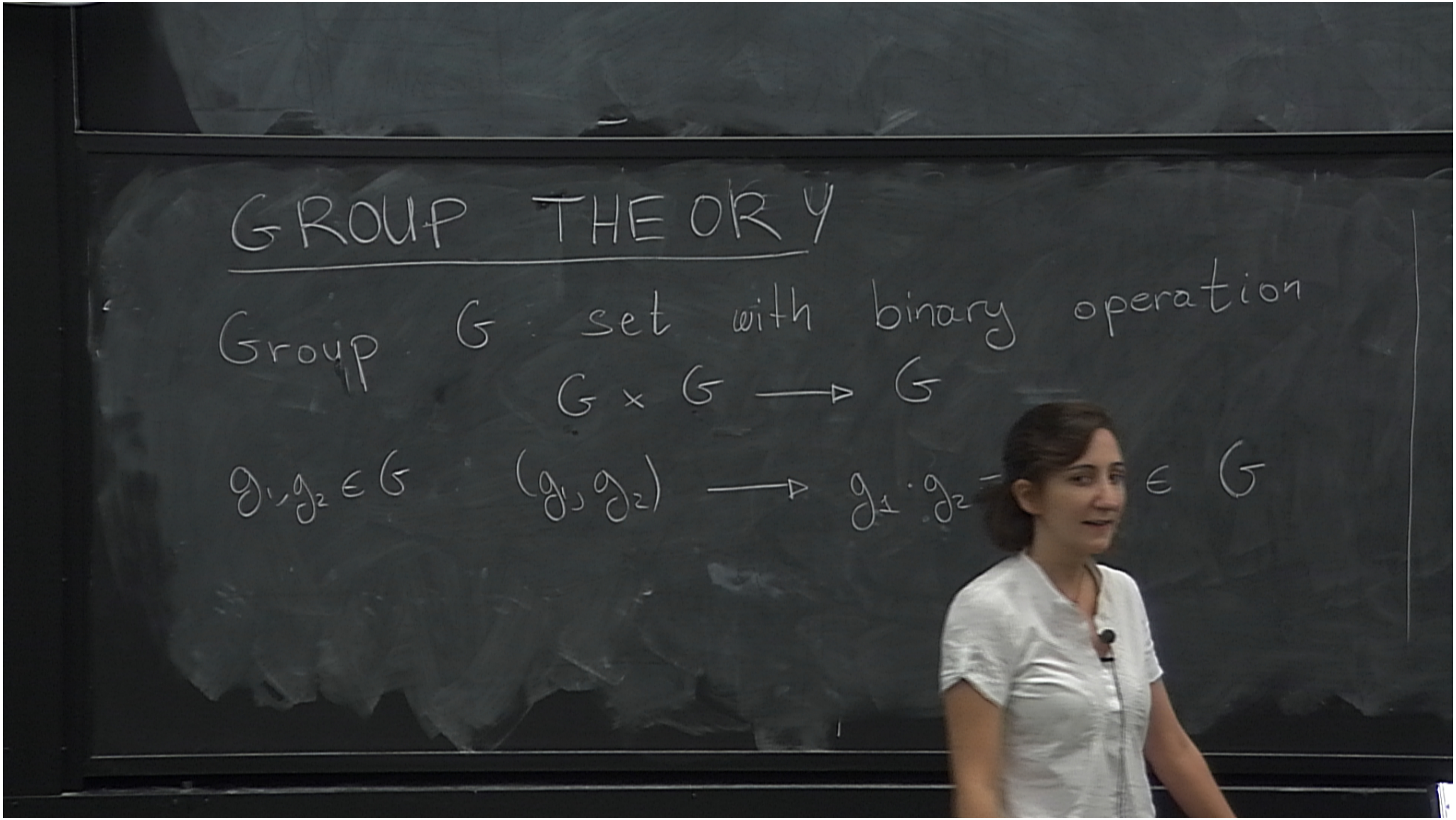
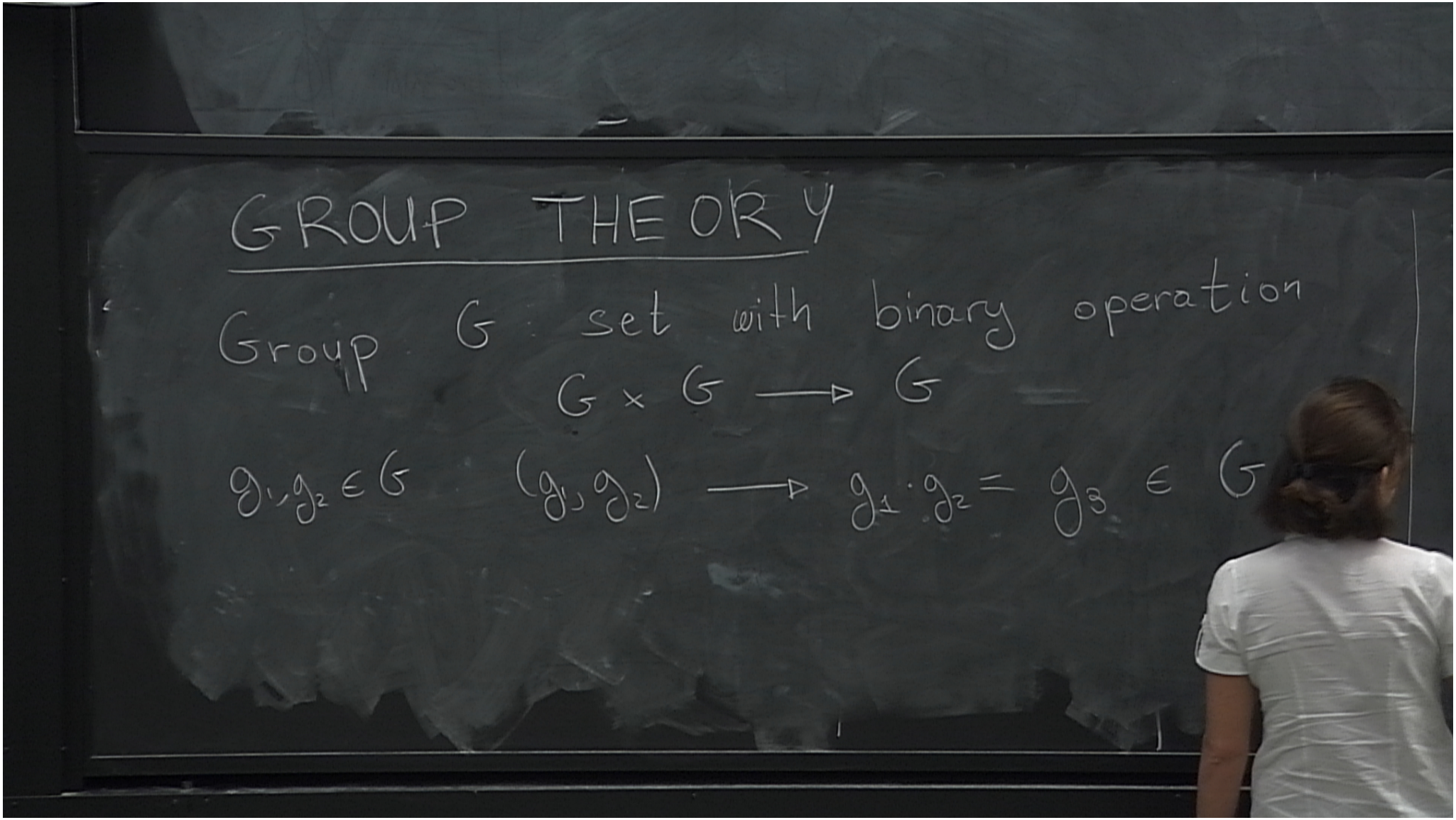
Title: Algebra - Lecture 3

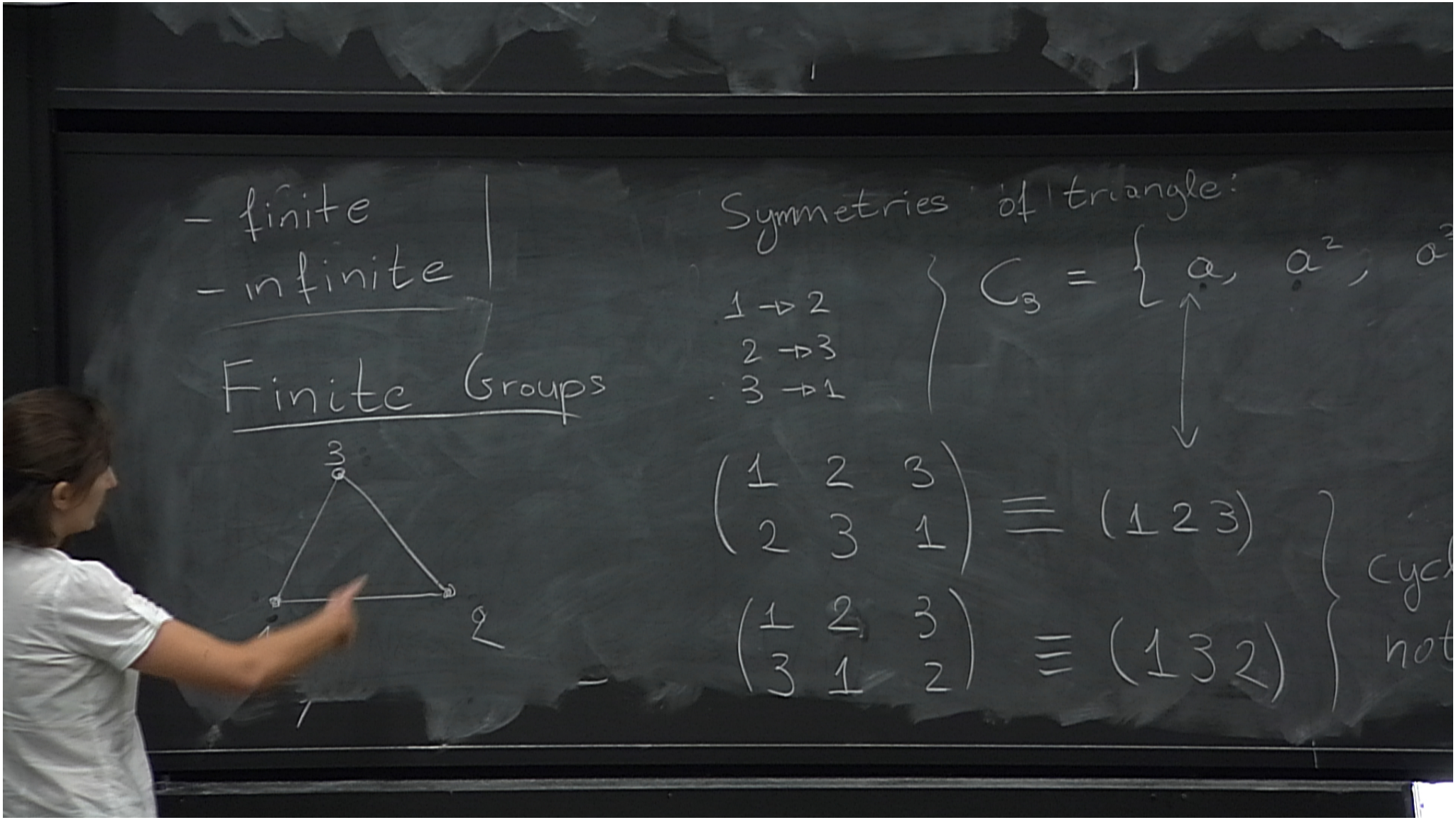
Date: Aug 18, 2011 11:00 AM

URL: http://pirsa.pi.local/11080084

Abstract:

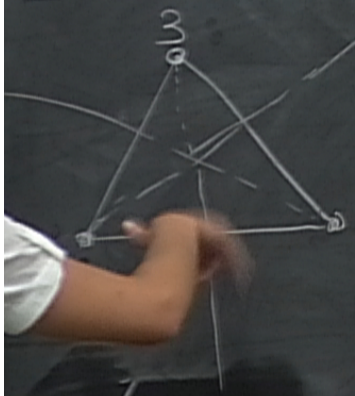

GROUP THEORY
Group G set with binary operation
$G \times G \longrightarrow G$
$g_1, g_2 \in G$
$(g_1, g_2) \longrightarrow g_1 \cdot g_2 \in G$


GROUP THEORY
Group G set with binary operation
$G \times G \longrightarrow G$
$g_1, g_2 \in G \quad (g_1, g_2) \longrightarrow g_1 \cdot g_2 = g_3 \in G$


- finite
- infinite
Finite Groups
Symmetries of triangle:
1 → 2
2 → 3
3 → 1
$C_3 = \{ a, a^2, a^3$
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \equiv (123)$
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \equiv (132)$
cycl
not
3
2

finite
infinite

Finite Groups



Symmetries of triangle:

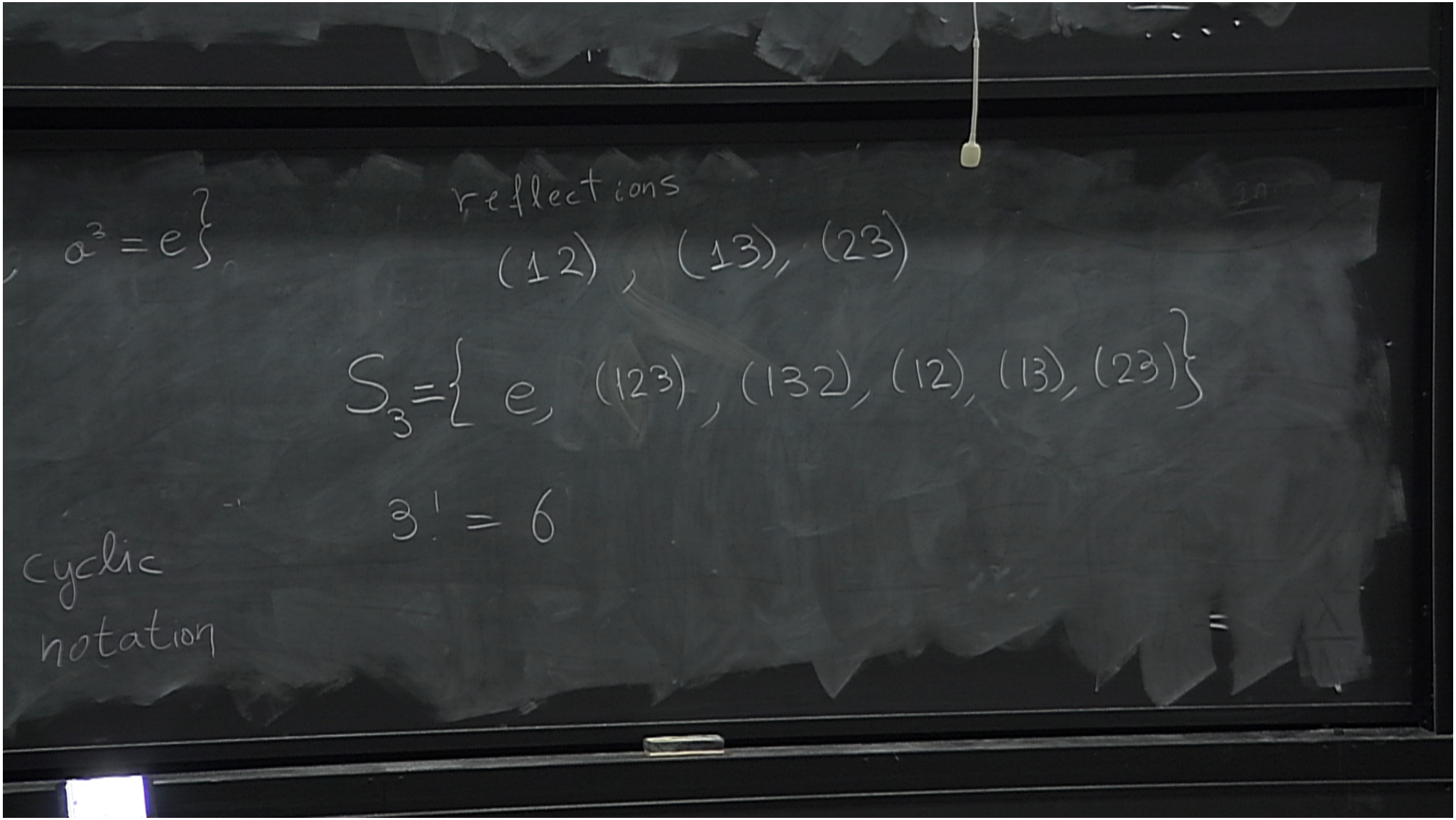
$1 \to 2$
$2 \to 3$
$3 \to 1$

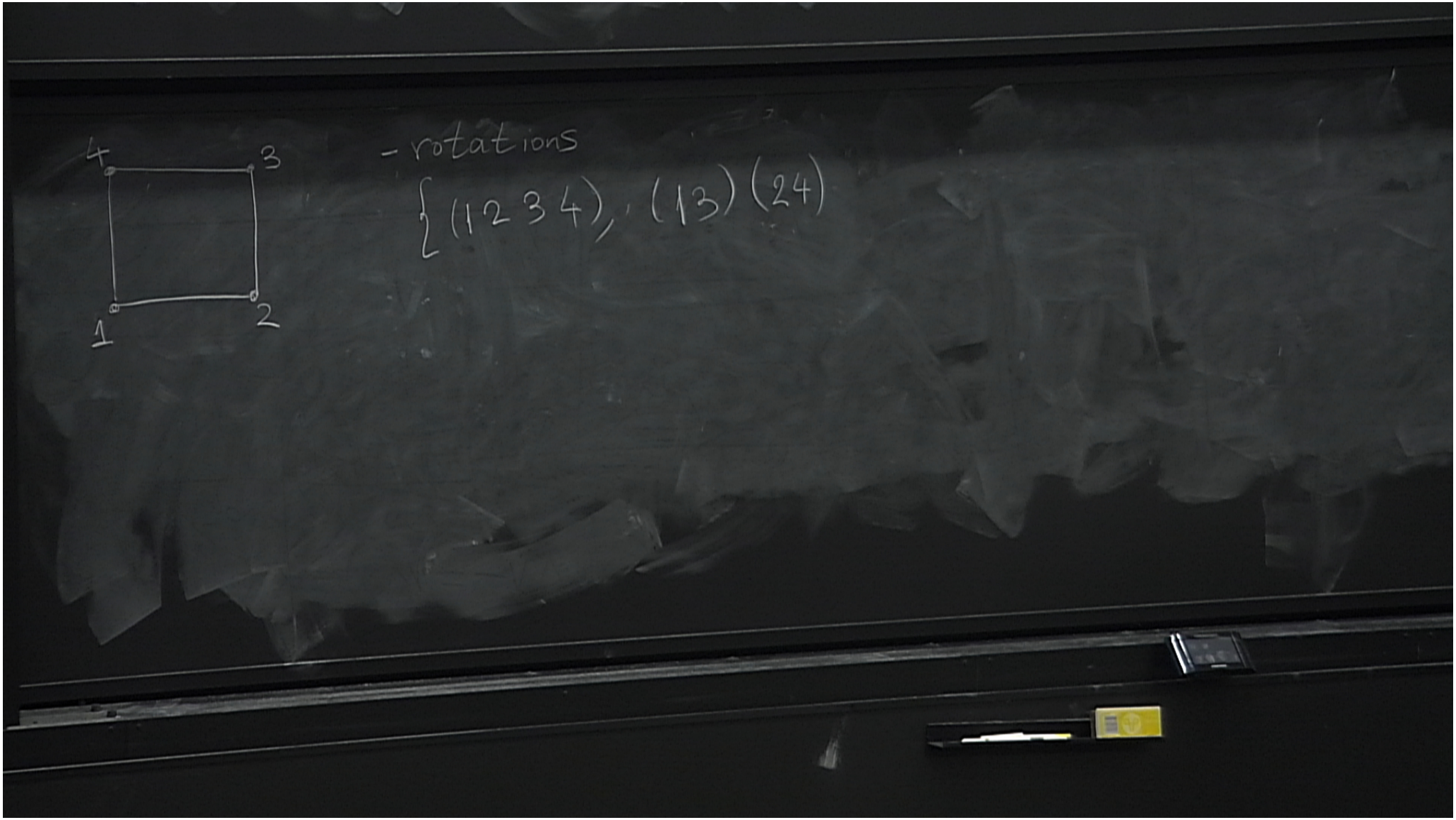
$C_3 = \{ a, a^2, a^3 = e \}$

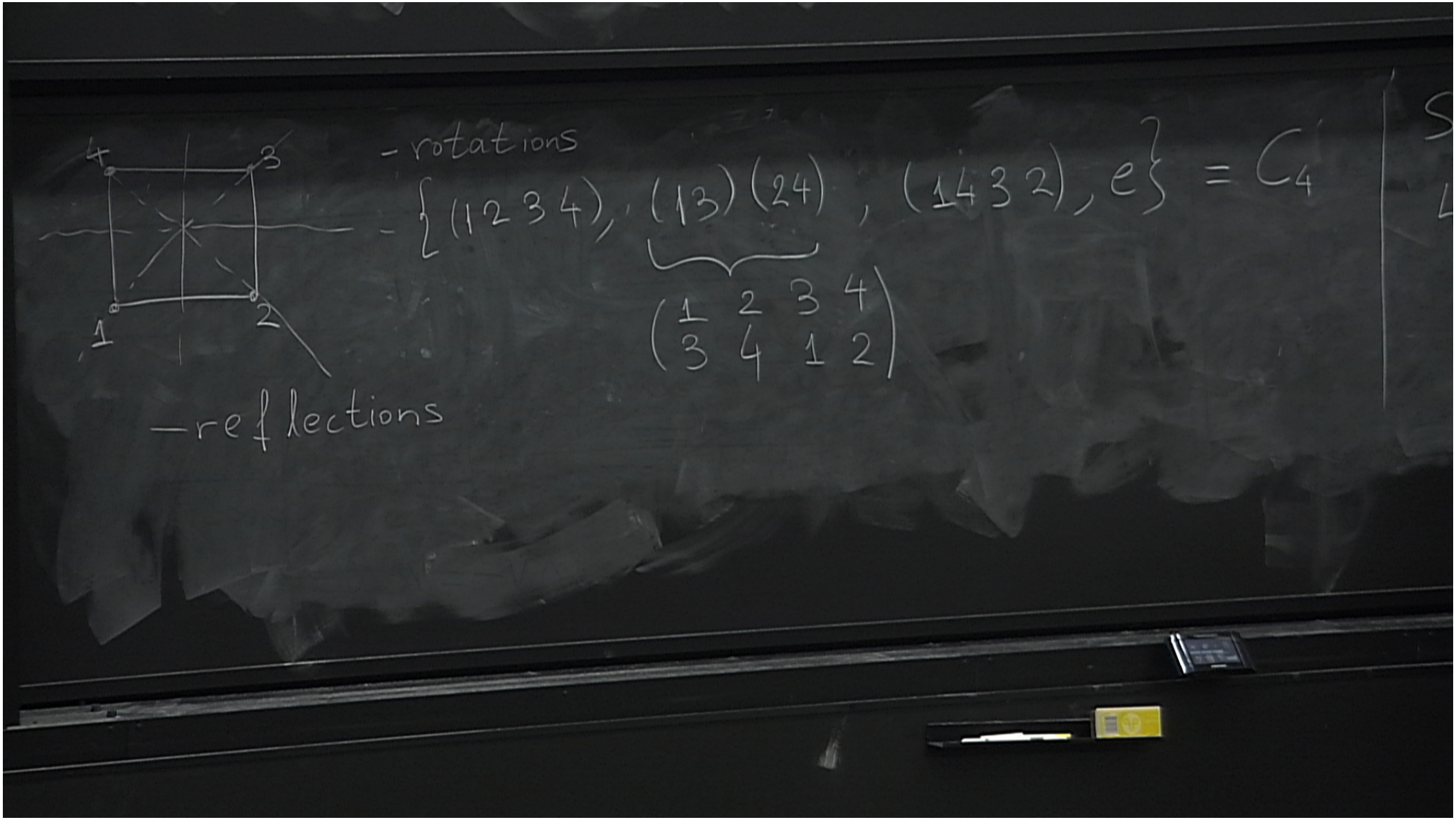
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \equiv (1\,2\,3)$

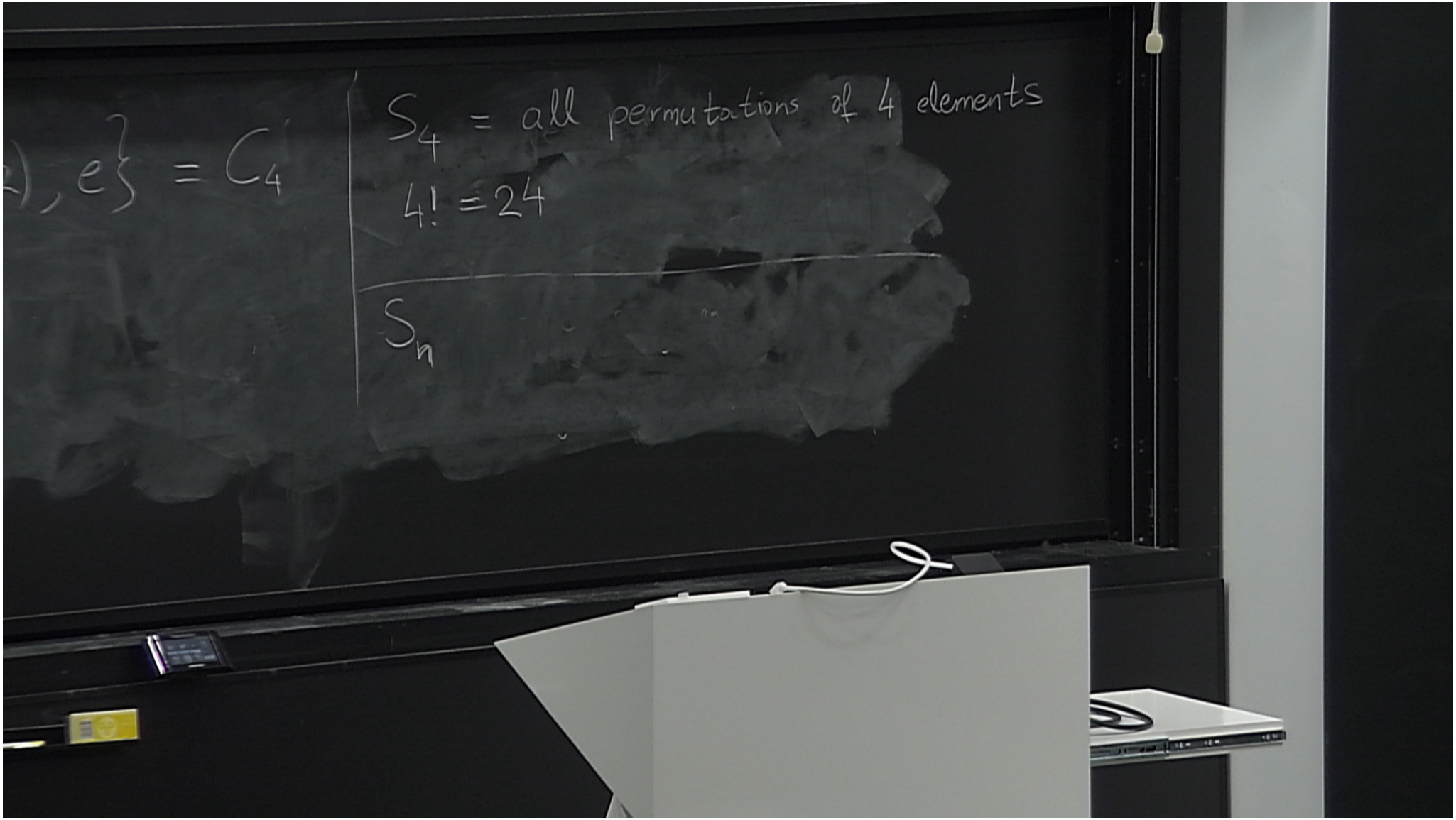
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \equiv (1\,3\,2)$

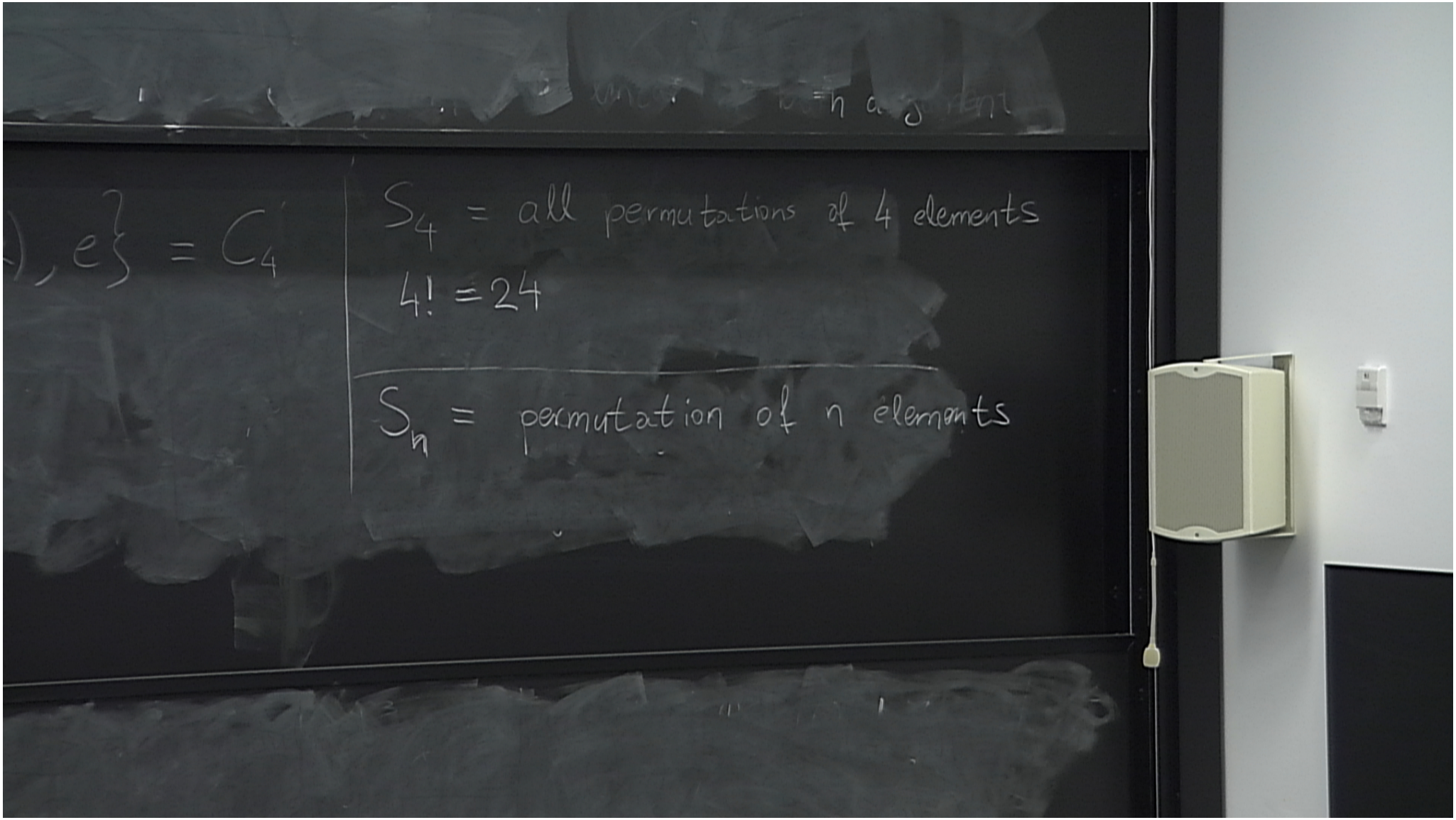
cyclic notation

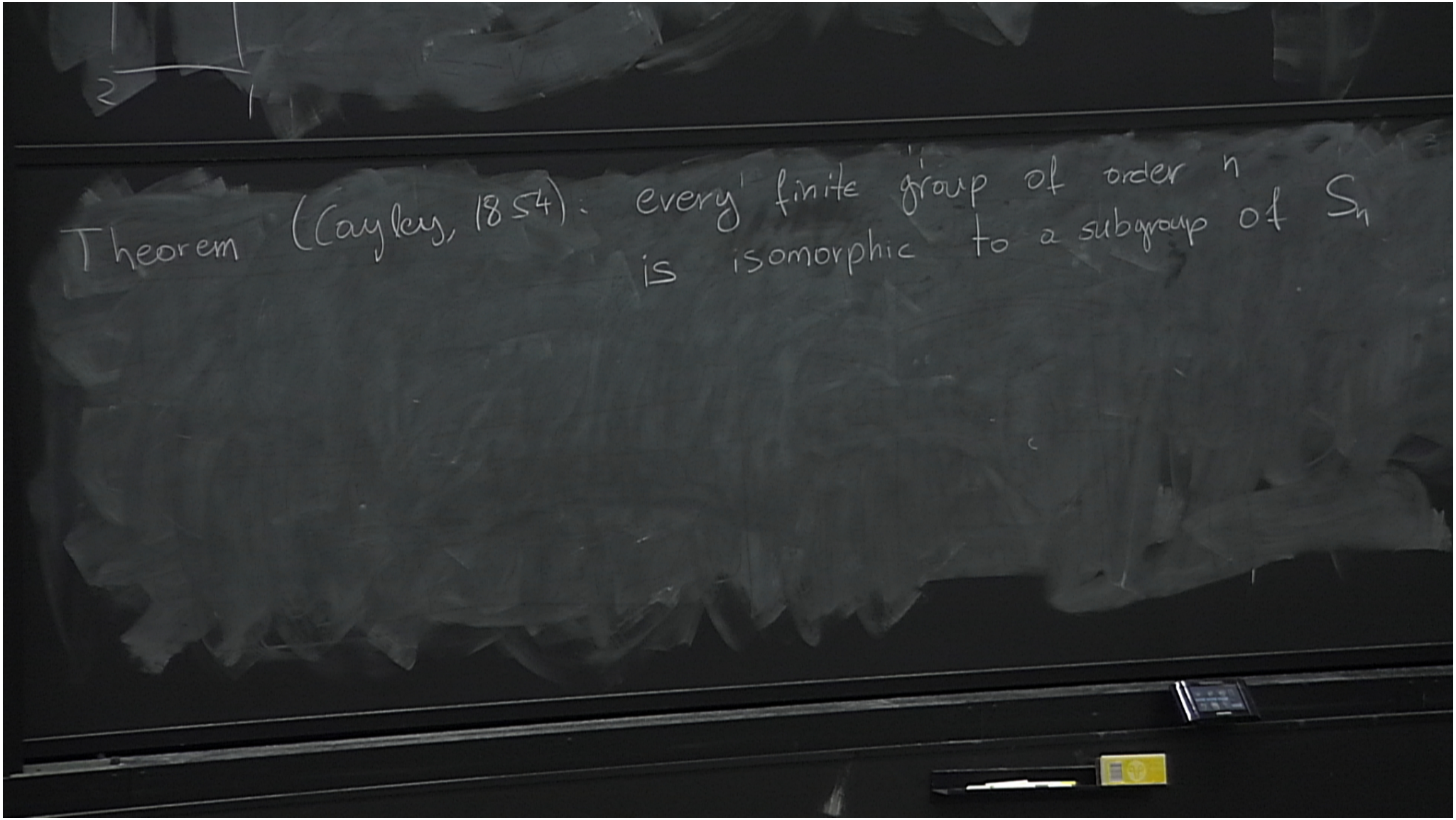

$a^3 = e\}$
reflections
(12), (13), (23)
$S_3 = \{e, (123), (132), (12), (13), (23)\}$
$3! = 6$
cyclic
notation

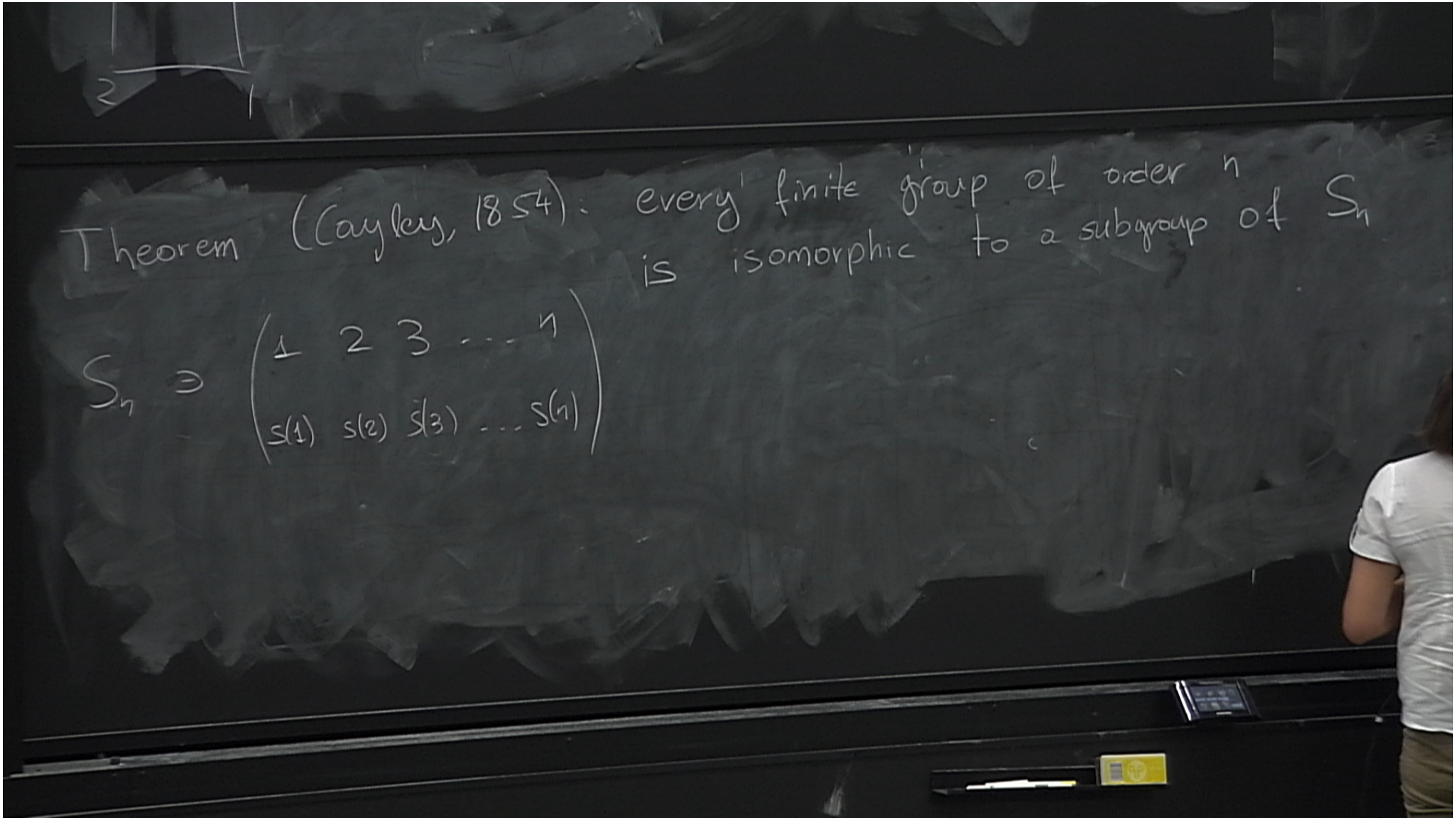

4
3
1
2
- rotations
{(1234), (13)(24)

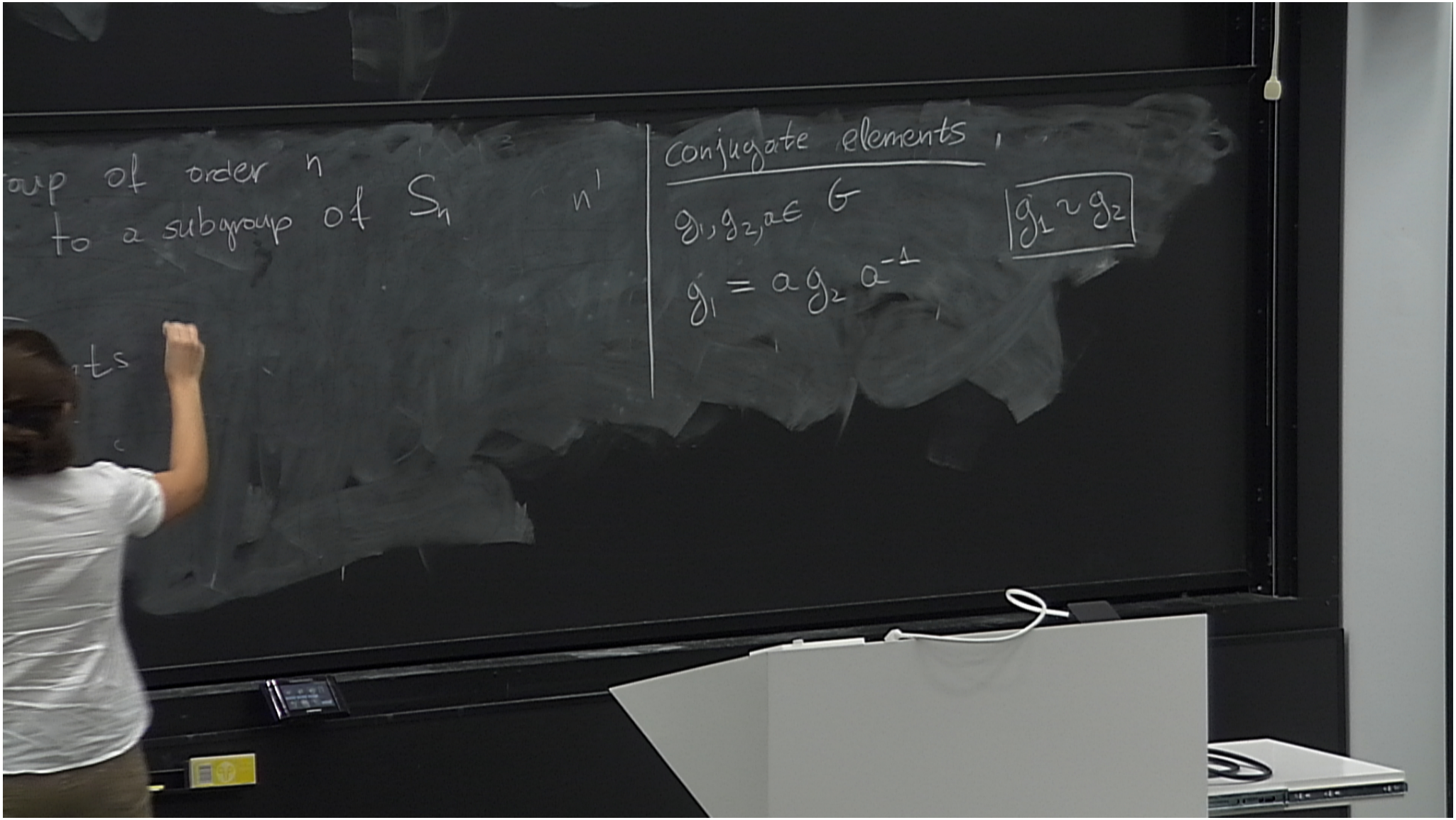

4
3
1
2
- rotations
- {(1234), (13)(24), (1432), e} = C4
(1 2 3 4)
(3 4 1 2)
- reflections

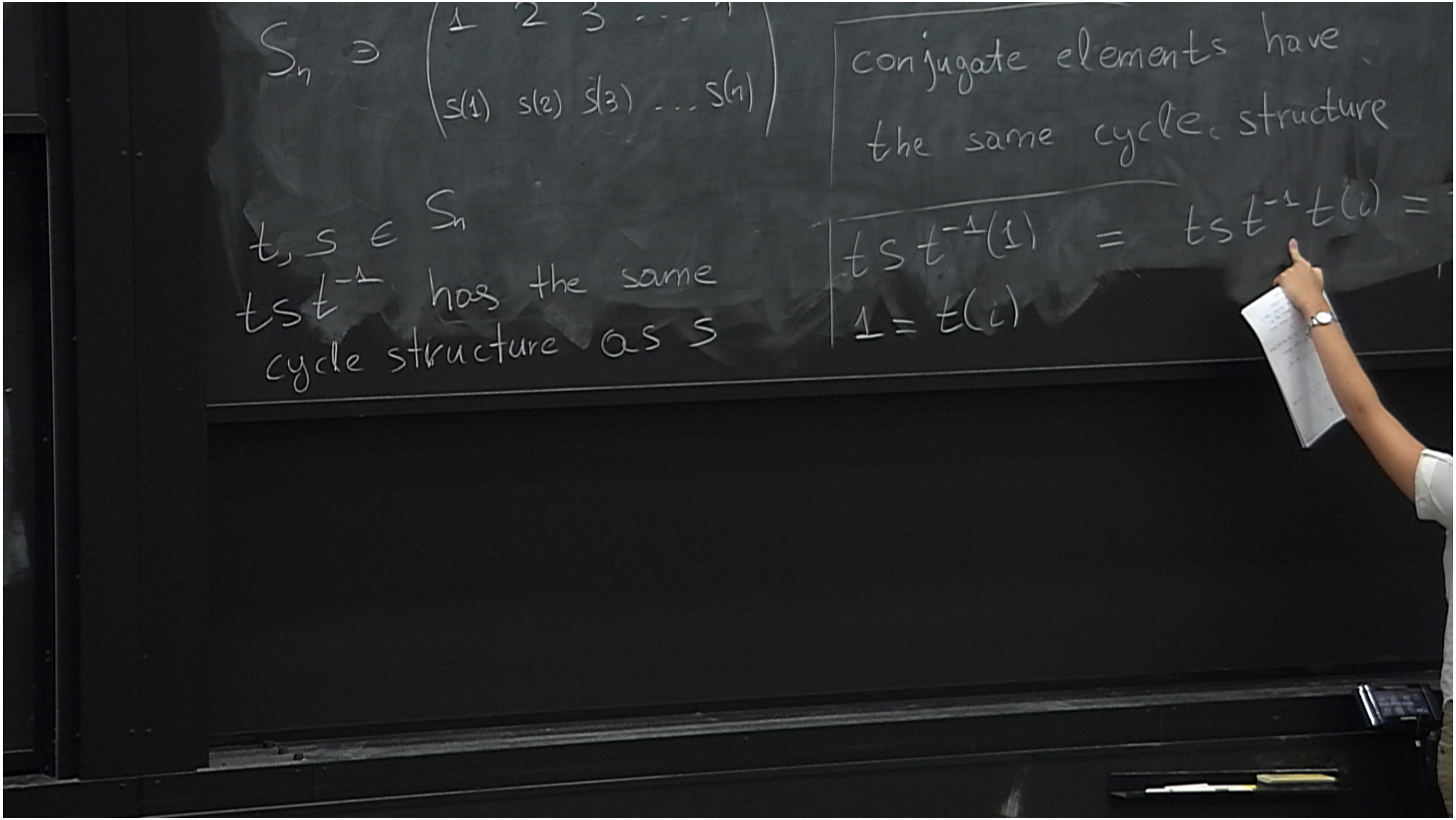

S_4 = all permutations of 4 elements
$4! = 24$
S_n

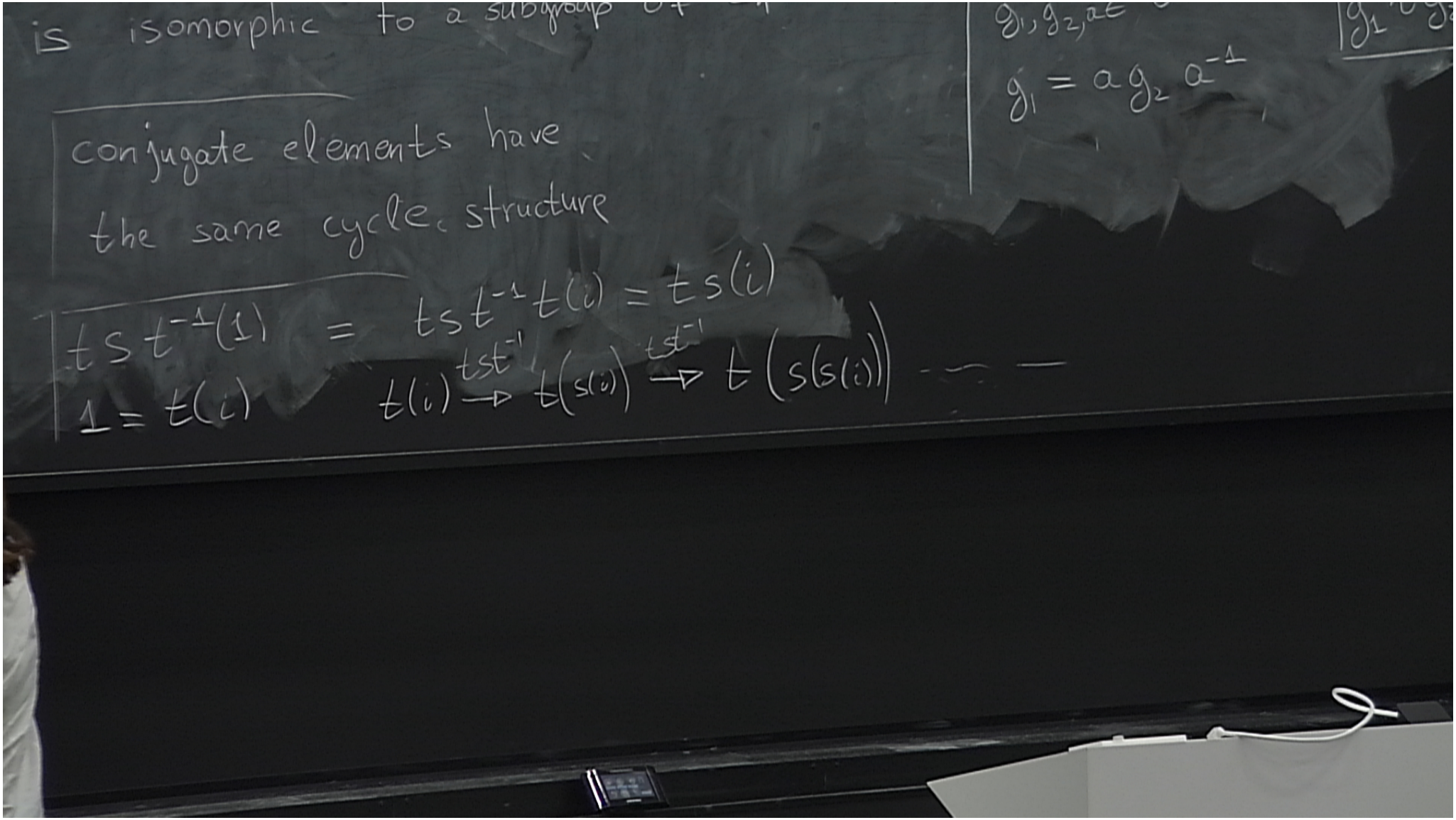

S_4 = all permutations of 4 elements
$4! = 24$
S_n = permutation of n elements


Theorem (Cayley, 1854): every finite group of order n
is isomorphic to a subgroup of S_n


Theorem (Cayley, 1854): every finite group of order n is isomorphic to a subgroup of S_n
$S_n \ni \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ s(1) & s(2) & s(3) & \dots & s(n) \end{pmatrix}$


oup of order n
to a subgroup of S_n
n'
Conjugate elements
$g_1, g_2, a \in G$
$g_1 = a g_2 a^{-1}$
$g_1 \sim g_2$


$S_n \ni \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ s(1) & s(2) & s(3) & \dots & s(n) \end{pmatrix}$
conjugate elements have the same cycle structure
$t, s \in S_n$
tst^{-1} has the same cycle structure as s
$tst^{-1}(1) = tst^{-1}t(i) =$
$1 = t(i)$


is isomorphic to a subgroup
conjugate elements have the same cycle structure
$tst^{-1}(1) = tst^{-1}t(i) = ts(i)$
$1 = t(i)$
$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{tst^{-1}} t(s(s(i)))$
$g_1, g_2, a \in$
$g_1 = a g_2 a^{-1}$

Theorem ... is isomorphic to a subgroup ...

$S_n \ni \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ s(1) & s(2) & s(3) & \dots & s(n) \end{pmatrix}$

$i \xrightarrow{t} t(i)$

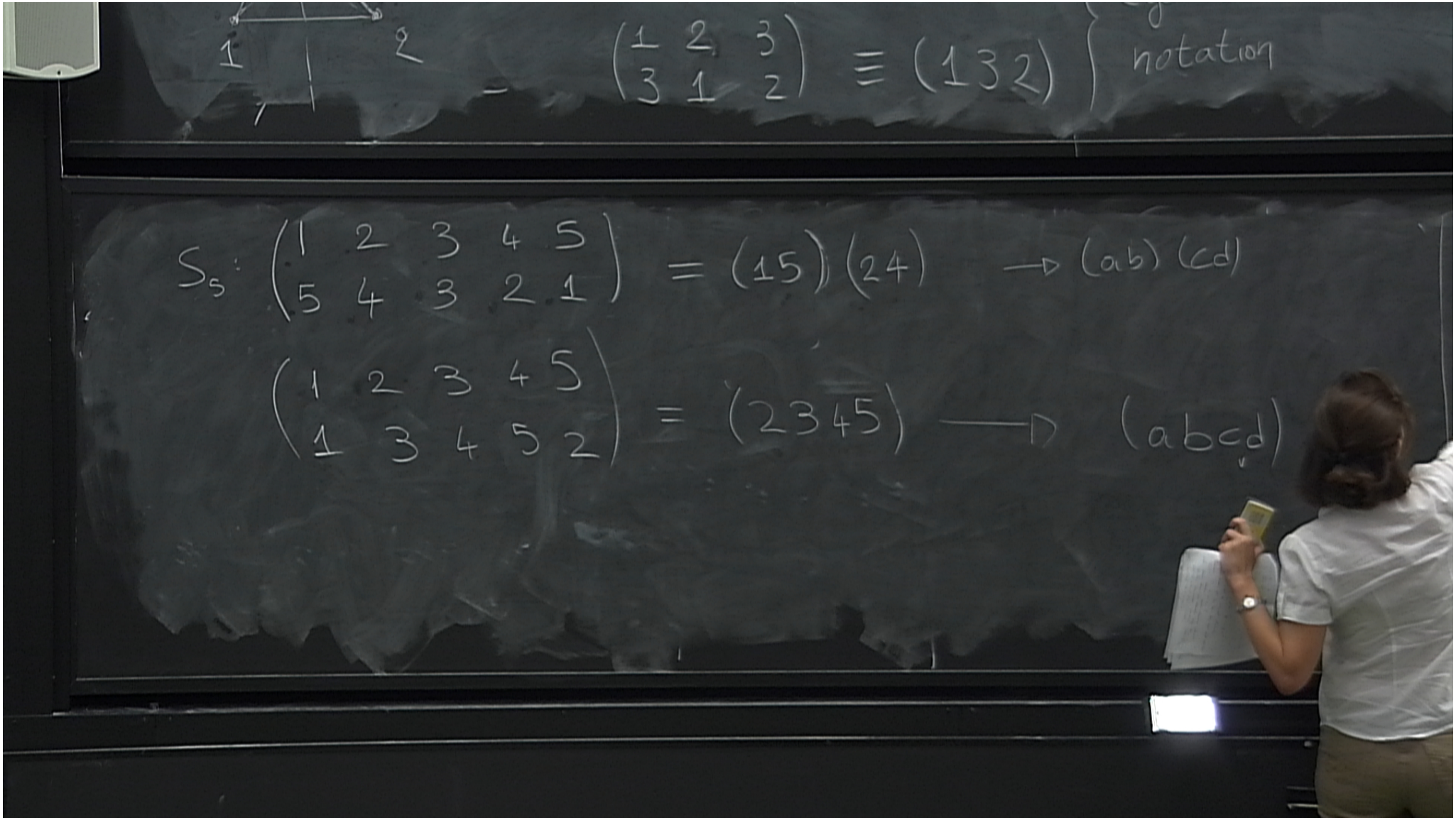
$t, s \in S_n$

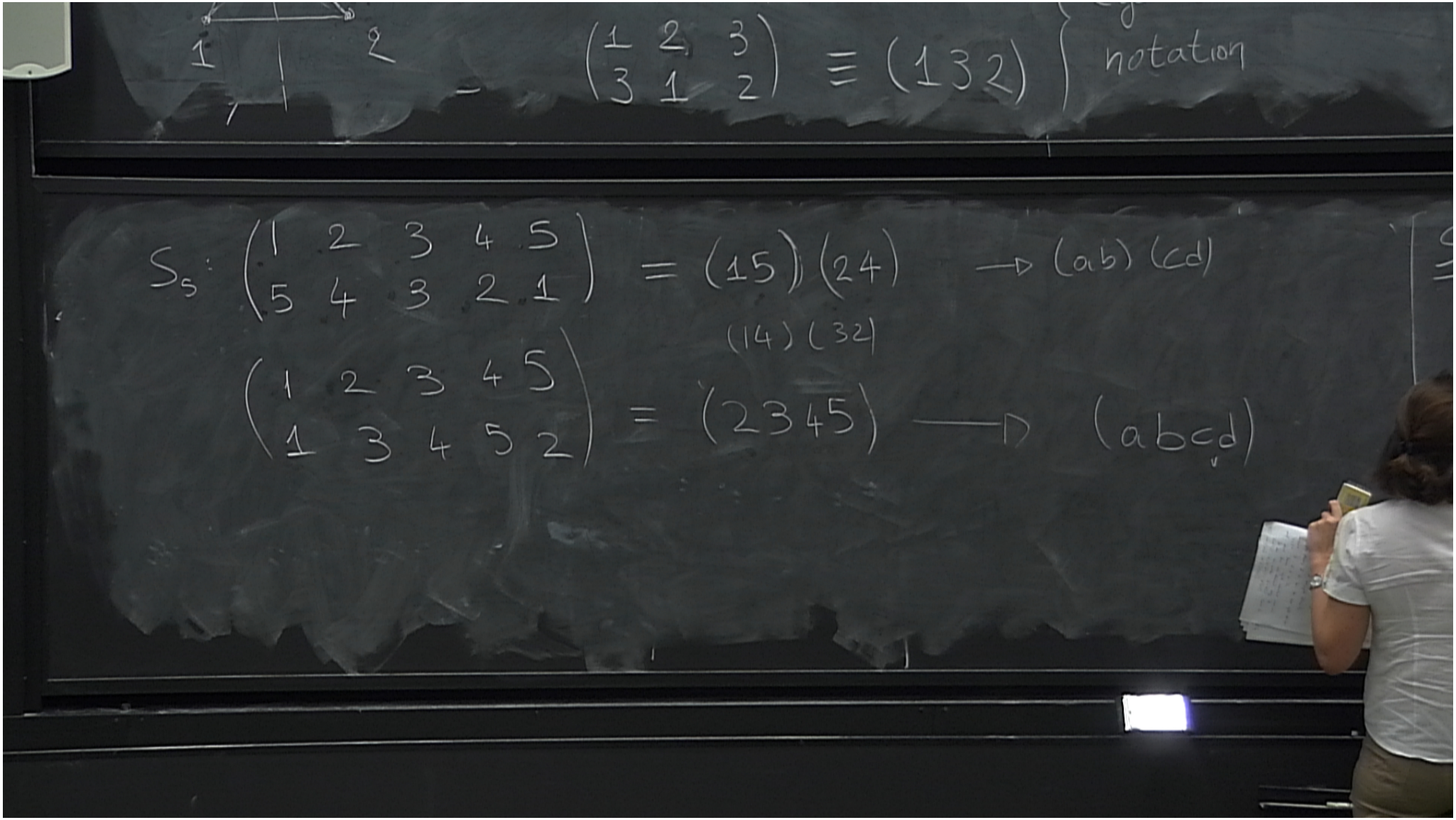
tst^{-1} has the same cycle structure as s

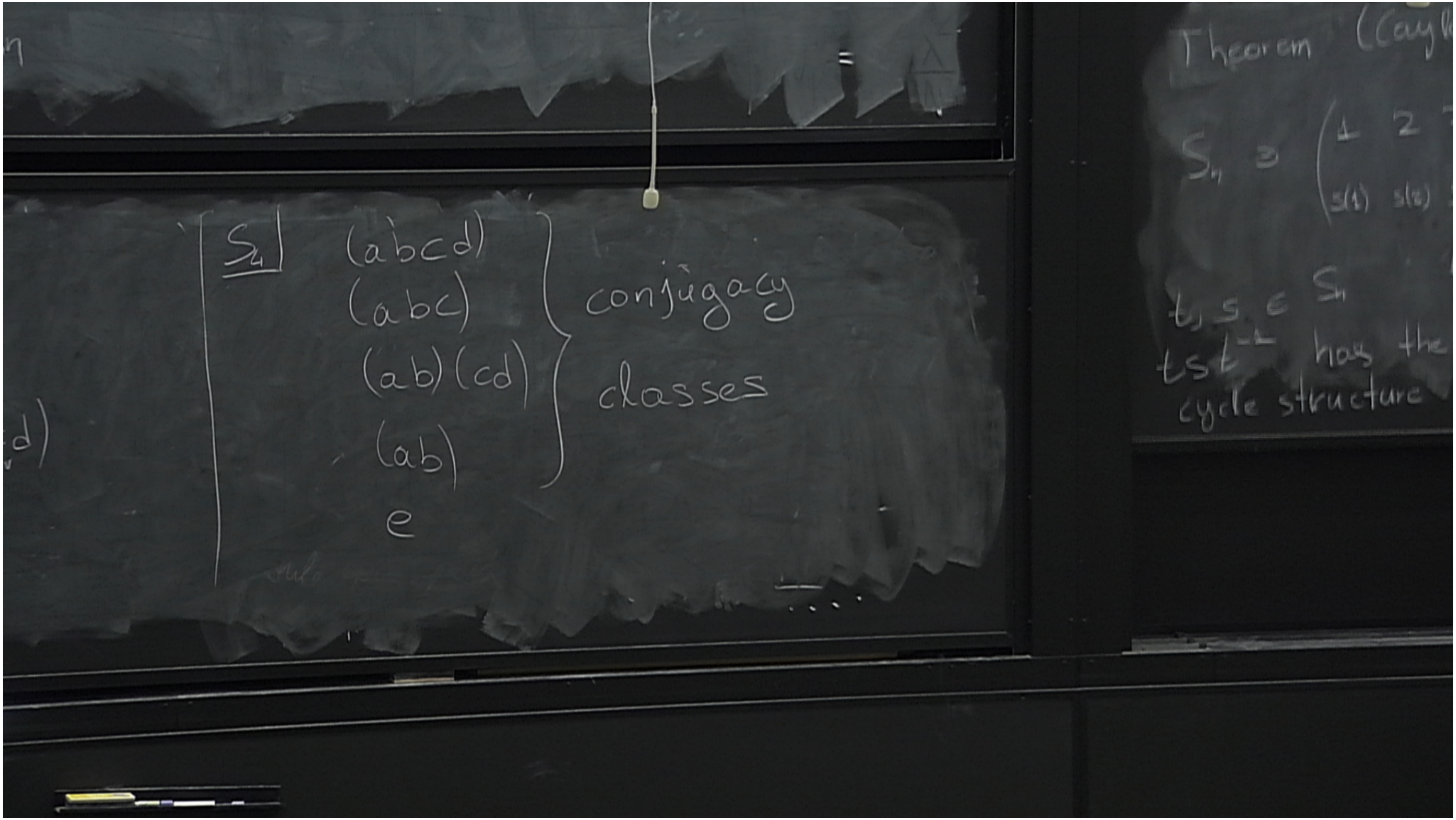
conjugate elements have the same cycle structure

$tst^{-1}(1) = tst^{-1}t(i) = ts($

$1 = t(i)$ $\quad t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{tst^{-1}} t$


1
2
$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \equiv (132)$
notation
$S_5: \begin{pmatrix}1&2&3&4&5\\5&4&3&2&1\end{pmatrix} = (15)(24) \longrightarrow (ab)(cd)$
$\begin{pmatrix}1&2&3&4&5\\1&3&4&5&2\end{pmatrix} = (2345) \longrightarrow (abcd)$


1
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$(14)(32)$
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Theorem
S_4
(abcd)
(abc)
(ab)(cd)
(ab)
e
conjugacy classes
$t, s \in S_4$
tst^{-1} has the
cycle structure

$t, s \in S_n$

tst^{-1} has the same cycle structure as s

$i \to t(i)$

$tst^{-1}(1) = tst^{-1}t(i) = ts(i)$

$j = t(i)$

$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{tst^{-1}} t(s(s(i)))$

$s(s(s(i))) = i$



$\{(1234), (13)(24), (1432), e\} = C_4$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

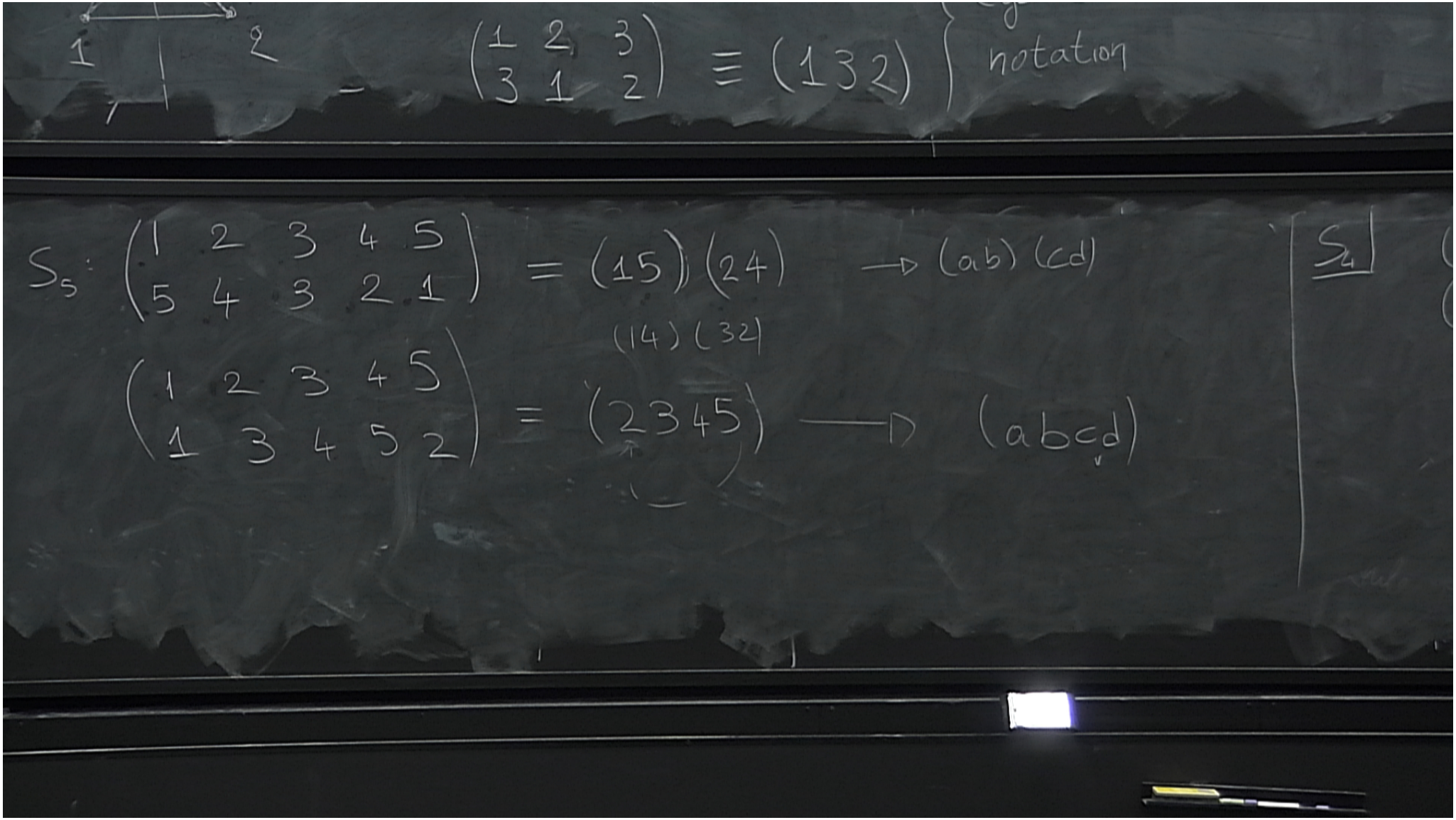
S_4 = all permutations

$4! = 24$

S_n = permutation of

- reflections




1
2
$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} \equiv (132)$
notation
S_5: $\begin{pmatrix}1&2&3&4&5\\5&4&3&2&1\end{pmatrix} = (15)(24) \longrightarrow (ab)(cd)$
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S_4

$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ s(1) & s(2) & s(3) & \dots & s(n) \end{pmatrix} \in S_n$

$i \xrightarrow{t} t(i)$

has the same structure as s

conjugate elements have the same cycle structure

$tst^{-1}(1) = tst^{-1}t(i) = ts(i)$

$j = t(i)$

$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{tst^{-1}} t(s(s(i)))$

$s(s(s(i))) = i$

$\longrightarrow t(s^m(i)) \equiv t(i)$

$g_1 = a g_2 a^{-1}$

S_4 = all permutations of 4 el

$4! = 24$

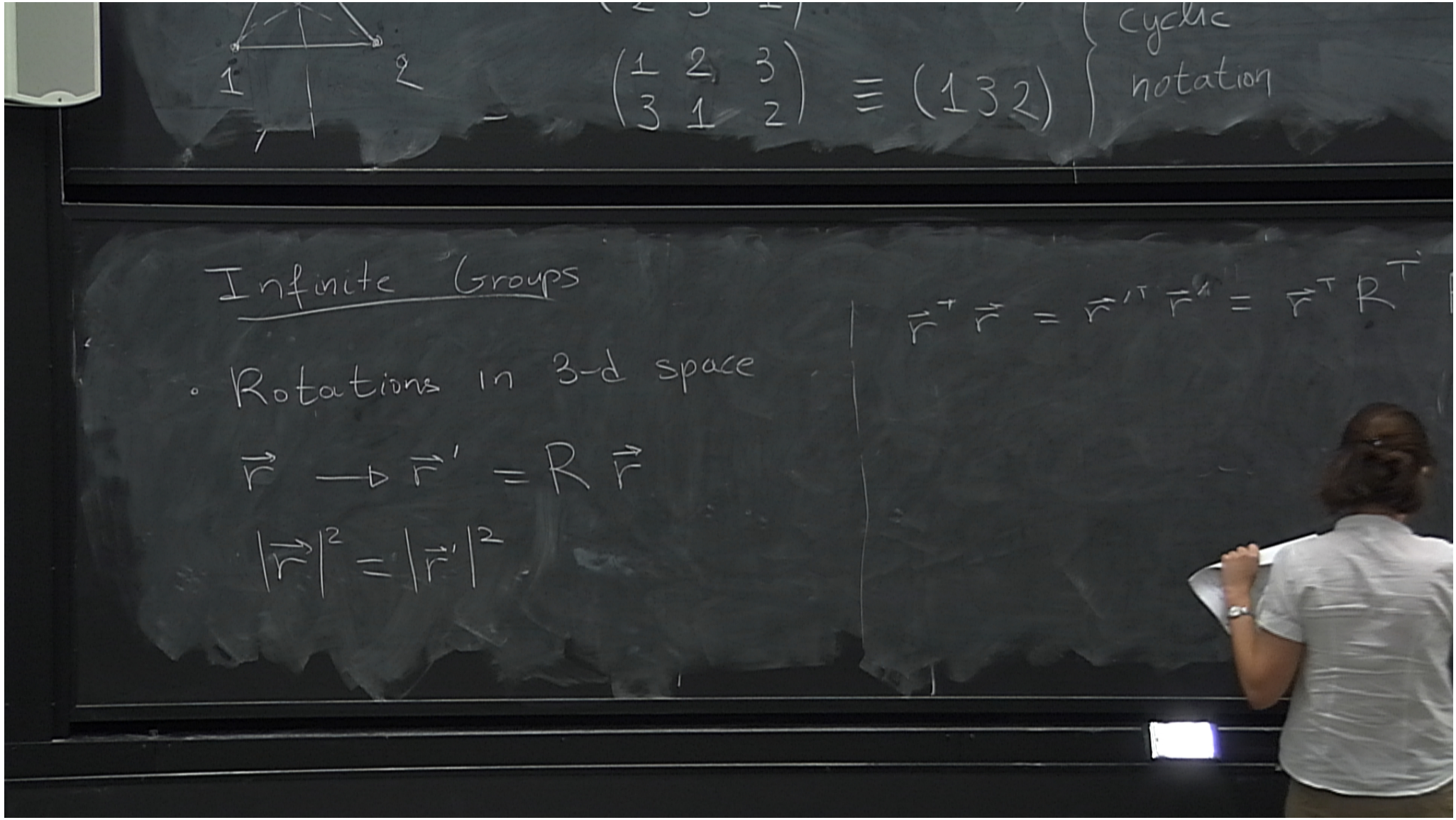
$\{(1234), (13)(24), (1432), e\} = C_4$

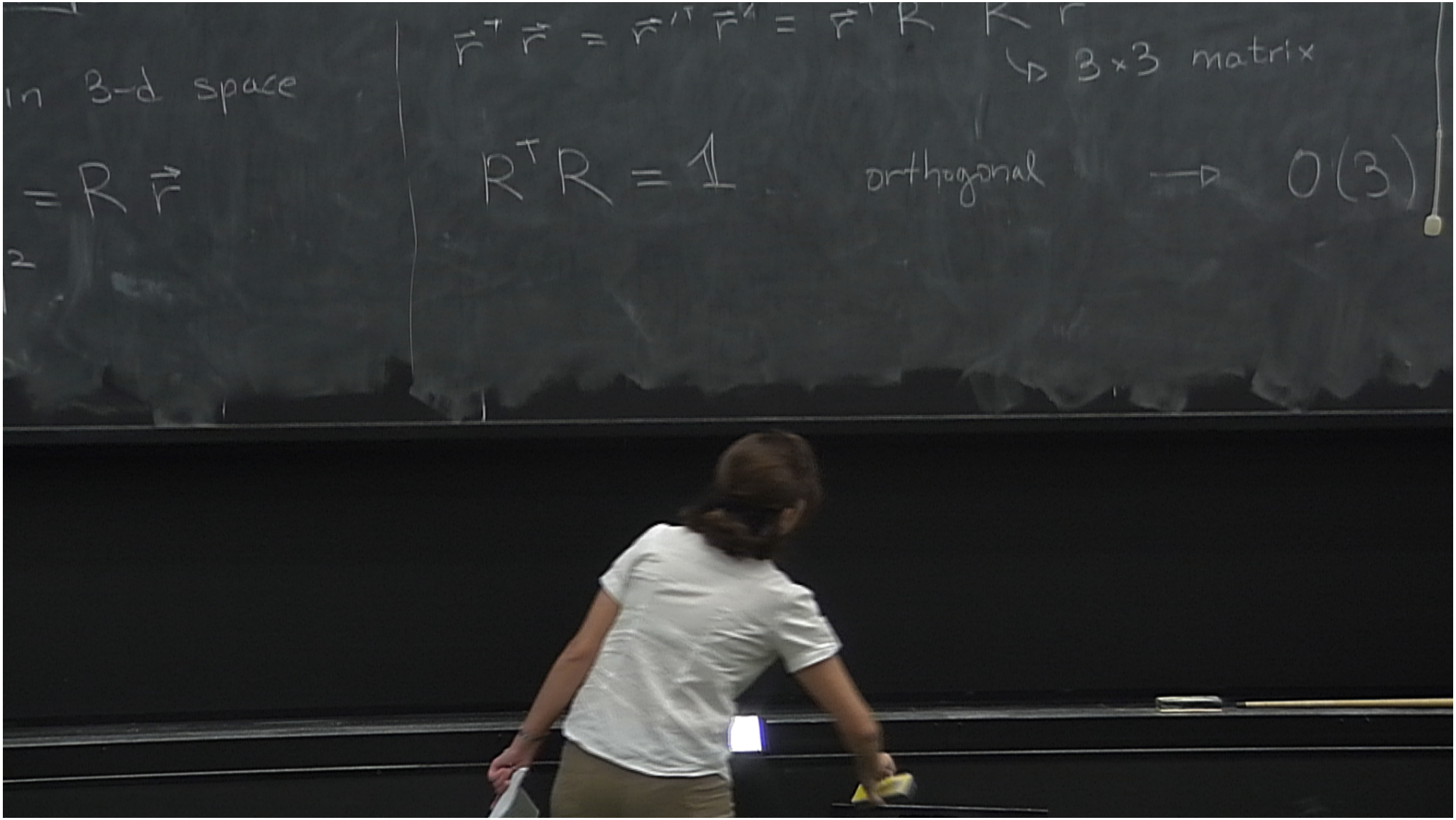
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

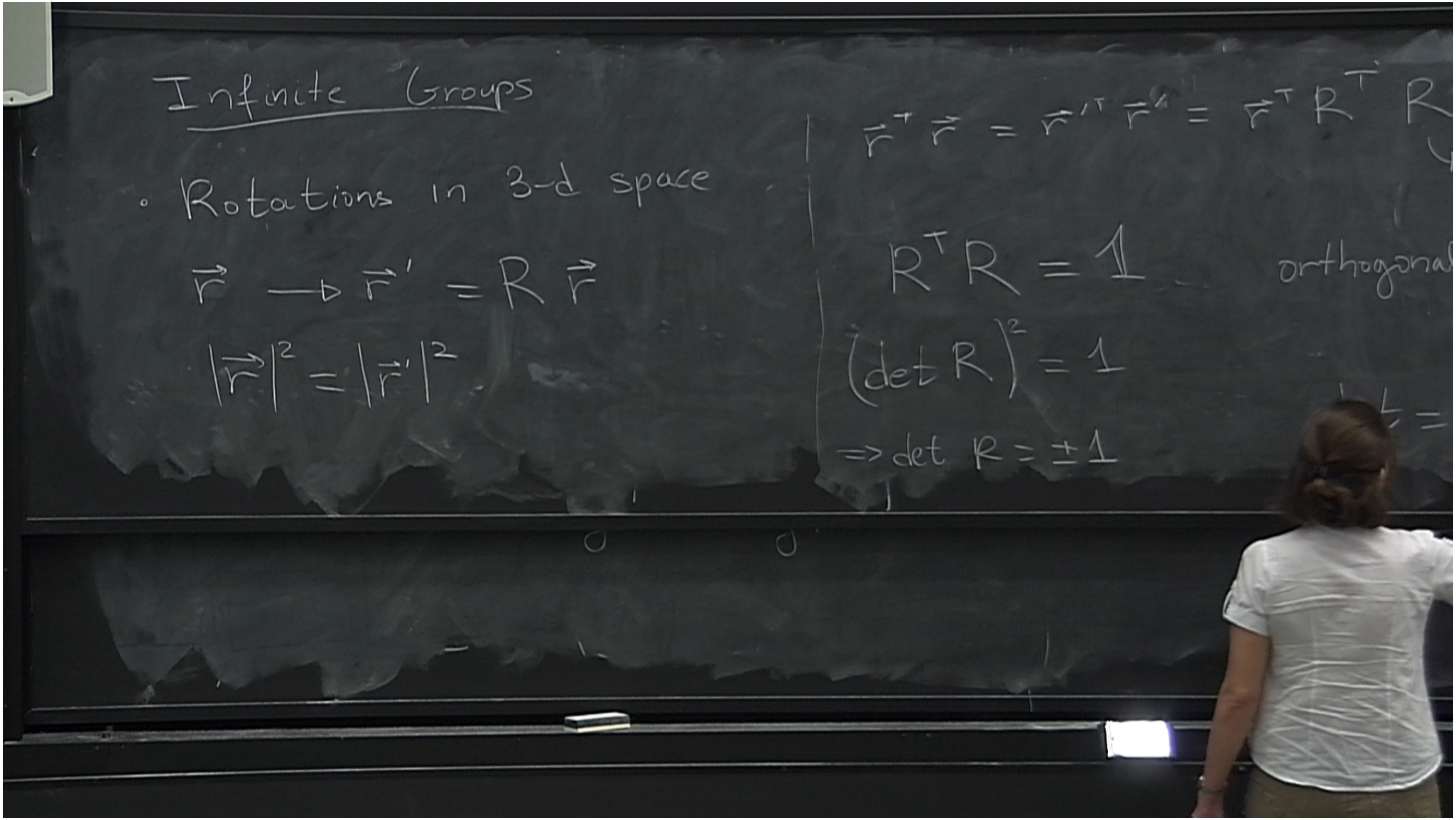
$\begin{cases} i \xrightarrow{s} s(i) \xrightarrow{s} s(s(i)) \\ \dots \xrightarrow{s} s^m(i) \equiv i \end{cases}$

S_n = permutation of n elem

reflections


1
2
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \equiv (132)$
cyclic notation
Infinite Groups
• Rotations in 3-d space
$\vec{r} \longrightarrow \vec{r}' = R\vec{r}$
$|\vec{r}|^2 = |\vec{r}'|^2$
$\vec{r}^T \vec{r} = \vec{r}'^T \vec{r}' = \vec{r}^T R^T$


in 3-d space
$= R \vec{r}$
$\vec{r}^T \vec{r} = \vec{r}'^T \vec{r}' = \vec{r}^T R^T R \vec{r}$
↳ 3×3 matrix
$R^T R = 1$
orthogonal
→ O(3)


Infinite Groups
• Rotations in 3-d space
$\vec{r} \to \vec{r}' = R\vec{r}$
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$\vec{r}^T\vec{r} = \vec{r}'^T\vec{r}' = \vec{r}^T R^T R$
$R^T R = 1$
orthogonal
$(\det R)^2 = 1$
$\Rightarrow \det R = \pm 1$

space

$\vec{r}^T \vec{r} = \vec{r}'^T \vec{r}' = \vec{r}^T R^T R \vec{r}$

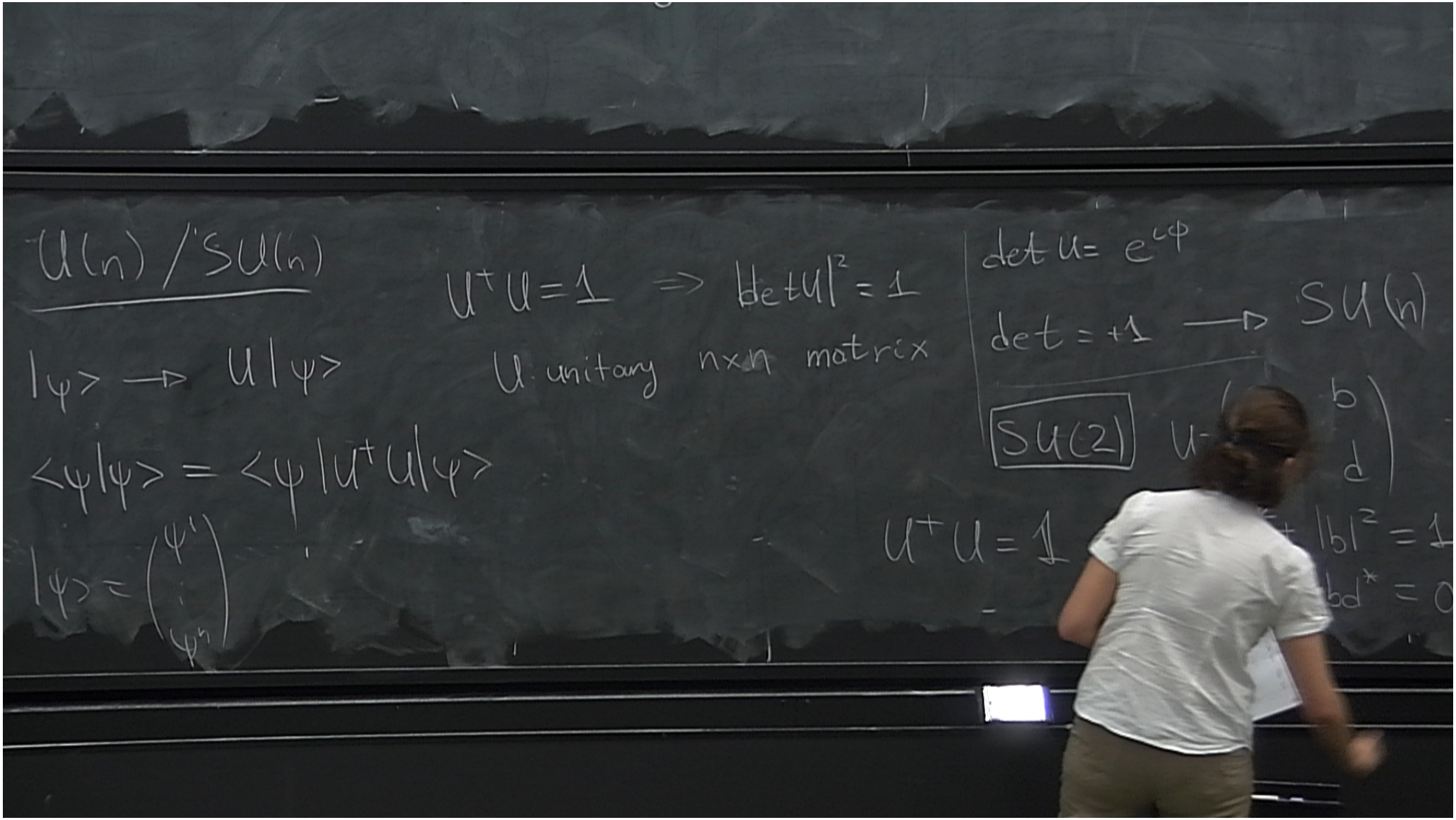
↳ 3×3 matrix

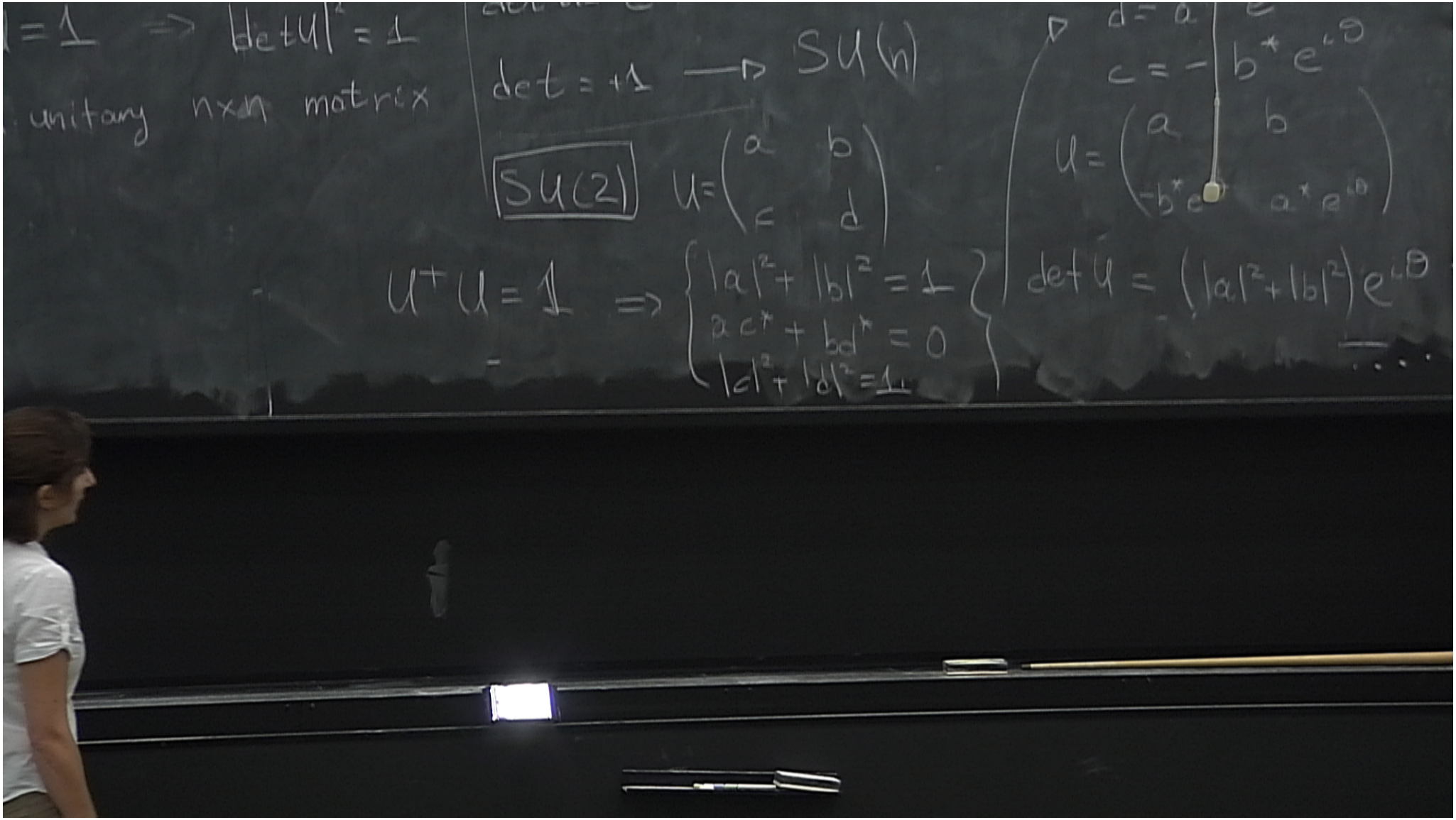
$R^T R = 1$ orthogonal → $O(3)$

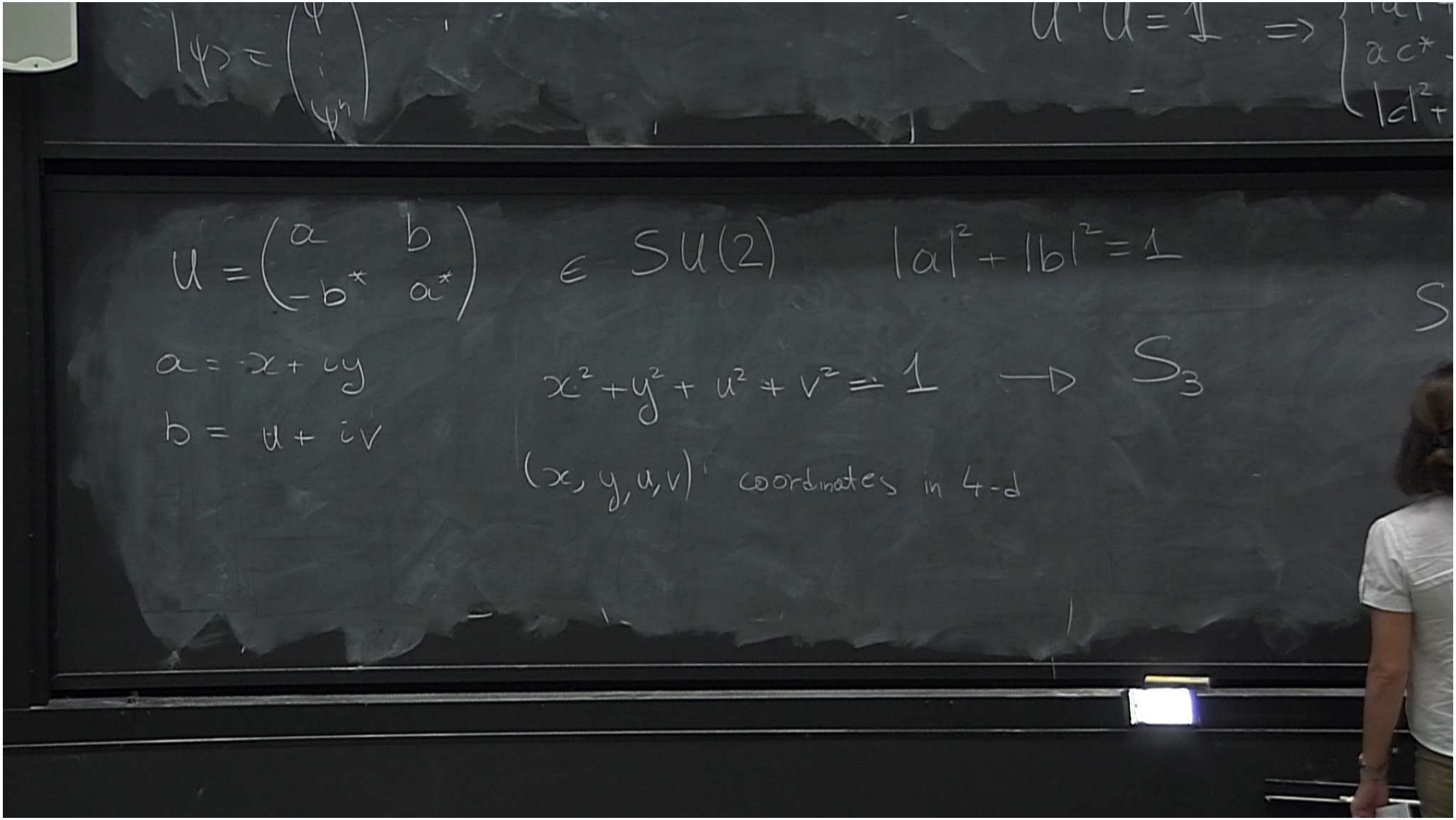
$(\det R)^2 = 1$

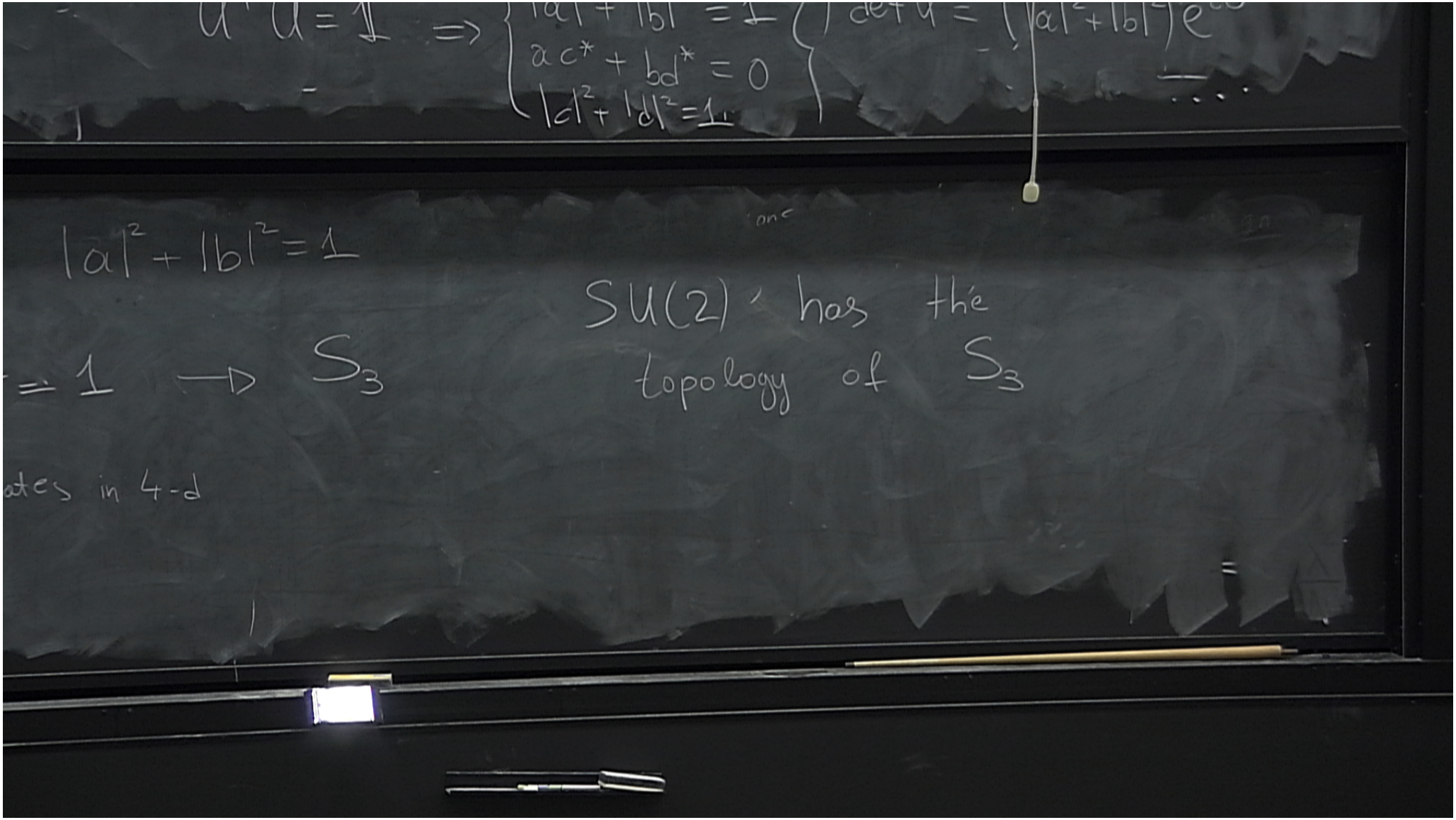
$\det = +1$ → $SO(3)$

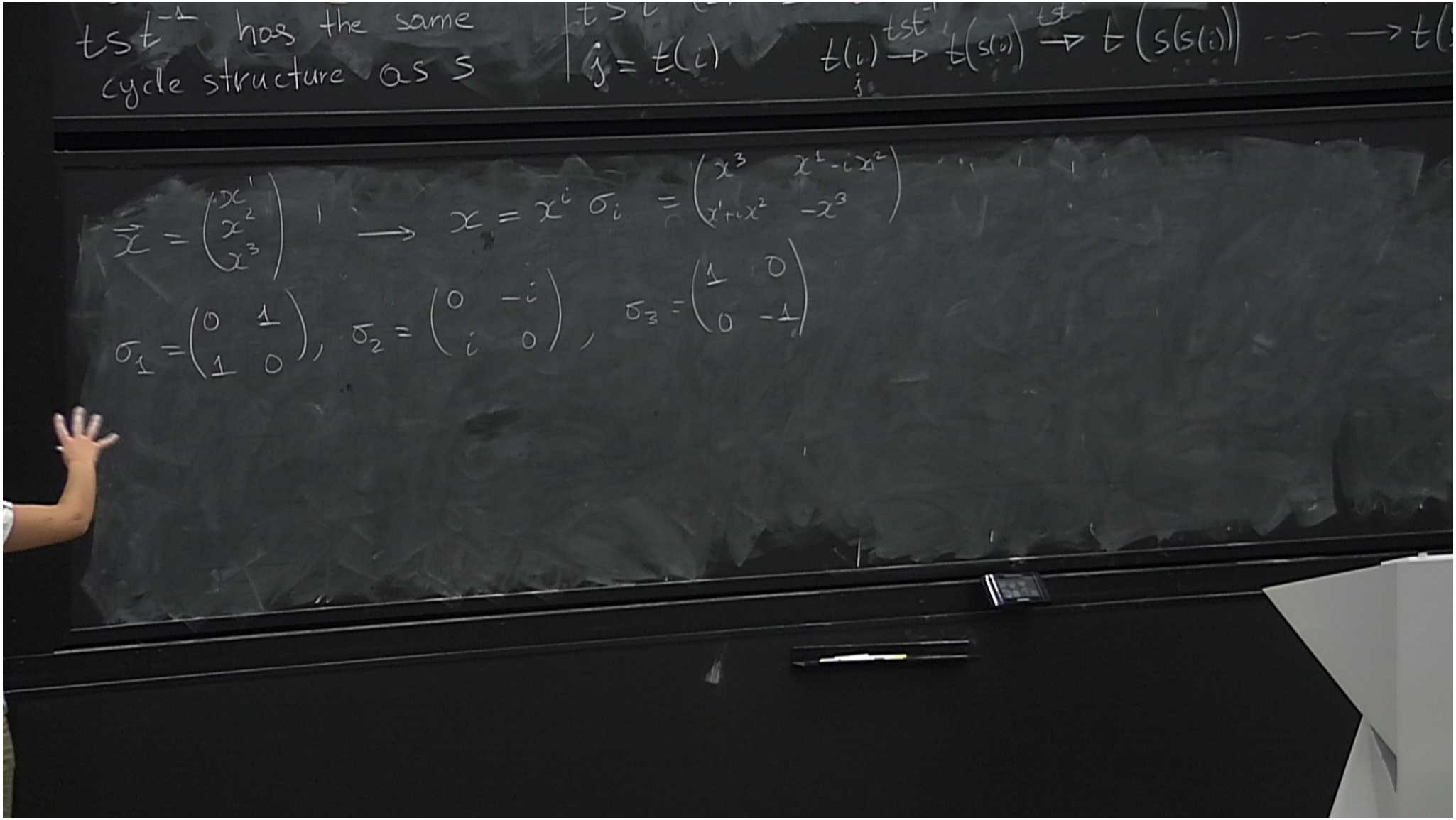
$\Rightarrow \det R = \pm 1$

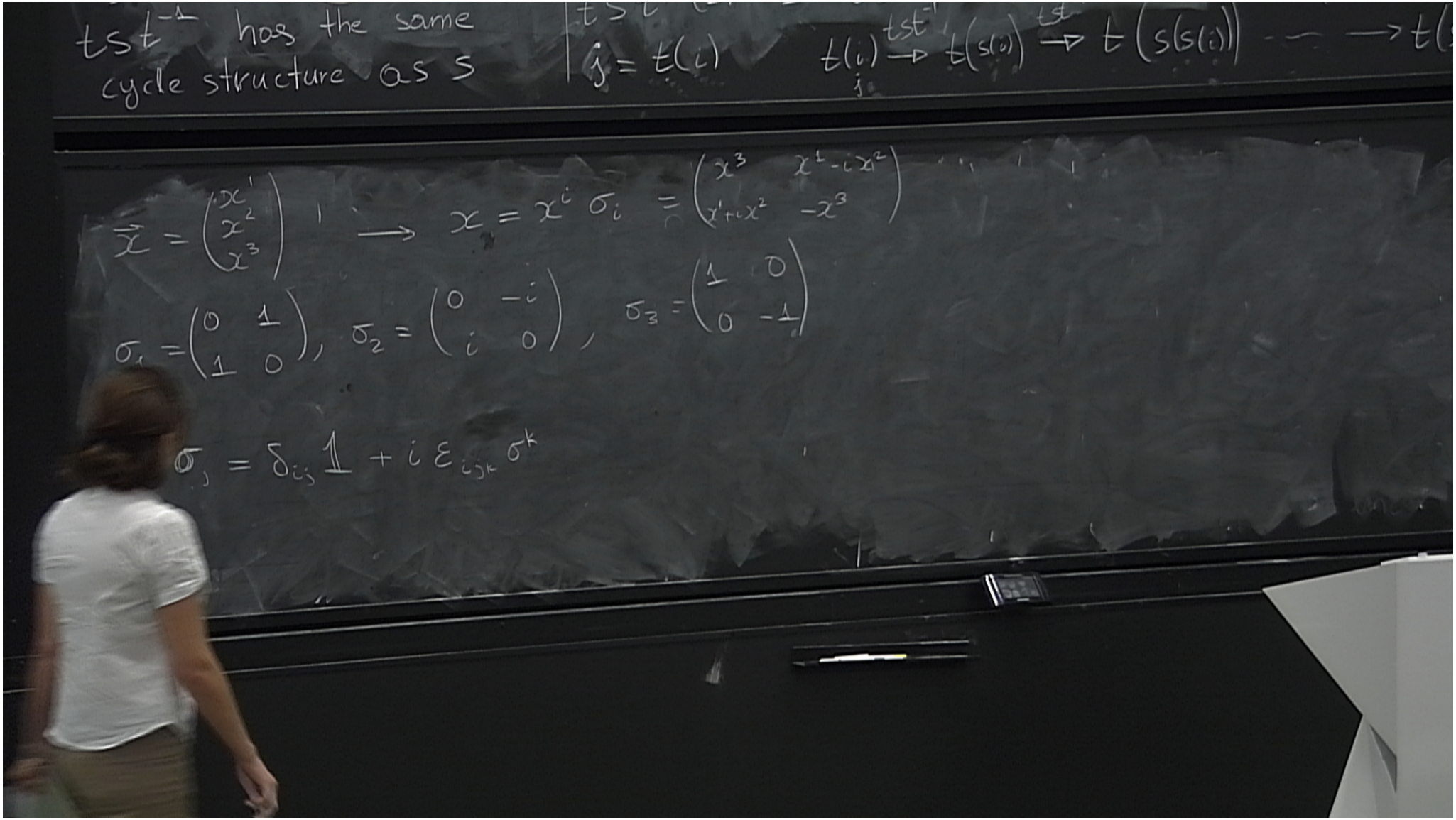

U(n)/SU(n)
$U^\dagger U = 1 \Rightarrow |\det U|^2 = 1$
U unitary n×n matrix
$\det U = e^{i\varphi}$
$\det = +1 \longrightarrow SU(n)$
$|\psi\rangle \longrightarrow U|\psi\rangle$
$\langle\psi|\psi\rangle = \langle\psi|U^\dagger U|\psi\rangle$
SU(2)
$U^\dagger U = 1$
$|\psi\rangle = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^n \end{pmatrix}$

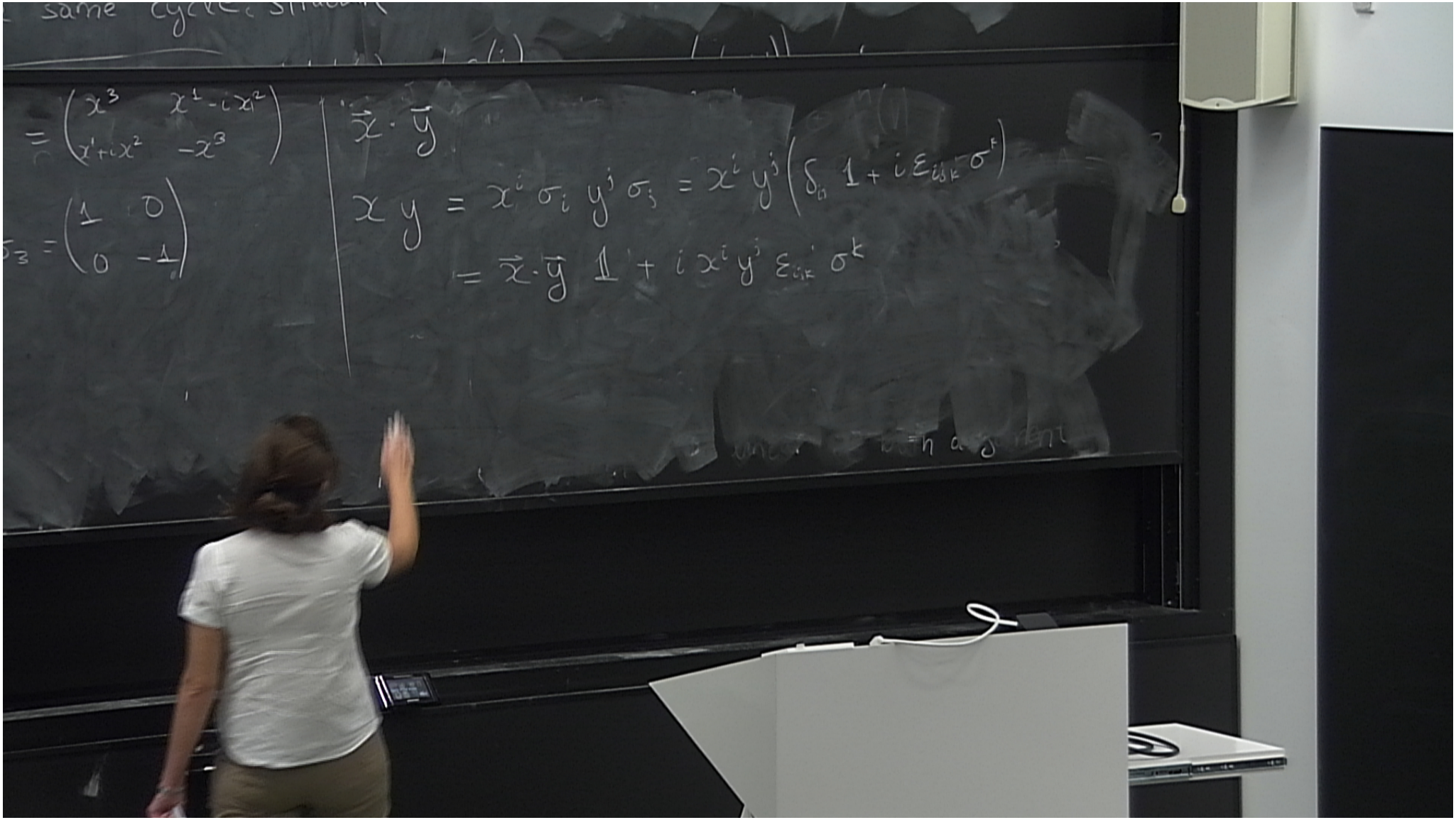

$|=1 \Rightarrow |\det U|^2 = 1$
unitary n×n matrix
$\det = +1 \longrightarrow SU(n)$
$SU(2)$
$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$U^\dagger U = 1 \Rightarrow \begin{cases} |a|^2 + |b|^2 = 1 \\ a c^* + b d^* = 0 \\ |c|^2 + |d|^2 = 1 \end{cases}$
$d = a^*$
$c = -b^* e^{i\theta}$
$U = \begin{pmatrix} a & b \\ -b^* e^{i\theta} & a^* e^{i\theta} \end{pmatrix}$
$\det U = (|a|^2 + |b|^2) e^{i\theta}$

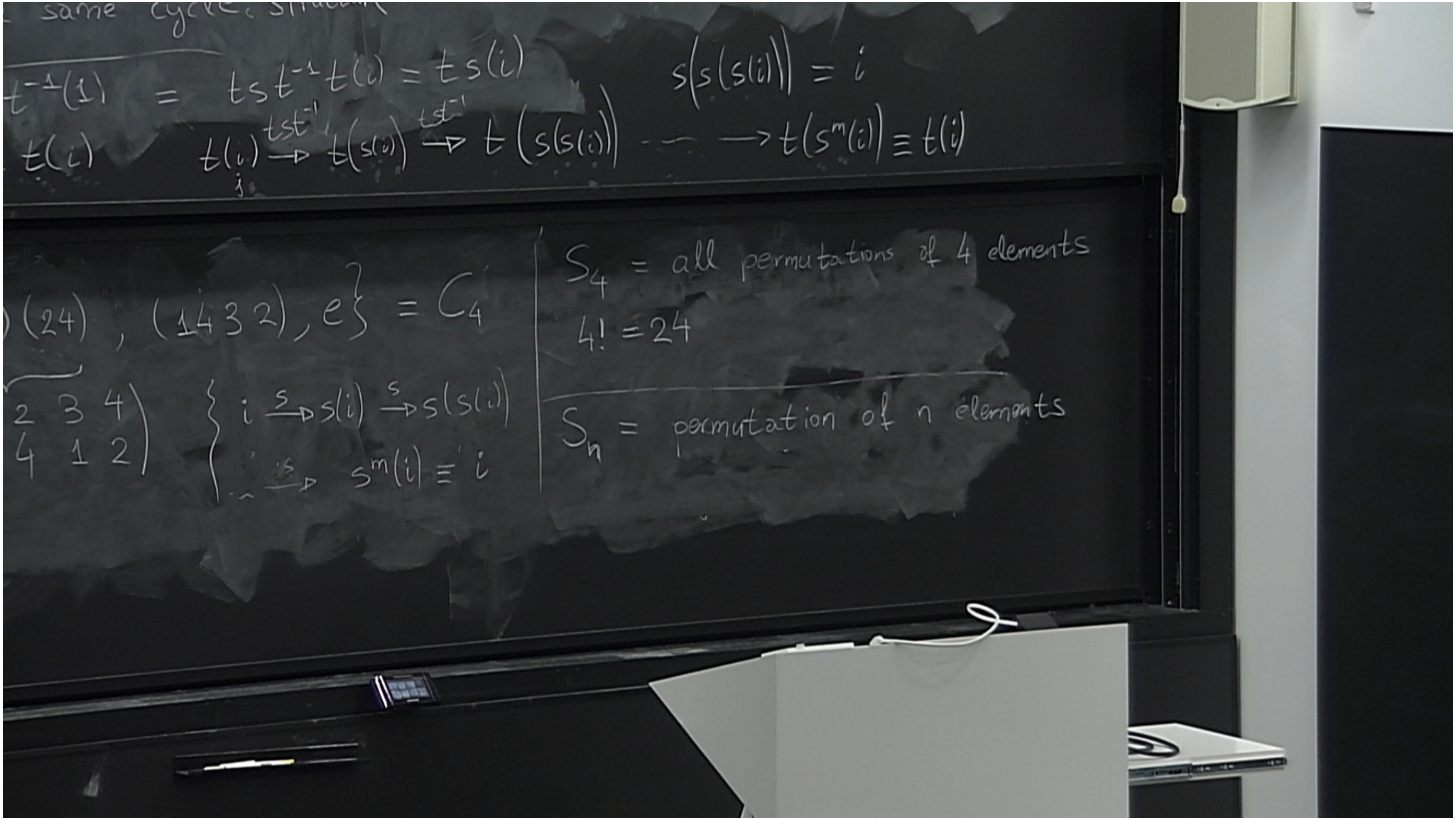

$|\psi\rangle = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^n \end{pmatrix}$
$U^\dagger U = 1 \Rightarrow$
$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2)$
$|a|^2 + |b|^2 = 1$
$a = x + iy$
$b = u + iv$
$x^2 + y^2 + u^2 + v^2 = 1 \longrightarrow S_3$
(x, y, u, v) coordinates in 4-d

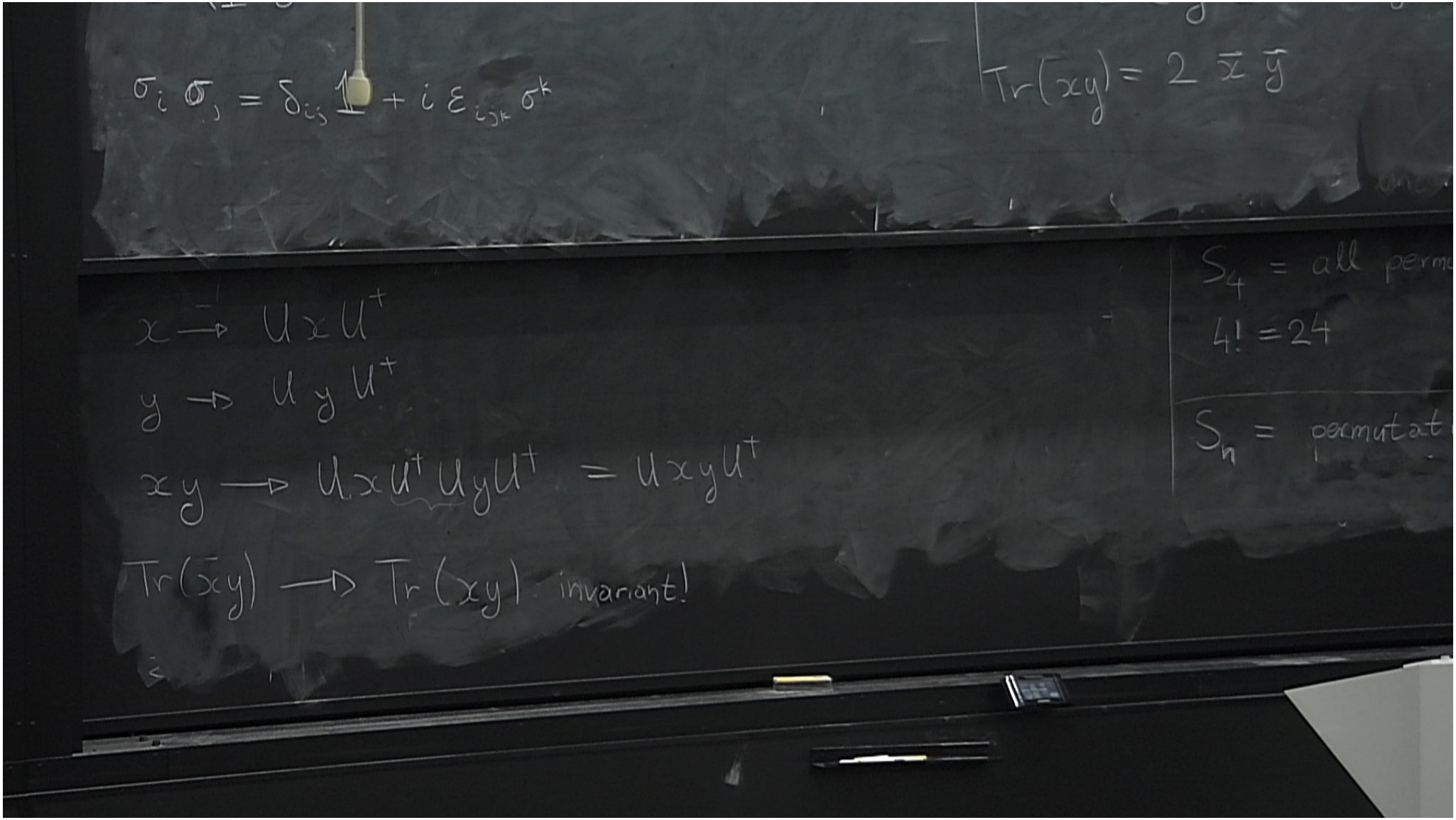

$U^\dagger U = \mathbb{1} \Rightarrow$
$|a|^2 + |b|^2 = 1$
$ac^* + bd^* = 0$
$|c|^2 + |d|^2 = 1$
$\det U = 1$
$|a|^2 + |b|^2 = 1$
one
$SU(2)$ has the topology of S_3
$= 1 \rightarrow S_3$
ates in 4-d

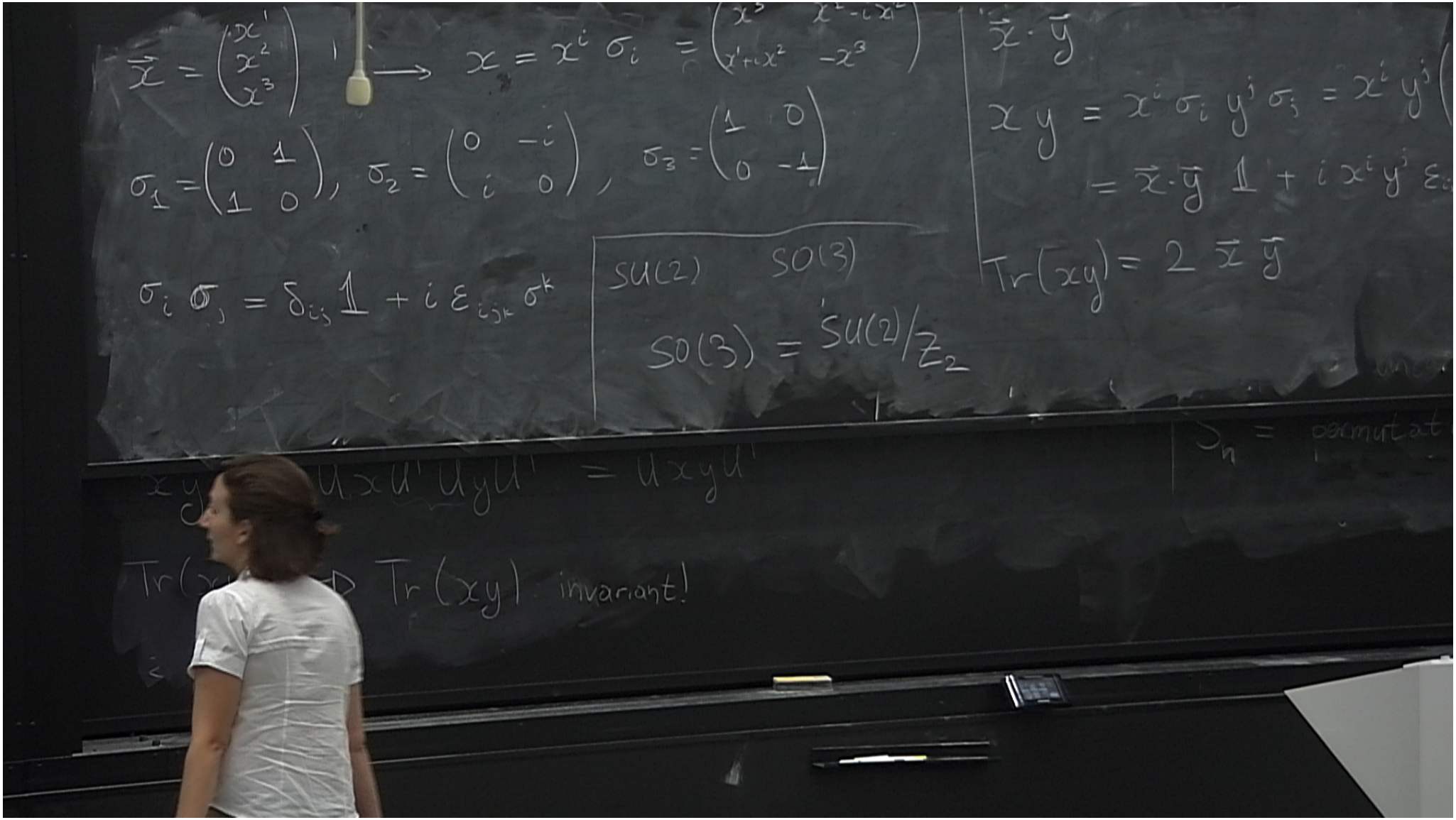

tst^{-1} has the same cycle structure as s
$j = t(i)$
$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{ts} t(s(s(i)))$
$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow x = x^i \sigma_i = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$


tst^{-1} has the same cycle structure as s
$j = t(i)$
$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{ts} t(s(s(i)))$
$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow x = x^i \sigma_i = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\sigma_j = \delta_{ij} 1 + i \varepsilon_{ijk} \sigma^k$


same cycle
$= \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$
$\vec{x} \cdot \vec{y}$
$x\,y = x^i \sigma_i\, y^j \sigma_j = x^i y^j \left(\delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma^k \right)$
$= \vec{x} \cdot \vec{y}\, \mathbb{1} + i\, x^i y^j \varepsilon_{ijk}\, \sigma^k$


same cycle: Student
$t^{-1}(1) = tst^{-1}t(i) = ts(i)$
$t(i)$
$t(i) \xrightarrow{tst^{-1}} t(s(i)) \xrightarrow{tst^{-1}} t(s(s(i)))$
$s(s(s(i))) = i$
$\longrightarrow t(s^m(i)) \equiv t(i)$
$(24), (1432), e\} = C_4$
$\begin{pmatrix} 2 & 3 & 4 \\ 4 & 1 & 2 \end{pmatrix}$
$\{ i \xrightarrow{s} s(i) \xrightarrow{s} s(s(i)) \xrightarrow{s} s^m(i) \equiv i$
S_4 = all permutations of 4 elements
$4! = 24$
S_n = permutation of n elements


$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma^k$
$Tr(xy) = 2 \vec{x} \vec{y}$
$x \to U x U^\dagger$
$y \to U y U^\dagger$
$xy \to U x U^\dagger U y U^\dagger = U x y U^\dagger$
$Tr(\vec{x}\vec{y}) \to Tr(\vec{x}\vec{y})$ invariant!
S_4 = all perm
$4! = 24$
S_n = permutat


$\vec{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow x = x^i \sigma_i = \begin{pmatrix} x^3 & x^1 - i x^2 \\ x^1 + i x^2 & -x^3 \end{pmatrix}$
$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \varepsilon_{ijk} \sigma^k$
SU(2) SO(3)
$SO(3) = SU(2)/Z_2$
$\vec{x} \cdot \vec{y}$
$xy = x^i \sigma_i y^j \sigma_j = x^i y^j$
$= \vec{x} \cdot \vec{y} \, \mathbb{1} + i x^i y^j \varepsilon$
$Tr(xy) = 2 \vec{x} \vec{y}$
$xy \to UxU^\dagger UyU^\dagger = UxyU^\dagger$
$Tr(xy) \to Tr(xy)$ invariant!
S_n = permutat

$U\,U^\dagger = \mathbb{1} \Rightarrow \begin{cases} ac^* + bd^* = 0 \\ |c|^2 + |d|^2 = 1 \end{cases}$

$|a|^2 + |b|^2 = 1$

$\mathbb{1} \to S_3$

$SU(2)$ has the topology of S_3

in 4-d

$SO(3)$ topology of S_3/Z_2

