Title: Hamiltonian Constraint in 3D Quantum Gravity

Date: Aug 09, 2011 02:30 PM

URL: http://pirsa.org/11080072

Abstract:

Hamiltonian constraint in 3d quantum gravity

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August 9, 2011

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Question

How to quantize gravity ?

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Two main approaches to quantize a theory

- Lagrangian (path-integral) approach $\rightsquigarrow$ non-renormalizable theory.

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- Hamiltonian approach ?

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |\,\psi\,\rangle = \hat{H}\,|\,\psi\,\rangle$$

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- Lagrangian (path-integral) approach $\rightsquigarrow$ non-renormalizable theory.
- Hamiltonian approach ?

$$i\hbar \frac{\partial}{\partial t} | \psi \rangle = \hat{H} | \psi \rangle$$

Problem: in General Relativity, there is no time!

- Hamiltonian formulation of GR $\Longrightarrow$ dynamics encoded in *constraints*.
- Quantization of those constraints?

Outline

1. From classical to quantum gravity
2. Spin network states
3. Hamiltonian constraint and recurrence relations

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Classical general relativity in 3d

- Dynamics of general relativity

$$S_{\mathrm{EH}}[g] = \int_{\mathcal{M}} R\sqrt{\det(g)}\,\mathrm{d}^3x \quad \overset{\delta S_{\mathrm{EH}}/\delta g_{\mu\nu}=0}{\Longrightarrow} \quad R_{\mu\nu} = 0.$$

- In 3 dimensions: linear relation between Riemann and Ricci tensors,

$$R_{\mu\nu} = 0 \implies R_{\mu\nu\rho\sigma} = 0 \qquad \text{(flat spacetime).}$$

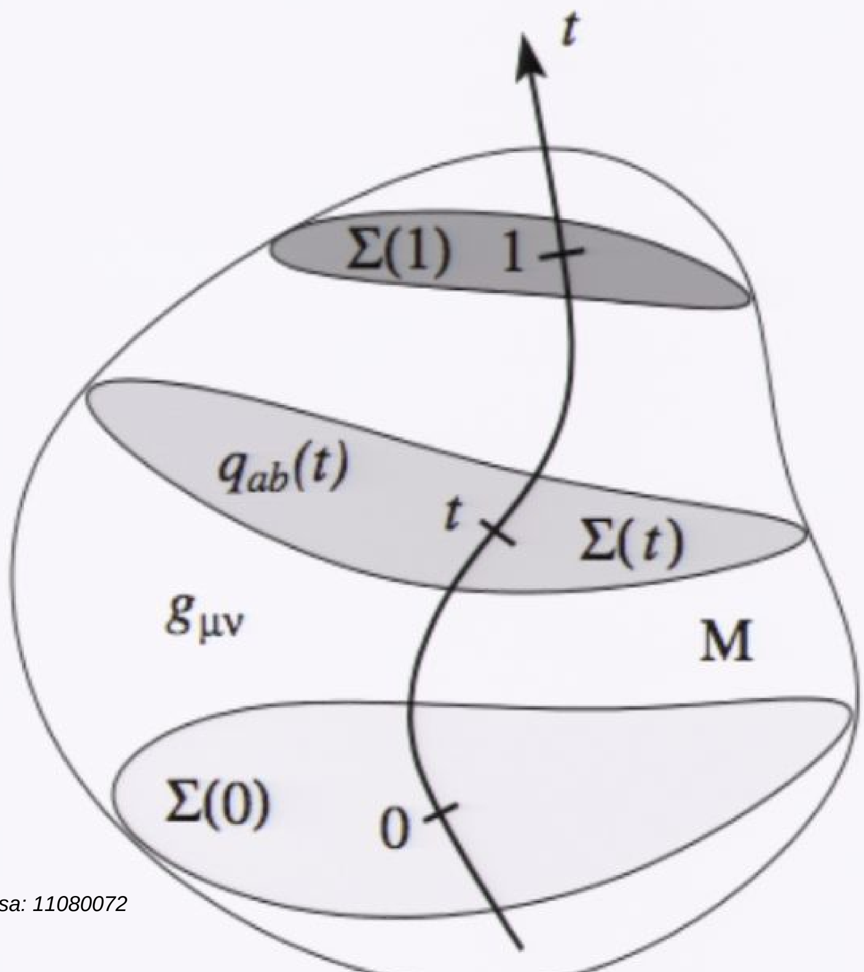
  Hence, 3d gravity has *no local degrees of freedom*.

- Physical consequences:
  - no gravitational force between point particles;
  - no gravitational waves.

Hamiltonian general relativity: ADM formalism

Fundamental equation of ADM formalism

Spacetime = Space + Time



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$g_{\mu\nu}$	q_{ab}, N, N^a
$R_{\mu\nu}$	R_{ab}, K_{ab}

- N = lapse, N^a = shift
- K_{ab} = extrinsic curvature

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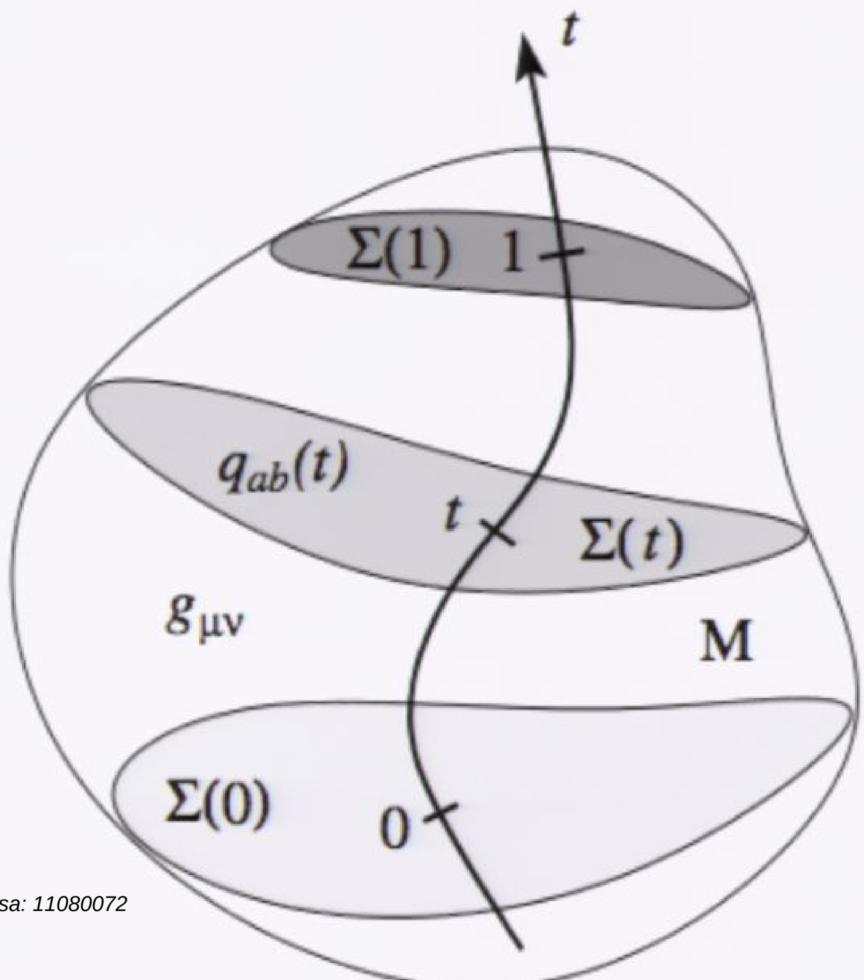

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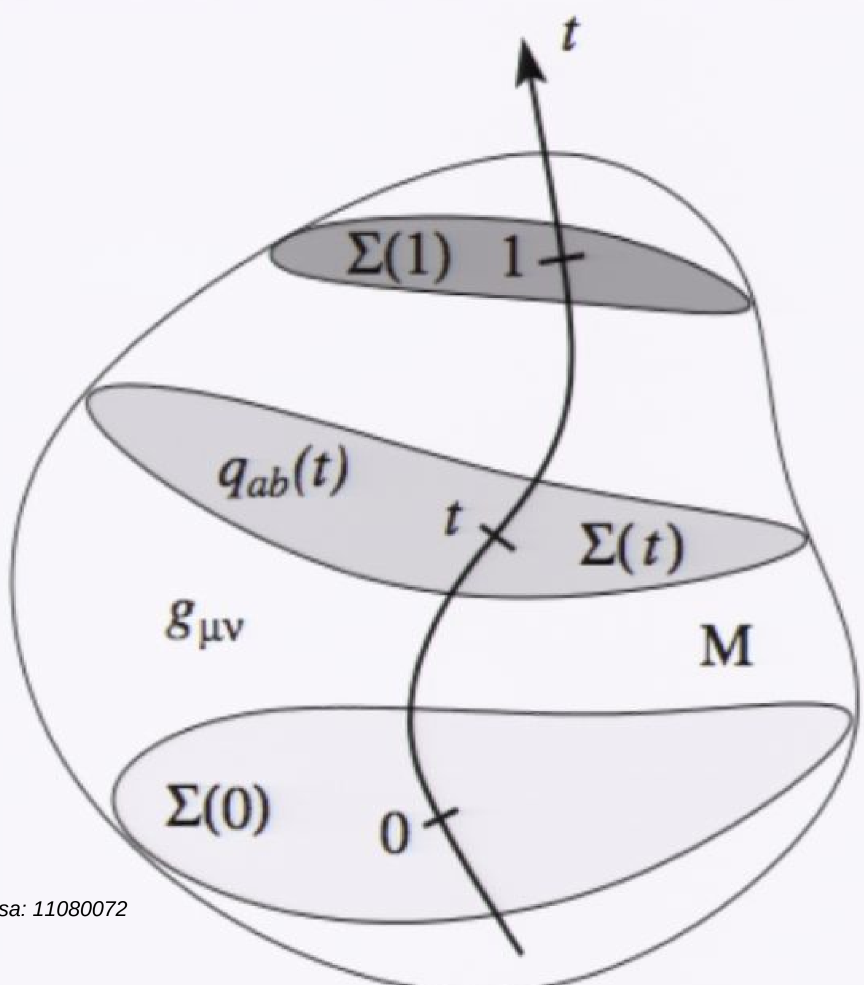

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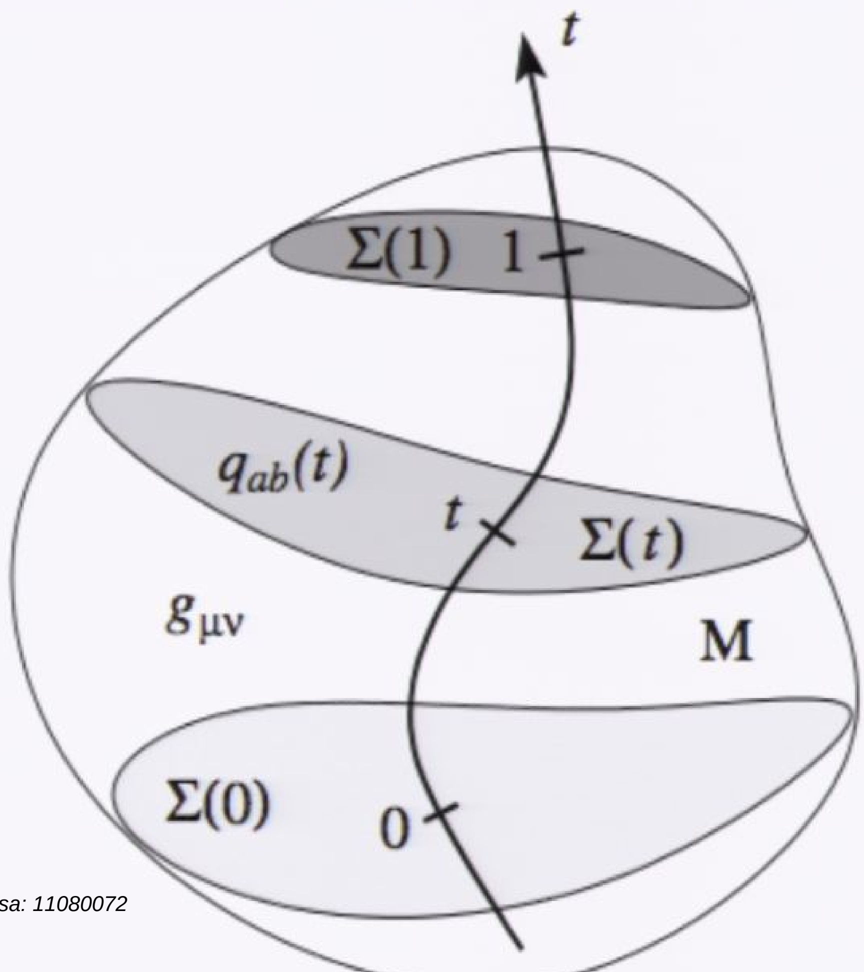

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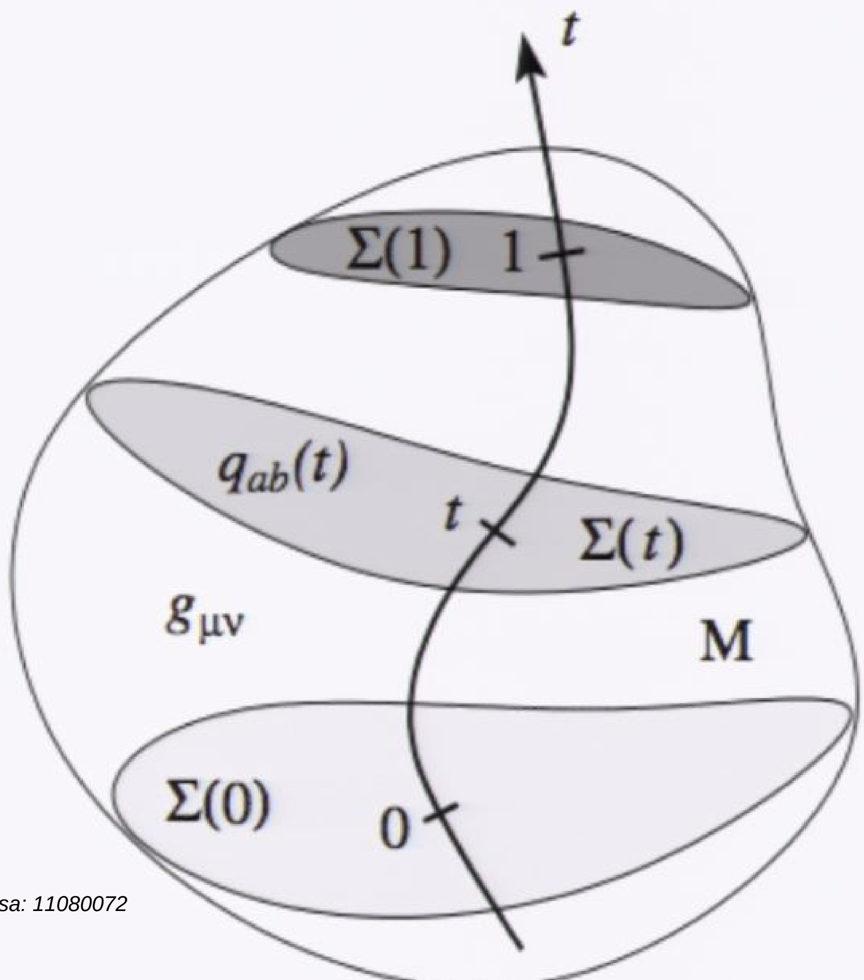

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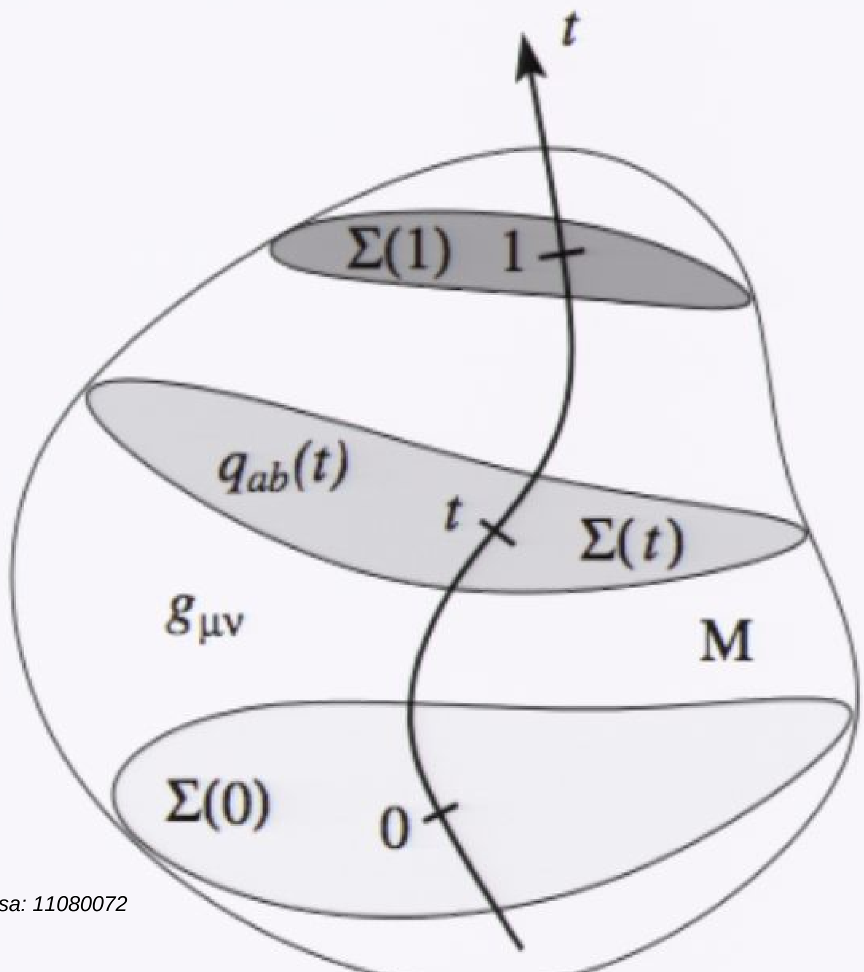

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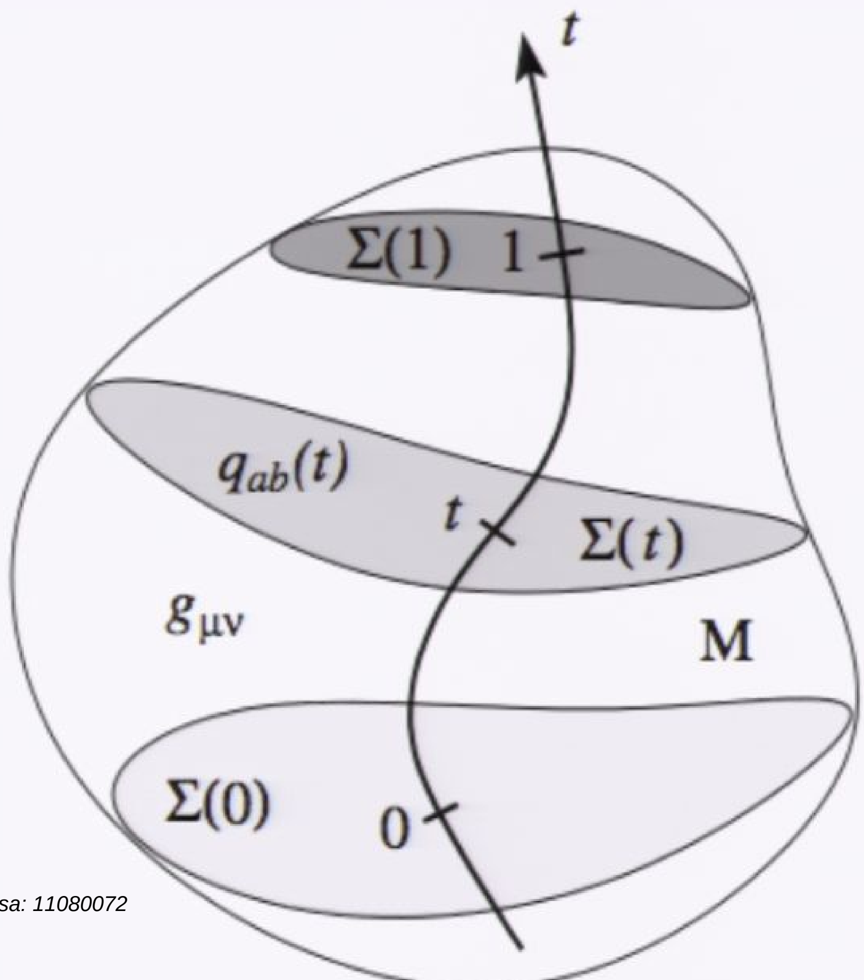

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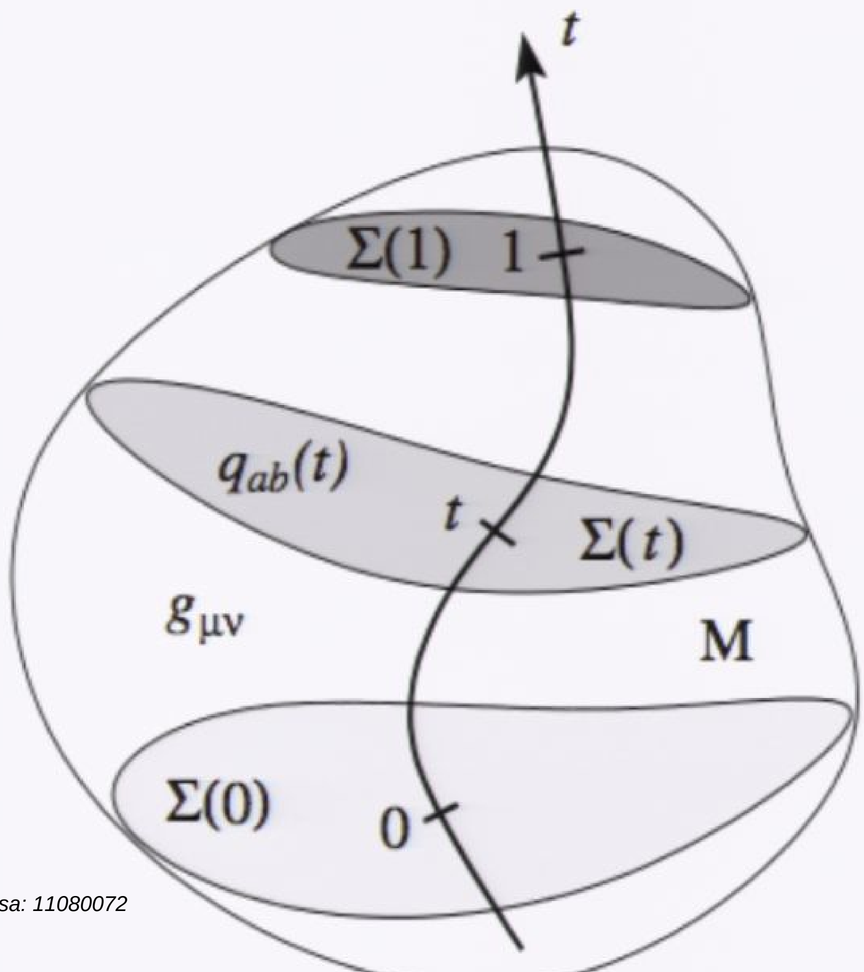

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Hamiltonian general relativity: constraints

The Lagrangian density of gravity $\mathcal{L}$ enables to define canonical momenta p and the Hamiltonian density $\mathcal{H}$

$$p^{ab} \overset{\text{def.}}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}}, \qquad \mathcal{H} \overset{\text{def.}}{=} p^{ab}\dot{q}_{ab} - \mathcal{L} = NC(q,p) + N^a V_a(q,p),$$

with

$$\frac{\delta S_{\text{EH}}}{\delta g_{\mu\nu}} = 0 \Longleftrightarrow \begin{cases} \dfrac{\delta S_{\text{EH}}}{\delta q_{ab}} = 0 & \text{Intrinsic dynamics} \\ C(q,p) = 0 & \textbf{Hamiltonian constraint} \\ V_a(q,p) = 0 & \text{Vector constraint} \end{cases}$$

$$C(q,p) = \frac{1}{\sqrt{\det q}}\left[\mathrm{tr}(p^2) - (\mathrm{tr}\, p)^2\right] - \sqrt{\det q}\,({}^2R)$$

Encodes crucial information *spacetime flatness*.

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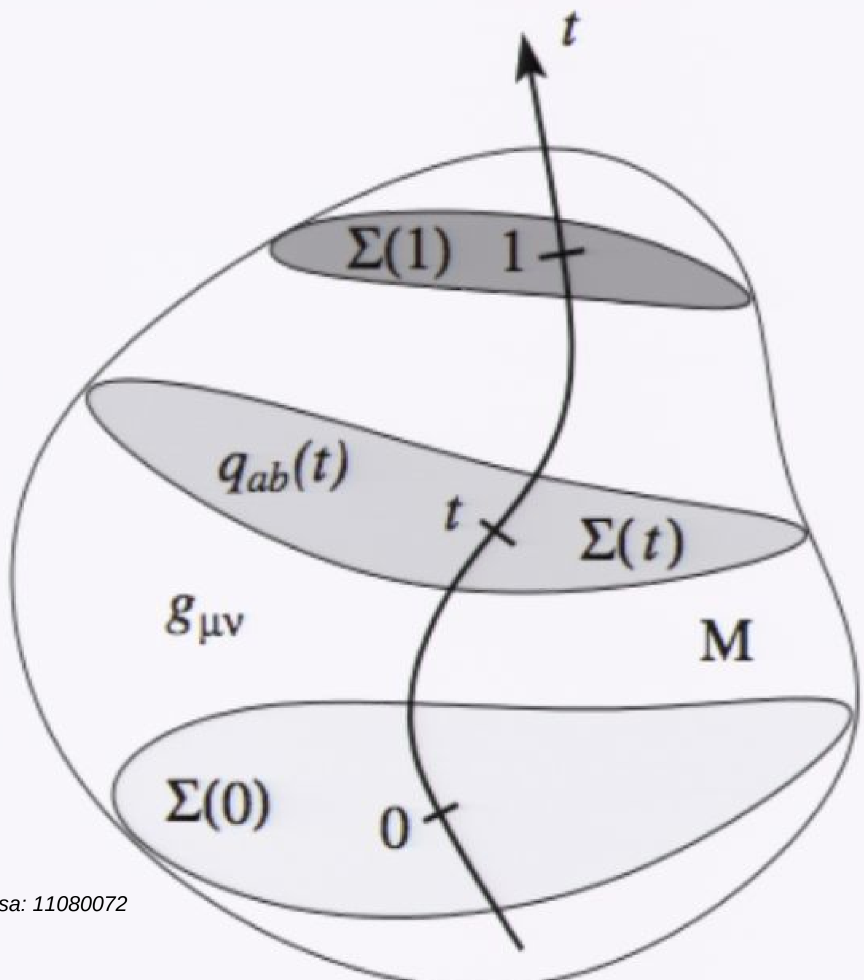

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Toward the quantization: triad and connection

Recall Einstein-Hilbert action

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Proposition

Promote the connection as an *independant* and *fundamental* variable.

New system of variables:

- **Connection one-form**: $A = A_\mu \mathrm{d}x^\mu, \quad A_\mu \in \mathfrak{so}(3) \cong \mathfrak{su}(2)$.
- **Triad / Dreibein**: $(E_1, E_2, E_3) = 3$ vector fields, local frame.

Einstein-Hilbert action $S_{\mathrm{EH}}[g] \longrightarrow S_{\mathrm{P}}[A, E]$ Palatini action.

Toward the quantization: triad and connection

Remarkable fact: connection and triad are *conjugate* in Palatini action

$$S_{\mathrm{P}}[A, E] = \frac{1}{8\pi\kappa} \int_0^1 \mathrm{d}t \int_{\Sigma(t)} \mathrm{d}^2x \left[E_i^a \dot{A}_a^i - \mathcal{H}(A, E) \right]$$

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The hamiltonian density remains a combination of constraints

$$\mathcal{H}(A,E) = NC + N^a V_a - A_t^i G_i$$

with

- C: Hamiltonian constraint, generates 'time' evolution. *Alternatively*: encodes embedding of 'space' in spacetime.
- V_a: Vector constraint, generates 'space' diffeomorphisms.
- G_i: Gauß constraint, generates $\mathrm{SU}(2)$ gauge transformations.

Loop quantization

Elementary QM	LQG
Position x	Connection A
Wavefunction $\psi(x)$	Cylindrical function $\psi_{\Gamma,f}[A]$
Position operator $\hat{X}\psi = x\,\psi(x)$	Connection operator $\hat{A} = A\,\psi_{\Gamma,f}[A]$
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Cylindrical functions

Γ is an oriented graph with N edges $e_1, e_2 \ldots, e_N$ and $f : (\mathrm{SU}(2))^N \to \mathbb{C}$,

$$\psi_{\Gamma,f}[A] \stackrel{\text{def.}}{=} f\Big(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_N}[A]\Big)$$

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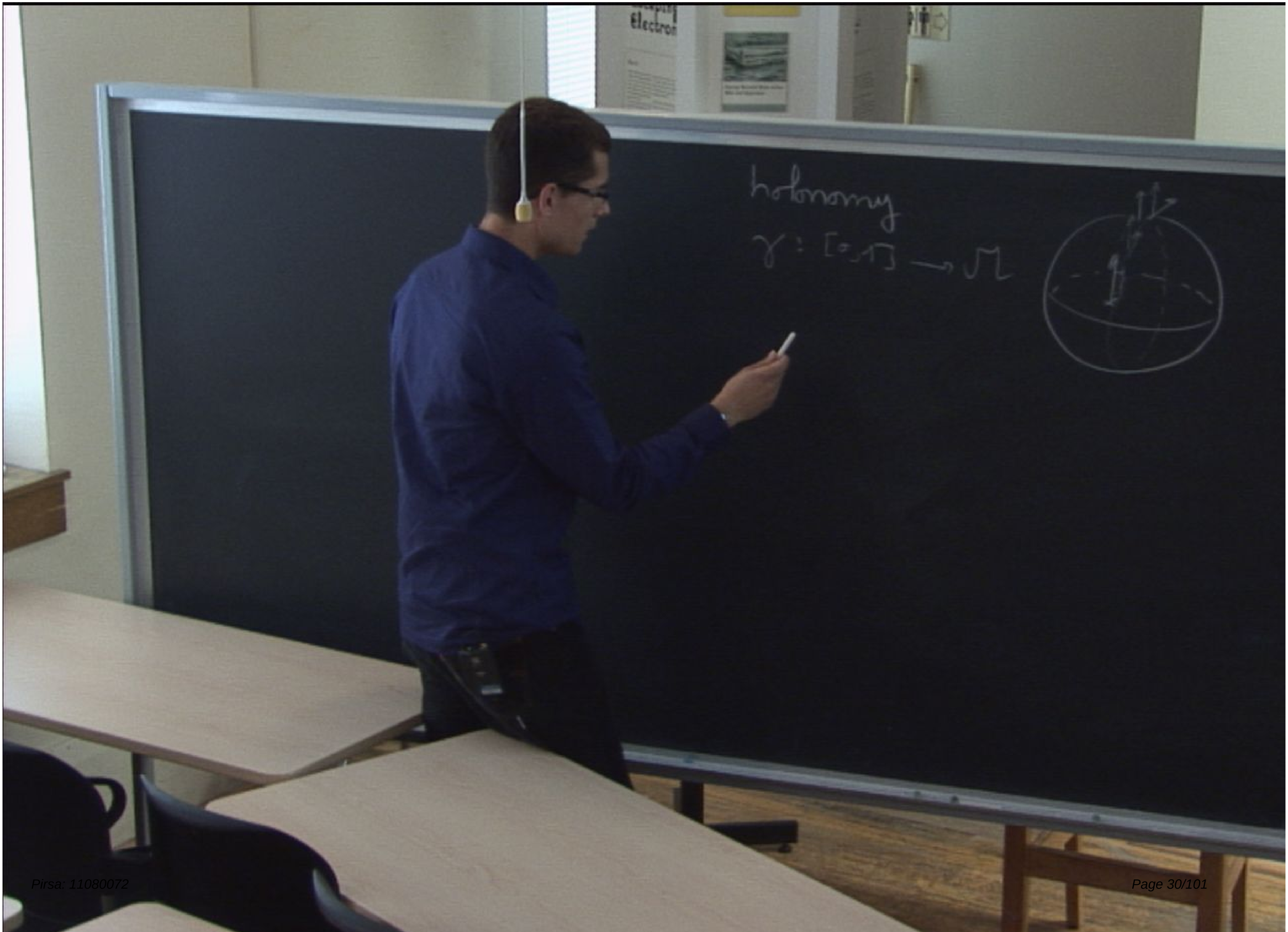
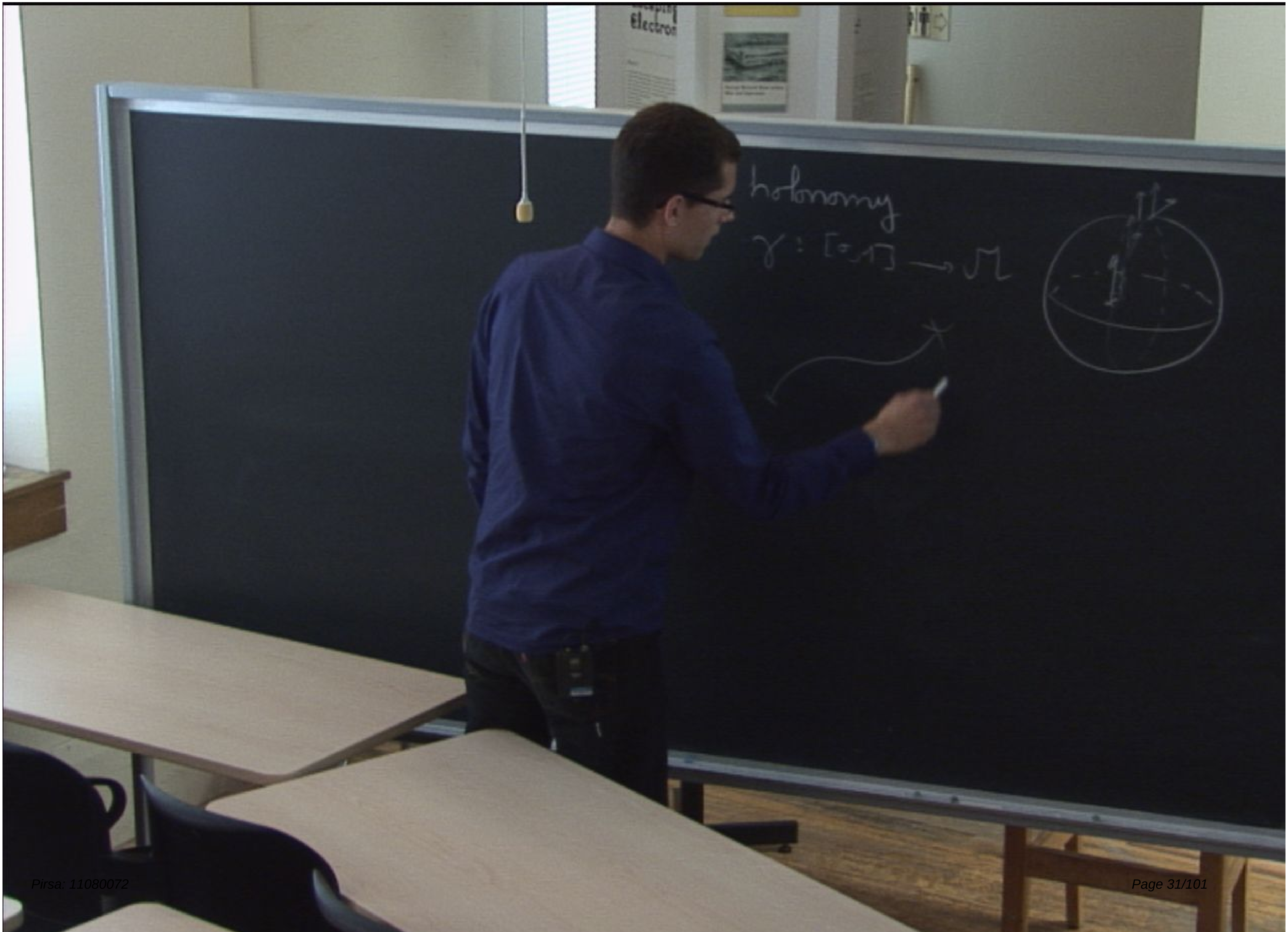

holonomy
γ : [0,1] → Ω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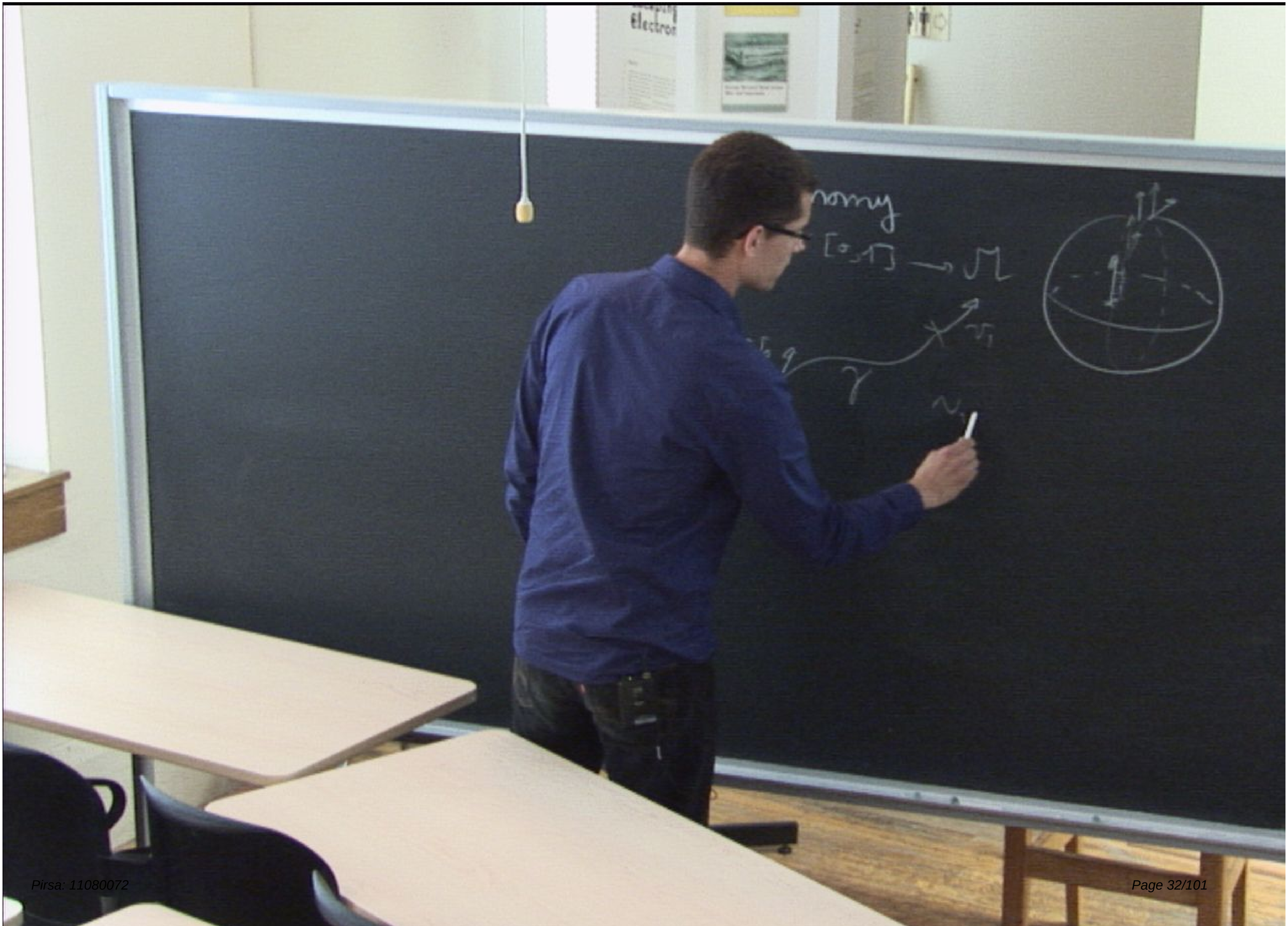

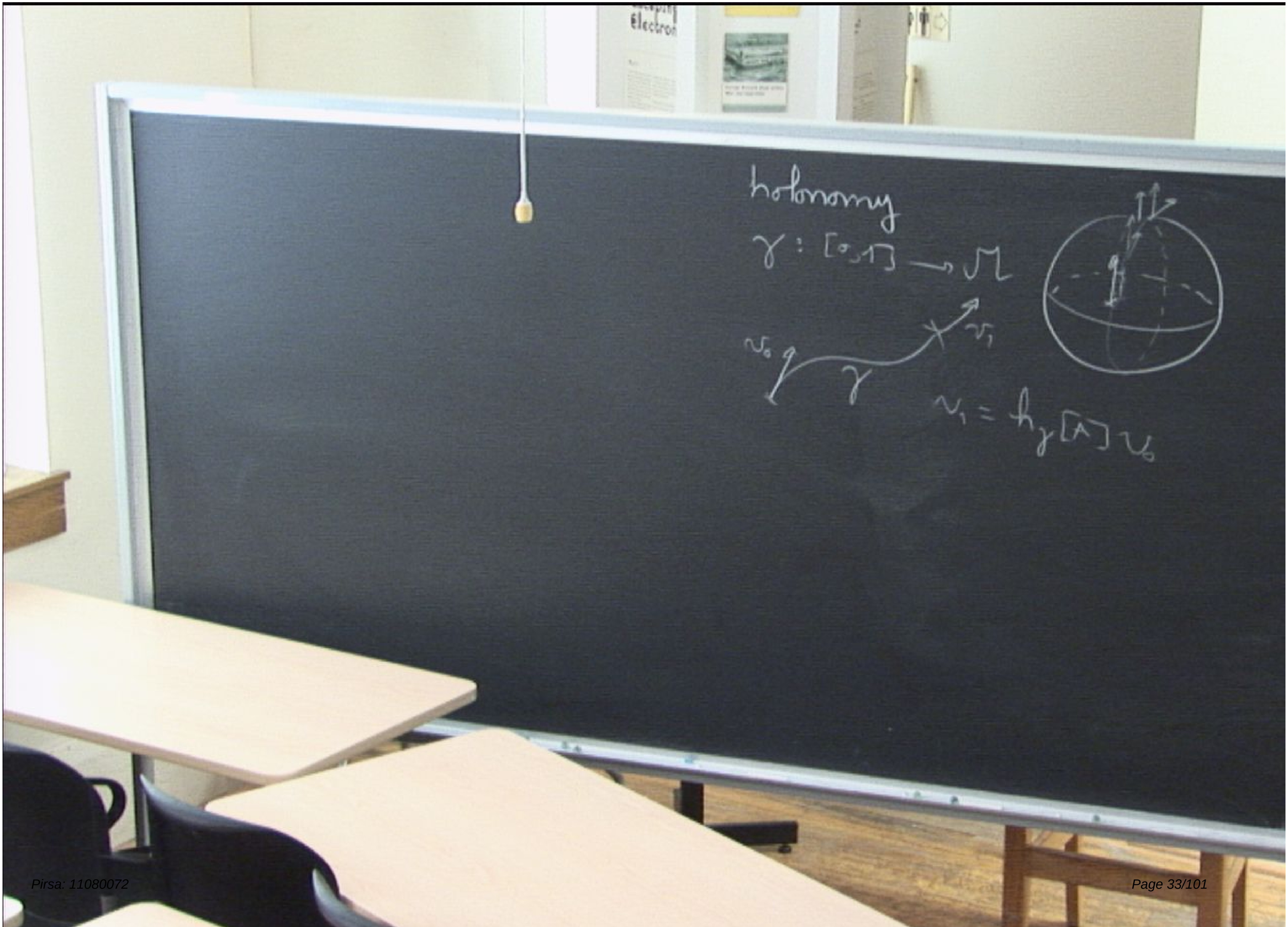
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$\gamma : [0,1] \to \mathcal{M}$
v_0
γ
v_1
$v_1 = h_\gamma[A] \, v_0$

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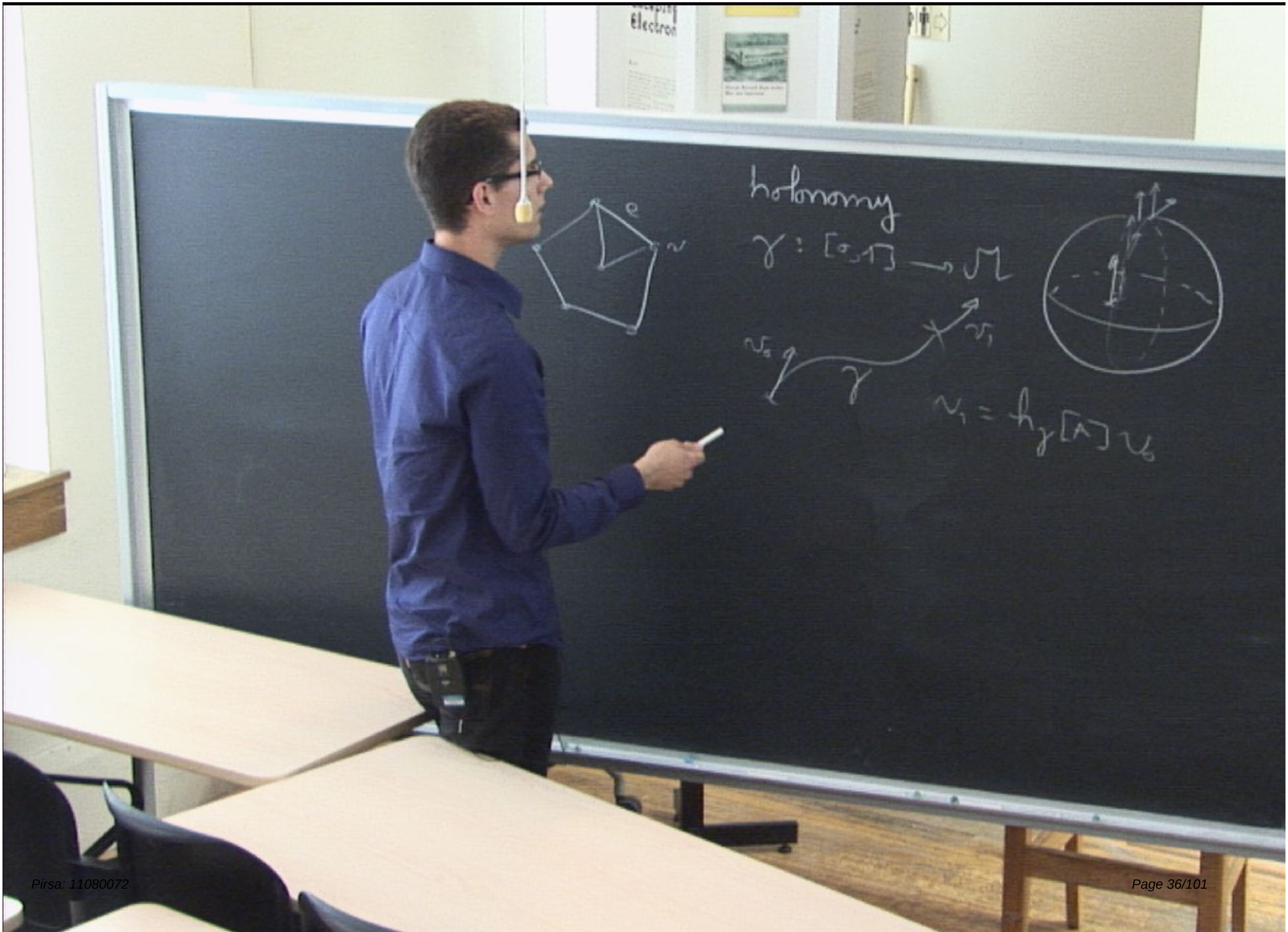
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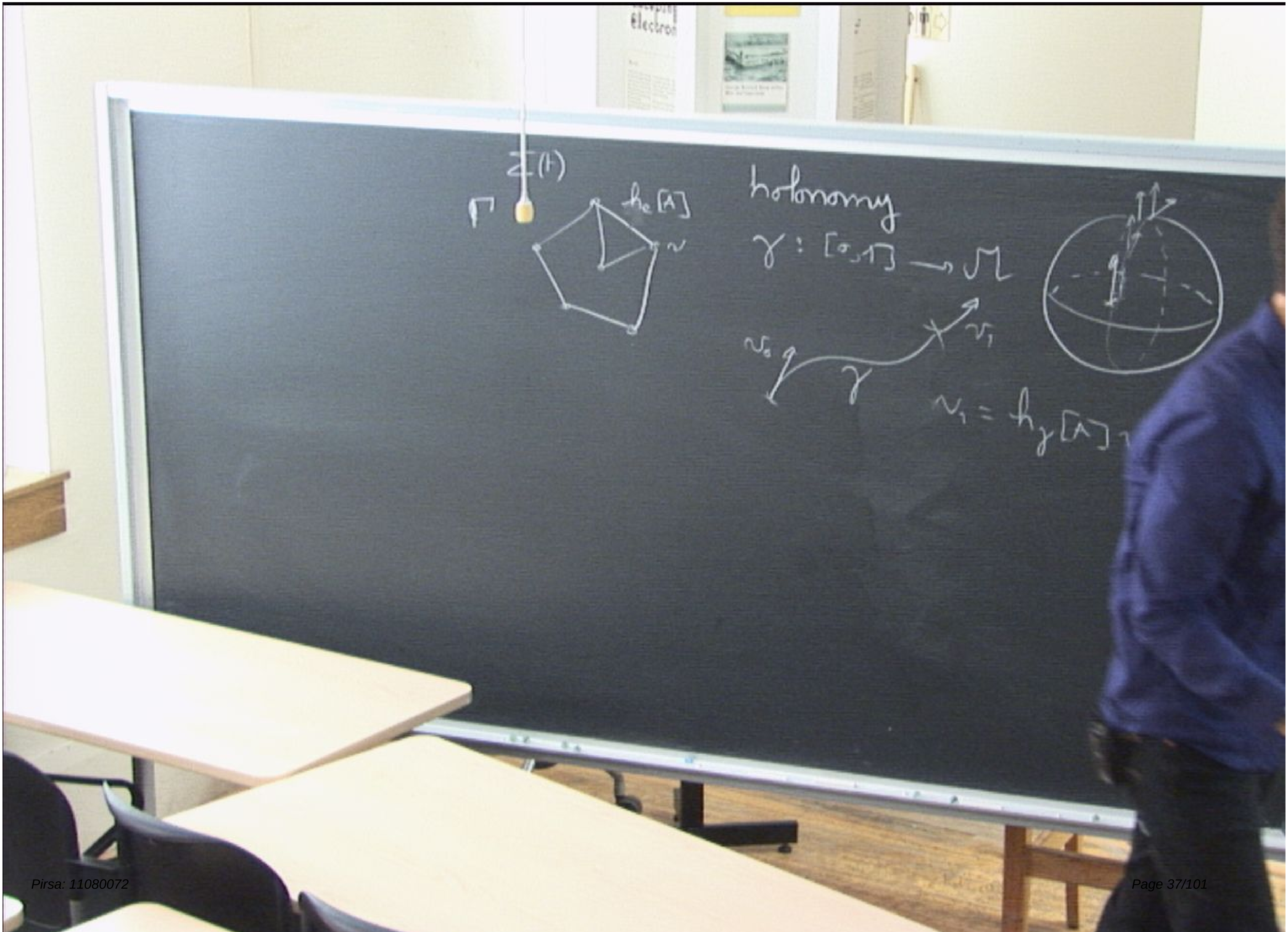
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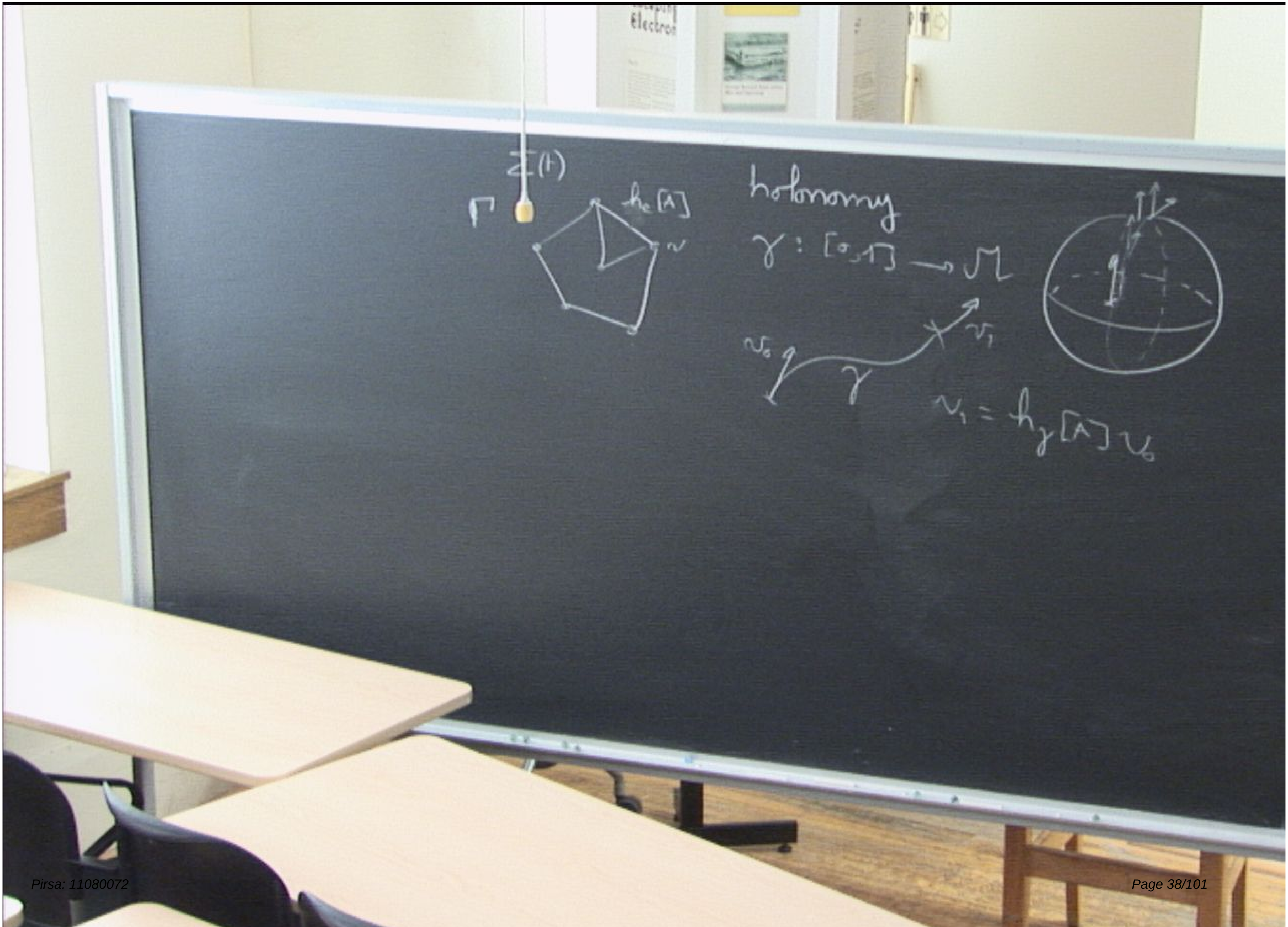

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h_e[A]

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Regularization of the triad operator: fluxes

Flux of the triad through a curve γ,

$$\hat{X}^i_\gamma \overset{\text{def.}}{=} \int_\gamma \varepsilon_{ab}\delta^{ij}\hat{E}^a_j \,\mathrm{d}x^b$$

Choosing γ conveniently,

$$\hat{X}^i_{s(e)}\psi^{\{j_e\}}_{\Gamma,f}[A] = f\Big(h_{e_1}[A], \ldots, h_e[A]\tau^i, \ldots, h_{e_N}[A]\Big)$$

$$\hat{X}^i_{t(e)}\psi^{\{j_e\}}_{\Gamma,f}[A] = f\Big(h_{e_1}[A], \ldots, \tau^i h_e[A], \ldots, h_{e_N}[A]\Big)$$

$$\text{with } \tau^i = -\tfrac{\mathrm{i}}{2}\sigma^i \in \mathfrak{su}(2)$$

On several states: very nice geometric interpretation, as vectors.

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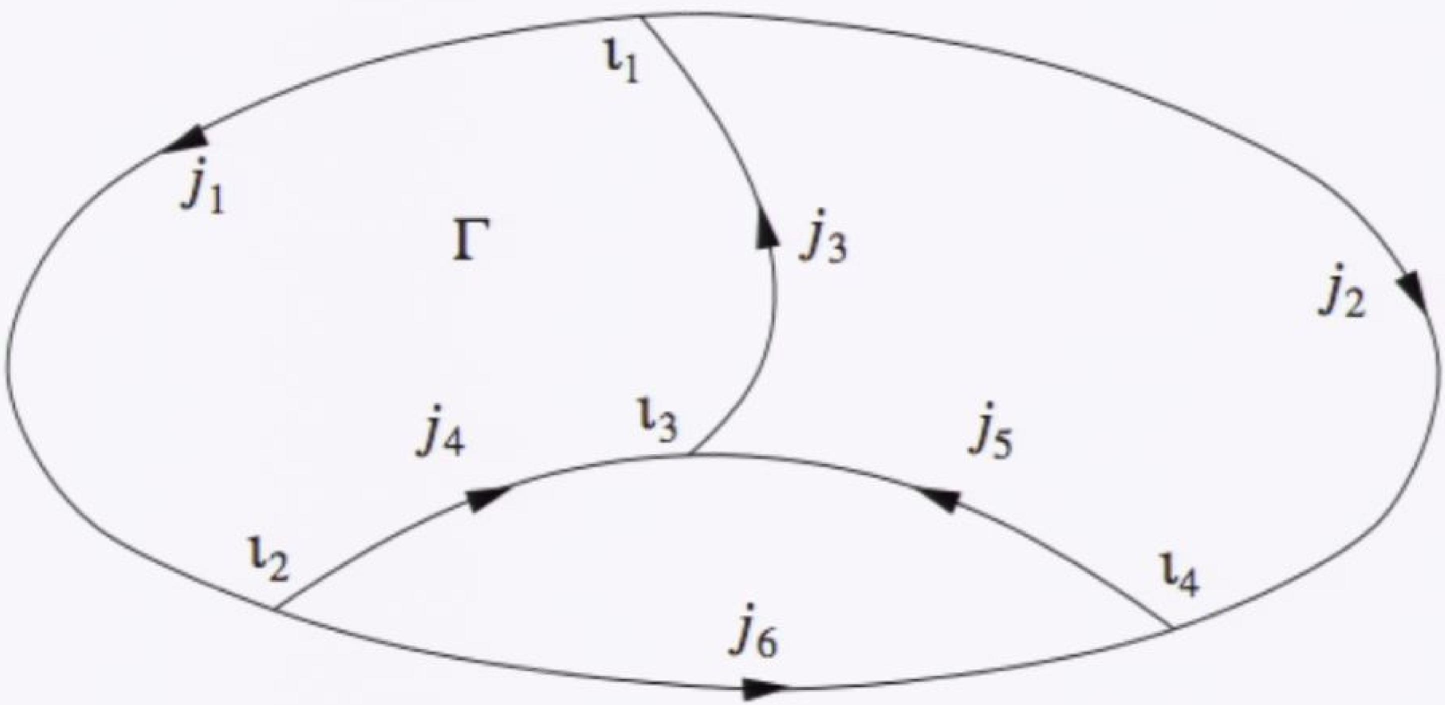
Spin networks

Spin network : oriented graph Γ, 'colored' by

- irreducible representations of $\mathrm{SU}(2)$ (spins) on edges;
- intertwiners on vertices.



Spin network state

Γ is a spin network with N edges, we define the corresponding

- spin network function

$$s_\Gamma^{\{j_e\}}(g_1, g_2, \ldots, g_N) \stackrel{\text{def.}}{=} \prod_e D^{(j_e)}(g_e) \cdot \prod_v \iota_v$$

- and spin network state

$$\psi_{\Gamma, s_\Gamma}^{\{j_e\}}[A] \stackrel{\text{def.}}{=} s_\Gamma^{\{j_e\}}\Big(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_N}[A]\Big).$$

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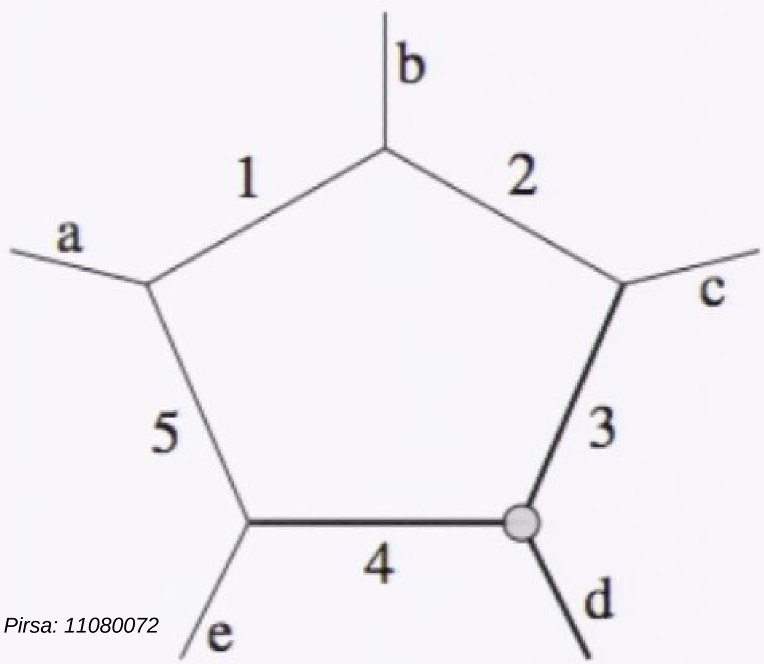
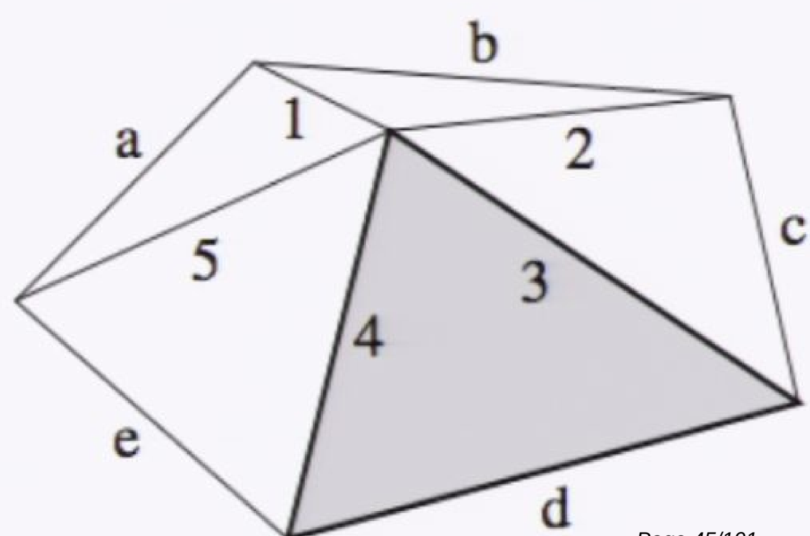
Interest in LQG

- automatically gauge–invariant states;
- nice geometric interpretation.

Spin network states and triangulations

Spin network	Triangulation
Vertex	Triangle
Link	Triangle's edge
Spin j_e	Length $\ell_e \propto \sqrt{j_e(j_e+1)}$

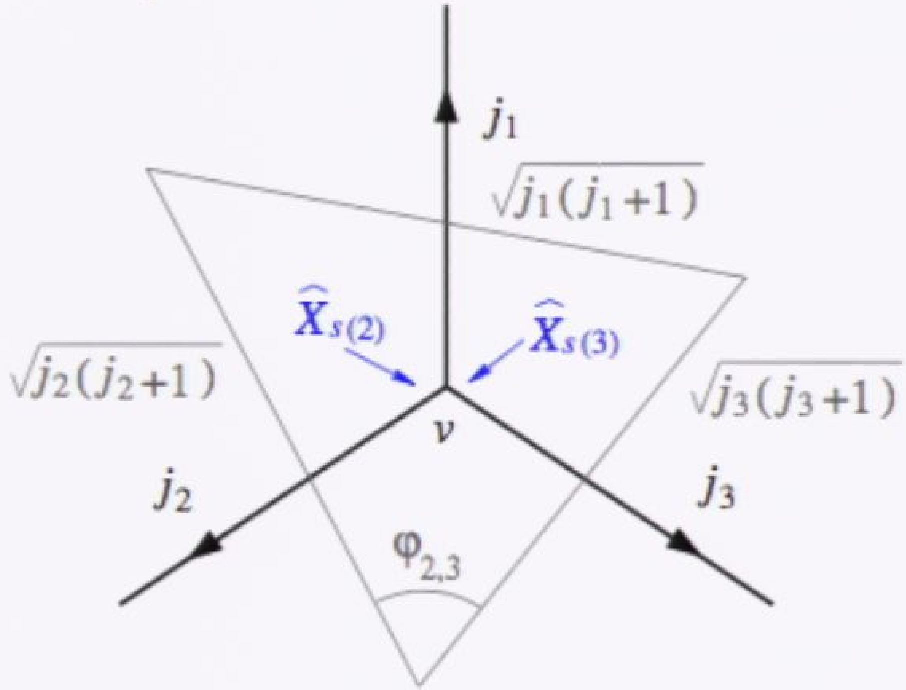




Spin network states and triangulation

Action of fluxes

Consider a vertex v of a spin network Γ and its dual triangle.



$$\left(\hat{X}_{s(2)} \cdot \hat{X}_{s(2)}\right) \psi_{\Gamma,s}^{\{j_e\}}[A] = \ell_2^2 \, \psi_{\Gamma,s}^{\{j_e\}}[A]$$

$$\left(\hat{X}_{s(2)} \cdot \hat{X}_{s(3)}\right) \psi_{\Gamma,s}^{\{j_e\}}[A] = \left(\ell_2 \ell_3 \cos \varphi_{2,3}\right) \psi_{\Gamma,s}^{\{j_e\}}[A]$$

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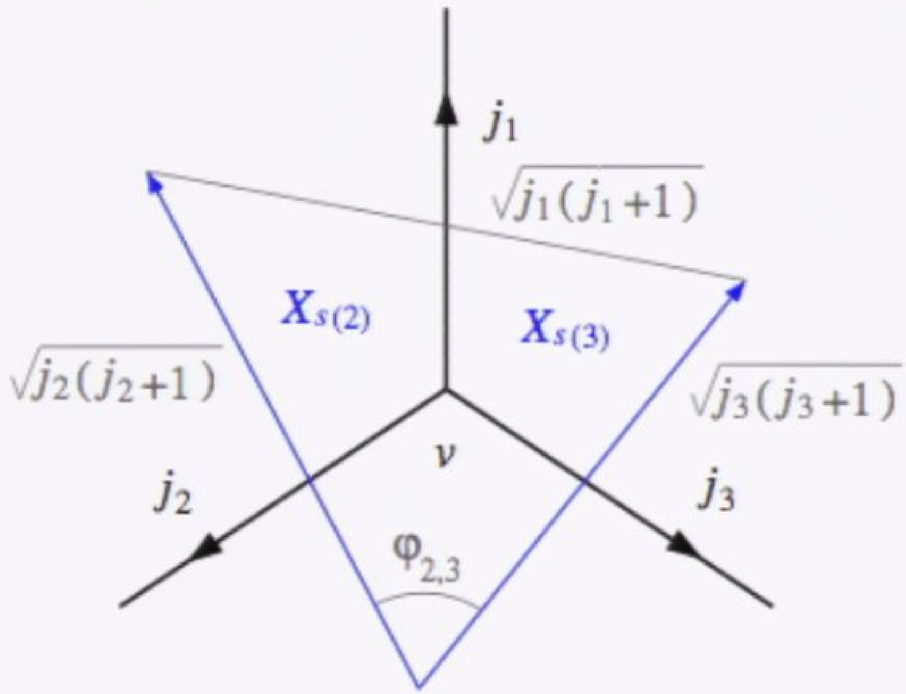

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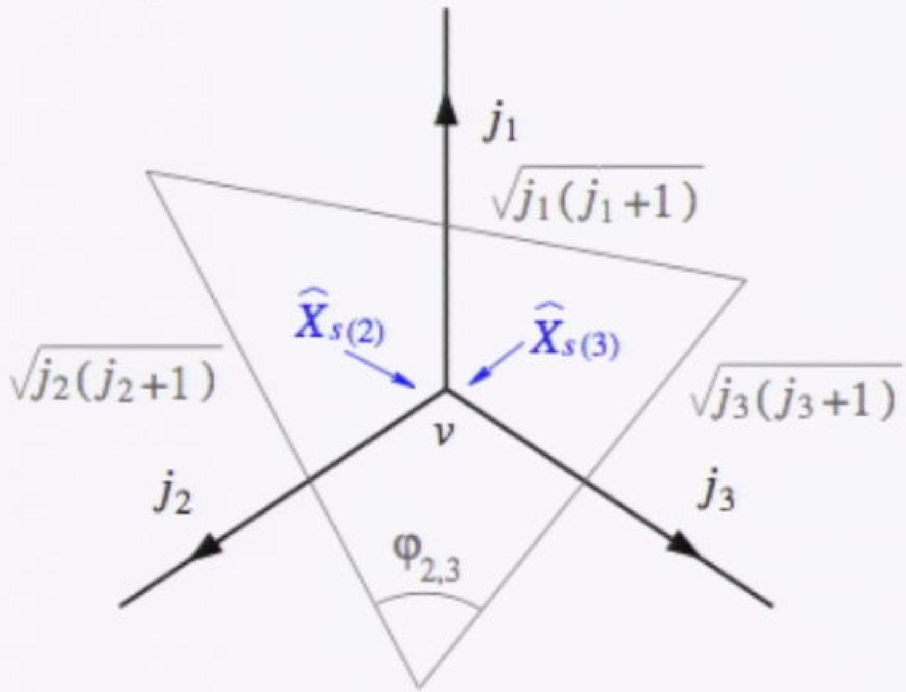
Spin network states and triangulation

Action of fluxes

Consider a vertex v of a spin network Γ and its dual triangle.



$$\left(\hat{X}_{s(2)} \cdot \hat{X}_{s(2)}\right)\psi_{\Gamma,s}^{\{j_e\}}[A] = \ell_2^2\,\psi_{\Gamma,s}^{\{j_e\}}[A]$$

$$\left(\hat{X}_{s(2)} \cdot \hat{X}_{s(3)}\right)\psi_{\Gamma,s}^{\{j_e\}}[A] = \left(\ell_2\ell_3\cos\varphi_{2,3}\right)\psi_{\Gamma,s}^{\{j_e\}}[A]$$

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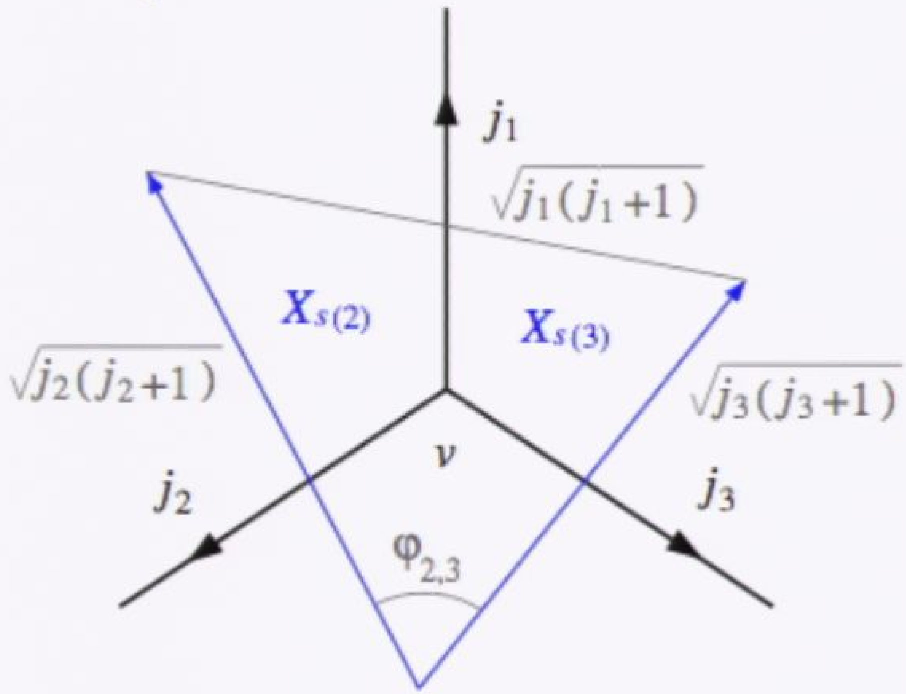

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Outline

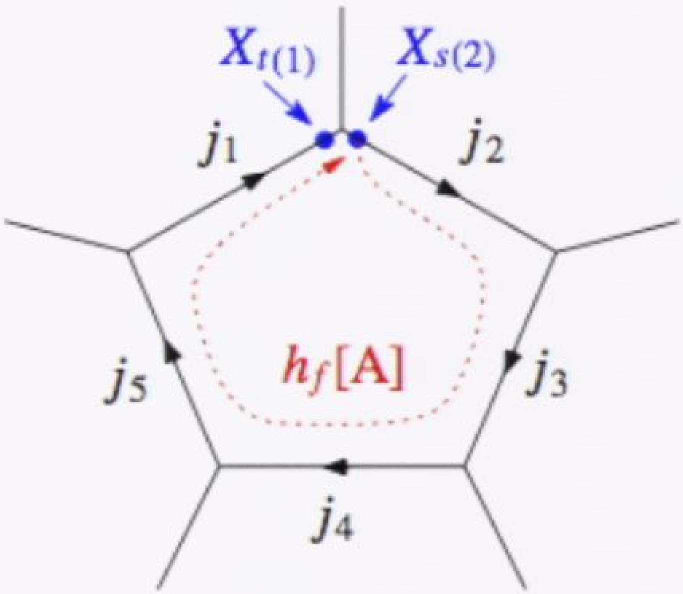
1. From classical to quantum gravity

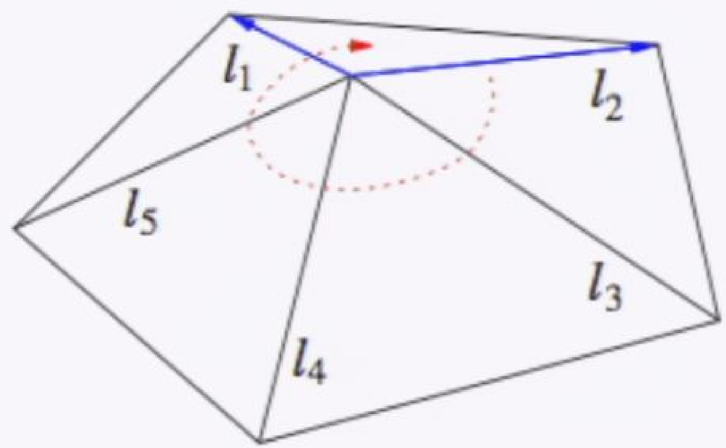
2. Spin network states

3. Hamiltonian constraint and recurrence relations

Quantization of the Hamiltonian constraint

Idea: use fluxes to probe the flatness of spacetime.





Quantization of the Hamiltonian constraint

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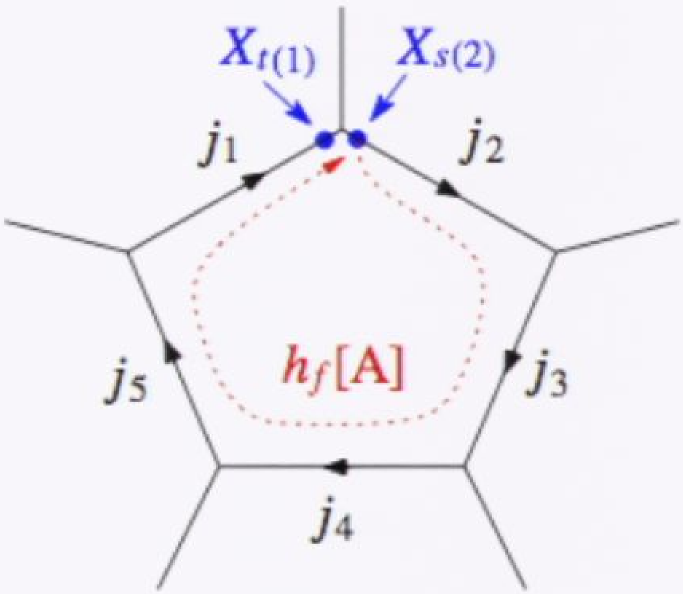
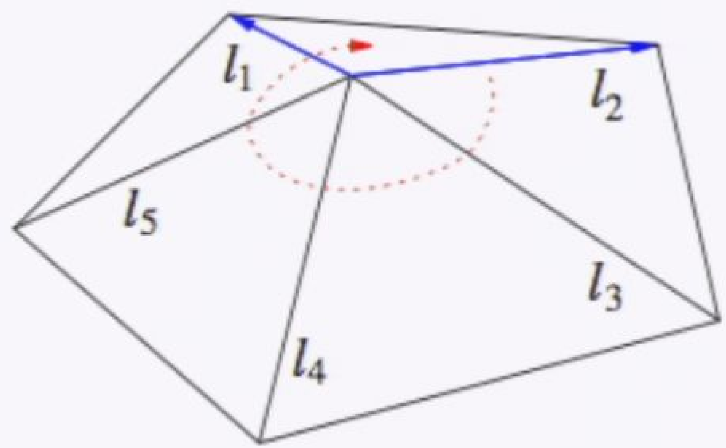




Proposition: quantum Hamiltonian contraint

$$\hat{C}_{12} \stackrel{\text{def.}}{=} \hat{X}_{t(1)} \cdot \hat{X}_{s(2)} - \hat{X}_{t(1)} \cdot R(h_f[A])\hat{X}_{s(2)}$$

$$h_f \stackrel{\text{def.}}{=} h_{e_1}\, h_{e_5}\, h_{e_4}\, h_{e_3}\, h_{e_2}$$

Quantization of the Hamiltonian constraint

Comparison with classical expression

Classical constraint $C = \left(\varepsilon_k{}^{ij} F^k_{ab}\right) E^a_i E^b_j$

Quantum contraint $\hat{C}_{12} = \left(\delta_{ij} - R(h_f[A])_{ij}\right) \hat{X}^i_{t(1)} \hat{X}^j_{s(2)}$

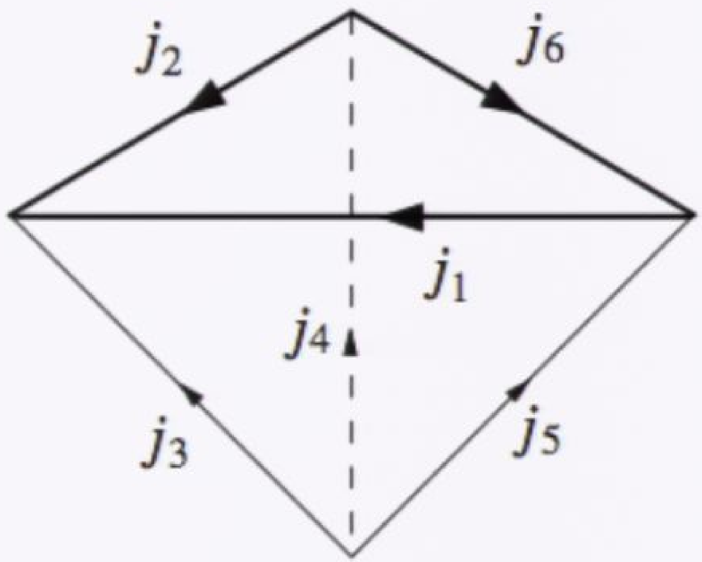
Quantization pattern

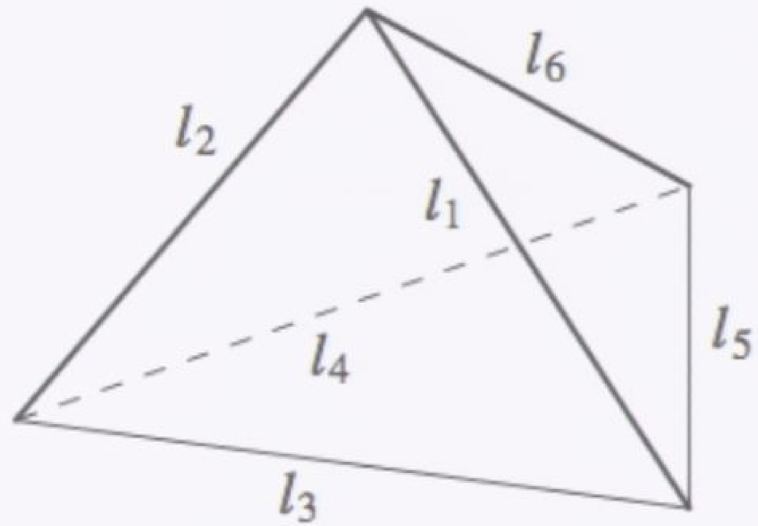
$$E^a_j \longrightarrow \hat{X}^i_{t(1)}$$

$$\varepsilon_k{}^{ij} F^k_{ab} \longrightarrow \delta_{ij} - R(h_f[A])_{ij}$$

Application to the tetrahedron

Consider the most simple triangulation of the 2-sphere: a tetrahedron.





Application to the tetrahedron

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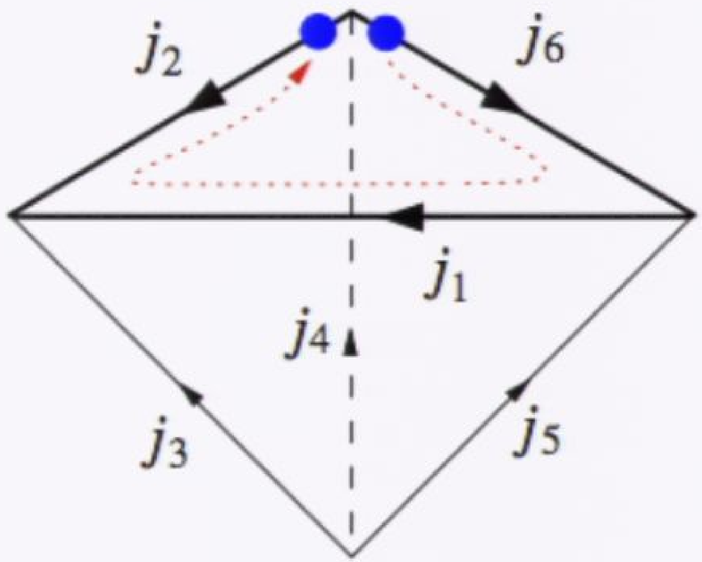
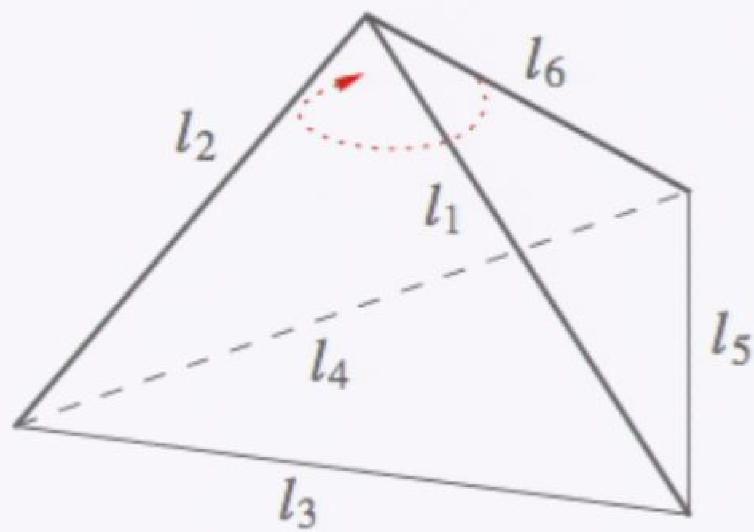




Question

For the constraint

$$\hat{C}_{26} = \hat{X}_{s(2)} \cdot \hat{X}_{s(6)} - \hat{X}_{s(2)} \cdot R(h_2^{-1}\, h_1\, h_6)\hat{X}_{s(6)},$$

what do we get imposing $\hat{C}_{26}\psi_{\text{tet}}^{\{j_e\}} = 0$?

Application to the tetrahedron

- Scalar product

$$\hat{X}_{s(2)} \cdot \hat{X}_{s(6)} \psi_{\text{tet}}^{\{j_e\}} = N_{j_2} N_{j_6} (-1)^{j_2+j_4+j_6} \begin{Bmatrix} j_2 & j_2 & 1 \\ j_6 & j_6 & j_4 \end{Bmatrix} \psi_{\text{tet}}^{\{j_e\}}.$$

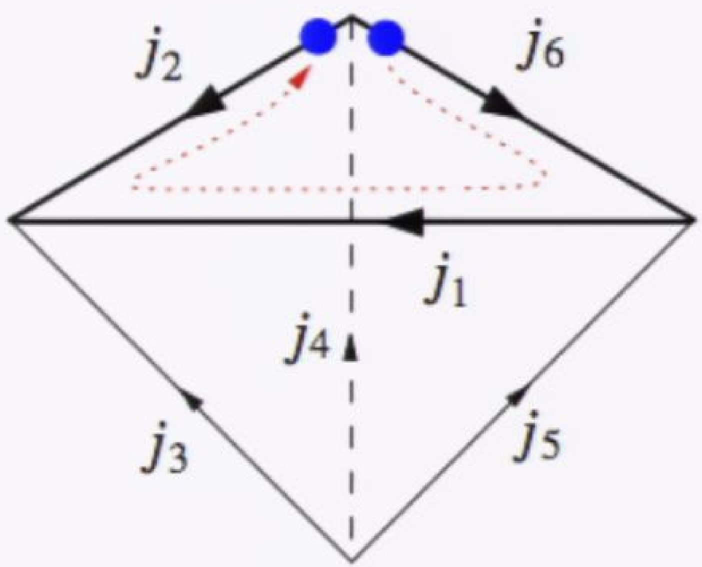
- Scalar product after parallel transport

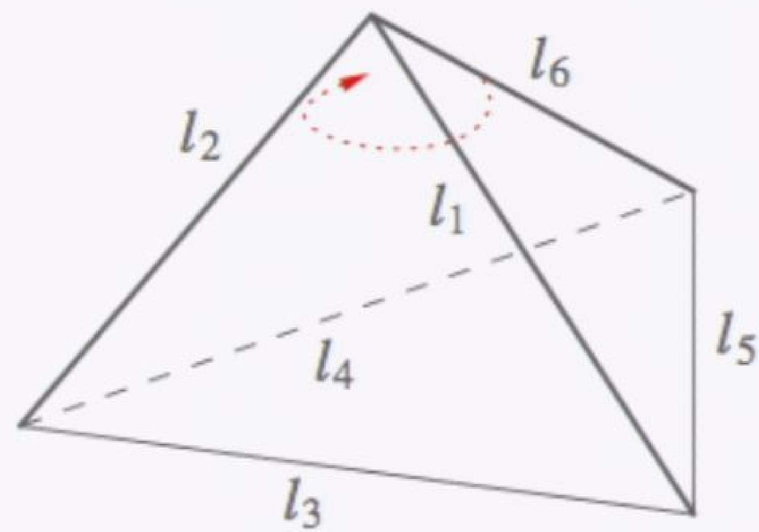
$$\begin{aligned} &\hat{X}_{s(2)} \cdot R(h_2^{-1}\, h_1\, h_6) \hat{X}_{s(6)} \psi_{\text{tet}}^{\{j_e\}} \\ &\quad = N_{j_2} N_{j_6} \sum_{\varepsilon_1=-1}^{1} (-1)^{1+\varepsilon_1} d_{j_1+\varepsilon_1} (-1)^{j_1+j_2+j_3} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{Bmatrix} \\ &\qquad\qquad \times (-1)^{j_1+j_5+j_6} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_6 & j_6 & j_5 \end{Bmatrix} \psi_{\text{tet}}^{j_1+\varepsilon_1,\{j_e\}}. \end{aligned}$$

We denoted $d_j \stackrel{\text{def.}}{=} 2j+1$ and $N_j \stackrel{\text{def.}}{=} \sqrt{j(j+1)d_j}$

Application to the tetrahedron

Consider the most simple triangulation of the 2-sphere: a tetrahedron.





Question

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- Scalar product after parallel transport

$$\hat{X}_{s(2)} \cdot R(h_2^{-1}\, h_1\, h_6) \hat{X}_{s(6)} \psi_{\text{tet}}^{\{j_e\}}$$

$$= N_{j_2} N_{j_6} \sum_{\varepsilon_1=-1}^{1} (-1)^{1+\varepsilon_1} d_{j_1+\varepsilon_1} (-1)^{j_1+j_2+j_3} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{Bmatrix}$$

$$\times (-1)^{j_1+j_5+j_6} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_6 & j_6 & j_5 \end{Bmatrix} \psi_{\text{tet}}^{j_1+\varepsilon_1,\{j_e\}}.$$

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Application to the tetrahedron

- We obtain a recurrence relation on $u(j_1) = \psi_{\text{tet}}^{j_1,\{j_e\}}[A]$,

$$A_+(j_1)\, u(j_1+1) + A_0(j_1)\, u(j_1) + A_-(j_1)\, u(j_1-1) = B\, u(j_1)$$

- The solution is quite well-known: $6j$-symbol

$$\psi_{\text{tet}}^{\{j_e\}}[A] = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

from Biedenharn-Elliott identity.

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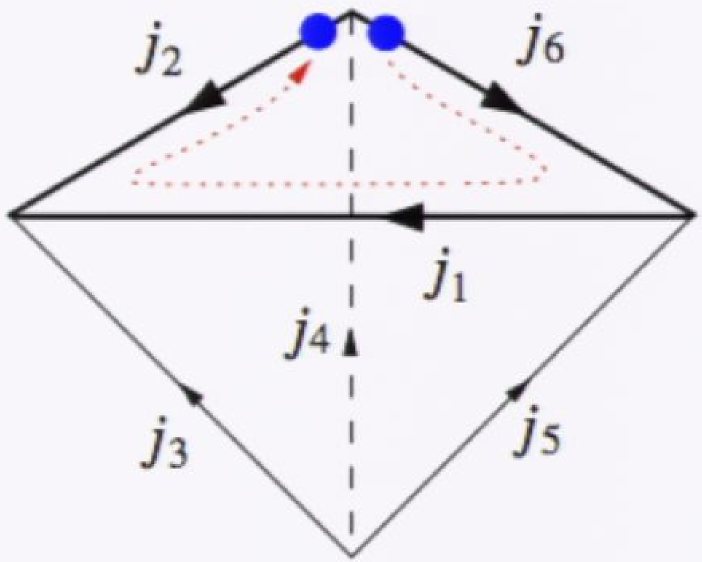
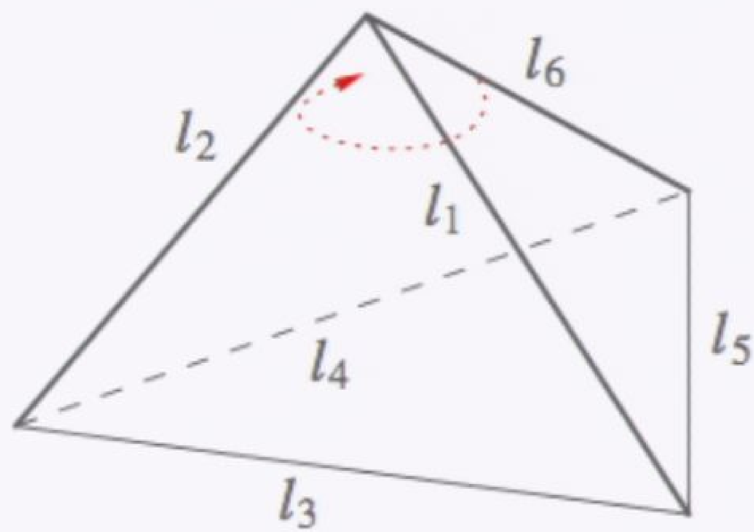
- We recover a flat connection.

Application to the tetrahedron

Consider the most simple triangulation of the 2-sphere: a tetrahedron.





Question

For the constraint

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$$\psi_{\text{tet}}^{\{j_e\}}[A] = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \psi_{\text{tet}}^{\{j_e\}}[A=0]$$

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Application to the tetrahedron

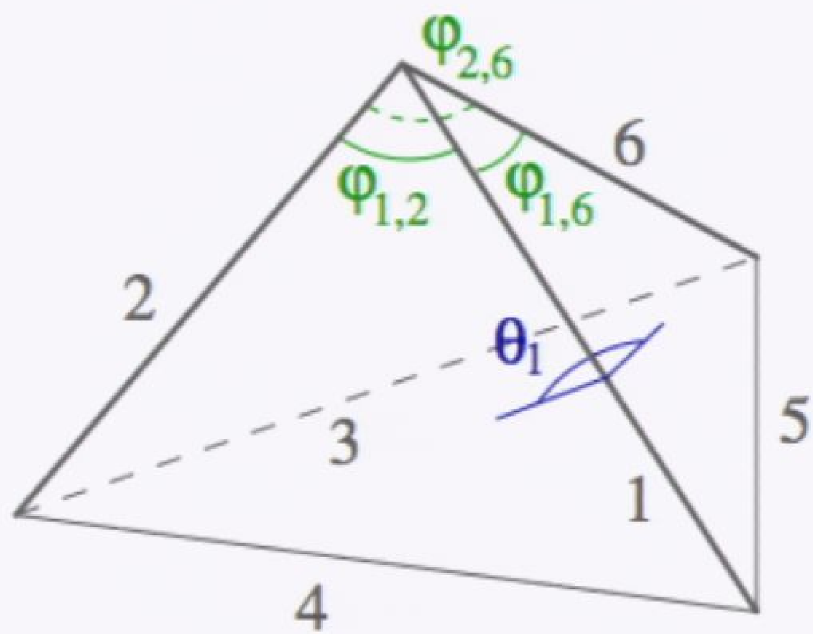
Asymptotics, geometric interpretation

$$\left(A_+(j_1)\,\hat{T}_1 + A_-(j_1)\,\hat{T}_1^{-1} + A_0(j_1)\,\mathbb{1}\right)u(j_1) = B\,u(j_1)$$

- Exact relations
  $A_0(j_1) = \cos\varphi_{1,2}\cos\varphi_{1,6}$
  $B = \cos\varphi_{2,6}$

- Large spin limit
  $A_+ \approx A_- \approx \frac{1}{2}\sin\varphi_{1,2}\sin\varphi_{1,6}$

- We write $\hat{T}_1 = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}(\hat{\theta}_1)}$

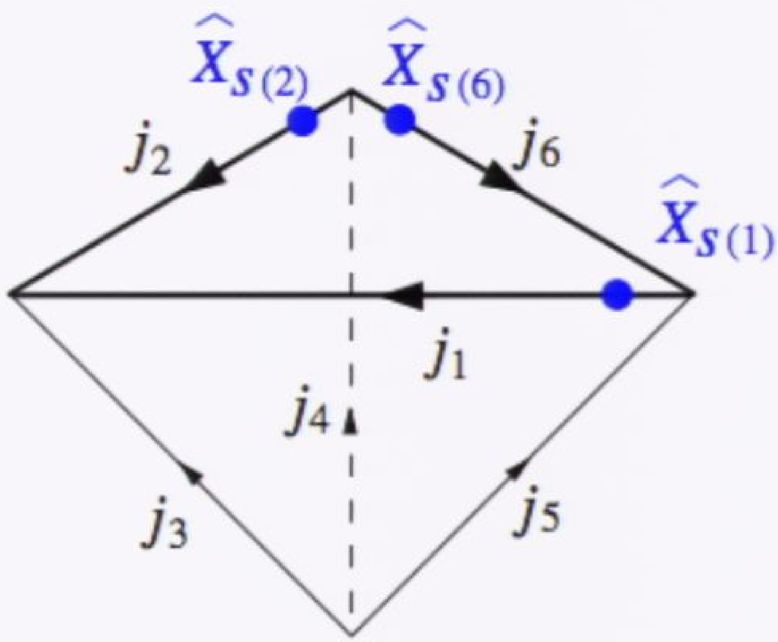


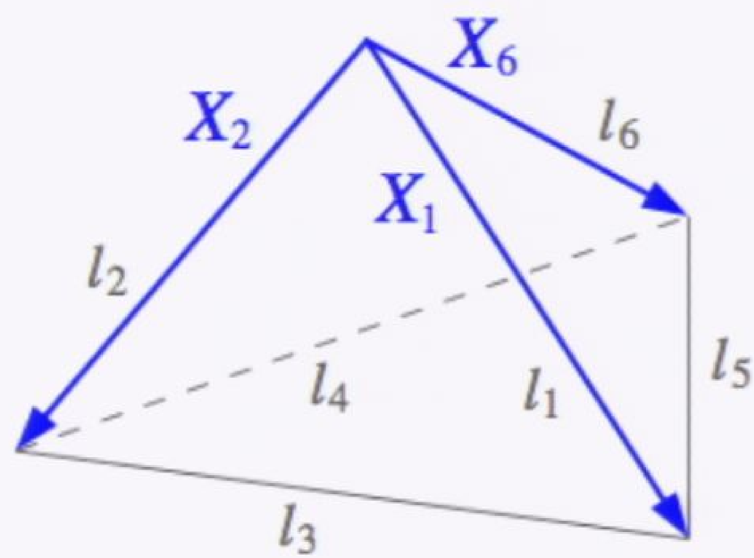
Asymptotics

$$-\sin\varphi_{1,2}\sin\varphi_{1,6}\cos\hat{\theta}_1 + \cos\varphi_{1,2}\cos\varphi_{1,6} = \cos\varphi_{2,6}$$

Other relations for the tetrahedron

Volume of the tetrahedron?





Constraint

Other relations for the tetrahedron

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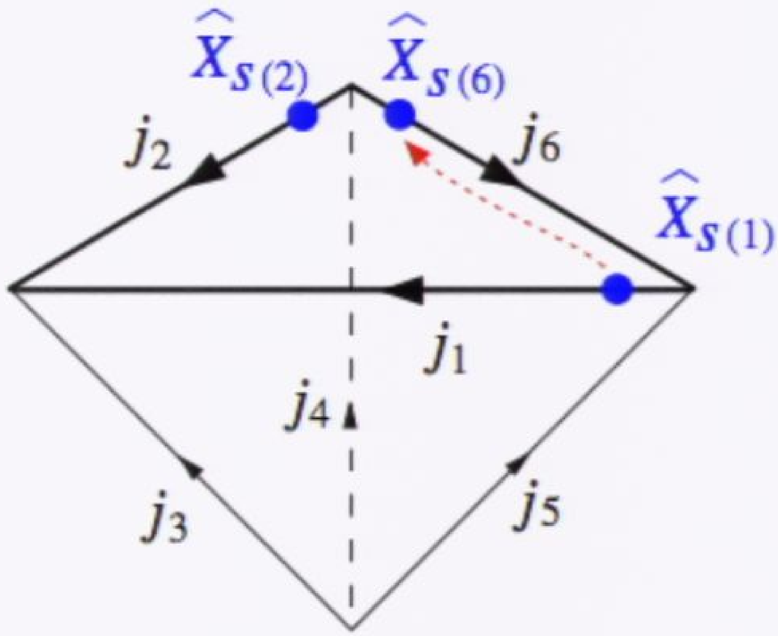

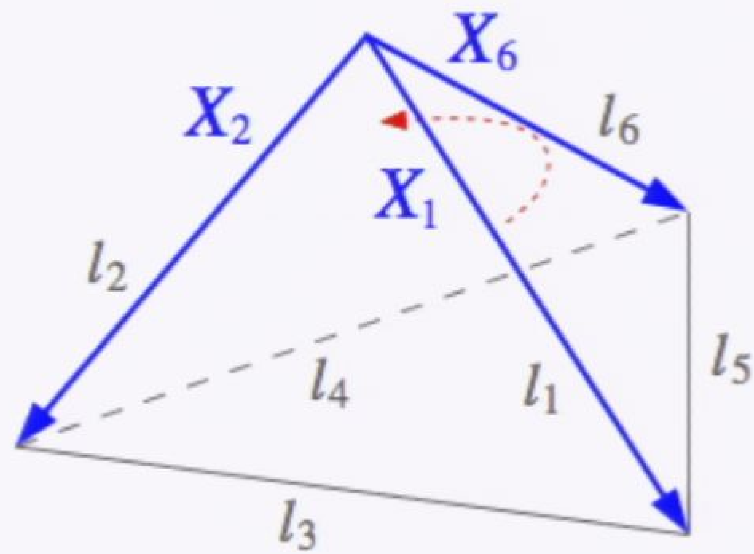

Constraint

$$\left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_6^{-1})\hat{X}_{s(1)}$$

Other relations for the tetrahedron

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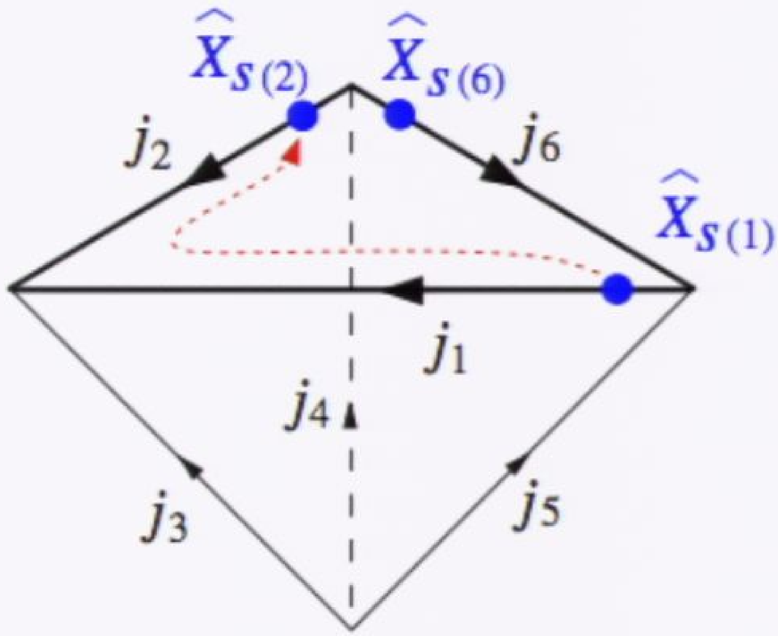

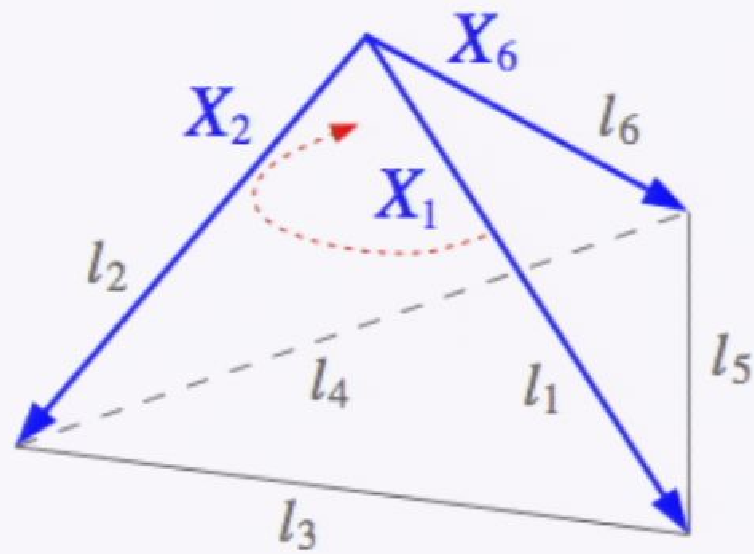

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Other relations for the tetrahedron

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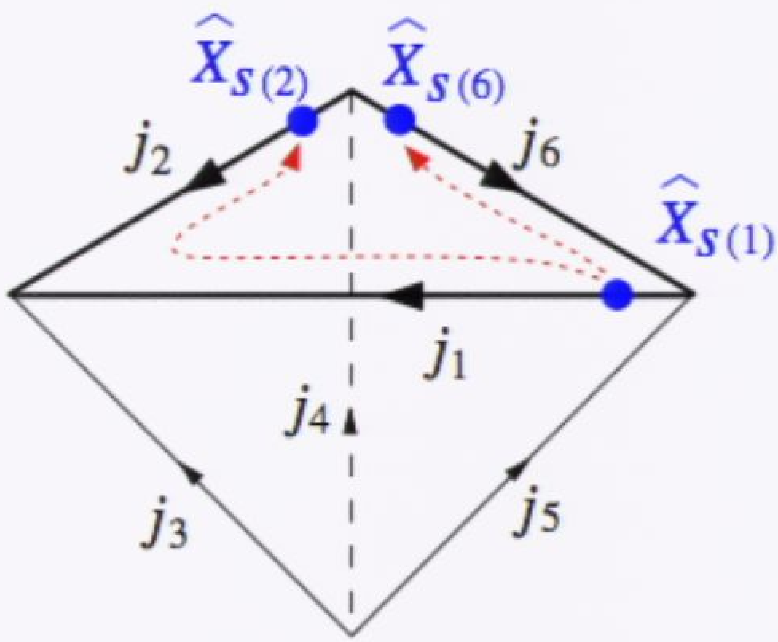

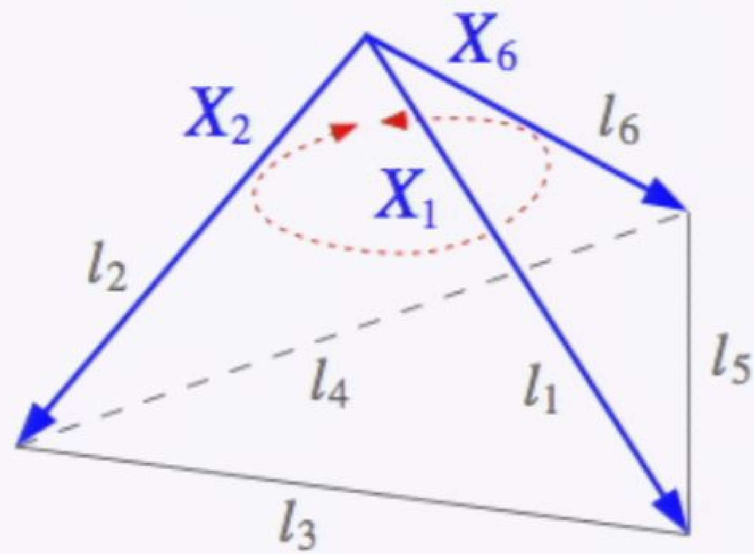

Constraint

$$\hat{C}_{126} = \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_6^{-1})\hat{X}_{s(1)} - \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_2^{-1} h_1)\hat{X}_{s(1)}$$

Other relations for the tetrahedron

Recursion relation

$$\sum_{\varepsilon_2=-1}^{1} d_{j_2+\varepsilon_2}(-1)^{1+2j_2} \begin{Bmatrix} 1 & 1 & 1 \\ j_2 & j_2 & j_2+\varepsilon_2 \end{Bmatrix} (-1)^{j_1+j_2+j_3} \begin{Bmatrix} j_2+\varepsilon_2 & j_2 & 1 \\ j_1 & j_1 & j_3 \end{Bmatrix}$$

$$\times(-1)^{j_2+j_4+j_6} \begin{Bmatrix} j_2+\varepsilon_2 & j_2 & 1 \\ j_6 & j_6 & j_4 \end{Bmatrix} \psi_{\text{tet}}^{j_2+\varepsilon_2,\{j_e\}}$$

$$= \sum_{\varepsilon_6=-1}^{1} d_{j_6+\varepsilon_6}(-1)^{1+2j_2} \begin{Bmatrix} 1 & 1 & 1 \\ j_6 & j_6 & j_6+\varepsilon_6 \end{Bmatrix} (-1)^{j_1+j_5+j_6} \begin{Bmatrix} j_6+\varepsilon_6 & j_6 & 1 \\ j_1 & j_1 & j_5 \end{Bmatrix}$$

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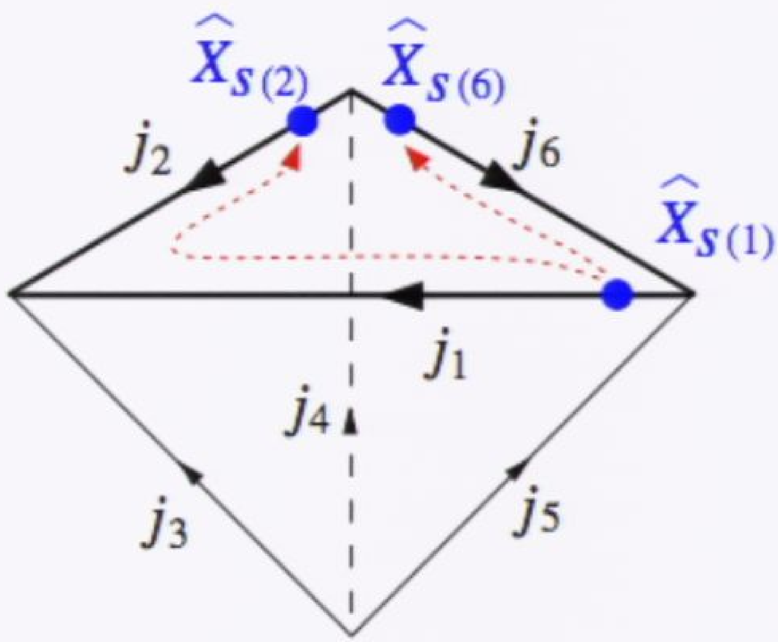
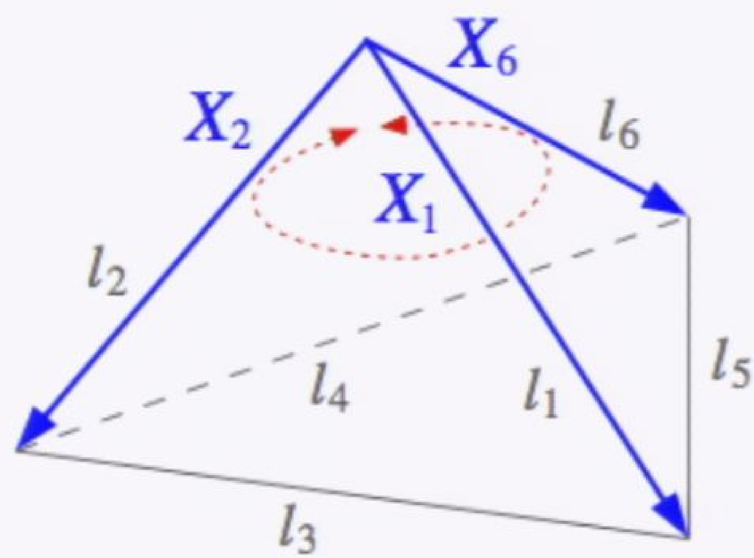
Other relations for the tetrahedron

Volume of the tetrahedron?





Constraint

$$\hat{C}_{126} = \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_6^{-1})\hat{X}_{s(1)} - \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_2^{-1} h_1)\hat{X}_{s(1)}$$

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$$\sum_{\varepsilon_2=-1}^{1} d_{j_2+\varepsilon_2}(-1)^{1+2j_2}\begin{Bmatrix}1 & 1 & 1\\ j_2 & j_2 & j_2+\varepsilon_2\end{Bmatrix}(-1)^{j_1+j_2+j_3}\begin{Bmatrix}j_2+\varepsilon_2 & j_2 & 1\\ j_1 & j_1 & j_3\end{Bmatrix}$$

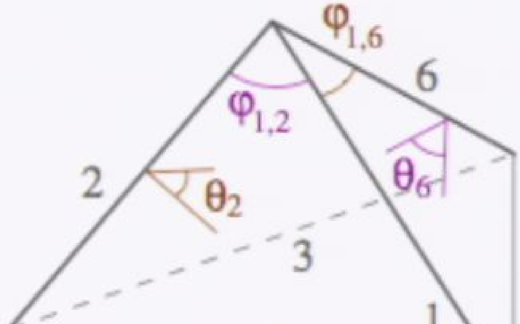
$$\times(-1)^{j_2+j_4+j_6}\begin{Bmatrix}j_2+\varepsilon_2 & j_2 & 1\\ j_6 & j_6 & j_4\end{Bmatrix}\psi_{\text{tet}}^{j_2+\varepsilon_2,\{j_e\}}$$

$$=\sum_{\varepsilon_6=-1}^{1} d_{j_6+\varepsilon_6}(-1)^{1+2j_2}\begin{Bmatrix}1 & 1 & 1\\ j_6 & j_6 & j_6+\varepsilon_6\end{Bmatrix}(-1)^{j_1+j_5+j_6}\begin{Bmatrix}j_6+\varepsilon_6 & j_6 & 1\\ j_1 & j_1 & j_5\end{Bmatrix}$$

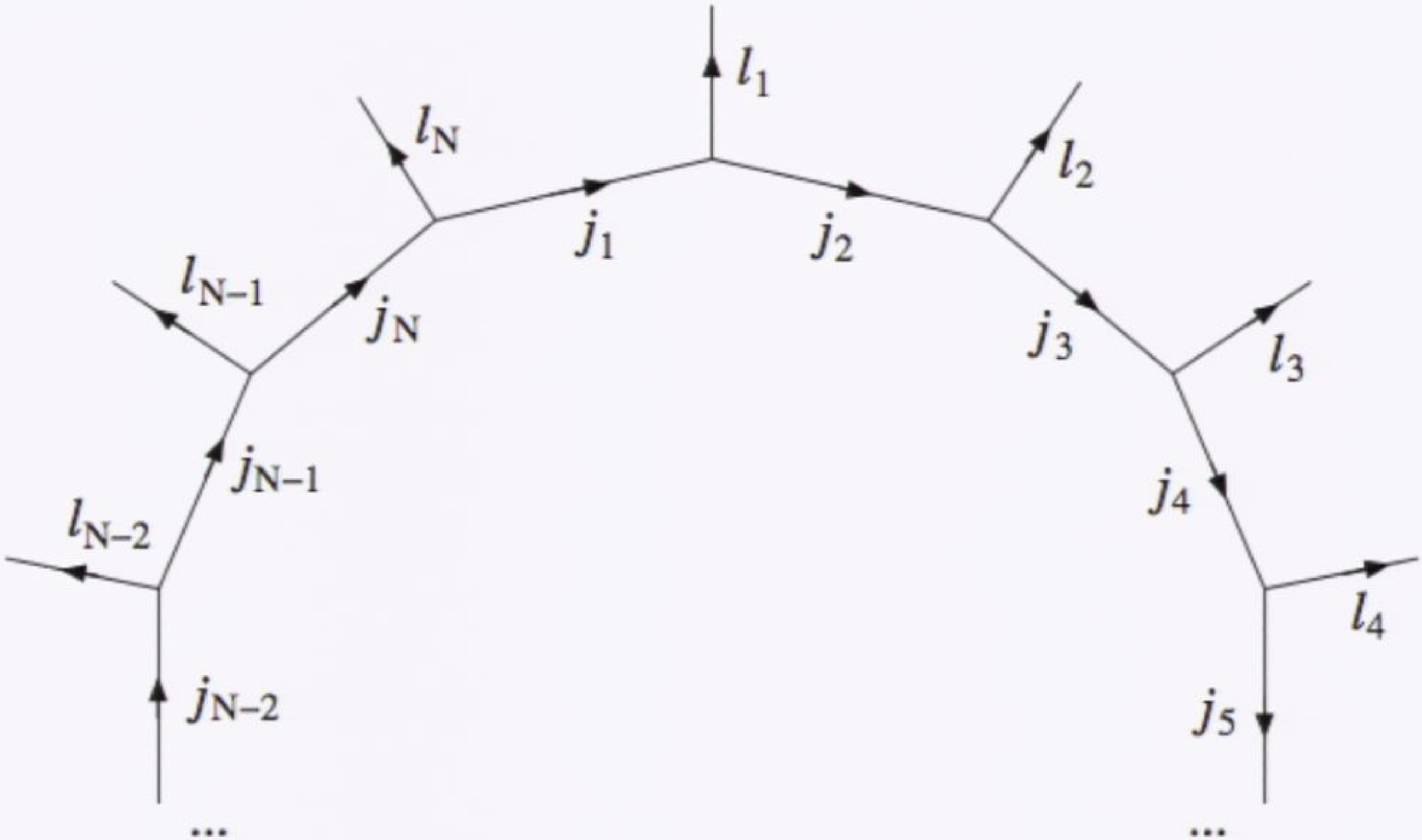
$$\times(-1)^{j_2+j_4+j_6}\begin{Bmatrix}j_6+\varepsilon_6 & j_6 & 1\\ j_2 & j_2 & j_4\end{Bmatrix}\psi_{\text{tet}}^{j_6+\varepsilon_6,\{j_e\}}$$

Asymptotics

$$\frac{\sin\hat\theta_2}{\sin\varphi_{1,6}}=\frac{\sin\hat\theta_6}{\sin\varphi_{1,2}}$$



Generalization: cycles with N egdes



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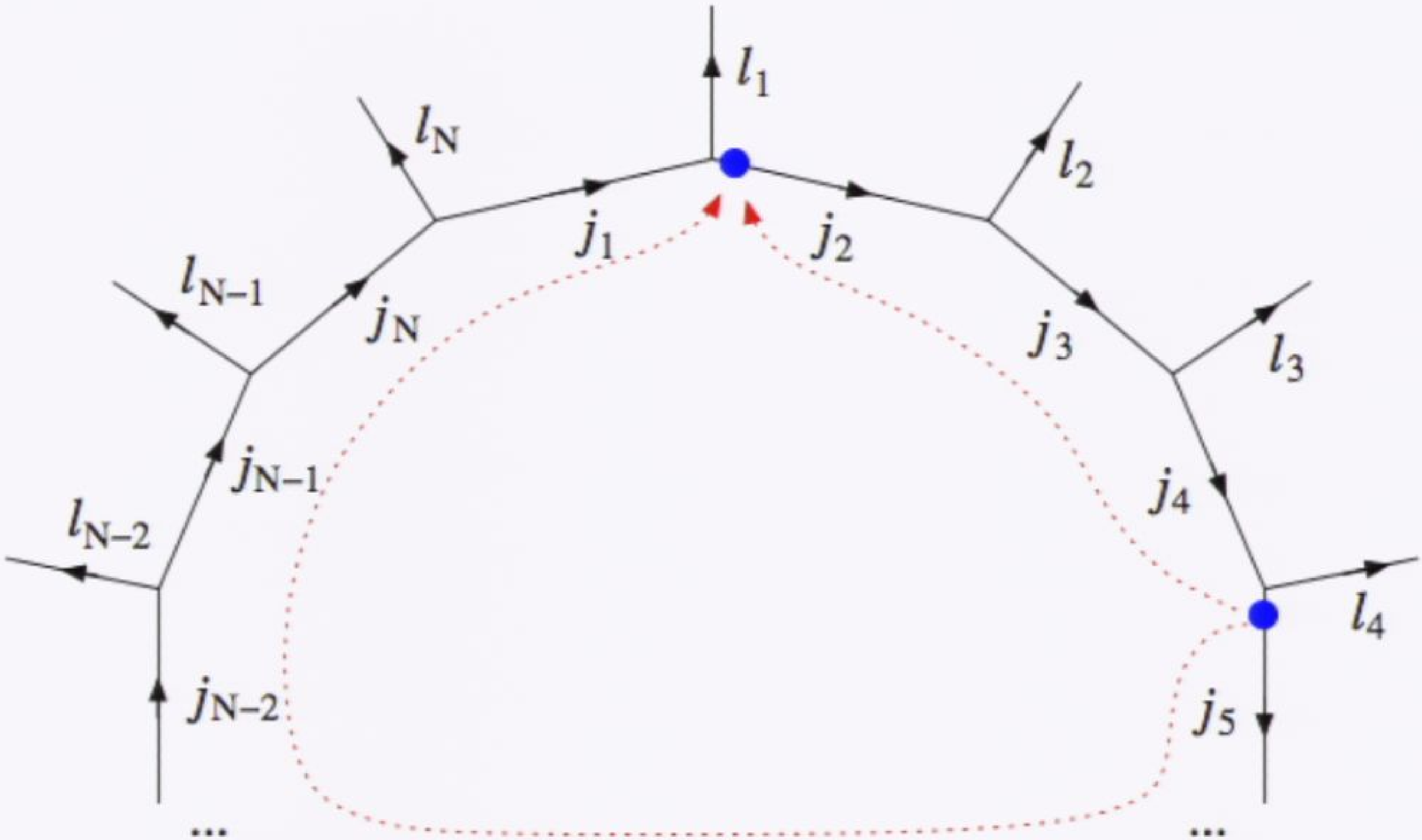
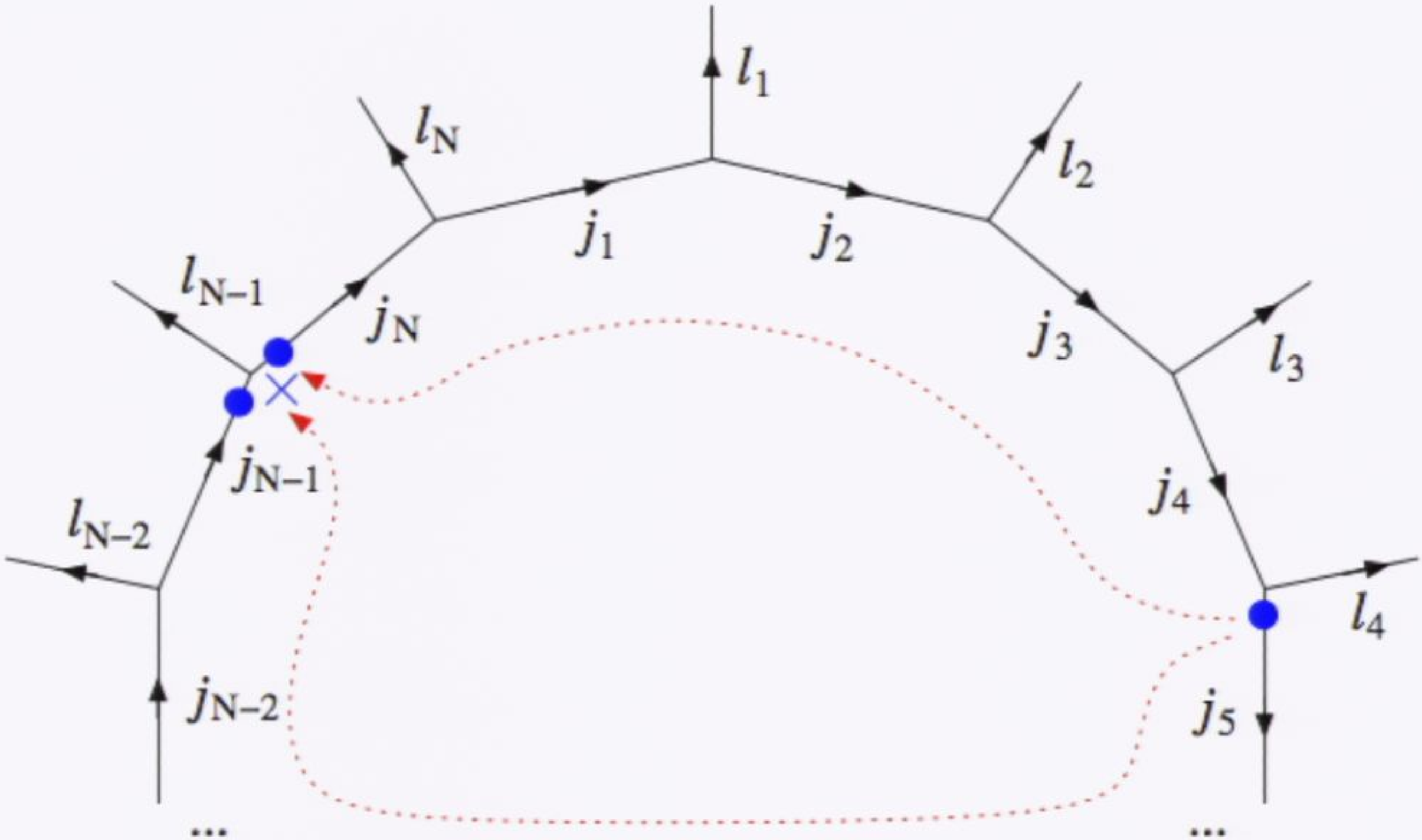


Constraint

$$\hat{C}_{25} = \hat{X}_{s(2)} \cdot R(h_{234}^{-1})\hat{X}_{s(5)} - \hat{X}_{s(2)} \cdot R(h_{56\ldots N1})\hat{X}_{s(5)}$$

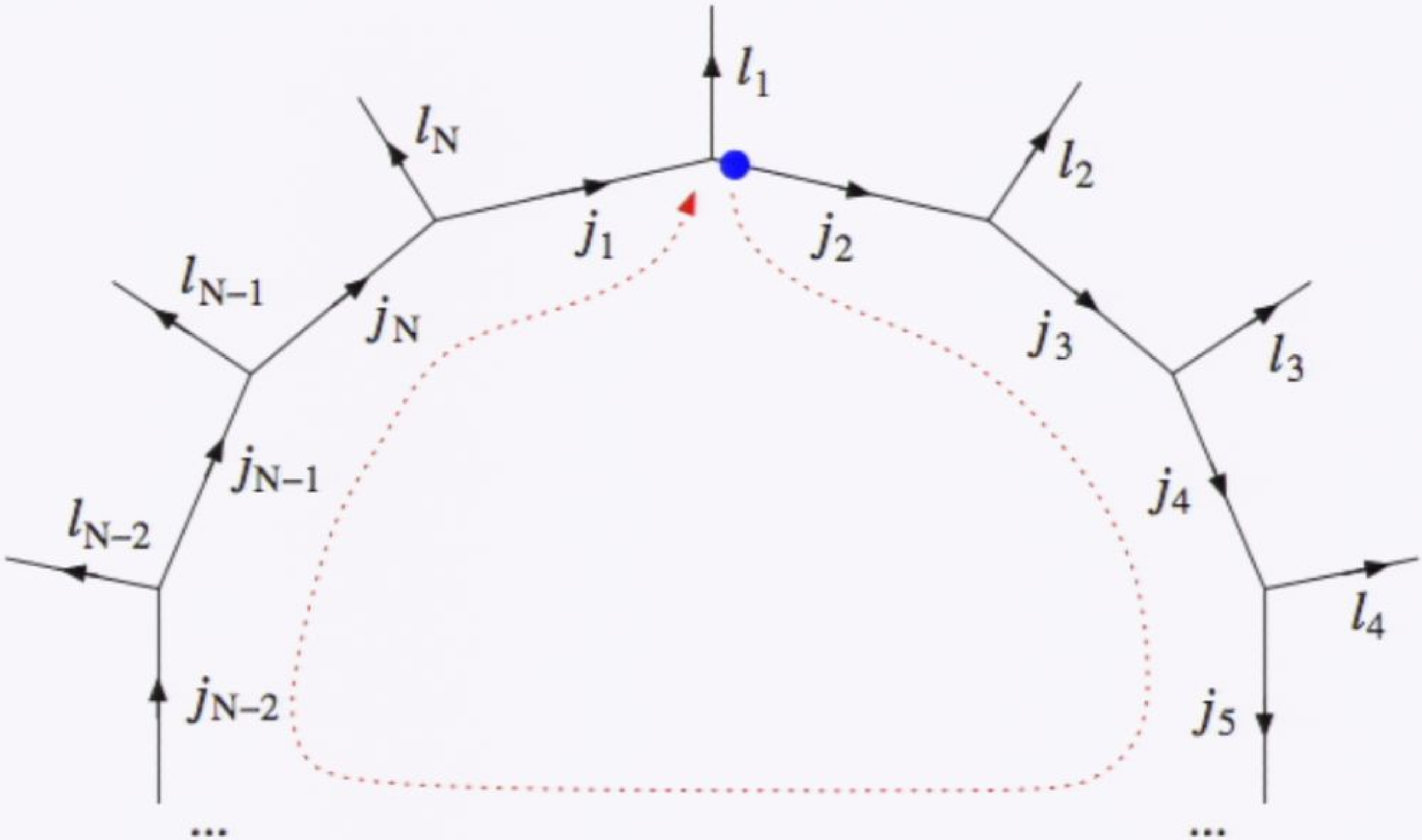
Generalization: cycles with N egdes



Constraint

$$\hat{C}_{(N-1)N5} = (\hat{X}_{i(N-1)} \times \hat{X}_{i(N)}) \cdot R(h^{-1}_{\ldots}) \hat{X}_{i(5)} - (\hat{X}_{i(N-1)} \times \hat{X}_{i(N)}) \cdot R(h_{5 \ldots (N-1)}) \hat{X}_{i(5)}$$

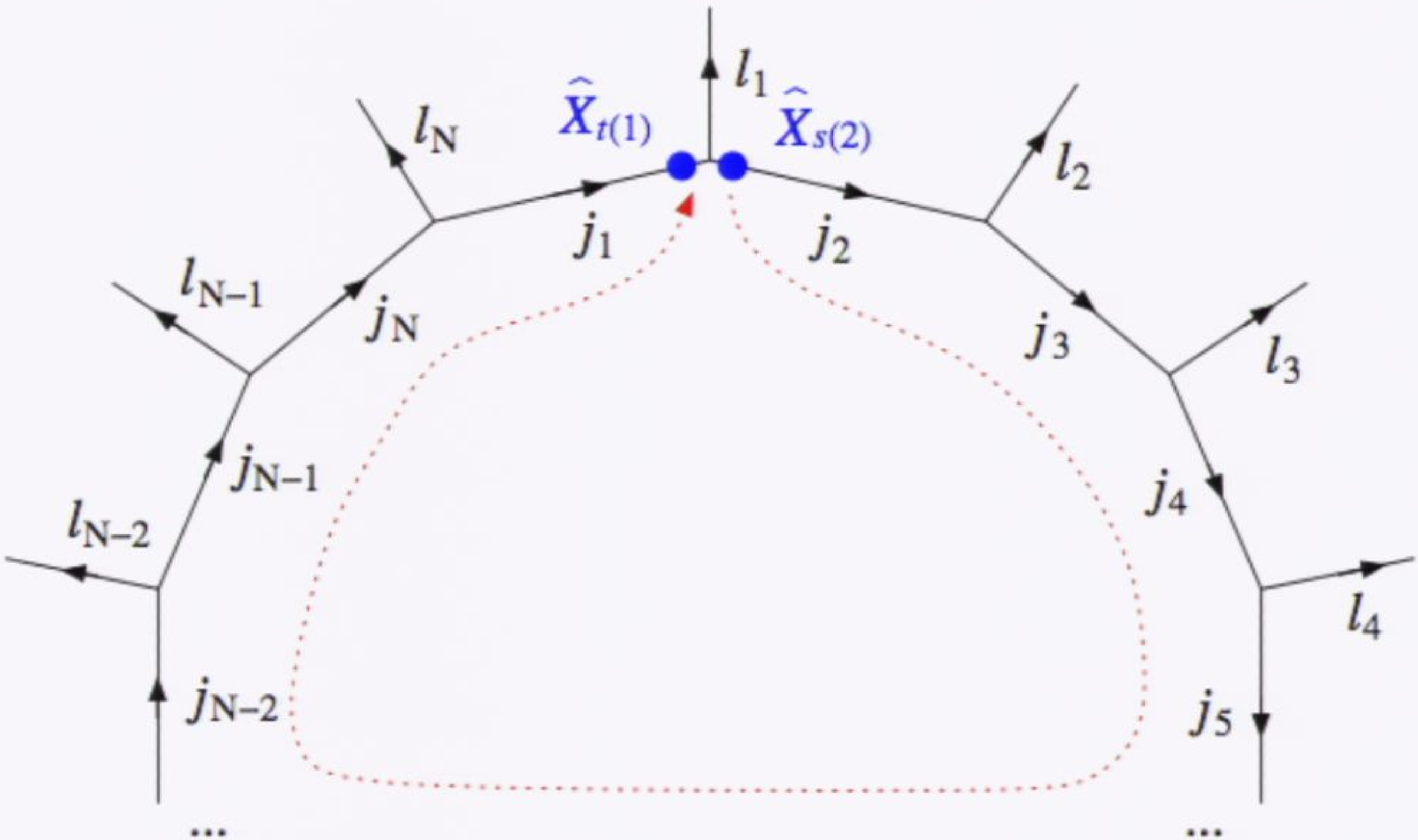
Generalization: cycles with N egdes



Constraint

$$\hat{C}_{22} = \hat{X}_{s(2)} \cdot \hat{X}_{s(2)} - \hat{X}_{s(2)} \cdot R(h_{23\ldots N1})\hat{X}_{s(2)}$$

An example



Constraint

$$\hat{C}_{12} = \hat{X}_{t(1)} \cdot \hat{X}_{s(2)} - \hat{X}_{t(1)} \cdot R(h_{23\ldots N1})\hat{X}_{s(2)}$$

An example

Recursion relation

$$(-1)^{j_1+j_2+l_2}\begin{Bmatrix} j_1 & j_1 & 1 \\ j_2 & j_2 & l_1 \end{Bmatrix}\psi_\Gamma^{\{j_e\}}$$

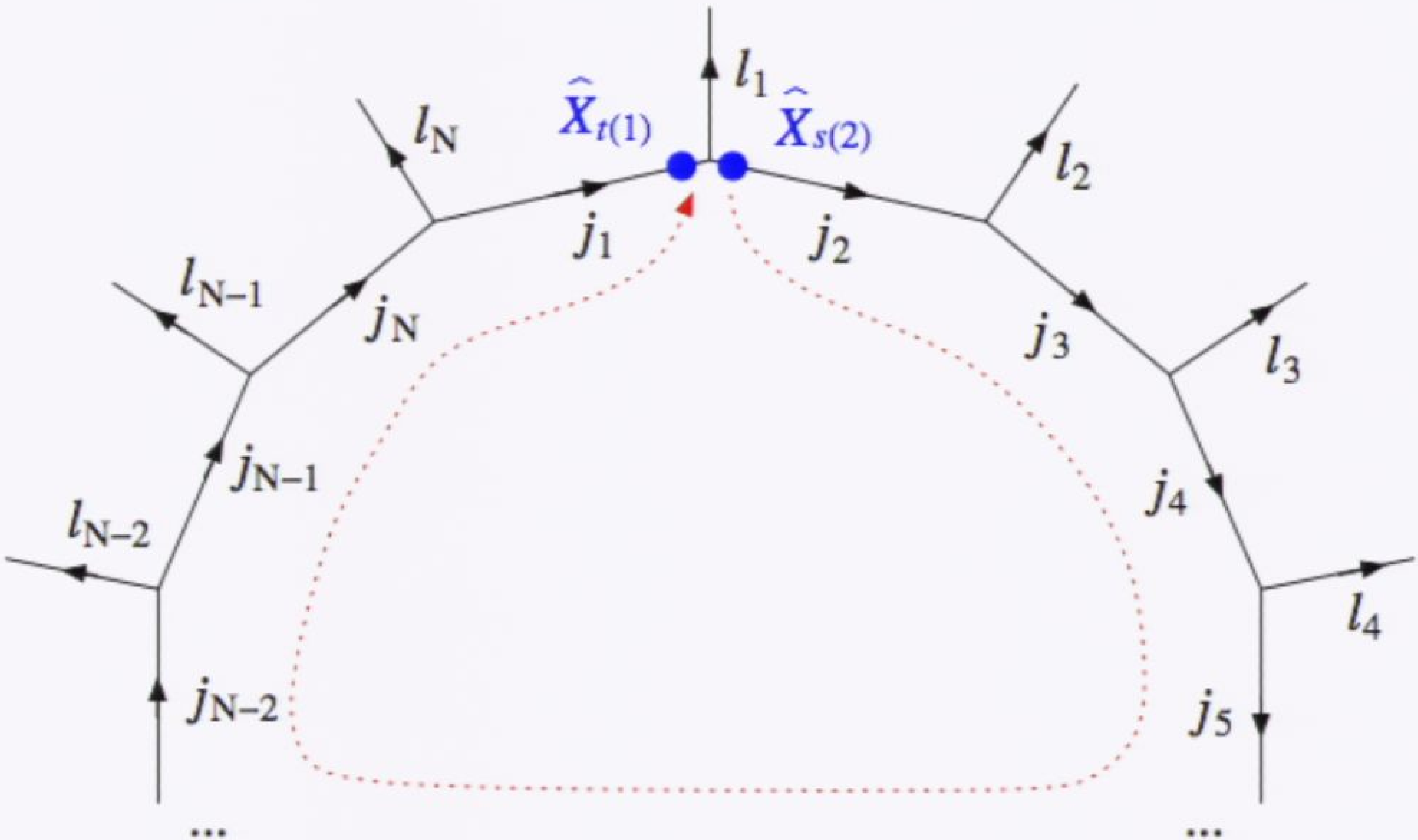
$$= \sum_{\varepsilon_{3,\dots,N}=-1}^{1}(-1)^{1+\varepsilon_N} d_{j_N+\varepsilon_N}$$

$$\times (-1)^{j_2+l_2+j_3}\begin{Bmatrix} j_2 & j_2 & 1 \\ j_3 & j_3 & l_2 \end{Bmatrix}(-1)^{j_N+l_N+j_1}\begin{Bmatrix} j_N & j_N+\varepsilon_N & 1 \\ j_1 & j_1 & l_N \end{Bmatrix}$$

$$\times \prod_{k=3}^{N-1}(-1)^{1+\varepsilon_k} d_{j_k+\varepsilon_k}(-1)^{j_k+l_k+j_{k+1}}\begin{Bmatrix} j_k & j_k+\varepsilon_k & 1 \\ j_{k+1}+\varepsilon_{k+1} & j_{k+1} & l_k \end{Bmatrix}\psi_\Gamma^{j_1,j_2,\{j_k+\varepsilon_k\}}$$

By construction, solutions are $3(N-1)j$-symbols.

An example



Constraint

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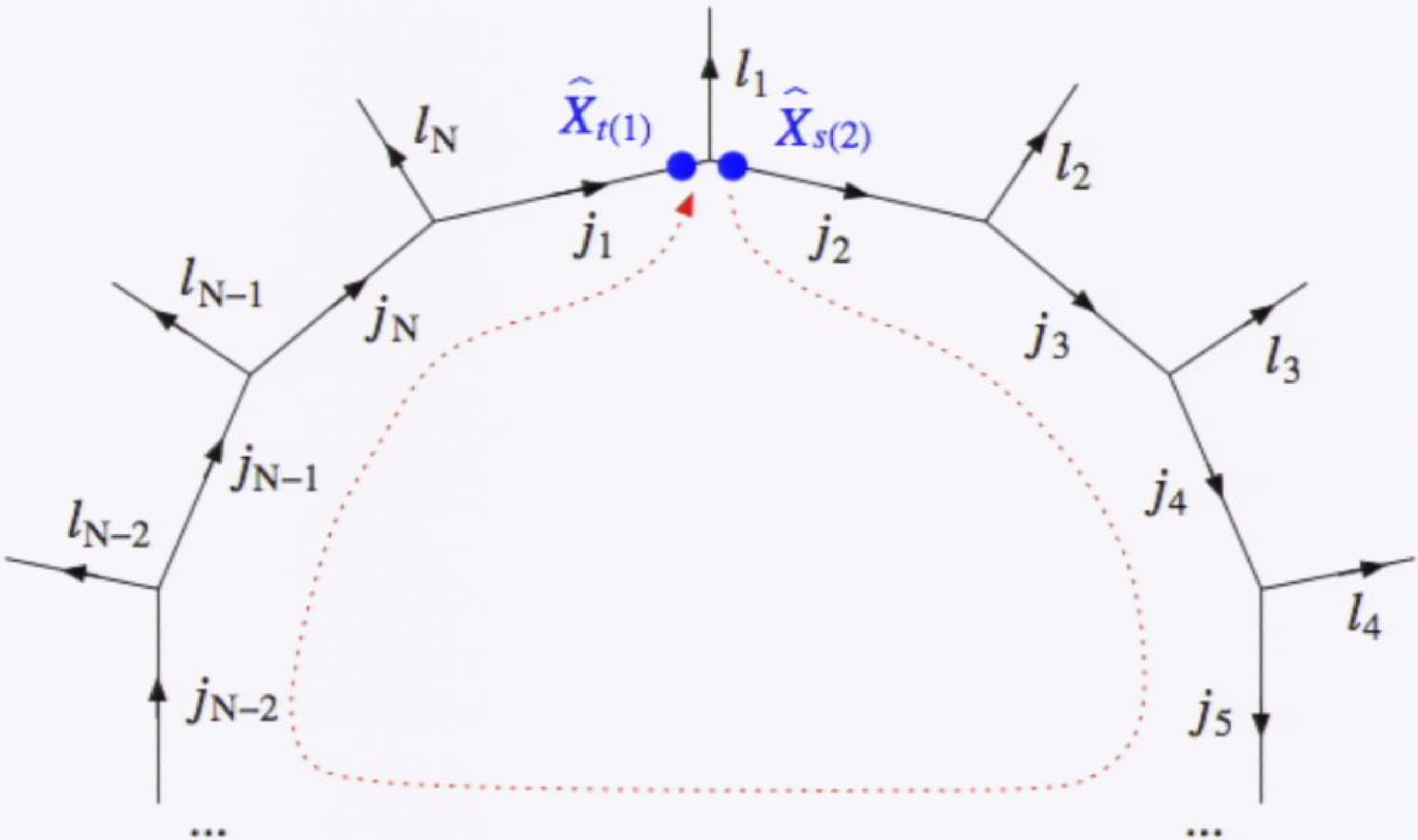
An example



Constraint

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An example

Recursion relation

$$(-1)^{j_1+j_2+l_2} \begin{Bmatrix} j_1 & j_1 & 1 \\ j_2 & j_2 & l_1 \end{Bmatrix} \psi_\Gamma^{\{j_e\}}$$

$$= \sum_{\varepsilon_{3,\ldots,N}=-1}^{1} (-1)^{1+\varepsilon_N} d_{j_N+\varepsilon_N}$$

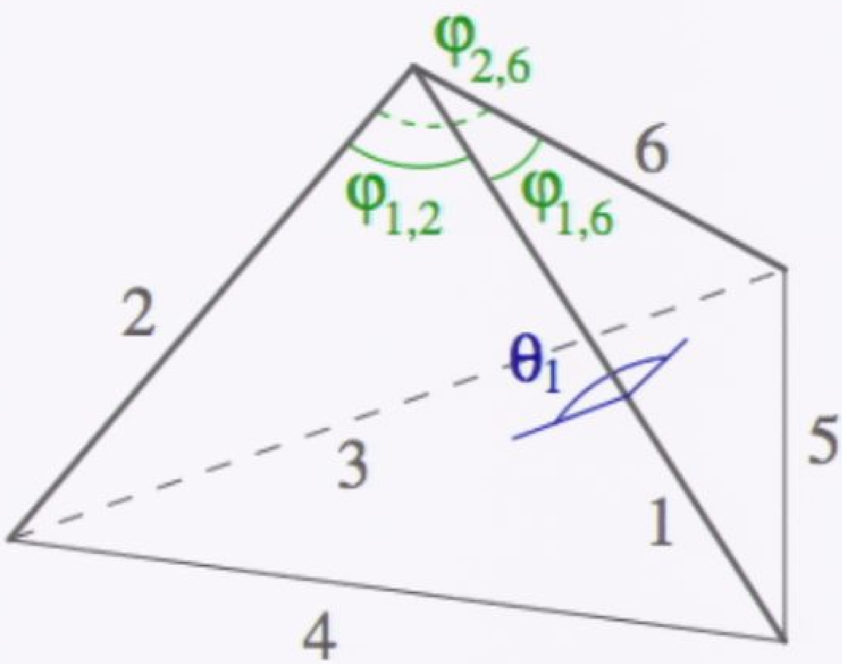
$$\times (-1)^{j_2+l_2+j_3} \begin{Bmatrix} j_2 & j_2 & 1 \\ j_3 & j_3 & l_2 \end{Bmatrix} (-1)^{j_N+l_N+j_1} \begin{Bmatrix} j_N & j_N+\varepsilon_N & 1 \\ j_1 & j_1 & l_N \end{Bmatrix}$$

$$\times \prod_{k=3}^{N-1} (-1)^{1+\varepsilon_k} d_{j_k+\varepsilon_k} (-1)^{j_k+l_k+j_{k+1}} \begin{Bmatrix} j_k & j_k+\varepsilon_k & 1 \\ j_{k+1}+\varepsilon_{k+1} & j_{k+1} & l_k \end{Bmatrix} \psi_\Gamma^{j_1,j_2,\{j_k+\varepsilon_k\}}$$

By construction, solutions are $3(N-1)j$-symbols.

An example

Geometric meaning in the large spin limit? Not always.



Tetrahedron

- dihedral angle θ_1 fixed, so
- $\hat{T}_1 = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\hat{\theta}_1}$ has a meaning.

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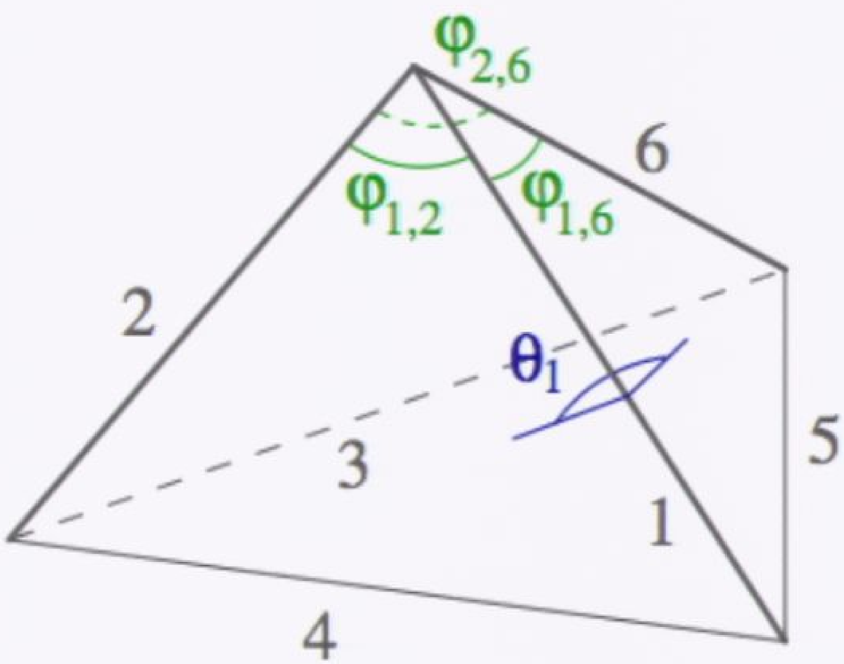

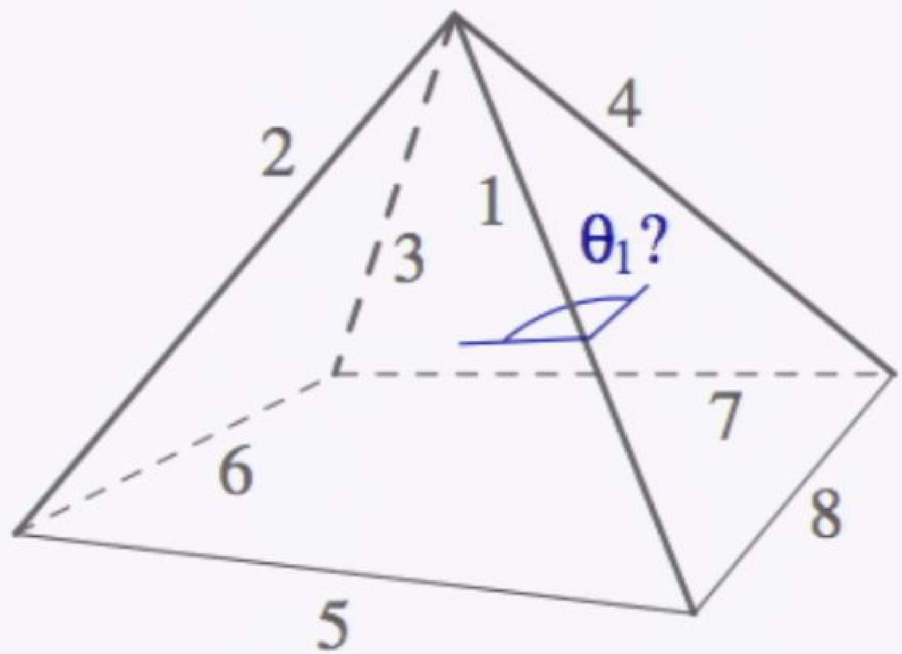

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Conclusion and outlook

Using LQG variables, a quantization of the Hamiltonian constraint has been proposed, it

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However

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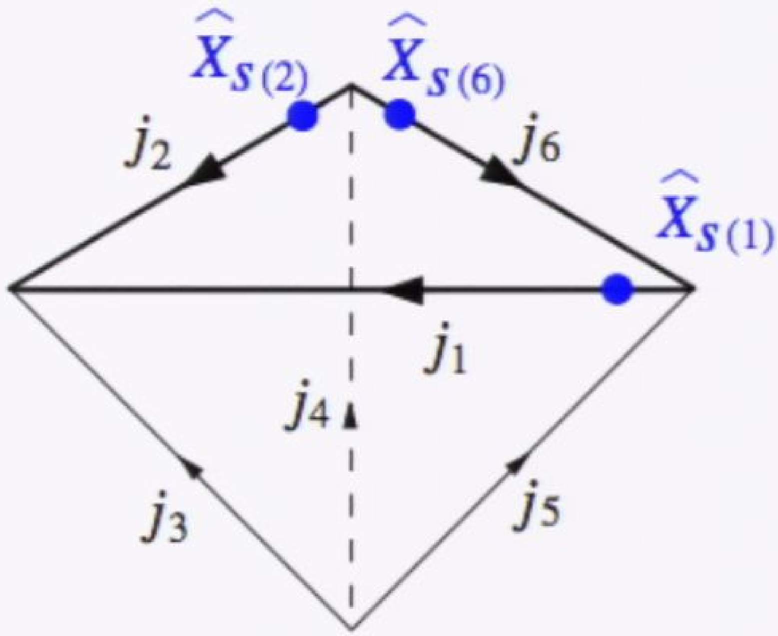
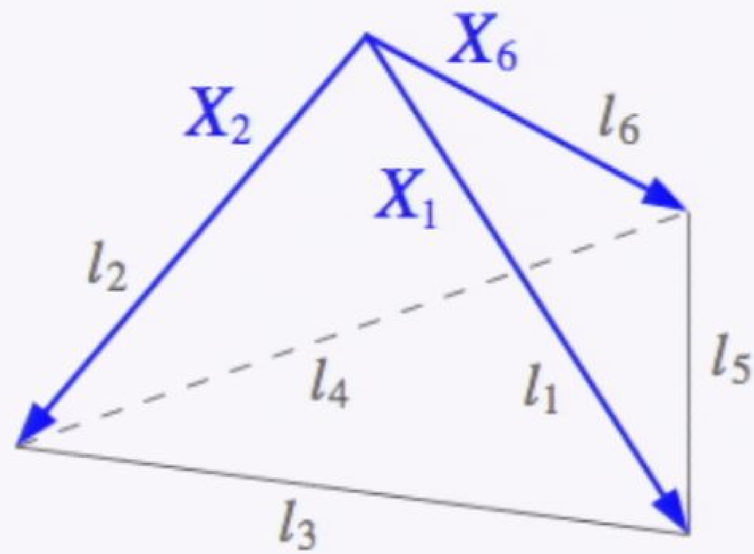
Thank you for your attention

Other relations for the tetrahedron

Volume of the tetrahedron?





Constraint

Loop quantization

Elementary QM	LQG
Position x	Connection A
Wavefunction $\psi(x)$	Cylindrical function $\psi_{\Gamma,f}[A]$
Position operator $\hat{X}\psi = x\,\psi(x)$	Connection operator $\hat{A} = A\,\psi_{\Gamma,f}[A]$
Momentum operator $\hat{P}\psi \propto \frac{\partial\psi}{\partial x}$	Triad operator $\hat{E}\psi \propto \frac{\delta\psi_{\Gamma,f}}{\delta A}$

Cylindrical functions

Γ is an oriented graph with N edges $e_1, e_2 \ldots, e_N$ and $f : (\mathrm{SU}(2))^N \to \mathbb{C}$,

$$\psi_{\Gamma,f}[A] \stackrel{\text{def.}}{=} f\left(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_N}[A]\right)$$

where $h_e[A] \in \mathrm{SU}(2)$ is the *holonomy* of A along e.

Hamiltonian general relativity: constraints

The Lagrangian density of gravity $\mathcal{L}$ enables to define canonical momenta p and the Hamiltonian density $\mathcal{H}$

$$p^{ab} \stackrel{\text{def.}}{=} \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}}, \qquad \mathcal{H} \stackrel{\text{def.}}{=} p^{ab}\dot{q}_{ab} - \mathcal{L} = NC(q,p) + N^a V_a(q,p),$$

with

$$\frac{\delta S_{\text{EH}}}{\delta g_{\mu\nu}} = 0 \Longleftrightarrow \begin{cases} \dfrac{\delta S_{\text{EH}}}{\delta q_{ab}} = 0 & \text{Intrinsic dynamics} \\ C(q,p) = 0 & \textbf{Hamiltonian constraint} \\ V_a(q,p) = 0 & \text{Vector constraint} \end{cases}$$

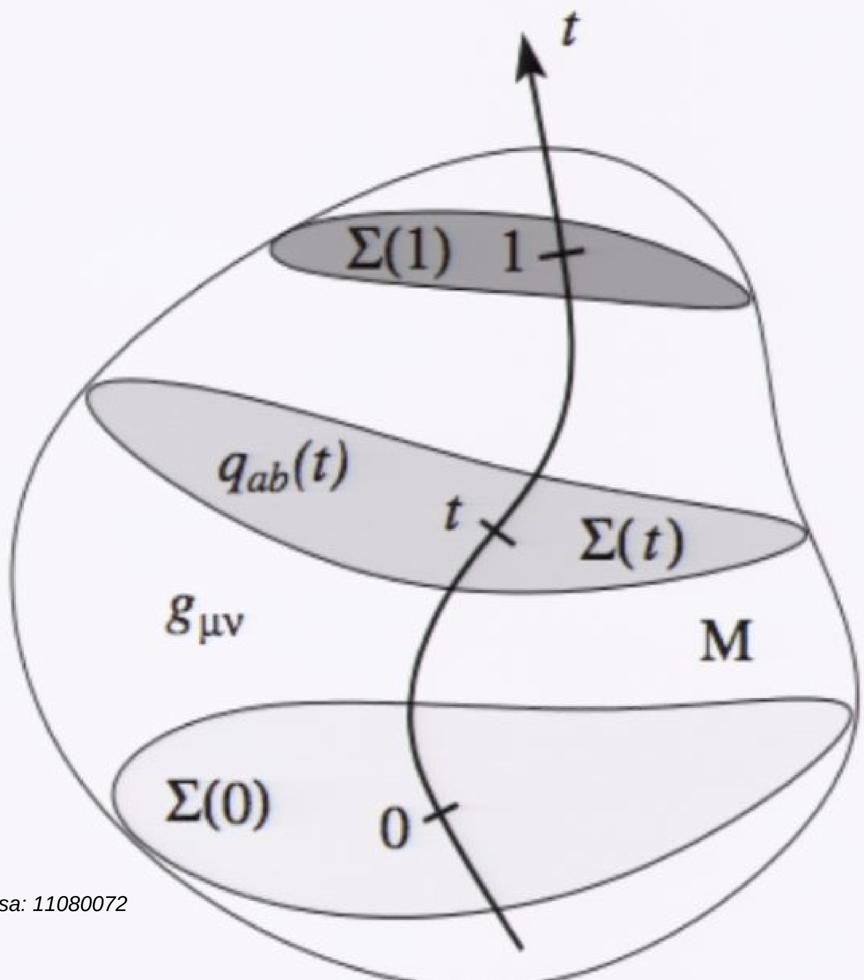
$$C(q,p) = \frac{1}{\sqrt{\det q}}\left[\operatorname{tr}(p^2) - (\operatorname{tr} p)^2\right] - \sqrt{\det q}\,({}^2R)$$

Encodes crucial information *spacetime flatness*.

Hamiltonian general relativity: ADM formalism

Fundamental equation of ADM formalism

Spacetime = Space + Time



Spacetime	Space + Time
x^μ	x^a, t
$g_{\mu\nu}$	q_{ab}, N, N^a
$R_{\mu\nu}$	R_{ab}, K_{ab}

- N = lapse, N^a = shift
- K_{ab} = extrinsic curvature

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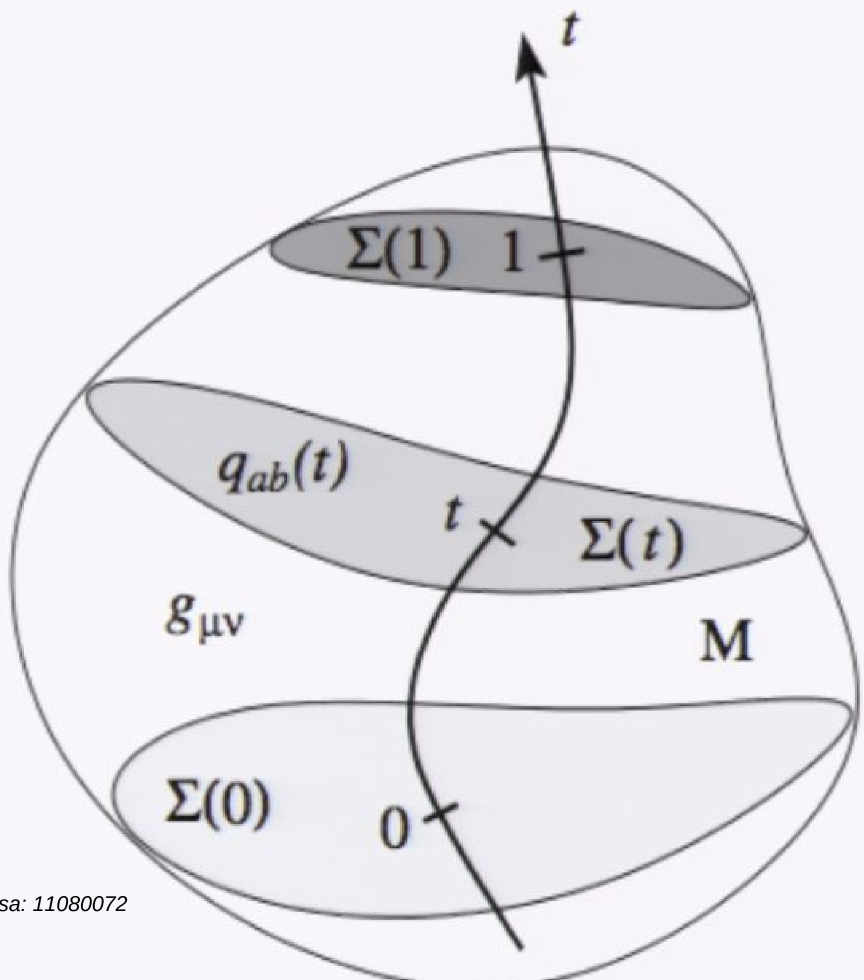

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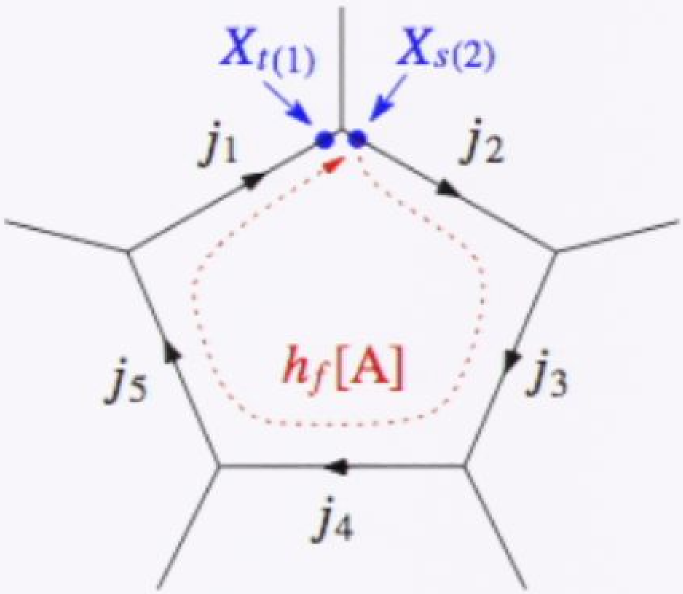
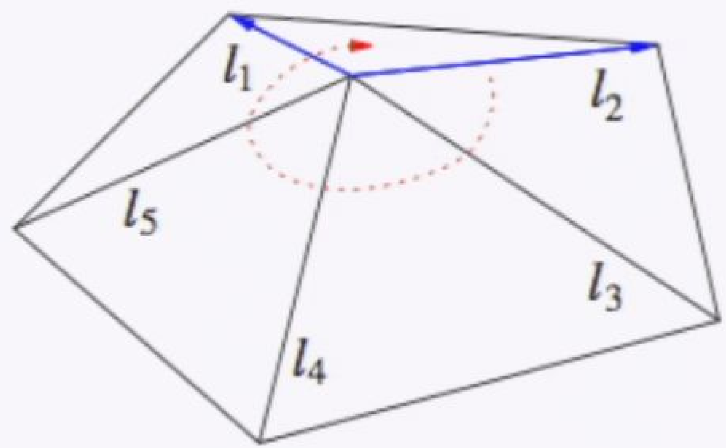
Outline

1. From classical to quantum gravity
2. Spin network states
3. Hamiltonian constraint and recurrence relations

Quantization of the Hamiltonian constraint

Idea: use fluxes to probe the flatness of spacetime.





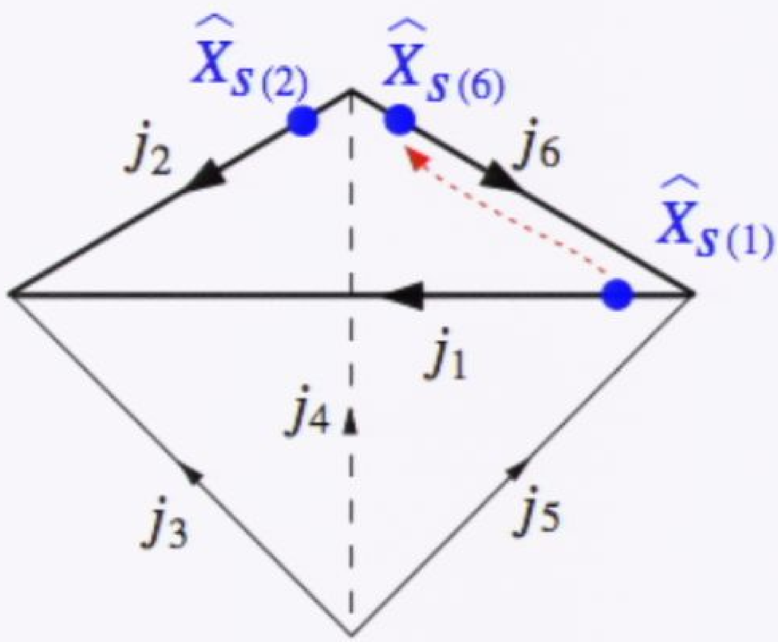
Proposition: quantum Hamiltonian contraint

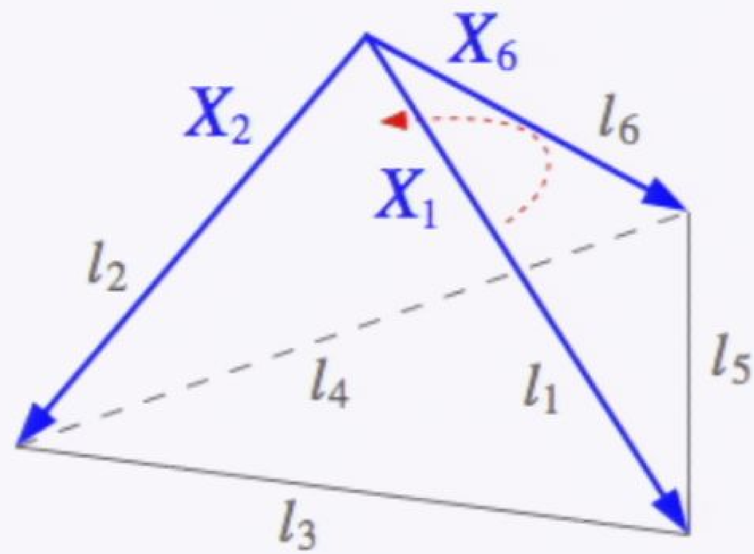
$$\hat{C}_{12} \stackrel{\text{def.}}{=} \hat{X}_{t(1)} \cdot \hat{X}_{s(2)} - \hat{X}_{t(1)} \cdot R(h_f[A])\hat{X}_{s(2)}$$

$$h_f \stackrel{\text{def.}}{=} h_{e_1}\, h_{e_5}\, h_{e_4}\, h_{e_3}\, h_{e_2}$$

Other relations for the tetrahedron

Volume of the tetrahedron?



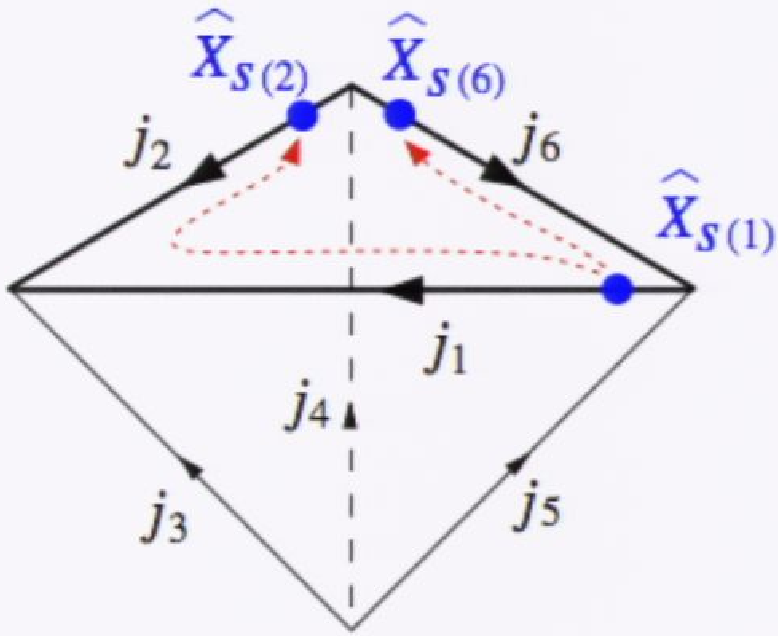


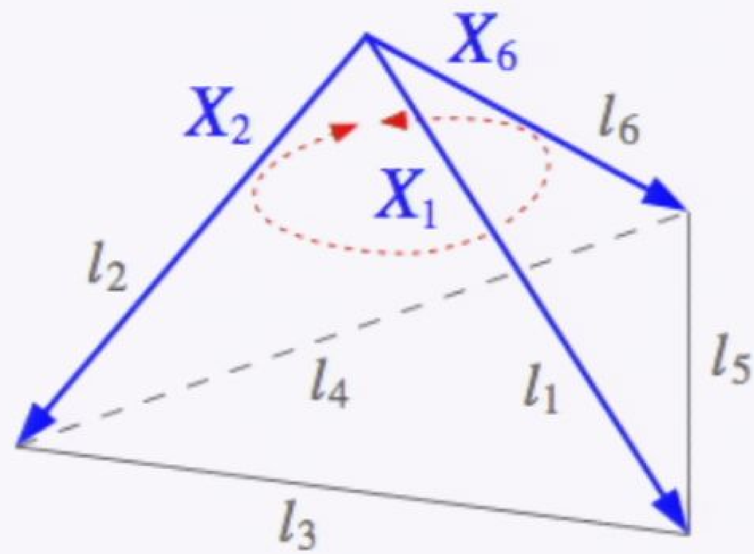
Constraint

$$\left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_6^{-1})\hat{X}_{s(1)}$$

Other relations for the tetrahedron

Volume of the tetrahedron?



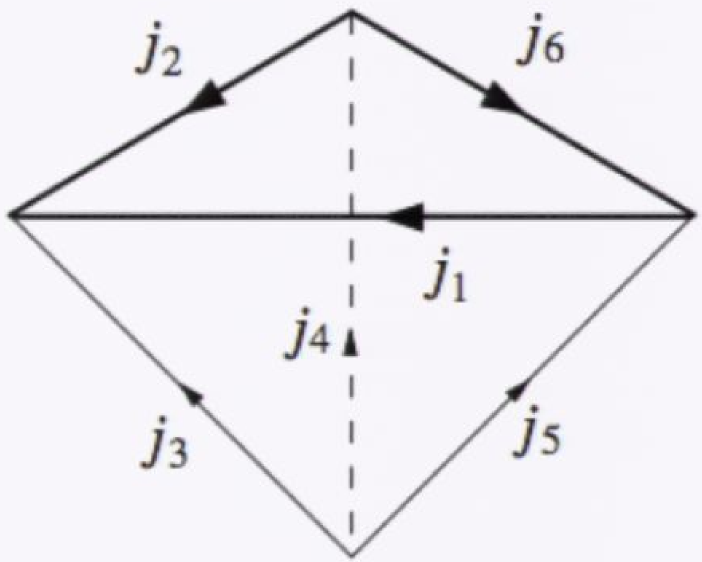


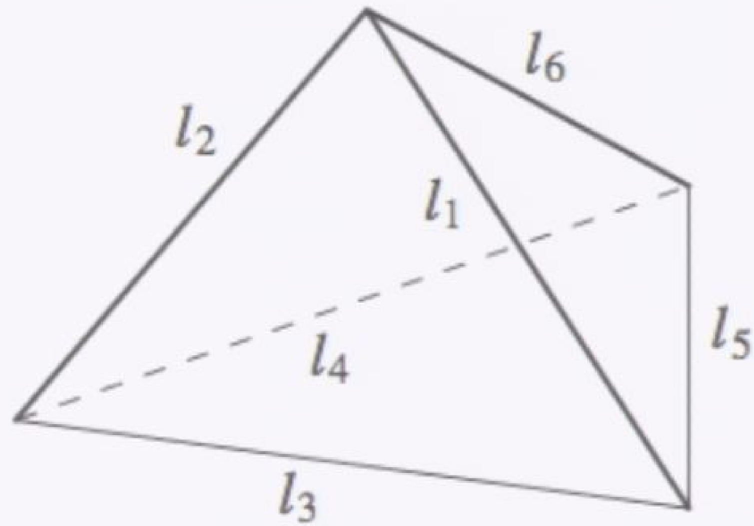
Constraint

$$\hat{C}_{126} = \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_6^{-1})\hat{X}_{s(1)} - \left(\hat{X}_{s(2)} \times \hat{X}_{s(6)}\right) \cdot R(h_2^{-1}h_1)\hat{X}_{s(1)}$$

Application to the tetrahedron

Consider the most simple triangulation of the 2-sphere: a tetrahedron.





Quantization of the Hamiltonian constraint

Comparison with classical expression

Classical constraint $C = \left(\varepsilon_k{}^{ij} F^k_{ab}\right) E^a_i E^b_j$

Quantum contraint $\hat{C}_{12} = \left(\delta_{ij} - R(h_f[A])_{ij}\right) \hat{X}^i_{t(1)} \hat{X}^j_{s(2)}$

Quantization pattern

$$E^a_j \longrightarrow \hat{X}^i_{t(1)}$$

$$\varepsilon_k{}^{ij} F^k_{ab} \longrightarrow \delta_{ij} - R(h_f[A])_{ij}$$

Application to the tetrahedron

- Scalar product

$$\hat{X}_{s(2)} \cdot \hat{X}_{s(6)} \psi_{\text{tet}}^{\{j_e\}} = N_{j_2} N_{j_6} (-1)^{j_2+j_4+j_6} \begin{Bmatrix} j_2 & j_2 & 1 \\ j_6 & j_6 & j_4 \end{Bmatrix} \psi_{\text{tet}}^{\{j_e\}}.$$

- Scalar product after parallel transport

$$\hat{X}_{s(2)} \cdot R(h_2^{-1}\, h_1\, h_6) \hat{X}_{s(6)} \psi_{\text{tet}}^{\{j_e\}}$$

$$= N_{j_2} N_{j_6} \sum_{\varepsilon_1=-1}^{1} (-1)^{1+\varepsilon_1} d_{j_1+\varepsilon_1} (-1)^{j_1+j_2+j_3} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{Bmatrix}$$

$$\times (-1)^{j_1+j_5+j_6} \begin{Bmatrix} j_1+\varepsilon_1 & j_1 & 1 \\ j_6 & j_6 & j_5 \end{Bmatrix} \psi_{\text{tet}}^{j_1+\varepsilon_1,\{j_e\}}.$$

We denoted $d_j \overset{\text{def.}}{=} 2j+1$ and $N_j \overset{\text{def.}}{=} \sqrt{j(j+1)d_j}$

Application to the tetrahedron

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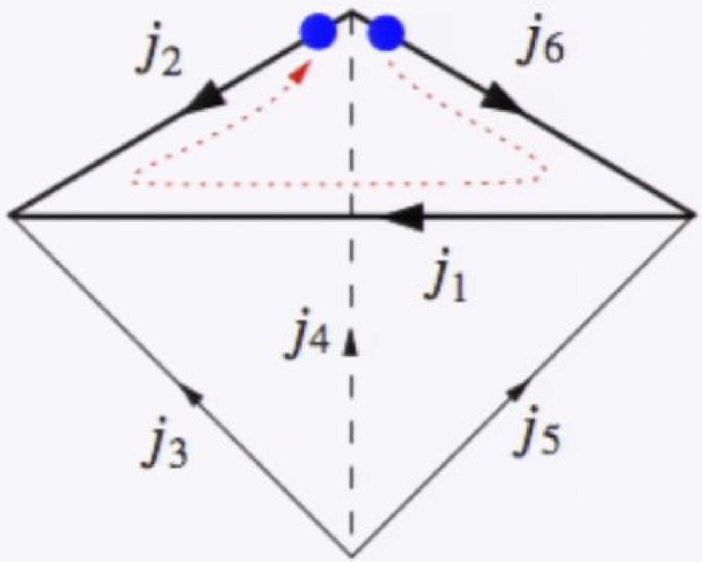
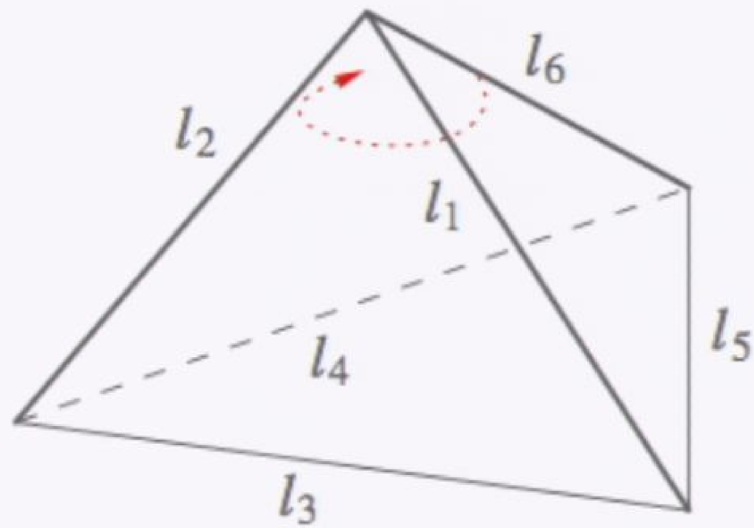




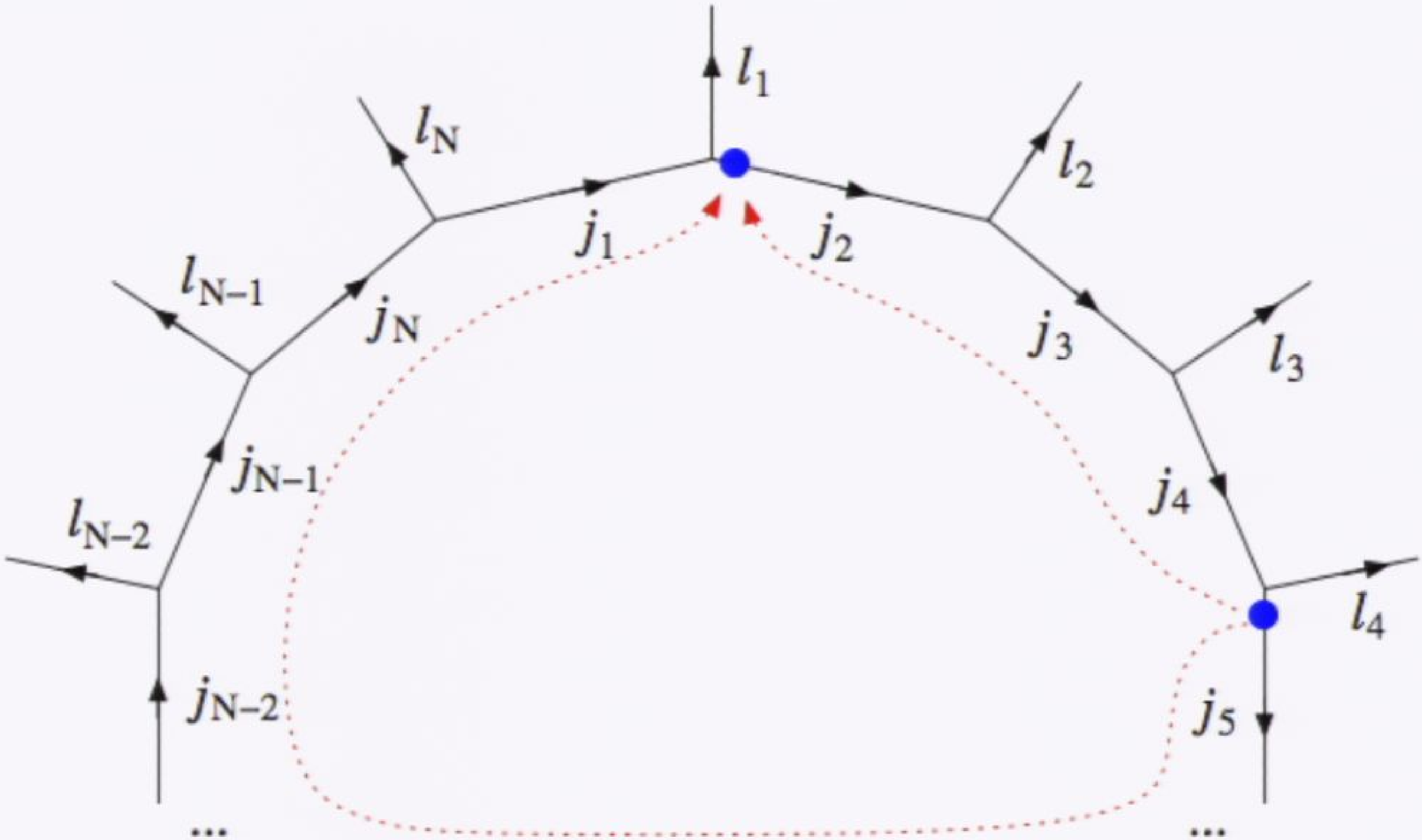
Question

For the constraint

$$\hat{C}_{26} = \hat{X}_{s(2)} \cdot \hat{X}_{s(6)} - \hat{X}_{s(2)} \cdot R(h_2^{-1}\, h_1\, h_6)\hat{X}_{s(6)},$$

what do we get imposing $\quad \hat{C}_{26}\psi_{\text{tet}}^{\{j_e\}} = 0$?

Generalization: cycles with N egdes



Constraint

$$\hat{C}_{25} = \hat{X}_{s(2)} \cdot R(h_{234}^{-1})\hat{X}_{s(5)} - \hat{X}_{s(2)} \cdot R(h_{56\ldots N1})\hat{X}_{s(5)}$$

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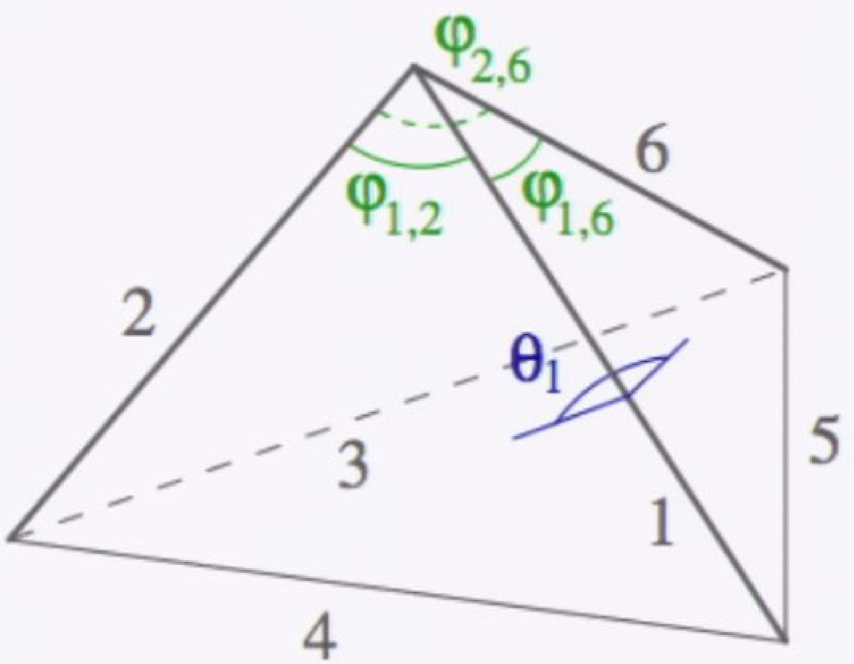

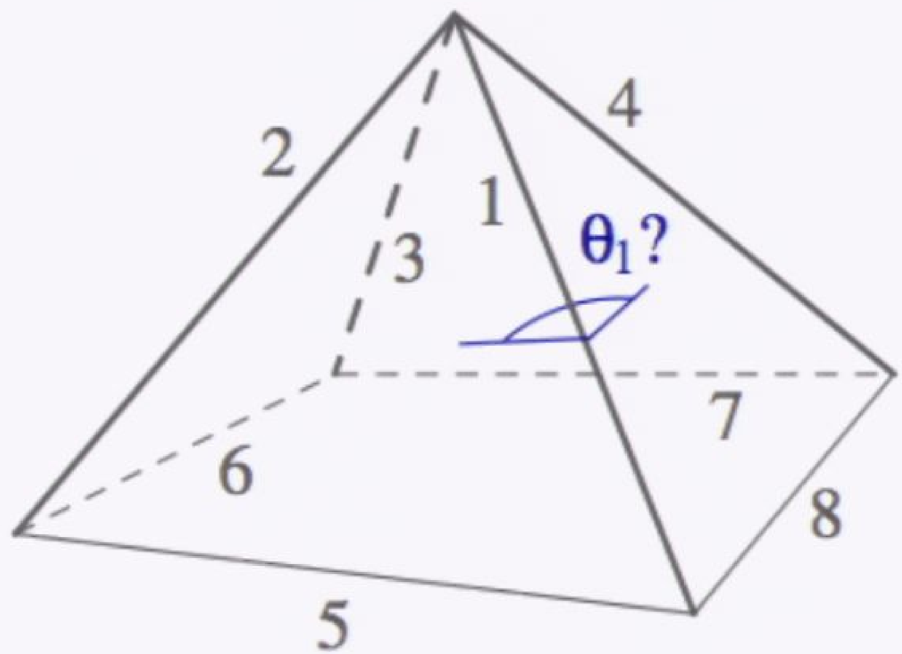

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