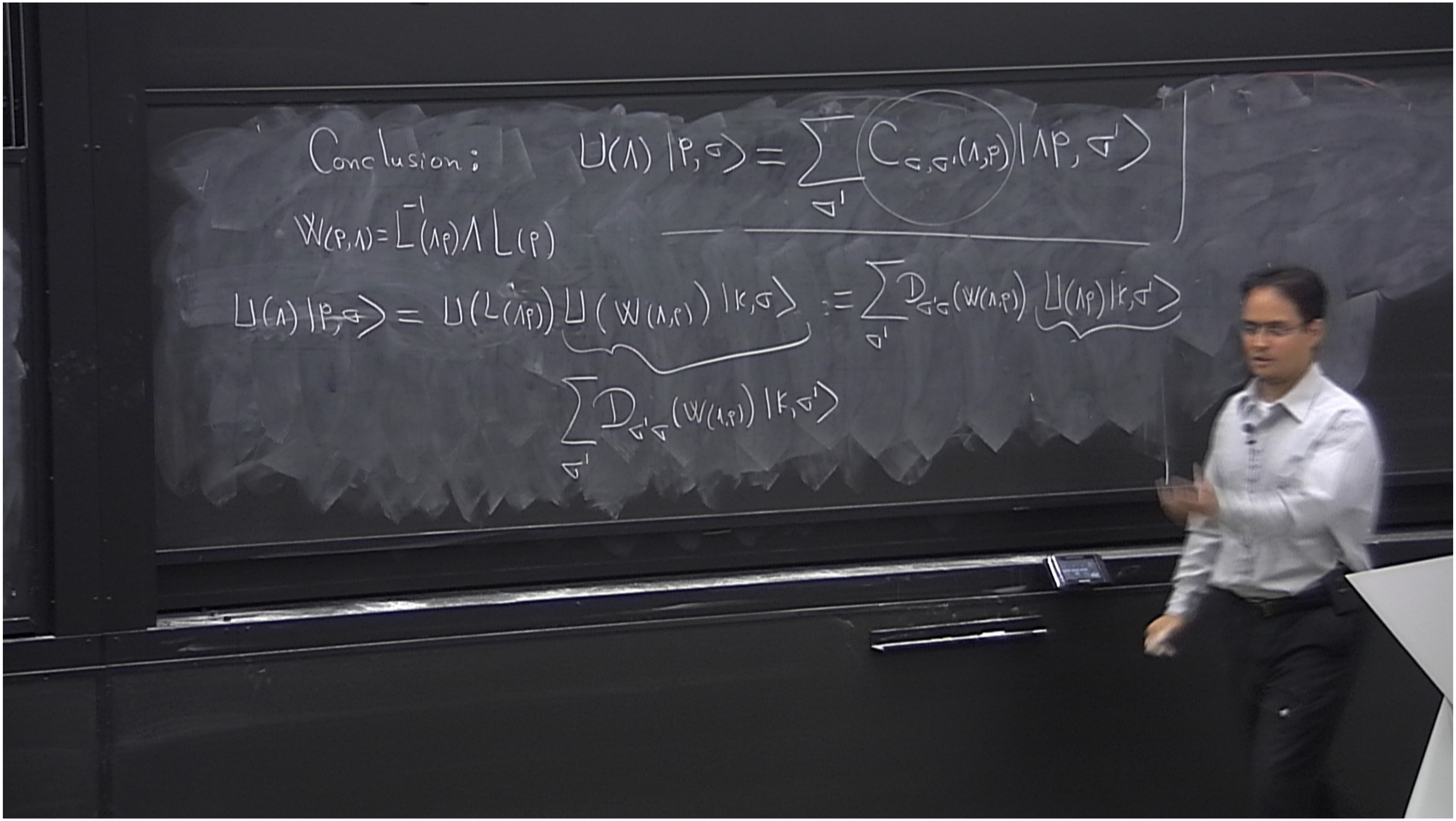
$D_{\sigma''\sigma'}(W_1) \sum_{\sigma'} D_{\sigma'\sigma}(W_2)\ |k,\sigma''\rangle$

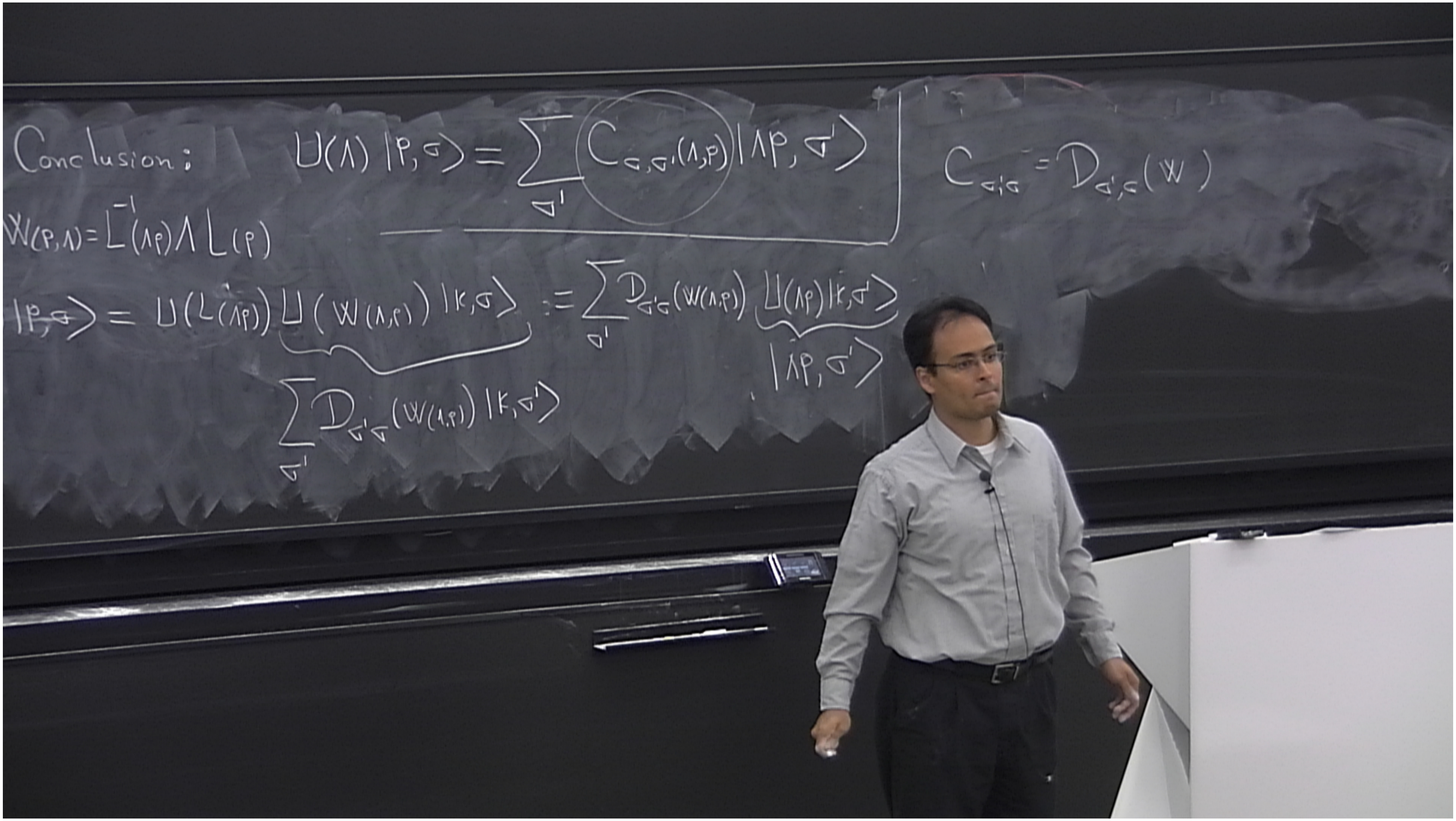
$D_{\sigma''\sigma}(W_1 W_2)\ |k,\sigma''\rangle$

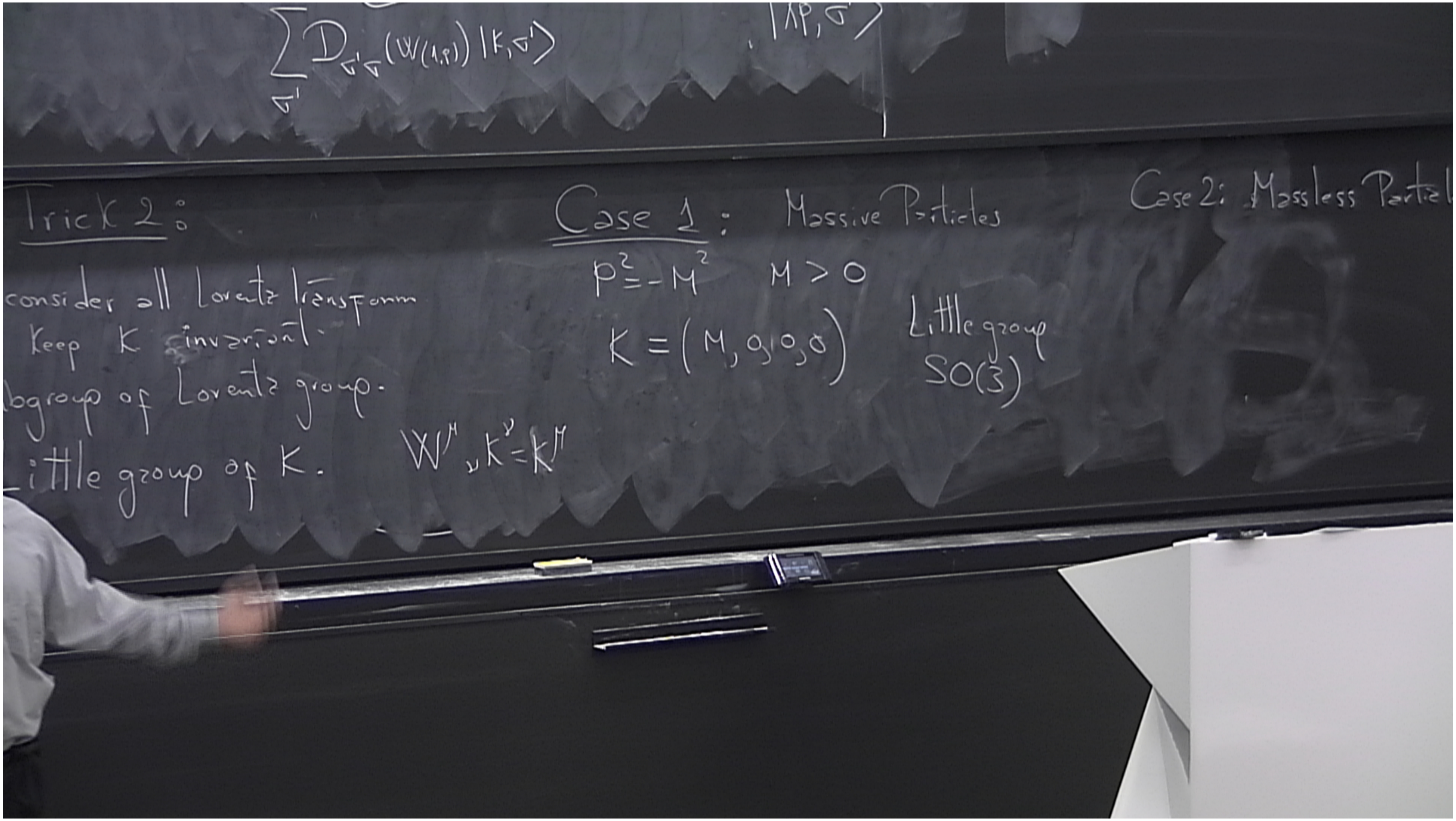
$$D_{\sigma''\sigma}(W_1 W_2) = \sum_{\sigma'} D_{\sigma''\sigma'}(W_1) D_{\sigma'\sigma}(W_2)$$

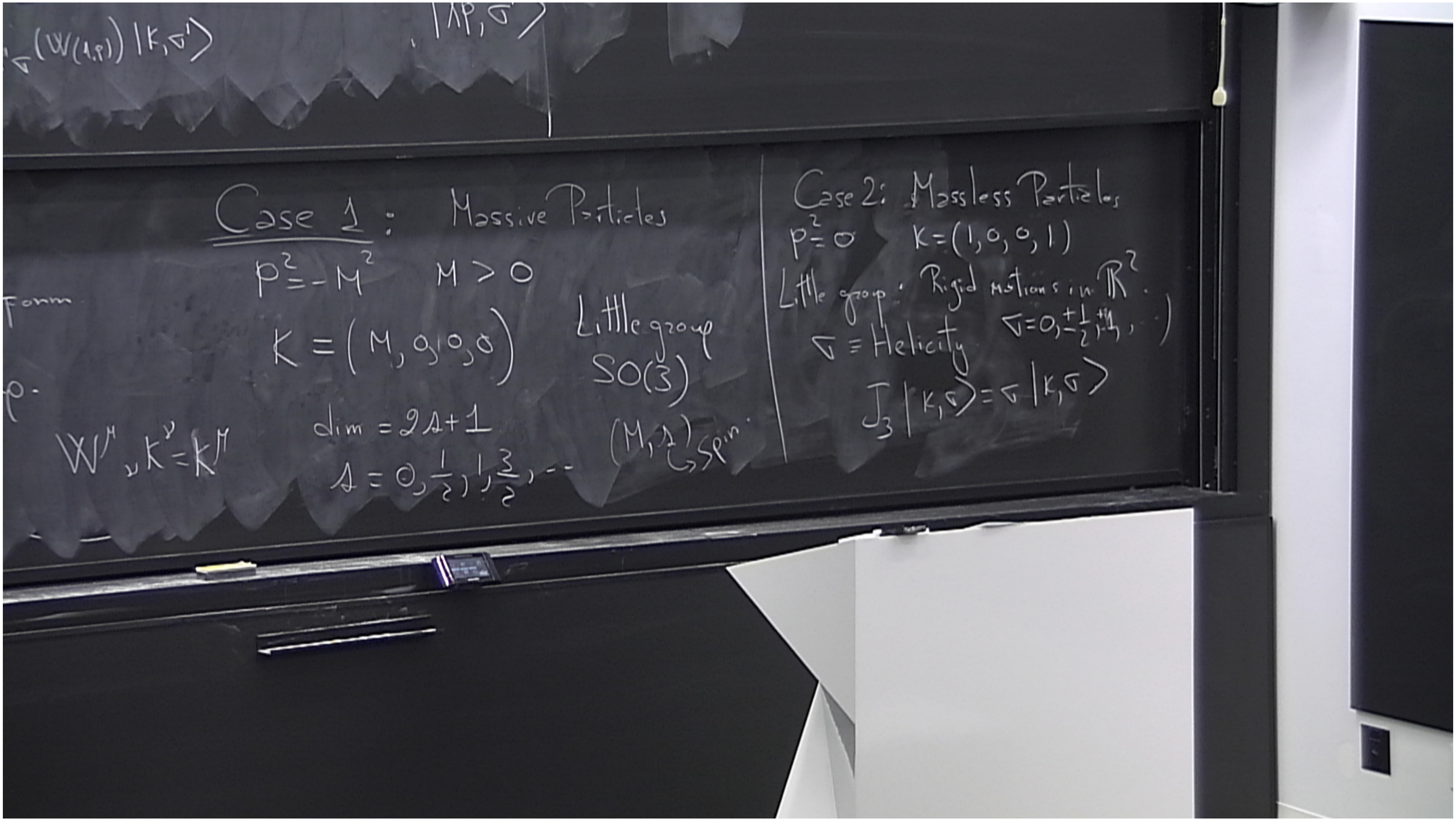
$$D(W_1 W_2) = D(W_1) D(W_2)$$

D are Matrix represent of the little group!


Conclusion: $U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$
$W(p,\Lambda) = L^{-1}(\Lambda p)\Lambda L(p)$
$U(\Lambda)|p,\sigma\rangle = U(L(\Lambda p))\underbrace{U(W(\Lambda,p))|k,\sigma\rangle}_{\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|k,\sigma'\rangle} := \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\,\underbrace{U(\Lambda p)|k,\sigma'\rangle}$


Conclusion: $U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$
$C_{\sigma'\sigma} = D_{\sigma',\sigma}(W)$
$W(p,\Lambda) = L^{-1}(\Lambda p)\Lambda L(p)$
$|p,\sigma\rangle = U(L(\Lambda p))\underbrace{U(W(\Lambda,p))|k,\sigma\rangle}_{\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|k,\sigma'\rangle} = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\underbrace{U(\Lambda p)|k,\sigma'\rangle}_{|\Lambda p,\sigma'\rangle}$


$\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\,|k,\sigma'\rangle$
$|\Lambda p,\sigma\rangle$
Trick 2:
consider all Lorentz transform
Keep K invariant.
bgroup of Lorentz group.
Little group of K.
$W^\mu{}_\nu k^\nu = k^\mu$
Case 1: Massive Particles
$p^2 = -M^2$ $M > 0$
$K = (M, 0, 0, 0)$
Little group
SO(3)
Case 2: Massless Partic


Case 1: Massive Particles
p² = -M²   M > 0
K = (M, 0, 0, 0)
Little group SO(3)
dim = 2s+1
s = 0, 1/2, 1, 3/2, ...
(M, s) spin
W^μ_ν K^ν = K^μ
Case 2: Massless Particles
p² = 0   K = (1, 0, 0, 1)
Little group: Rigid motions in ℝ².
σ ≡ Helicity
J₃ |K, σ⟩ = σ |K, σ⟩