Title: Lie Groups & Lie Algebras - Lecture 2

Date: Aug 23, 2011 09:00 AM

URL: http://pirsa.org/11080122

Abstract:

Commutator $\leftarrow$ $\underbrace{[\hat{t}_a, \hat{t}_b]}_{\hat{t}_a * \hat{t}_b} = f^c{}_{ab}\, \hat{t}_c$

$\hat{t}_a \hat{t}_b - \hat{t}_b \hat{t}_a$

$\hat{t}_a * \hat{t}_b = f^c{}_{ab}\, \hat{t}_c$

$\underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,b,c} + t * (t * t$

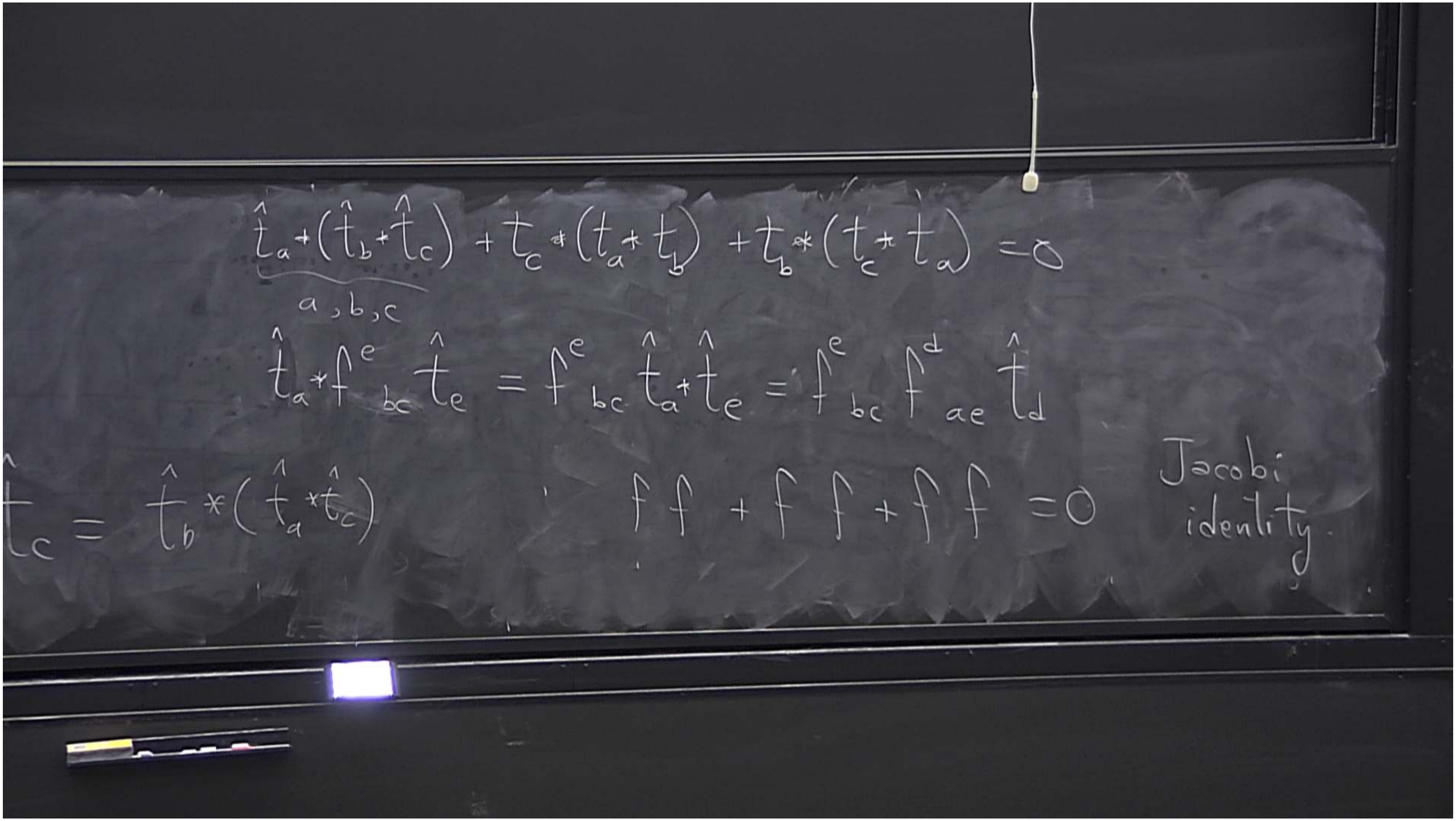
- $\hat{t}_a * \hat{t}_b = -\hat{t}_b * \hat{t}_a$
- $\hat{t}_a * (\hat{t}_b * \hat{t}_c) - (\hat{t}_a * \hat{t}_b) * \hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$

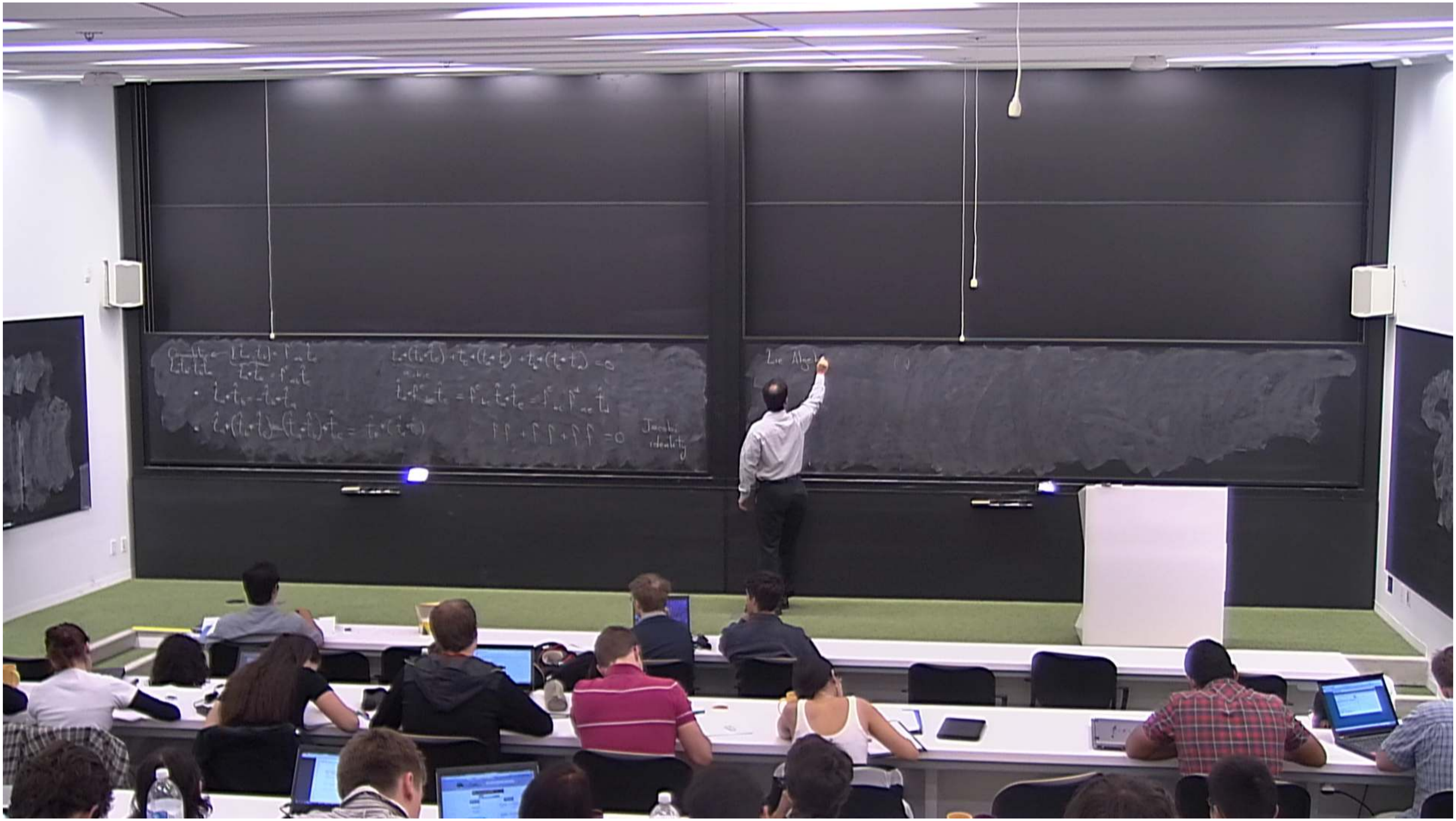
$$] = f^{c}{}_{ab}\, t_c \qquad \underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,b,c} + t_c * (t_a * t_b) + t_b * (t_c * t_a) = 0$$

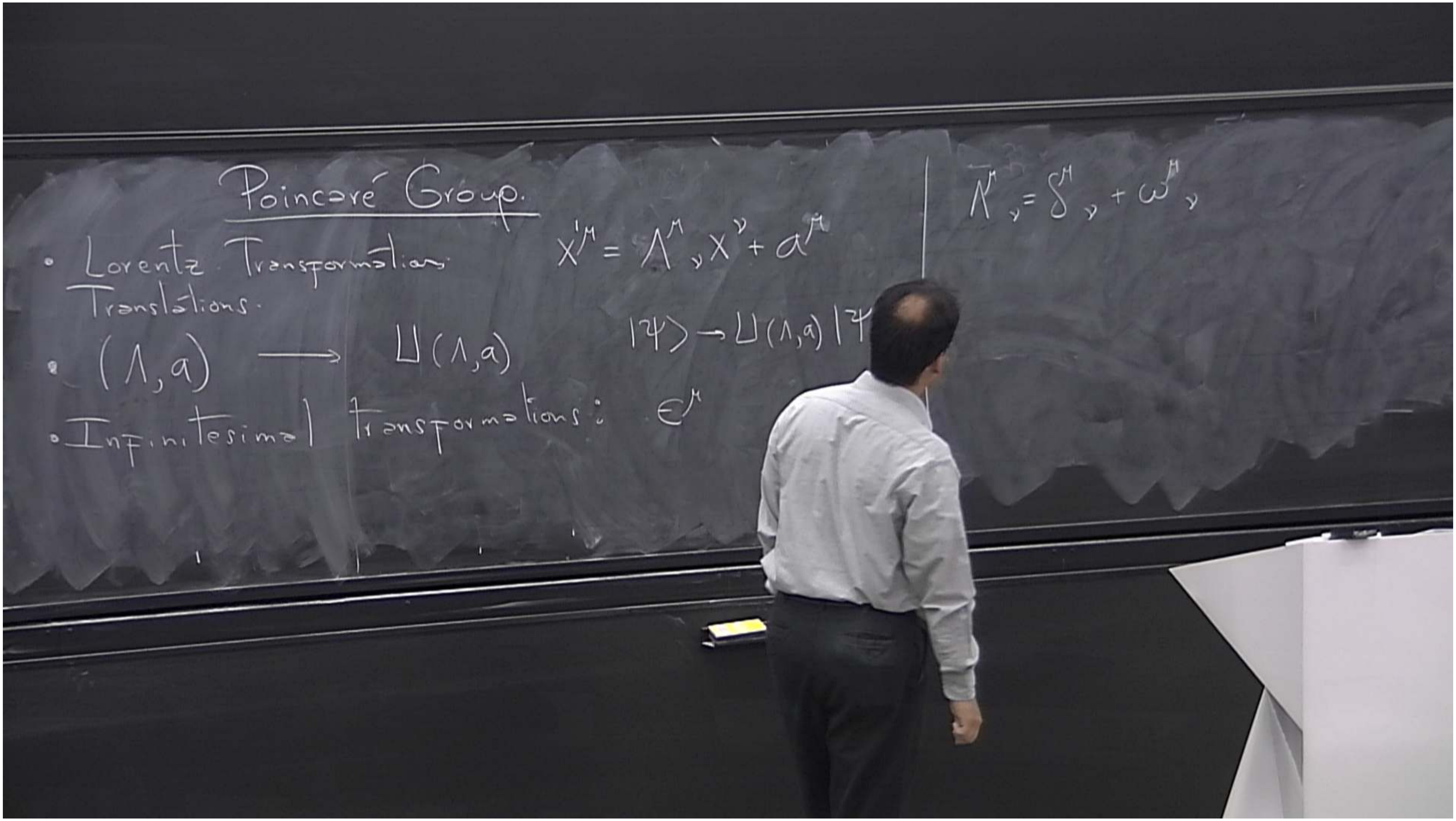
$$= f^{c}{}_{ab}\, \hat{t}_c$$

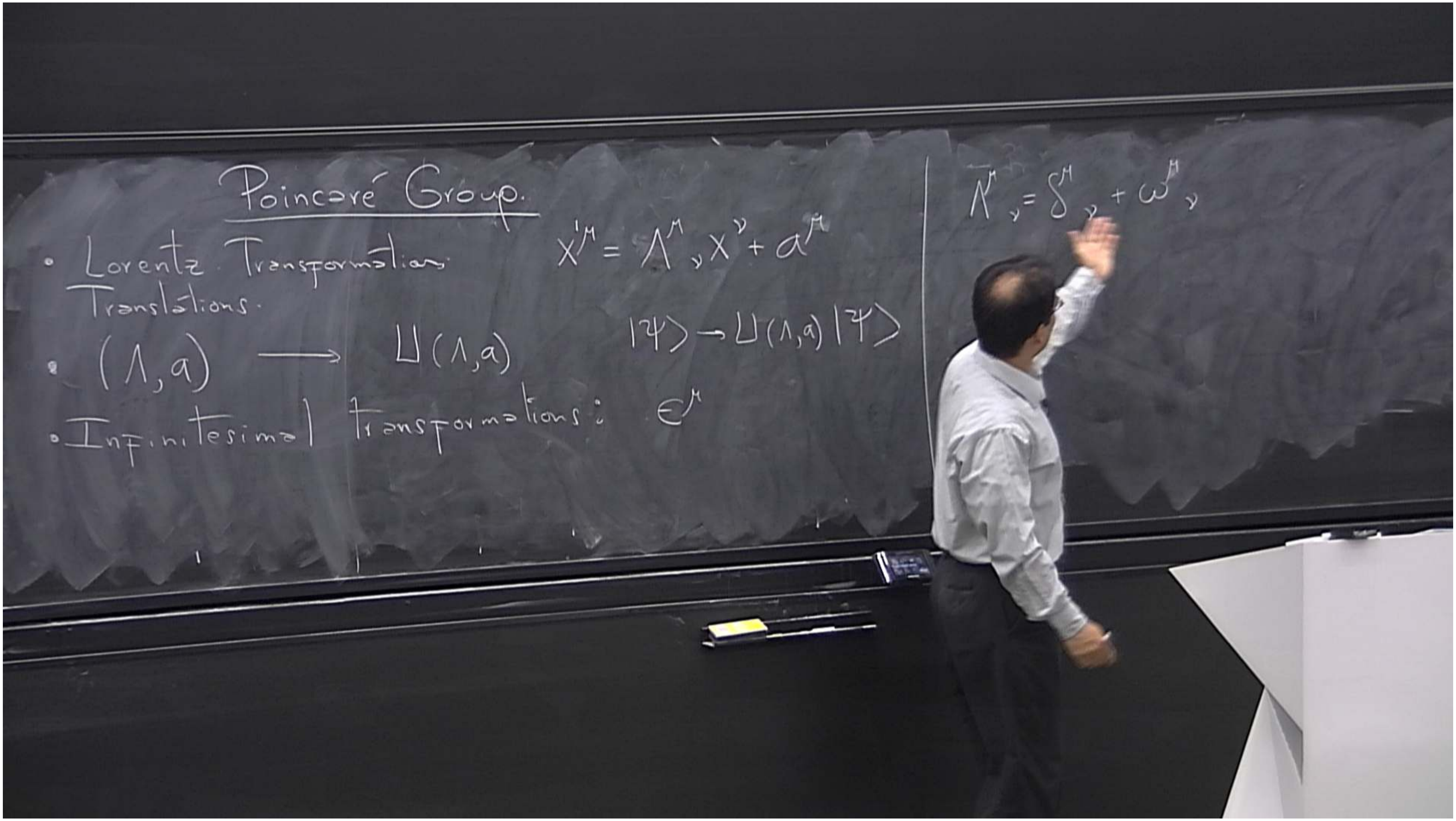
$$\hat{t}_b * \hat{t}_a$$

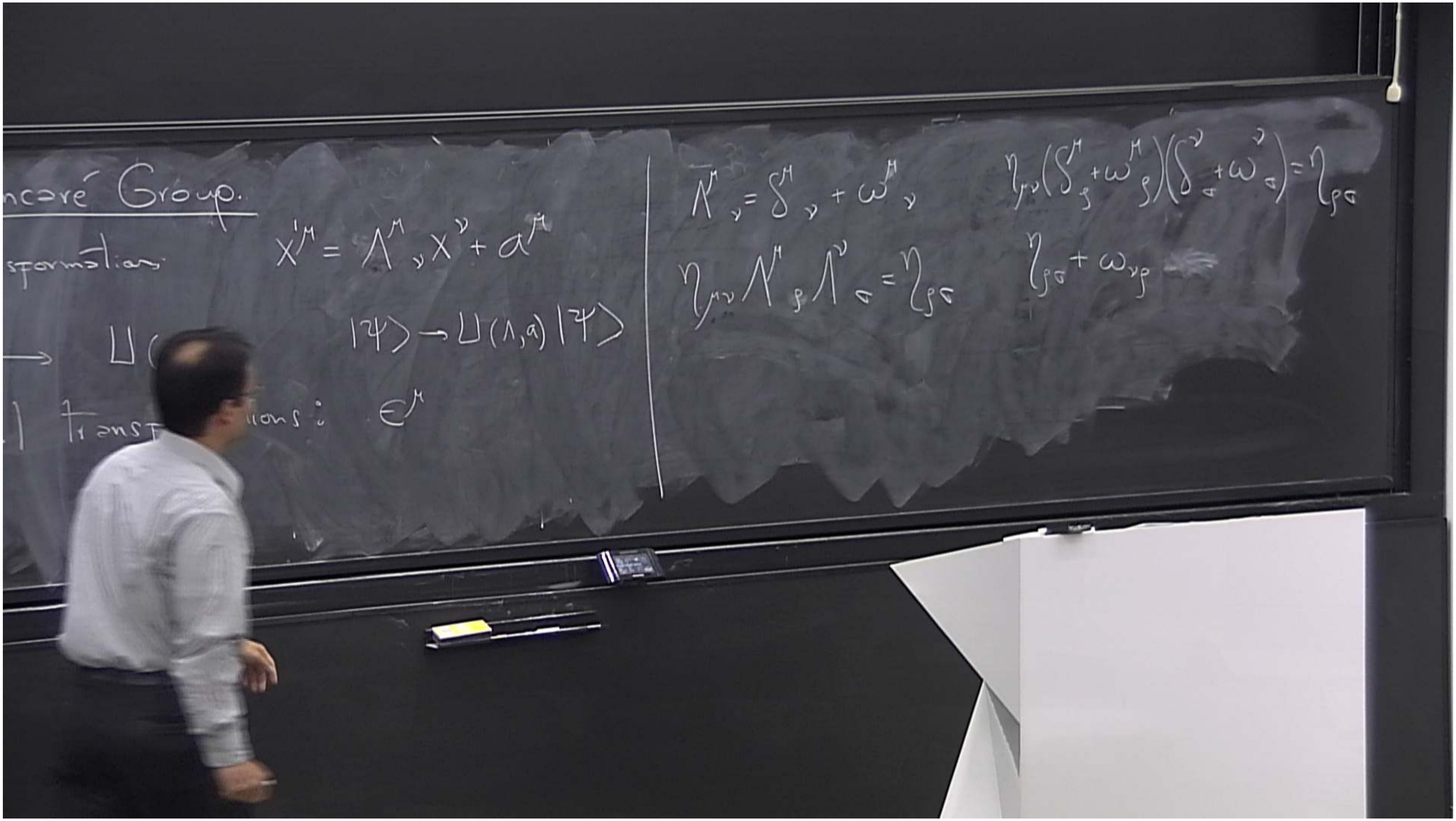
$$) - (\hat{t}_a * \hat{t}_b) * \hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$$

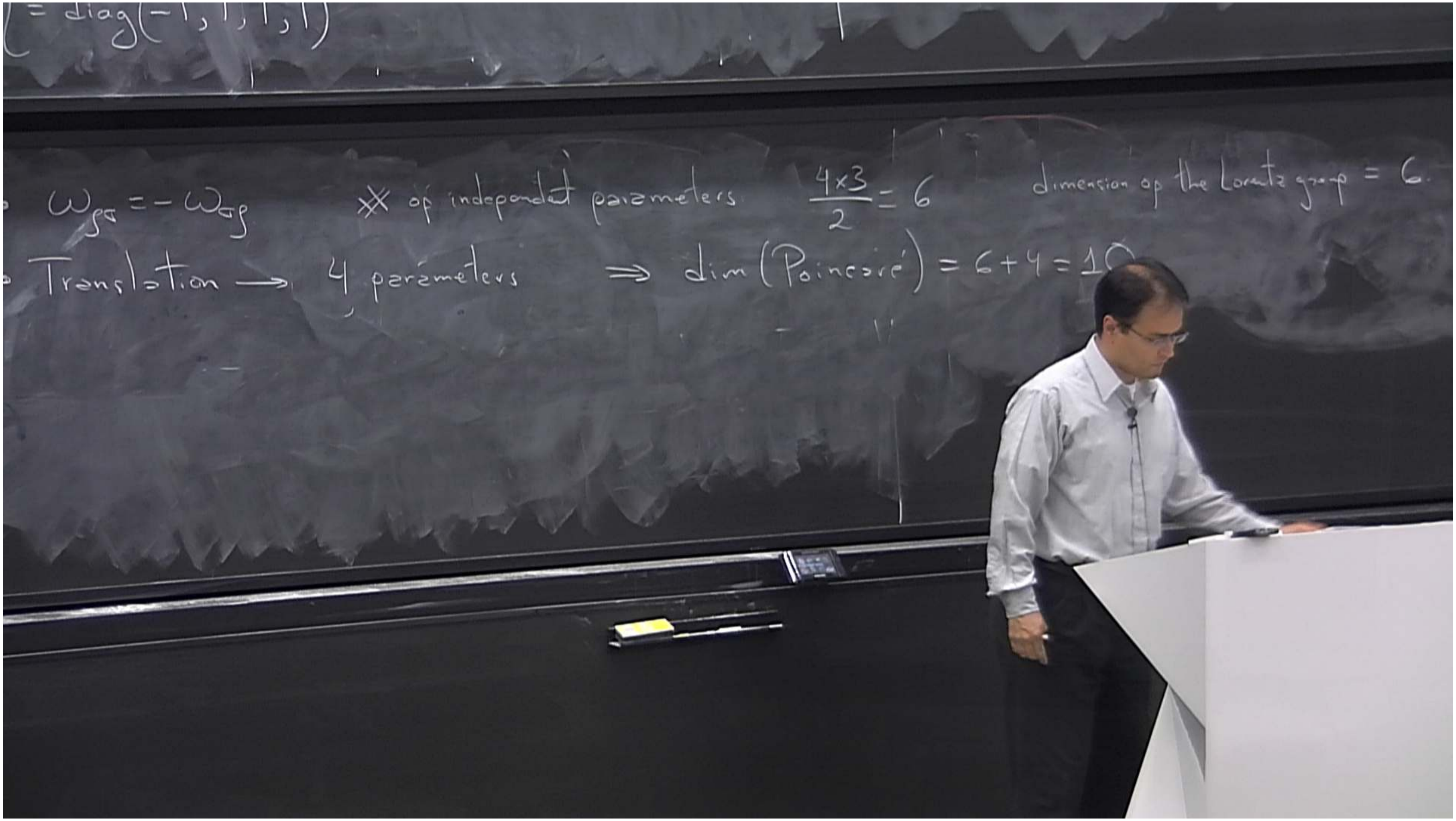

$\underbrace{\hat{t}_a * (\hat{t}_b * \hat{t}_c)}_{a,b,c} + t_c * (t_a * t_b) + t_b * (t_c * t_a) = 0$
$\hat{t}_a * f^e{}_{bc} \hat{t}_e = f^e{}_{bc} \hat{t}_a * \hat{t}_e = f^e{}_{bc} f^d{}_{ae} \hat{t}_d$
$\hat{t}_c = \hat{t}_b * (\hat{t}_a * \hat{t}_c)$
$f f + f f + f f = 0$
Jacobi identity




Poincaré Group.
• Lorentz Transformations
Translations.
$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$
• $(\Lambda, a) \longrightarrow U(\Lambda, a)$
$|\psi\rangle \to U(\Lambda,a)|\psi\rangle$
• Infinitesimal transformations: ϵ^μ
$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$


Poincaré Group.
• Lorentz Transformations
Translations.
$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$
• $(\Lambda, a) \longrightarrow U(\Lambda, a)$
$|\Psi\rangle \to U(\Lambda,a)|\Psi\rangle$
• Infinitesimal transformations: ϵ^μ
$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$


ncaré Group.
sformation
$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$
$|\psi\rangle \to U(\Lambda, a)|\psi\rangle$
$\to U($
transl ions : ϵ^μ
$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$
$\eta_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$
$\eta_{\mu\nu}(\delta^\mu{}_\rho + \omega^\mu{}_\rho)(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma) = \eta_{\rho\sigma}$
$\eta_{\rho\sigma} + \omega_{\nu\rho}$


= diag(−1, 1, 1, 1)
ωρσ = −ωσρ
# of independent parameters 4×3/2 = 6
dimension of the Lorentz group = 6.
Translation → 4 parameters
⇒ dim (Poincaré) = 6+4 = 10

$\eta = \mathrm{diag}(-1,1,1,1)$

- $\omega_{\rho\sigma} = -\omega_{\sigma\rho}$ # of independent parameters $\frac{4\times 3}{2} = 6$ dimension of the Lorentz group =
- Translation $\longrightarrow$ 4 parameters $\Rightarrow$ $\dim(\text{Poincaré}) = 6+4 = 10$

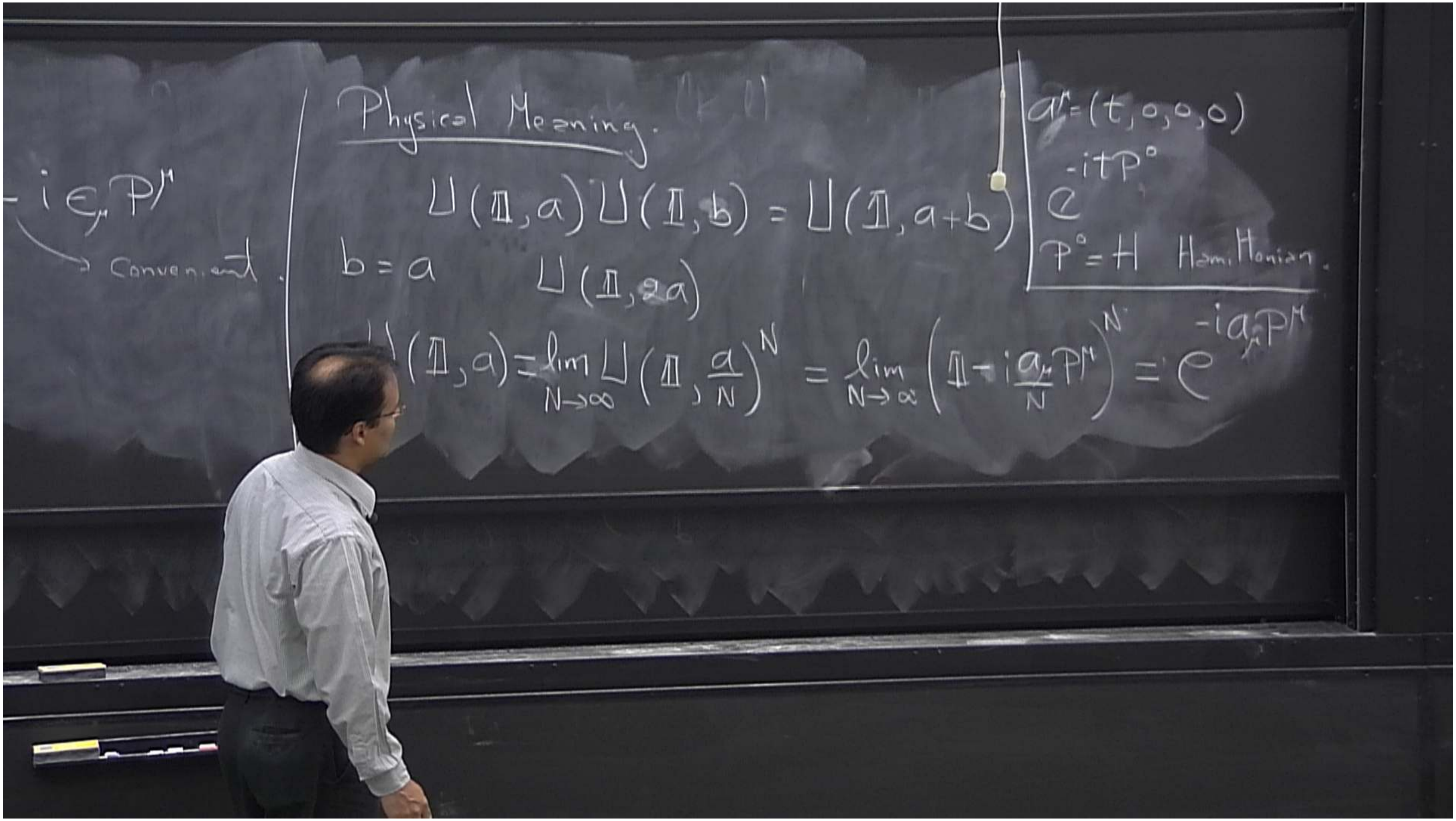
  $\hookrightarrow$ Abelian $U(\mathbb{1}, a)$
- Lorentz $U(\Lambda, 0)$

Unitary Operator

$$U(\mathbb{1}+\omega, \epsilon) = \mathbb{1} + i\omega_{\mu\nu} J^{\mu\nu} - i\epsilon_\mu P^\mu$$

Convenient.

$J^{\mu\nu}, P^\mu$ Hermitian.


$-i\epsilon_\mu P^\mu$
Convenient.
Physical Meaning.
$U(\mathbb{1},a)U(\mathbb{1},b) = U(\mathbb{1},a+b)$
$b = a \quad U(\mathbb{1},2a)$
$U(\mathbb{1},a) = \lim_{N\to\infty} U(\mathbb{1},\frac{a}{N})^N = \lim_{N\to\infty}\left(\mathbb{1} - i\frac{a_\mu}{N}P^\mu\right)^N = e^{-ia_\mu P^\mu}$
$a^\mu = (t,0,0,0)$
e^{-itP^0}
$P^0 = H$ Hamiltonian.

$$\square(\mathbb{1}, a) = \lim_{N\to\infty} \square\left(\mathbb{1}, \frac{a}{N}\right)$$

$\vec{\mathbb{P}} = (P^1, P^2, P^3) \longrightarrow$ Momentum Operators

$\vec{\mathbb{J}} = (J^{23}, J^{31}, J^{12}) \longrightarrow$ Angular momentum operators

$$U(\mathbb{1}, a) = \lim_{N\to\infty} U\left(\mathbb{1}, \frac{a}{N}\right)^N = \lim_{N\to\infty}\left(\mathbb{1} - i\frac{a_\mu P^\mu}{N}\right)^N = e$$

→ Momentum Operators

→ Angular momentum operators

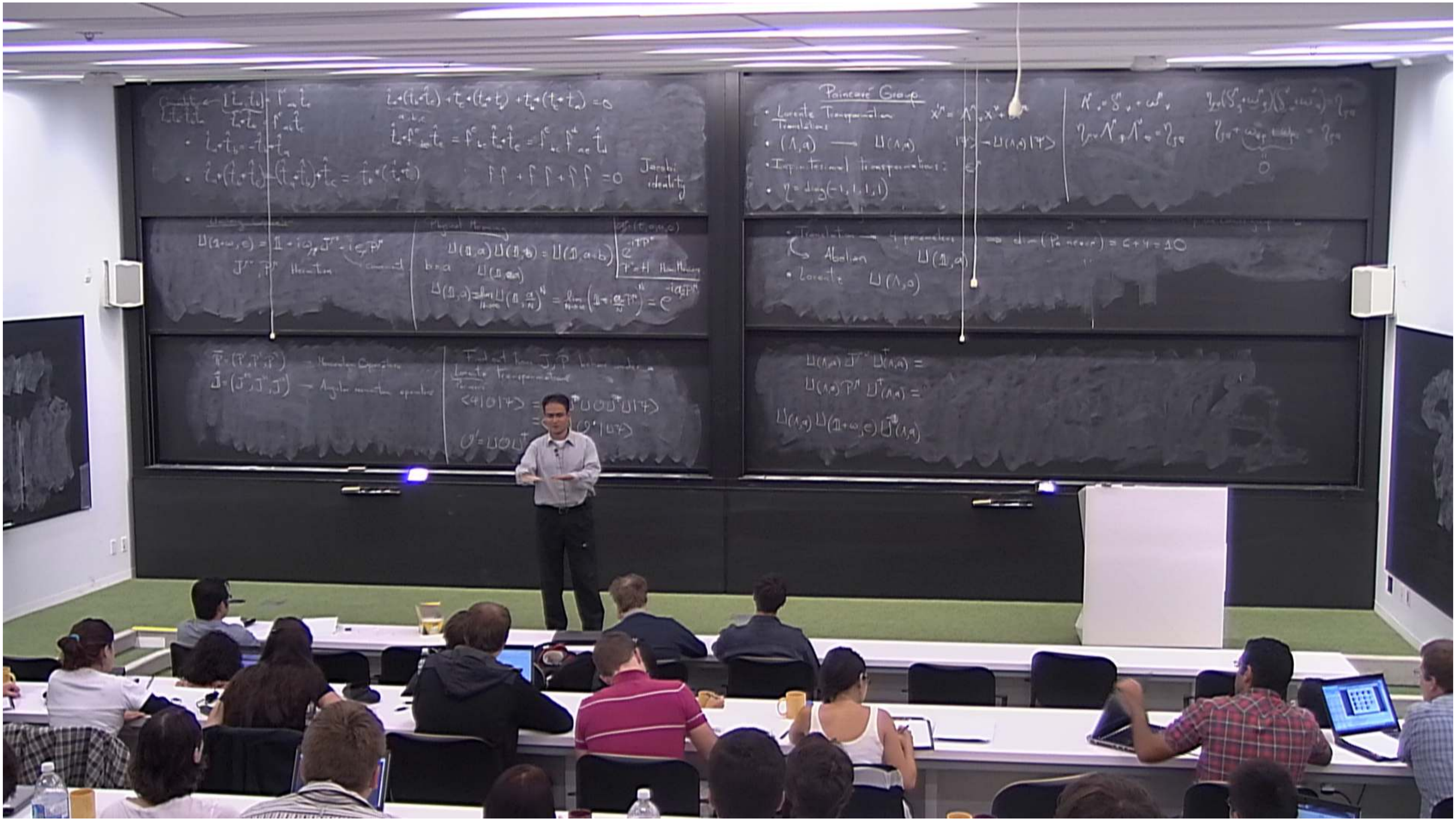
Find out how J, P behave under a Lorentz transformation

Poincaré

$$\langle\varphi|\mathcal{O}|\psi\rangle = \langle\varphi|U^\dagger U\,\mathcal{O}\,U^\dagger U|\psi\rangle$$

$$= \langle U\varphi|\,\mathcal{O}'\,|U\psi\rangle$$

$$\mathcal{O}' = U\mathcal{O}U^\dagger$$



$$U(\Lambda,a)\, J^{\mu\nu}\, U^{\dagger}(\Lambda,a) =$$

$$U(\Lambda,a)\, P^{\mu}\, U^{\dagger}(\Lambda,a) =$$

$$U(\Lambda,a)\, U(\mathbb{1}+\omega,\epsilon)\, U^{-1}(\Lambda,a) = U(\circ\,,\ \circ)$$

$$U(\Lambda,a)\left(\mathbb{1} + i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,a) = U($$

$$U(\Lambda,a)\, J^{\mu\nu}\, U^{\dagger}(\Lambda,a) =$$

$$U(\Lambda,a)\, P^{\mu}\, U^{\dagger}(\Lambda,a) =$$

$$U(\Lambda,a)\, U(\mathbb{1}+\omega,\epsilon)\, U^{-1}(\Lambda,a) = U($$

$$U(\Lambda,a)\left(\mathbb{1} + i\omega_{\mu\nu}J^{\mu\nu} - i\epsilon_\mu P^\mu\right)U^{-1}(\Lambda,a) = U\left(\mathbb{1} + \Lambda\omega\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a\right)$$

Ex:

$$U(\Lambda,a)\, J^{\mu\nu}\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right)$$

$$U(\Lambda,a)\, P^\mu\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu P^\rho$$

Find the Poincaré Algebra

$$\Lambda = \mathbb{1} + \tilde{\omega} \qquad a^\mu = \tilde{\epsilon}^\mu$$

Poincaré Algebra

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}$$

$$i[P^{\mu}, J^{\rho\sigma}] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}$$

$$[P^{\mu}, P^{\nu}] = 0$$

Ex:

$$U(\Lambda,a)\, J^{\mu\nu}\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right)$$

$$U(\Lambda,a)\, P^\mu\, U^\dagger(\Lambda,a) = \Lambda_\rho{}^\mu P^\rho$$

Find the Poincaré Algebra

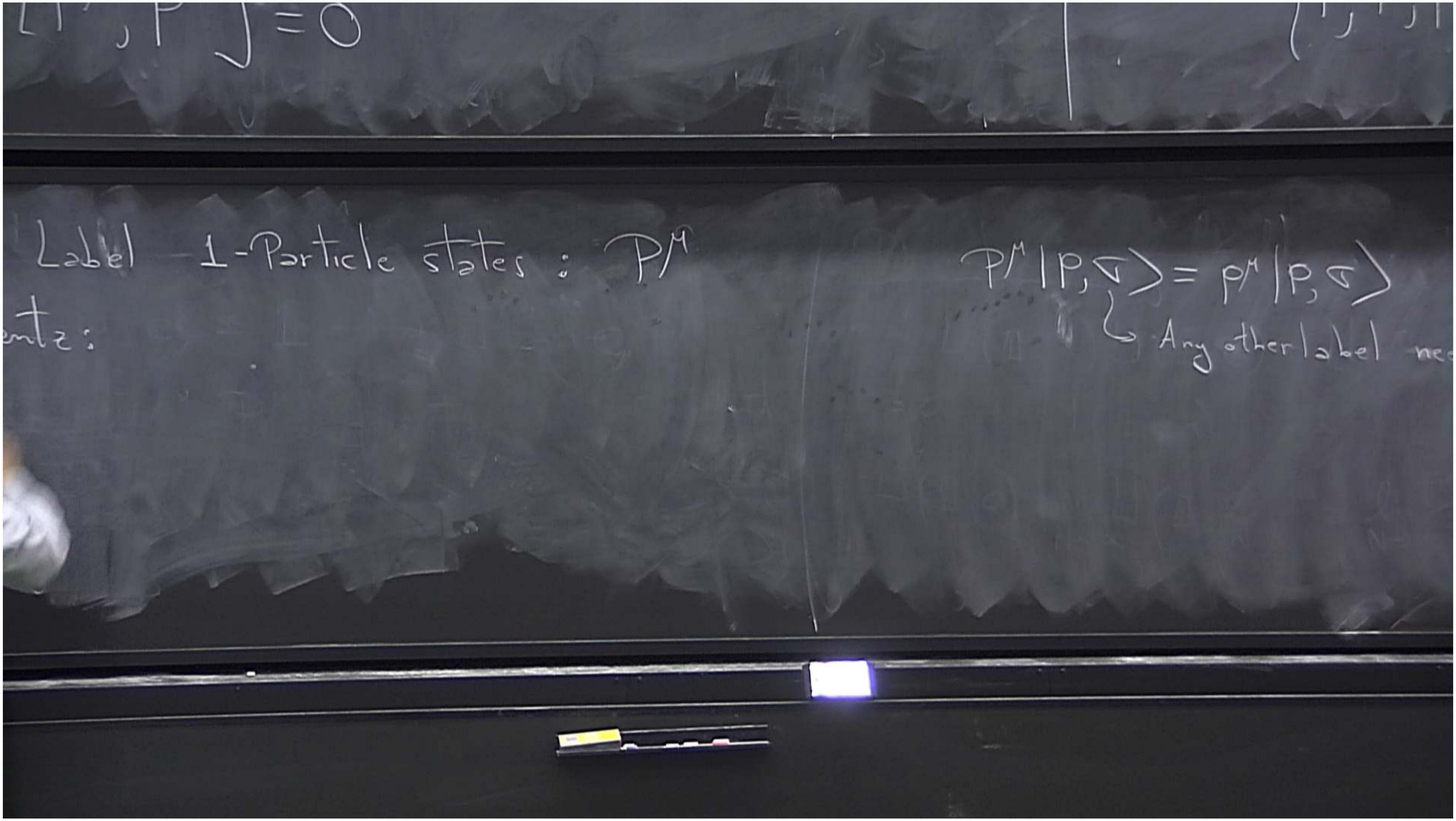
$$\Lambda = \mathbb{1} + \tilde{\omega} \qquad a^\mu = \tilde{\epsilon}^\mu$$

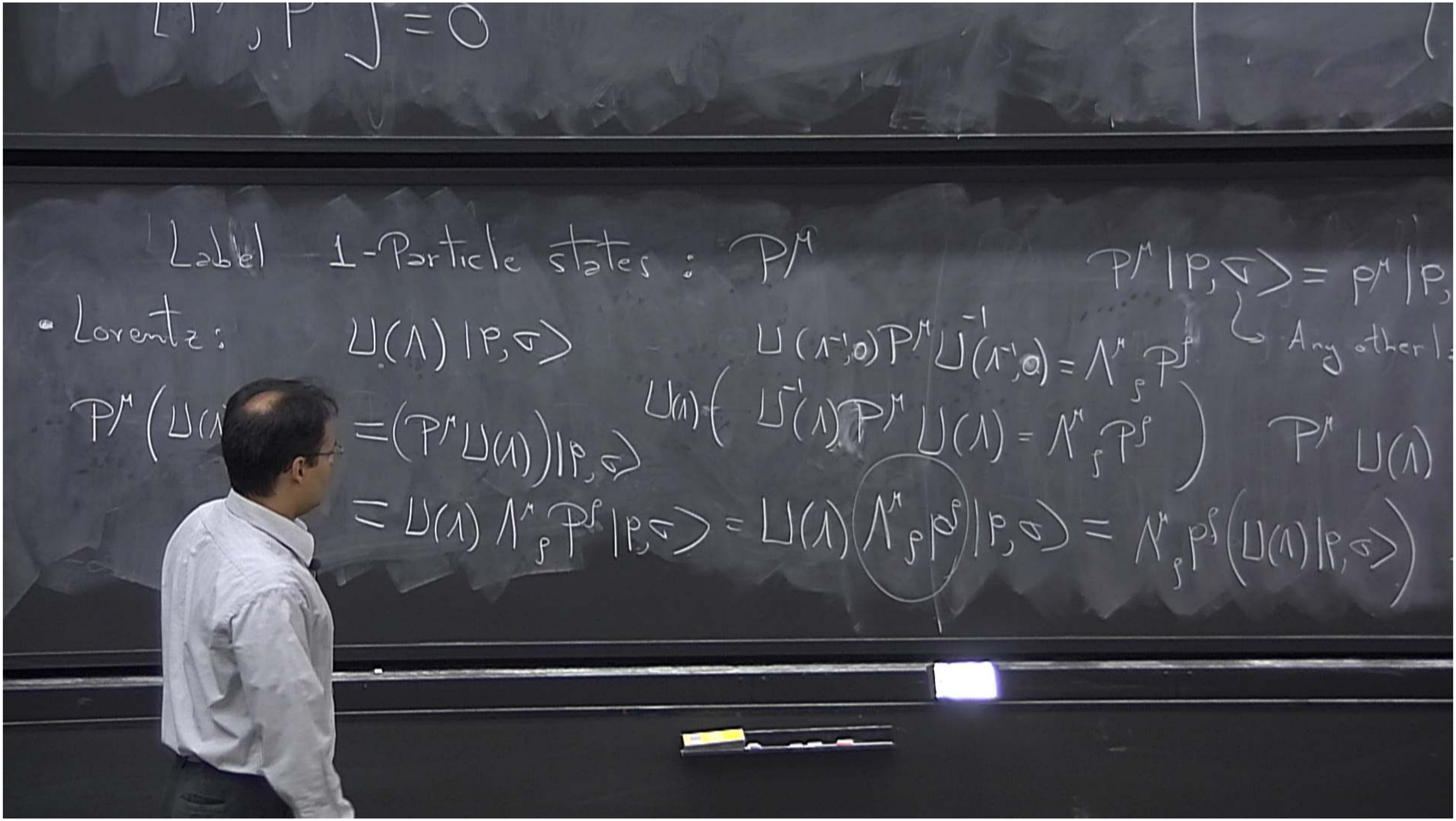
Poincaré Algebra

$$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\sigma\mu} J^{\rho\nu} + \eta^{\sigma\nu} J^{\rho\mu}$$

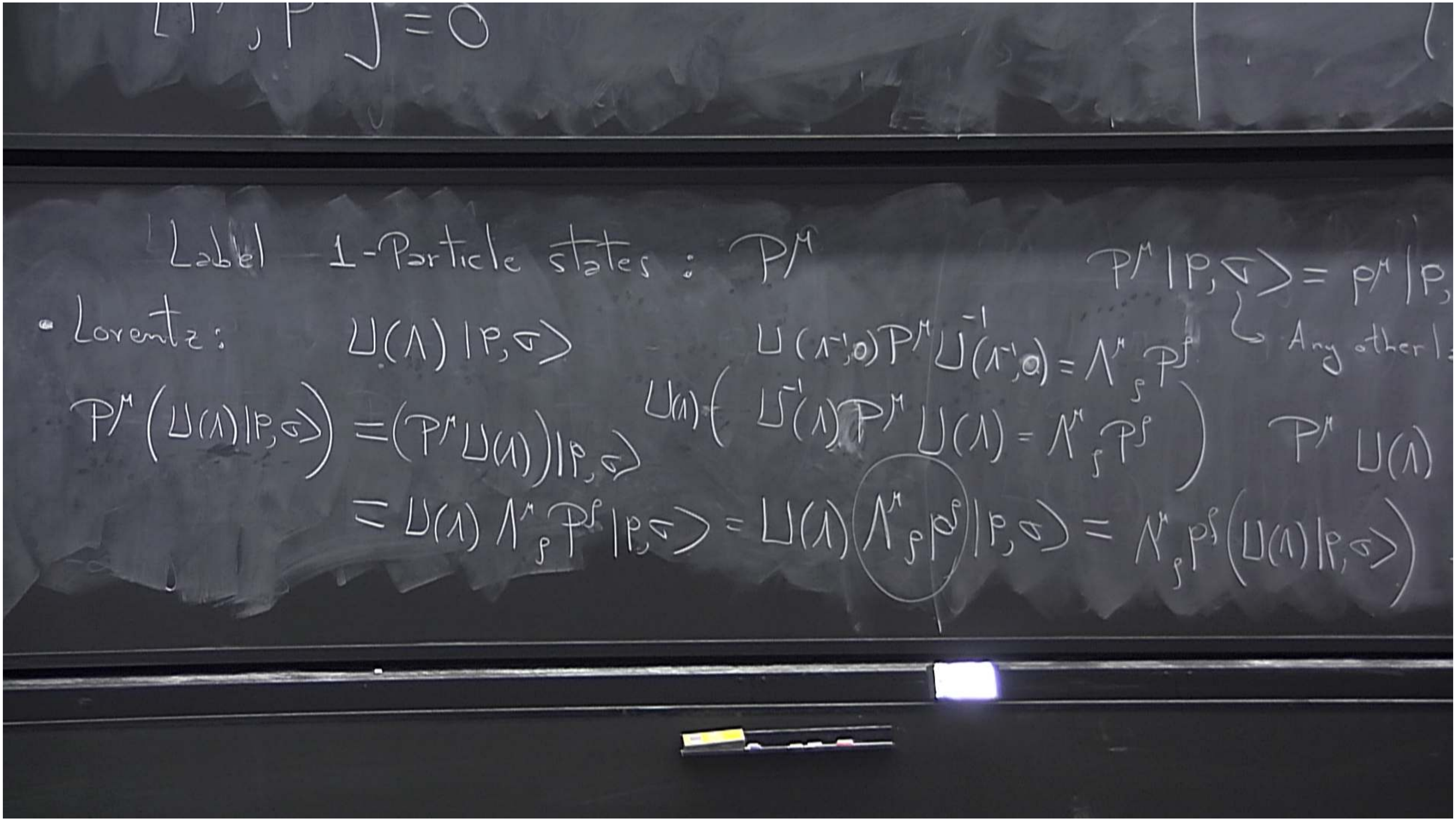
$$i[P^{\mu}, J^{\rho\sigma}] = \eta^{\mu\rho} P^{\sigma} - \eta^{\mu\sigma} P^{\rho}$$

$$[P^{\mu}, P^{\nu}] = 0$$


Label 1-Particle states : P^μ
$P^\mu |p,\sigma\rangle = p^\mu |p,\sigma\rangle$
Any other label



Label 1-Particle states : P^μ
P^μ |p, σ> = p^μ |p,
↳ Any other l
• Lorentz: U(Λ) |p, σ>
U(Λ^{-1}, 0) P^μ U^{-1}(Λ^{-1}, 0) = Λ^μ_ρ P^ρ
U(Λ) (U^{-1}(Λ) P^μ U(Λ) = Λ^μ_ρ P^ρ)
P^μ U(Λ)
P^μ (U(Λ
= (P^μ U(Λ)) |p, σ>
= U(Λ) Λ^μ_ρ P^ρ |p, σ> = U(Λ) Λ^μ_ρ p^ρ |p, σ> = Λ^μ_ρ p^ρ (U(Λ) |p, σ>)


Label 1-Particle states : P^μ
$P^\mu |p,\sigma\rangle = p^\mu |p,$
Any other
• Lorentz: $U(\Lambda)|p,\sigma\rangle$
$U(\Lambda^{-1},0) P^\mu U^{-1}(\Lambda^{-1},0) = \Lambda^\mu{}_\rho P^\rho$
$U(\Lambda)\left(U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\rho P^\rho \right)$
$P^\mu U(\Lambda)$
$P^\mu \left(U(\Lambda)|p,\sigma\rangle\right) = \left(P^\mu U(\Lambda)\right)|p,\sigma\rangle$
$= U(\Lambda)\Lambda^\mu{}_\rho P^\rho |p,\sigma\rangle = U(\Lambda)\Lambda^\mu{}_\rho p^\rho |p,\sigma\rangle = \Lambda^\mu{}_\rho p^\rho \left(U(\Lambda)|p,\sigma\rangle\right)$

Conclusion: $\boxed{U(\Lambda)\,|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)\,|\Lambda p,\sigma'\rangle}$

Trick: Note that $p^\mu p_\mu = p^2 = -M^2$ First label.

M → Mass

$p^2 < 0$ sign(p^0) is Lorentz invariant.

Let's consider a reference vector $\underline{k^\mu}$ $\quad k^2 = -M^2$

$$\boxed{p^\mu = L^\mu{}_\nu(p)\, k^\nu}$$

$\hookrightarrow$ 4×4 matrix that leaves η invariant.

$, P] = 0$

nsider a reference vector $\underline{\underline{K^\mu}}$

$K^2 = -M^2$

$$\boxed{p^\mu = L^\mu{}_\nu(p)\, K^\nu}$$

$\hookrightarrow$ 4x4 matrix that leaves η invariant.

$\underline{\text{Def}}$: $|p, \sigma\rangle \equiv U(L(p))\, |k, \sigma\rangle$

Apply

Trick 2:

$$U(\Lambda)|p,\sigma\rangle = U(\Lambda L(p))|k,\sigma\rangle = \underbrace{U^{-1}(L^{-1}(\Lambda p))\,U(L^{-1}(\Lambda p))}_{\mathbb{1}}\,U(\Lambda L(p))|k,\sigma\rangle$$

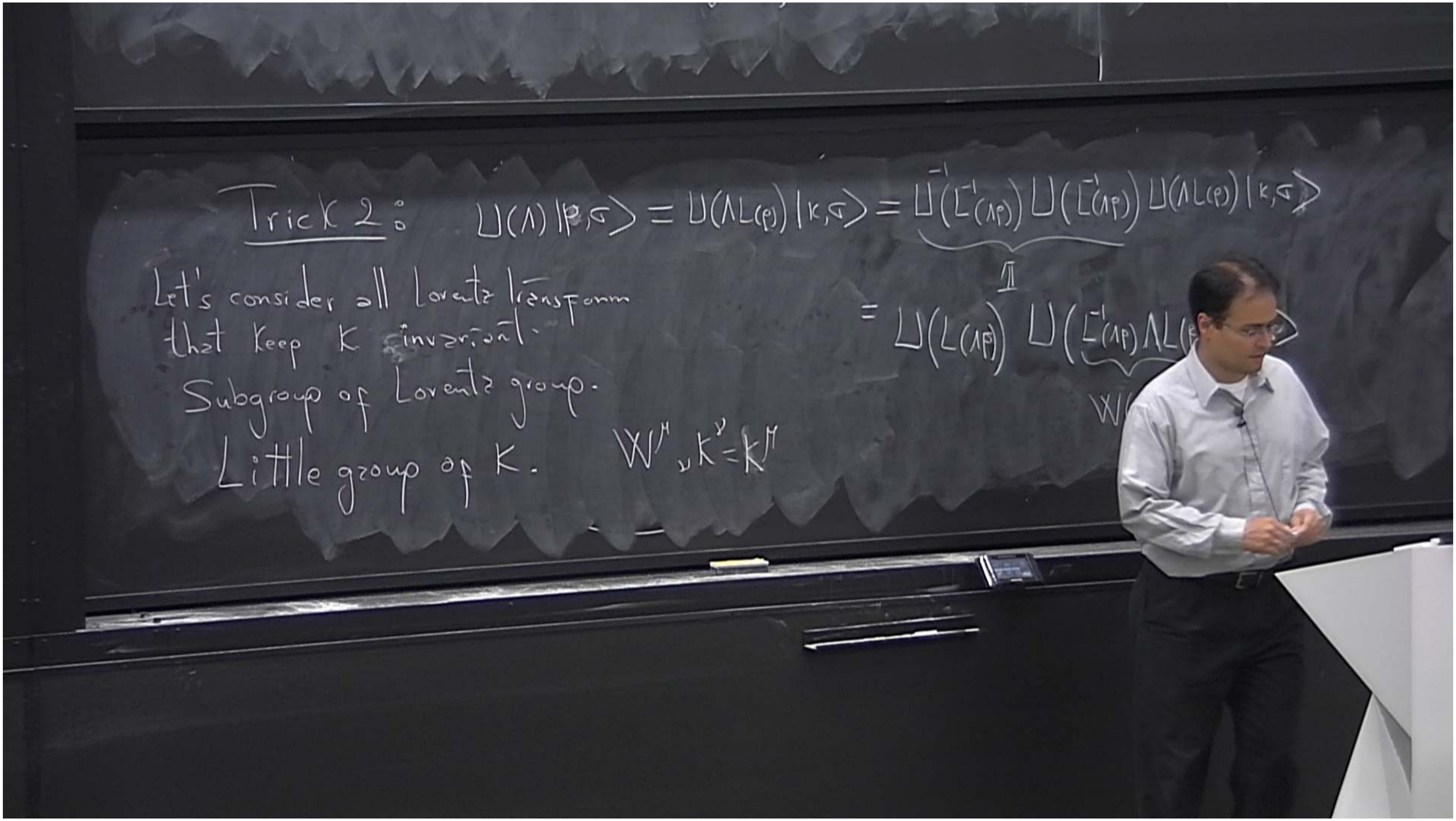
$$= U(L(\Lambda p))\,U(\underbrace{L^{-1}(\Lambda p)\Lambda L(p)})|k,\sigma\rangle$$

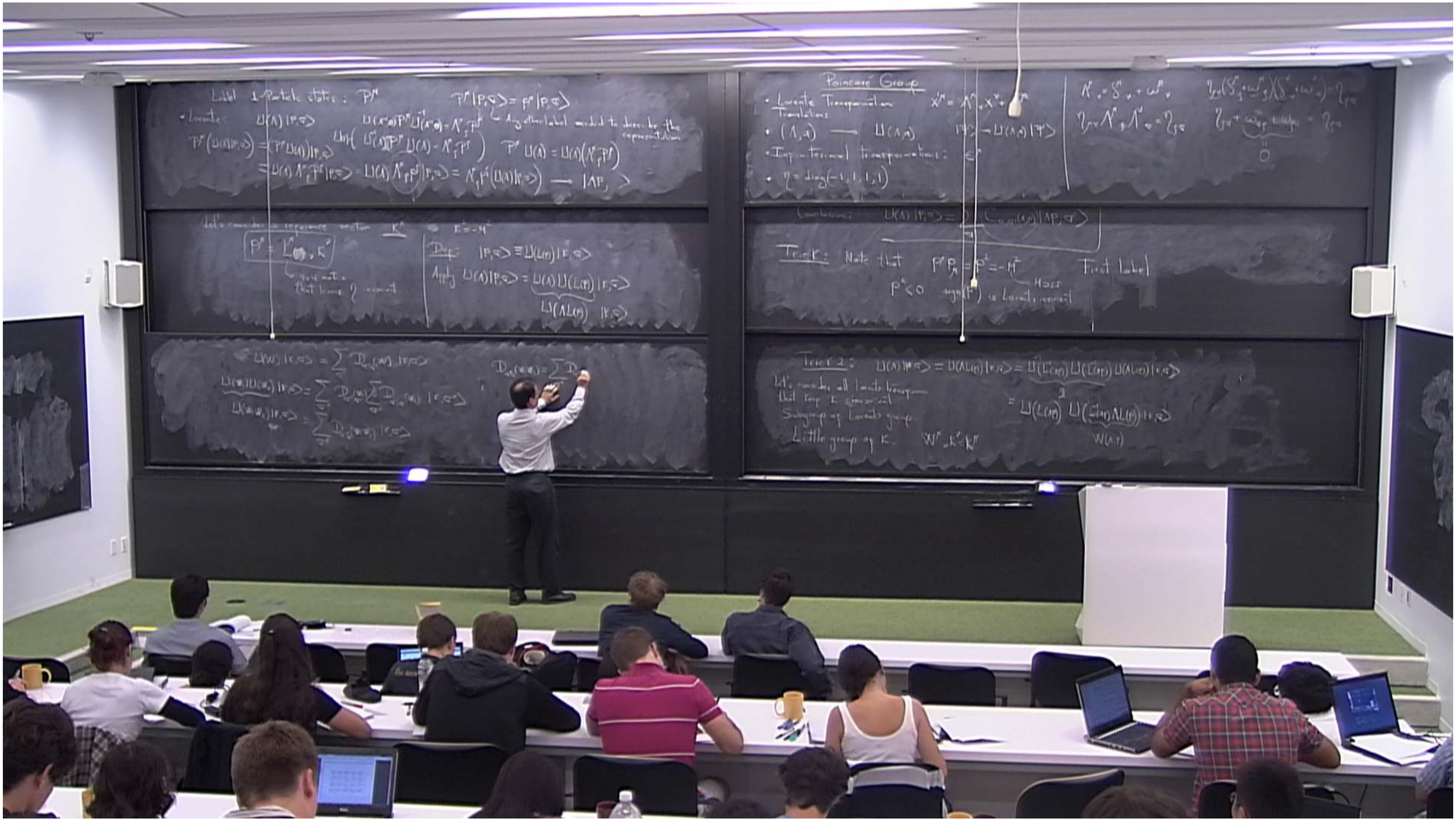
Let's consider all Lorentz transform that keep k invariant.

$$L(\Lambda p)k = \Lambda p$$

$$L(p)k = p$$

$$\Lambda L(p)k = \Lambda p \qquad L^{-1}(\Lambda p)\Lambda L(p)k = k$$


Trick 2: U(Λ)|p,σ> = U(ΛL(p))|k,σ> = U⁻¹(L⁻¹(Λp)) U(L⁻¹(Λp)) U(ΛL(p))|k,σ>
= U(L(Λp)) U(L⁻¹(Λp)ΛL(p
Let's consider all Lorentz transform
that keep k invariant.
Subgroup of Lorentz group.
Little group of k.
W^μ_ν k^ν = k^μ



$$\underbrace{\qquad\qquad}\quad U(\Lambda L(p)) \quad |k,\sigma\rangle$$

$$= \sum_{\sigma'} D_{\sigma'\sigma}(W) \; |k,\sigma'\rangle$$

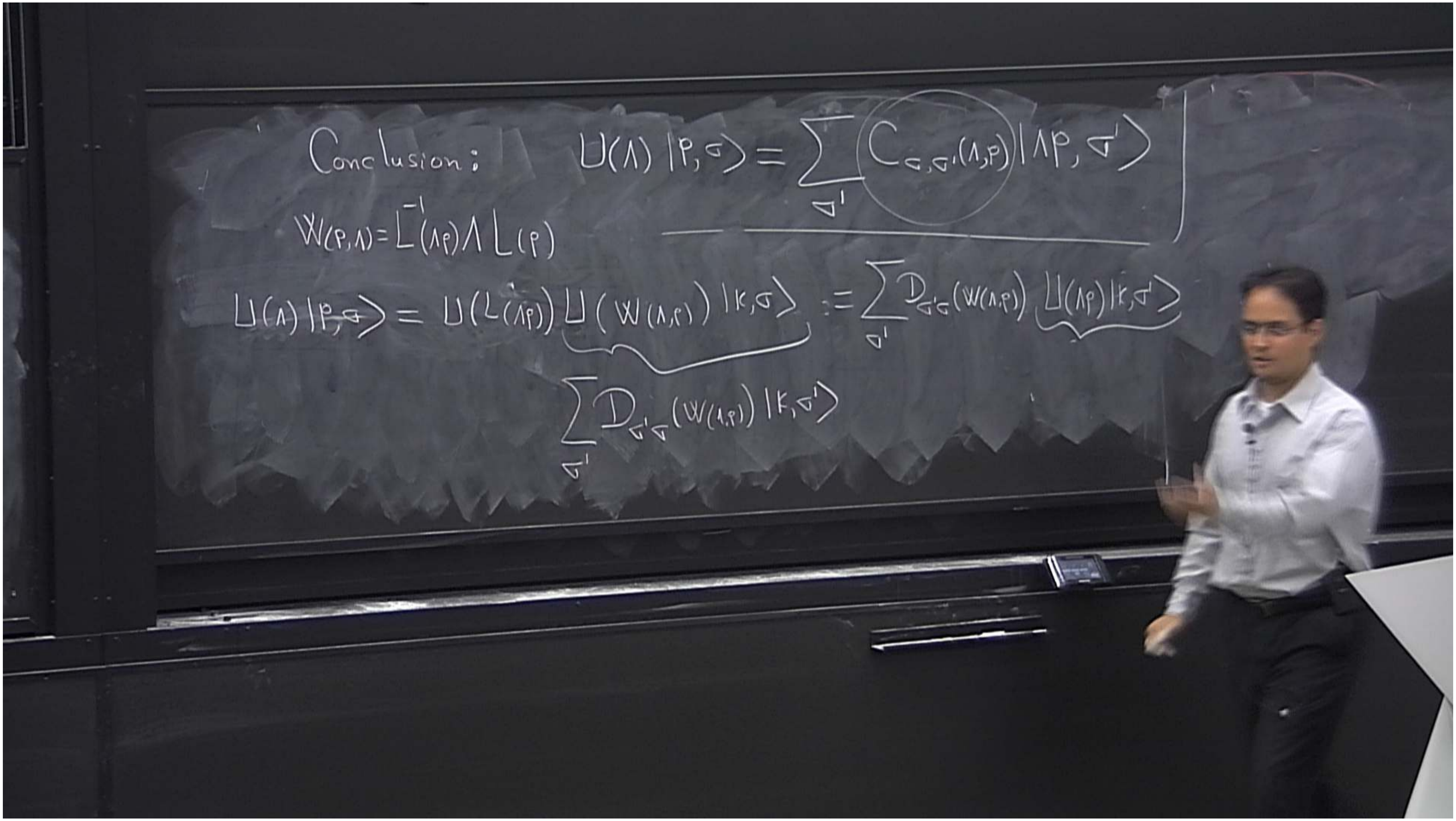
$$= \sum_{\sigma''} D_{\sigma''\sigma'}(W_1) \sum_{\sigma'} D_{\sigma'\sigma}(W_2) \; |k,\sigma''\rangle$$

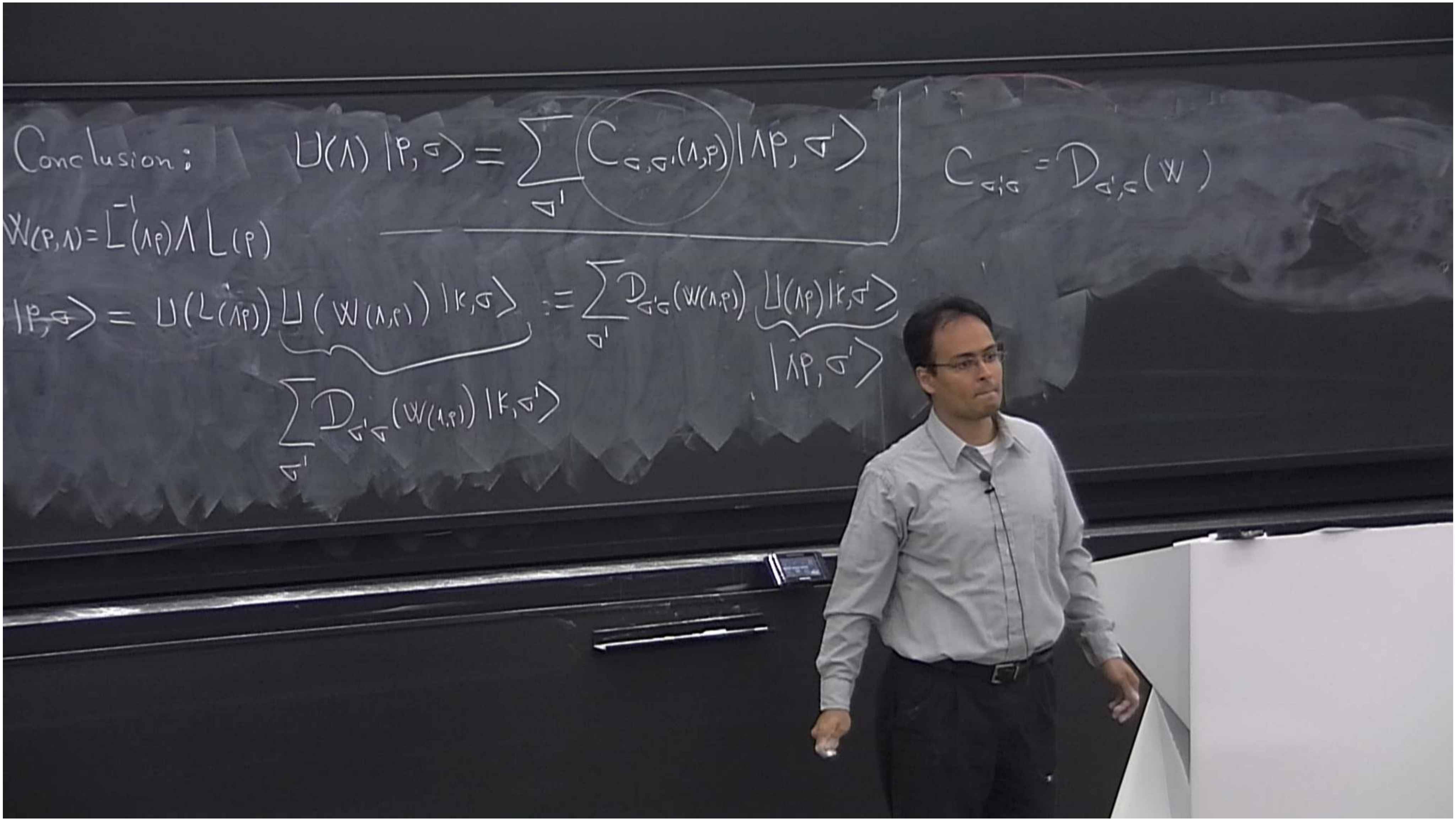
$$= \sum_{\sigma''} D_{\sigma''\sigma}(W_1 W_2) \; |k,\sigma''\rangle$$

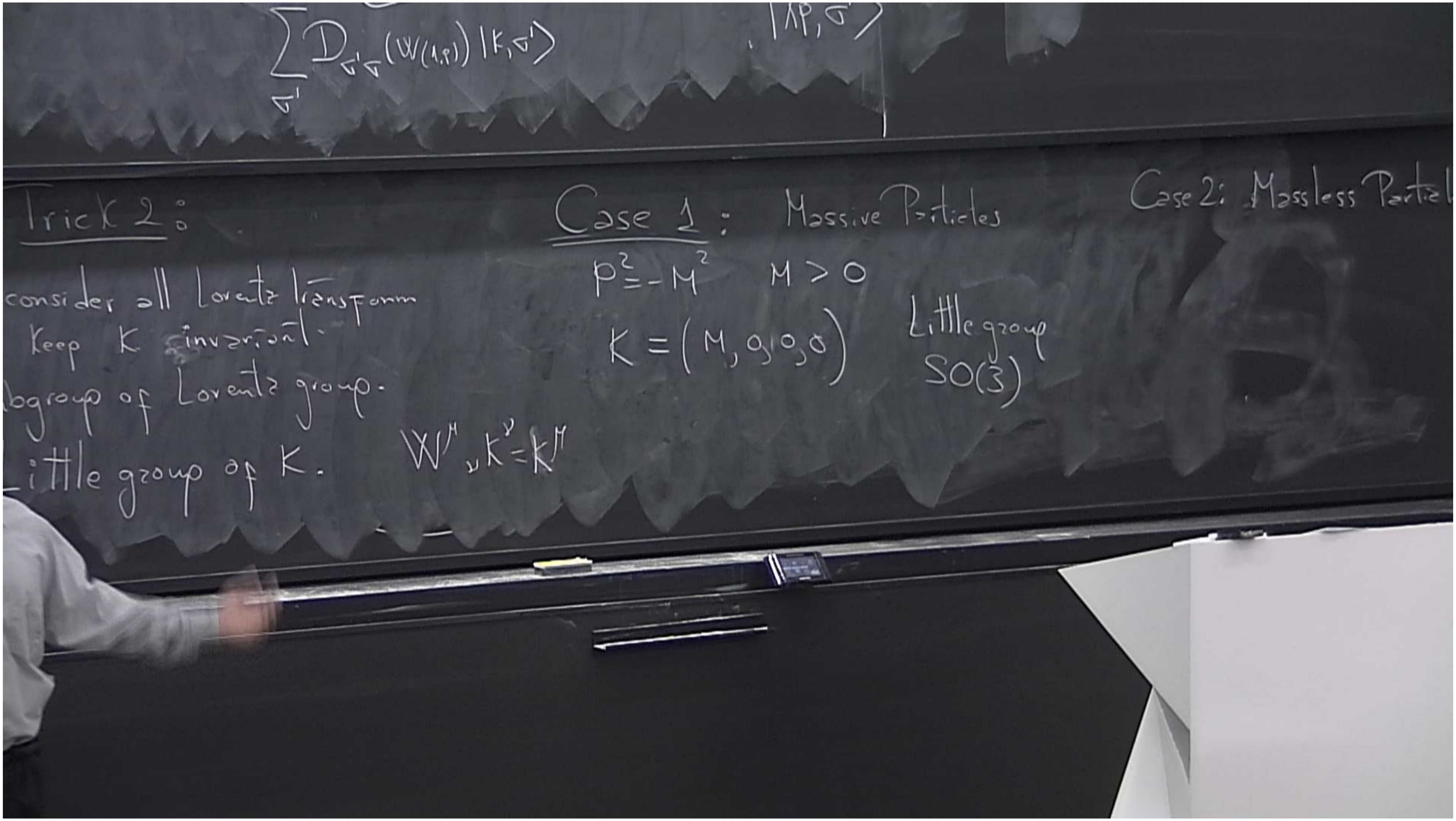
$$D_{\sigma''\sigma}(W_1 W_2) = \sum_{\sigma'} D_{\sigma''\sigma'}(W_1) D_{\sigma'\sigma}(W_2)$$

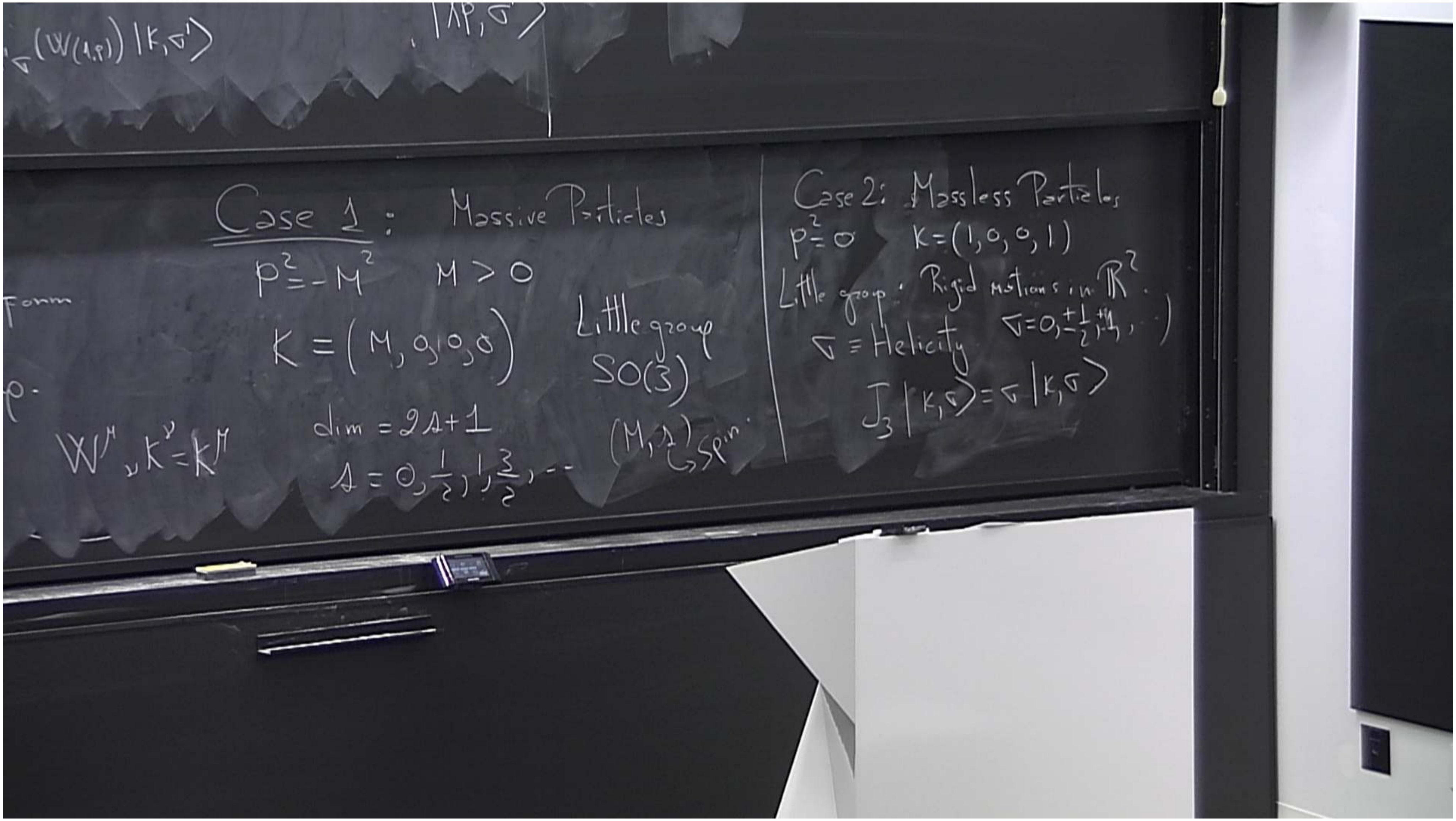
$$D(W_1 W_2) = D(W_1) D(W_2)$$

D are Matrix represent of the little group!


Conclusion:
$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$
$W(p,\Lambda) = L^{-1}(\Lambda p)\Lambda L(p)$
$U(\Lambda)|p,\sigma\rangle = U(L(\Lambda p))U(W(\Lambda,p))|k,\sigma\rangle := \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\,U(\Lambda p)|k,\sigma'\rangle$
$\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|k,\sigma'\rangle$


Conclusion:
$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma,\sigma'}(\Lambda,p)|\Lambda p,\sigma'\rangle$
$C_{\sigma'\sigma} = D_{\sigma',\sigma}(W)$
$W(p,\Lambda) = L^{-1}(\Lambda p)\Lambda L(p)$
$|p,\sigma\rangle = U(L(\Lambda p))\underbrace{U(W(\Lambda,p))|k,\sigma\rangle}_{\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|k,\sigma'\rangle} = \sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))\underbrace{U(\Lambda p)|k,\sigma'\rangle}_{|\Lambda p,\sigma'\rangle}$


$\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p)) |k,\sigma'\rangle$
$|\Lambda p,\sigma\rangle$
Trick 2:
consider all Lorentz transform
Keep K invariant.
bgroup of Lorentz group.
Little group of K.
$W^{\mu}{}_{\nu} k^{\nu} = k^{\mu}$
Case 1: Massive Particles
$p^2 = -M^2$ $M > 0$
$K = (M, 0, 0, 0)$
Little group
SO(3)
Case 2: Massless Partic


Case 1: Massive Particles
p² = −M²   M > 0
K = (M, 0, 0, 0)
Little group SO(3)
dim = 2s+1
s = 0, 1/2, 1, 3/2, ...
(M, s)
spin
Case 2: Massless Particles
p² = 0   K = (1, 0, 0, 1)
Little group: Rigid motions in ℝ².
σ ≡ Helicity
J₃ |K, σ⟩ = σ |K, σ⟩