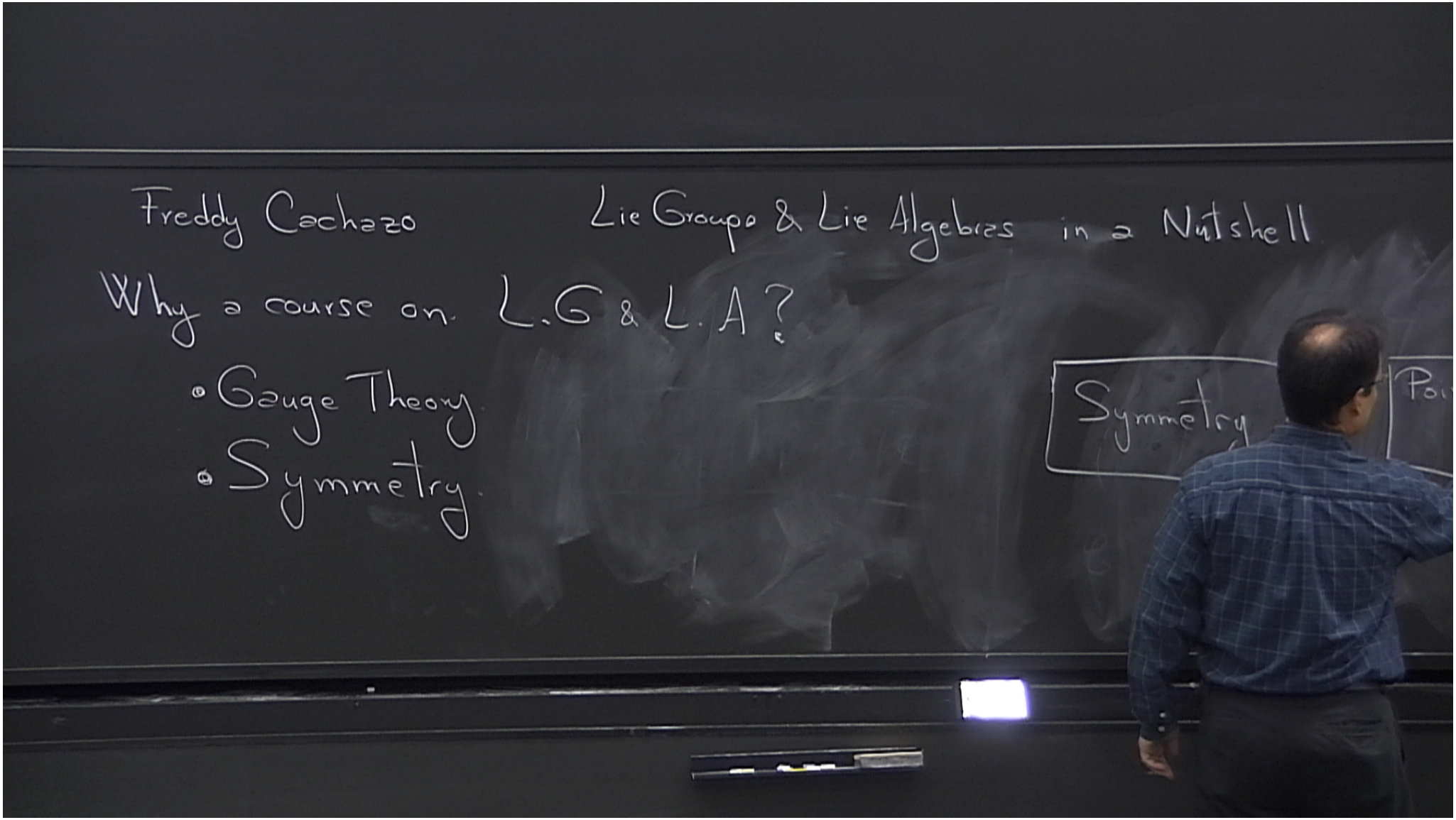
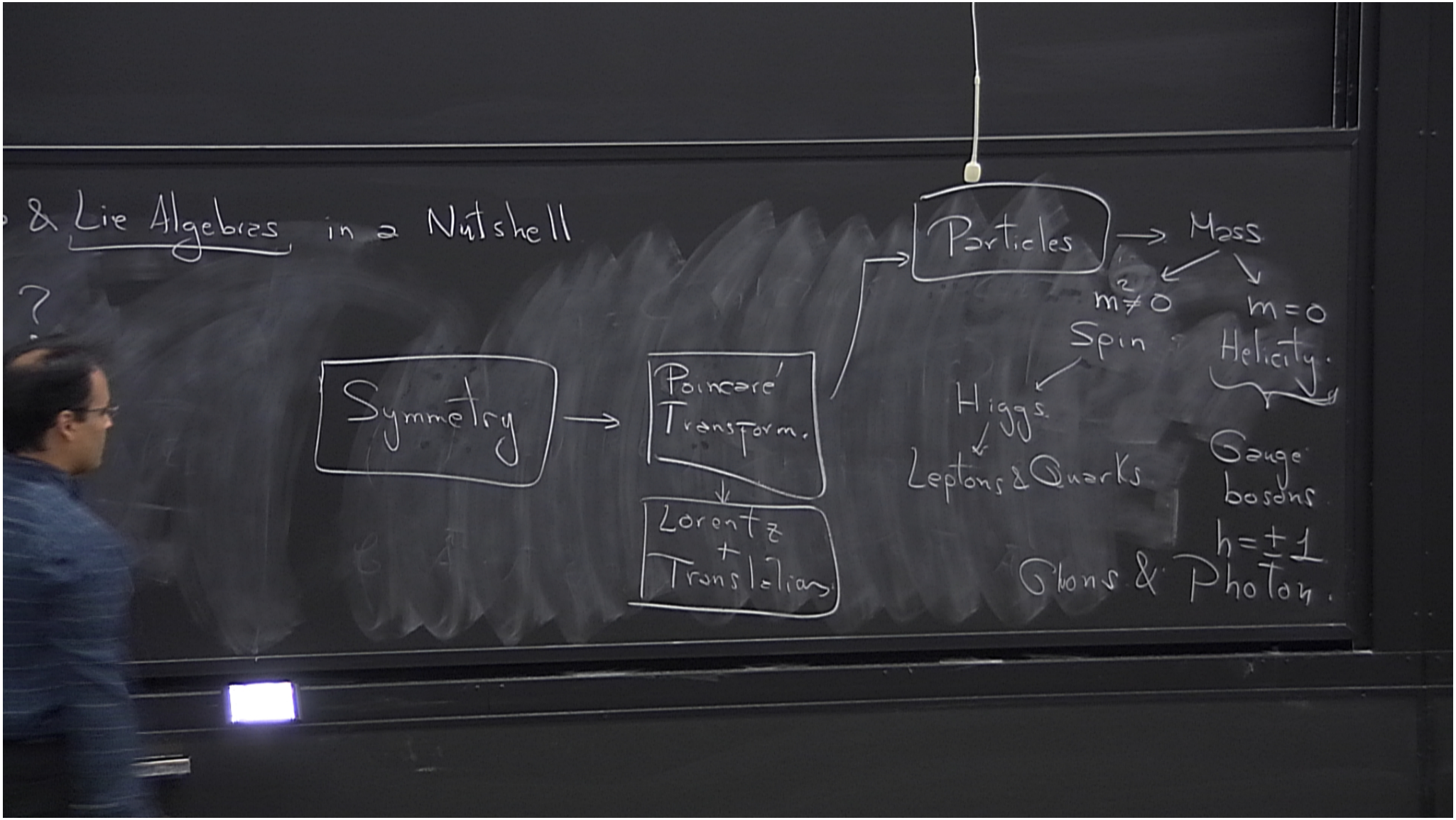
Title: Lie Groups & Lie Algebras - Lecture 1

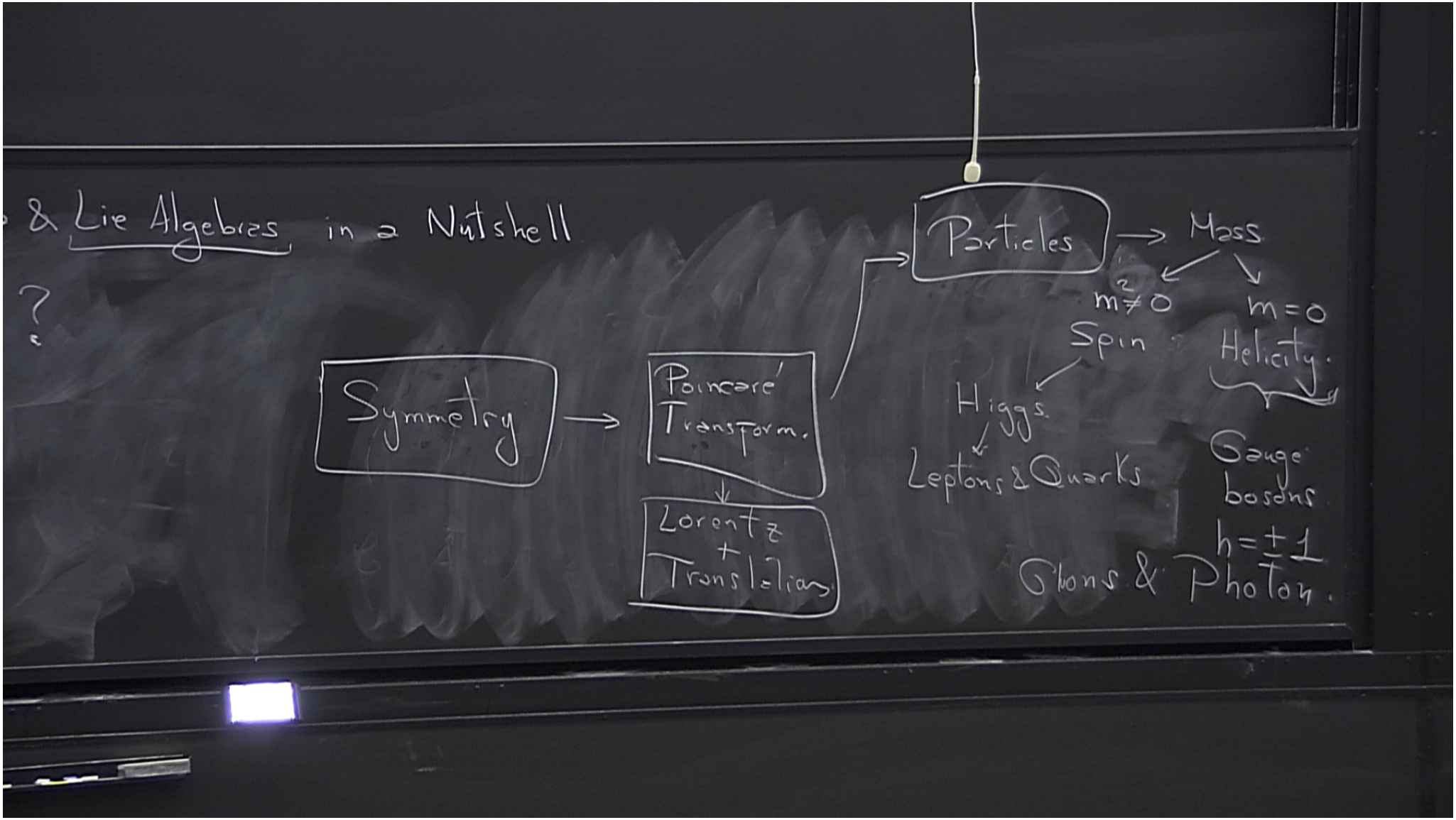
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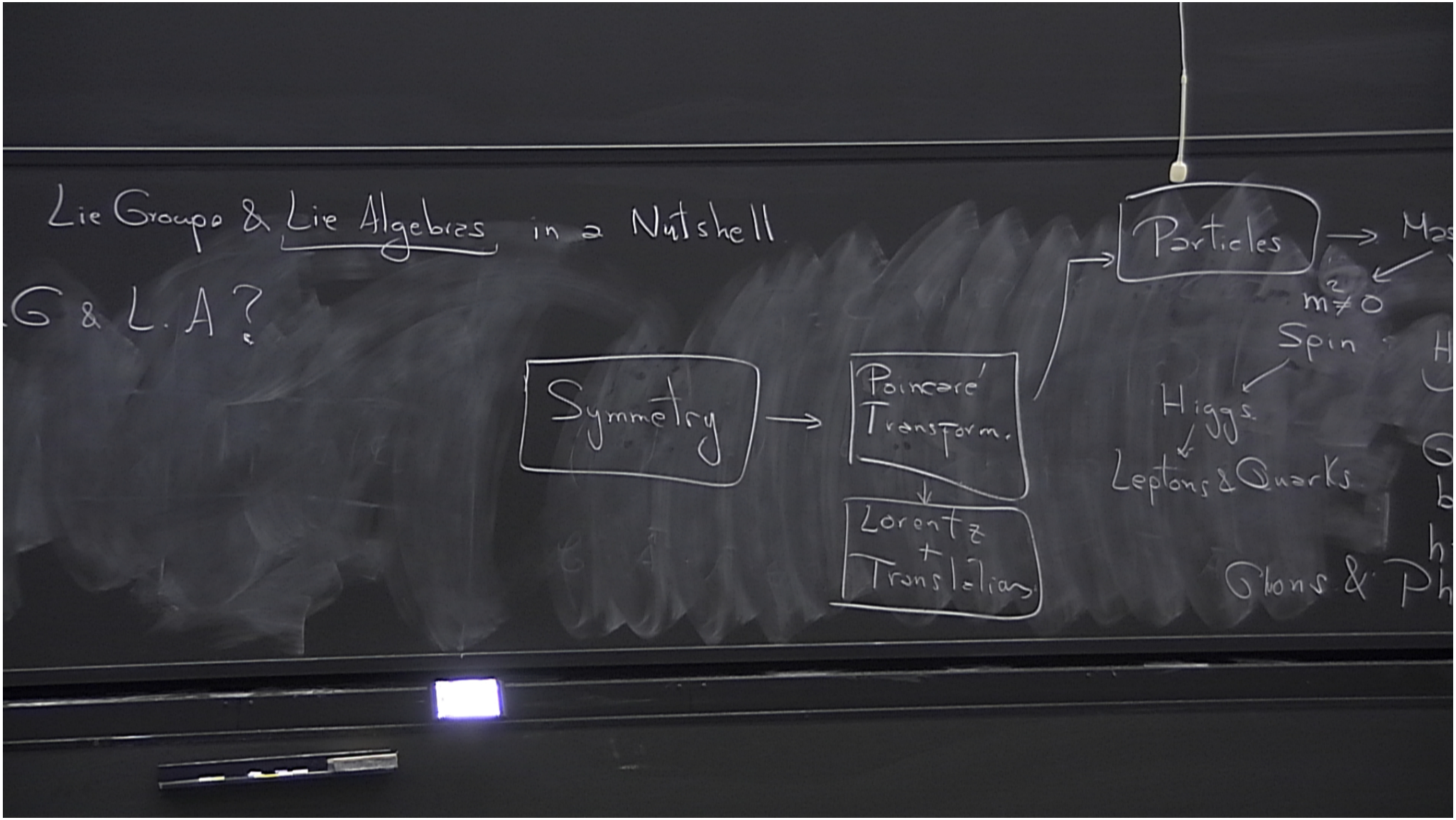
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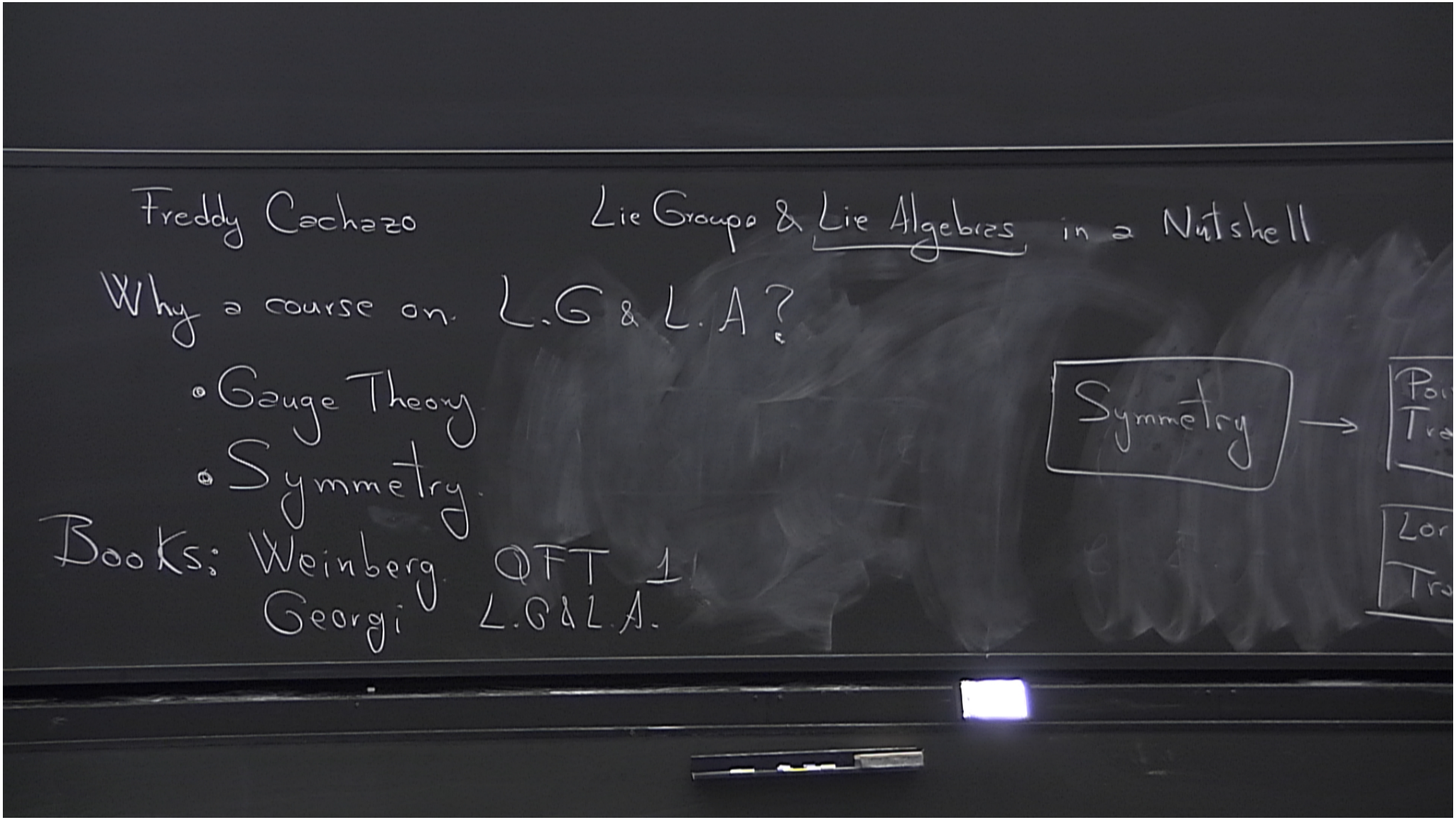
Abstract:

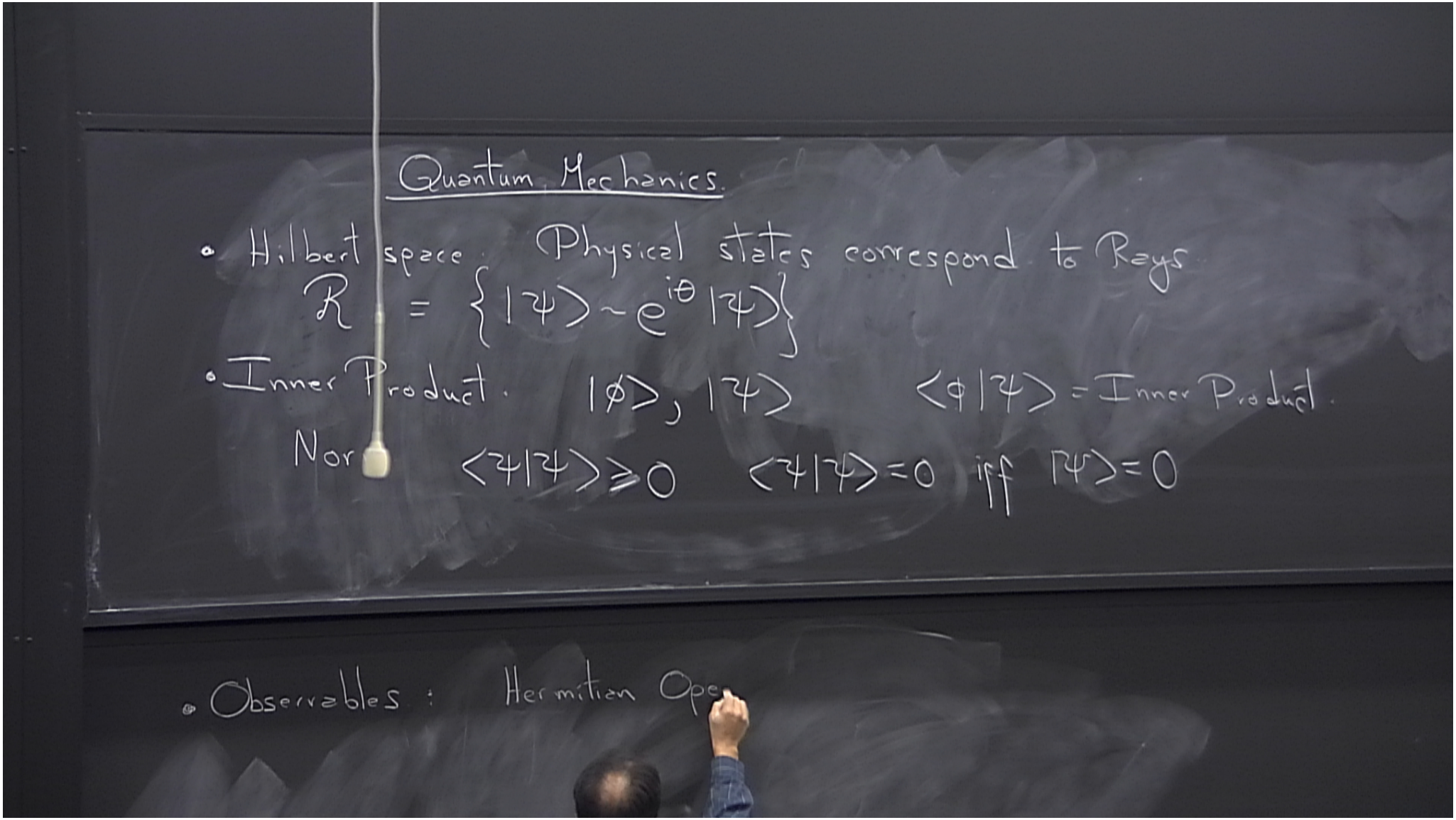

Freddy Cachazo
Lie Groups & Lie Algebras in a Nutshell
Why a course on L.G & L.A?
• Gauge Theory
• Symmetry
Symmetry

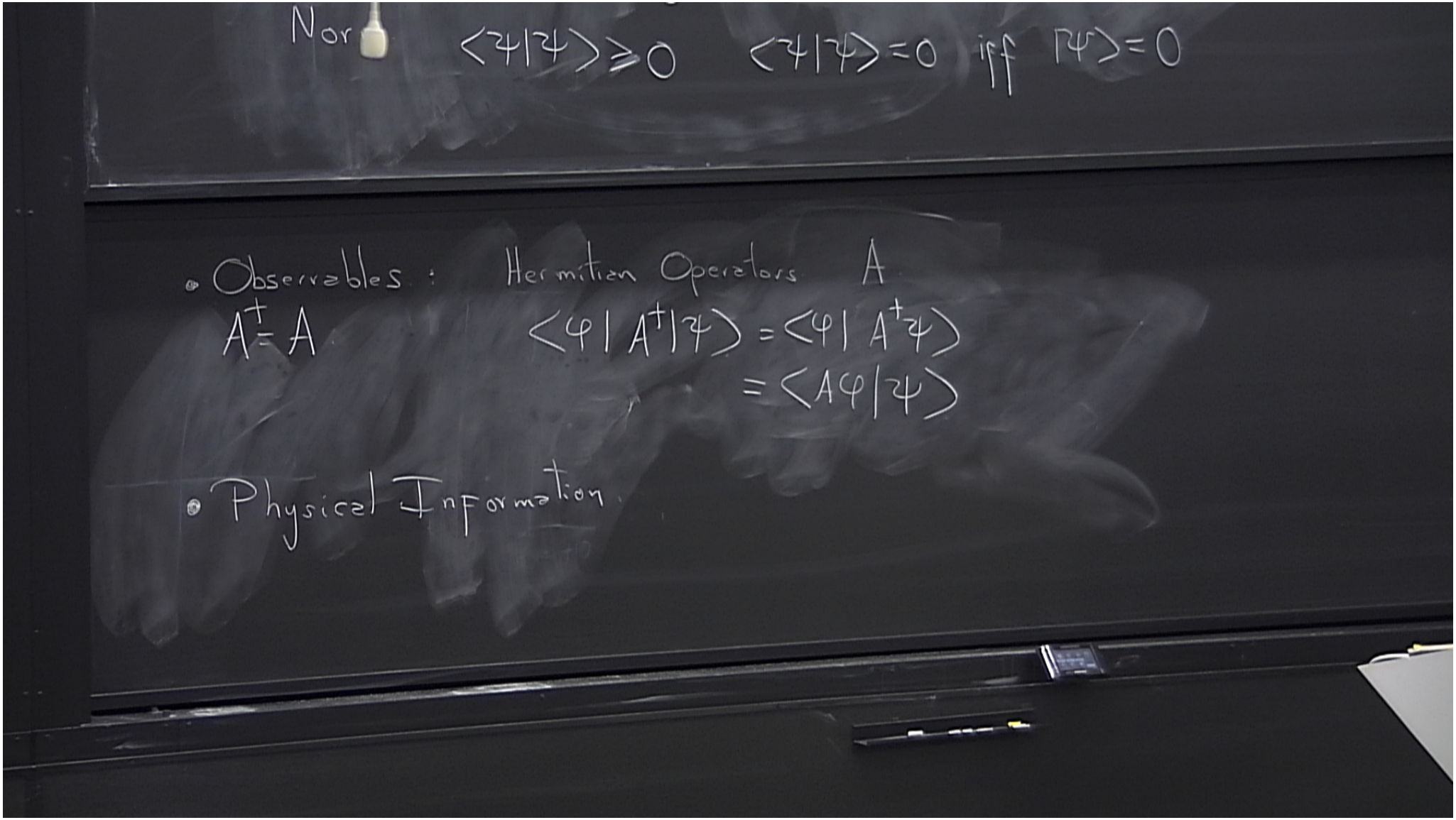

& Lie Algebras in a Nutshell
?
Symmetry
Poincaré Transform.
Lorentz + Translations
Particles
Mass
$m^2 \neq 0$
Spin
Higgs
Leptons & Quarks
$m = 0$
Helicity.
Gauge bosons
$h = \pm 1$
Gluons & Photon.

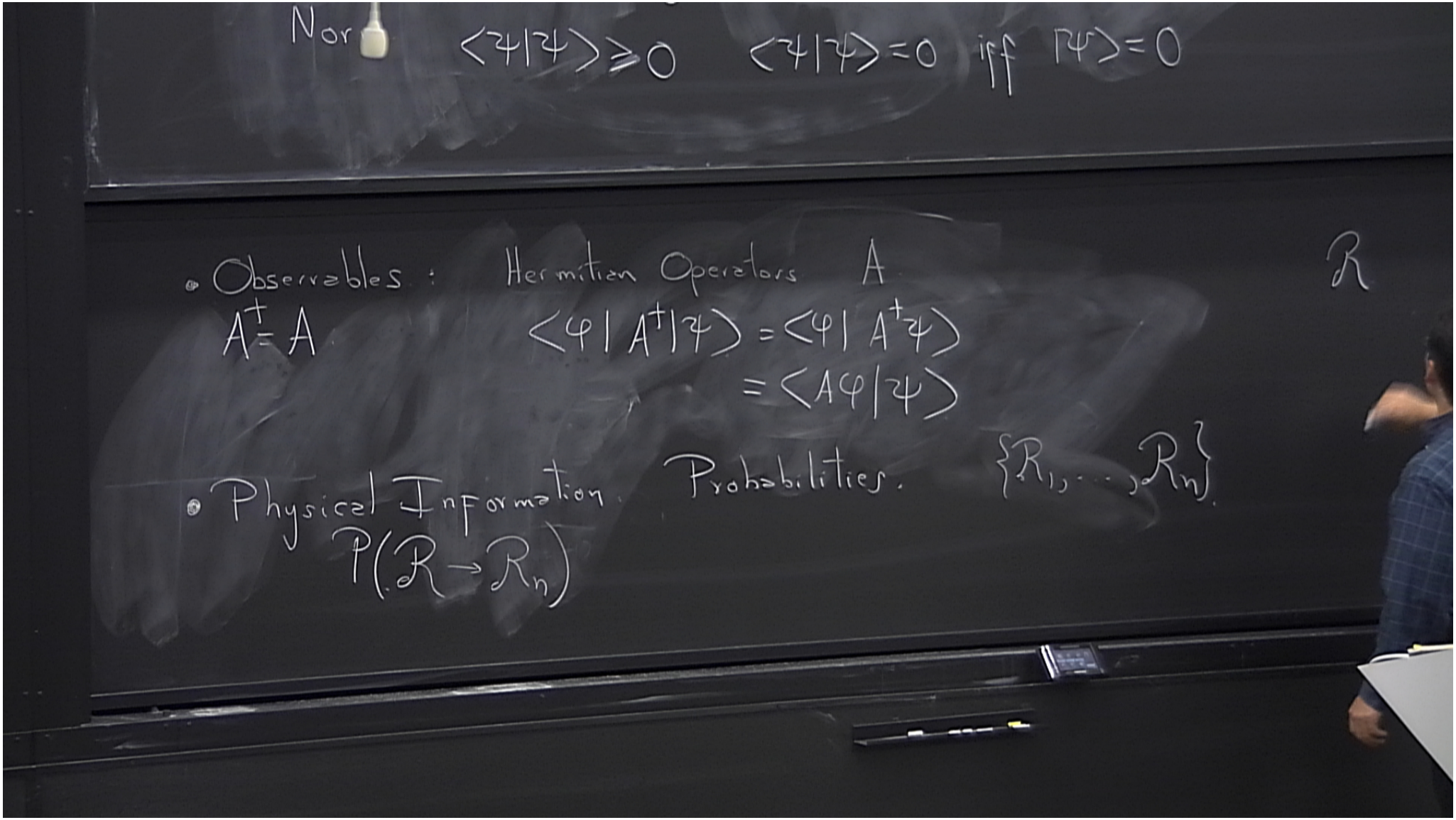

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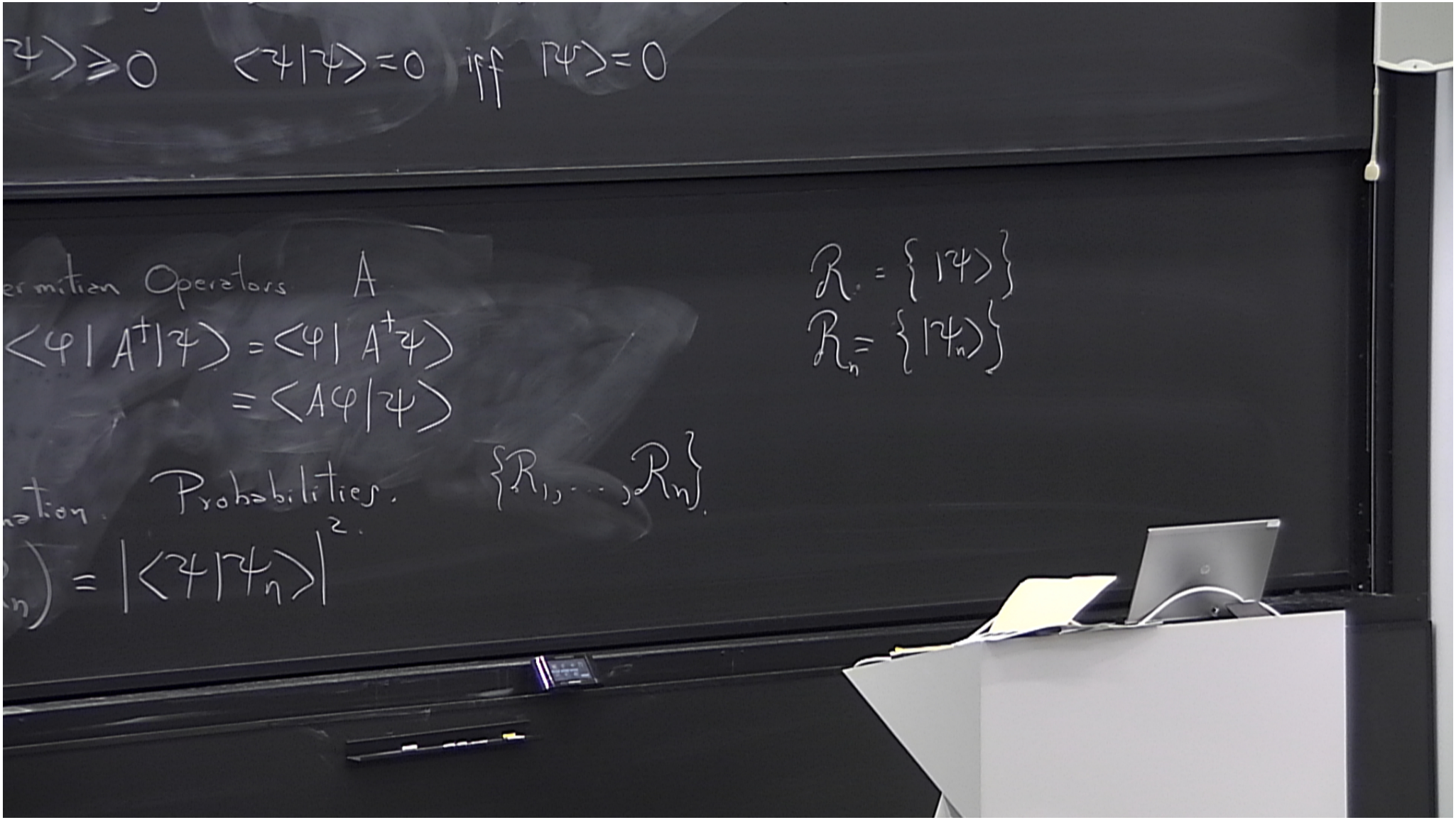

Lie Groups & Lie Algebras in a Nutshell
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Books: Weinberg QFT 1
Georgi L.G&L.A.
Symmetry


Quantum Mechanics.
• Hilbert space. Physical states correspond to Rays
$\mathcal{R} = \{|\psi\rangle \sim e^{i\theta}|\psi\rangle\}$
• Inner Product. $|\phi\rangle, |\psi\rangle$ $\langle\phi|\psi\rangle$ = Inner Product.
Nor $\langle\psi|\psi\rangle \geq 0$ $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = 0$
• Observables : Hermitian Ope


Nor $\langle\psi|\psi\rangle \geq 0$ $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = 0$
• Observables : Hermitian Operators A
$A^\dagger = A$
$\langle\varphi|A^\dagger|\psi\rangle = \langle\varphi|A^\dagger\psi\rangle$
$= \langle A\varphi|\psi\rangle$
• Physical Information


Nor
$\langle\psi|\psi\rangle \geq 0$ $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = 0$
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$\mathcal{R}$
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$\langle\varphi|A^\dagger|\psi\rangle = \langle\varphi|A^\dagger\psi\rangle$
$= \langle A\varphi|\psi\rangle$
• Physical Information. Probabilities. $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$
$P(\mathcal{R} \to \mathcal{R}_n)$


$|\psi\rangle \geq 0 \quad \langle\psi|\psi\rangle = 0 \text{ iff } |\psi\rangle = 0$
ermitian Operators A
$\langle\varphi|A^\dagger|\psi\rangle = \langle\varphi|A^\dagger\psi\rangle$
$= \langle A\varphi|\psi\rangle$
$\mathcal{R}_1 = \{|\psi\rangle\}$
$\mathcal{R}_n = \{|\psi_n\rangle\}$
ation. Probabilities. $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$
$= |\langle\psi|\psi_n\rangle|^2$

Nor $\langle\psi|\psi\rangle \geq 0$ $\langle\psi|\psi\rangle = 0$ iff $|\psi\rangle = 0$

- Observables : Hermitian Operators A

$A^\dagger = A$ $\langle\varphi|A^\dagger|\psi\rangle = \langle\varphi|A^\dagger\psi\rangle$

$= \langle A\varphi|\psi\rangle$

$\mathcal{R}_1 =$

$\mathcal{R}_n =$

- Physical Information Probabilities. $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$

$$P(\mathcal{R} \to \mathcal{R}_n) = |\langle\psi|\psi_n\rangle|^2.$$

• Symmetry.

Books: Weinberg QFT 1
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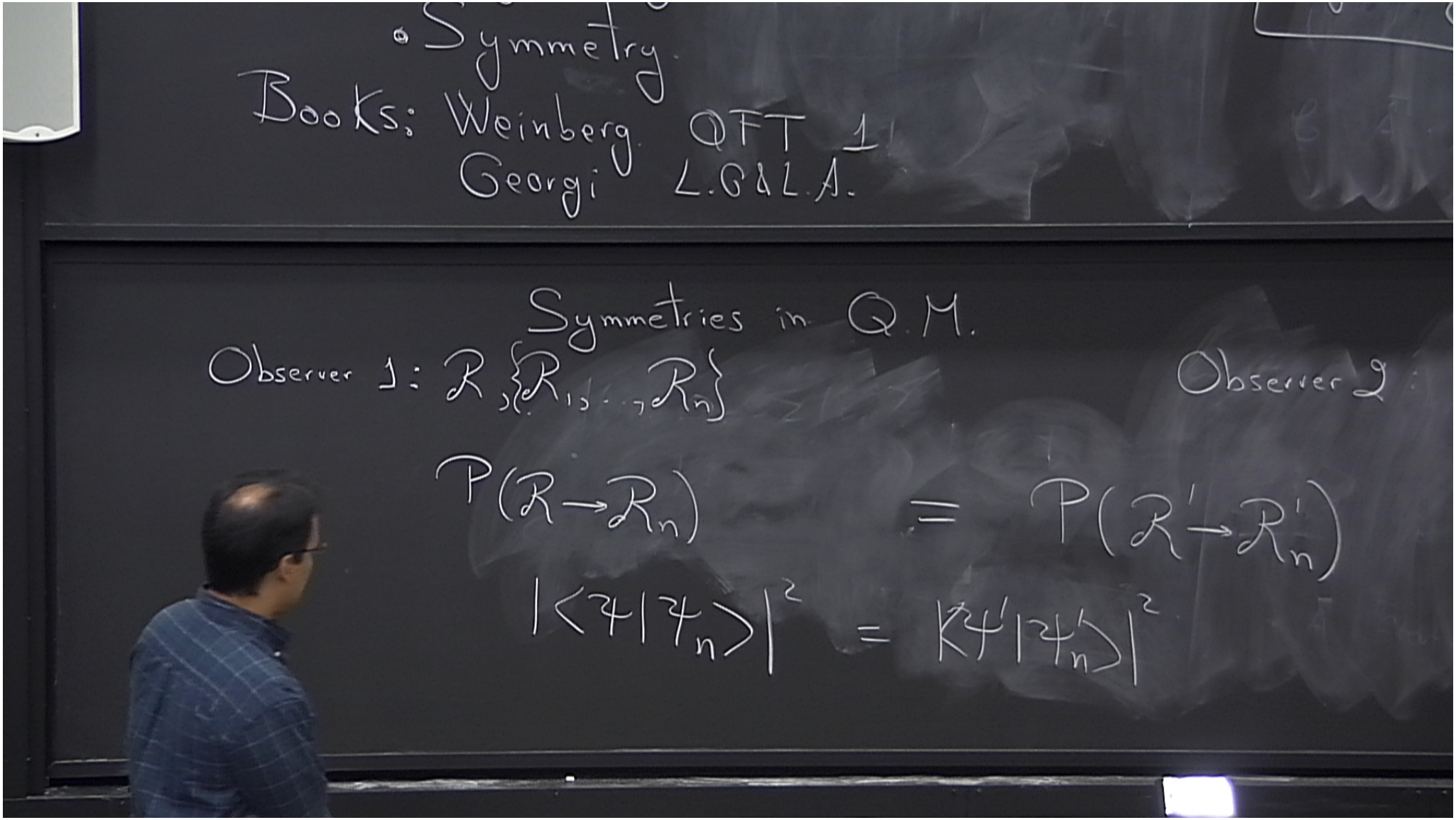
Symmetries in Q.M.

Observer 1: $\mathcal{R}, \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ Observer 2:

$$P(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$$

$$|\langle \psi | \psi_n \rangle|^2 = |\langle \psi' | \psi'_n \rangle|^2$$




Symmetry
Books: Weinberg QFT 1
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Symmetries in Q.M.
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Observer 2:
$P(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$
$|\langle \psi | \psi_n \rangle|^2 = |\langle U\psi | \psi'_n \rangle|^2$

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Wigner's theorem: (1930's) $\exists$ an operator U such that if $|\psi\rangle \in \mathcal{R}$ then $U|\psi\rangle \in \mathcal{R}'$ & U is either

- Unitary & Linear

or

- Anti Unitary & Anti L

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then $U|\psi\rangle \in \mathcal{R}'$ & U is either

* Unitary & Linear $U^\dagger U = \mathbb{I}$ $U^\dagger = U^{-1}$

$$\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$$

- Unitary & Linear

or

- Anti Unitary & Ant

$$U(\alpha|\varphi\rangle + \beta|\psi\rangle) = \alpha U|\varphi\rangle + \beta U|\psi\rangle$$

$(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$

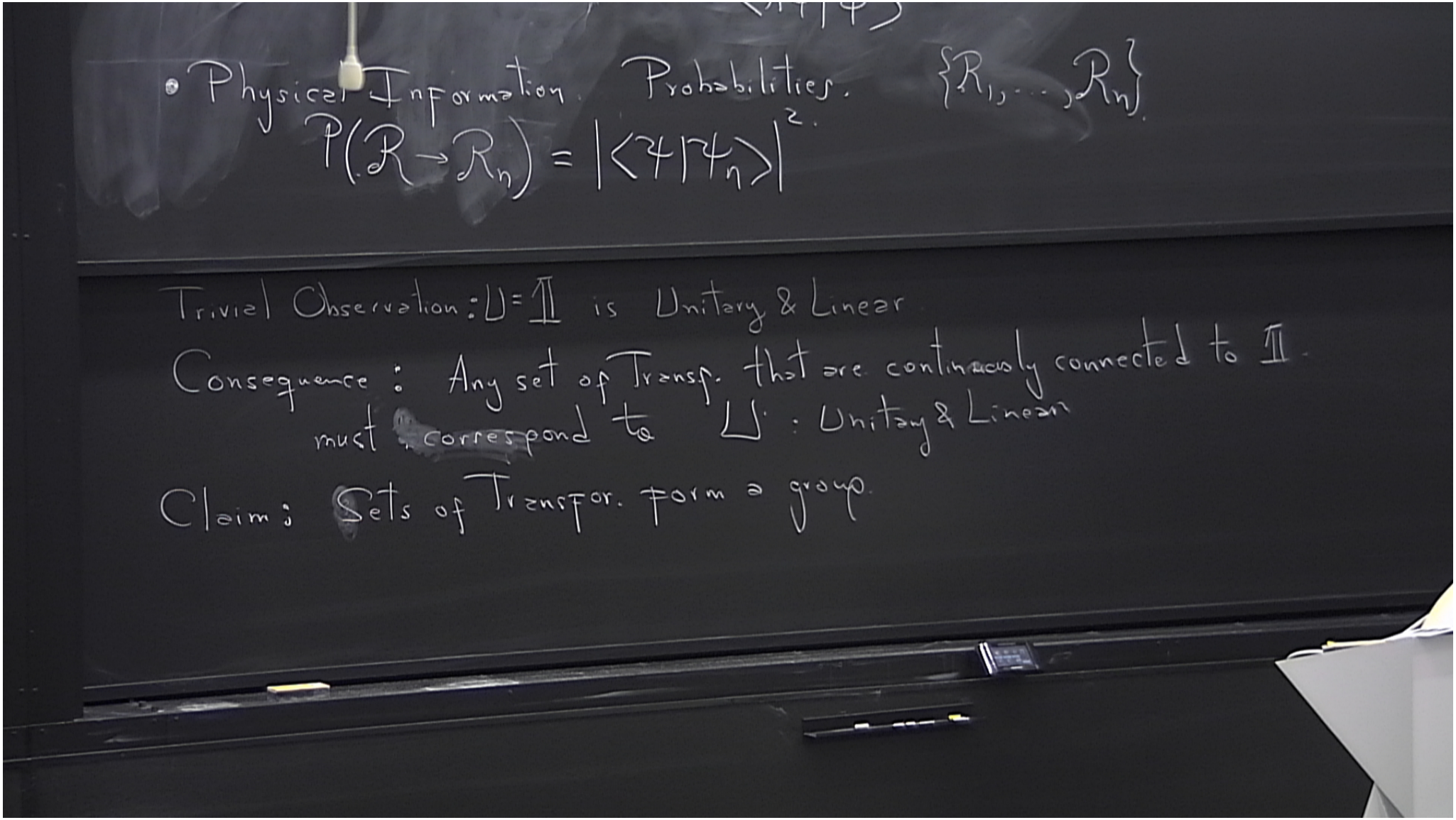
$|\langle \psi | \psi_n \rangle|^2 = |\langle U\psi | U\psi_n \rangle|^2$

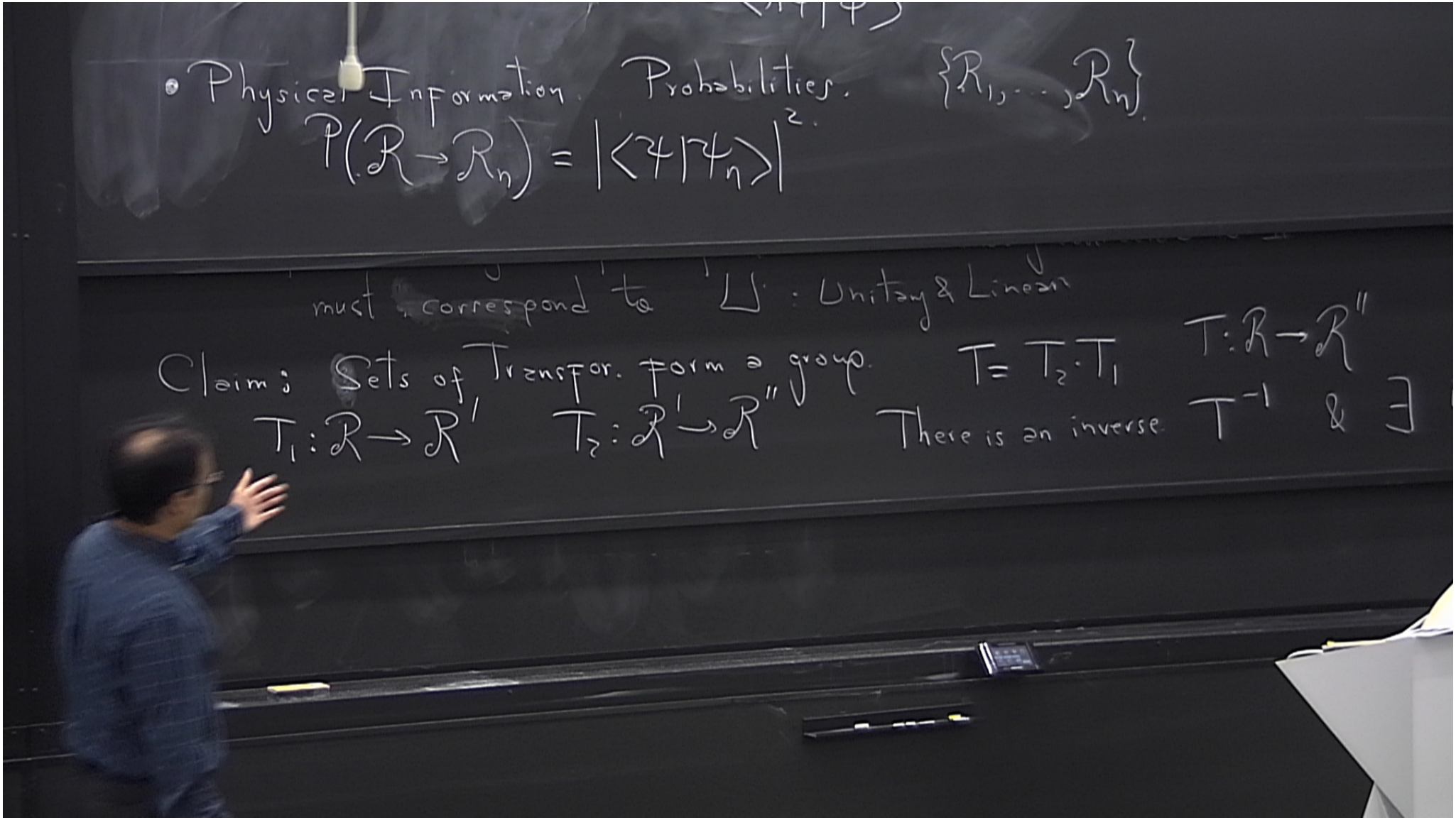
Theorem : (1930's) $\exists$ an operator U such that if $|\psi\rangle \in \mathcal{R}$ $U|\psi\rangle \in \mathcal{R}'$ & U is either

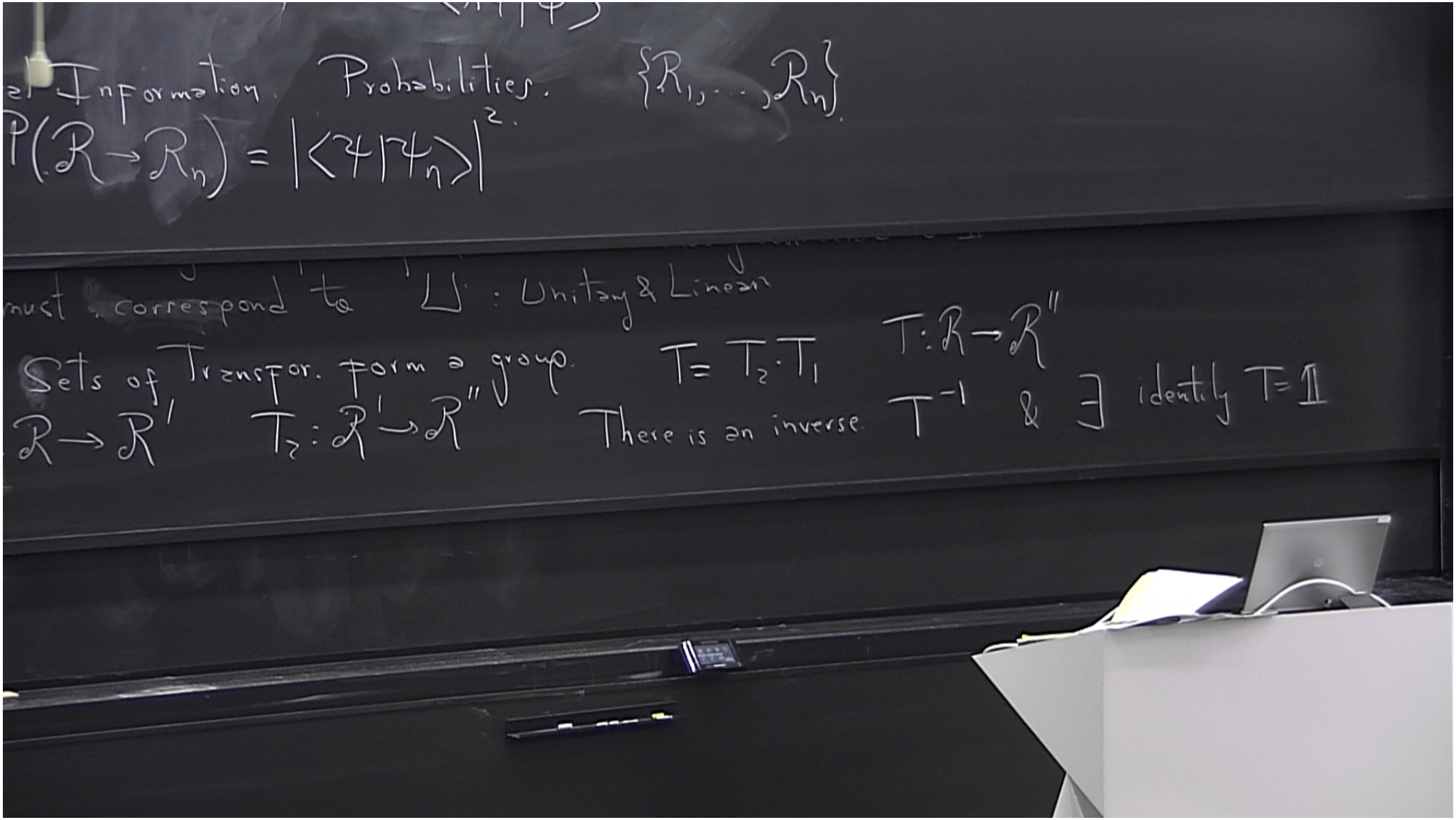
- Unitary & Linear
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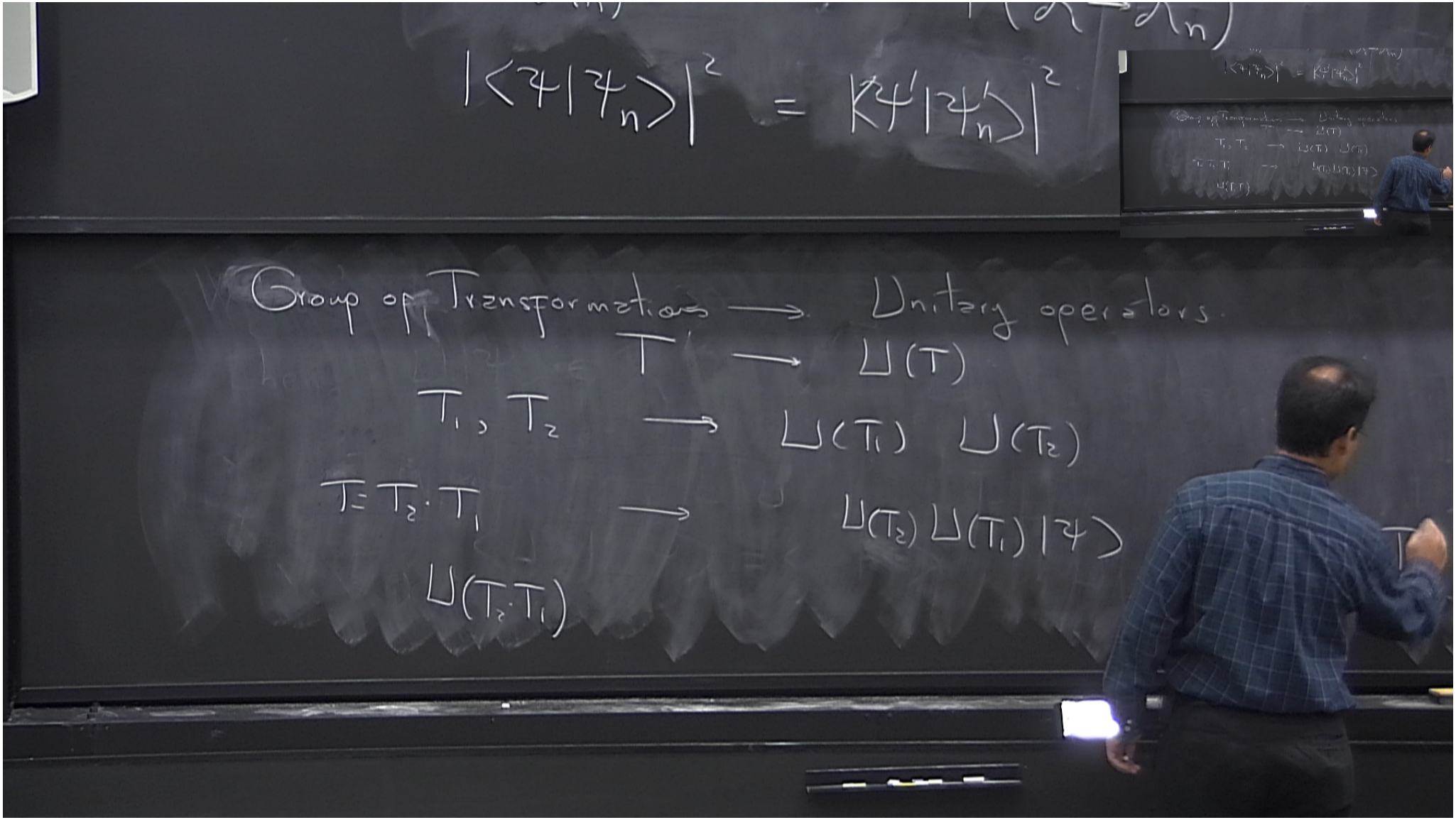
& Linear $U^\dagger U = \mathbb{1}$ $U^\dagger = U^{-1}$

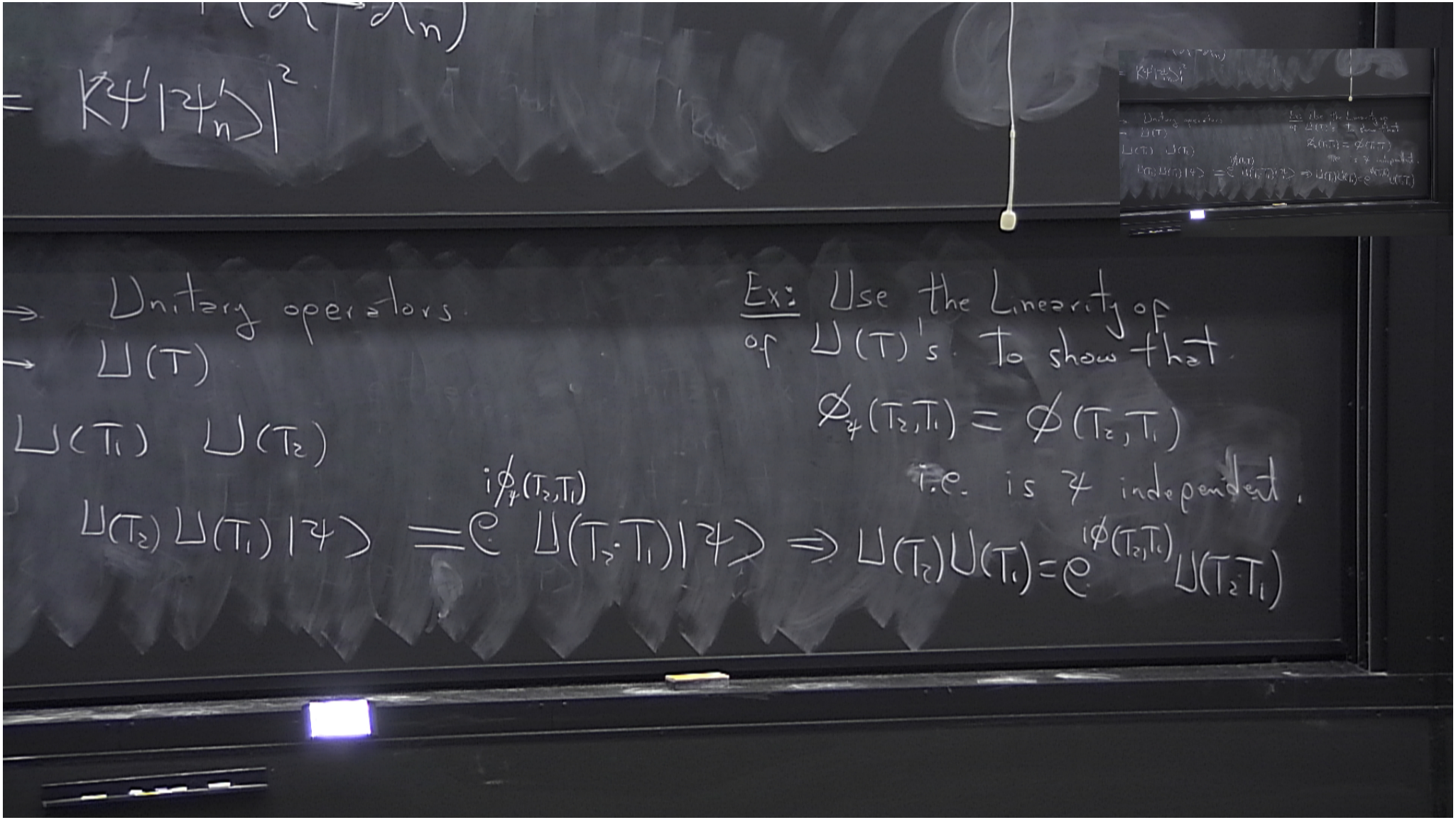
$\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$ $U(\alpha|\varphi\rangle + \beta|\psi\rangle) = \alpha U|\varphi\rangle + \beta U|\psi\rangle$

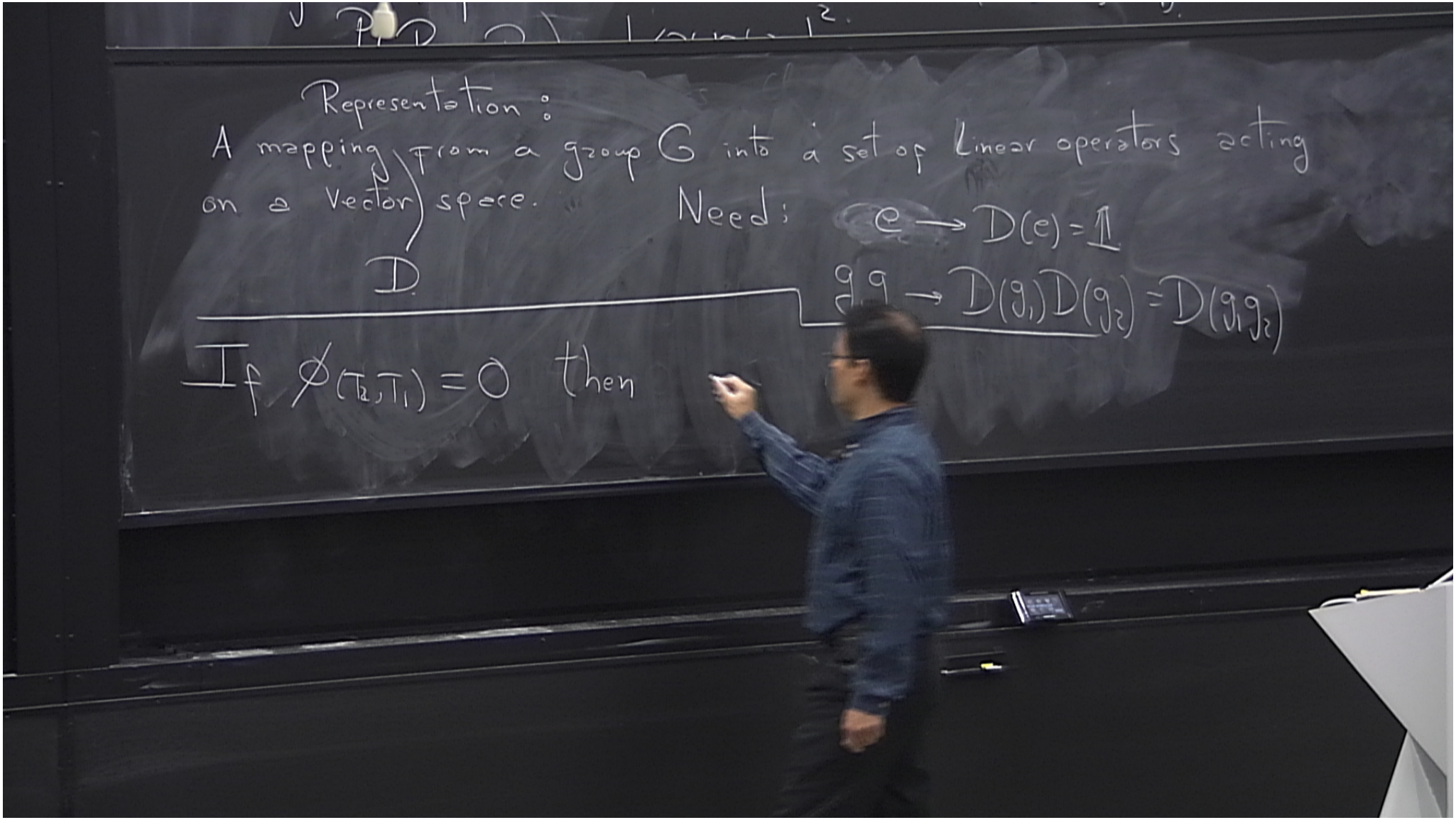

• Physical Information. Probabilities. $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$
$P(\mathcal{R} \to \mathcal{R}_n) = |\langle \psi | \psi_n \rangle|^2$.
Trivial Observation: $U = \mathbb{1}$ is Unitary & Linear
Consequence: Any set of Transf. that are continuously connected to $\mathbb{1}$
must correspond to U: Unitary & Linear
Claim: Sets of Transfor. form a group


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Claim: Sets of Transfor. form a group. $T = T_2 \cdot T_1$ $T: \mathcal{R} \to \mathcal{R}''$
$T_1: \mathcal{R} \to \mathcal{R}'$ $T_2: \mathcal{R}' \to \mathcal{R}''$ There is an inverse T^{-1} & $\exists$


Information. Probabilities. $\{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$
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$\mathcal{R} \to \mathcal{R}'$ $T_2: \mathcal{R}' \to \mathcal{R}''$ There is an inverse T^{-1} & $\exists$ identity $T = \mathbb{1}$


$|\langle\psi|\psi_n\rangle|^2 = |\langle\psi|\psi_n\rangle|^2$
Group of Transformations $\longrightarrow$ Unitary operators.
$T \longrightarrow U(T)$
$T_1, T_2 \longrightarrow U(T_1)\ U(T_2)$
$T = T_2 \cdot T_1 \longrightarrow U(T_2)\,U(T_1)\,|\psi\rangle$
$U(T_2 \cdot T_1)$


$= |\langle \psi | \psi_n \rangle|^2$
Unitary operators
$U(T)$
$U(T_1)\ U(T_2)$
$U(T_2)\,U(T_1)|\psi\rangle = e^{i\phi_\psi(T_2,T_1)}\,U(T_2 T_1)|\psi\rangle \Rightarrow U(T_2)U(T_1) = e^{i\phi(T_2,T_1)}U(T_2 T_1)$
Ex: Use the Linearity of of $U(T)$'s. To show that
$\phi_\psi(T_2,T_1) = \phi(T_2,T_1)$
i.e. is ψ independent.


Representation:
A mapping D from a group G into a set of linear operators acting on a vector space.
D
Need: $e \rightarrow D(e) = \mathbb{1}$
$g_1 g_2 \rightarrow D(g_1)D(g_2) = D(g_1 g_2)$
If $\phi(T_2, T_1) = 0$ then

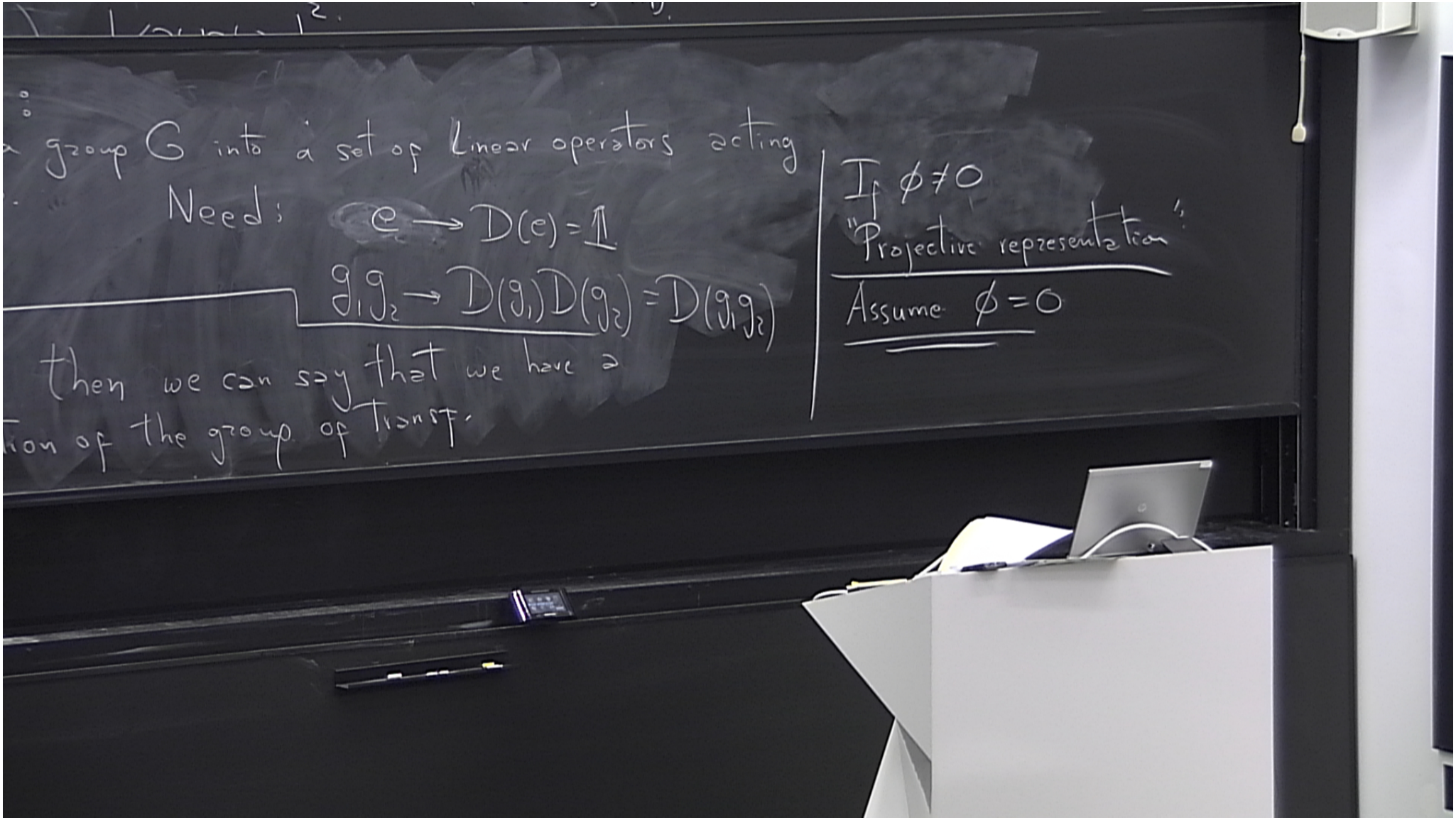
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If $\phi(T_2, T_1) = 0$ then we can say that we have a unitary representation of the group of transf.


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tion of the group of Transf.
If $\phi \neq 0$
"Projective representation"
Assume $\phi = 0$

tary representation of the group of Transf.

Lie Group: Groups whose elements are labelled by a continuous set of parameters.

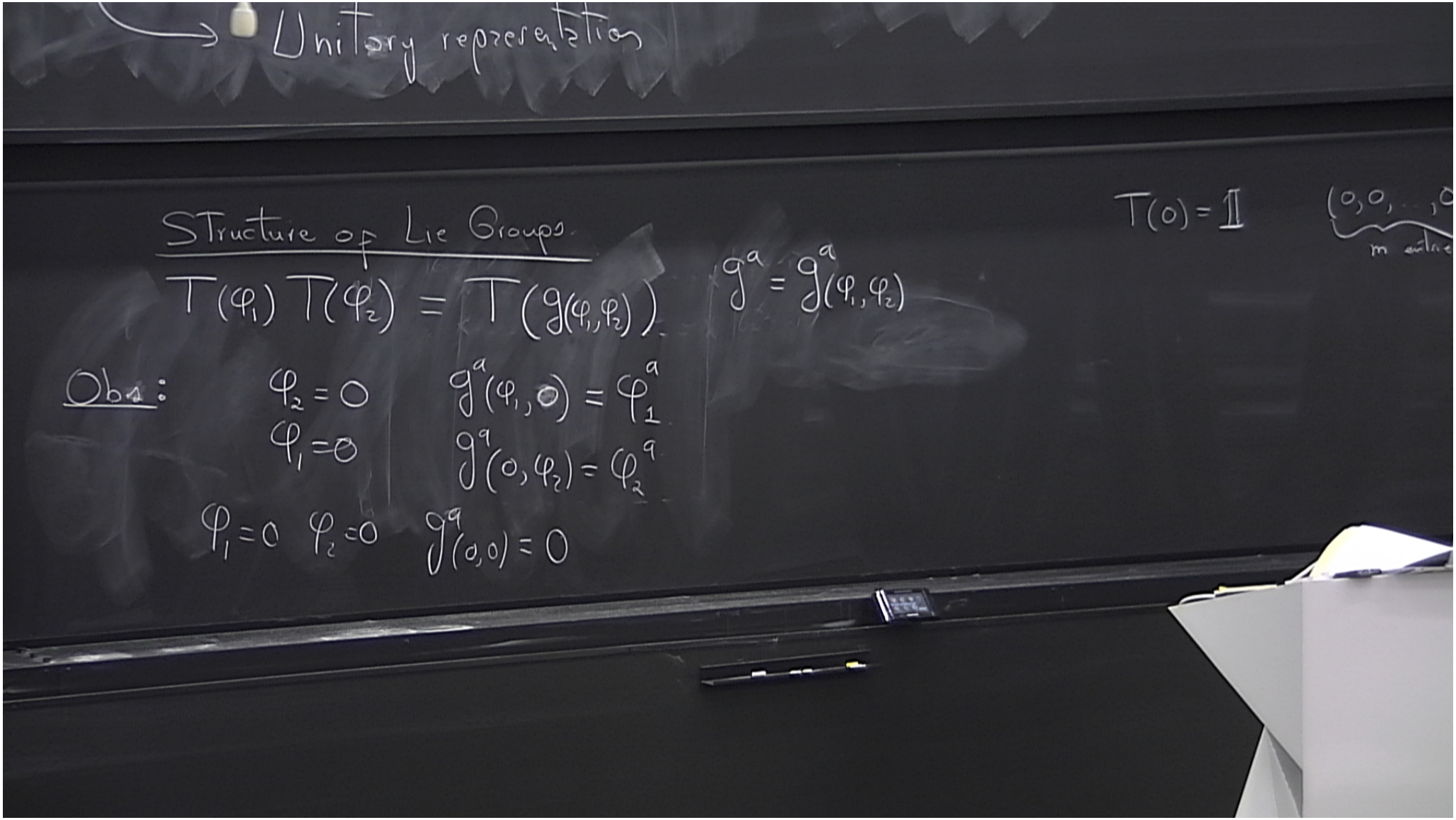
$T(\varphi)$ → Real parameters $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^m)$ φ^a

...tary representation of the group of Transf.

Lie Group: Groups whose elements are labelled by a continuous set of parameters.

$T(\varphi)$ → Real parameters $\varphi = (\varphi^1, \varphi^2, \ldots, \varphi^m)$ φ^a

Connected Lie Groups: Every element can be reach from the $\mathbb{I}$ by following a path in parameter space


Unitary representation
Structure of Lie Groups
$T(\varphi_1)\,T(\varphi_2) = T(g(\varphi_1,\varphi_2))$
$g^a = g^a(\varphi_1,\varphi_2)$
$T(0) = \mathbb{I}$
$(0,0,\ldots,0)$
m entries
Obs:
$\varphi_2 = 0 \quad g^a(\varphi_1,0) = \varphi_1^a$
$\varphi_1 = 0 \quad g^a(0,\varphi_2) = \varphi_2^a$
$\varphi_1 = 0 \quad \varphi_2 = 0 \quad g^a(0,0) = 0$

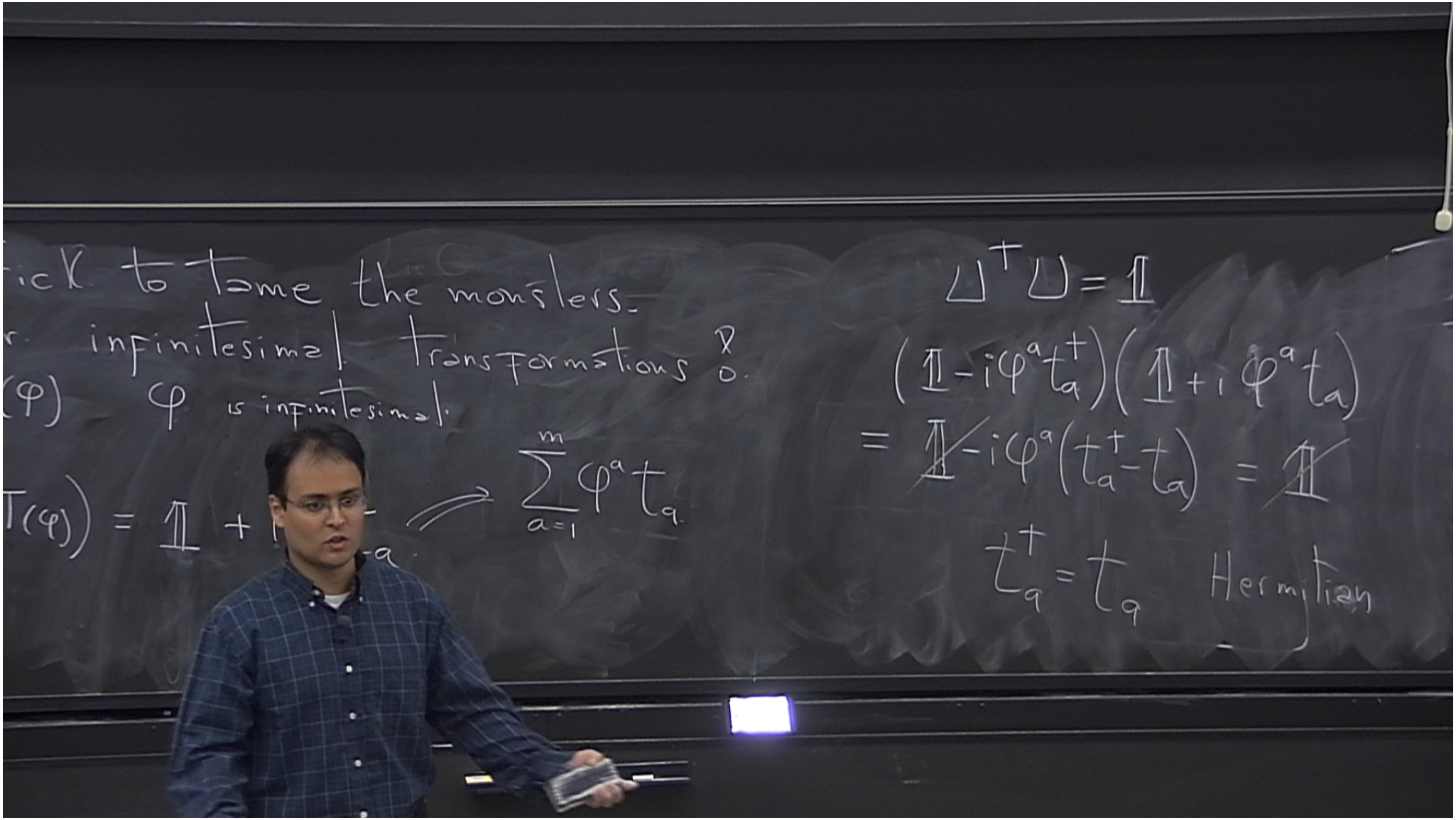
Trick to tame the monsters.

Consider infinitesimal transformations $T(\varphi)$ φ is infinitesimal.

$$U(T(\varphi)) = \mathbb{1} + i\varphi^a t_a \rightarrow \sum_{a=1}^{m} \varphi^a t_a$$

$U^\dagger U = \mathbb{1}$

$(\mathbb{1} - \; t_a^\dagger)($


ick to tame the monsters.
infinitesimal transformations ?
(φ) φ is infinitesimal.
$\sum_{a=1}^{m} \varphi^a t_a$
$U^\dagger U = \mathbb{1}$
$(\mathbb{1} - i\varphi^a t_a^\dagger)(\mathbb{1} + i\varphi^a t_a)$
$= \mathbb{1} - i\varphi^a (t_a^\dagger - t_a) = \mathbb{1}$
$t_a^\dagger = t_a$ Hermitian

$t_a^\dagger = t_a$ Hermi

Small φ's

$$U(T(\varphi)) = \mathbb{1} + i\varphi^a t_a + \frac{1}{2}\varphi^a\varphi^b t_{ab} + \mathcal{O}(\varphi^3) \qquad g^a(\varphi_1,\varphi_2) = A^a + B$$

$$\boxed{t_{ab} = t_{ba}}$$

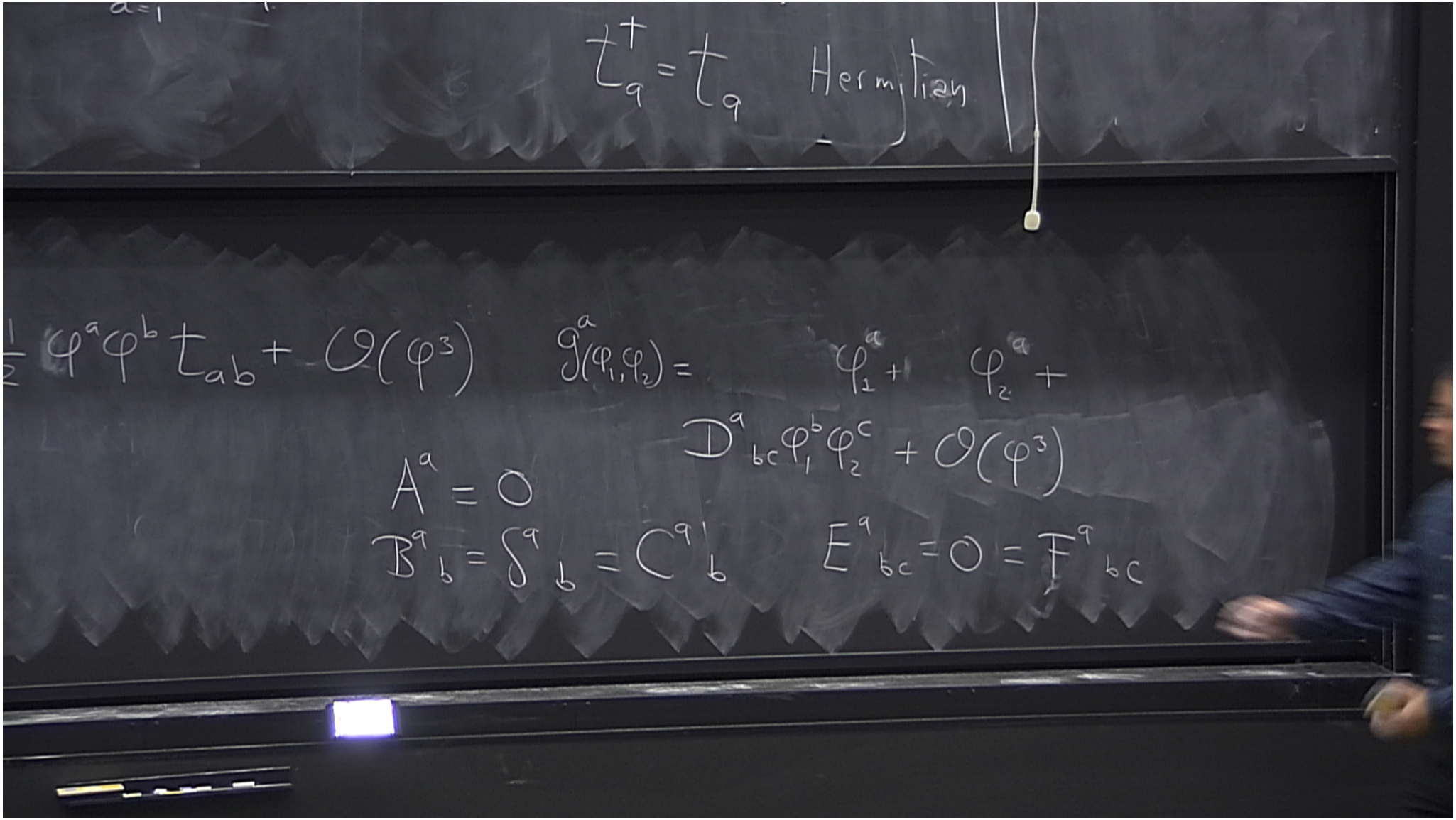
$t_a^\dagger = t_a$ Hermitian

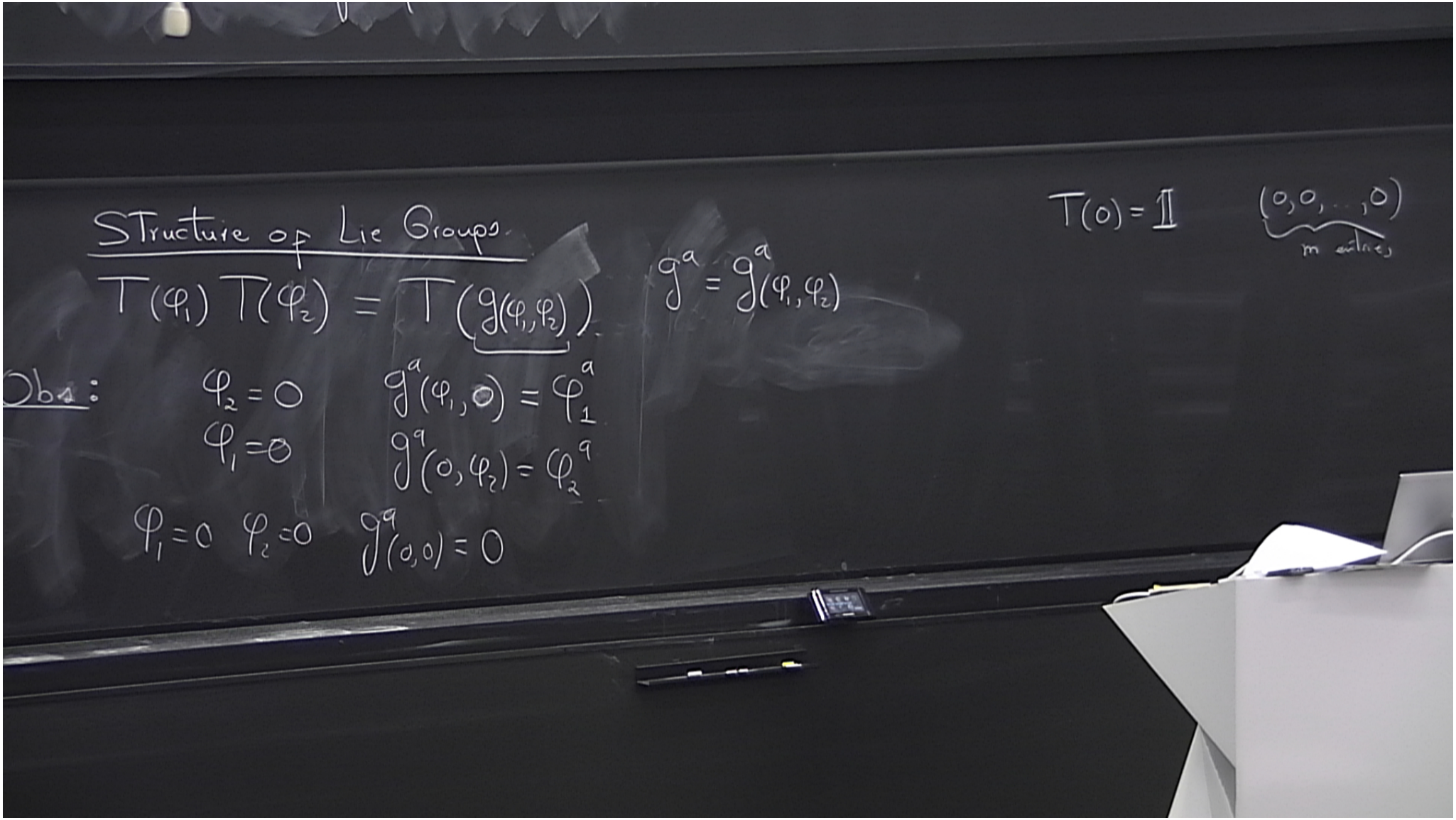
$i\varphi^a t_a + \frac{1}{2}\varphi^a\varphi^b t_{ab} + \mathcal{O}(\varphi^3)$

t_{ba}

$g^a(\varphi_1,\varphi_2) = A^a + B^a{}_b\varphi_1^b + C^a{}_b\varphi_2^b +$

$D^a{}_{bc}\varphi_1^b\varphi_2^c + E^a{}_{bc}\varphi_1^b\varphi_1^c + \ldots\varphi_2^b$


$t_a^\dagger = t_a$ Hermitian
$\frac{1}{2}\varphi^a\varphi^b t_{ab} + \mathcal{O}(\varphi^3)$
$g^a(\varphi_1,\varphi_2) = \varphi_1^a + \varphi_2^a +$
$D^a{}_{bc}\varphi_1^b\varphi_2^c + \mathcal{O}(\varphi^3)$
$A^a = 0$
$B^a{}_b = \delta^a{}_b = C^a{}_b$
$E^a{}_{bc} = 0 = F^a{}_{bc}$


Structure of Lie Groups
$T(\varphi_1)\,T(\varphi_2) = T(g(\varphi_1,\varphi_2))$
$g^a = g^a(\varphi_1,\varphi_2)$
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$\varphi_1 = 0$ $\varphi_2 = 0$ $g^a(0,0) = 0$

$$P(\mathcal{R} \to \mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$$

$$U(T(\varphi_1))\,U(T(\varphi_2)) = U(T(g(\varphi_1,\varphi_2))) = \mathbb{1} + i\, g^a(\varphi_1,\varphi_2)\, t_a + \frac{1}{2}\, g^a(\varphi_1,\varphi_2)\, g^b(\varphi_1,\varphi_2)\, t_{ab}$$

$$\left(\mathbb{1} + i\varphi_1^a t_a + \frac{1}{2}\varphi_1^a\varphi_1^b t_{ab}\right)\left(\mathbb{1} + i\varphi_2^a t_a + \frac{1}{2}\varphi_2^a\varphi_2^b t_{ab}\right)$$

$$= \mathbb{1} + i\left(\varphi_1^a + \varphi_2^a + D^a{}_{ef}\,\varphi_1^e\varphi_2^f\right) t_a + \frac{1}{2}\left(\varphi_1^a + \varphi_2^a\right)$$

$$\mathbb{1} + i\varphi_1^a t_a + i\varphi_2^a t_a + \frac{1}{2}\left(\varphi_1^a\varphi_1^b + \varphi_2^a\varphi_2^b\right) t_{ab} - \varphi_1^a\varphi_2^b\, t_a t_b$$



$$\mathcal{R}_n) = P(\mathcal{R}' \to \mathcal{R}'_n)$$

$$= U(T(g(\varphi_1,\varphi_2)) = \mathbb{I} + i\, g^a(\varphi_1,\varphi_2)\, t_a + \tfrac{1}{2}\, g^a(\varphi_1,\varphi_2)\, g^b(\varphi_1,\varphi_2)\, t_{ab}$$

$$= \mathbb{I} + i\left(\varphi_1^a + \varphi_2^a + D^a_{ef}\,\varphi_1^e\varphi_2^f\right) t_a + \tfrac{1}{2}\left(\varphi_1^a + \varphi_2^a + D^a_{ef}\,\varphi_1^e\varphi_2^f\right)\left(\varphi_1^b + \varphi_2^b + D^b_{mn}\,\varphi_1^m$$

$$i\varphi_2^a t_a + \tfrac{1}{2}\varphi_2^a\varphi_2^b t_{ab})$$

$$+\varphi_2^a\varphi_2^b)\, t_{ab} - \varphi_1^a\varphi_2^b\, t_a t_b$$

$$= \mathcal{P}(\mathcal{R}' \to \mathcal{R}'_n)$$

$$\varphi_2)) = \mathbb{1} + i\, g^a(\varphi_1,\varphi_2)\, t_a + \frac{1}{2}\, g^a(\varphi_1,\varphi_2)\, g^b(\varphi_1,\varphi_2)\, t_{ab}$$

$$= \mathbb{1} + i\left(\varphi_1^a + \varphi_2^a + D^a{}_{ef}\,\varphi_1^e\varphi_2^f\right) t_a + \frac{1}{2}\left(\varphi_1^a + \varphi_2^a + D^a{}_{ef}\,\varphi_1^e\varphi_2^f\right)\left(\varphi_1^b + \varphi_2^b + D^b{}_{mn}\,\varphi_1^m\varphi_2^n\right) t_{ab}$$

$$= i D^a{}_{ef}\,\varphi_1^e\varphi_2^f + \underbrace{\frac{1}{2}\,\varphi_1^a\varphi_2^b\, t_{ab} + \frac{1}{2}\,\varphi_2^a\varphi_1^b\, t_{ab}}_{\varphi_1^a\varphi_2^b\, t_{a\underline{b}}}$$

$$\varphi_1^a\varphi_2^b\, t_a t_b$$

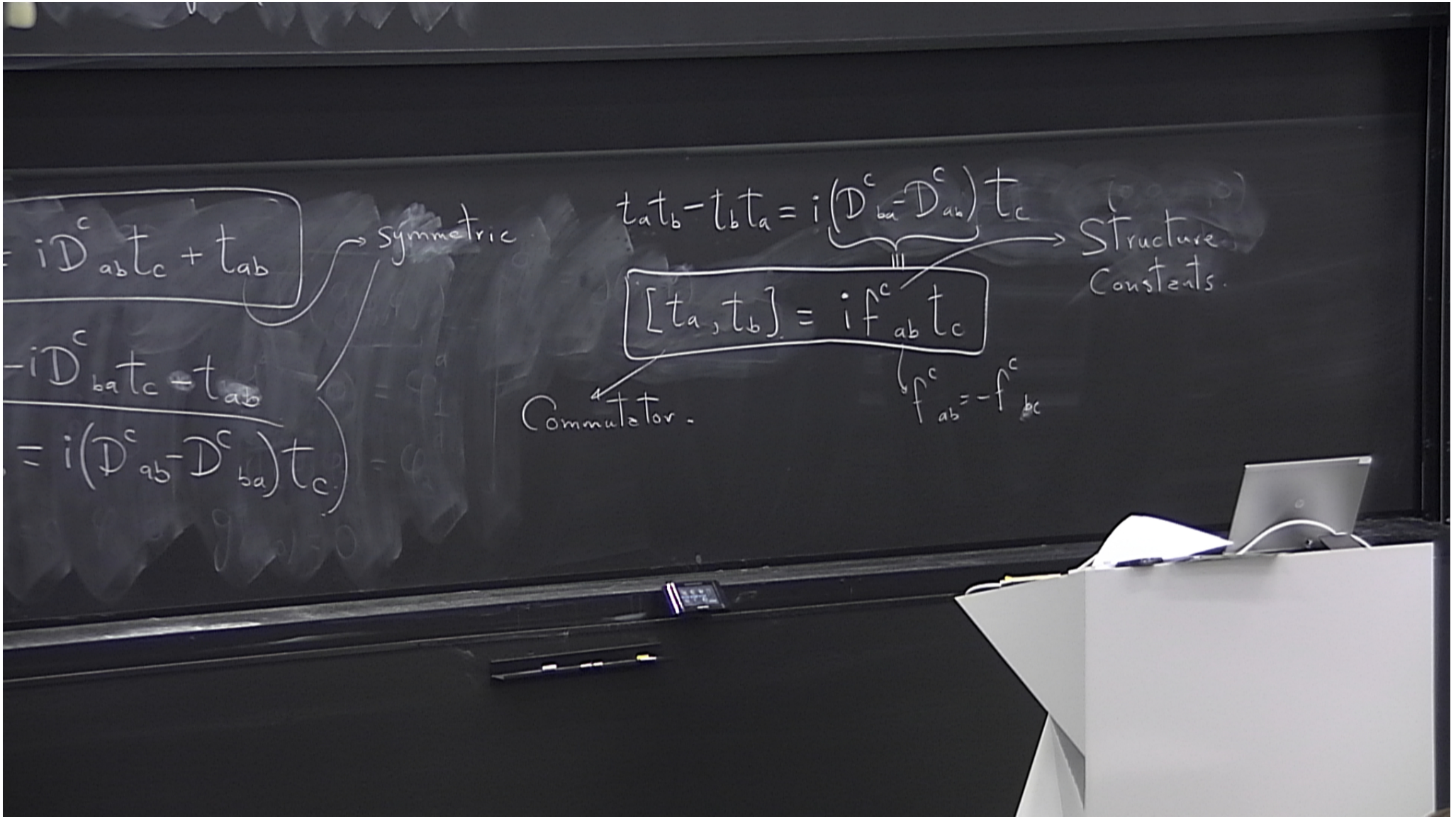
$$= P(\mathcal{R}' \to \mathcal{R}'_n)$$

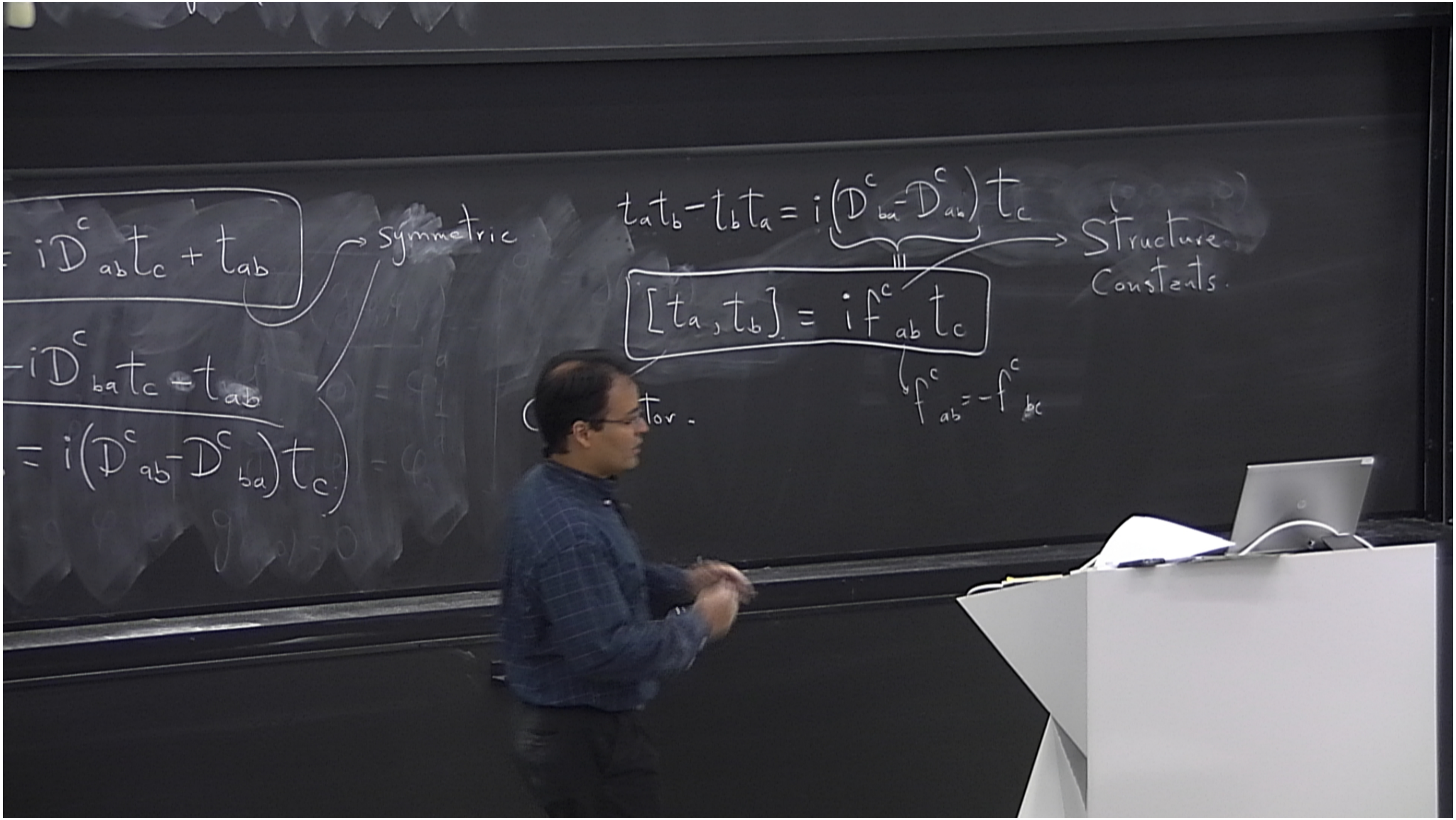
$$\varphi_2)) = \mathbb{1} + i\, g^a(\varphi_1,\varphi_2)\, t_a + \frac{1}{2}\, g^a(\varphi_1,\varphi_2)\, g^b(\varphi_1,\varphi_2)\, t_{ab}$$

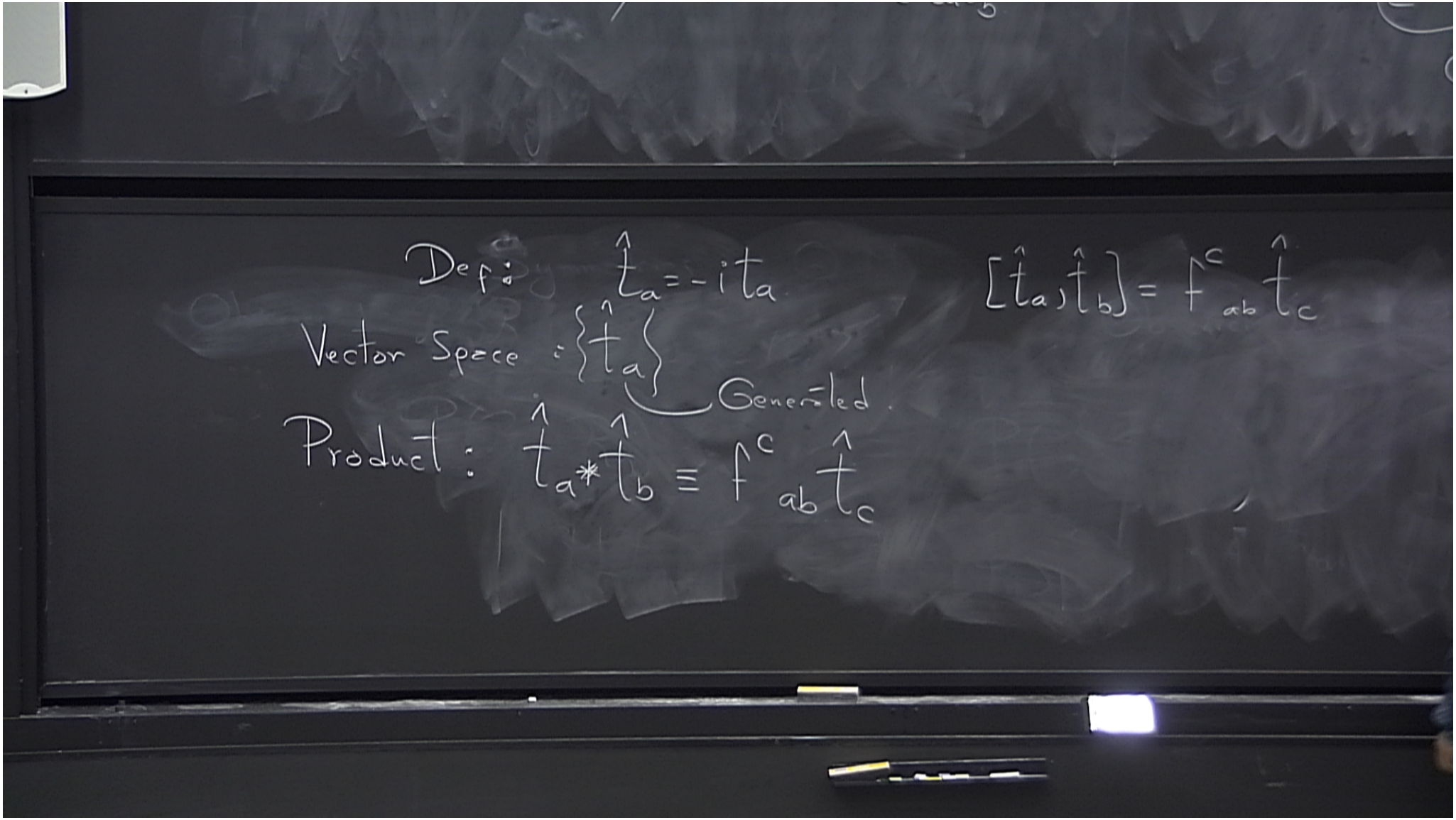
$$= \mathbb{1} + i\left(\varphi_1^a + \varphi_2^a + D^a{}_{ef}\,\varphi_1^e\varphi_2^f\right) t_a + \frac{1}{2}\left(\varphi_1^a + \varphi_2^a + D^a{}_{ef}\,\varphi_1^e\varphi_2^f\right)\left(\varphi_1^b + \varphi_2^b + D^b{}_{mn}\,\varphi_1^m\varphi_2^n\right) t_{ab}$$

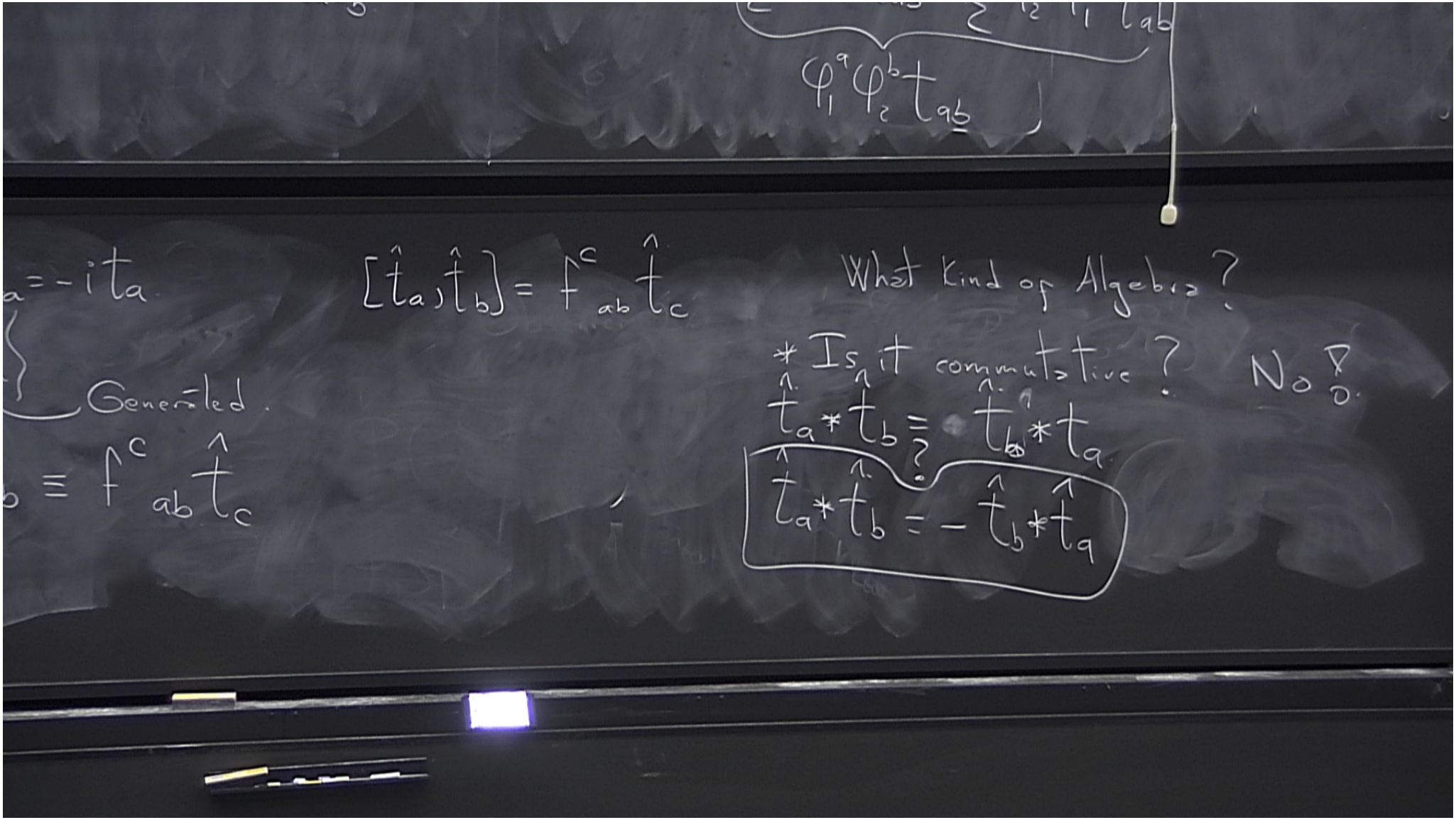
$$= i D^a{}_{ef}\,\varphi_1^e\varphi_2^f t_a + \frac{1}{2}\,\varphi_1^a\varphi_2^b t_{ab} + \frac{1}{2}\,\varphi_2^a\varphi_1^b t_{ab}$$

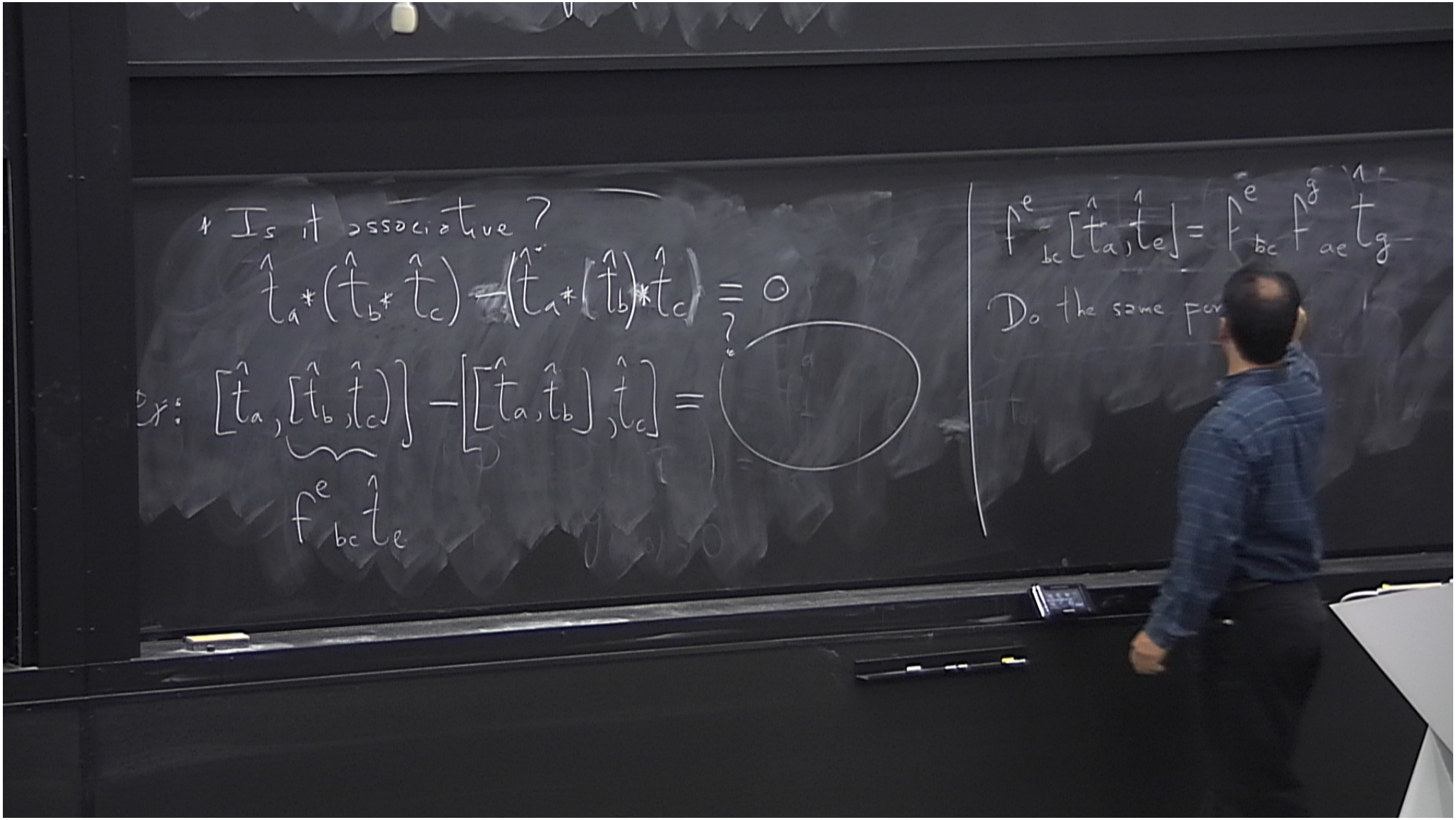
$$\varphi_1^a\varphi_2^b t_{a\underline{b}}$$

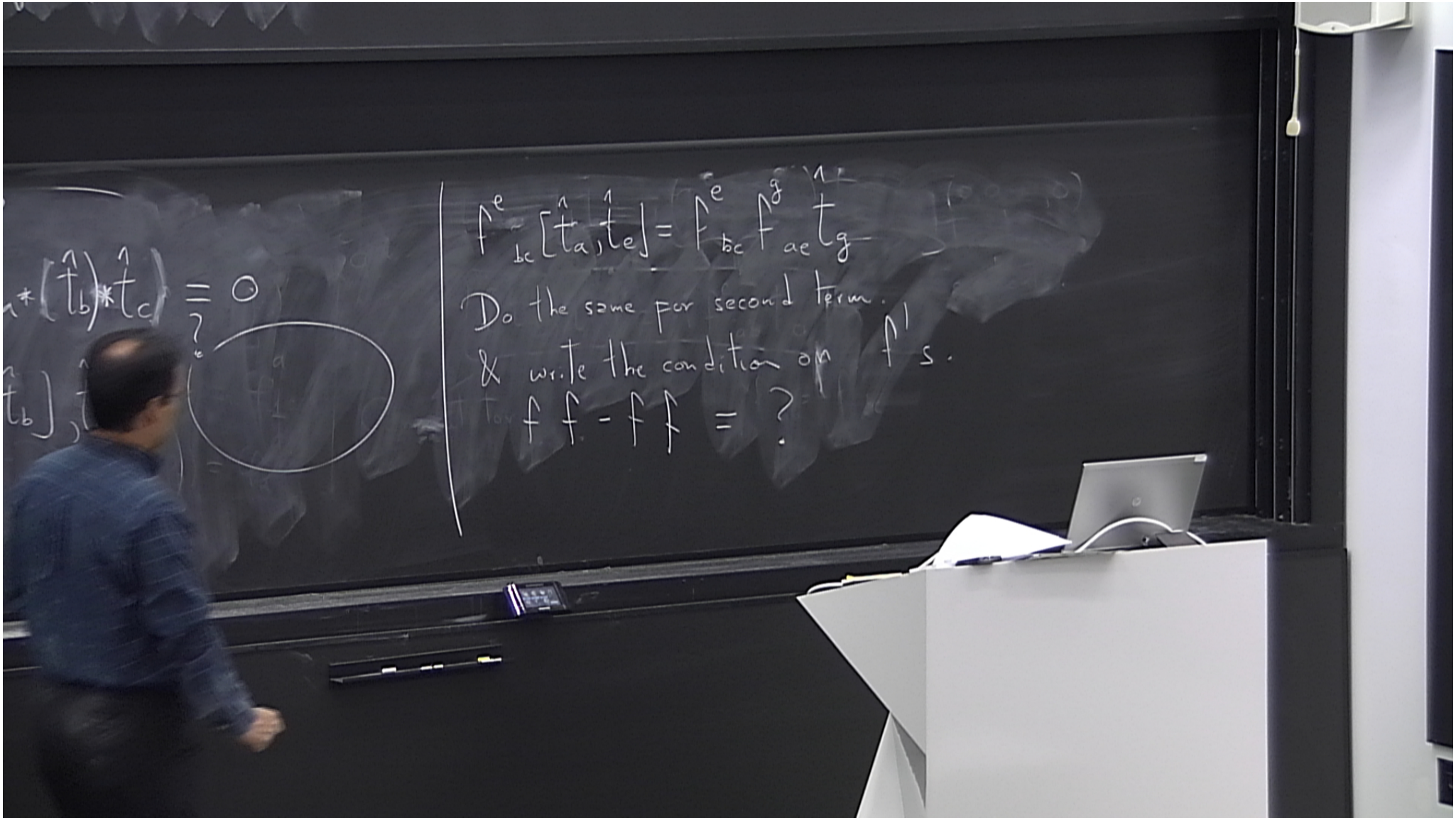

$= iD^c_{ab} t_c + t_{ab}$
$-iD^c_{ba} t_c - t_{ab}$
$= i(D^c_{ab} - D^c_{ba}) t_c$
symmetric
$t_a t_b - t_b t_a = i(D^c_{ba} - D^c_{ab}) t_c$
$[t_a, t_b] = i f^c_{ab} t_c$
Commutator.
Structure Constants.
$f^c_{ab} = -f^c_{bc}$


$= iD^c_{ab} t_c + t_{ab}$
symmetric
$-iD^c_{ba} t_c - t_{ab}$
$= i(D^c_{ab} - D^c_{ba}) t_c$
$t_a t_b - t_b t_a = i(D^c_{ba} - D^c_{ab}) t_c$
Structure Constants.
$[t_a, t_b] = i f^c_{ab} t_c$
$f^c_{ab} = -f^c_{bc}$


Def: $\hat{t}_a = -it_a$
$[\hat{t}_a, \hat{t}_b] = f^c_{ab}\hat{t}_c$
Vector Space: $\{\hat{t}_a\}$ Generated
Product: $\hat{t}_a * \hat{t}_b \equiv f^c_{ab}\hat{t}_c$


$\varphi_1^a \varphi_2^b t_{ab}$
$= -i t_a$
Generalized
$\equiv f^c_{ab} \hat{t}_c$
$[\hat{t}_a, \hat{t}_b] = f^c_{ab} \hat{t}_c$
What Kind of Algebra?
* Is it commutative? No!
$\hat{t}_a * \hat{t}_b \stackrel{?}{=} \hat{t}_b * \hat{t}_a$
$\hat{t}_a * \hat{t}_b = -\hat{t}_b * \hat{t}_a$


Is it associative?
$\hat{t}_a * (\hat{t}_b * \hat{t}_c) - (\hat{t}_a * \hat{t}_b) * \hat{t}_c = 0$?
Pf: $[\hat{t}_a, [\hat{t}_b, \hat{t}_c]] - [[\hat{t}_a, \hat{t}_b], \hat{t}_c] =$
$f^e_{bc} \hat{t}_e$
$f^e_{bc} [\hat{t}_a, \hat{t}_e] = f^e_{bc} f^g_{ae} \hat{t}_g$
Do the same for


$f^e_{bc}[\hat{t}_a, \hat{t}_e] = f^e_{bc} f^g_{ae} \hat{t}_g$
Do the same for second term.
& write the condition on f^ls.
$ff - ff = ?$