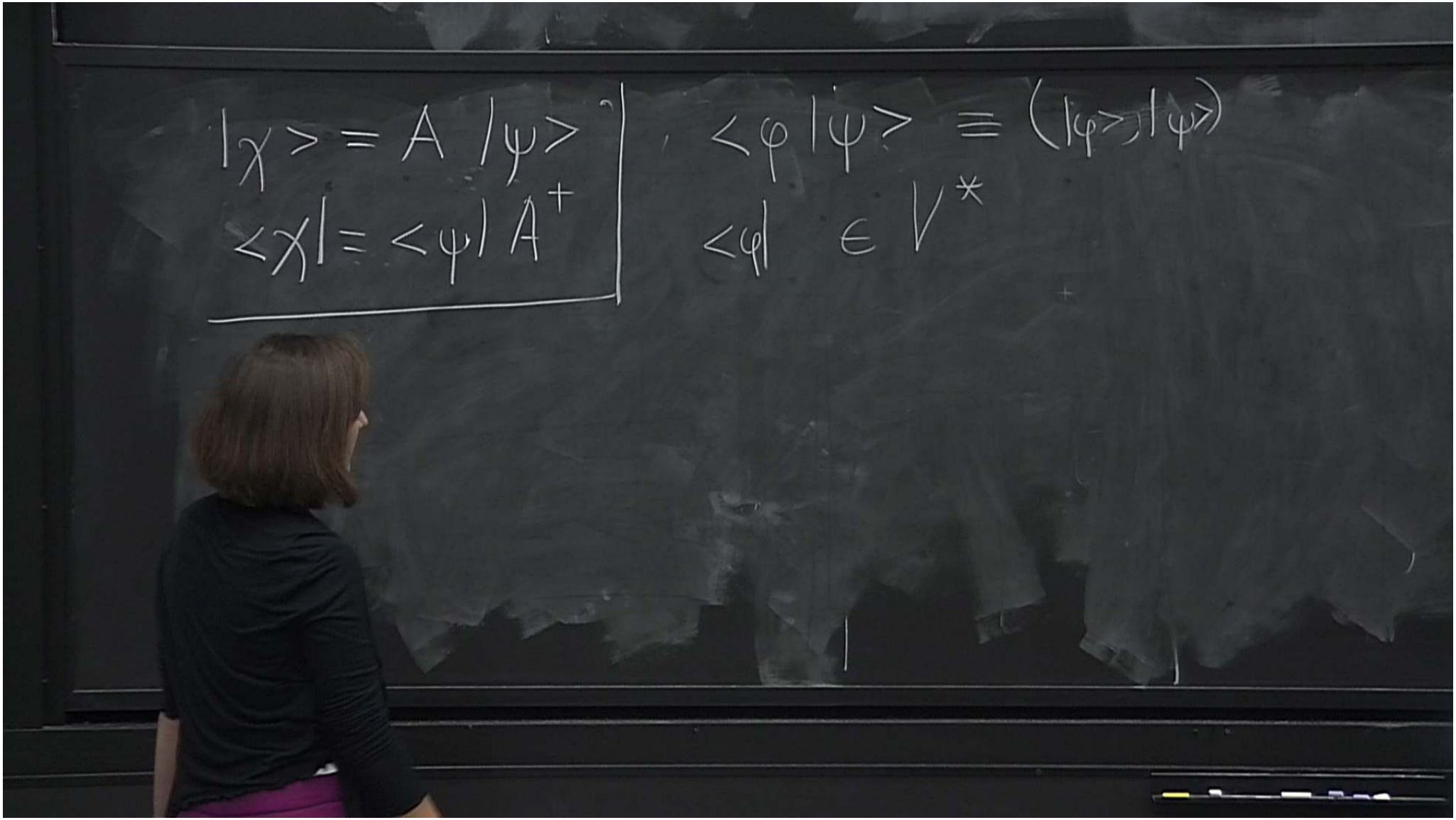
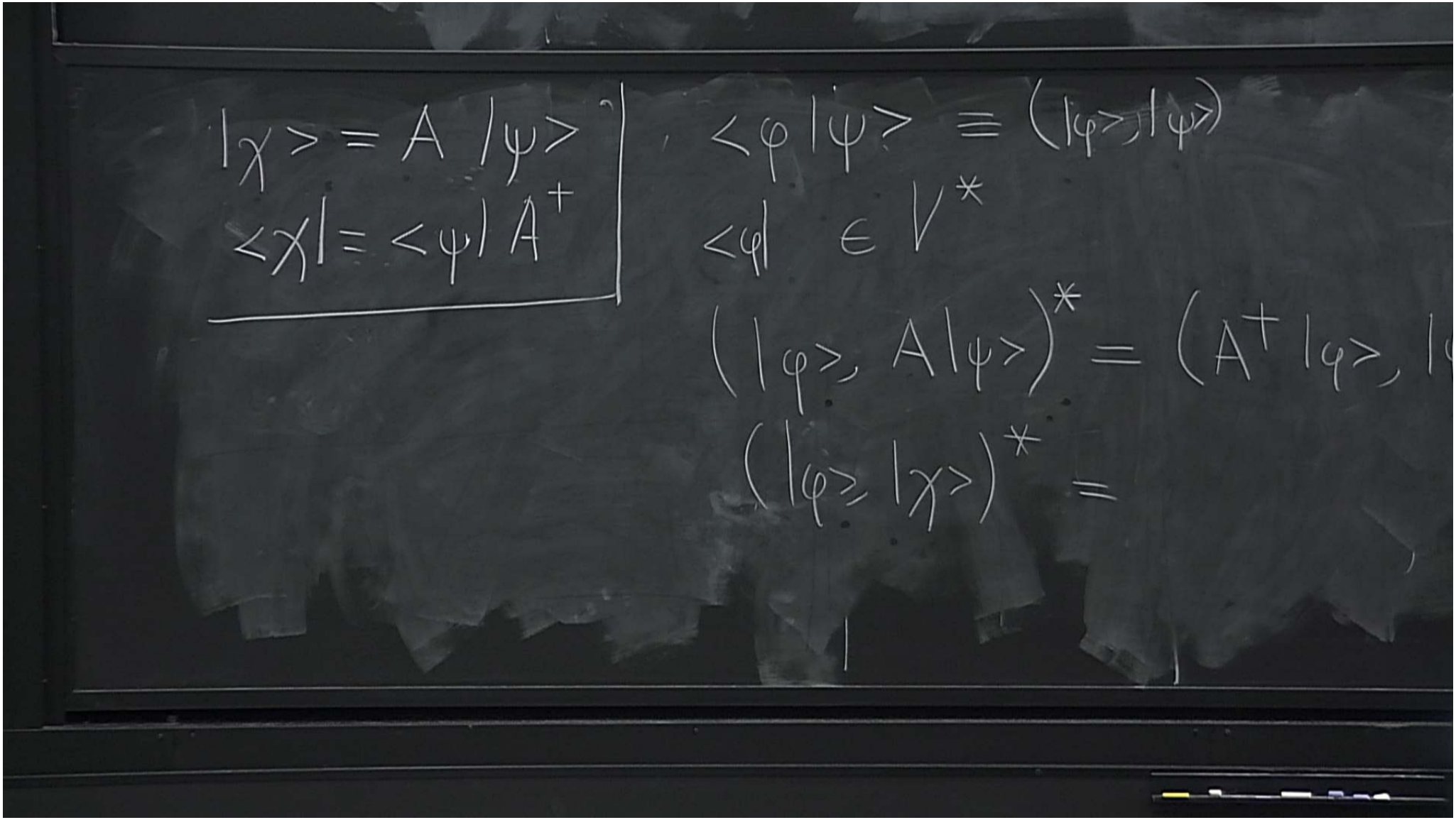
Title: Algebra - Lecture 2a

Date: Aug 17, 2011 11:00 AM

URL: http://pirsa.pi.local/11080079

Abstract:


$|\chi\rangle = A\,|\psi\rangle$
$\langle\chi| = \langle\psi|\,A^{+}$
$\langle\varphi|\psi\rangle \equiv (|\varphi\rangle, |\psi\rangle)$
$\langle\varphi| \in V^{*}$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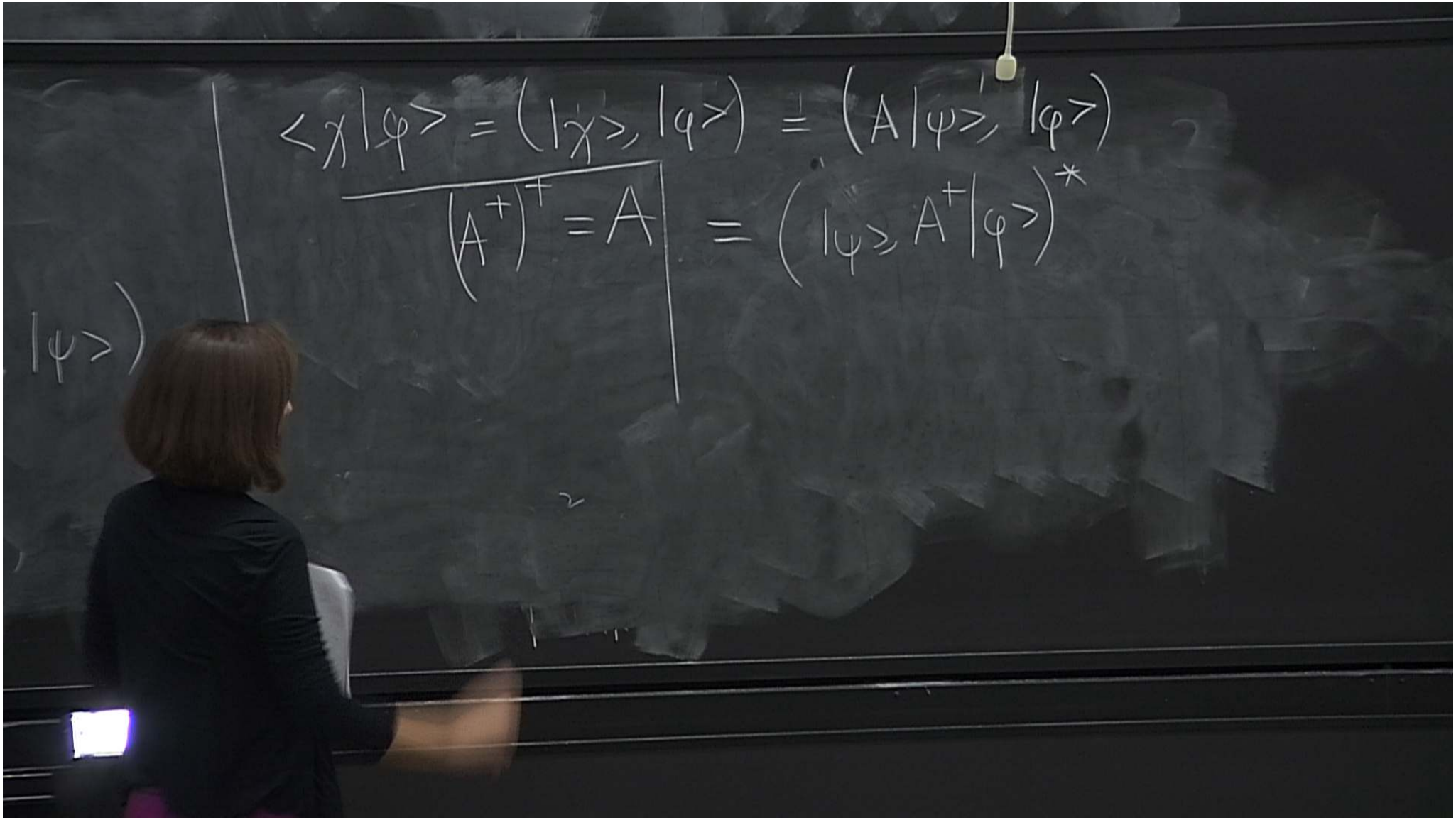
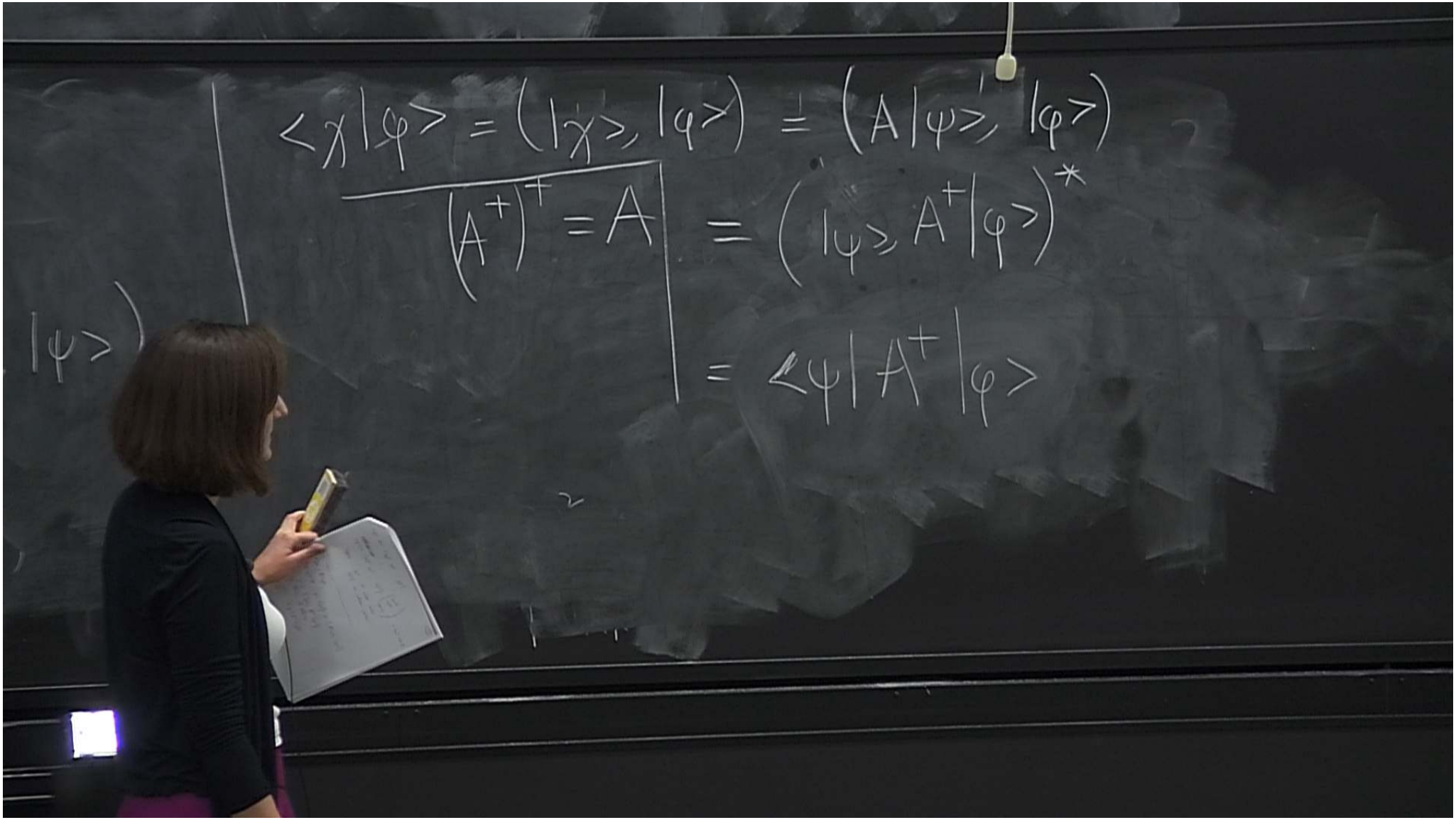
$\langle\chi| = \langle\psi|\,A^{+}$
$\langle\varphi|\psi\rangle \equiv (|\varphi\rangle, |\psi\rangle)$
$\langle\varphi| \in V^{*}$
$(|\varphi\rangle, A|\psi\rangle)^{*} = (A^{+}|\varphi\rangle, |$
$(|\varphi\rangle, |\chi\rangle)^{*} =$


$\langle\chi|\varphi\rangle = (|\chi\rangle, |\varphi\rangle) = (A|\psi\rangle, |\varphi\rangle)$
$(A^+)^+ = A$
$= (|\psi\rangle, A^+|\varphi\rangle)^*$
$|\psi\rangle)$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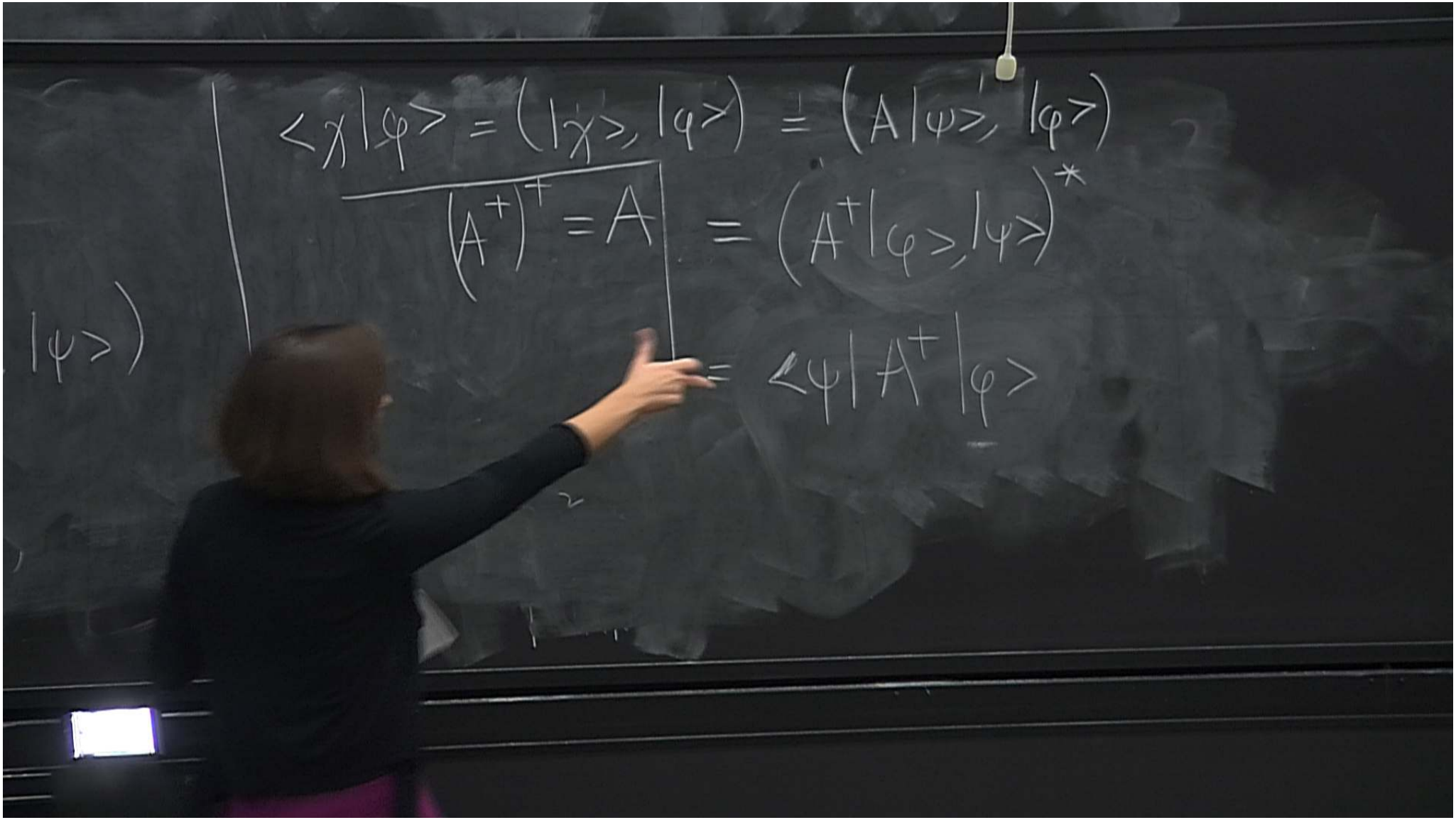
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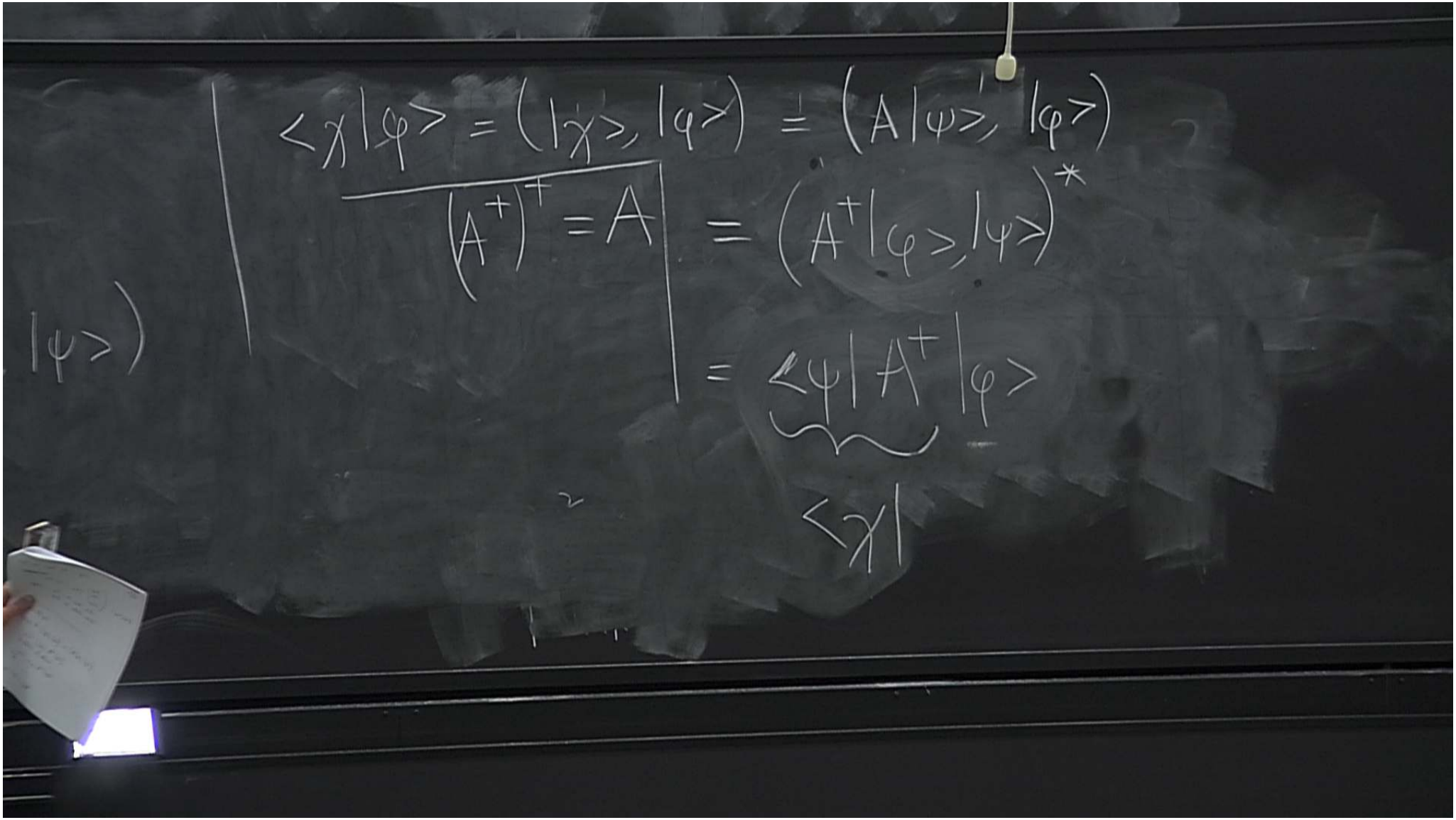
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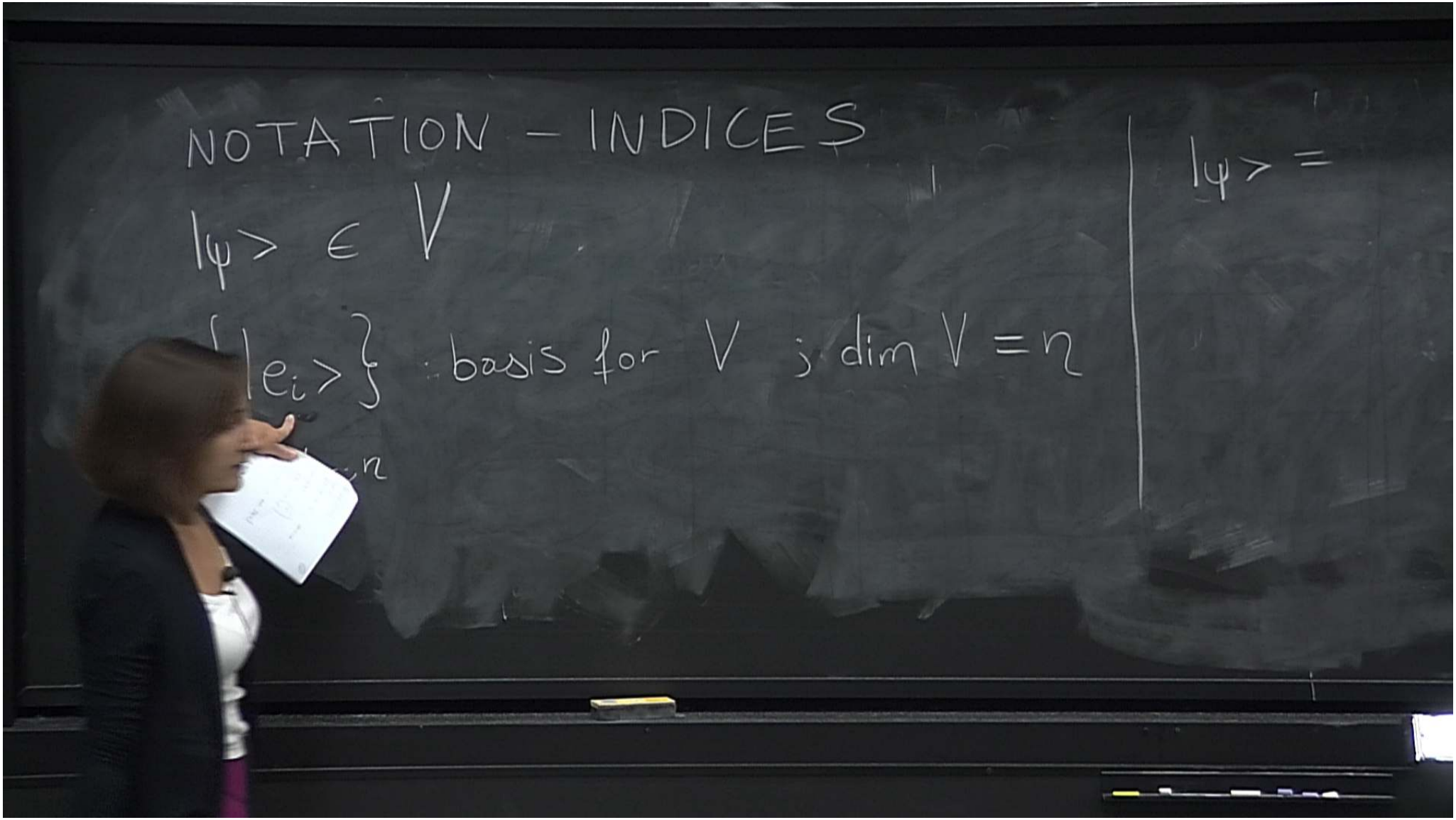
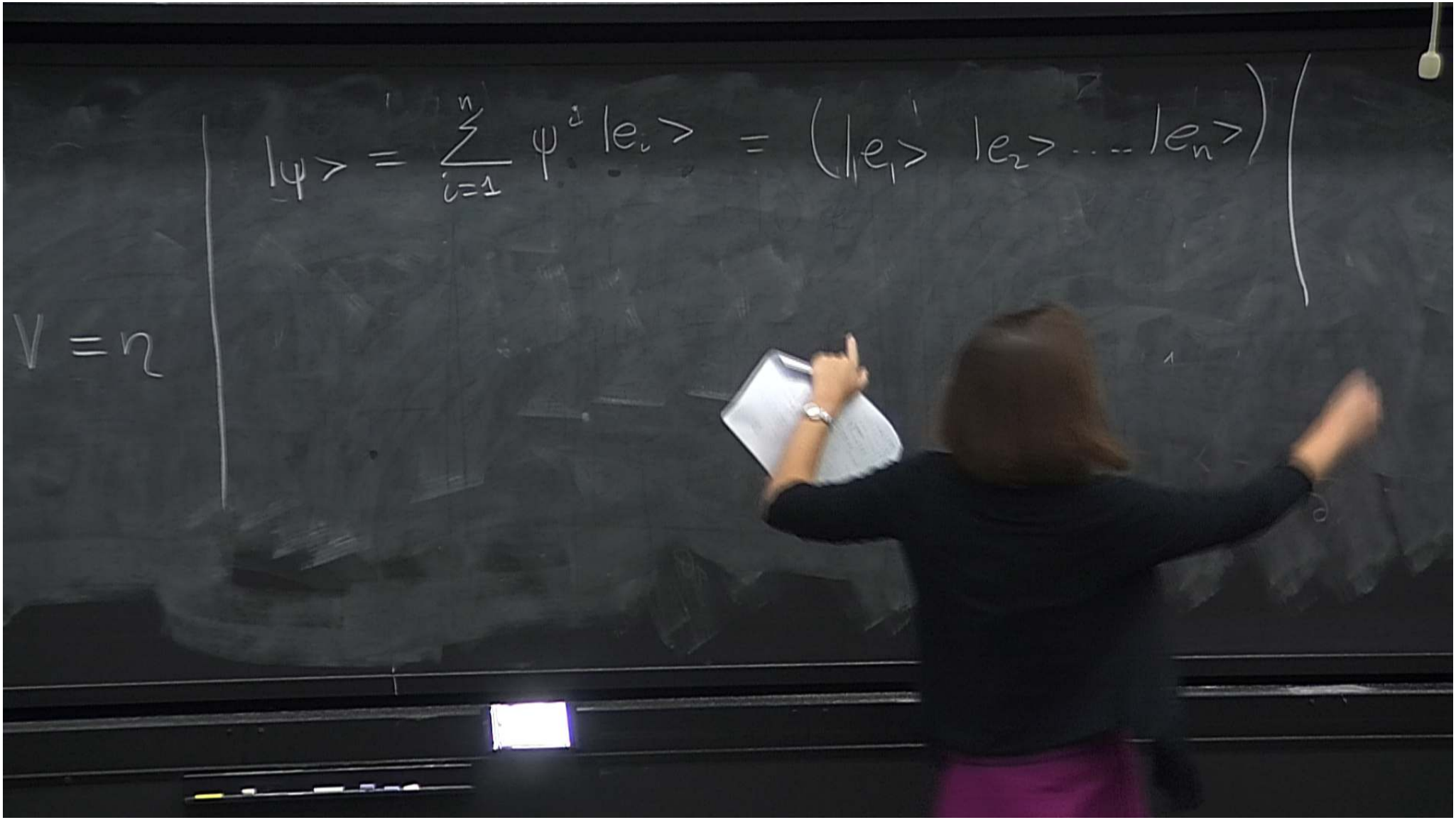
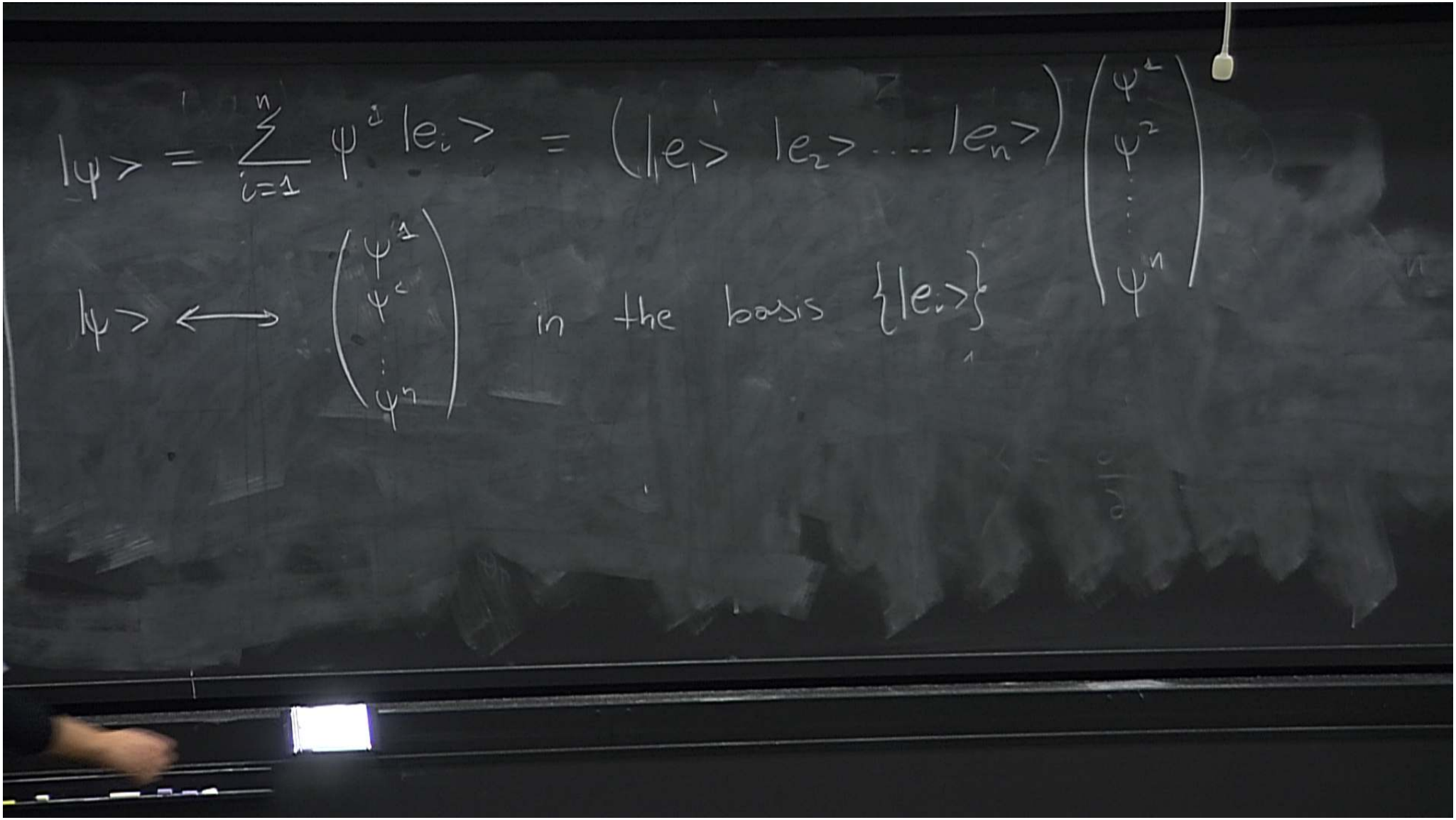
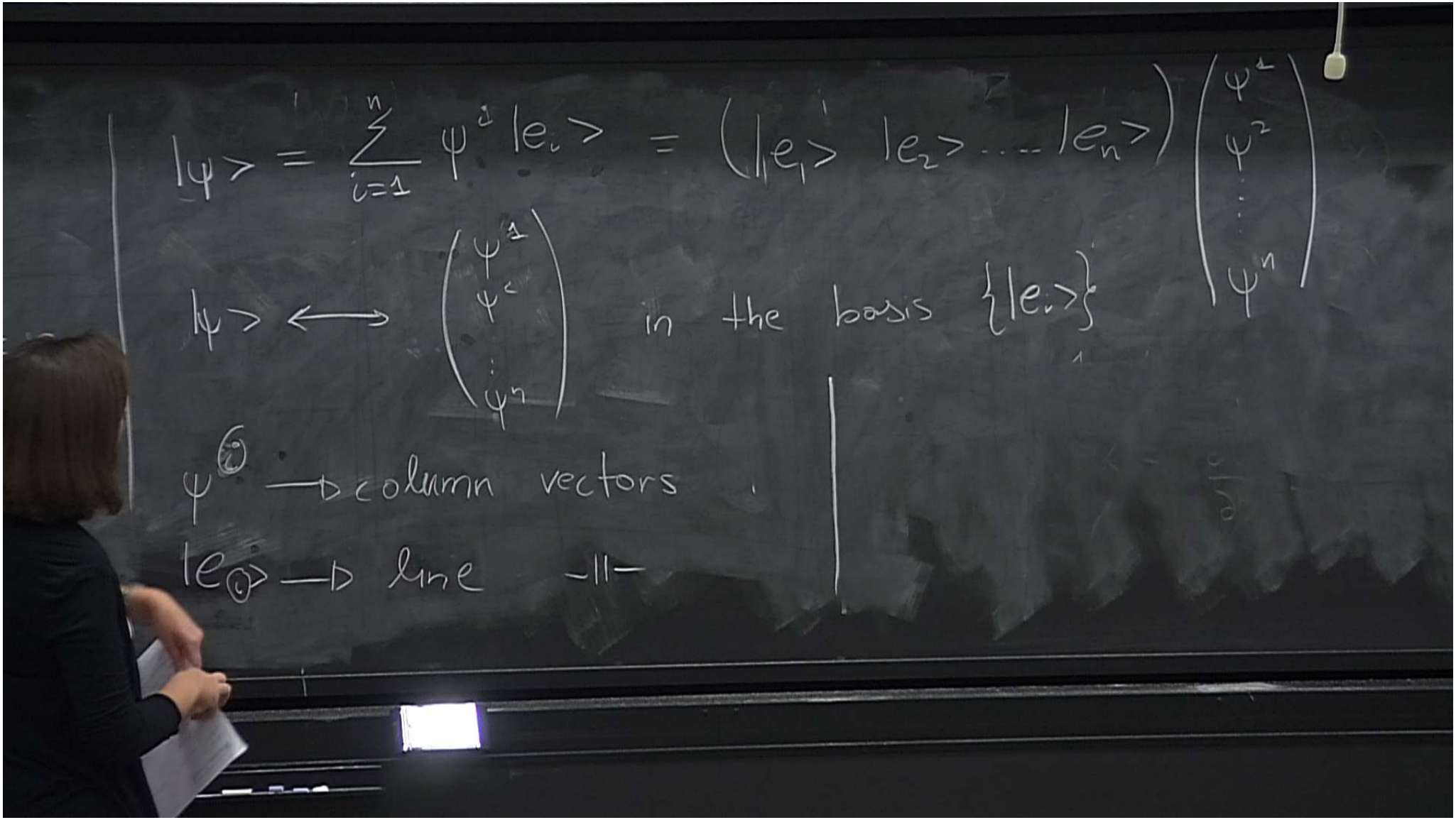

NOTATION – INDICES
$|\psi\rangle \in V$
$\{|e_i\rangle\}$: basis for V ; $\dim V = n$
$|\psi\rangle =$


$|\psi\rangle = \sum_{i=1}^{n} \psi^i |e_i\rangle = (|e_1\rangle \; |e_2\rangle \ldots |e_n\rangle)$
$V = n$


$|\psi\rangle = \sum_{i=1}^{n} \psi^i |e_i\rangle = (|e_1\rangle\ |e_2\rangle \dots |e_n\rangle) \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^n \end{pmatrix}$
$|\psi\rangle \longleftrightarrow \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^n \end{pmatrix}$ in the basis $\{|e_i\rangle\}$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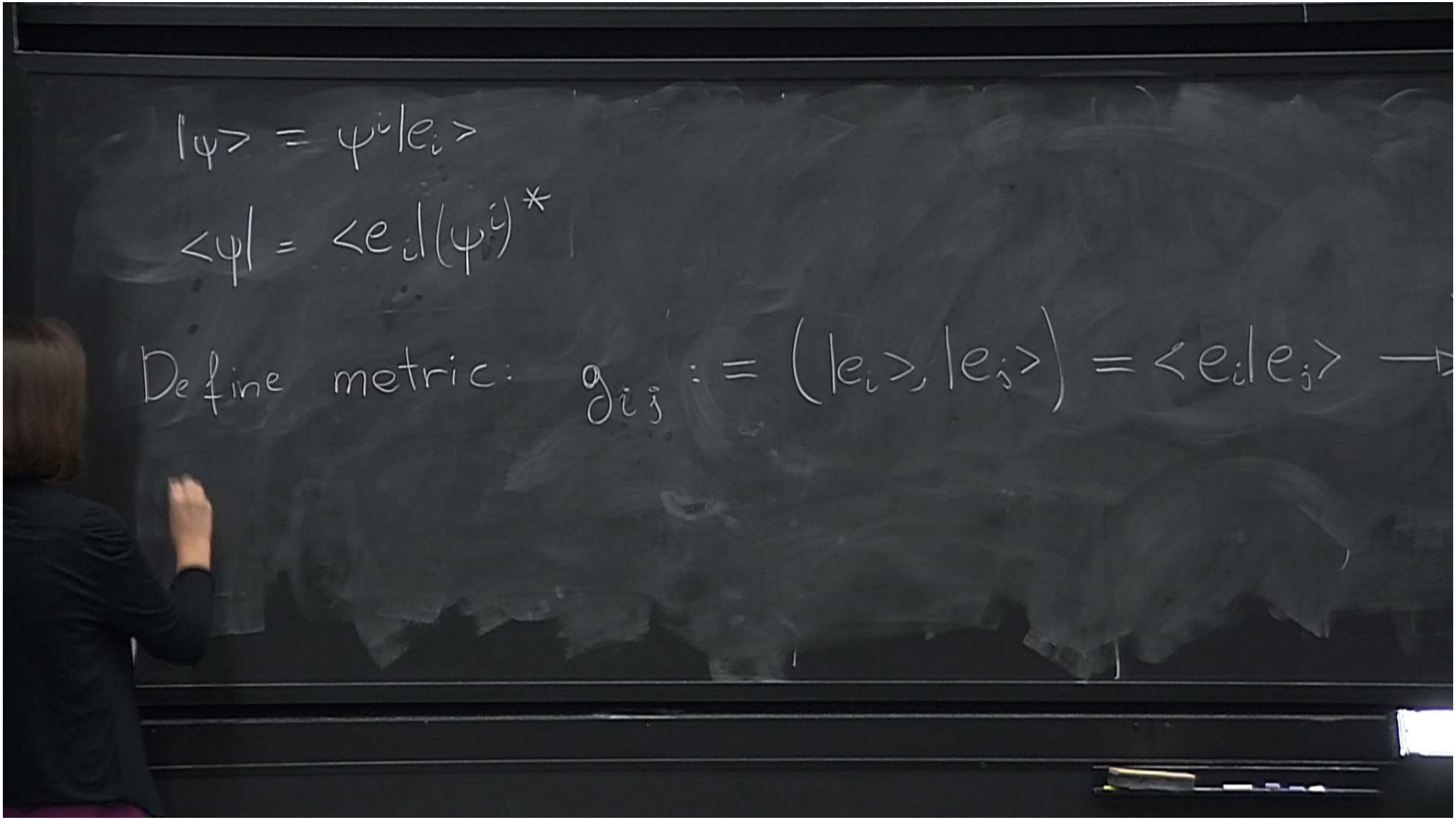
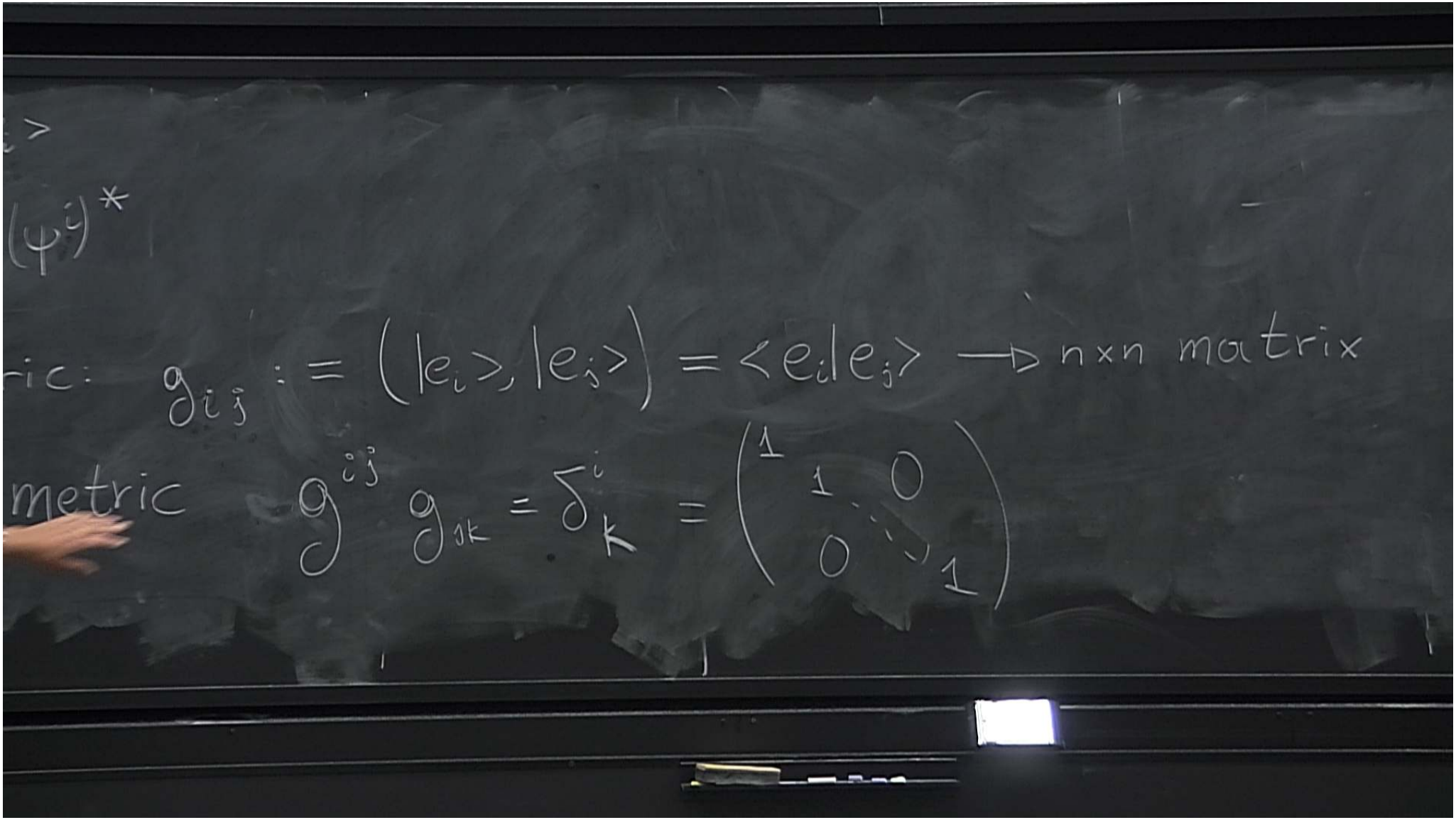
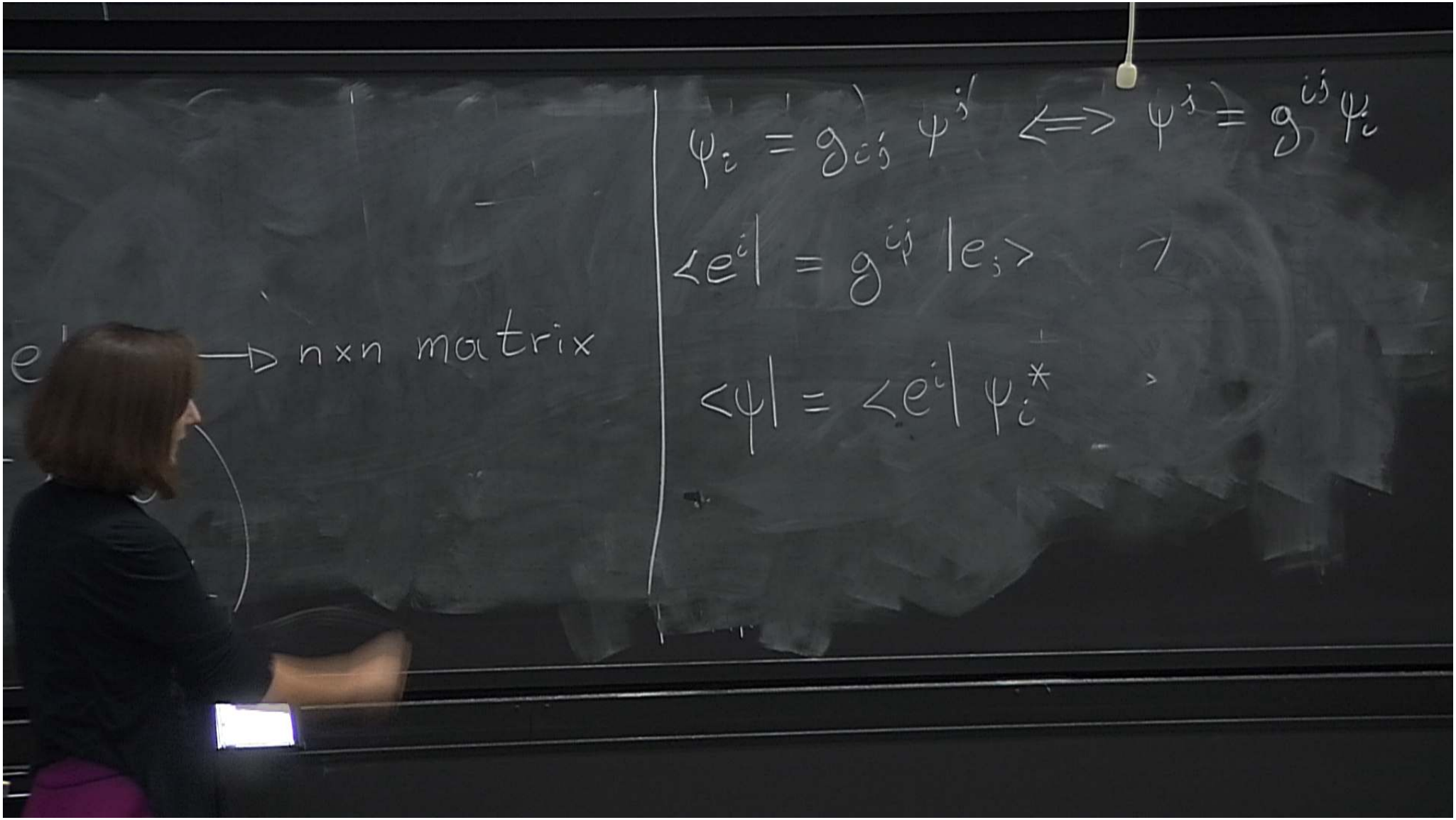
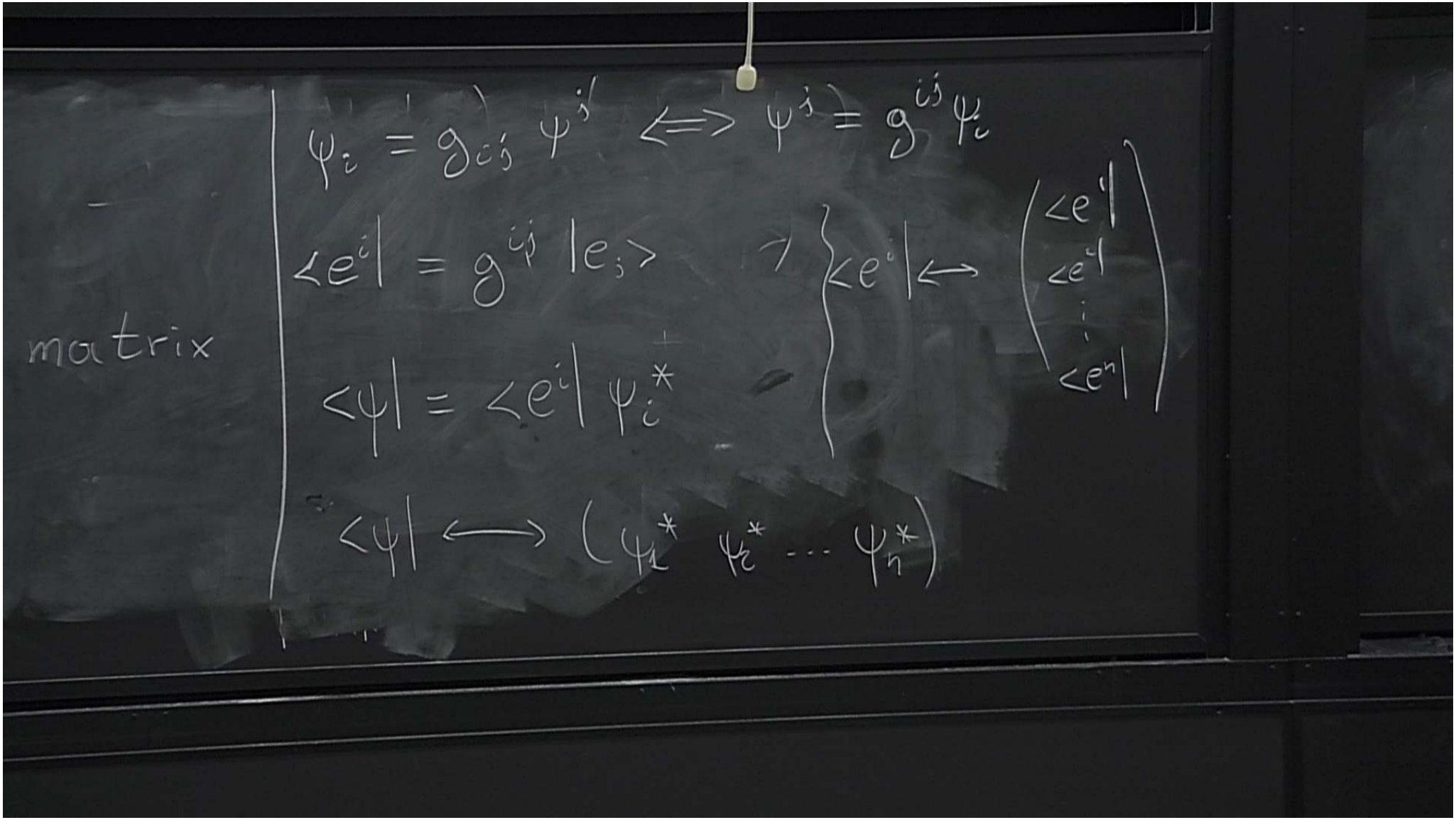
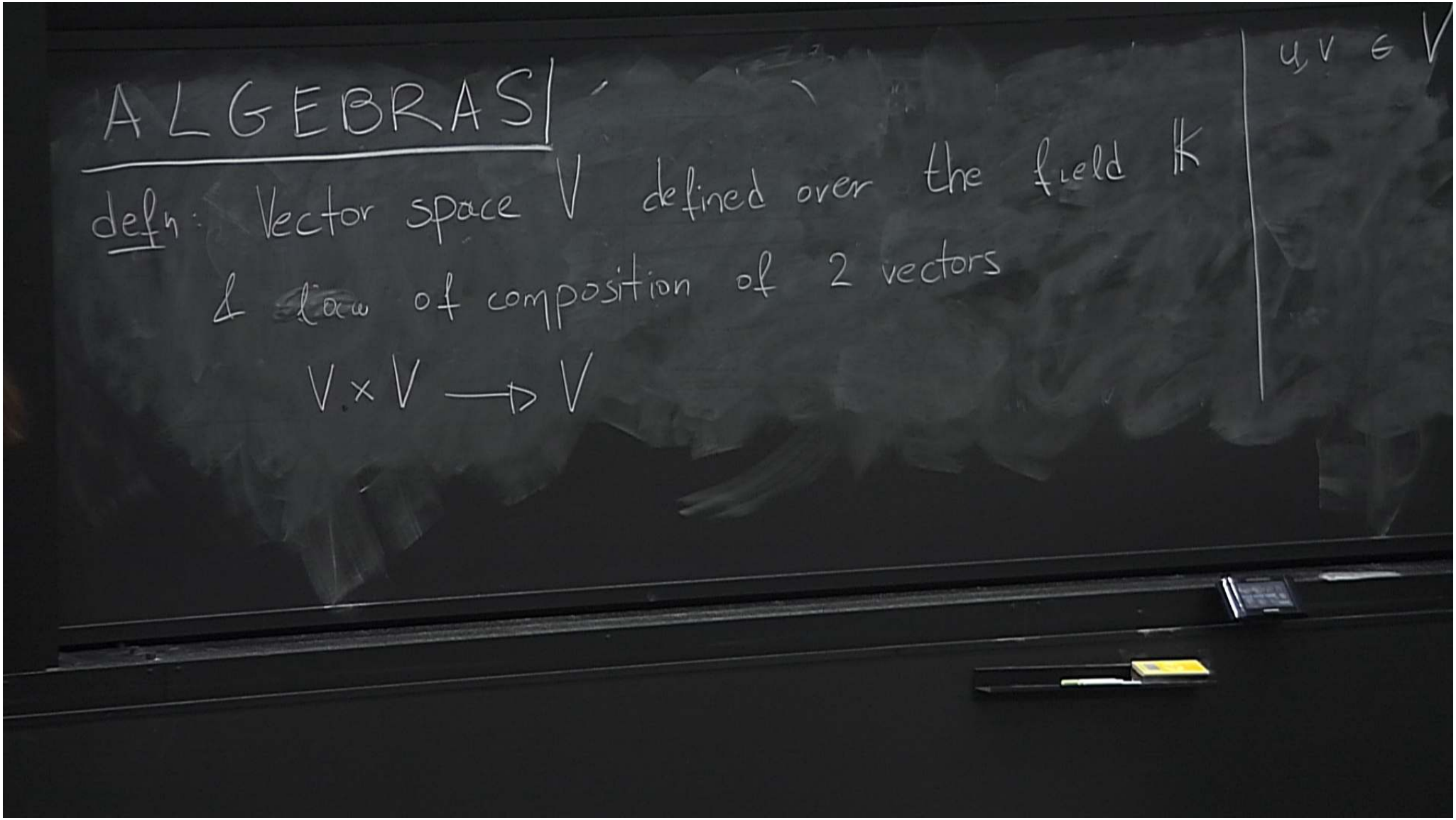
$|\psi\rangle \longleftrightarrow \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^n \end{pmatrix}$ in the basis $\{|e_i\rangle\}$
$\psi^{(i)}$ → column vectors
$|e_{(i)}\rangle$ → line -11-


$|\psi\rangle = \psi^i |e_i\rangle$
$\langle\psi| = \langle e_i| (\psi^i)^*$
Define metric: $g_{ij} := (|e_i\rangle, |e_j\rangle) = \langle e_i | e_j \rangle \rightarrow$

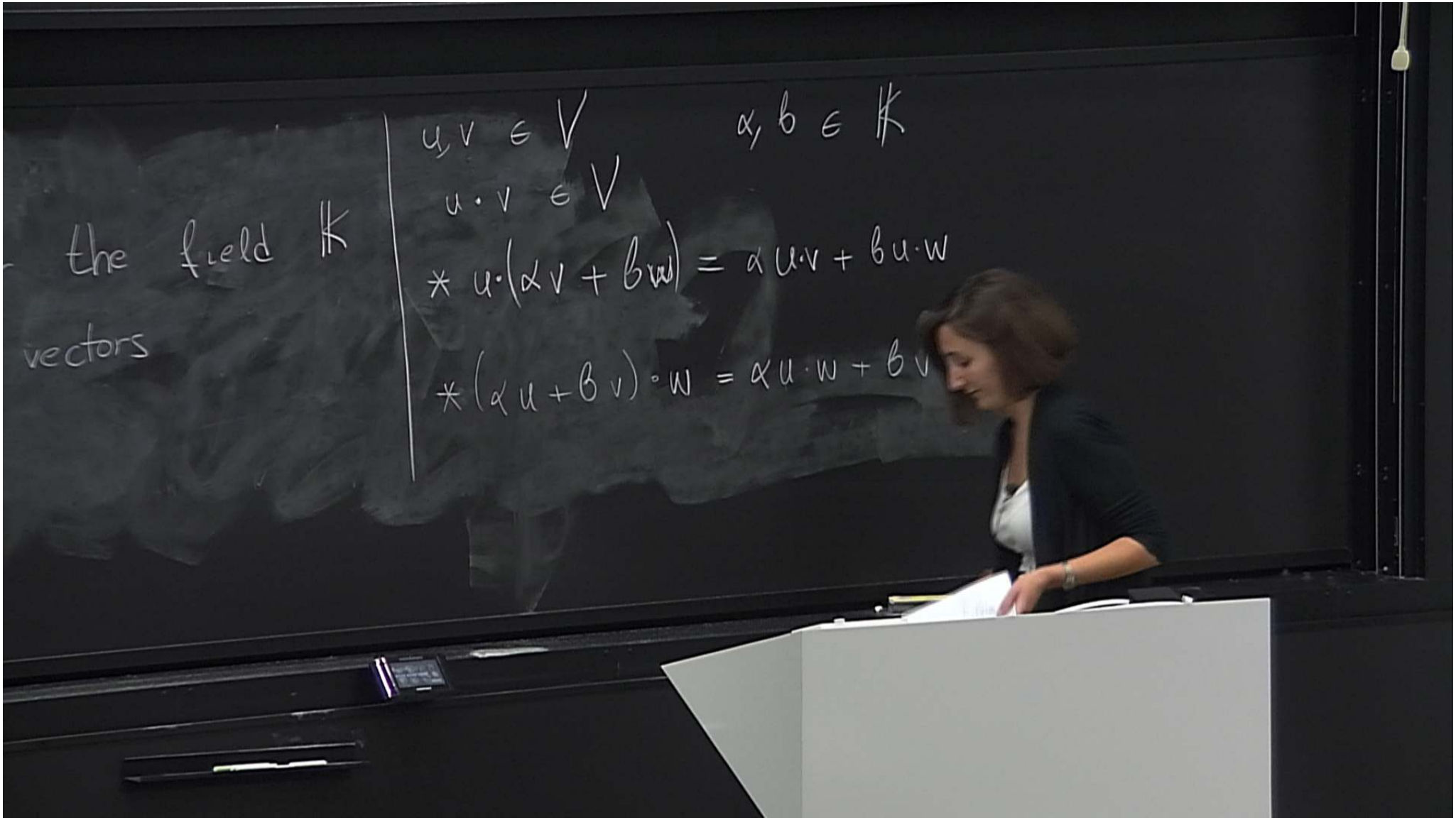

$|\rangle$
$(\psi^i)^*$
ric: $g_{ij} := (|e_i\rangle, |e_j\rangle) = \langle e_i | e_j \rangle \longrightarrow n \times n$ matrix
metric $g^{ij}\, g_{jk} = \delta^i_k = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

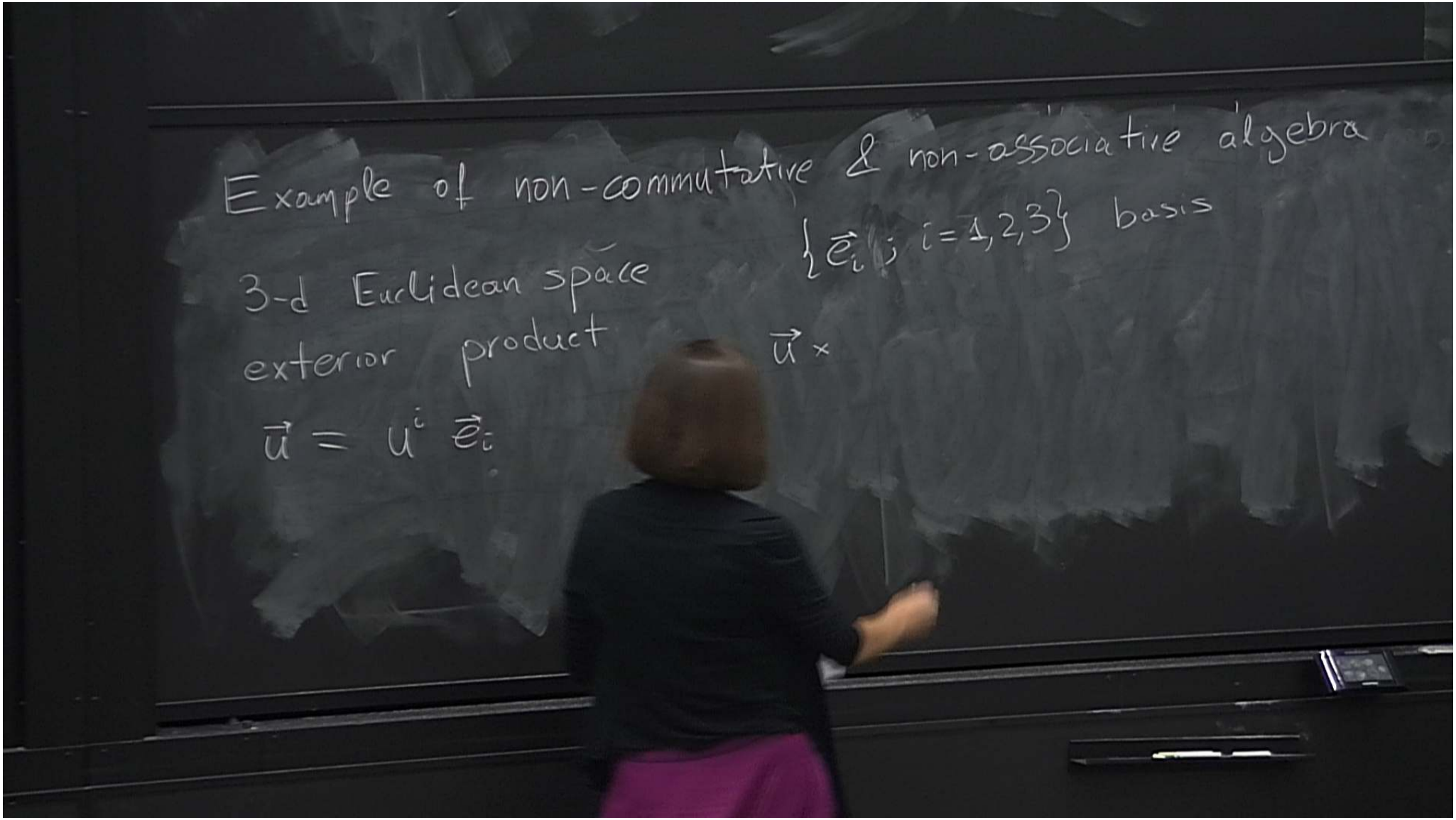

$\psi_i = g_{ij}\,\psi^j \iff \psi^i = g^{ij}\psi_i$
$\langle e^i| = g^{ij}\,|e_j\rangle$
$\langle\psi| = \langle e^i|\,\psi_i^*$
$e^i \longrightarrow n \times n$ matrix

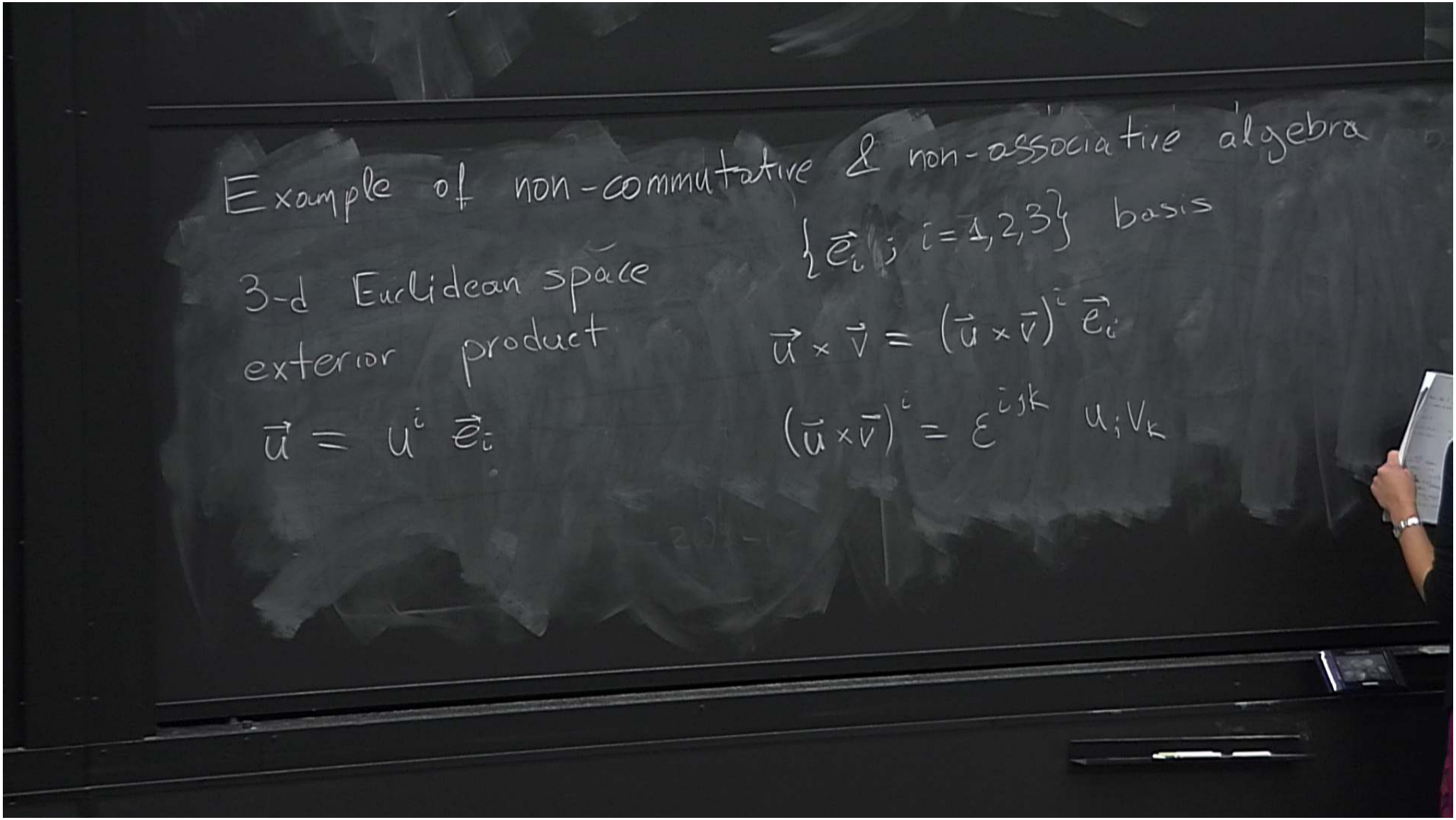

matrix
$\psi_i = g_{ij}\,\psi^j \iff \psi^i = g^{ij}\psi_j$
$\langle e^i| = g^{ij}\,|e_j\rangle$
$\langle\psi| = \langle e^i|\,\psi_i^*$
$\langle\psi| \longleftrightarrow (\psi_1^* \;\psi_2^* \cdots \psi_n^*)$
$\langle e^i| \leftrightarrow \begin{pmatrix} \langle e^1| \\ \langle e^2| \\ \vdots \\ \langle e^n| \end{pmatrix}$


ALGEBRAS
defn: Vector space V defined over the field K
& law of composition of 2 vectors
V × V → V
u, v ∈ V

ALGEBRAS


the field K
vectors
u, v ∈ V
α, b ∈ K
u · v ∈ V
* u·(αv + βw) = αu·v + βu·w
* (αu + βv)·w = αu·w + βv


Example of non-commutative & non-associative algebra
3-d Euclidean space
$\{\vec{e}_i ; i=1,2,3\}$ basis
exterior product
$\vec{u} \times$
$\vec{u} = u^i \vec{e}_i$


Example of non-commutative & non-associative algebra
3-d Euclidean space
exterior product
$\vec{u} = u^i \vec{e}_i$
$\{\vec{e}_i ; i = 1,2,3\}$ basis
$\vec{u} \times \vec{v} = (\vec{u} \times \vec{v})^i \vec{e}_i$
$(\vec{u} \times \vec{v})^i = \varepsilon^{ijk} u_j v_k$

$$\left[(\vec{u}\times\vec{v})\times\vec{w}-\vec{u}\times(\vec{v}\times\vec{w})\right]^i =$$

$$= \varepsilon^{ijk}\varepsilon^{jlm}u^l v^m w^k - \varepsilon^{ilj}\varepsilon^{jmk}u^l v^m w^k$$

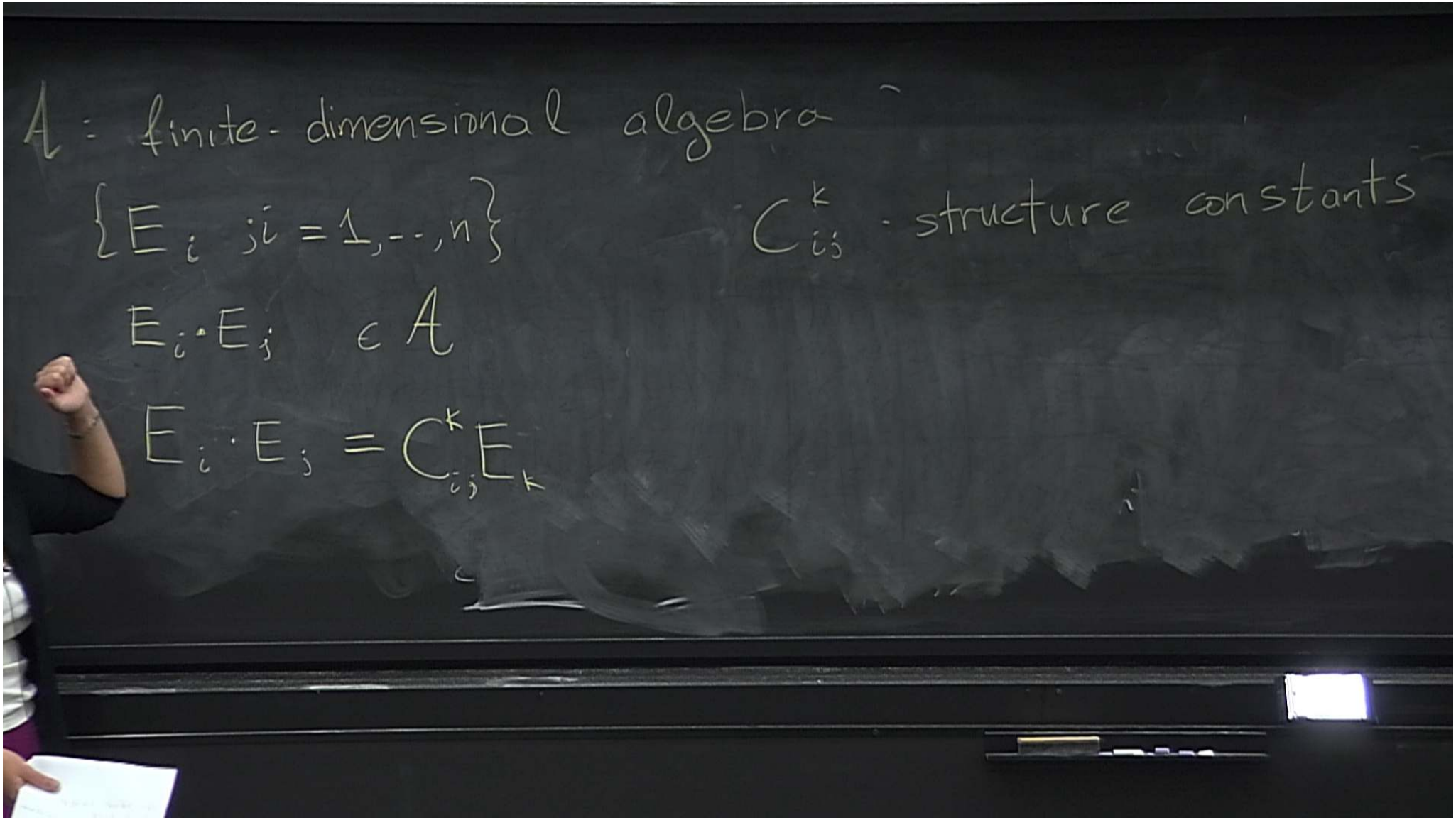
$$= (\varepsilon^{ijk}\varepsilon^{jlm} - \varepsilon^{ilj}\varepsilon^{jmk})\,u^l v^m w^k$$

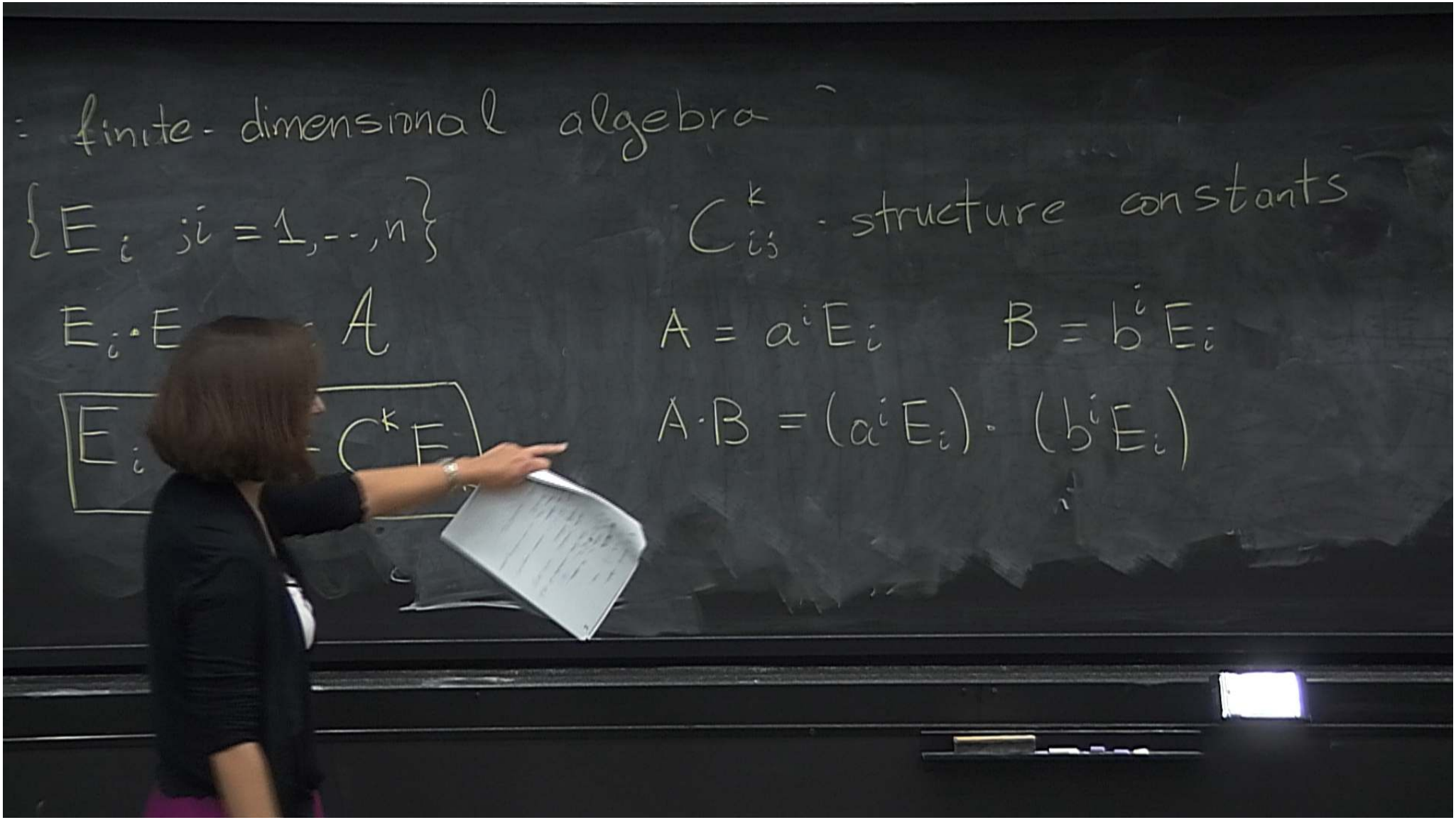
$$= (-\varepsilon^{jik}\varepsilon^{jlm} + \varepsilon^{jli}\varepsilon^{jmk})\,u^l v^m w^k$$

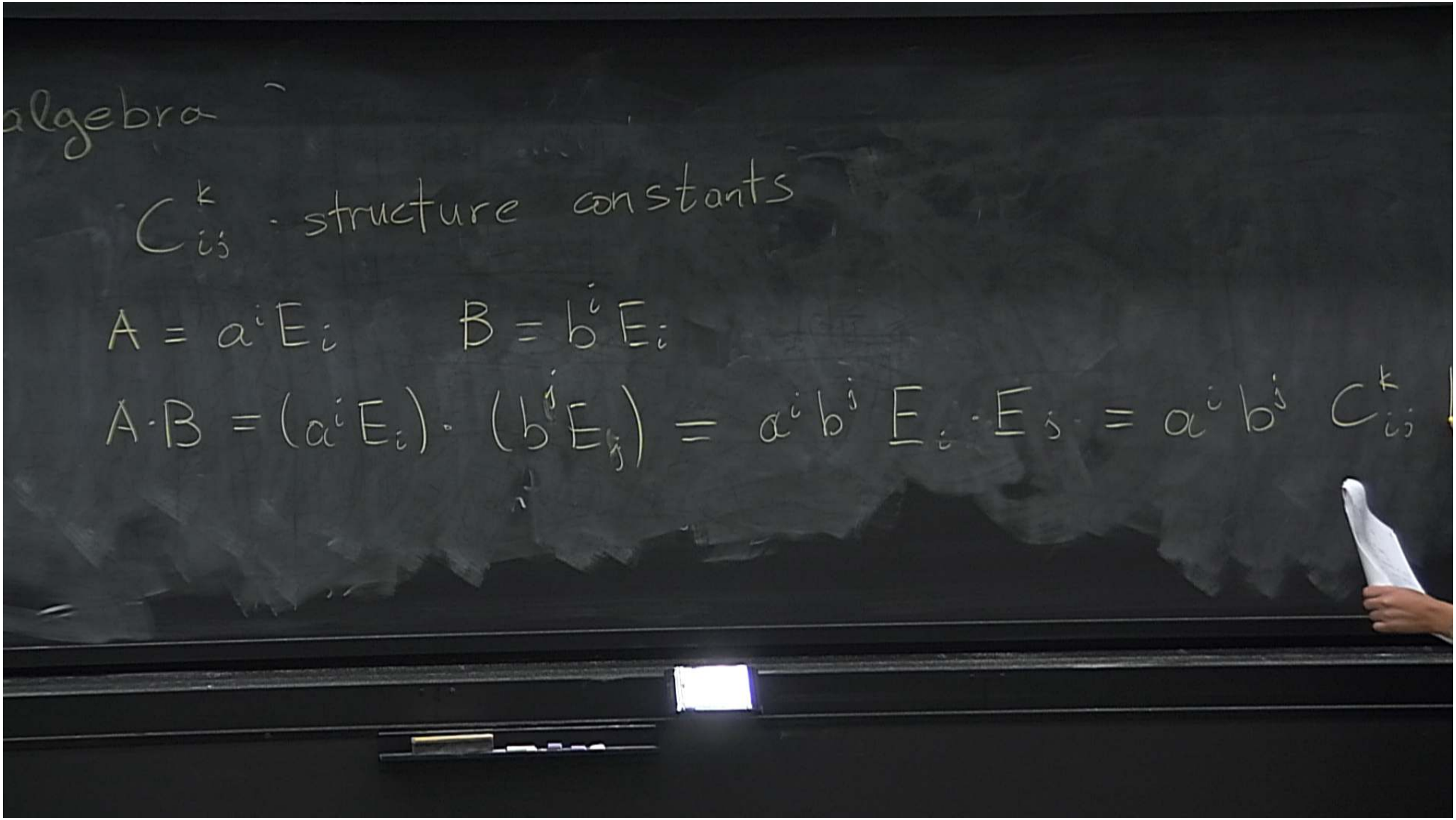
$$= (-\delta^{il}\delta^{km} + \delta^{im}\delta^{kl} + \delta^{lm}\delta^{ik} - \delta^{lk}\delta^{im})\,u^l v^m w^k$$

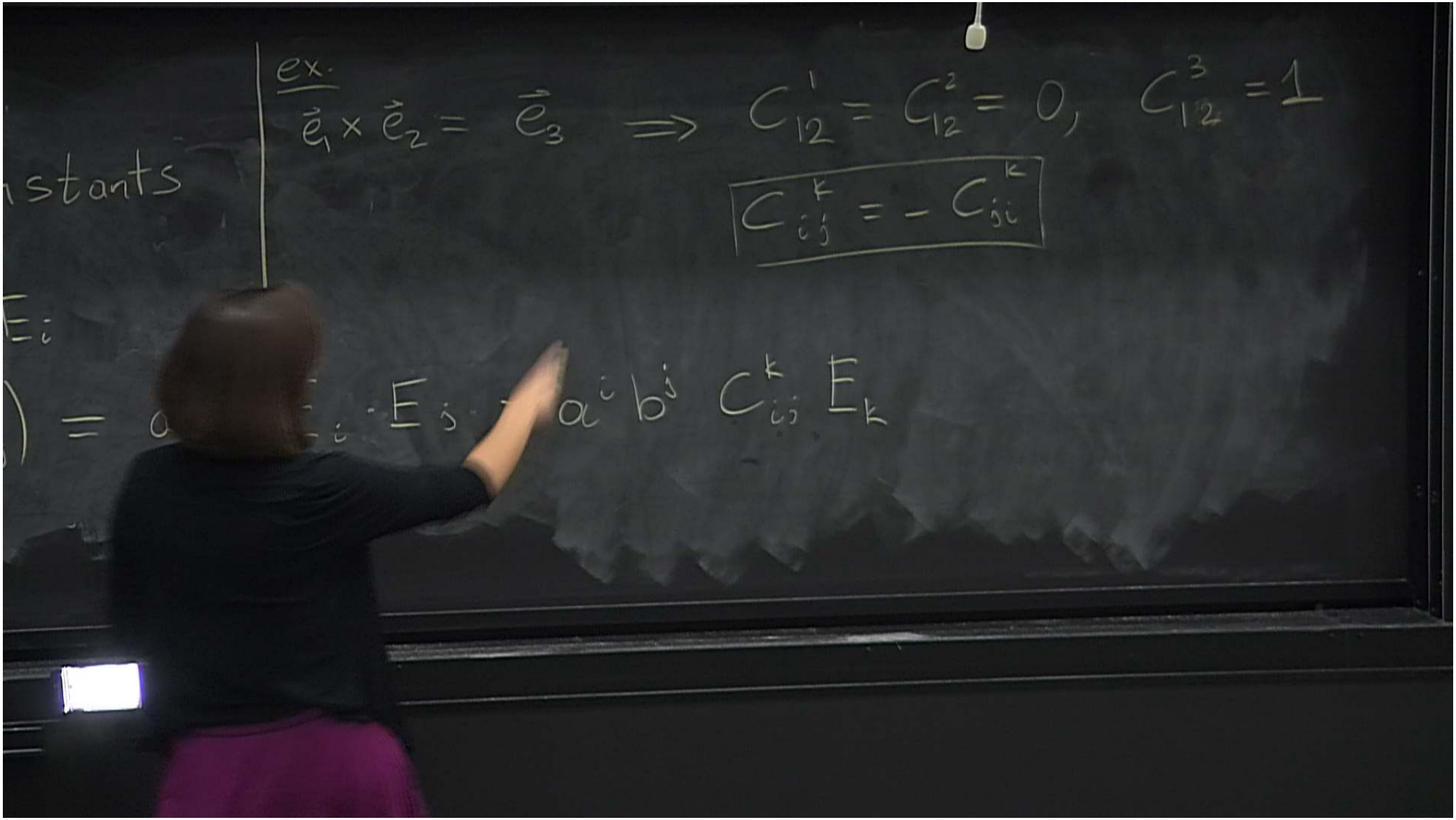
$$= -u^i v\cdot w + \cancel{v^i u\cdot w} + w^i u\cdot v - \cancel{v^i u\cdot w}$$

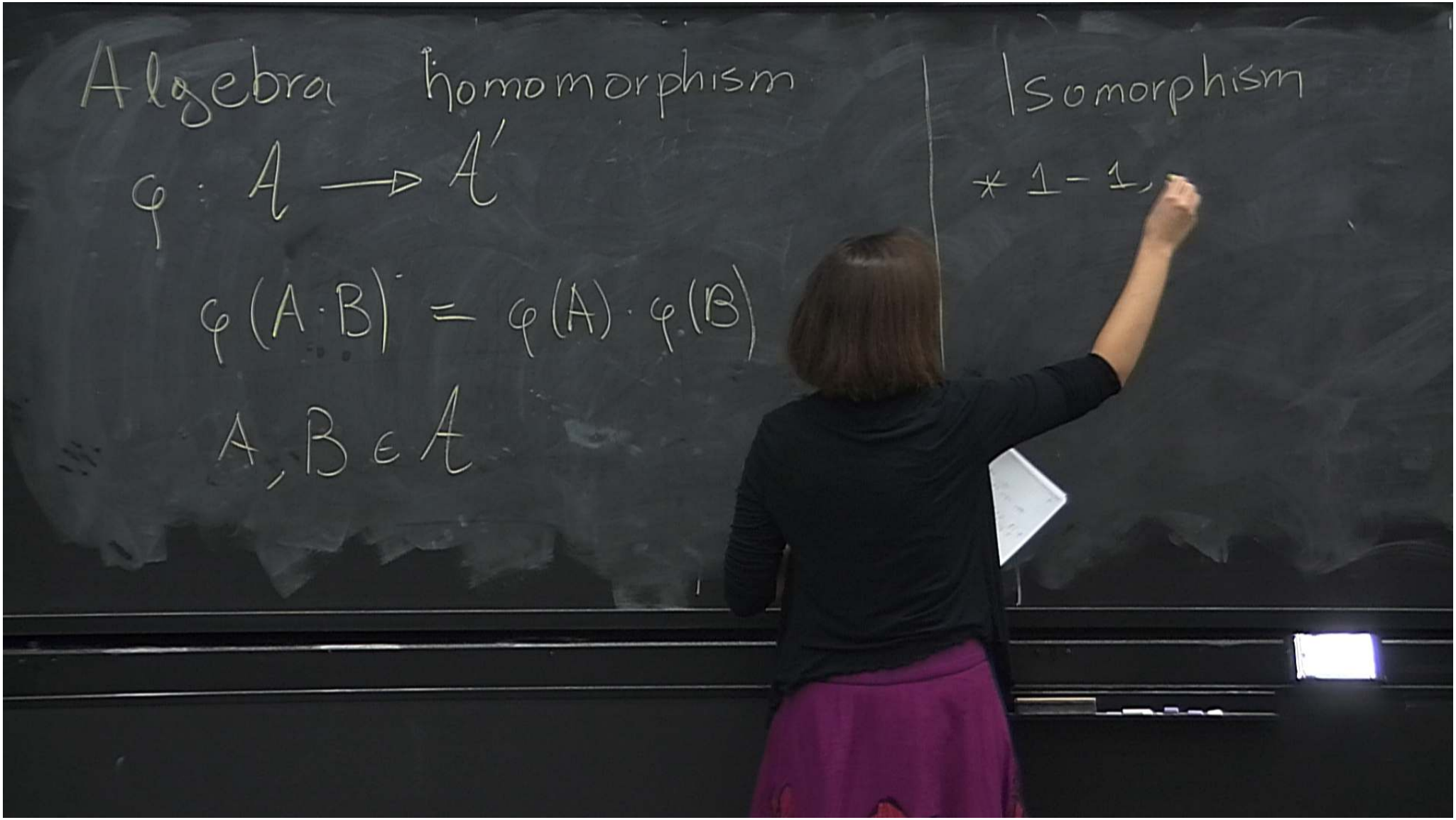
$$\varepsilon^{ijk}\varepsilon^{ijk} = \delta^{ii}\delta^{jj} - \delta$$

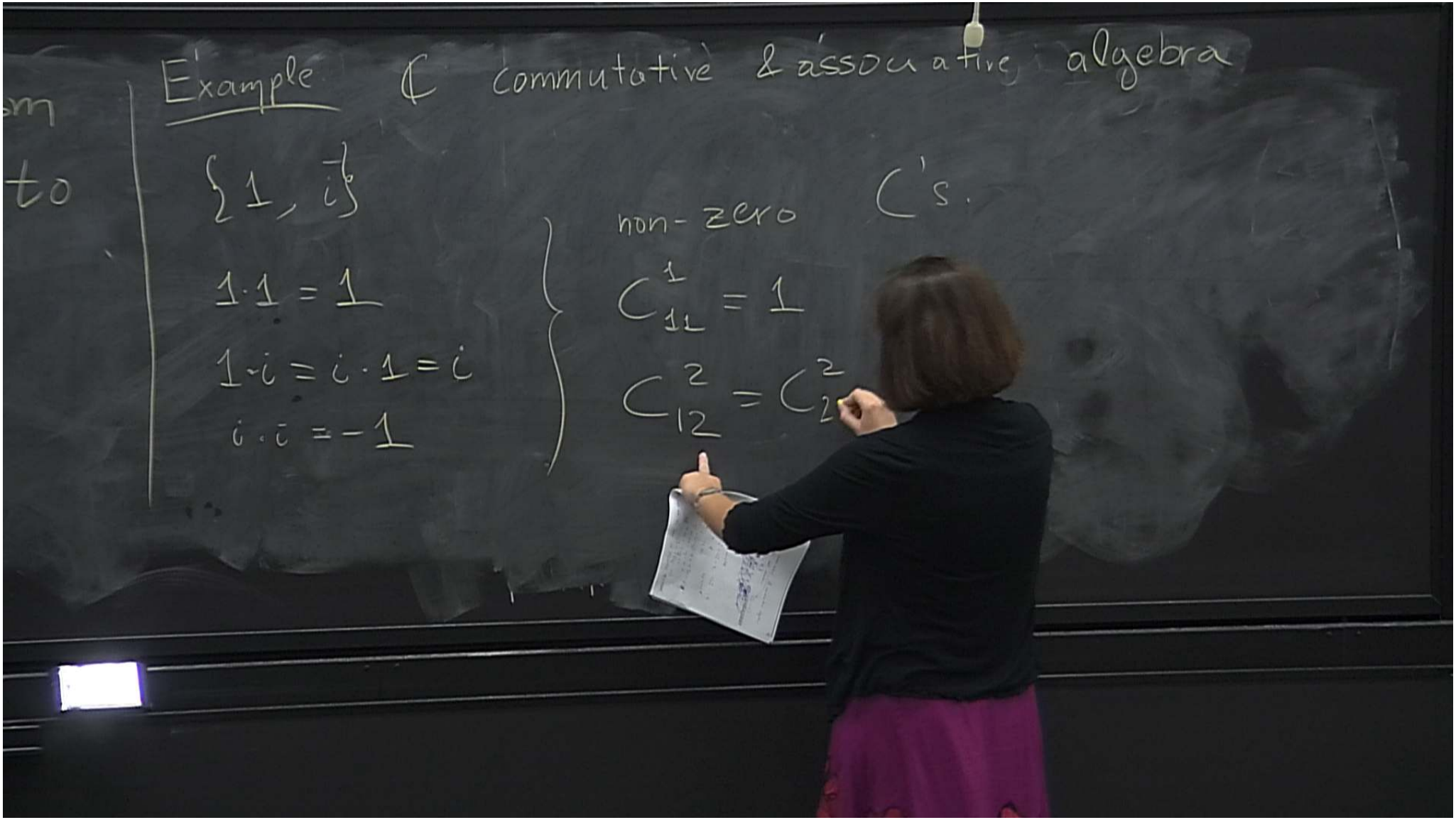

A : finite-dimensional algebra
$\{E_i \; ; i = 1, \dots, n\}$
C_{ij}^{k} - structure constants
$E_i \cdot E_j \in A$
$E_i \cdot E_j = C_{ij}^{k} E_k$

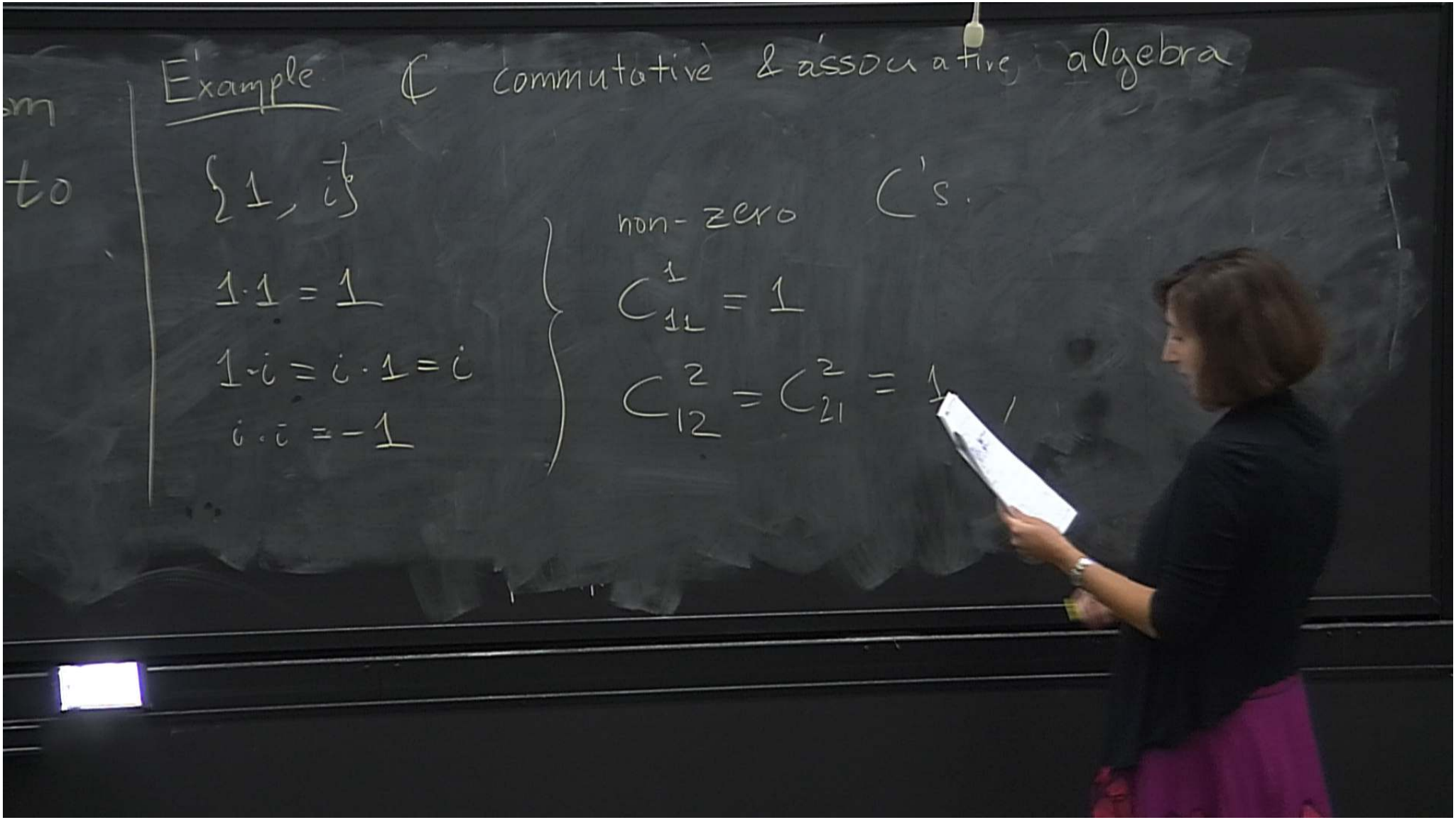

finite-dimensional algebra
$\{E_i \; ; i = 1, \dots, n\}$
C^k_{ij} - structure constants
$A = a^i E_i$
$B = b^i E_i$
$A \cdot B = (a^i E_i) \cdot (b^i E_i)$


algebra
C^k_{ij} - structure constants
$A = a^i E_i \qquad B = b^i E_i$
$A \cdot B = (a^i E_i) \cdot (b^j E_j) = a^i b^j \underline{E}_i \cdot \underline{E}_j = a^i b^j C^k_{ij}$


ex.
$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3 \Rightarrow C^1_{12} = C^2_{12} = 0, \quad C^3_{12} = 1$
$C^k_{ij} = -C^k_{ji}$
E_i
$= a^i b^j C^k_{ij} E_k$


Algebra homomorphism
$\varphi : \mathcal{A} \longrightarrow \mathcal{A}'$
$\varphi(A \cdot B) = \varphi(A) \cdot \varphi(B)$
$A, B \in \mathcal{A}$
Isomorphism
* 1 - 1,


Example
commutative & associative algebra
{1, i}
1·1 = 1
1·i = i·1 = i
i·i = −1
non-zero
's.


Example ℂ commutative & associative algebra
{1, i}
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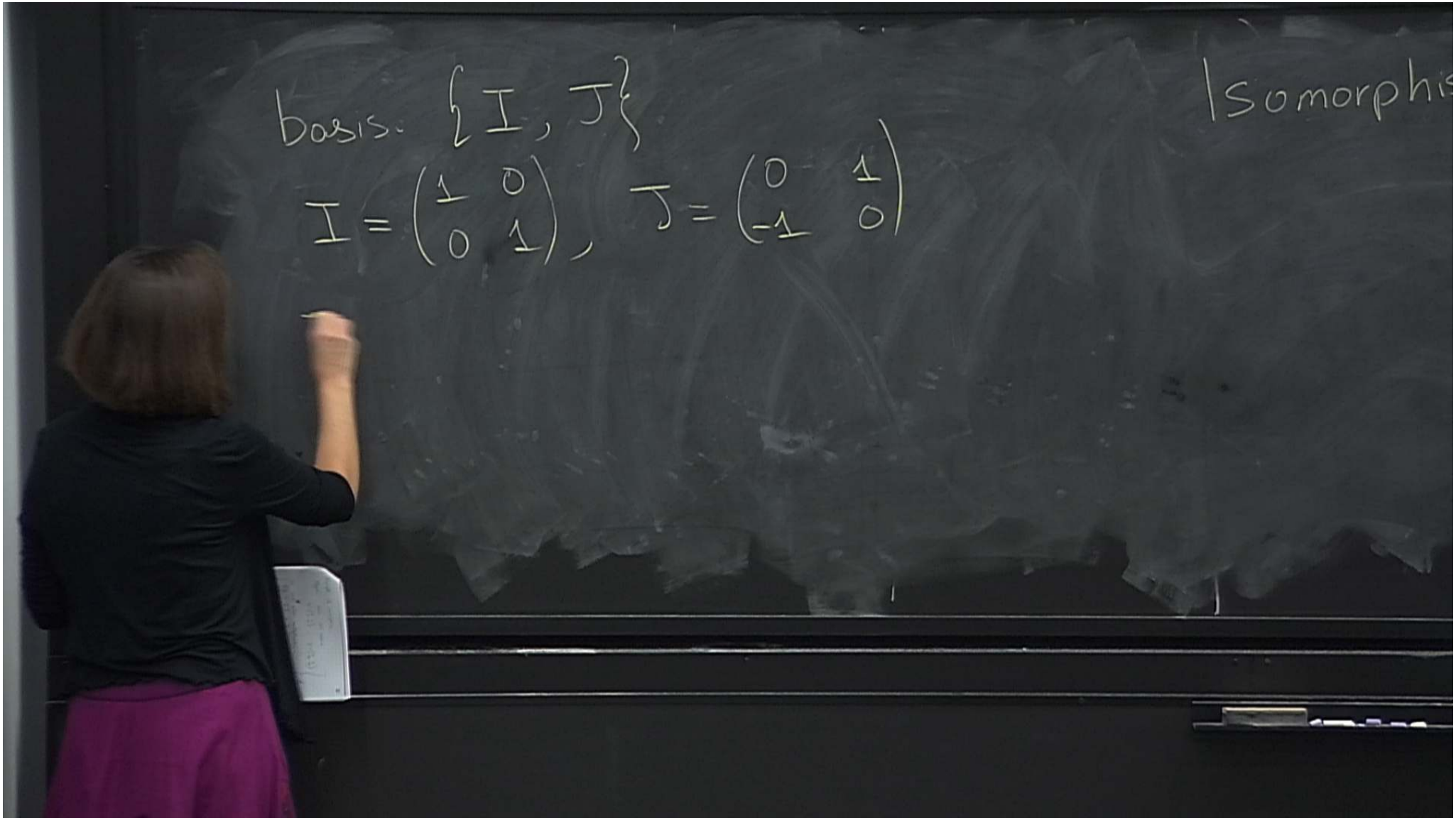
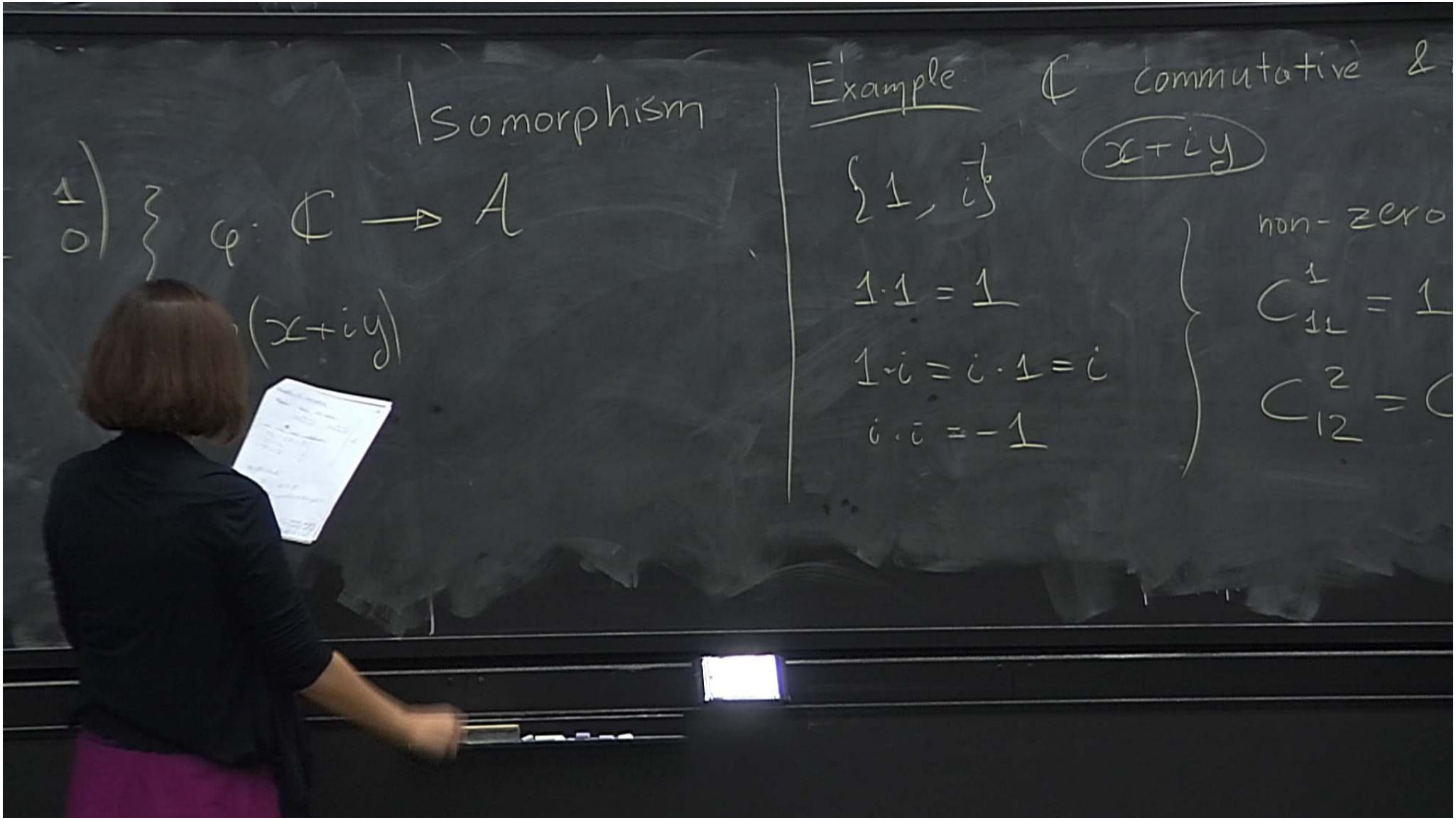
1·i = i·1 = i
i·i = −1
non-zero
$C_{11}^1 = 1$
$C_{12}^2 = C_{21}^2 = 1$


basis: {I, J}
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
Isomorphis


Isomorphism
1)
0
φ: ℂ → A
(x+iy)
Example
ℂ commutative &
x+iy
{1, i}
1·1 = 1
1·i = i·1 = i
i·i = −1
non-zero
$C_{11}^{1} = 1$
$C_{12}^{2} =$