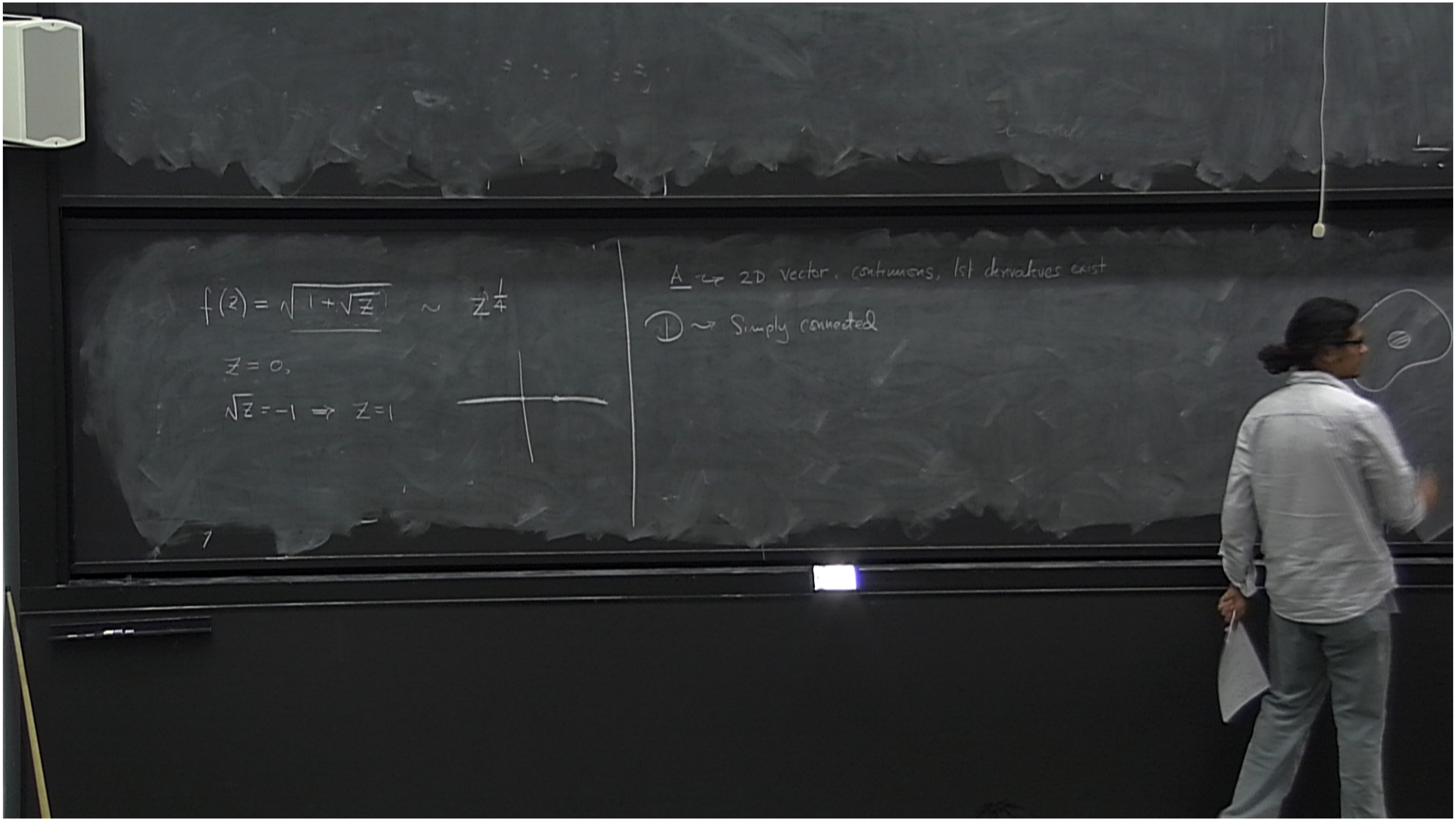
Title: Complex Analysis - Lecture 3

Date: Aug 18, 2011 09:00 AM

URL: http://pirsa.pi.local/11080083

Abstract:


f(z) = √(1 + √z) ~ z^(1/4)
z = 0,
√z = −1 ⇒ z = 1
A ⇝ 2D vector, continuous, 1st derivatives exist
D ⇝ Simply connected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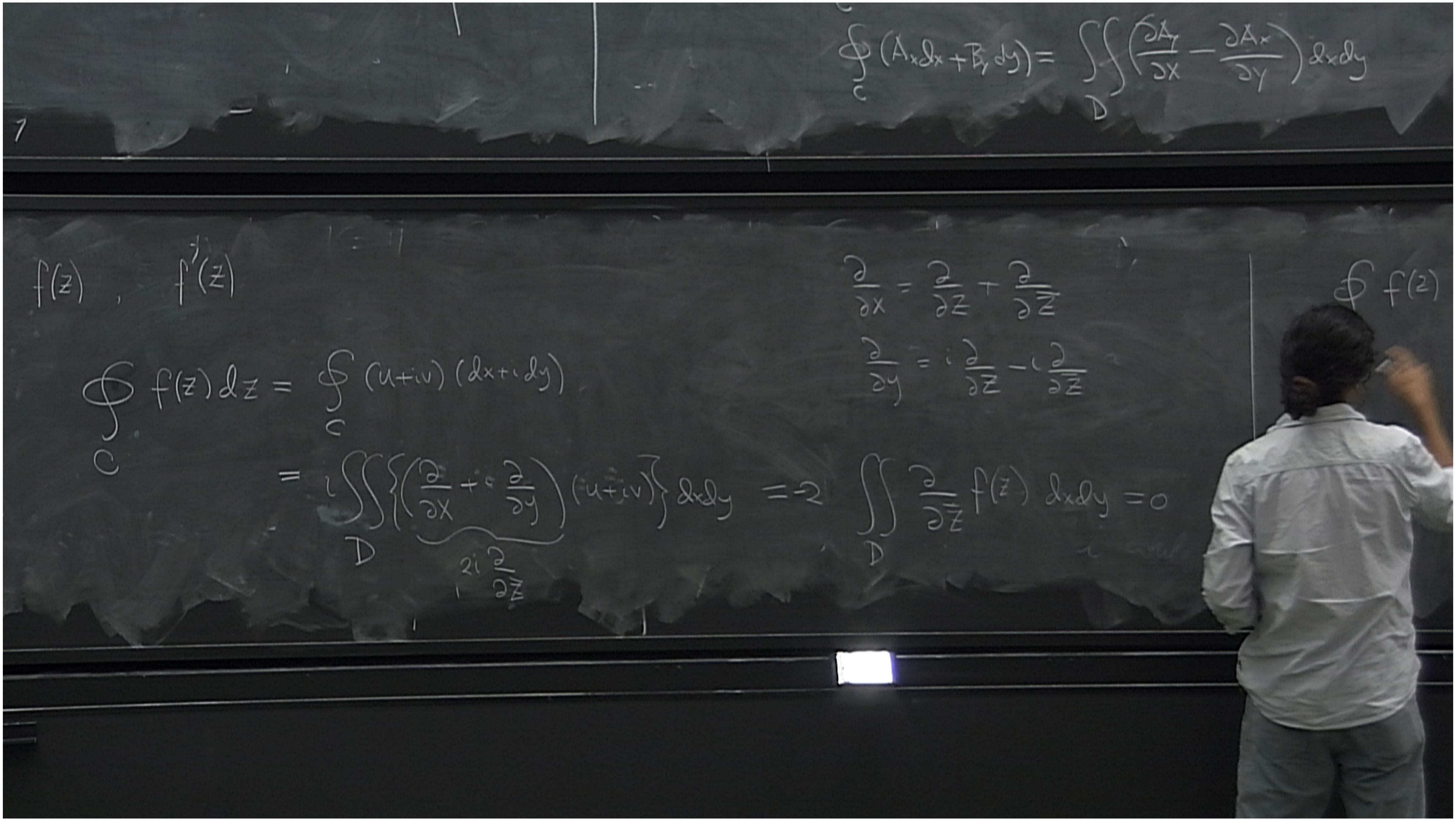

∮_C (A_x dx + B_y dy) = ∬_D (∂A_y/∂x − ∂A_x/∂y) dx dy
f(z) , f′(z)
∂/∂x = ∂/∂z + ∂/∂z̄
∂/∂y = i ∂/∂z − i ∂/∂z̄
∮_C f(z) dz = ∮_C (u+iv)(dx+i dy)
= i ∬_D {(∂/∂x + i ∂/∂y)(u+iv)} dx dy = 2 ∬_D ∂/∂z̄ f(z) dx dy = 0
2i ∂/∂z̄
∮ f(z)

$$\oint_C (A_x dx + B_y dy) = \iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

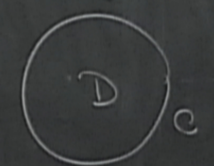
$$\frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \bar{z}}$$

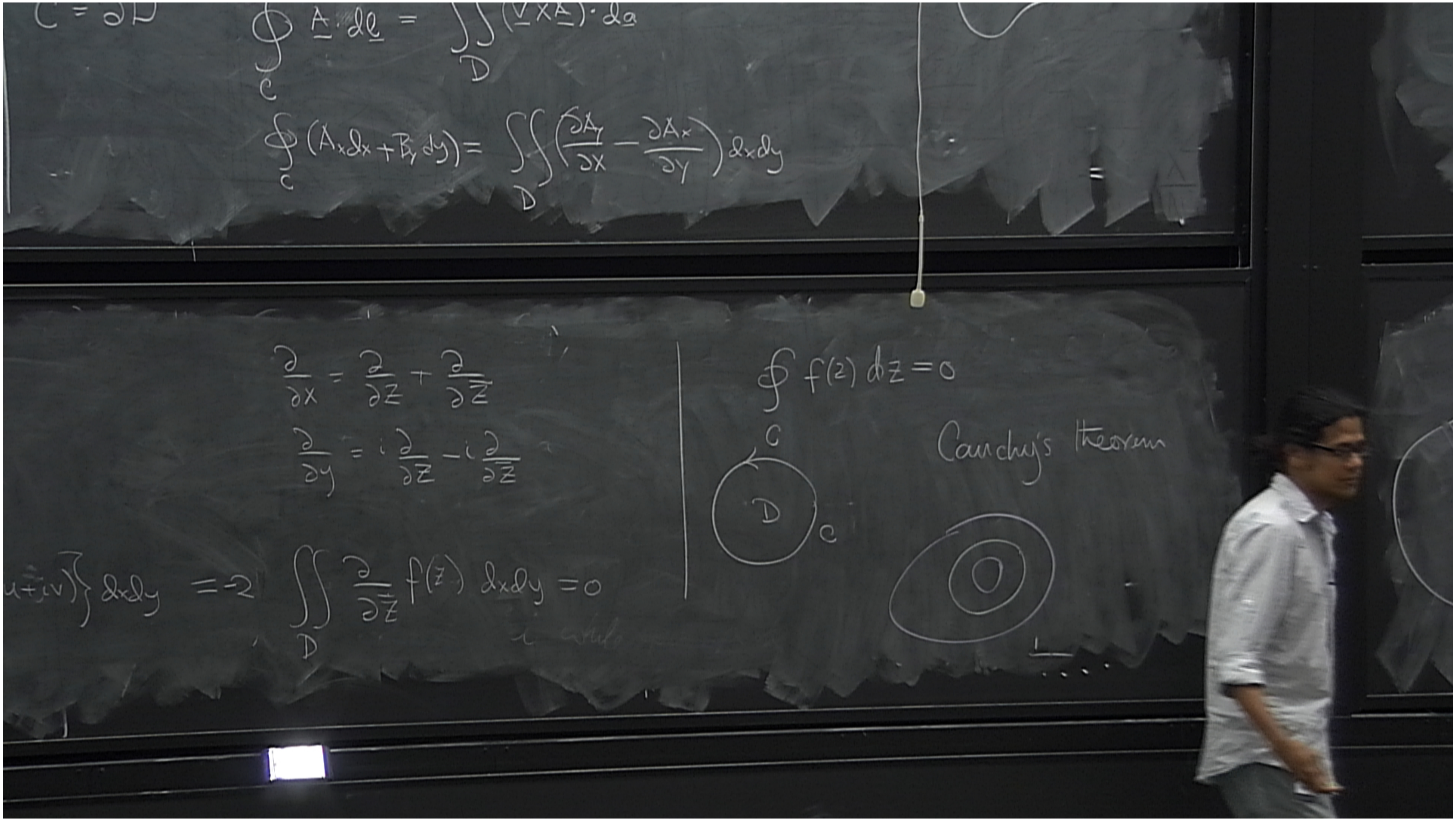
$$\ldots iv)\,(dx + i\,dy)$$

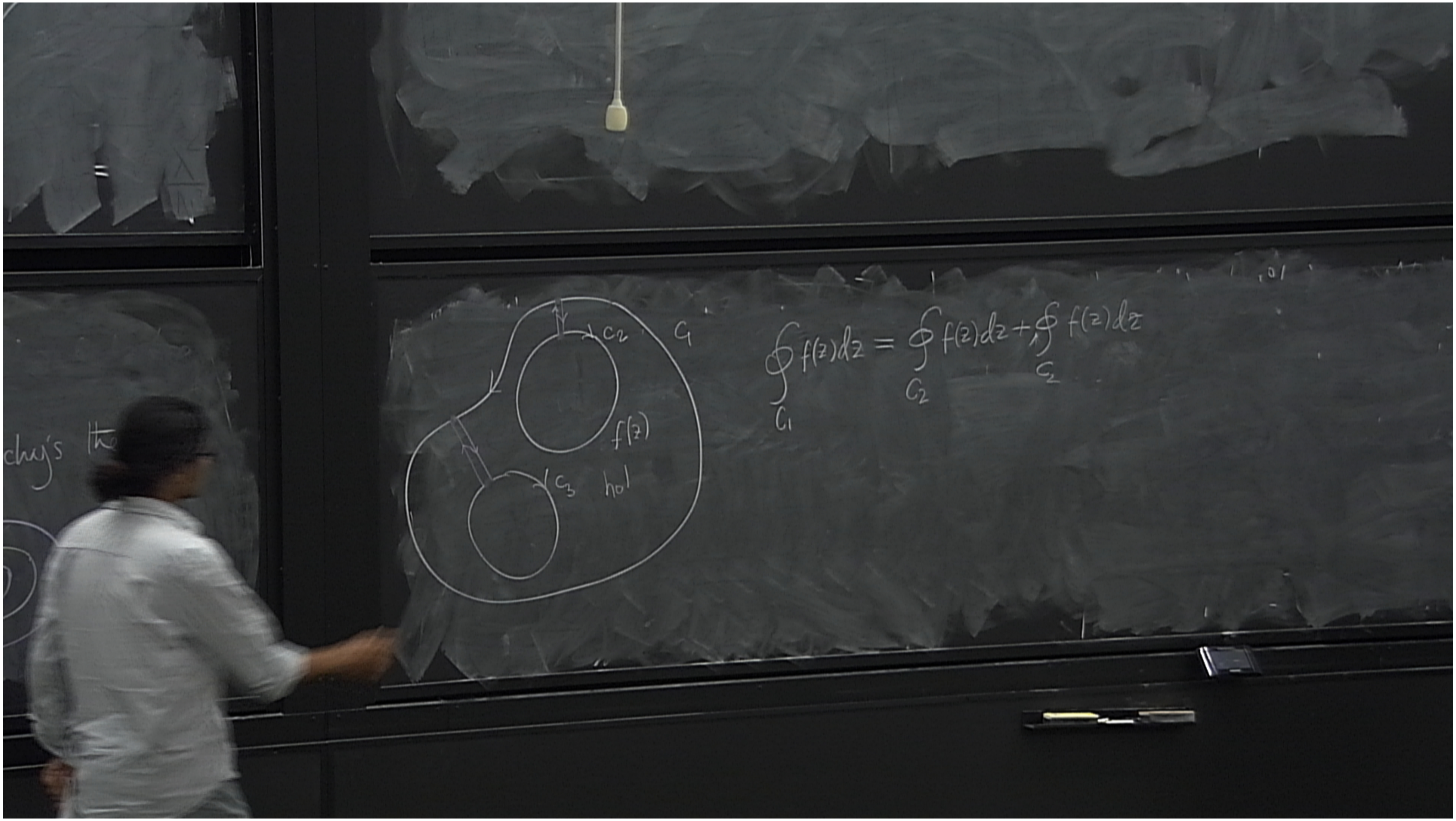
$$\ldots \underbrace{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)}_{2i\frac{\partial}{\partial \bar{z}}}(u+iv)\Big\}\,dx\,dy = 2\iint_D \frac{\partial}{\partial \bar{z}} f(z)\,dx\,dy = 0$$

$$\oint_C f(z)\,dz = 0$$

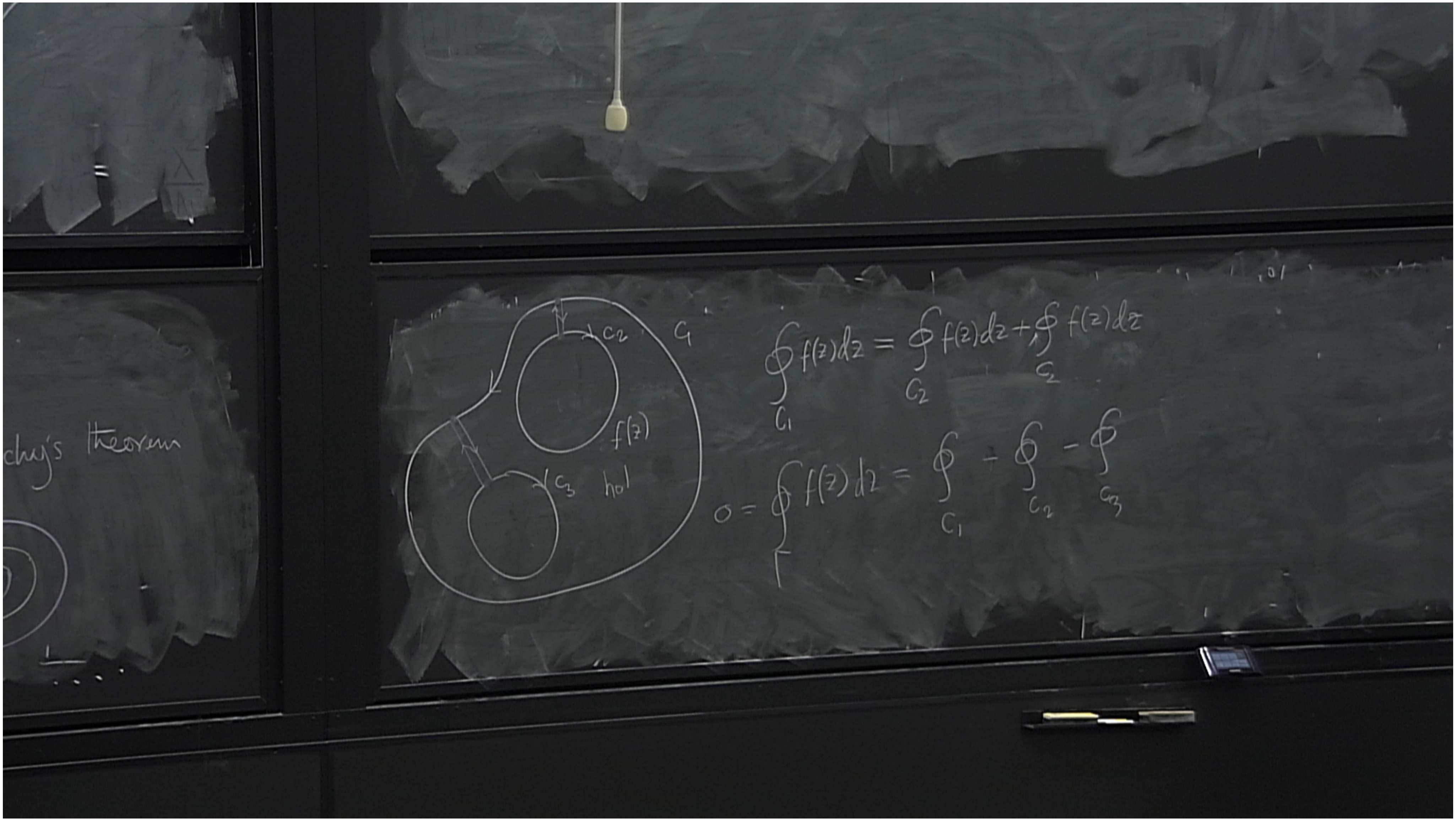
Cauchy's theorem

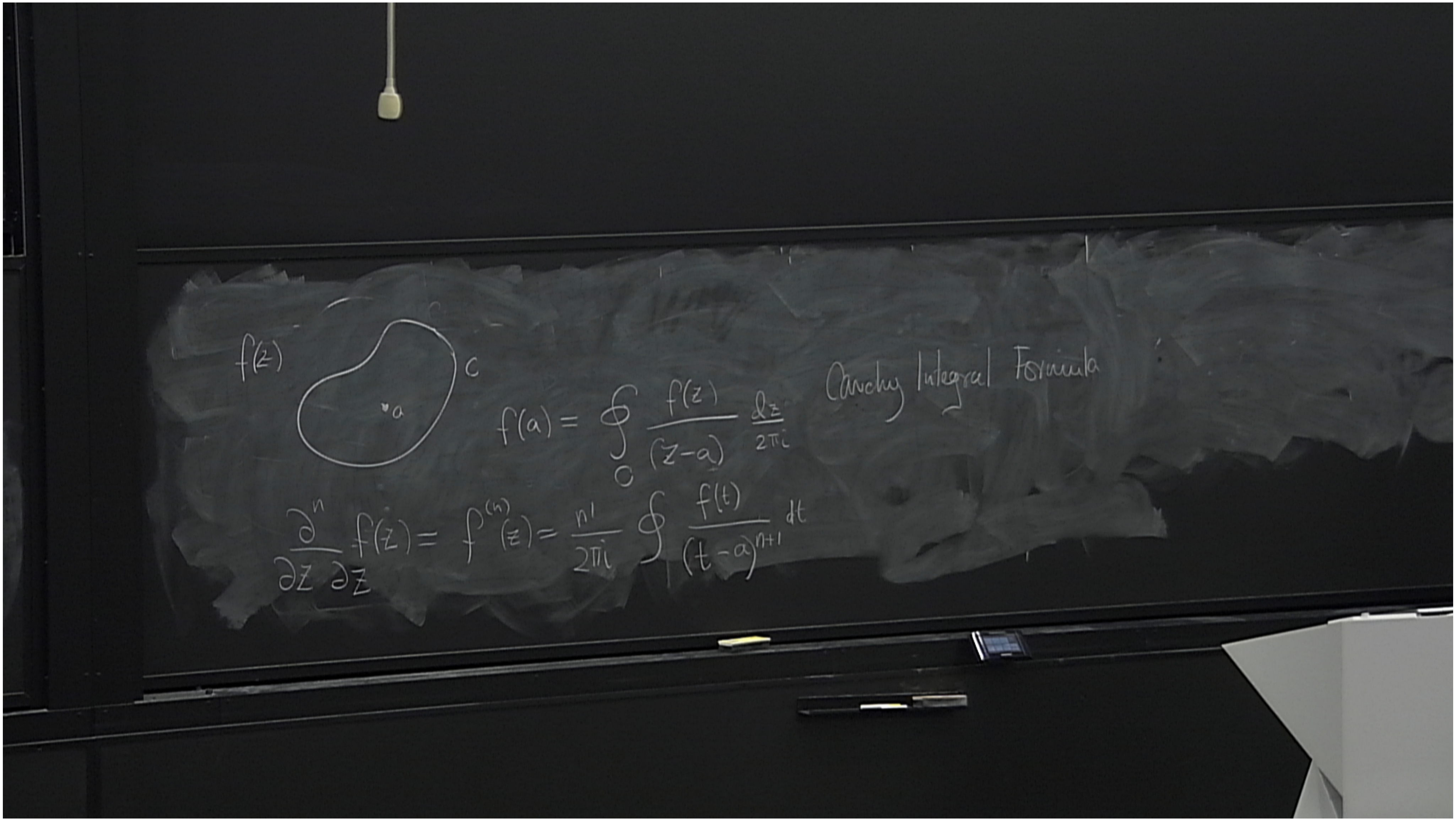


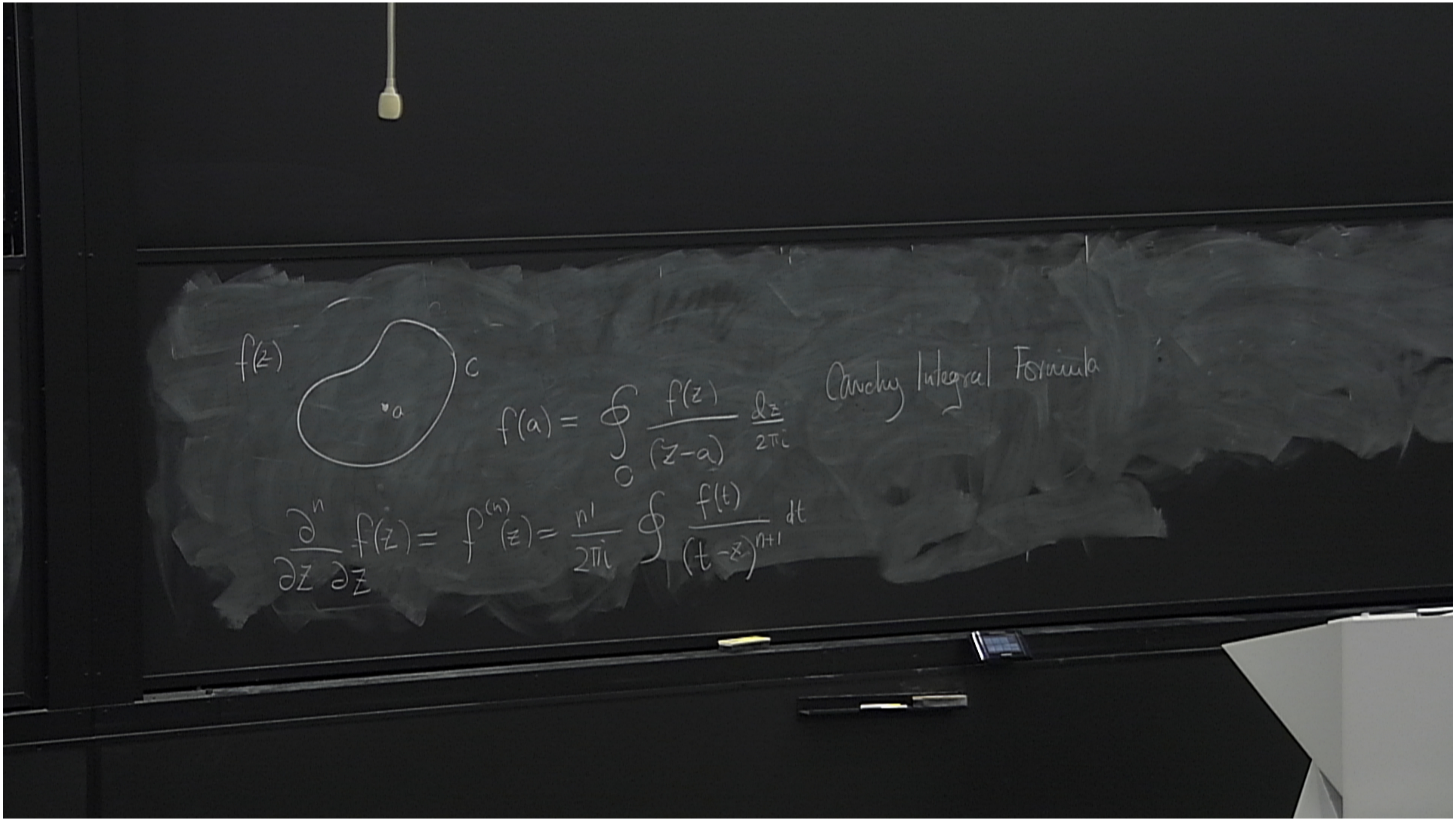

Cauchy's theorem
∮ f(z) dz = 0
D
C

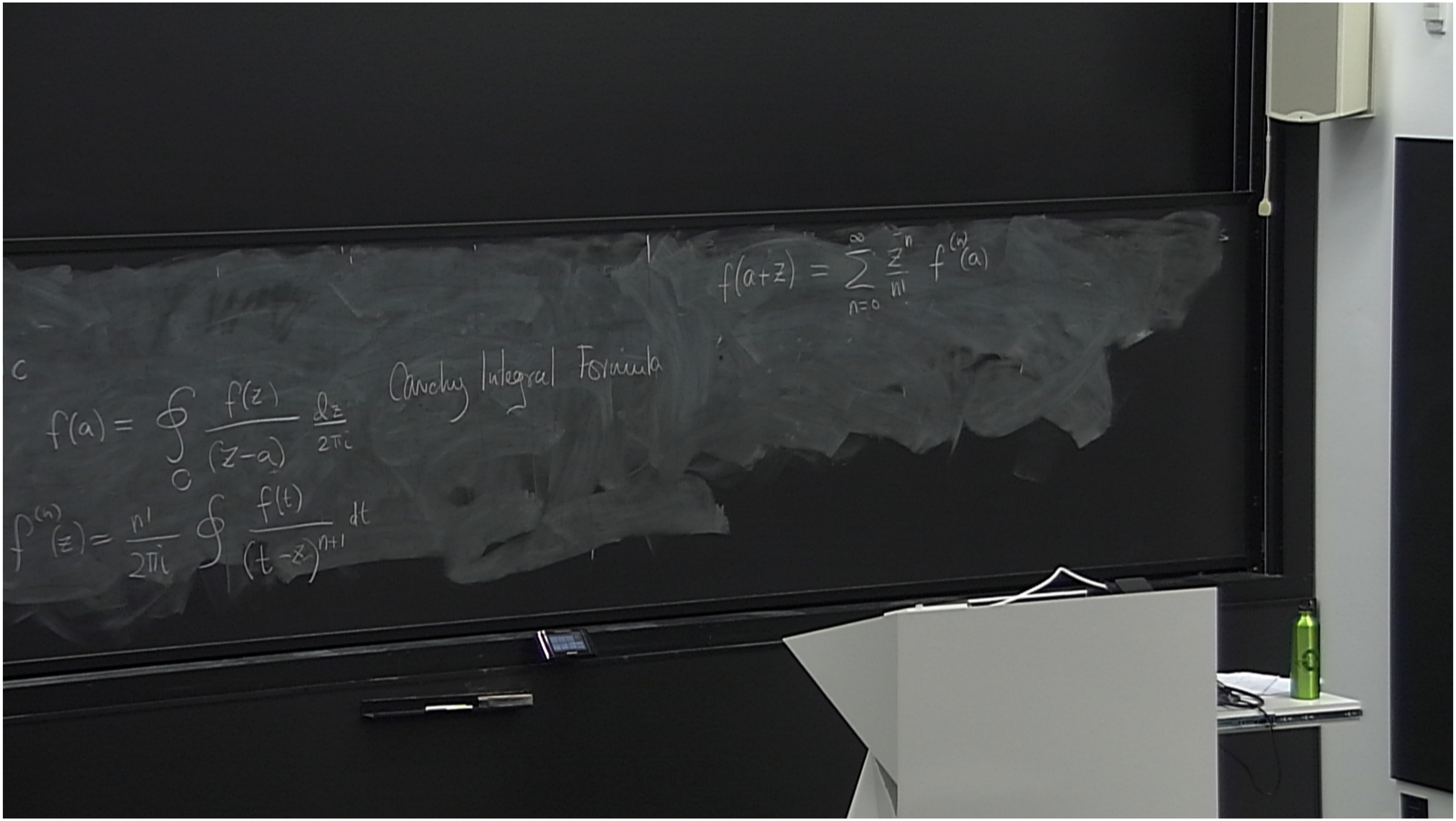


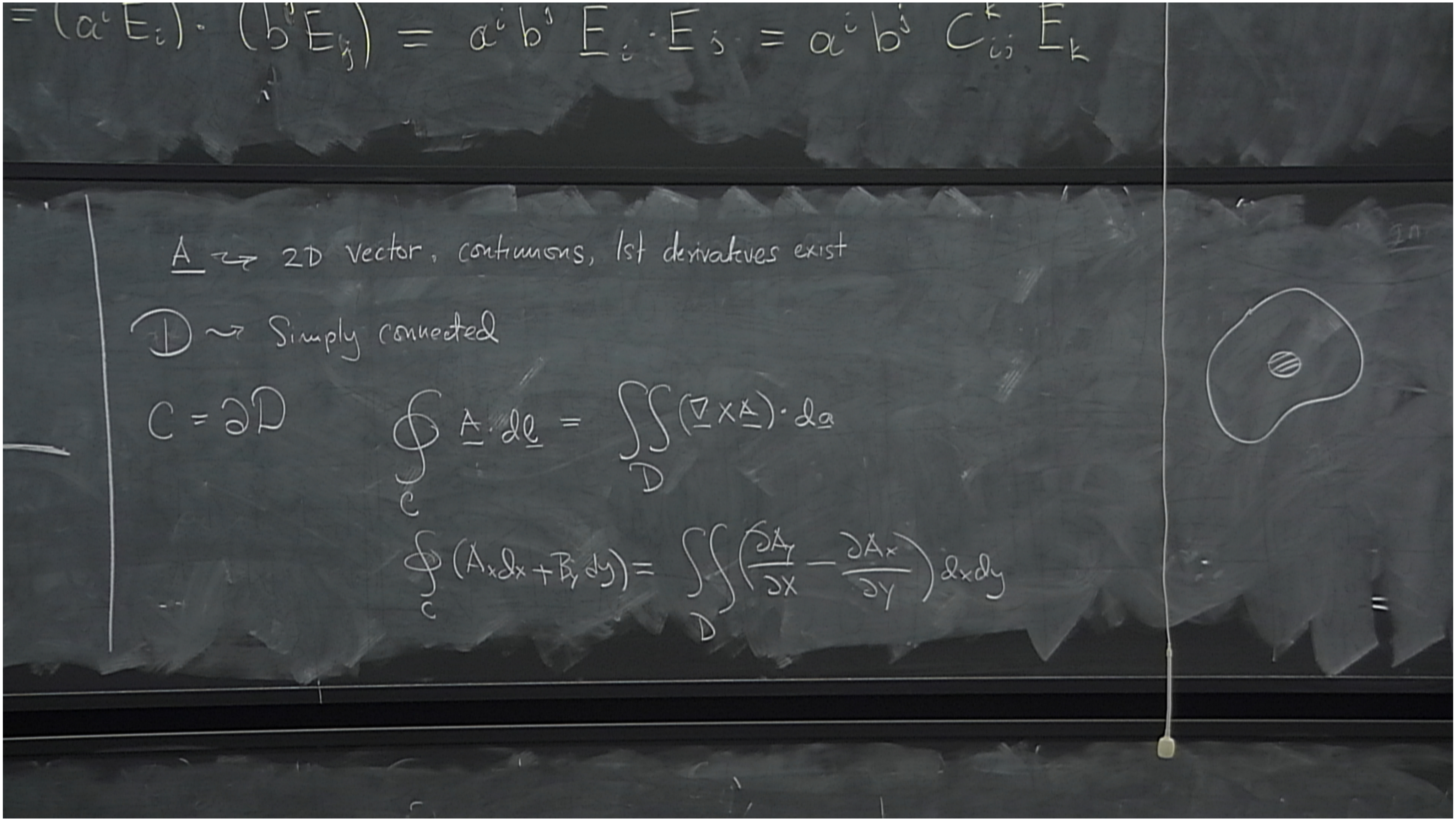
$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$
C_1
C_2
C_3
f(z)
hol

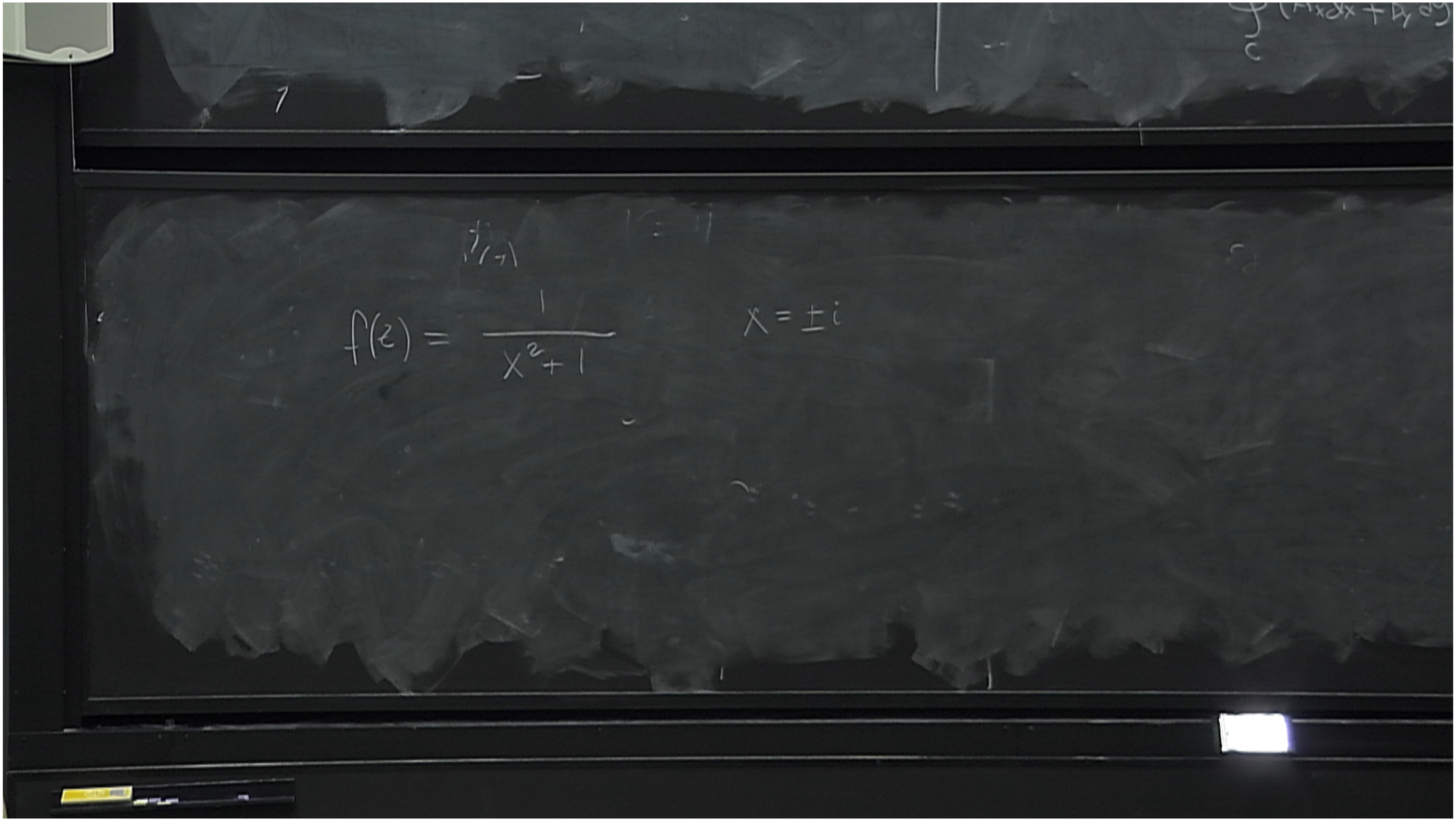

chy's theorem
C2
C1
f(z)
C3
hol
∮C1 f(z)dz = ∮C2 f(z)dz + ∮C2 f(z)dz
0 = ∮ f(z)dz = ∮C1 − ∮C2 − ∮C3

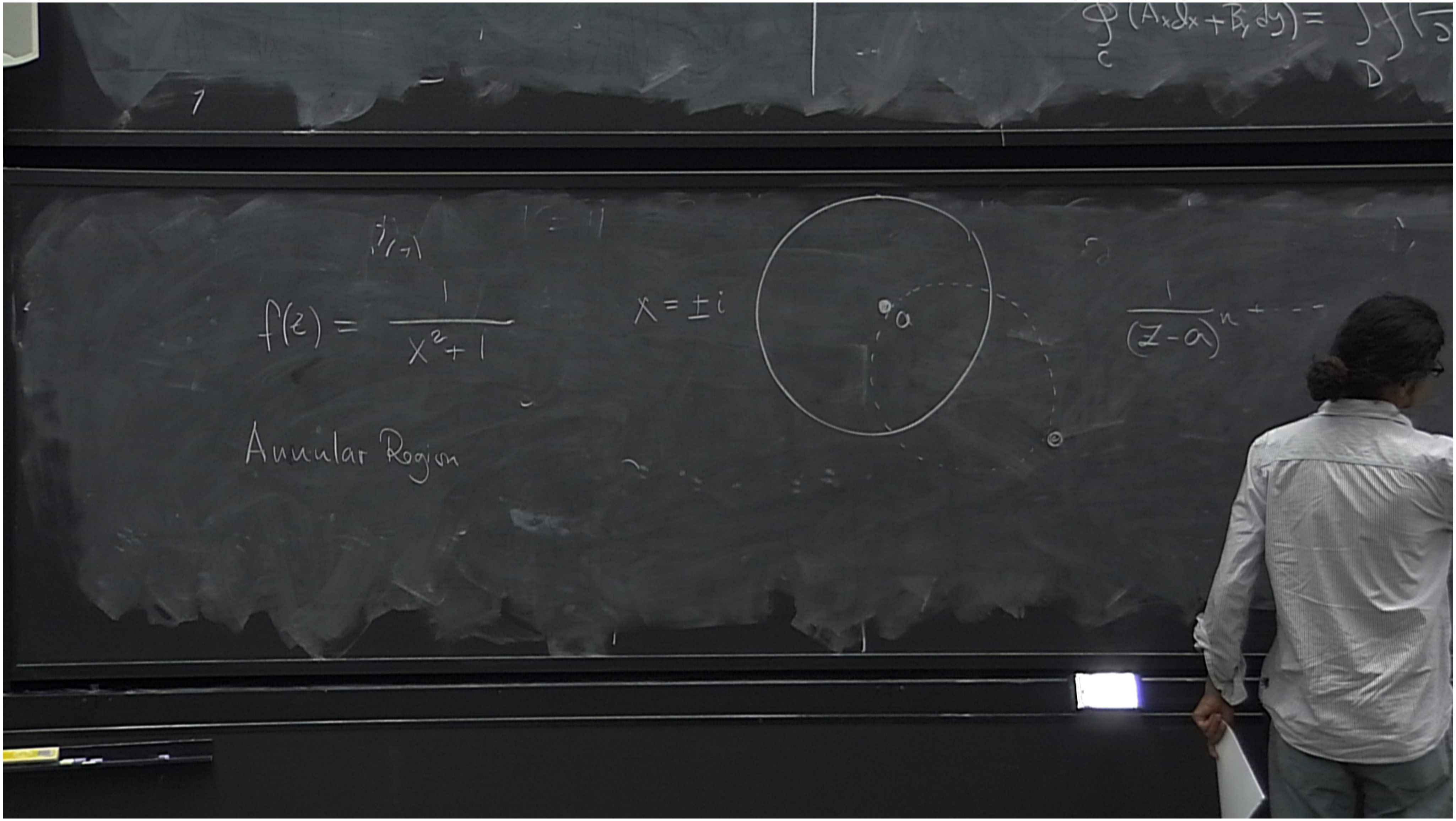

f(z)
C
a
$f(a) = \oint_C \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$
Cauchy Integral Formula
$\frac{\partial^n}{\partial z \, \partial z} f(z) = f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(t)}{(t-a)^{n+1}} dt$

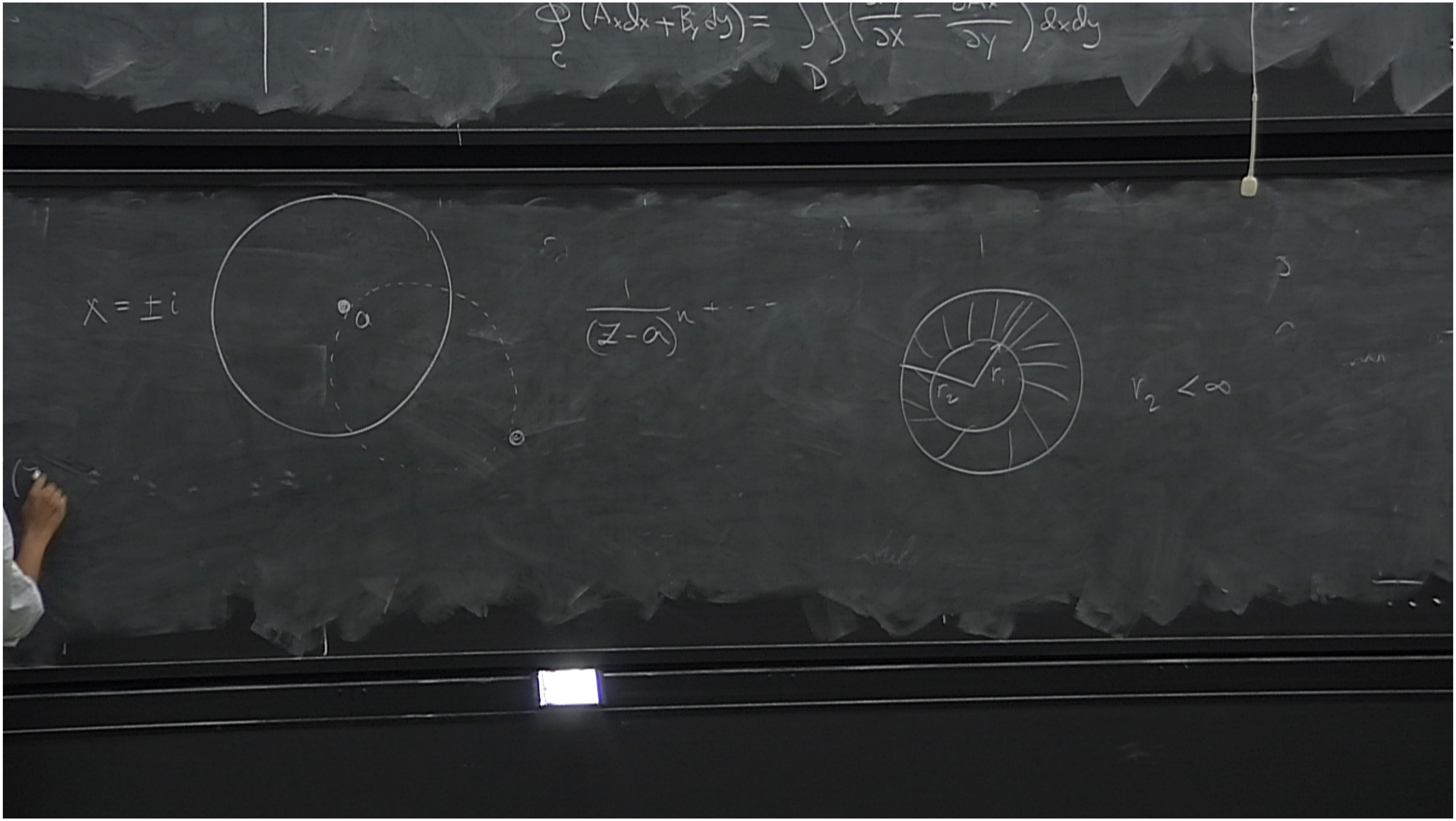

f(z)
C
a
f(a) = ∮_C f(z)/(z−a) dz/2πi
Cauchy Integral Formula
∂^n/∂z…∂z f(z) = f^(n)(z) = n!/2πi ∮ f(t)/(t−z)^(n+1) dt


$f(a+z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} f^{(n)}(a)$
C
Cauchy Integral Formula
$f(a) = \oint_C \frac{f(z)}{(z-a)} \frac{dz}{2\pi i}$
$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(t)}{(t-z)^{n+1}} dt$

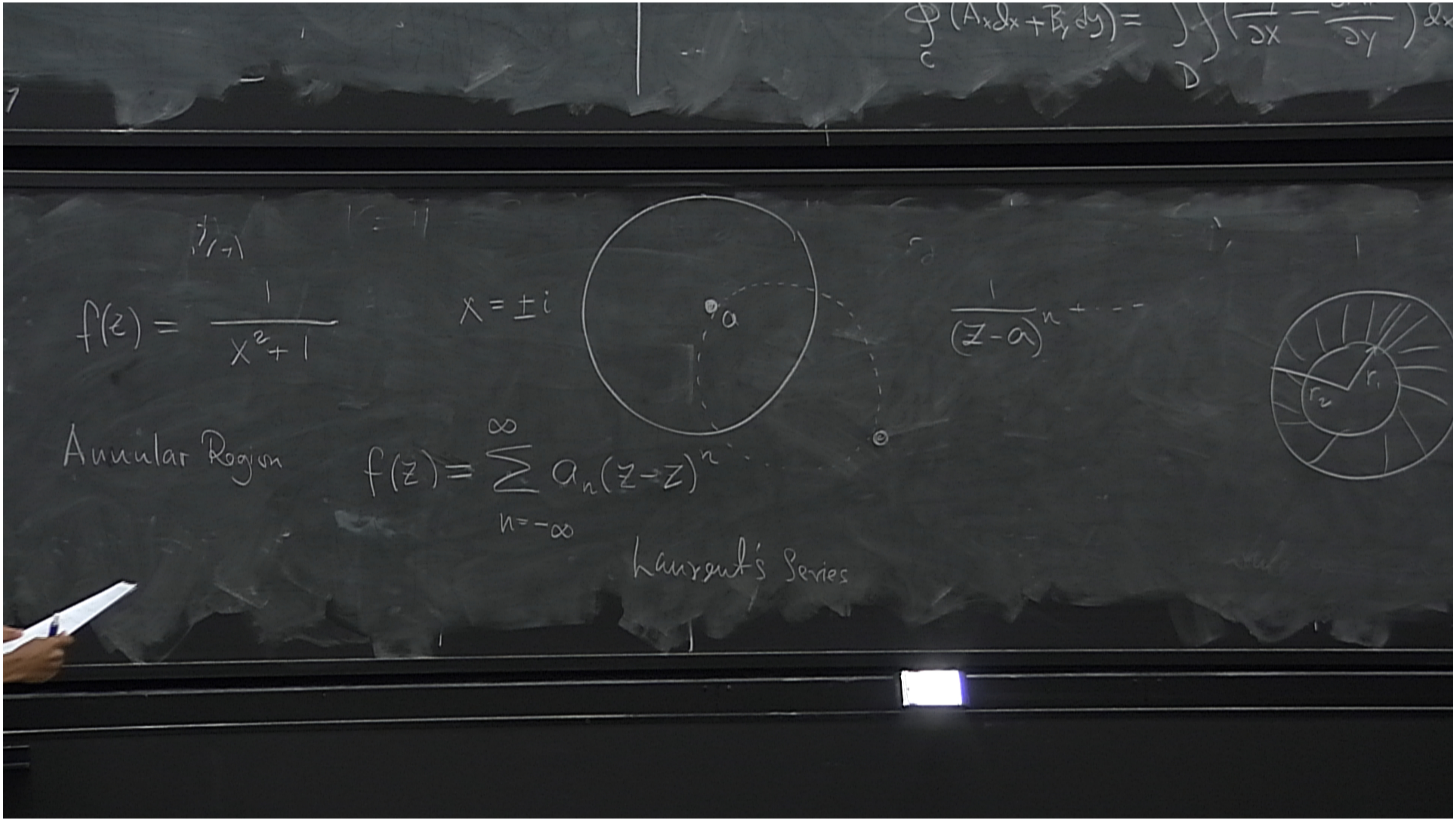

$= (a^i E_i) \cdot (b^j E_j) = a^i b^j \underline{E}_i \cdot \underline{E}_j = a^i b^j C^k_{ij} \underline{E}_k$
$\underline{A}$ ⇝ 2D vector, continuous, 1st derivatives exist
D ⇝ Simply connected
$C = \partial D$
$\oint_C \underline{A} \cdot d\underline{l} = \iint_D (\underline{\nabla} \times \underline{A}) \cdot d\underline{a}$
$\oint_C (A_x dx + A_y dy) = \iint_D \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx\, dy$

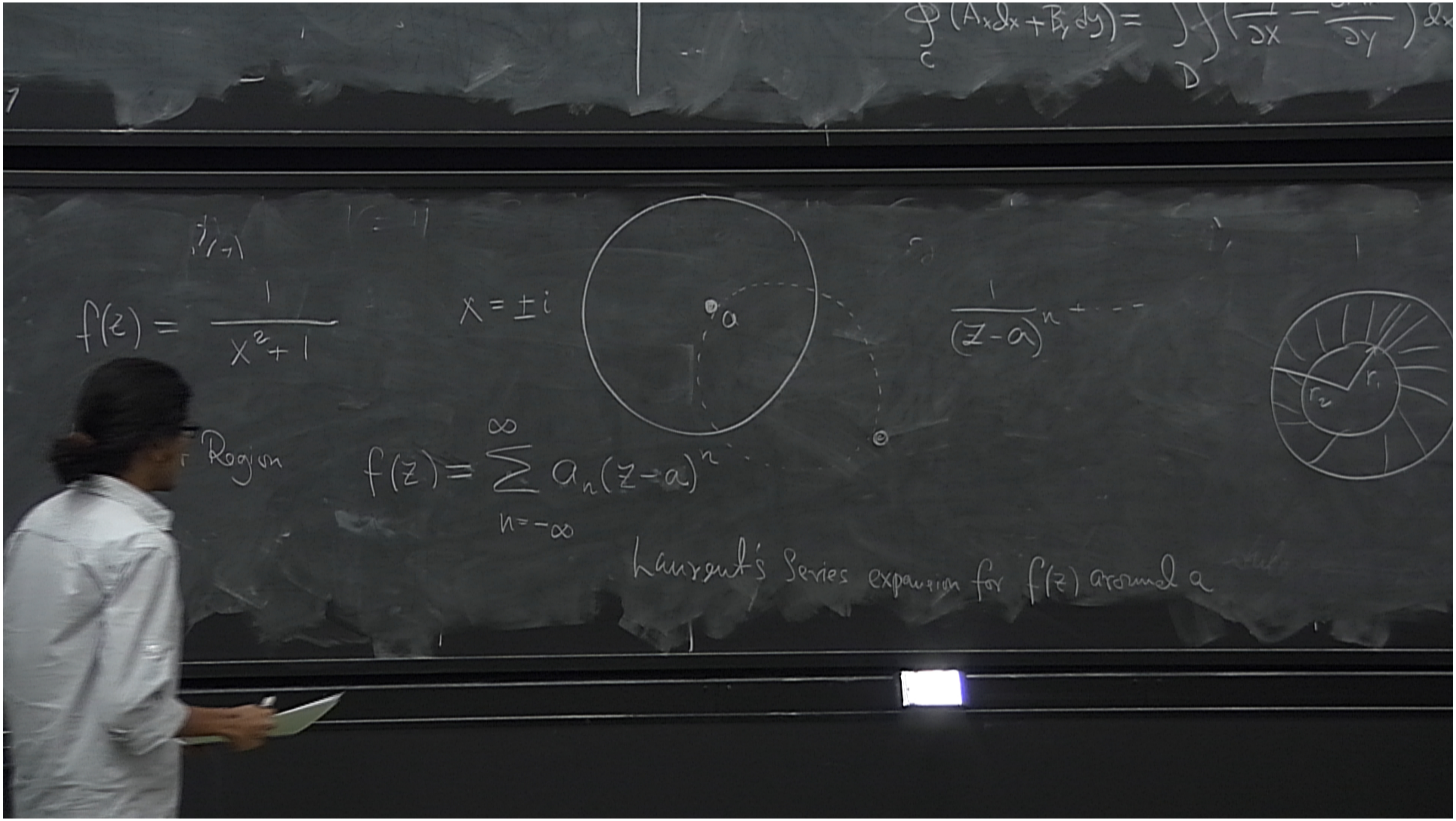

f(z) = 1 / (x²+1)
x = ±i

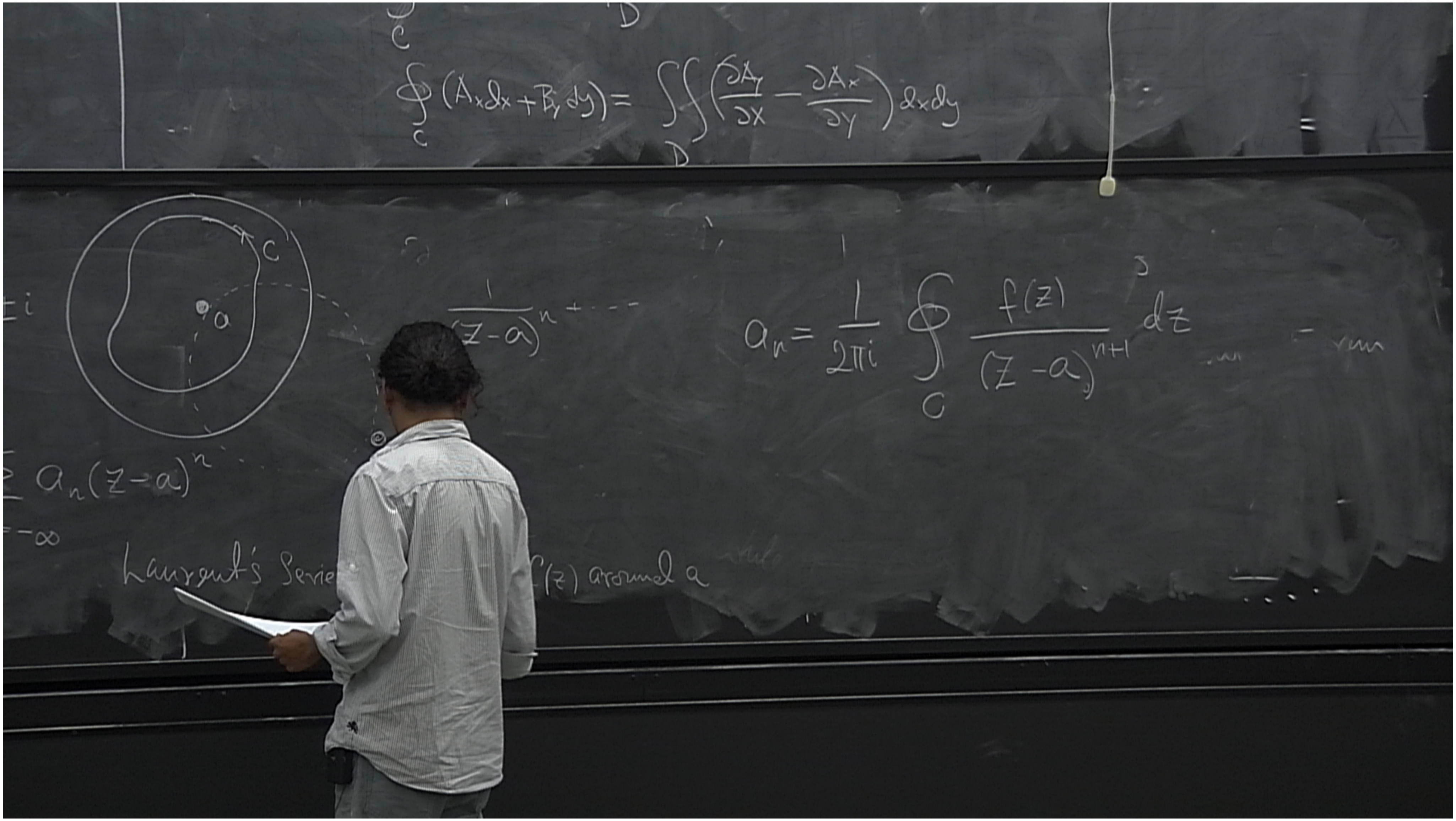

∮_C (A_x dx + B_y dy) = ∫∫_D
f(z) = 1/(x²+1)
x = ±i
a
1/(z-a)^n + - - -
Annular Region

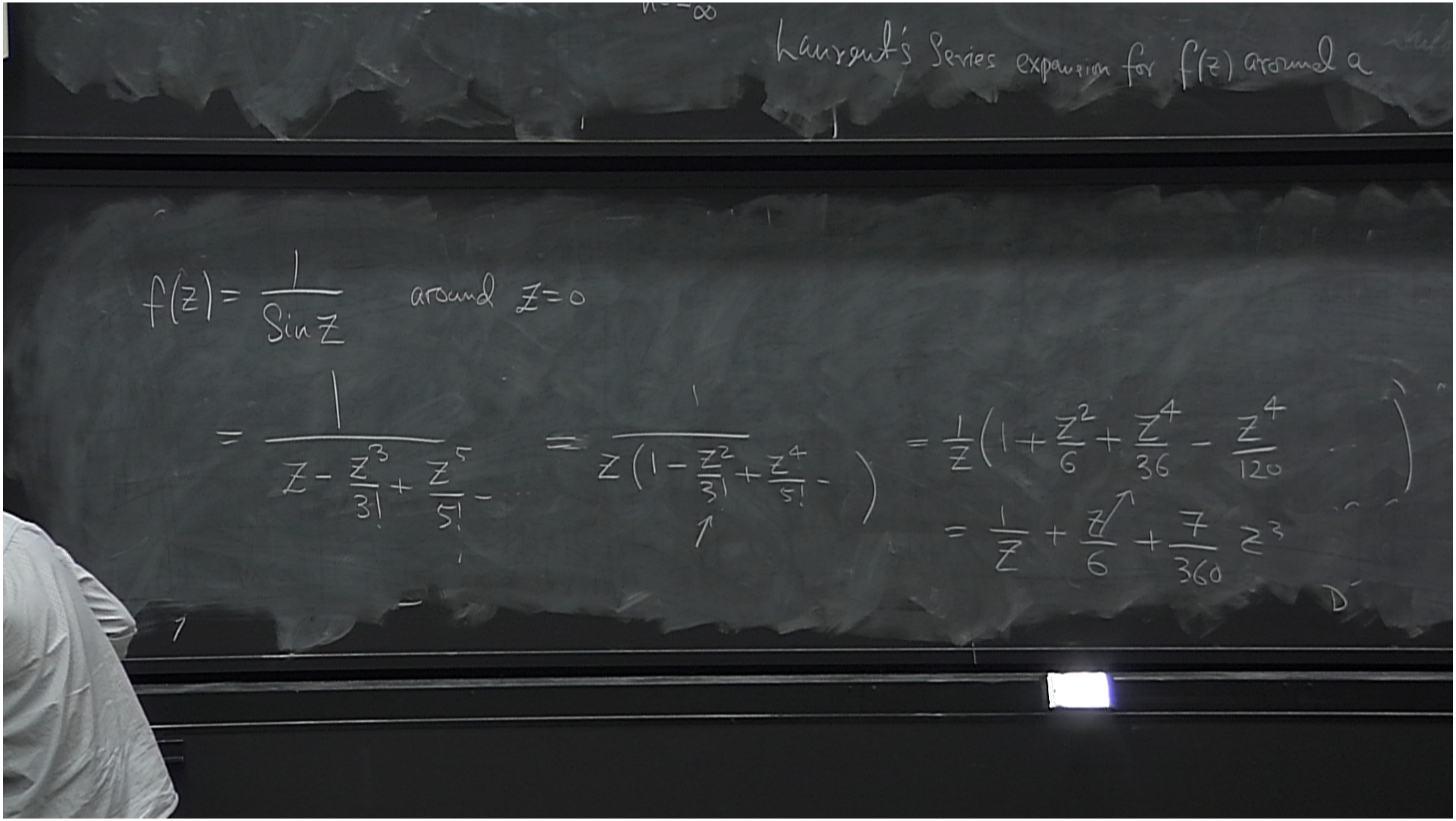


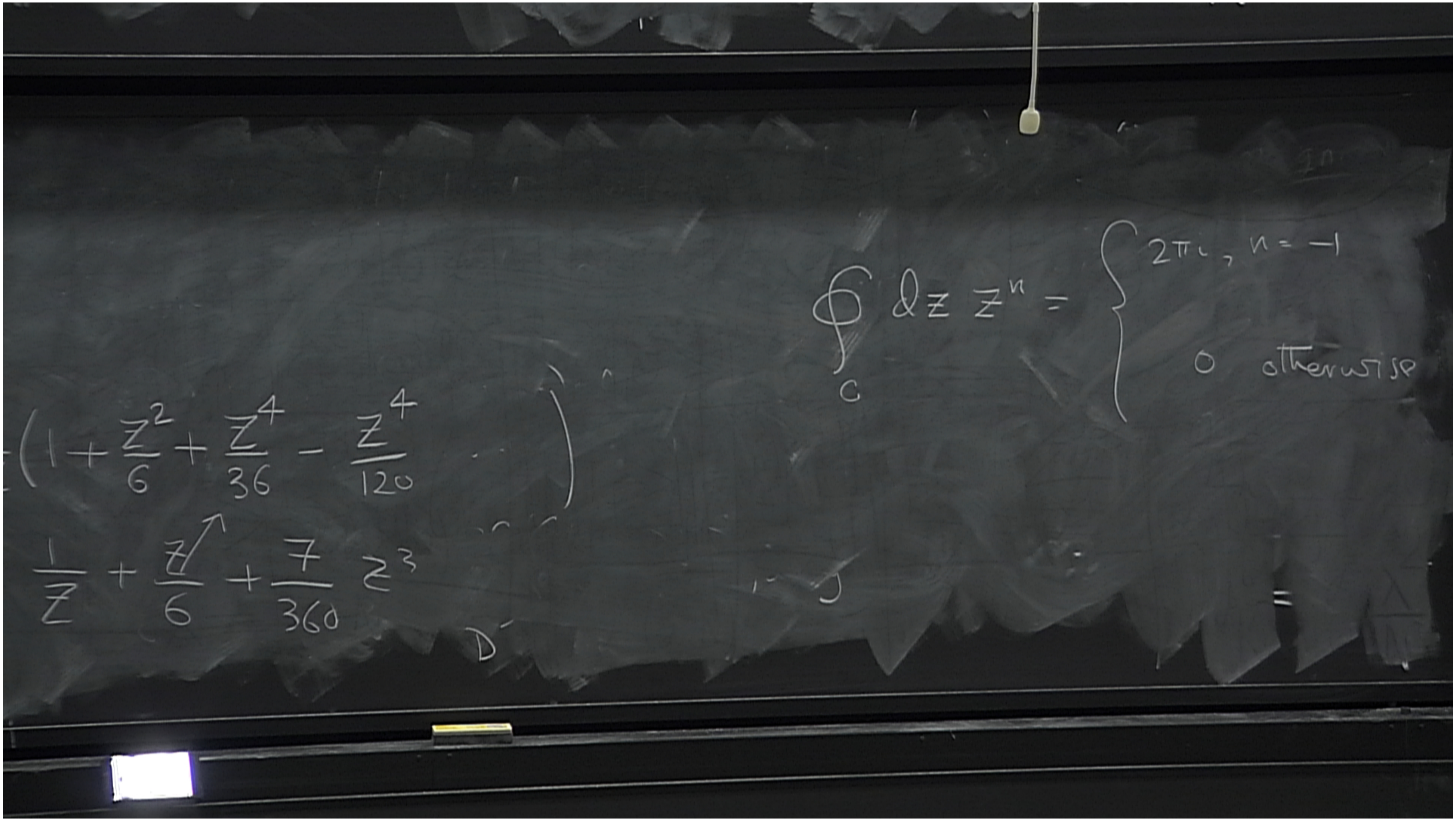
(Ax dx + By dy) =
C
D
dx dy
x = ±i
a
1
(z - a)^n + . . .
r1
r2
r2 < ∞

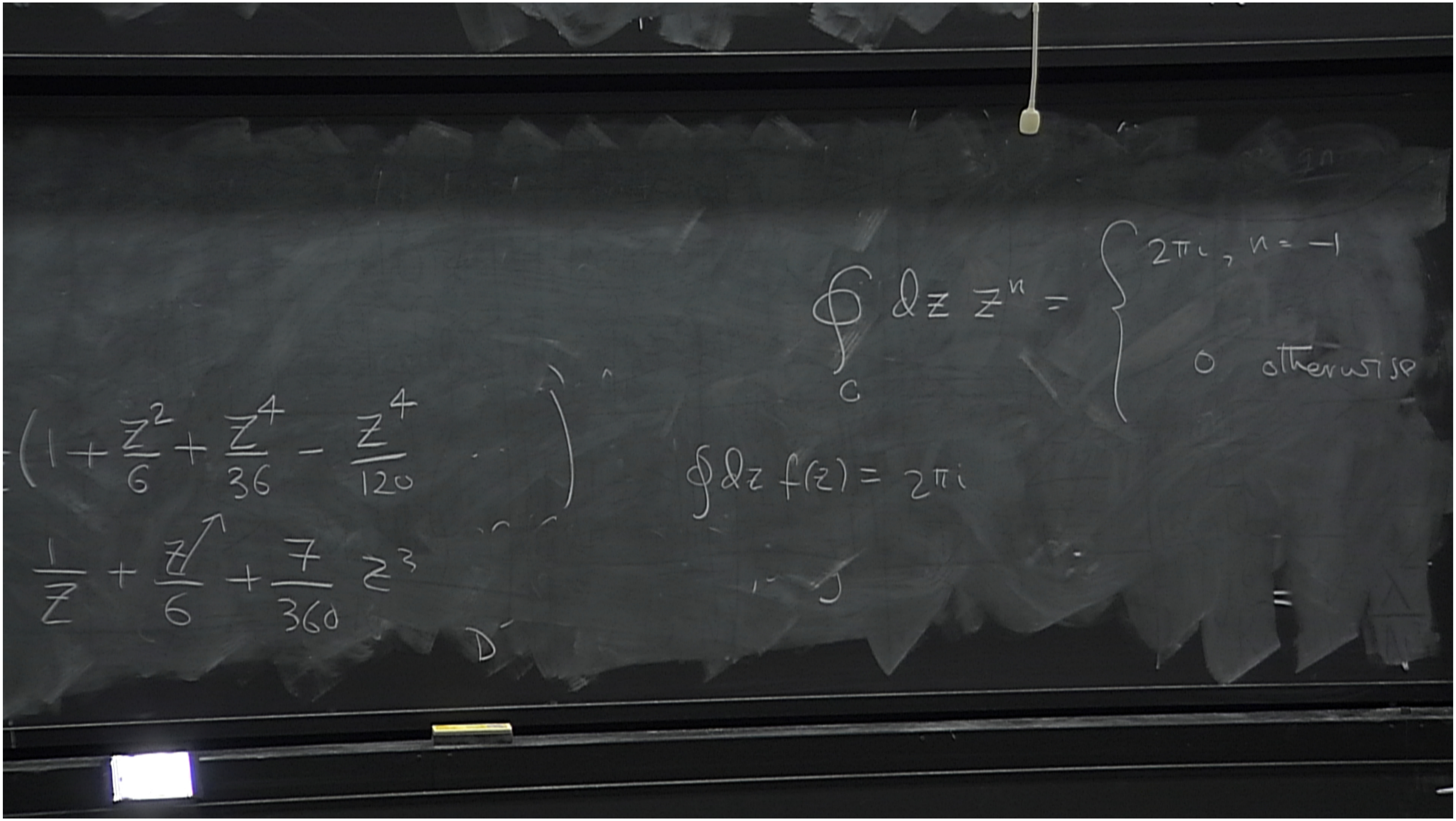

f(z) = 1 / (x^2 + 1)
x = ±i
1 / (z - a)^n + - - -
Annular Region
f(z) = Σ_{n=-∞}^{∞} a_n (z - z)^n
Laurent's Series

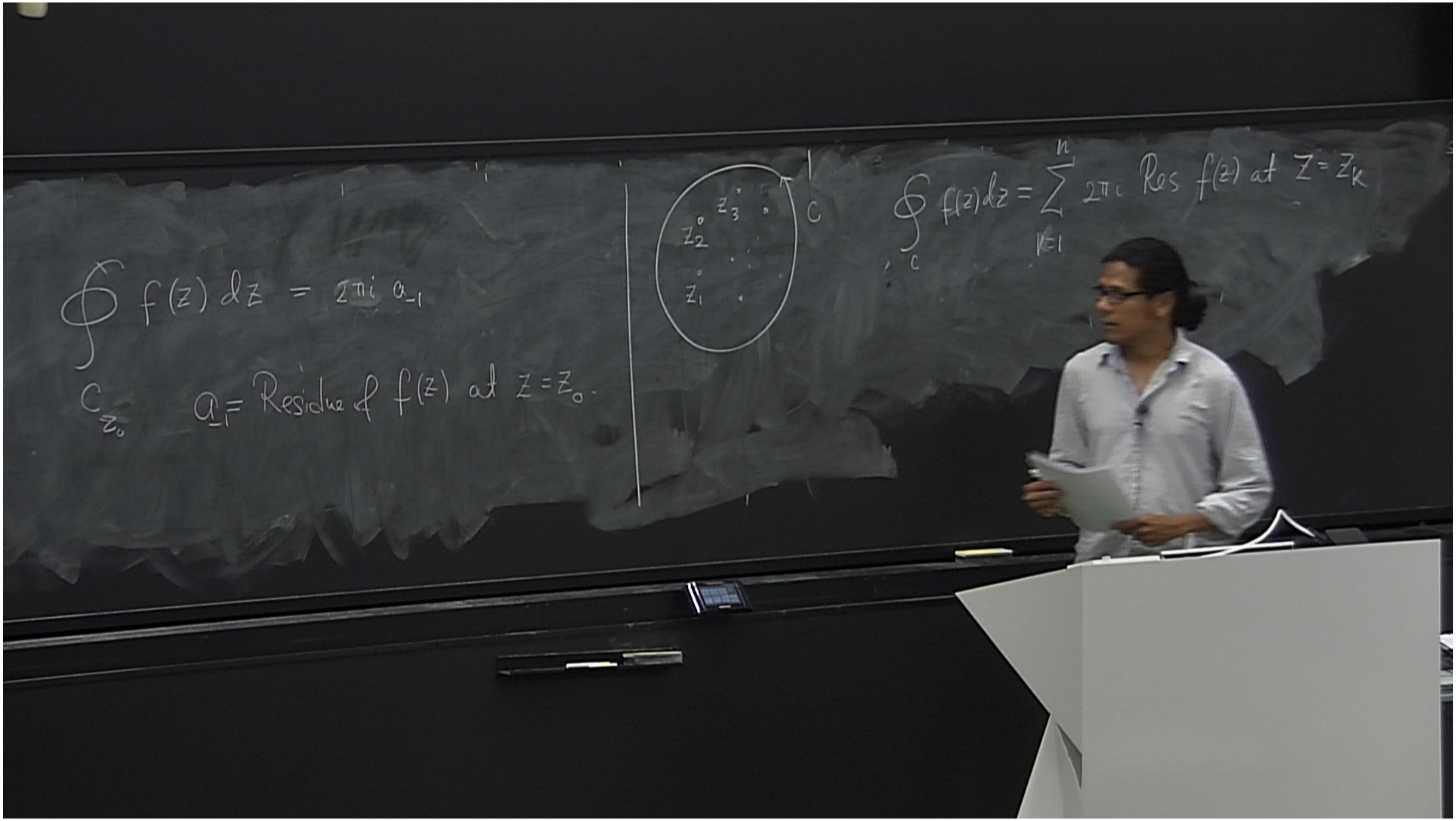

f(z) = 1 / (x^2 + 1)
x = ±i
1 / (z - a)^n + ...
Region
f(z) = Σ_{n=-∞}^{∞} a_n (z - a)^n
r_1
r_2
Laurent's Series expansion for f(z) around a

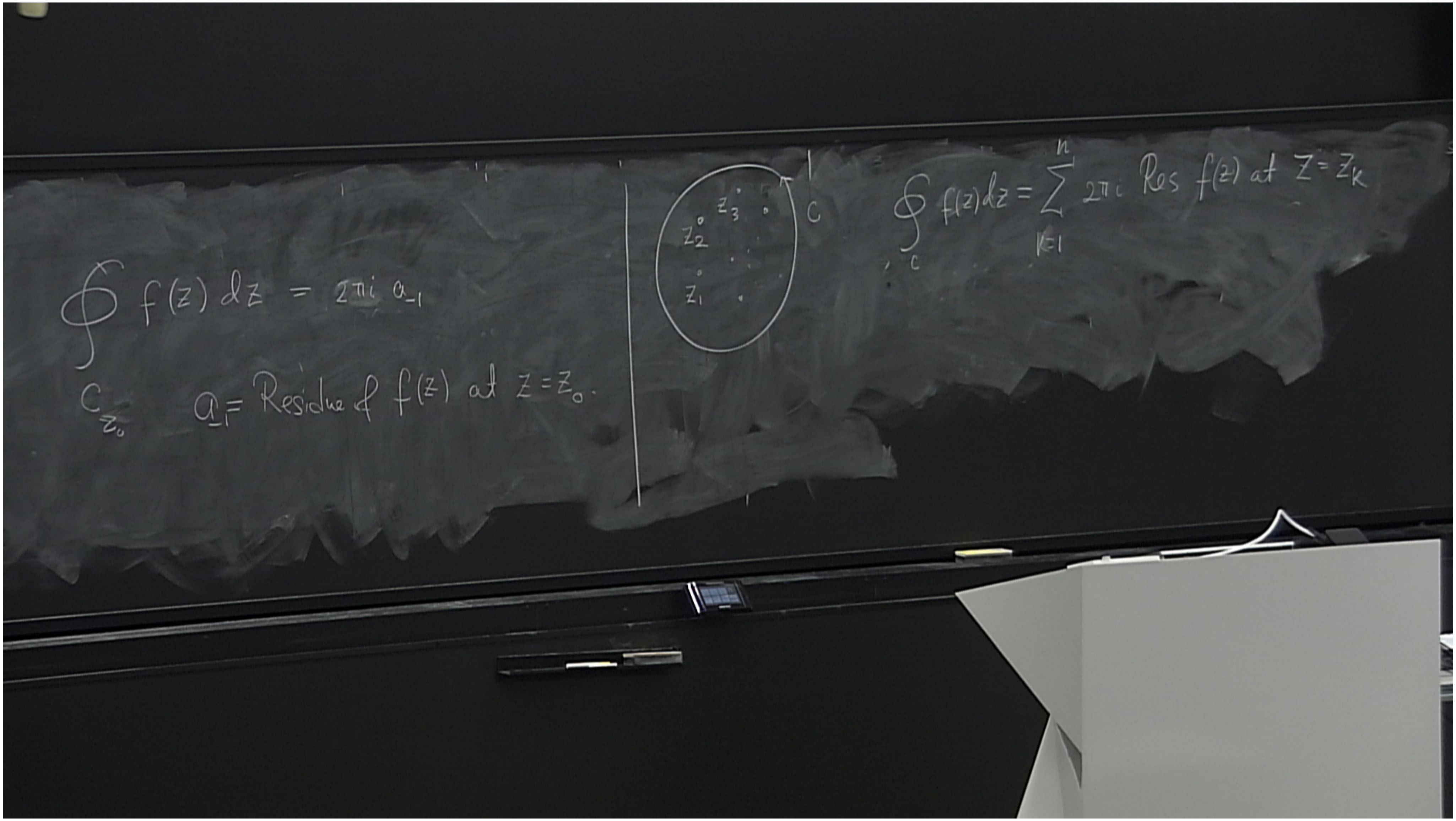

∮ (Ax dx + By dy) = ∬ (∂Ay/∂x − ∂Ax/∂y) dx dy
C
D
an = 1/2πi ∮ f(z)/(z − a)^(n+1) dz
an(z − a)^n
Laurent's Series
around a

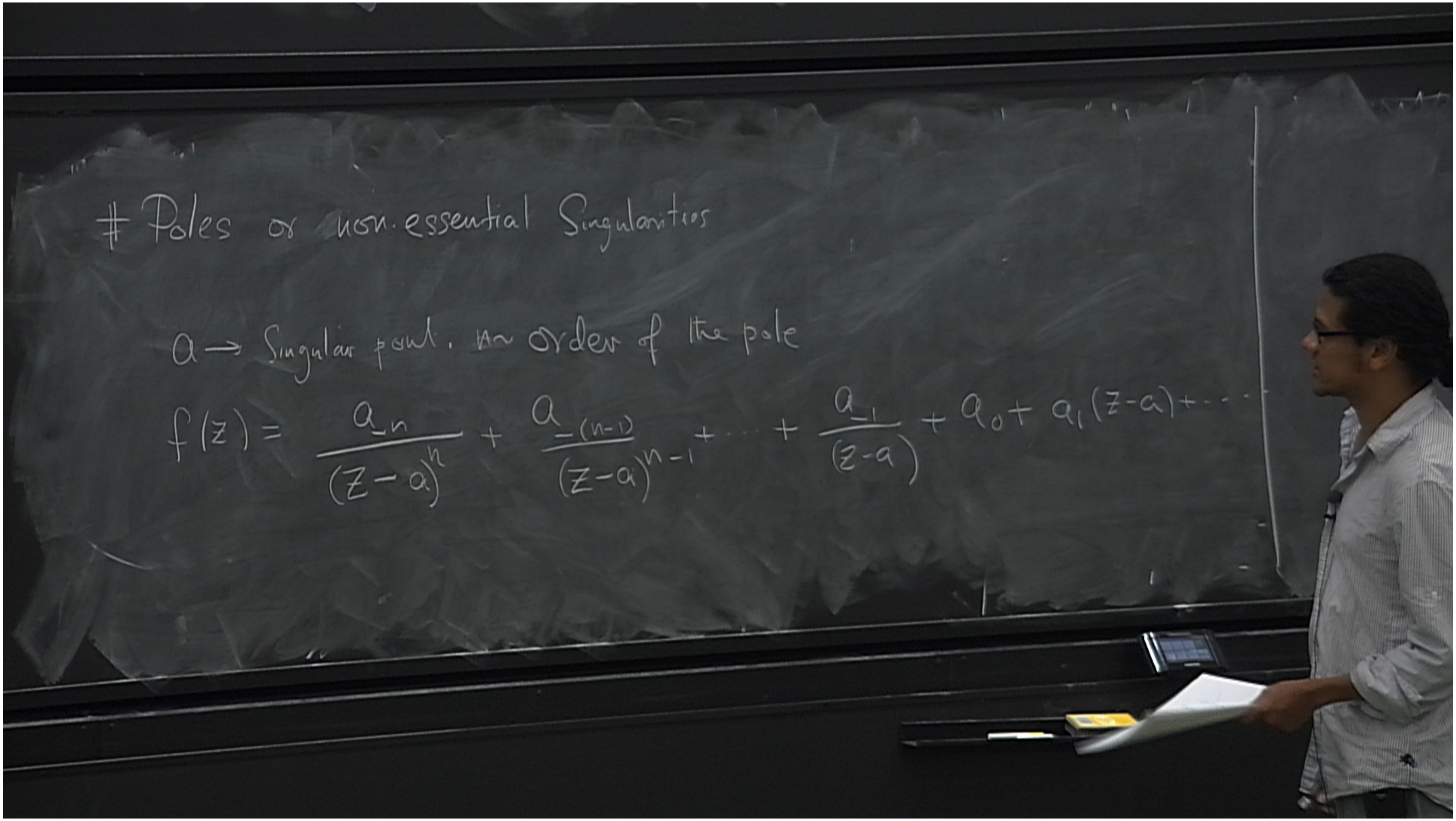

Laurent's Series expansion for f(z) around a
f(z) = 1/Sin z around z = 0
= 1/(z − z³/3! + z⁵/5! − ...)
= 1/(z(1 − z²/3! + z⁴/5! − ...))
= 1/z (1 + z²/6 + z⁴/36 − z⁴/120 ...)
= 1/z + z/6 + 7/360 z³

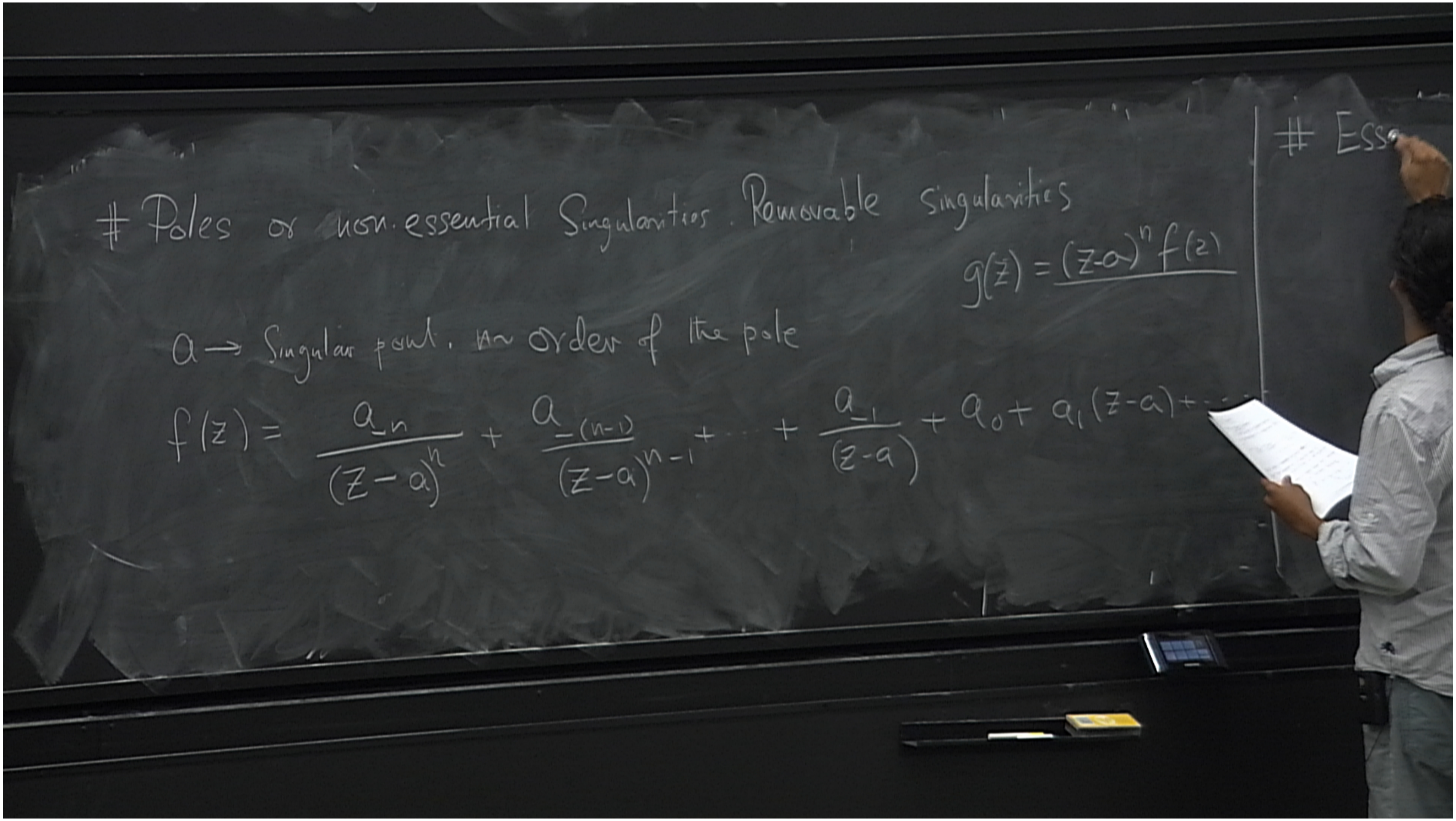

$\oint_C dz\, z^n = \begin{cases} 2\pi i, & n = -1 \\ 0 & \text{otherwise} \end{cases}$
$\left(1 + \frac{z^2}{6} + \frac{z^4}{36} - \frac{z^4}{120} \cdots \right)$
$\frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3$


$\oint_C dz\, z^n = \begin{cases} 2\pi i, & n = -1 \\ 0 & \text{otherwise} \end{cases}$
$(1 + \frac{z^2}{6} + \frac{z^4}{36} - \frac{z^4}{120} \cdots)$
$\oint dz\, f(z) = 2\pi i$
$\frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3$

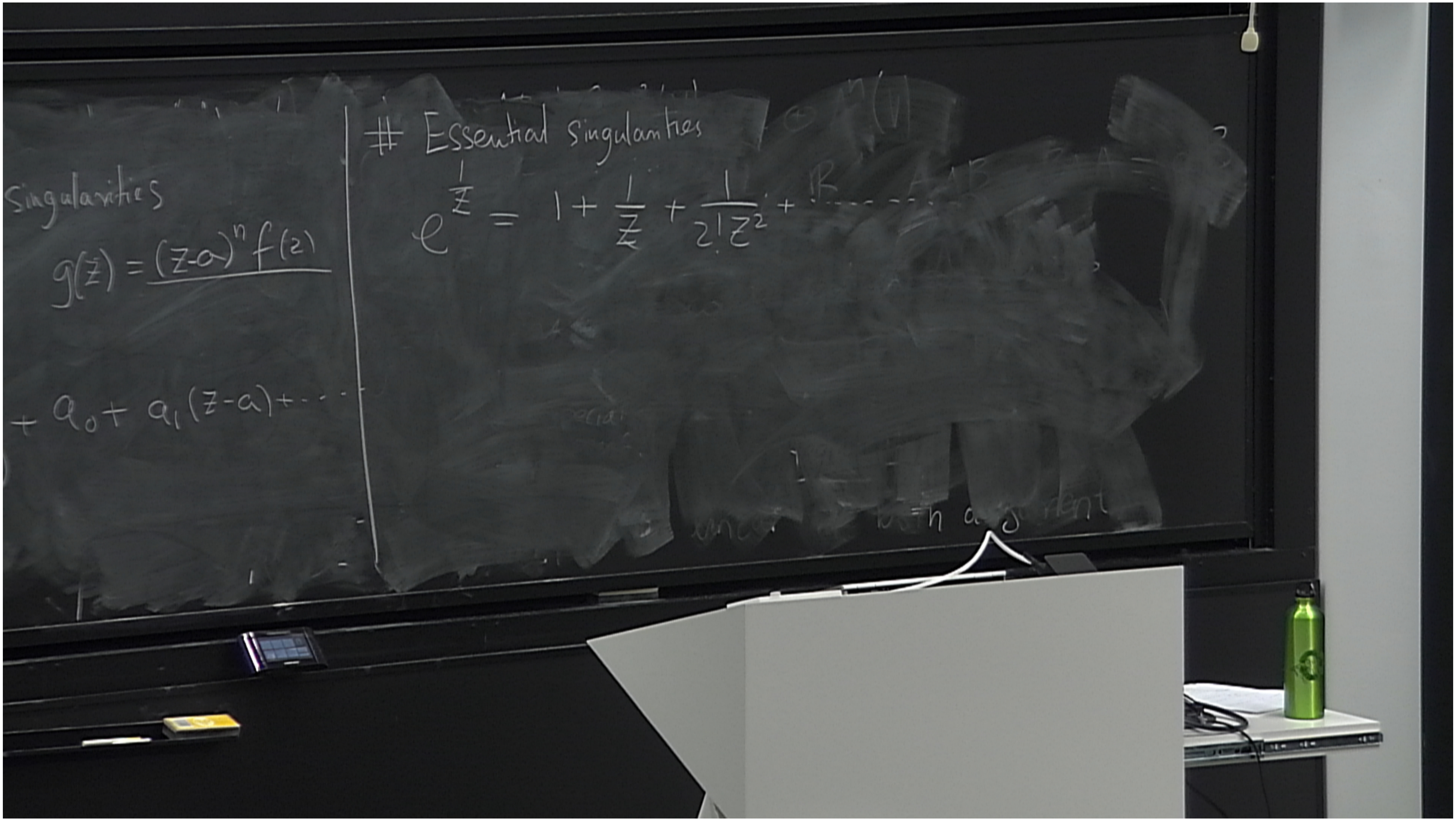

∮ f(z) dz = 2πi a₋₁
C_{z₀}
a₋₁ = Residue of f(z) at z = z₀.
z₃
z₂
z₁
C
∮_C f(z) dz = Σ_{k=1}^{n} 2πi Res f(z) at z = z_k


∮ f(z) dz = 2πi a₋₁
C_{z₀}
a₋₁ = Residue of f(z) at z = z₀.
z₃
z₂
z₁
C
∮ f(z) dz = Σ_{k=1}^{n} 2πi Res f(z) at z = z_k
C

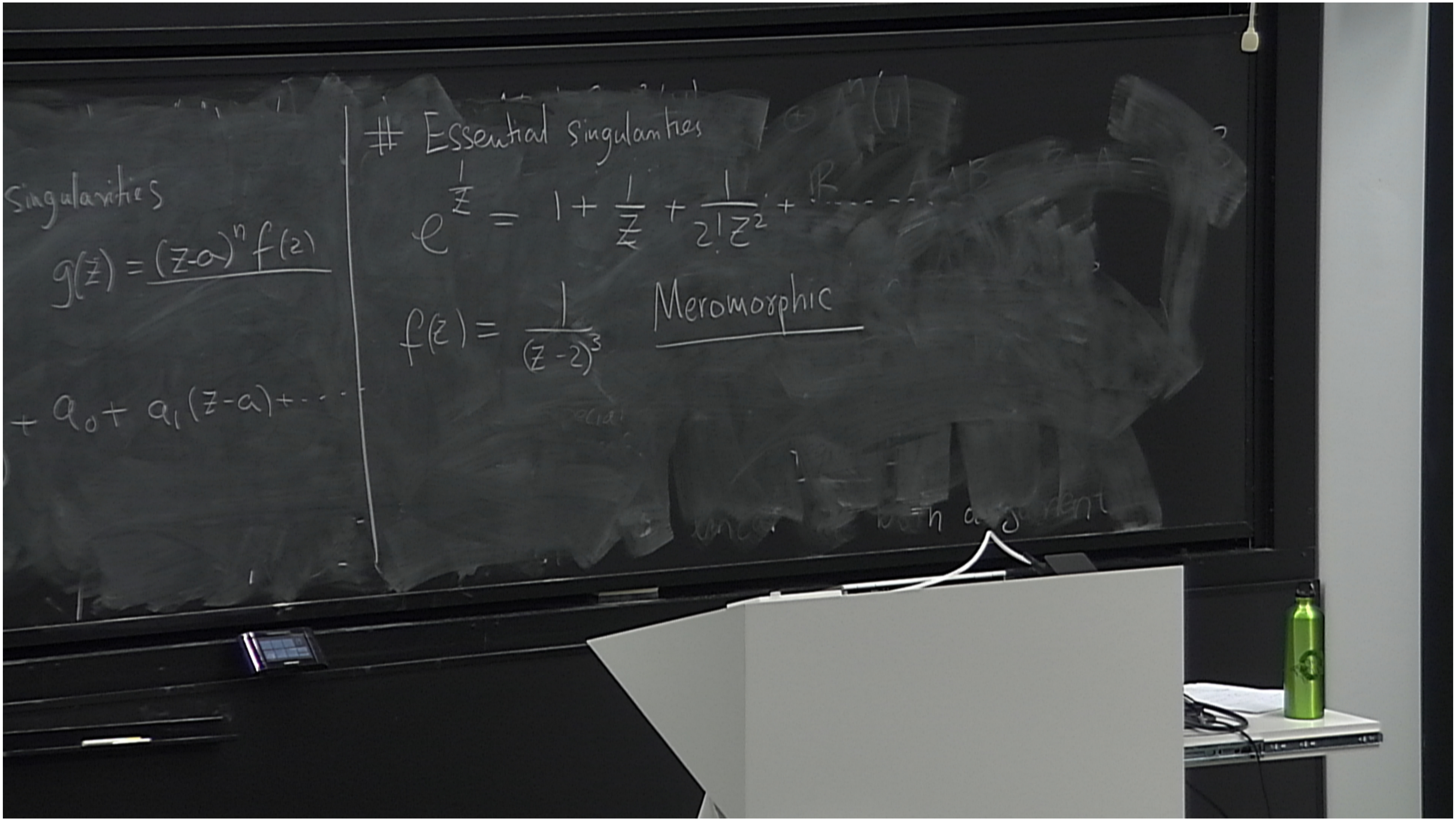

Poles or non-essential Singularities
a → Singular point. n ~ order of the pole
f(z) = a_{-n}/(z−a)^n + a_{-(n-1)}/(z−a)^{n-1} + ... + a_{-1}/(z−a) + a_0 + a_1(z−a) + ...



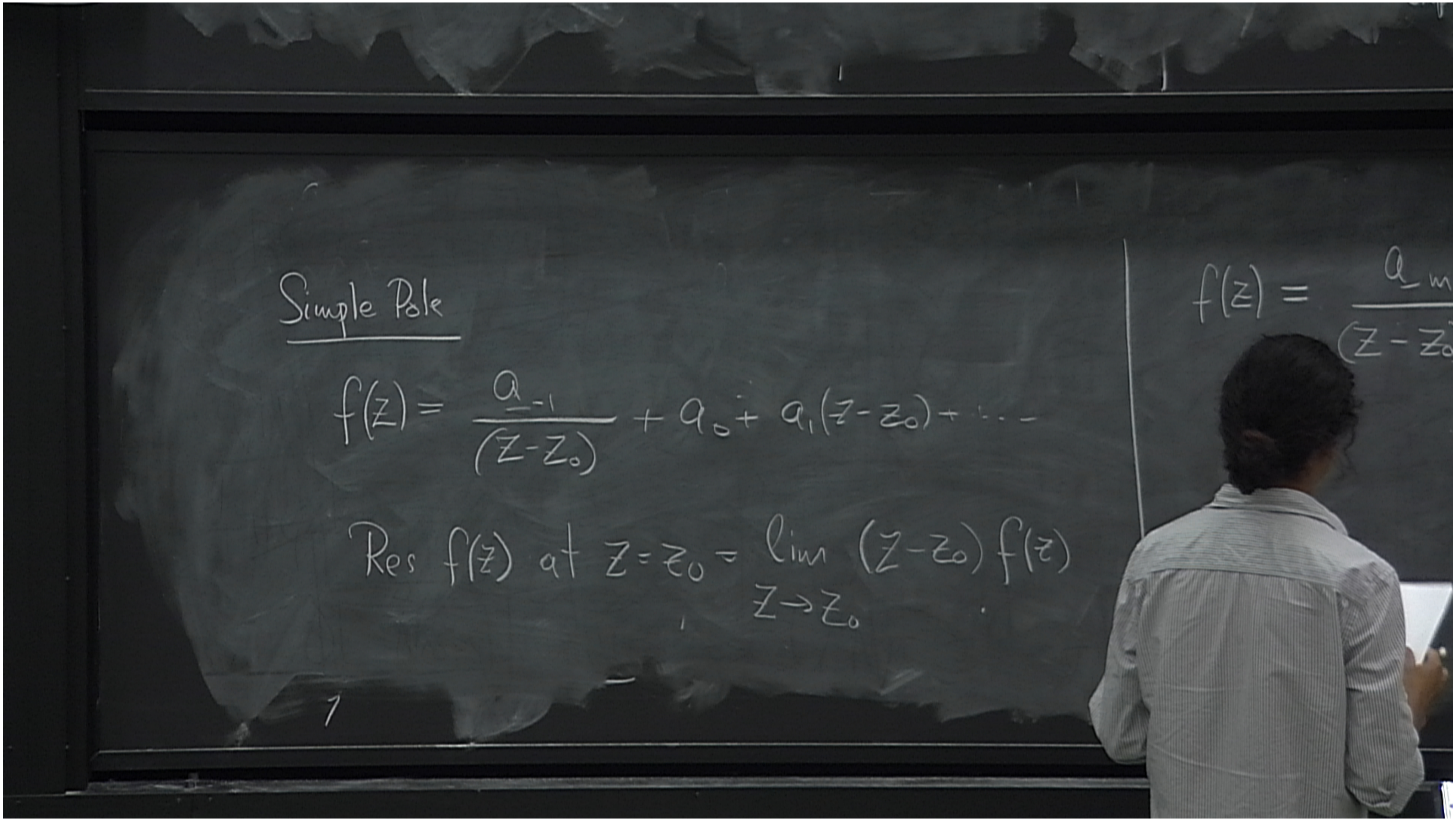
Poles or non.essential Singularities . Removable singularities
$g(z) = \frac{(z-a)^n f(z)}{}$
$a \to$ Singular point. $n \sim$ order of the pole
$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-(n-1)}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + \cdots$
Ess

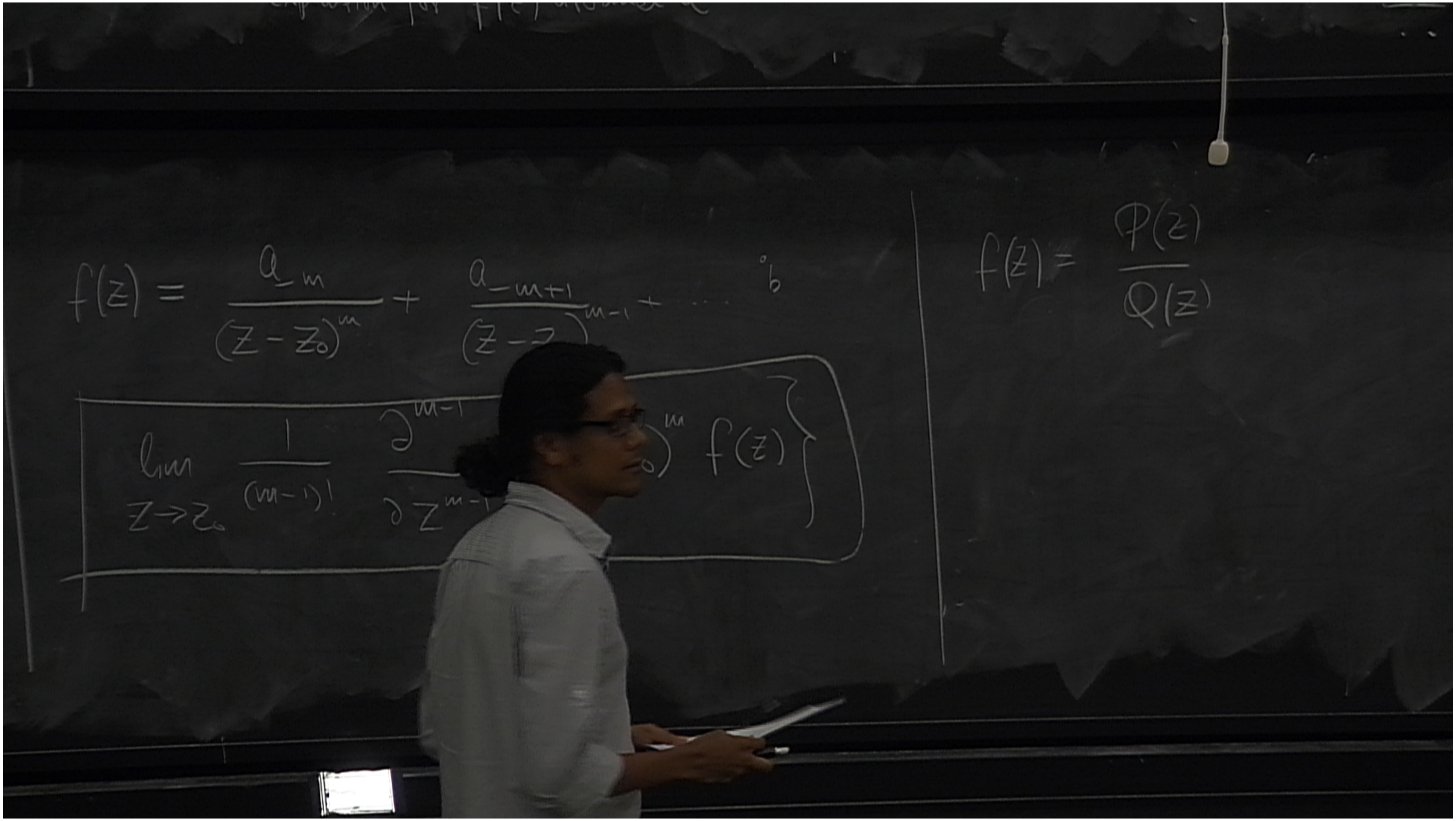


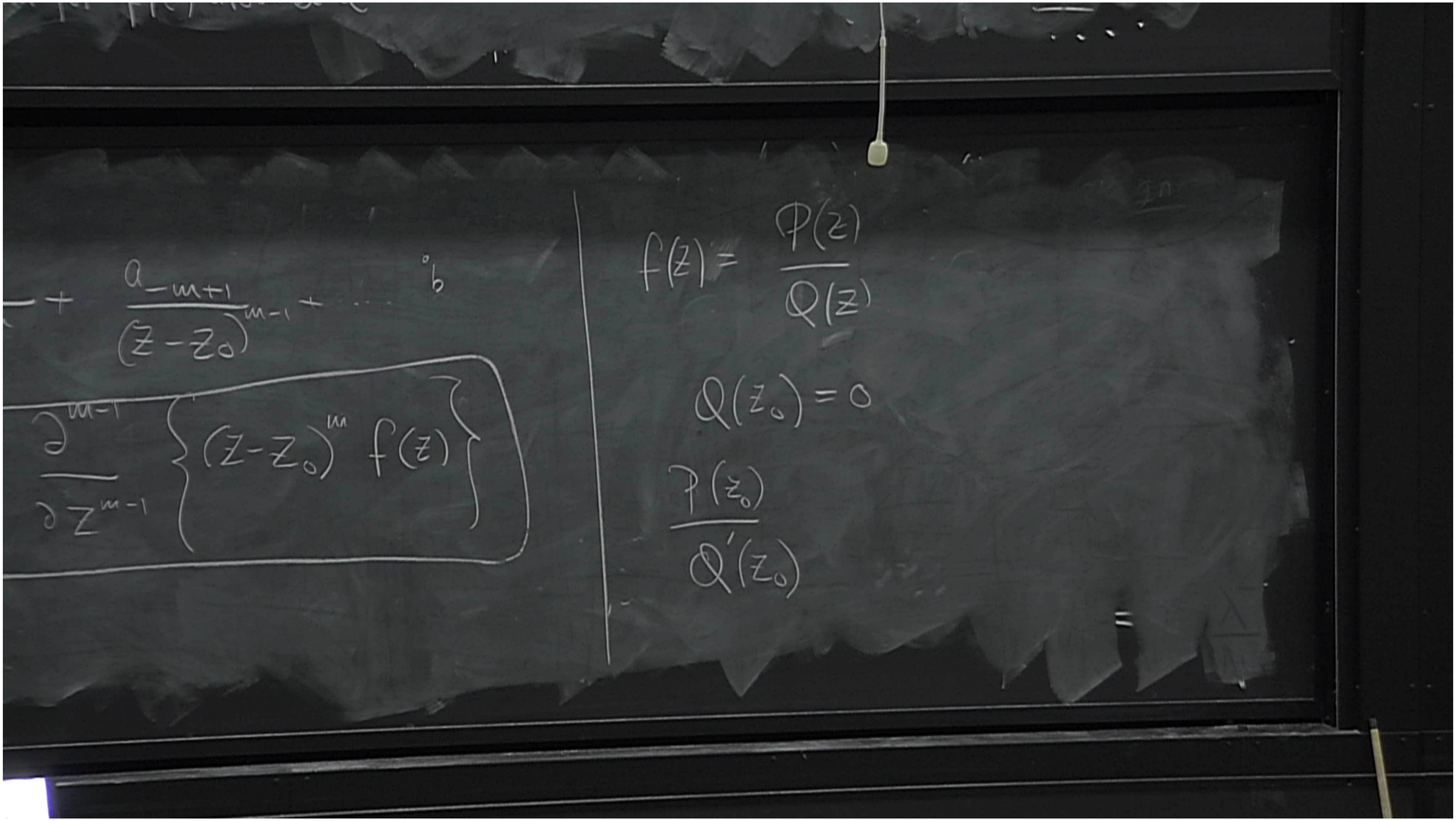
singularities
g(z) = (z-a)^n f(z)
+ a_0 + a_1 (z-a) + . . .
# Essential singularities
e^{1/z} = 1 + 1/z + 1/(2! z^2) + . . .

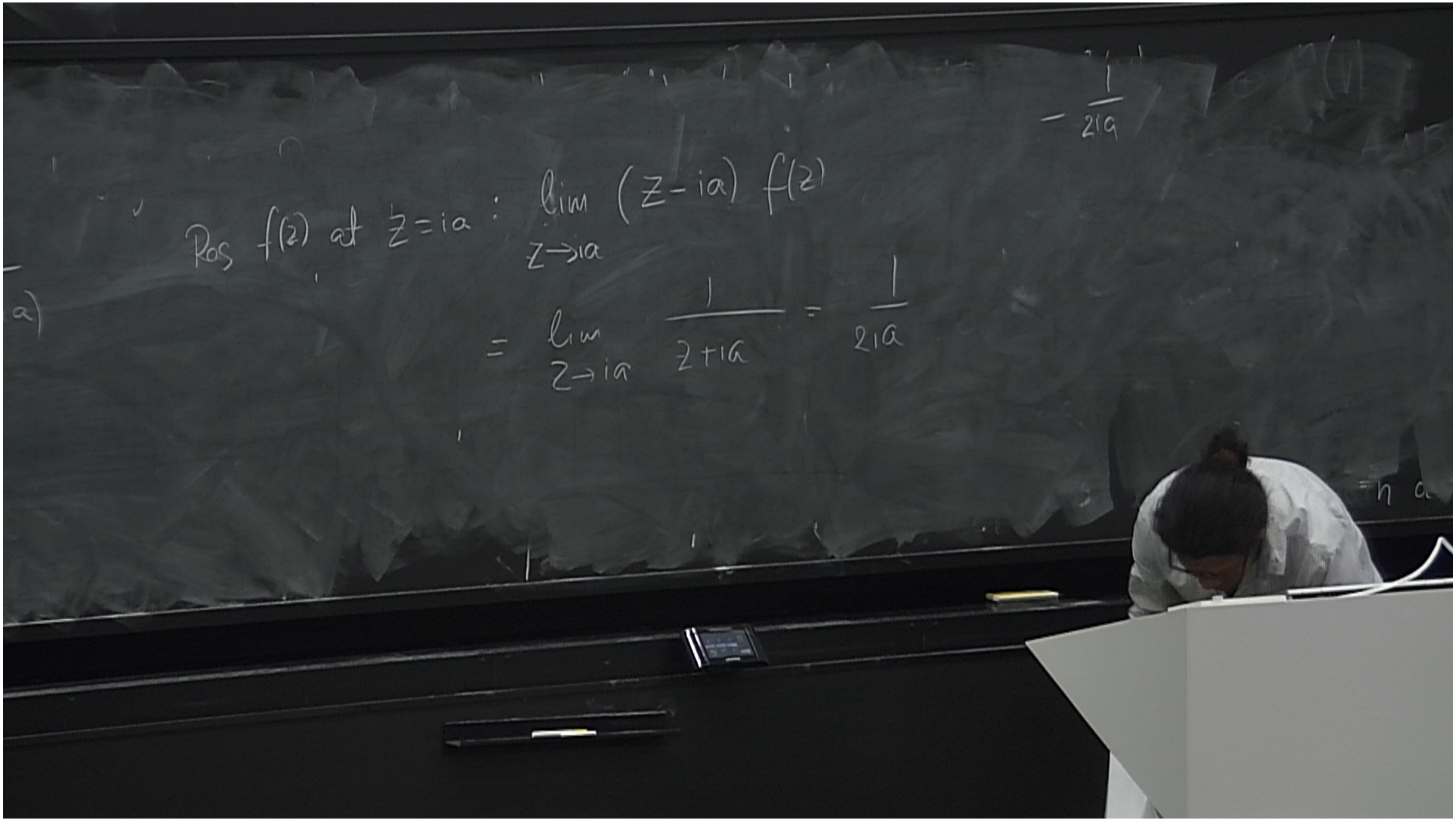


Essential singularities
e^(1/Z) = 1 + 1/Z + 1/(2! Z²) + ...
f(z) = 1/(Z - 2)³
Meromorphic
singularities
g(Z) = (Z-a)ⁿ f(z)
+ a₀ + a₁(Z-a) + . . .


Simple Pole
$f(z) = \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots$
$\text{Res } f(z) \text{ at } z=z_0 = \lim_{z\to z_0} (z-z_0) f(z)$
$f(z) =$


f(z) = P(z) / Q(z)


$+ \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots$
$\frac{\partial^{m-1}}{\partial z^{m-1}} \left\{ (z-z_0)^m f(z) \right\}$
$f(z) = \frac{P(z)}{Q(z)}$
$Q(z_0) = 0$
$\frac{P(z_0)}{Q'(z_0)}$


Res f(z) at z = ia : lim (z − ia) f(z)
z→ia
= lim 1/(z+ia) = 1/(2ia)
z→ia
− 1/(2ia)