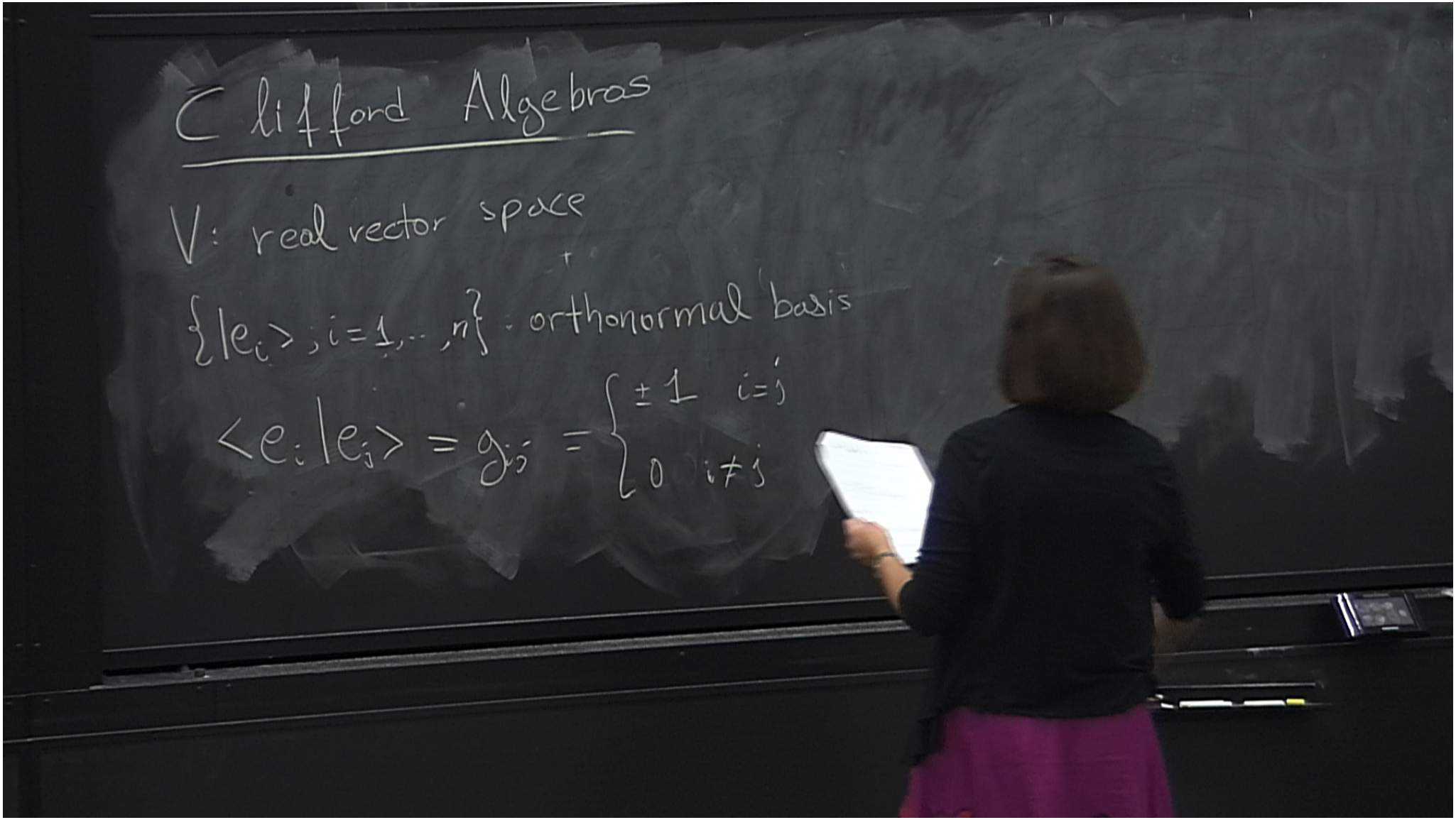
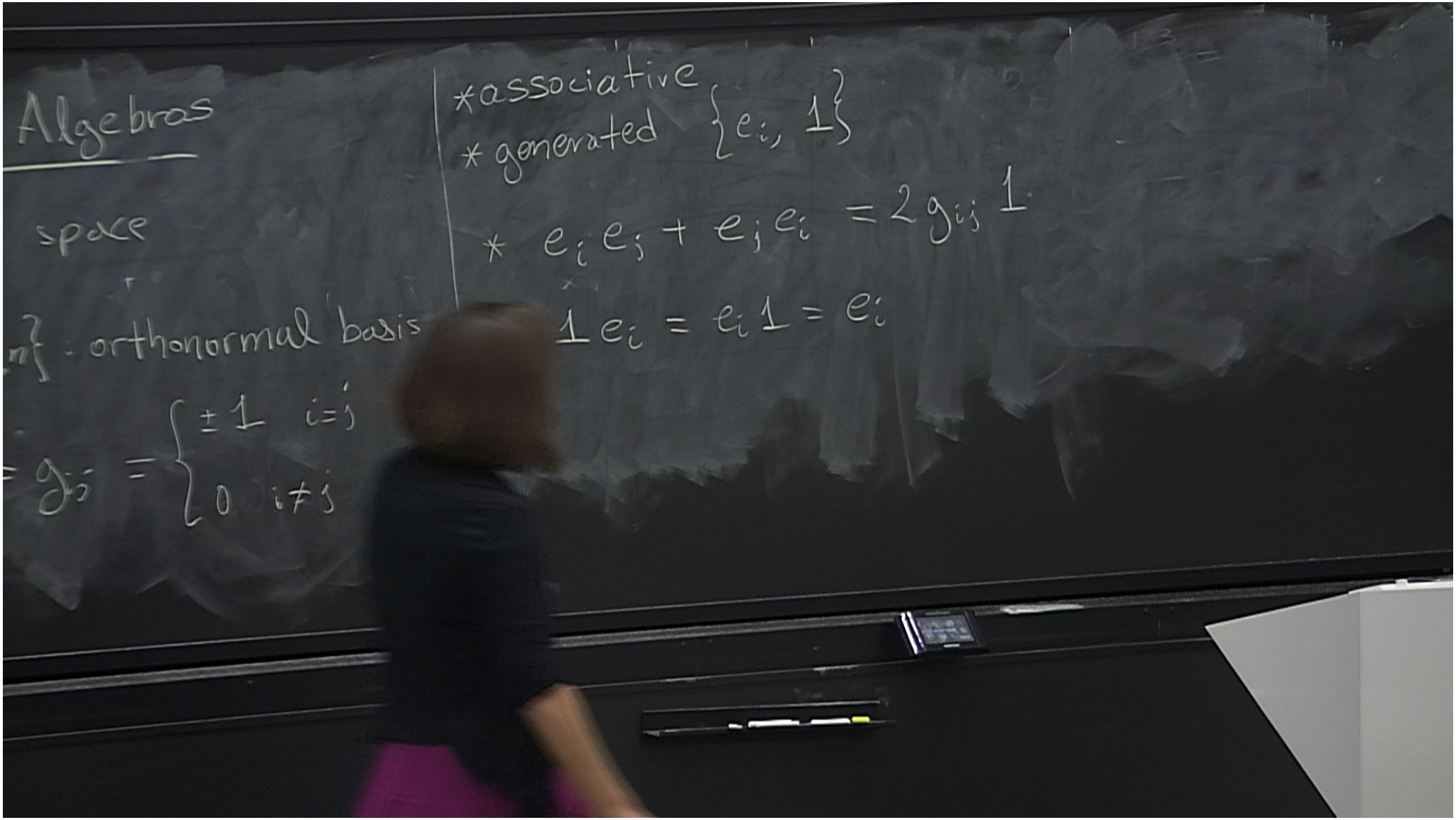
Title: Algebra - Lecture 2b

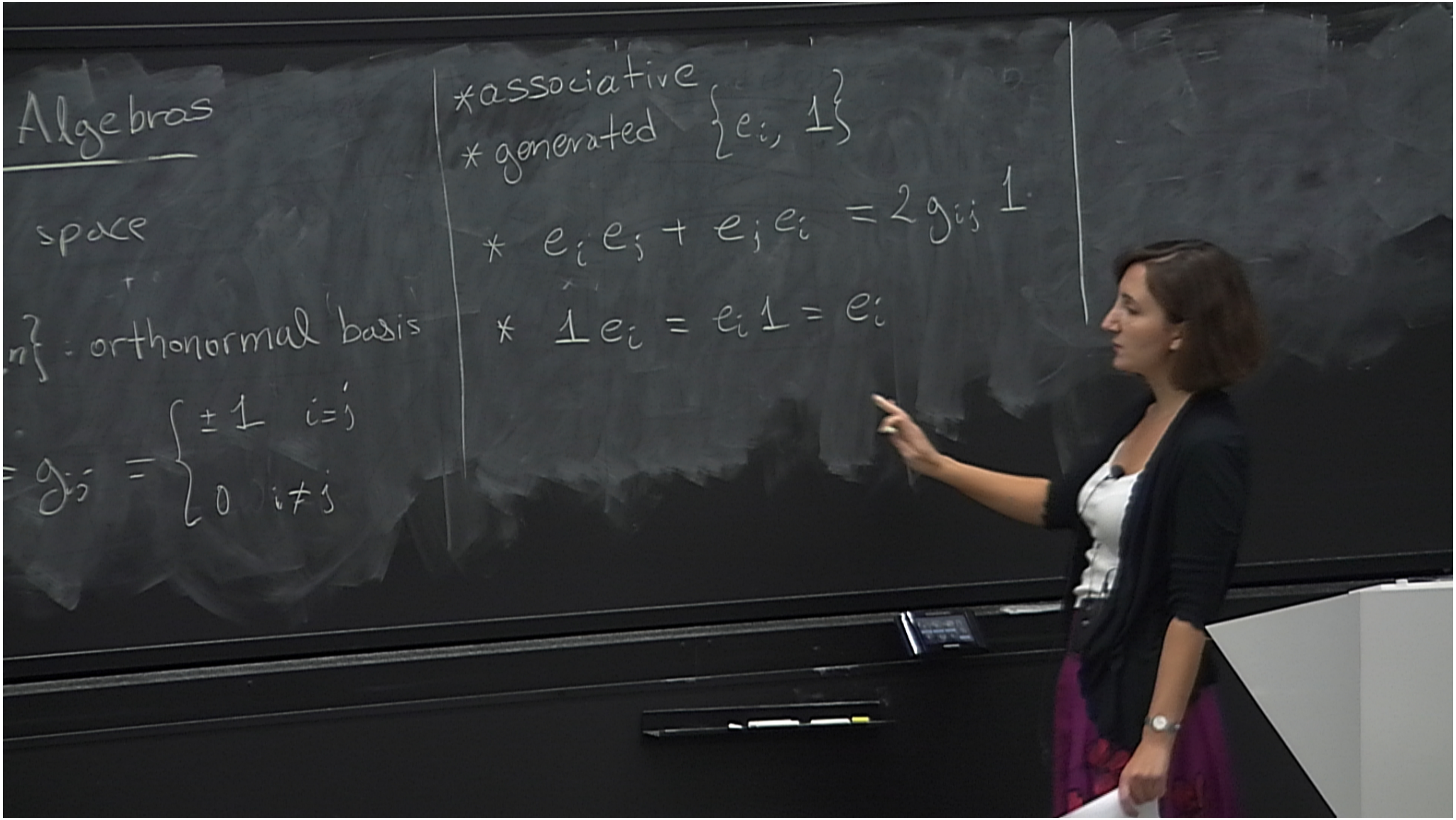
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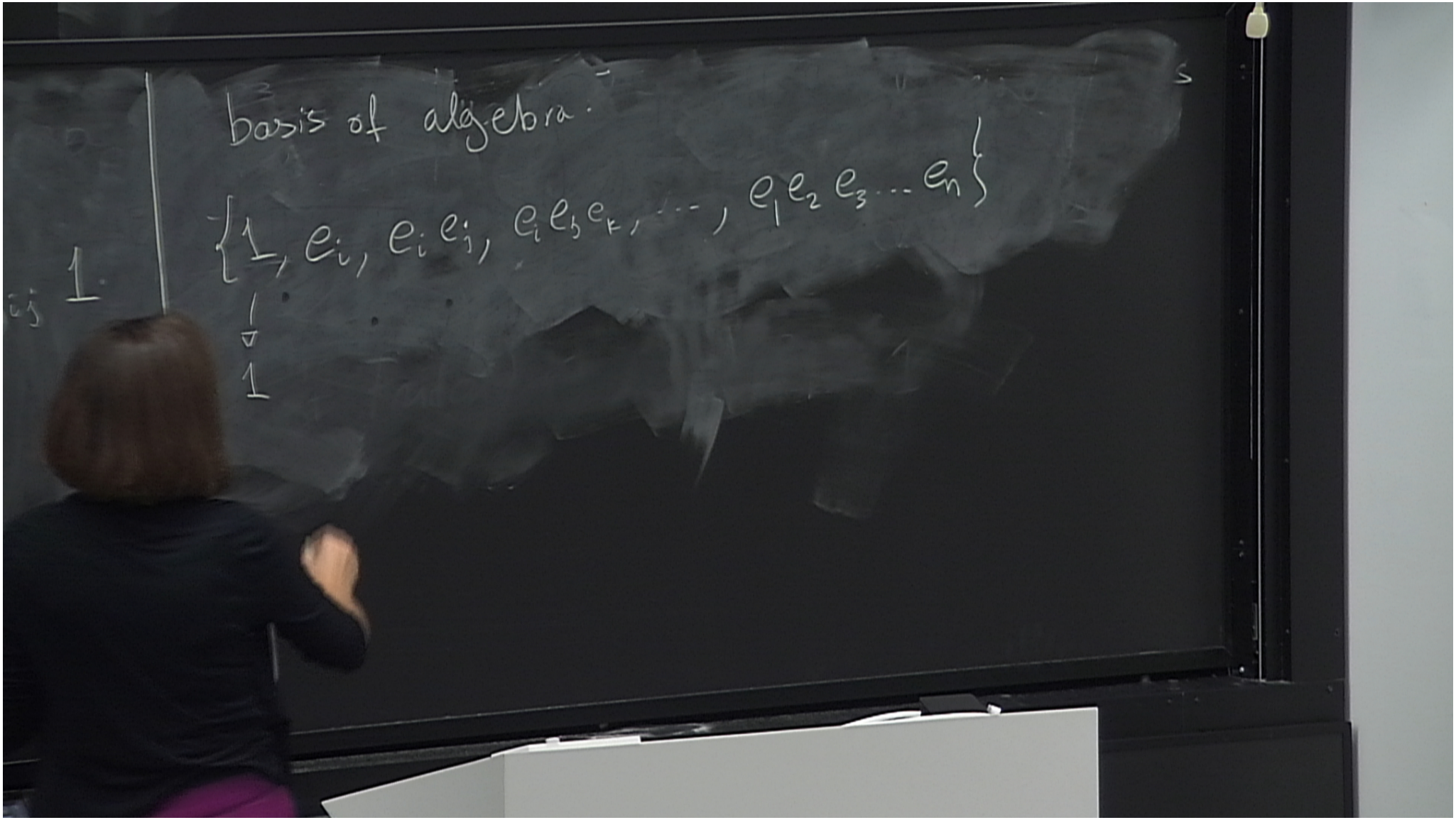
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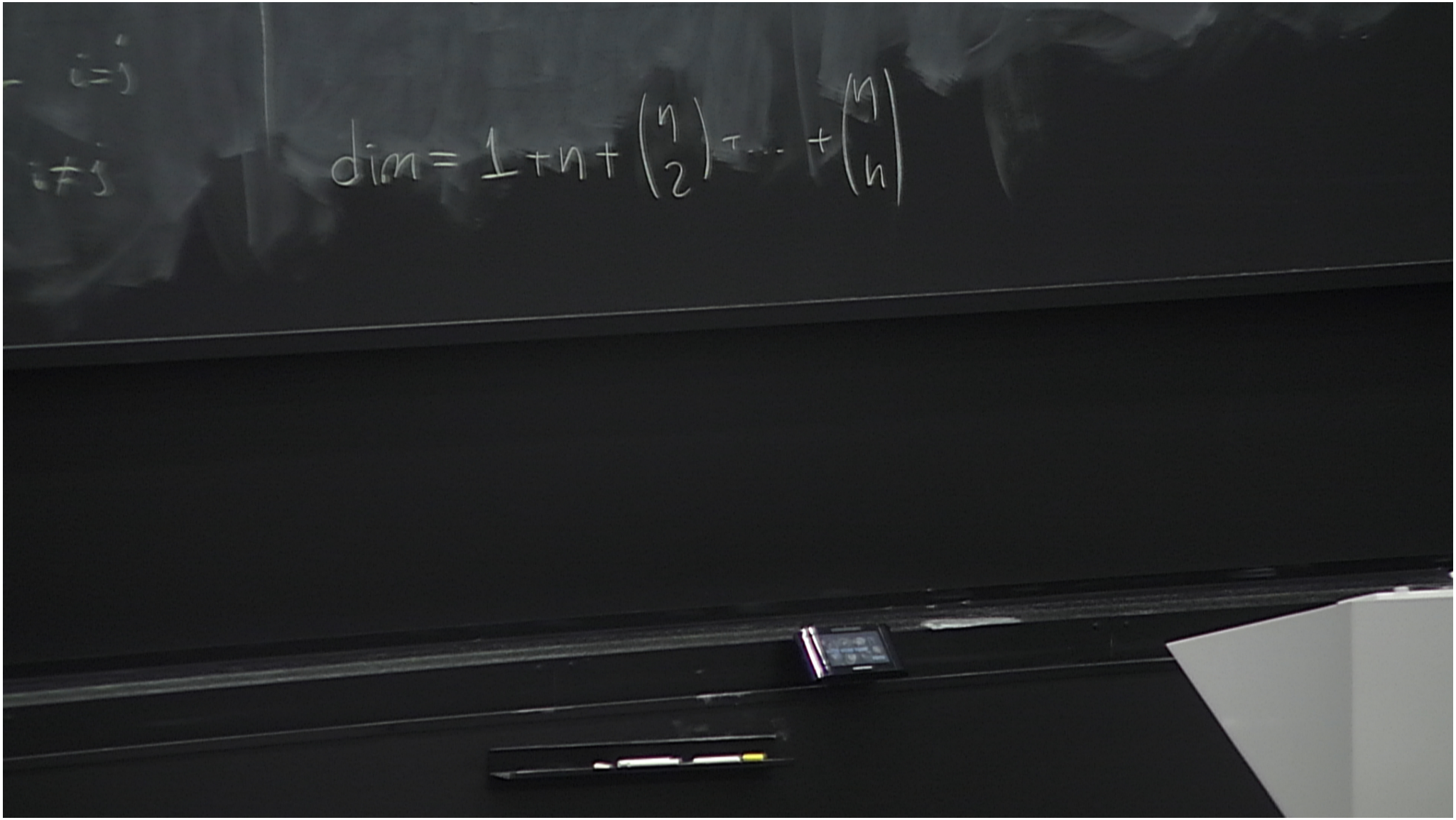
Abstract:

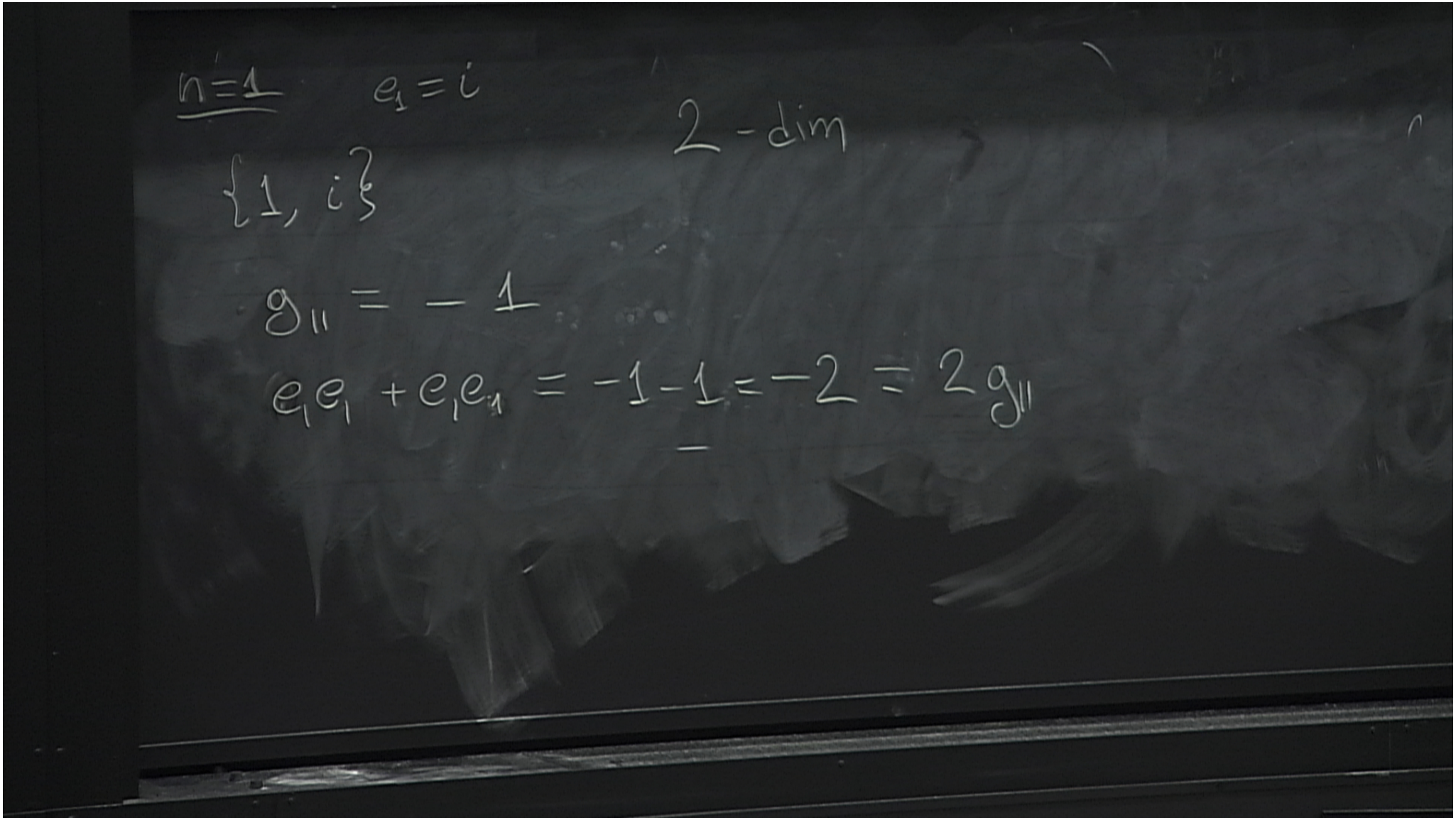

Clifford Algebras
V: real vector space
$\{|e_i\rangle, i=1,\dots,n\}$: orthonormal basis
$\langle e_i|e_j\rangle = g_{ij} = \begin{cases} \pm 1 & i=j \\ 0 & i\neq j \end{cases}$

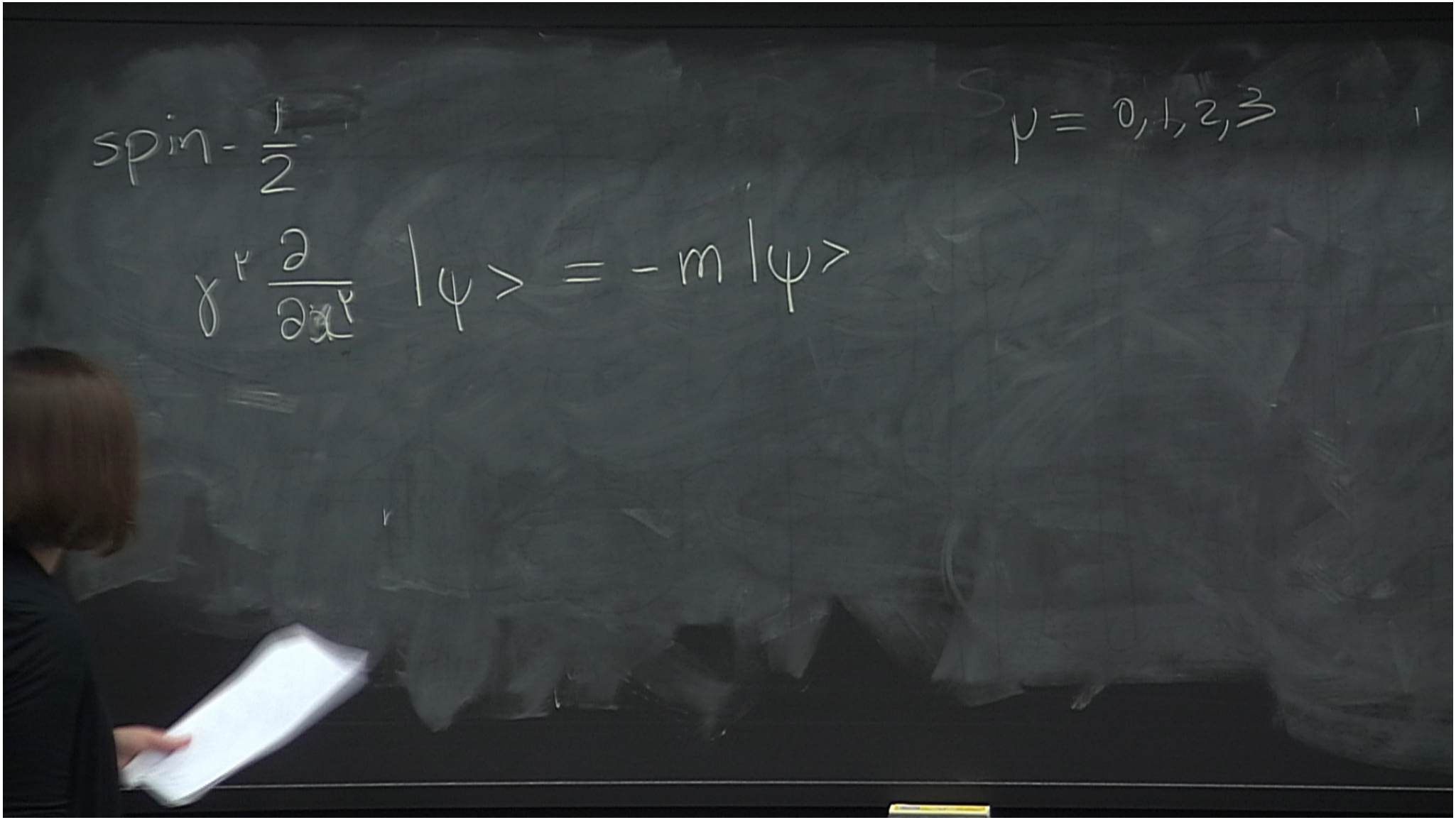

Algebras
space
orthonormal basis
$g_{ij} = \begin{cases} \pm 1 & i=j \\ 0 & i \neq j \end{cases}$
*associative
*generated $\{e_i, 1\}$
* $e_i e_j + e_j e_i = 2 g_{ij} 1$
$1 e_i = e_i 1 = e_i$

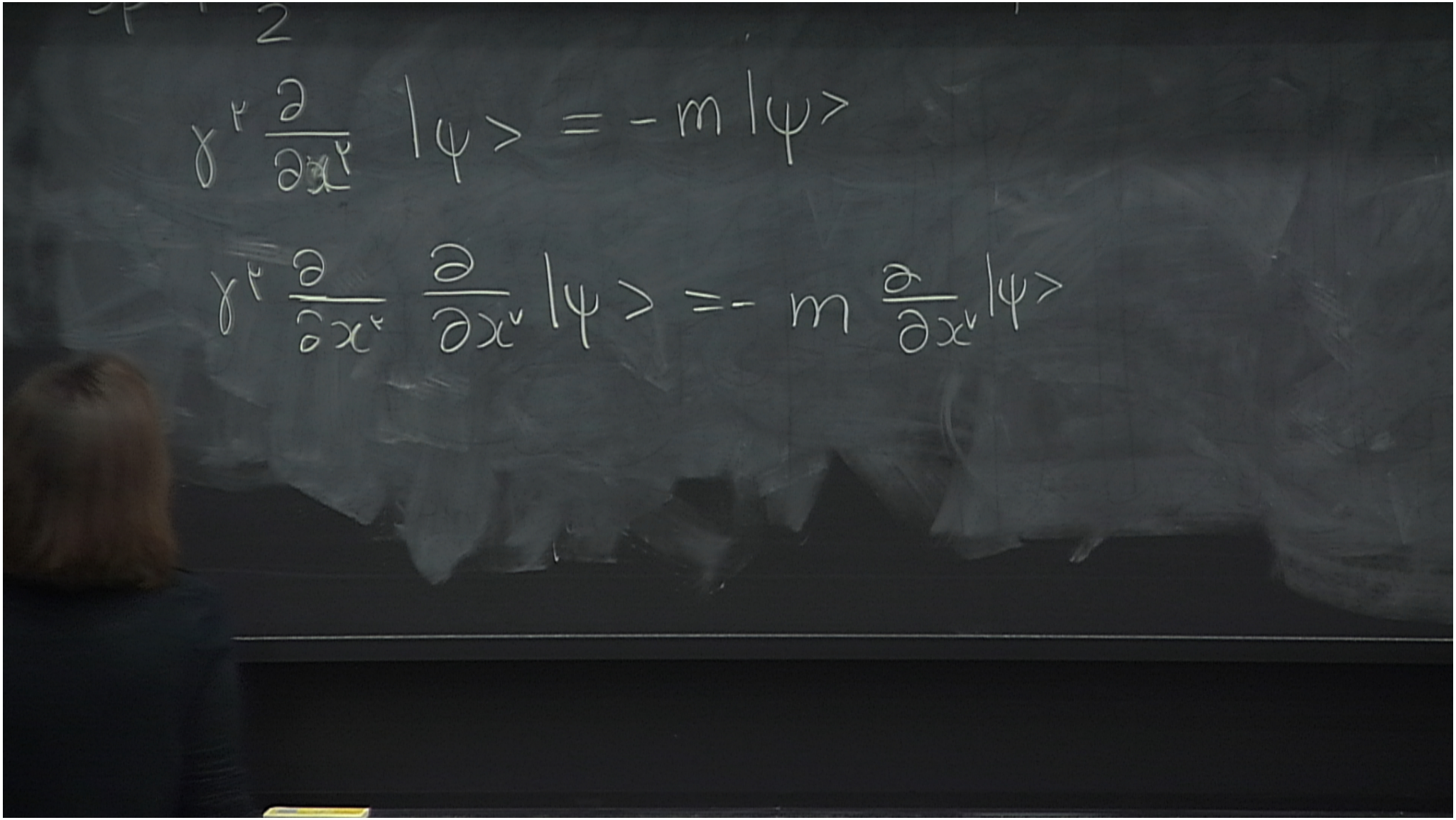

Algebras
space
n} : orthonormal basis
= g_ij = { ±1 i=j ; 0 i≠j
*associative
*generated {e_i, 1}
* e_i e_j + e_j e_i = 2 g_ij 1
* 1 e_i = e_i 1 = e_i

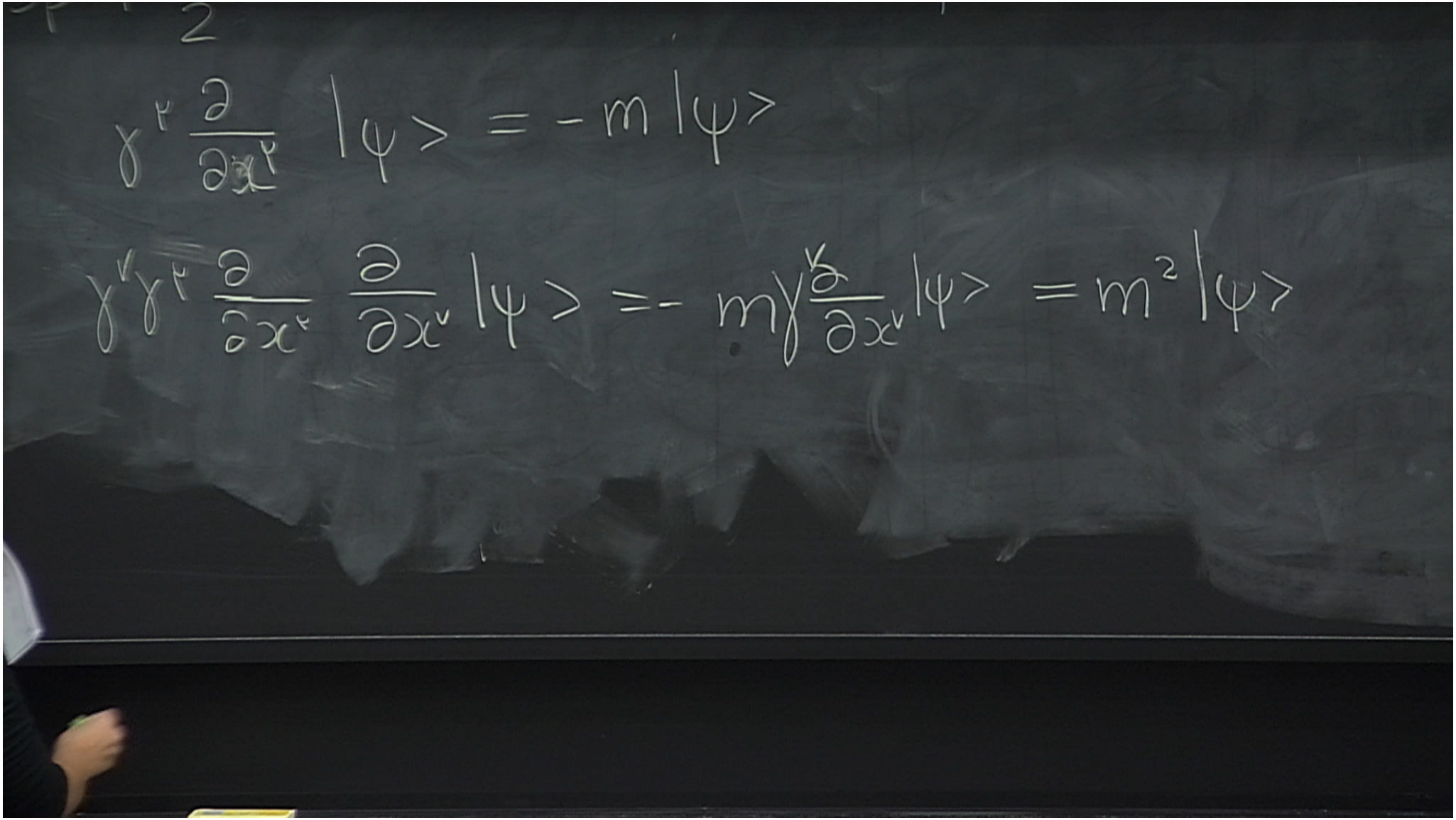

basis of algebra.
$\{1, e_i, e_i e_j, e_i e_j e_k, \cdots, e_1 e_2 e_3 \cdots e_n\}$
1

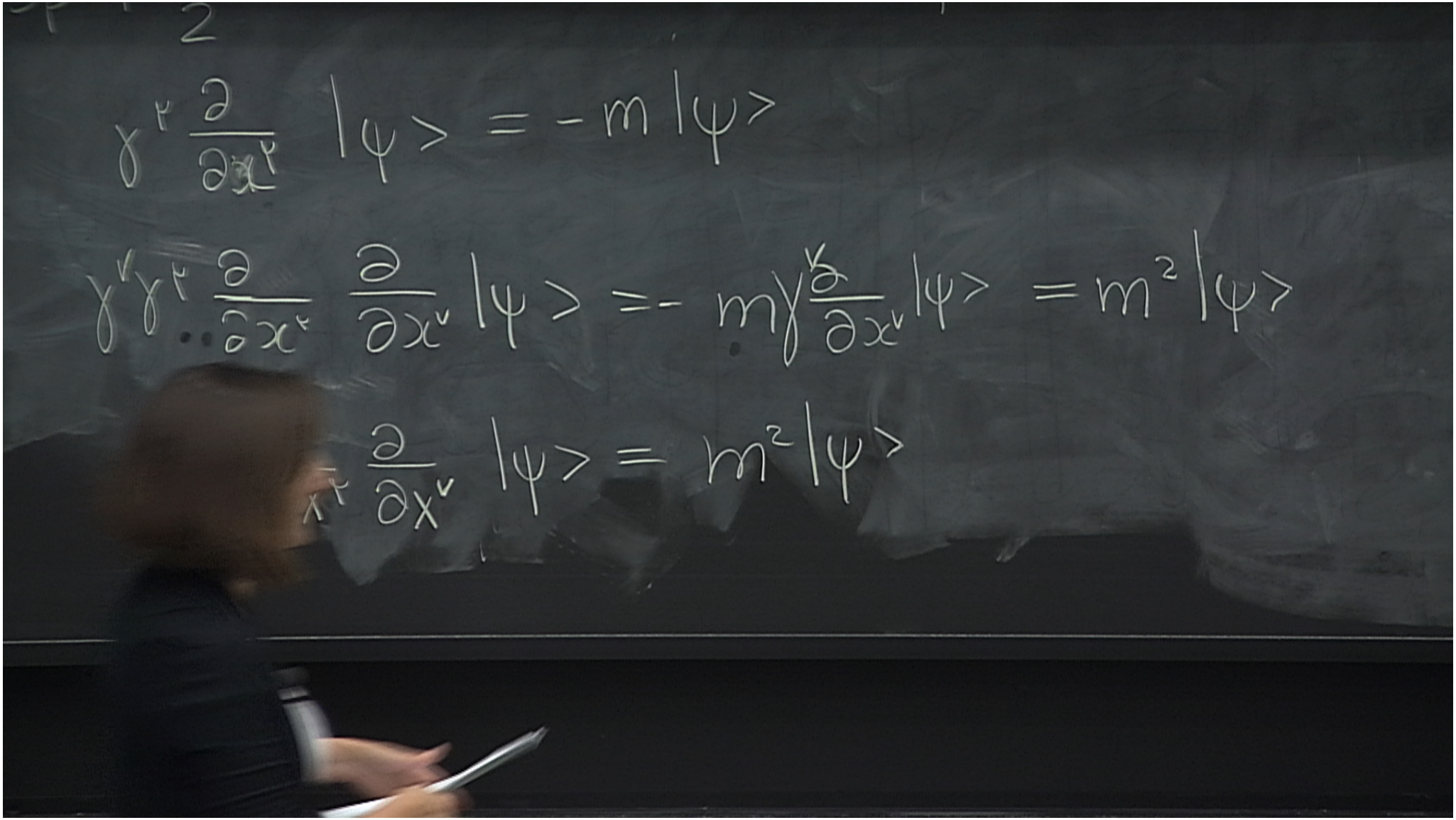

i=j
i≠j
dim = 1 + n + $\binom{n}{2}$ + ... + $\binom{n}{n}$


n=1
$e_1 = i$
2-dim
$\{1, i\}$
$g_{11} = -1$
$e_1 e_1 + e_1 e_1 = -1 - 1 = -2 = 2 g_{11}$


spin-$\frac{1}{2}$
$\mu = 0, 1, 2, 3$
$\gamma^\mu \frac{\partial}{\partial x^\mu} |\psi\rangle = -m|\psi\rangle$


2
$\gamma^\nu \frac{\partial}{\partial x^\nu} |\psi\rangle = -m|\psi\rangle$
$\gamma^\nu \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\nu} |\psi\rangle = -m \frac{\partial}{\partial x^\nu} |\psi\rangle$

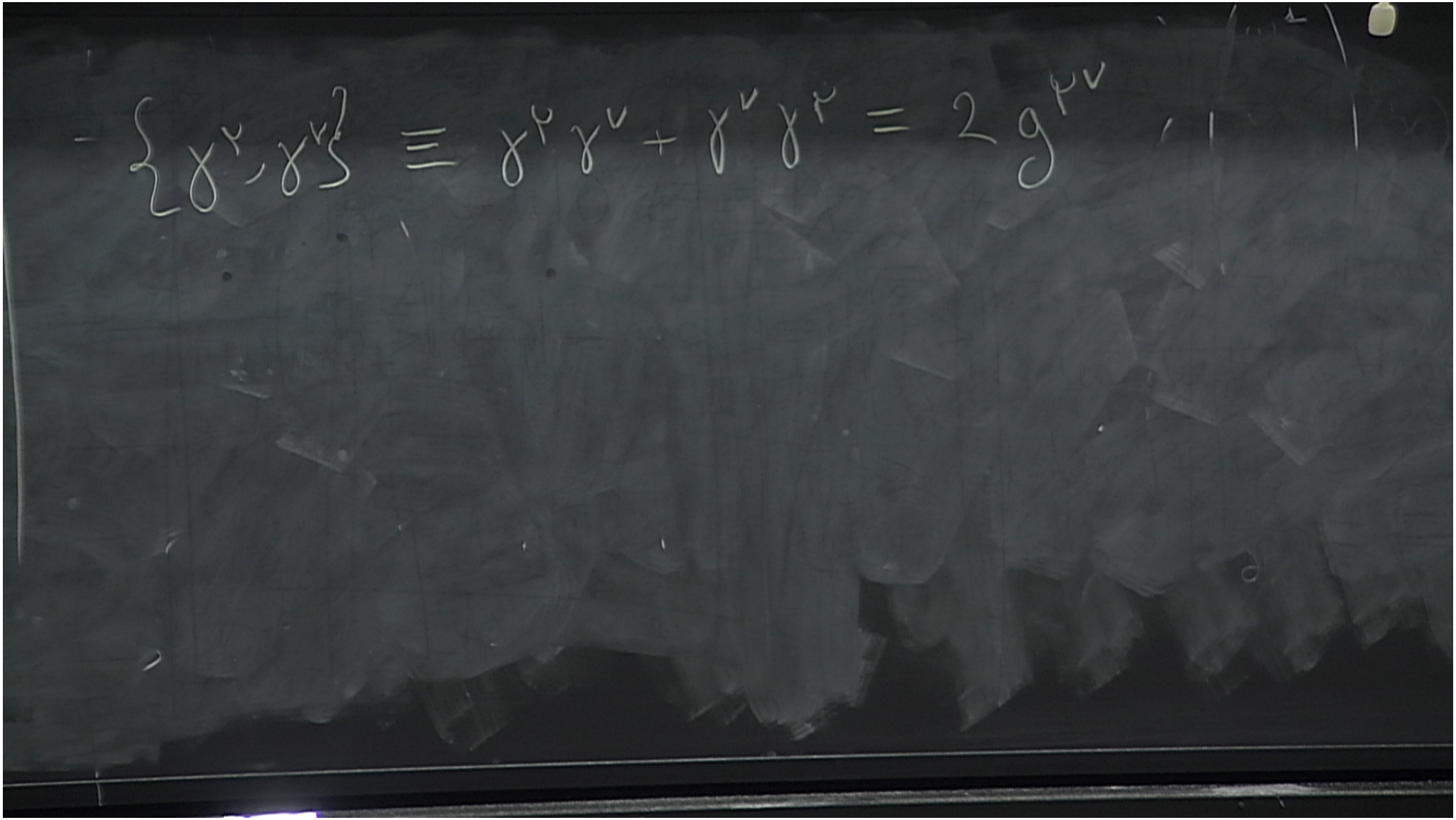

2
$\gamma^\mu \frac{\partial}{\partial x^\mu} |\psi\rangle = -m|\psi\rangle$
$\gamma^\nu \gamma^\mu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} |\psi\rangle = -m\gamma^\nu \frac{\partial}{\partial x^\nu} |\psi\rangle = m^2 |\psi\rangle$


2
$\gamma^\nu \frac{\partial}{\partial x^\nu} |\psi\rangle = -m|\psi\rangle$
$\gamma^\nu\gamma^\nu \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\nu} |\psi\rangle = -m\gamma^\nu \frac{\partial}{\partial x^\nu}|\psi\rangle = m^2|\psi\rangle$
$\frac{\partial}{\partial x^\nu} |\psi\rangle = m^2|\psi\rangle$

$$\gamma^\mu \frac{\partial}{\partial x^\mu} |\psi\rangle = -m|\psi\rangle$$

$$\gamma^\nu \gamma^\mu \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} |\psi\rangle = -m\gamma^\nu \frac{\partial}{\partial x^\nu} |\psi\rangle = m^2 |\psi\rangle$$

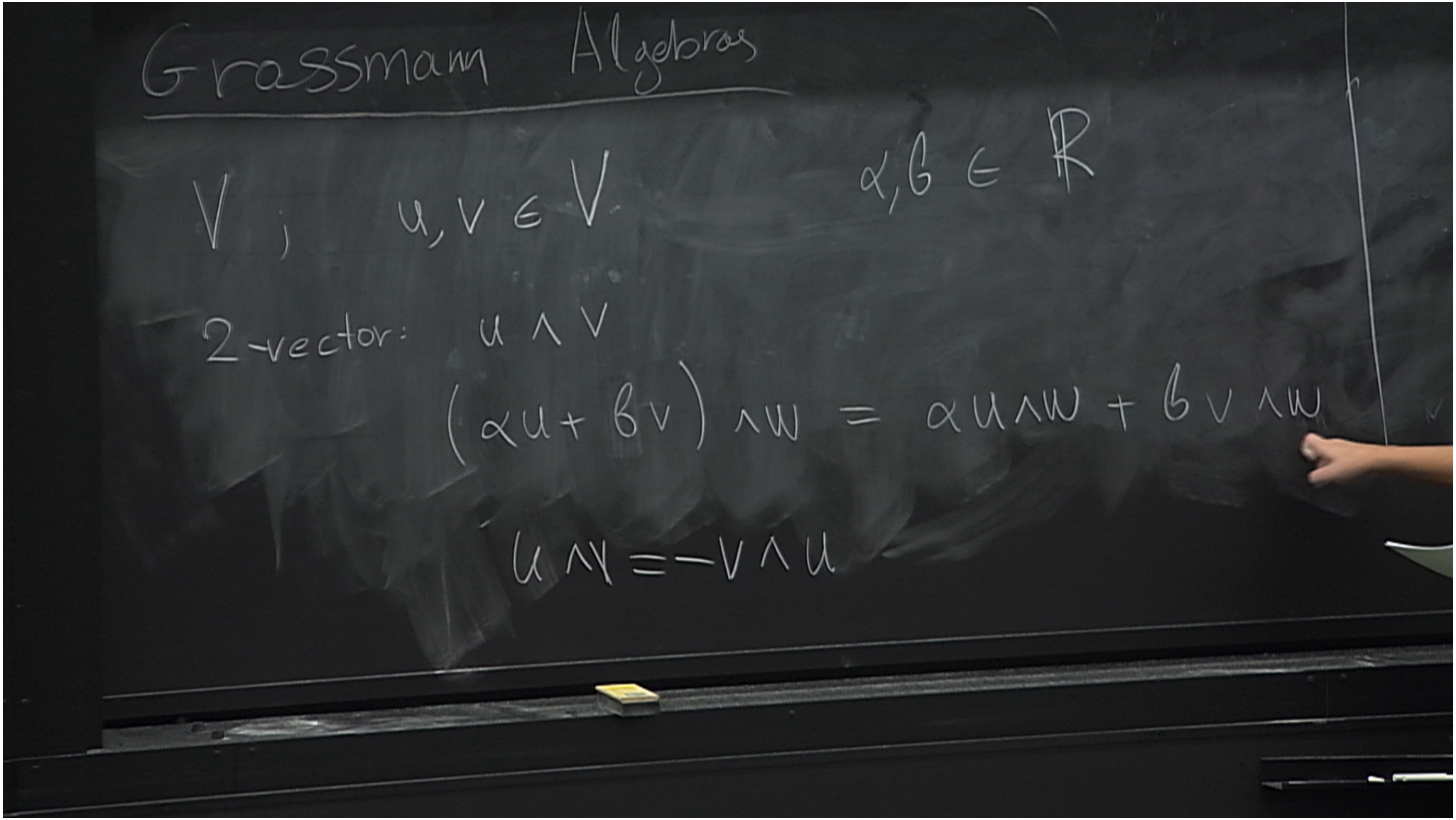
$$g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} |\psi\rangle = m^2 |\psi\rangle$$


$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$2^4 = 16$ - dimensional algebra

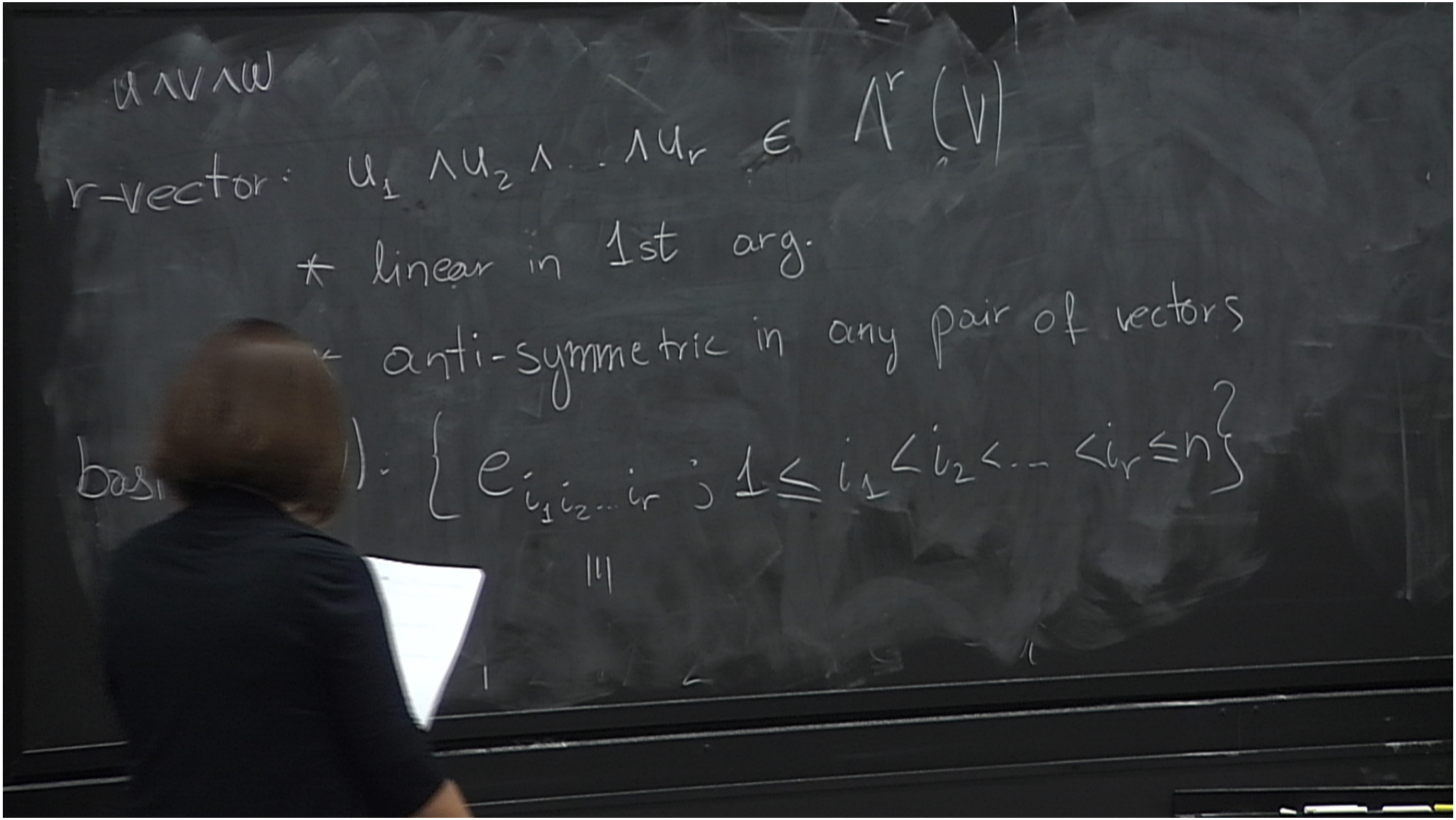
$|\psi\rangle$

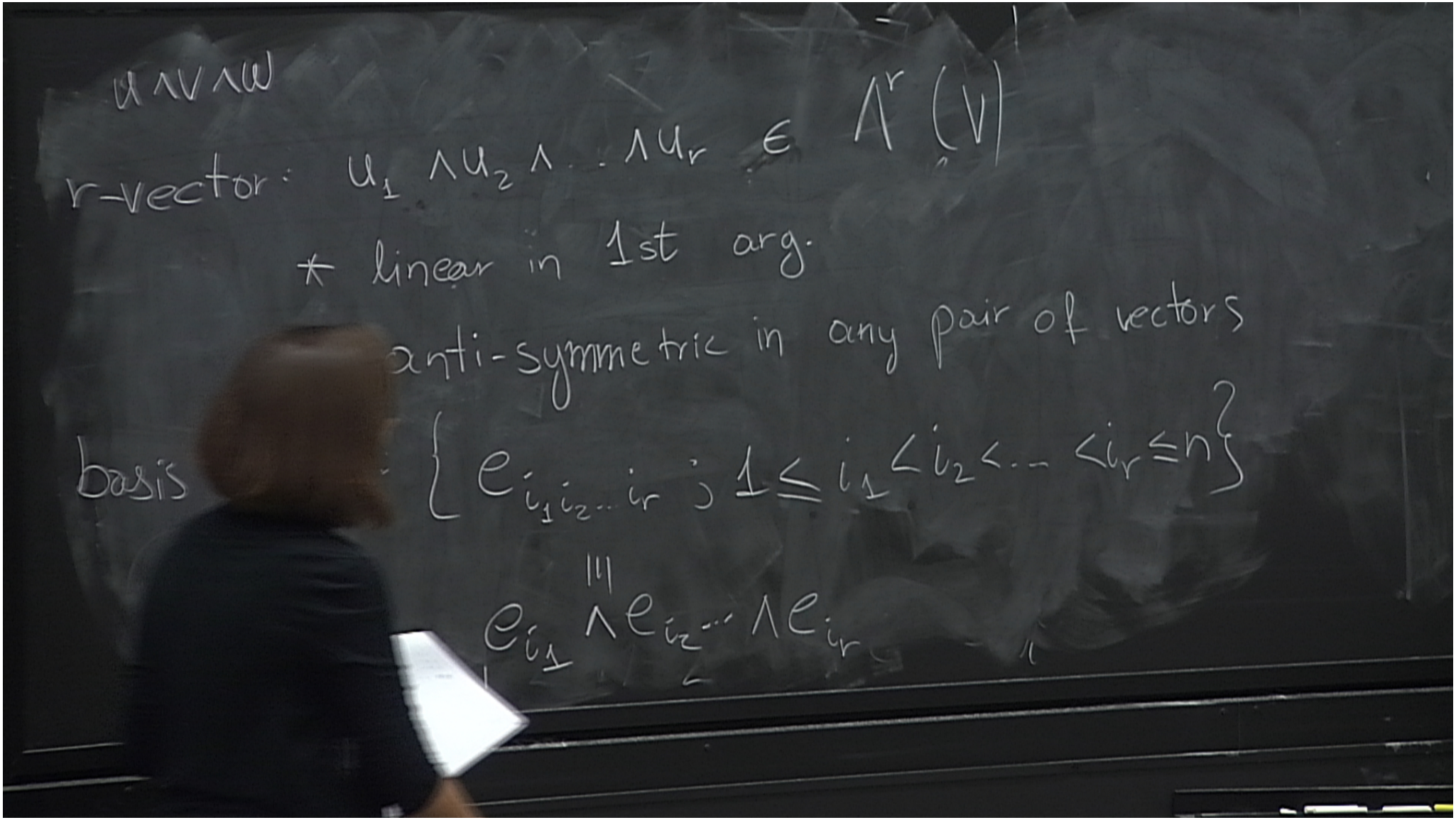

Grassmann Algebras
V ; u, v ∈ V
α, β ∈ ℝ
2-vector: u ∧ v
(αu + βv) ∧ w = α u∧w + β v ∧ w
u ∧ v = − v ∧ u

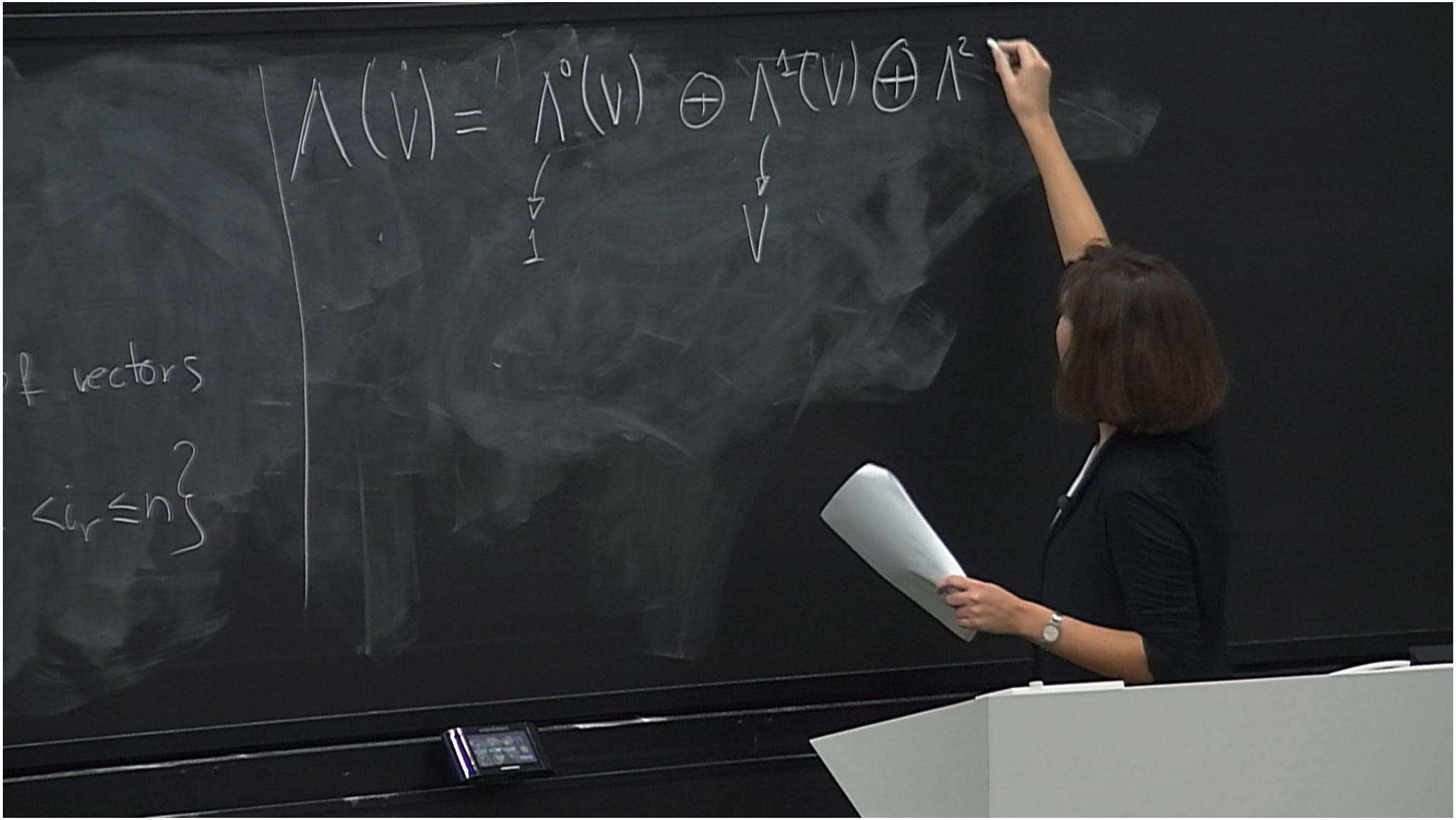
$u \wedge v \wedge w$

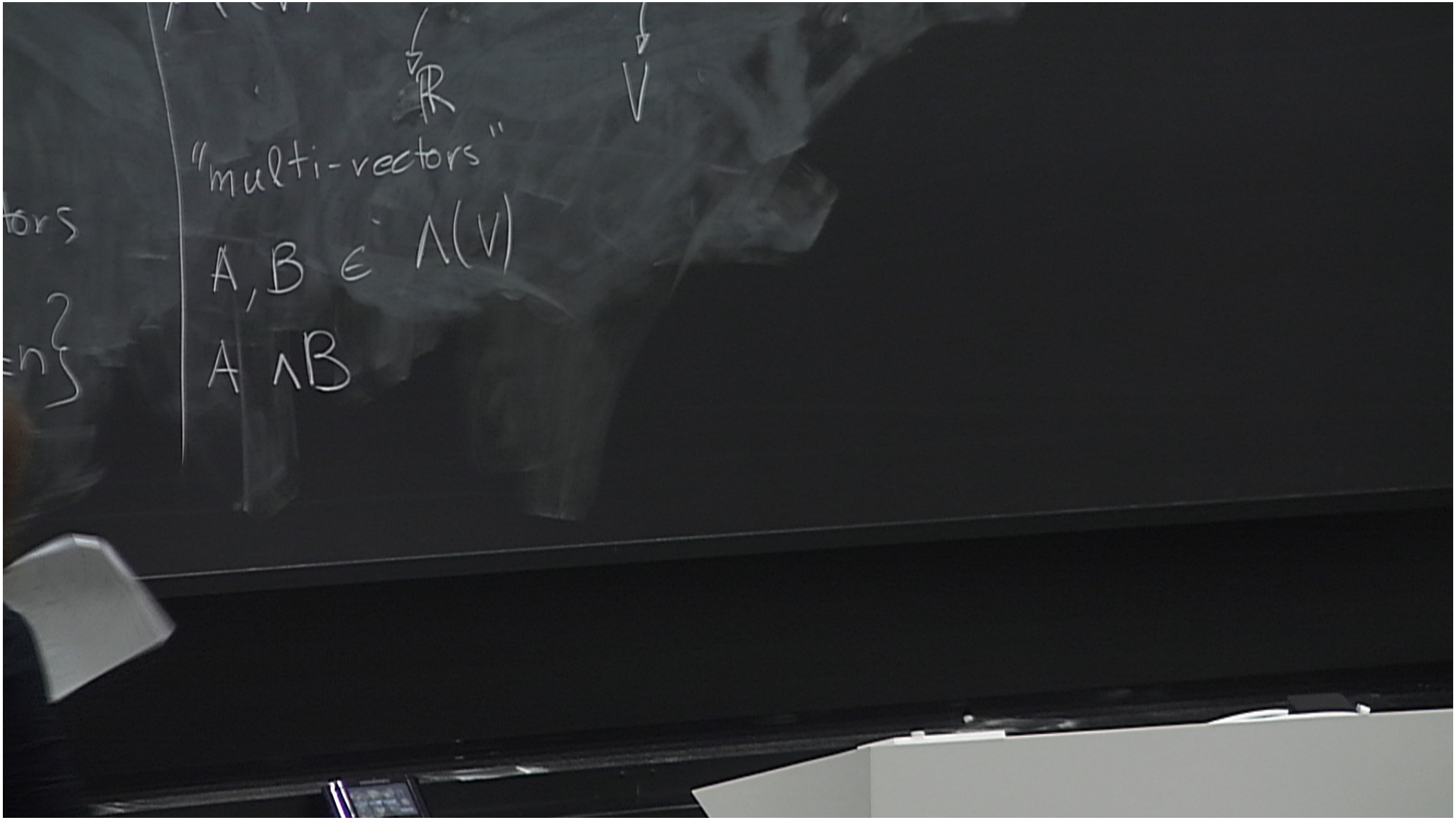
r-vector: $u_1 \wedge u_2 \wedge \ldots \wedge u_r \in \Lambda^r(V)$

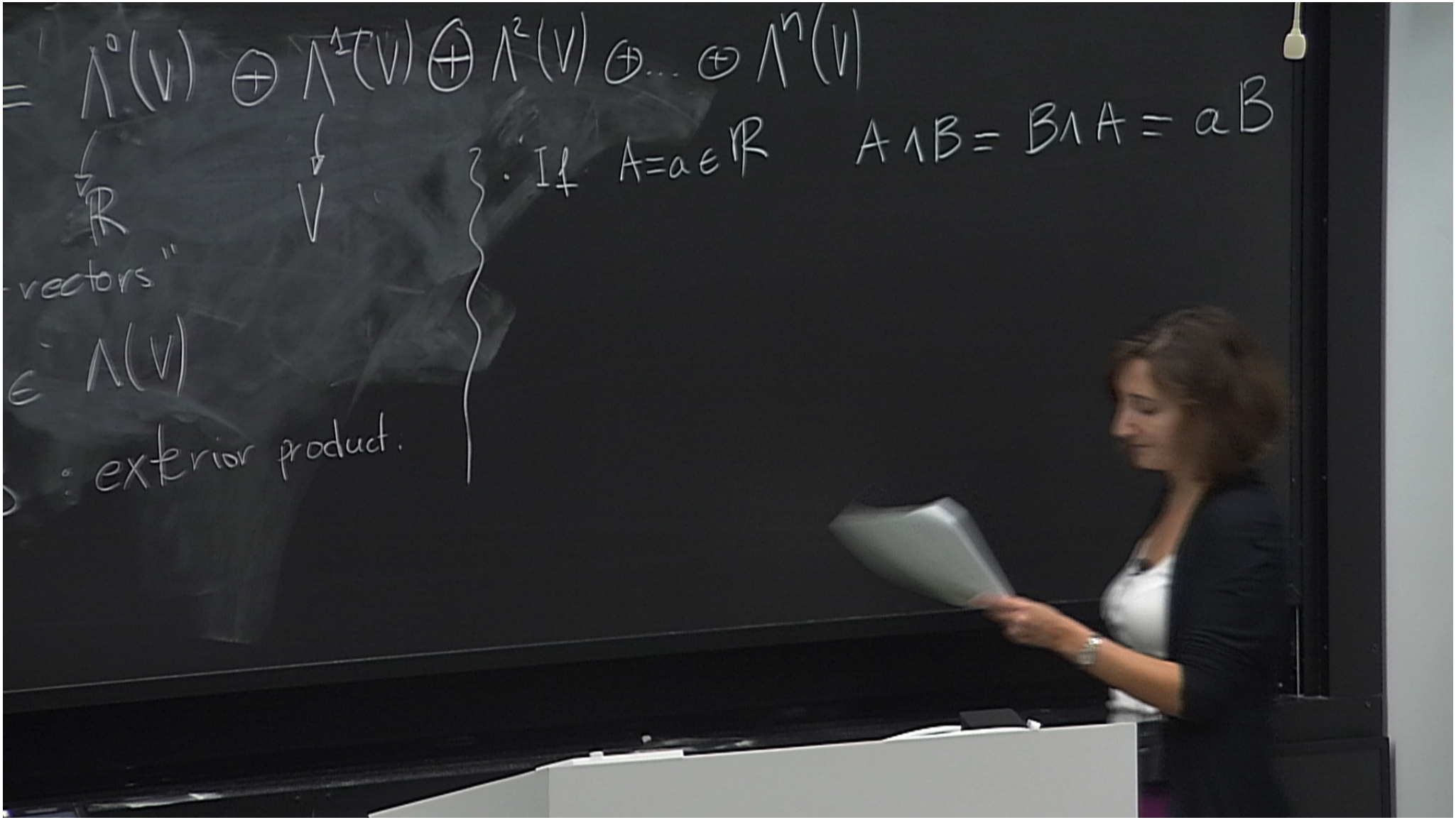
* linear in 1st arg.
* anti-symmetric in any pair of vectors

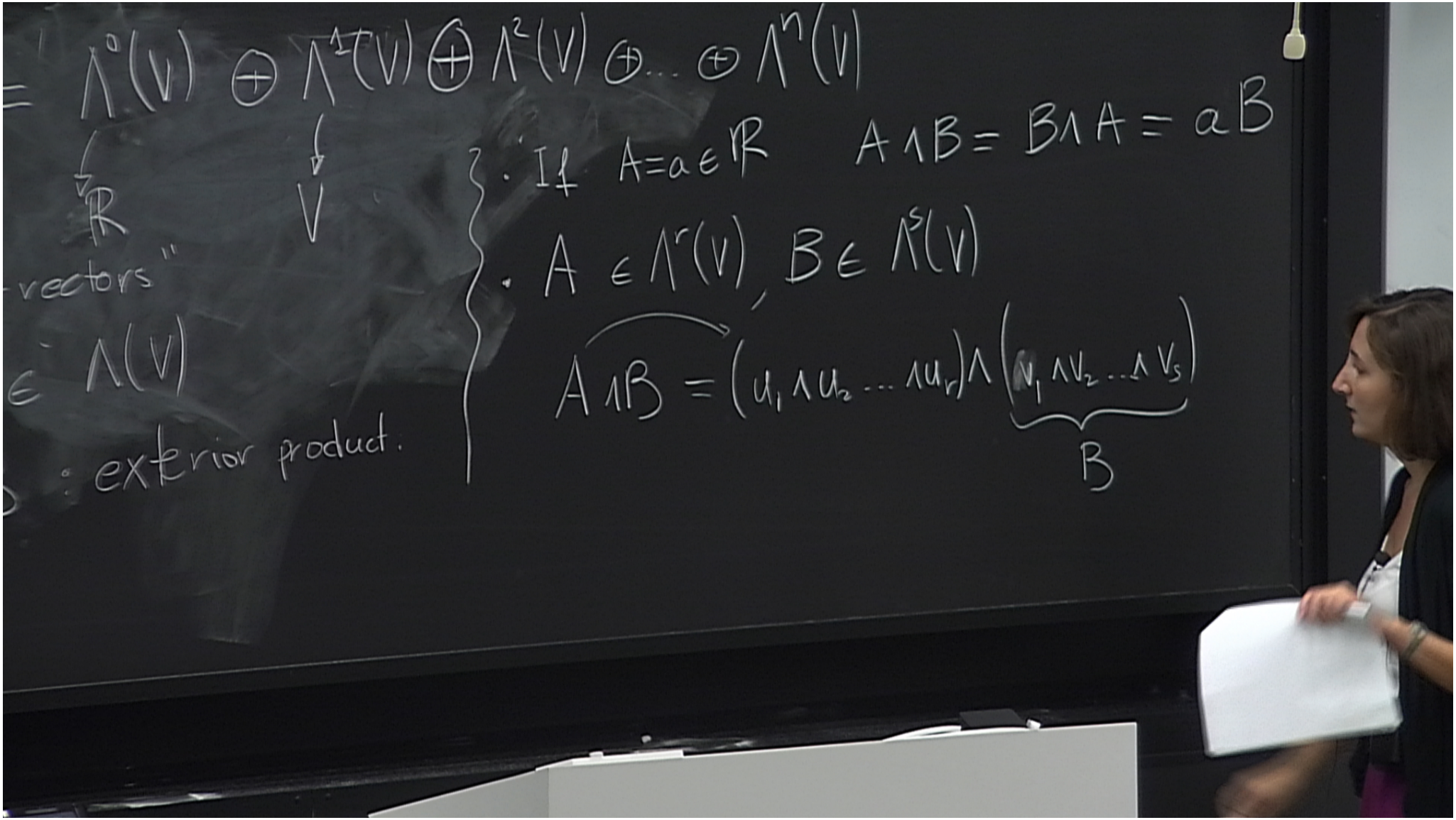

u∧v∧w
r-vector: $u_1 \wedge u_2 \wedge \dots \wedge u_r \in \Lambda^r(V)$
* linear in 1st arg.
anti-symmetric in any pair of vectors
basis: $\{e_{i_1 i_2 \dots i_r} ; 1 \le i_1 < i_2 < \dots < i_r \le n\}$

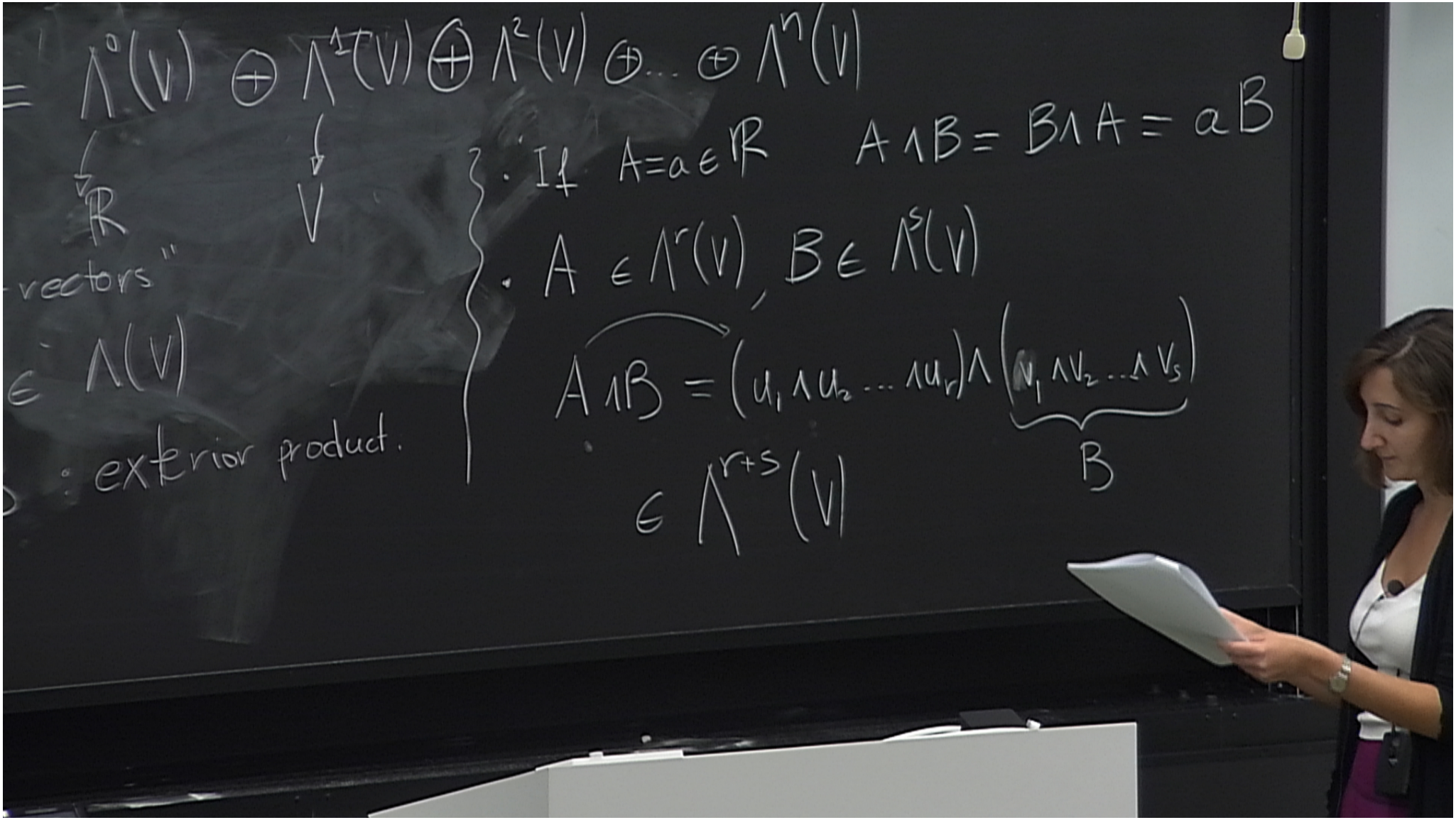

u ∧ v ∧ w
r-vector: u1 ∧ u2 ∧ ... ∧ ur ∈ Λ^r(V)
* linear in 1st arg.
anti-symmetric in any pair of vectors
basis
{ e_{i1 i2 ... ir} ; 1 ≤ i1 < i2 < ... < ir ≤ n }
|||
e_{i1} ∧ e_{i2} ... ∧ e_{ir}

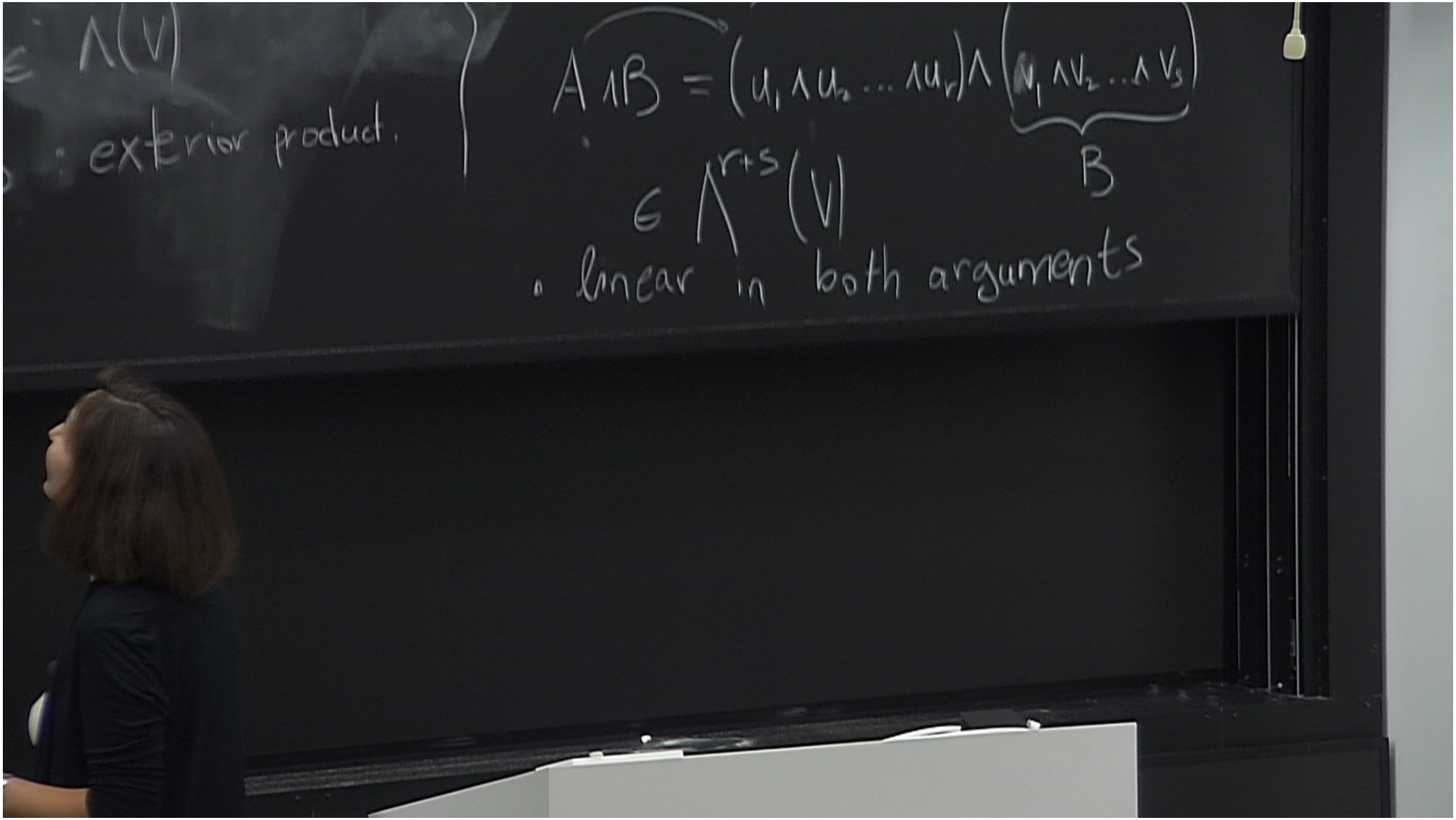

$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2$
1
V
of vectors
$\{i_r \leq n\}$?

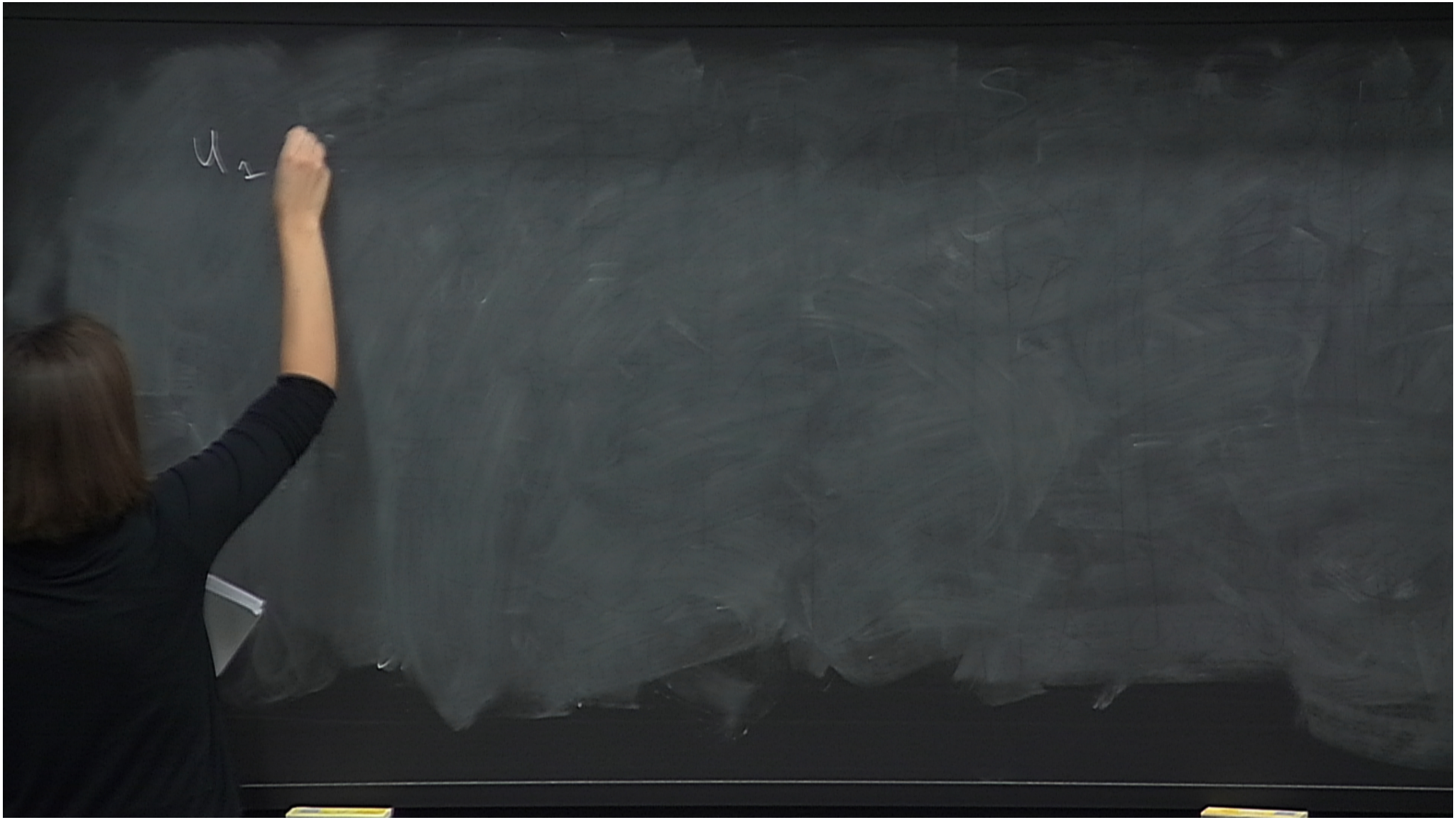

R
V
"multi-vectors"
A, B ∈ Λ(V)
A ∧ B

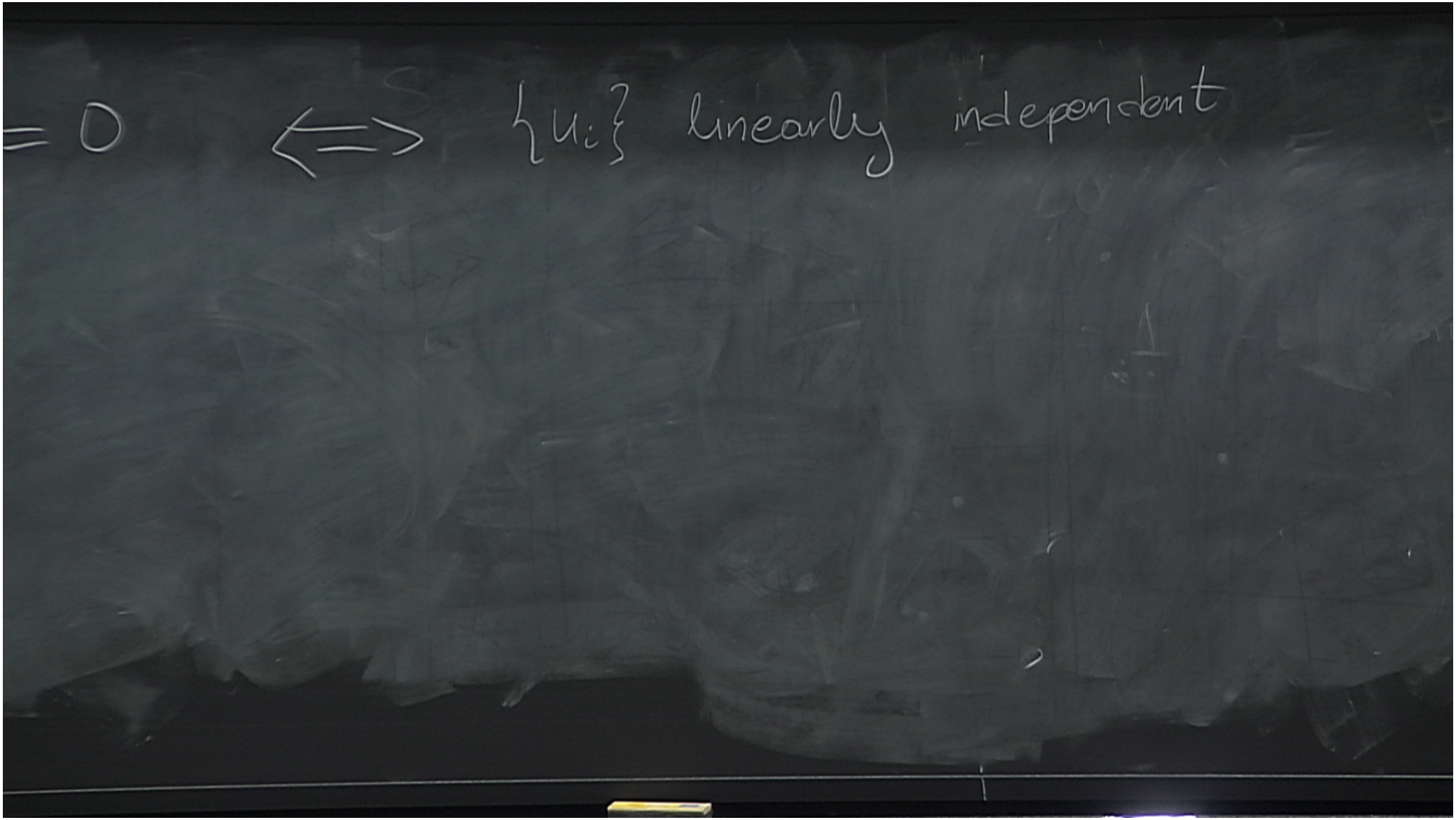

$\Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \ldots \oplus \Lambda^n(V)$
R
V
If $A = a \in \mathbb{R}$ $A \wedge B = B \wedge A = aB$
vectors"
$\Lambda(V)$
: exterior product.

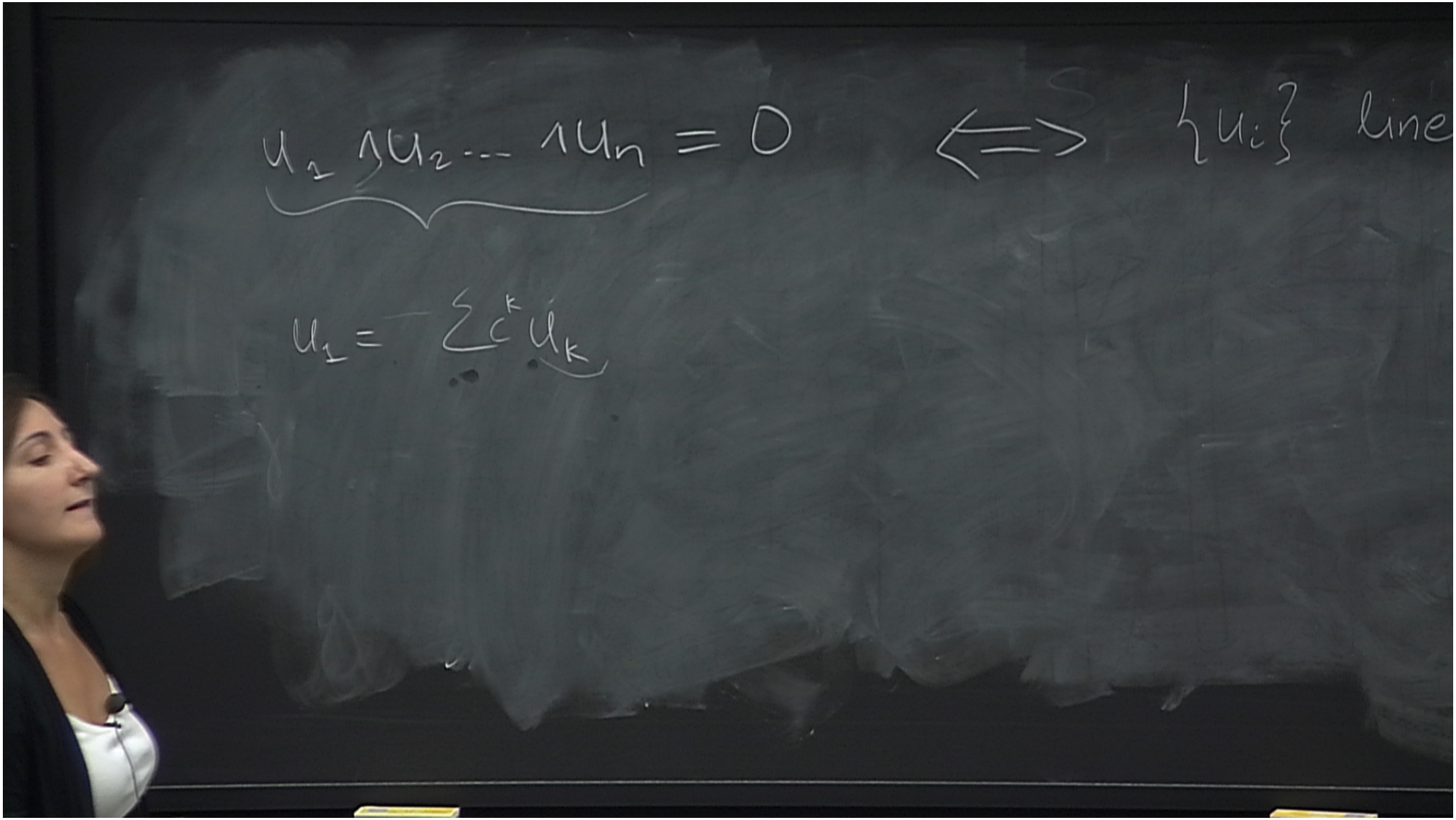

$\Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \dots \oplus \Lambda^n(V)$
R
V
vectors"
$\Lambda(V)$
: exterior product.
If $A = a \in \mathbb{R}$ $A \wedge B = B \wedge A = aB$
$A \in \Lambda^r(V)$, $B \in \Lambda^s(V)$
$A \wedge B = (u_1 \wedge u_2 \dots \wedge u_r) \wedge \underbrace{(v_1 \wedge v_2 \dots \wedge v_s)}_{B}$

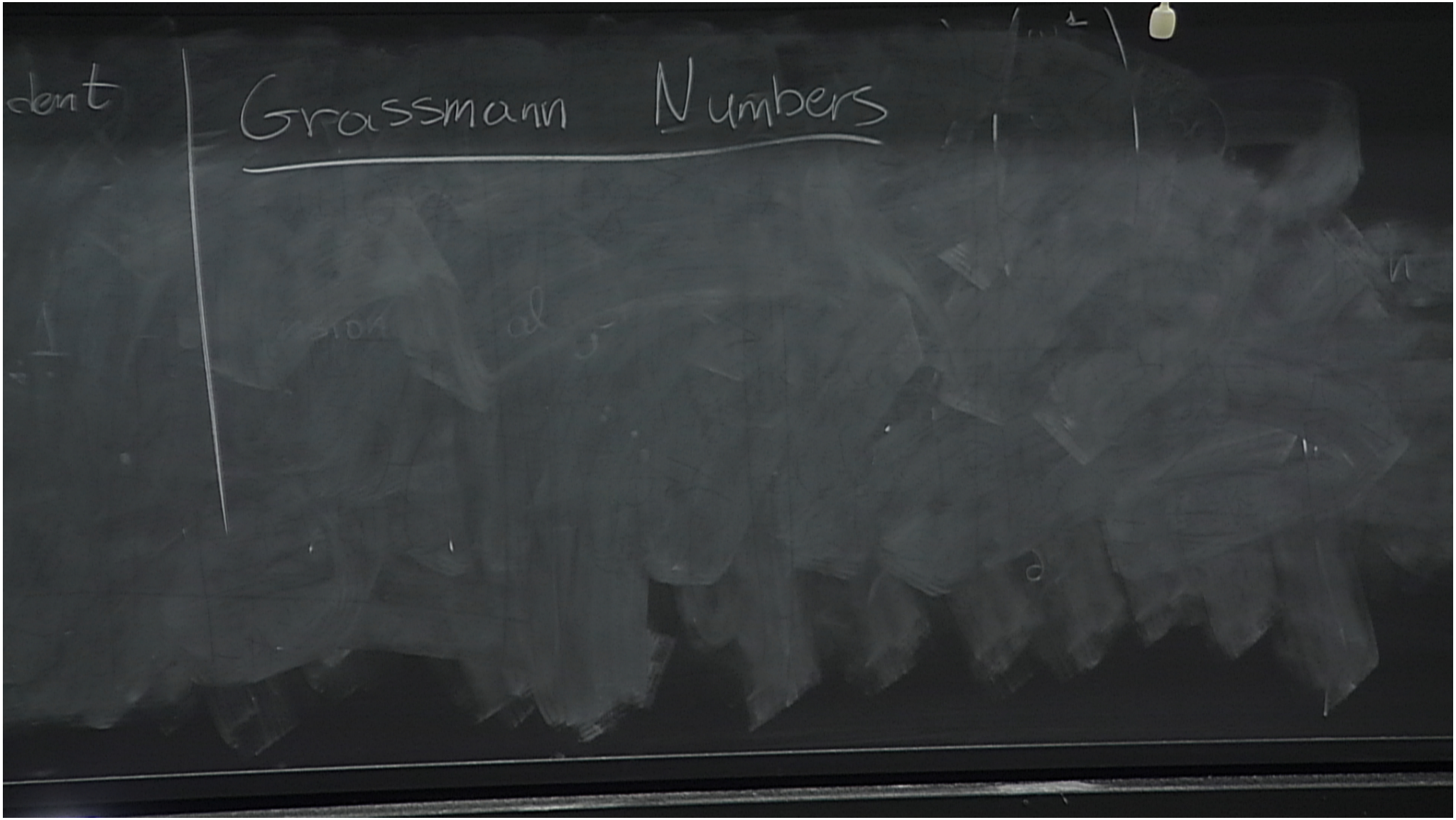

$\Lambda^0(V) \oplus \Lambda^1(V) \oplus \Lambda^2(V) \oplus \dots \oplus \Lambda^n(V)$
$\mathbb{R}$
V
vectors"
$\Lambda(V)$
: exterior product.
If $A = a \in \mathbb{R}$ $A \wedge B = B \wedge A = aB$
$A \in \Lambda^r(V)$, $B \in \Lambda^s(V)$
$A \wedge B = (u_1 \wedge u_2 \dots \wedge u_r) \wedge \underbrace{(v_1 \wedge v_2 \dots \wedge v_s)}_{B}$
$\in \Lambda^{r+s}(V)$

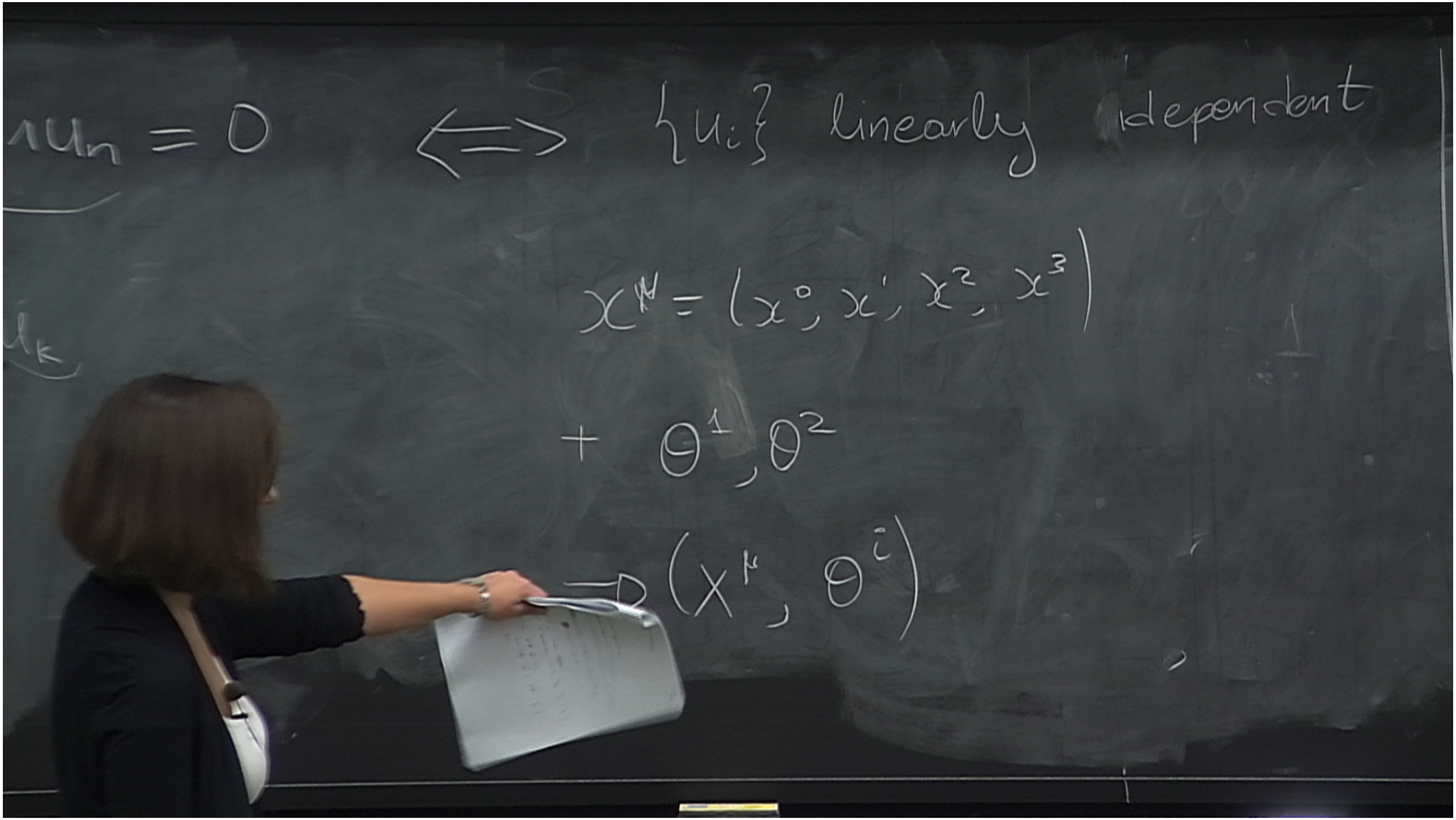

$\Lambda(V)$
: exterior product.
$A \wedge B = (u_1 \wedge u_2 \dots \wedge u_r) \wedge \underbrace{(v_1 \wedge v_2 \dots \wedge v_s)}_{B}$
$\in \Lambda^{r+s}(V)$
• linear in both arguments




=0 <=> {ui} linearly independent


$u_1 \wedge u_2 \cdots \wedge u_n = 0 \iff \{u_i\}$ line
$u_1 = \sum c^k u_k$


dent
Grassmann Numbers

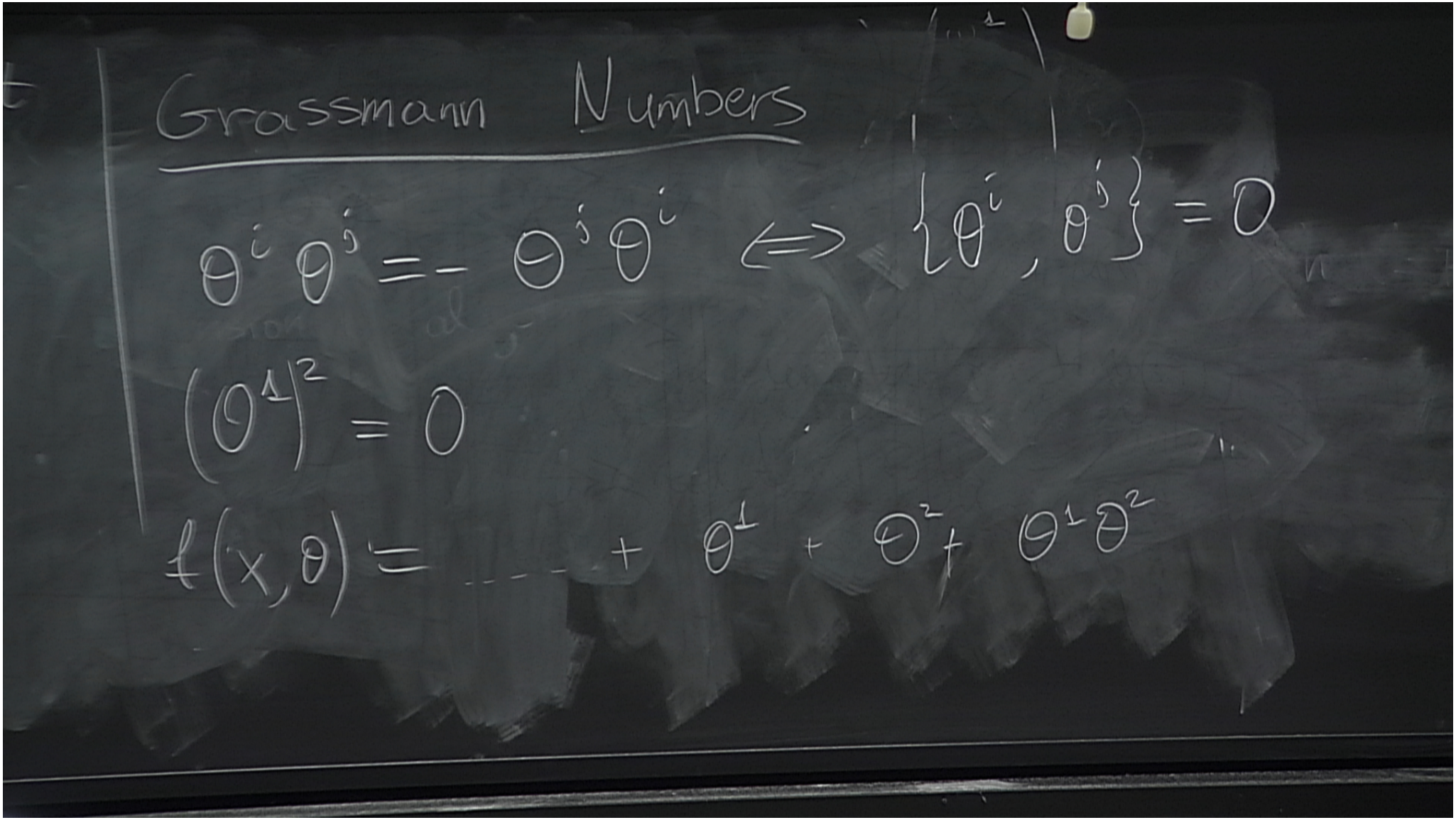

$\iota u_n = 0 \iff \{u_i\}$ linearly independent
$x^\mu = (x^0, x^1, x^2, x^3)$
$+ \theta^1, \theta^2$
$\rightarrow (x^\mu, \theta^i)$

$\det u_n = 0 \iff \{u_i\}$ linearly dependent

$x^\mu = (x^0, x^1, x^2, x^3)$

$+ \theta^1, \theta^2$

$\rightarrow (x^\mu, \theta^i)$


Grassmann Numbers
$\theta^i \theta^j = -\theta^j \theta^i \iff \{\theta^i, \theta^j\} = 0$
$(\theta^1)^2 = 0$
$f(x,\theta) = \ldots + \theta^1 + \theta^2 + \theta^1\theta^2$