Title: N=1 SQCD and the Transverse Field Ising Model

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Abstract: We study the dimensions of non-chiral operators in the Veneziano limit of N=1 supersymmetric QCD in the conformal window. We show that when acting on gauge-invariant operators built out of scalars, the 1-loop dilatation operator is equivalent to the spin chain Hamiltonian of the 1D Ising model in a transverse magnetic field, which is a nontrivial integrable system that is exactly solvable at finite length. Solutions with periodic boundary conditions give the anomalous dimensions of flavor-singlet operators and solutions with fixed boundary conditions give the anomalous dimensions of operators whose ends contain open flavor indices.

Motivation

4D CFTs and SCFTs are interesting for many reasons:

- Basic building blocks of 4D QFTs,
- Dual to theories of quantum gravity in AdS,
- Could play a role in BSM physics!
  - Walking/Conformal Technicolor [Many people...]
  - Conformal Sequestering [Luty, Sundrum '01; Schmaltz, Sundrum '06]
  - Solution to $\mu/B\mu$ problem [Roy, Schmaltz '07; Murayama, Nomura, Poland '07]
  - Flavor Hierarchies [Georgi, Nelson, Manohar '83; Nelson, Strassler '00; Poland, DSD '09; Craig '10] ...

Operator dimensions at $\lambda \sim O(1)$

Unfortunately,

- Phenomenological applications often involve statements about operator dimensions that are difficult to check.
- In $\mathcal{N}=1$ SCFTs, we know lots about chiral operators, but not much about non-chiral operators. Gravity duals useful at very strong coupling $\lambda \gg 1$.

Hard to say things at $\lambda \sim O(1)$. Major exception is in (planar) $\mathcal{N}=4$ SYM (and related theories), using integrability.

- So far, dilatation operator studied in somewhat narrow range of theories ($\mathcal{N}=4$, orbifolds, β-deformation, recently $\mathcal{N}=2$ [Gadde, Pomoni, Rastelli '10])
- None of them phenomenologically relevant.
- Success suggests interesting structures might lie hidden in a wider range of theories (not necessarily all-loops integrability)

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Why $\mathcal{N}=1$ SQCD?

Let's gather some data!

Some early evidence for interesting structure in $\mathcal{N}=4$: Minahan, Zarembo '02 showed that the 1-loop dilatation operator acting on scalars $\mathrm{Tr}(X_i \ldots X_j)$ is equivalent to an exactly solvable Heisenberg spin chain.

This talk: follow Minahan and Zarembo for (planar) $\mathcal{N}=1$ SQCD

- Simplest nontrivial $\mathcal{N}=1$ superconformal theory, outside the $\mathcal{N}=4$ family.
- *Two* weak coupling (Banks–Zaks) regimes. For now, we'll focus on the weakly-coupled electric description. Eventually, we might hope to learn more about Seiberg duality.

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2. 1-loop Dilatation Operator
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$\mathcal{N}=1$ SQCD

- Matter content:

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$	$U(1)_B$	$U(1)_R$
Q_{ai}	$\square$	$\square$	1	1	$1-\frac{N_c}{N_f}$
$\widetilde{Q}^{\tilde{\imath}a}$	$\overline{\square}$	1	$\overline{\square}$	-1	$1-\frac{N_c}{N_f}$

$$
\begin{aligned}
i,\tilde{\imath} &= 1,\ldots,N_f \\
a &= 1,\ldots,N_c
\end{aligned}
$$

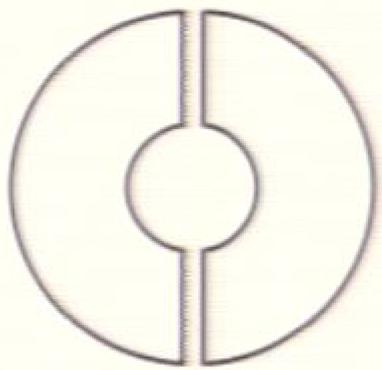
- Conformal window: $\frac{3}{2} < \frac{N_f}{N_c} < 3$
- Veneziano limit: $N_f, N_c \to \infty$, with $\frac{N_f}{N_c}$ fixed.
  - Electric weak-coupling regime: $\epsilon = \frac{3N_c}{N_f} - 1 \ll 1$,

$$
\lambda \equiv \frac{g^2 N_c}{8\pi^2} = \epsilon + \frac{\epsilon^2}{2} + \frac{9}{4}(1+2\zeta_3)\epsilon^3 \ldots
$$

  - Magnetic weak-coupling regime $\tilde{\epsilon} = 1 - \frac{3N_c}{2N_f} \ll 1$.

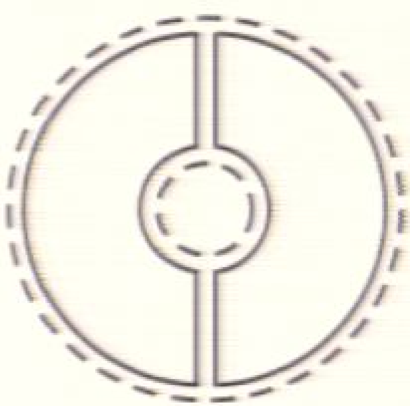
Large-N Counting

- 't Hooft limit: Double lines for adjoints; Single lines for fundamentals



$$\sim N_c^{2-2h-b}$$

- Veneziano limit: Double (gauge+flavor) lines for fundamentals



$$\sim N_c^{2-2h-b} N_f^b \sim N^{2-2h}$$

Generalized Single Trace Operators

Basic objects are "generalized single-trace" (GST) operators. Using only scalars, we have: flavor singlets,

$$Q_{ai}\, Q^{\dagger ib}\, \widetilde{Q}^{\dagger}_{b\tilde{\jmath}} \widetilde{Q}^{\tilde{\jmath}c} \ldots Q^{ka} \quad = \quad \mathrm{Tr}(XY\ldots X) \quad \text{“closed”}$$

where,

$$X \equiv (QQ^{\dagger})^{a}_{b}, \quad Y \equiv (\widetilde{Q}^{\dagger}\widetilde{Q})^{a}_{b}$$

and flavor adjoint+bifundamentals,

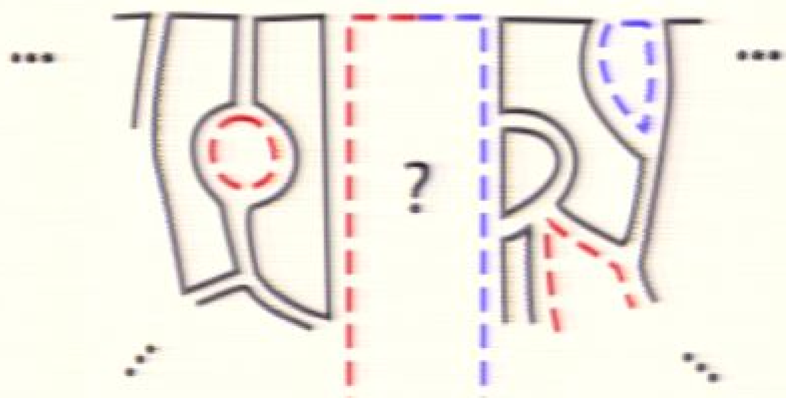
$$\left.\begin{array}{c} Q^{\dagger}XY\ldots XQ \\ \widetilde{Q}XY\ldots XQ \\ Q^{\dagger}XY\ldots X\widetilde{Q}^{\dagger} \\ \widetilde{Q}XY\ldots X\widetilde{Q}^{\dagger} \end{array}\right\} \quad \text{“open”}$$

Factorization

What about big chiral operators,

$$\text{e.g.:} \qquad \text{Tr}(MM\ldots M) \;=\; Q_{ai}\,\widetilde{Q}^{ib}\,Q_{bj}\,\widetilde{Q}^{jc}\ldots\widetilde{Q}^{ka}?$$

Left-right flavor indices contracted are generalized *multi*-trace



Dimensions and correlators of GMT operators factorize at large N:

$$\begin{aligned}\dim(\mathcal{O}_1\mathcal{O}_2) &= \dim\mathcal{O}_1+\dim\mathcal{O}_2+O(1/N^2)\\ \langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle &= \sum_{\text{pairings}}\prod_{\text{pairs}}\langle\mathcal{O}_i\mathcal{O}_j\rangle+O(1/N)\end{aligned}$$

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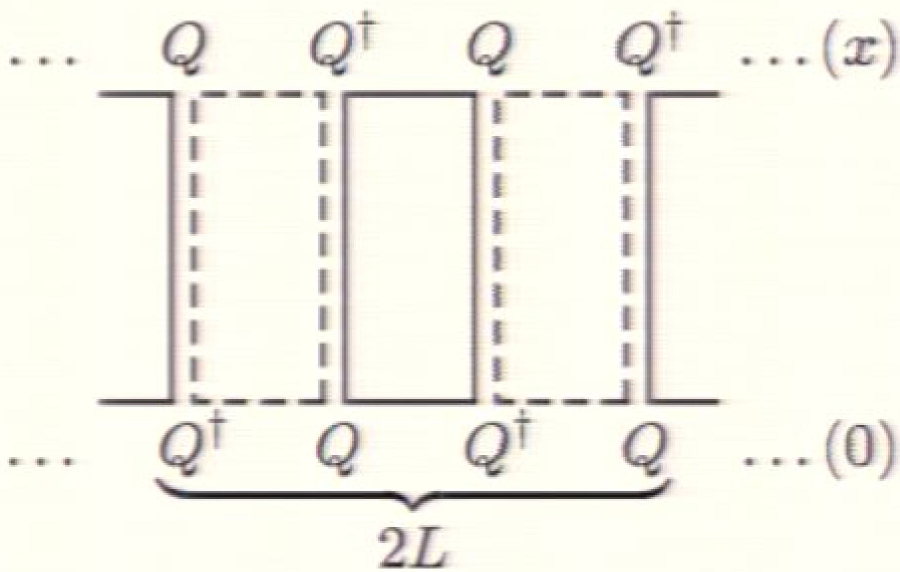
Tree-Level

To compute anomalous dimensions, study 2-pt functions

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim x^{-2(\Delta_0+\lambda\Delta_1+\dots)} = x^{-2\Delta_0}(1-\lambda\Delta_1\log(x^2)+\dots)$$

At tree level,



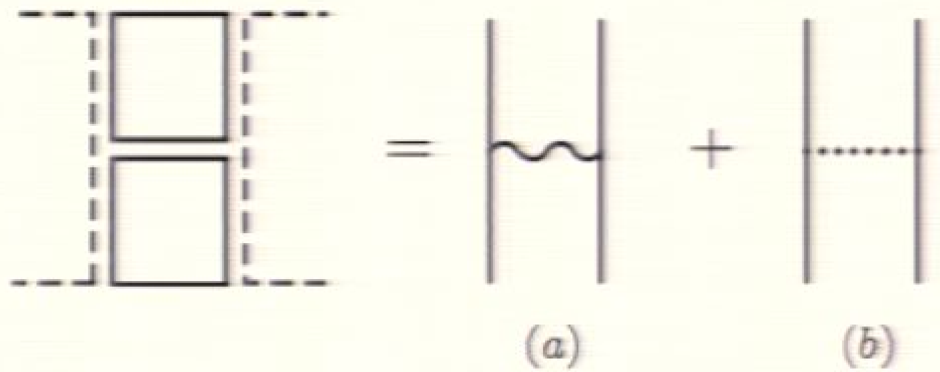
$$= N_c^L N_f^L x^{-2(2L)}$$

One Loop Contributions

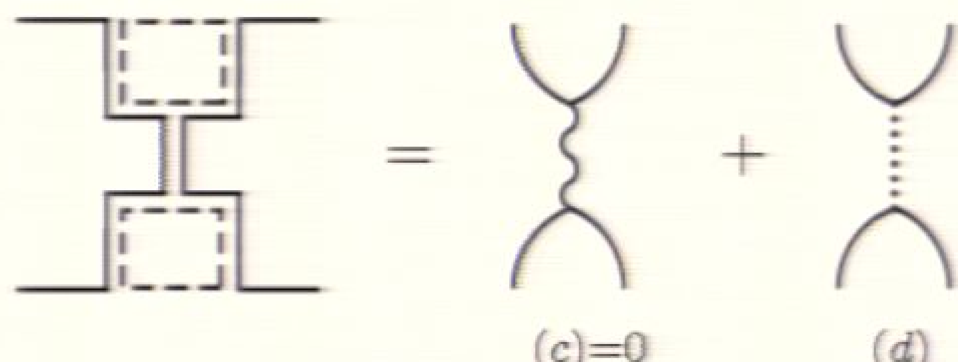
wavefunction:

$= 0$ (Feynman gauge)

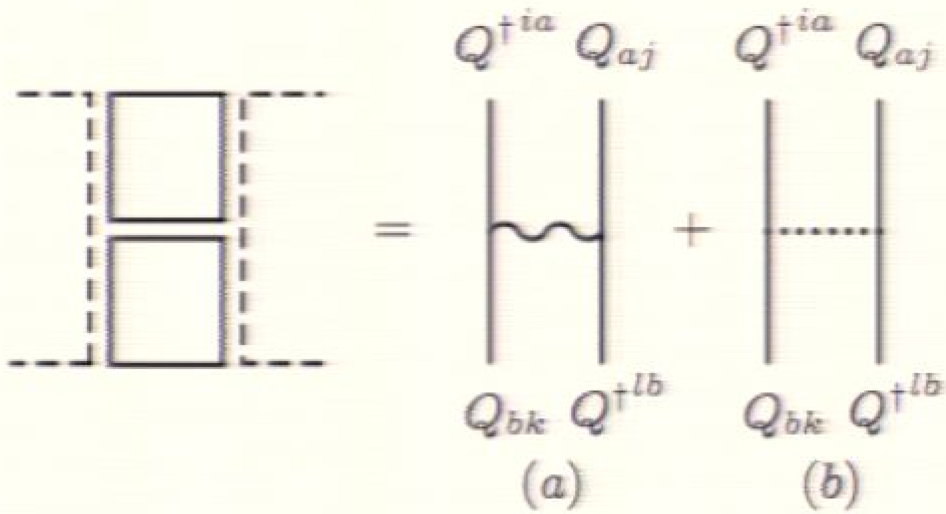
color-loop:



flavor-loop:



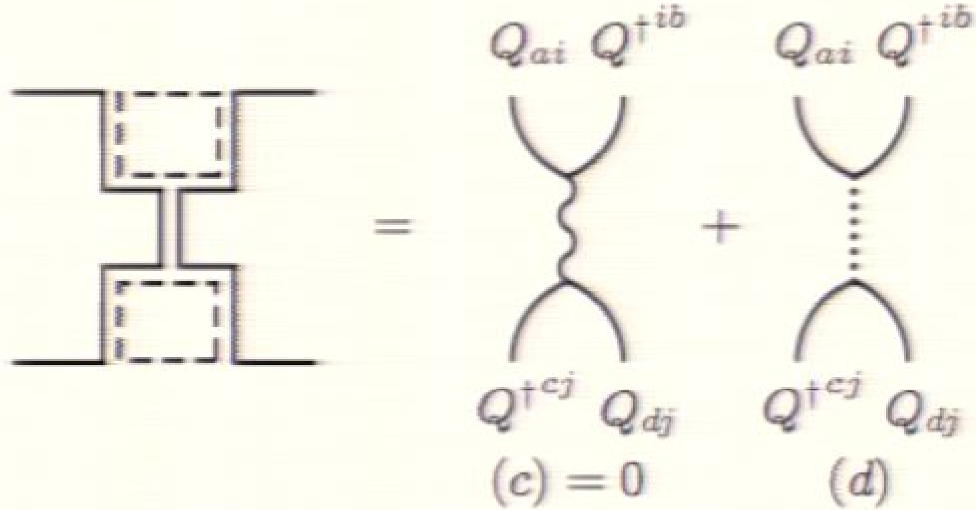
Color Loops



Graph (b) comes from the D-term potential: $(Q^\dagger T^A Q - \widetilde{Q} T^A \widetilde{Q}^\dagger)^2$

$$(a)+(b) \;=\; \left(\begin{array}{c|cccc} & Q^\dagger Q & Q^\dagger \widetilde{Q}^\dagger & \widetilde{Q} Q & \widetilde{Q}\widetilde{Q}^\dagger \\ \hline Q Q^\dagger & A+B & & & \\ \widetilde{Q} Q & & A-B & & \\ Q^\dagger \widetilde{Q}^\dagger & & & A-B & \\ \widetilde{Q}\widetilde{Q}^\dagger & & & & A+B \end{array}\right) \times \lambda \, \mathbb{I}_{\text{flavor}}$$

Flavor Loops



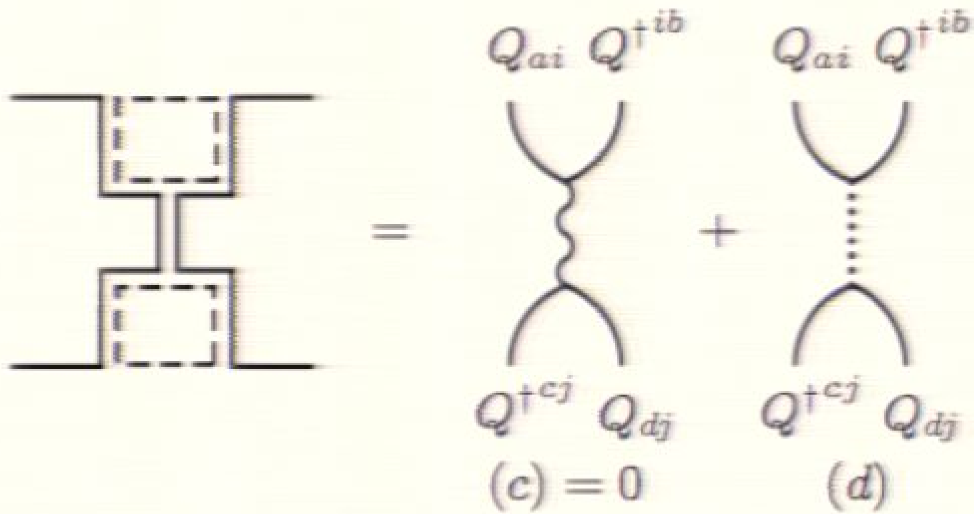
Graph (d) is a 4-scalar contact interaction, so equals (b) up to overall factors

$$(c)+(d) \;=\; \left(\begin{array}{c|cc} & QQ^\dagger & \widetilde{Q}^\dagger\widetilde{Q} \\ \hline Q^\dagger Q & B & -B \\ \widetilde{Q}\widetilde{Q}^\dagger & -B & B \end{array}\right) \times \frac{N_f}{N_c}\lambda \times 2T^A\otimes T^A$$

In the planar limit, $2T^A\otimes T^A$ is equivalent to $\mathbb{I}_{\text{color}}$.

Exception: 2-field operators, where $\text{Tr}(T^A)\otimes\text{Tr}(T^A)=0$.

Flavor Loops



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Fixing A, B from Consistency

$$\gamma_{\text{two-field}} = \left(\begin{array}{c|cccc} & Q^\dagger Q & Q^\dagger \widetilde{Q}^\dagger & \widetilde{Q}Q & \widetilde{Q}\widetilde{Q}^\dagger \\ \hline QQ^\dagger & A+B & & & \\ \widetilde{Q}Q & & A-B & & \\ Q^\dagger\widetilde{Q}^\dagger & & & A-B & \\ \widetilde{Q}\widetilde{Q}^\dagger & & & & A+B \end{array}\right)\lambda$$

No need to evaluate any diagrams:

$$\begin{aligned} \gamma(\widetilde{Q}Q) &= (A-B)\lambda = -\lambda && \text{(chiral: } \Delta = 3R/2) \\ \gamma(\mathrm{Tr}(Q^\dagger Q - \widetilde{Q}\widetilde{Q}^\dagger)) &= (A+B)\lambda = 0 && \text{(conserved current)} \end{aligned}$$

(remember $\lambda = \frac{3N_c}{N_f} - 1$ at one loop)

So $A = -B = -\frac{1}{2}$.

Spin Chain Notation

- We have gauge-adjoint dimers $X = QQ^\dagger, Y = \widetilde{Q}^\dagger \widetilde{Q}$.
- Useful notation: $XYX\ldots \longrightarrow |\uparrow\downarrow\uparrow\ldots\rangle$ (spin chain state). $\mathrm{Tr}(XY\ldots X)$ corresponds to a shift-invariant state.

$$\begin{aligned}\text{color-loop} &= \lambda(A\,\mathbb{I}_i \otimes \mathbb{I}_{i+1} + B\,\sigma_i^z \otimes \sigma_{i+1}^z) \\ \text{flavor-loop} &= \frac{\lambda N_f}{N_c}(B\mathbb{I}_i - B\sigma_i^x)\end{aligned}$$

$$\gamma_{\text{closed}} = \frac{N_f - N_c}{2N_c}\lambda L + \frac{\lambda}{2}\underbrace{\sum_{i=0}^{L-1}(\sigma_i^z\sigma_{i+1}^z - \frac{N_f}{N_c}\sigma^x)}_{H}$$

- H is Hamiltonian for Transverse Field Ising Model ($h = \frac{N_f}{N_c} \approx 3$): exactly solvable (Pfeuty '70)

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Jordan-Wigner Transformation

The Hamiltonian is

$$H = \sum_{n=0}^{L-1} \sigma_n^z \sigma_{n+1}^z - h\sigma_n^x \qquad (\text{with } h = \frac{N_f}{N_c})$$

- Jordan-Wigner transformation ('28):

$$c_n^\dagger = \left(\prod_{m=0}^{n-1} \sigma_m^x \right) \sigma_n^+ \qquad c_n = \left(\prod_{m=0}^{n-1} \sigma_m^x \right) \sigma_n^-$$

$$\{c_n^\dagger, c_m\} = \delta_{nm}, \qquad \{c_n, c_m\} = \{c_n^\dagger, c_m^\dagger\} = 0$$

- Flip σ^x is parity symmetry of the gauge theory $Q \leftrightarrow \widetilde{Q}^\dagger$.

Diagonalizing the Fermion System

Key fact: Hamiltonian quadratic in $c, c^\dagger$,

$$
\begin{aligned}
H &= \sum_{n=0}^{L-1}(c_n^\dagger + c_n)(c_{n+1}^\dagger - c_{n+1}) - h(2c_n^\dagger c_n - 1) \\
&= \sum_k \left[-(2\cos k + 2h)c_k^\dagger c_k - i\sin k(c_{-k}^\dagger c_k^\dagger + c_{-k}c_k)\right] + Lh \\
&= \sum_k \epsilon(k)\left(b_k^\dagger b_k - \frac{1}{2}\right) \qquad \text{(Bogoliubov)}
\end{aligned}
$$

$\Longrightarrow$ A system of free fermions, with dispersion relation

$$
\epsilon(k) = 2\sqrt{h^2 + 1 + 2h\cos k} \qquad \left(h = \frac{N_f}{N_c} = 3 + O(\lambda)\right)
$$

Quasimomenta Quantization

For closed chains, boundary condition is twisted by $-(-1)^P$:

$$P = \prod_{n=0}^{L-1} \sigma_n^x = \begin{cases} -1: & k = \frac{2\pi m}{L} \\ +1: & k = \frac{(2m+1)\pi}{L} \end{cases}$$

We can also think of operators with open flavor indices as states in an open chain with "Dirichlet" boundary conditions,

$$Q^\dagger XY \ldots YQ \quad \Longrightarrow \quad |\uparrow \uparrow\downarrow \ldots \downarrow \uparrow\rangle$$

This system is still solvable [Douçot, Feigel'man, Ioffe, Ioselevich, cond-mat/0403712]. Same H as above, with an interesting quantization condition,

$$\frac{\sin(k(L+2))}{\sin(k(L+1))} = -h \qquad \text{(open chains).}$$

Example: Closed 4-field Operators

4-field flavor singlet operators: $\mathrm{Tr}(X^2)$, $\mathrm{Tr}(Y^2)$, $\mathrm{Tr}(XY)$.

- Odd parity

$$b_\pi^\dagger|0\rangle \qquad \underbrace{b_0^\dagger|0\rangle}_{\sum k=0} \longleftrightarrow \mathrm{Tr}(X^2-Y^2)$$

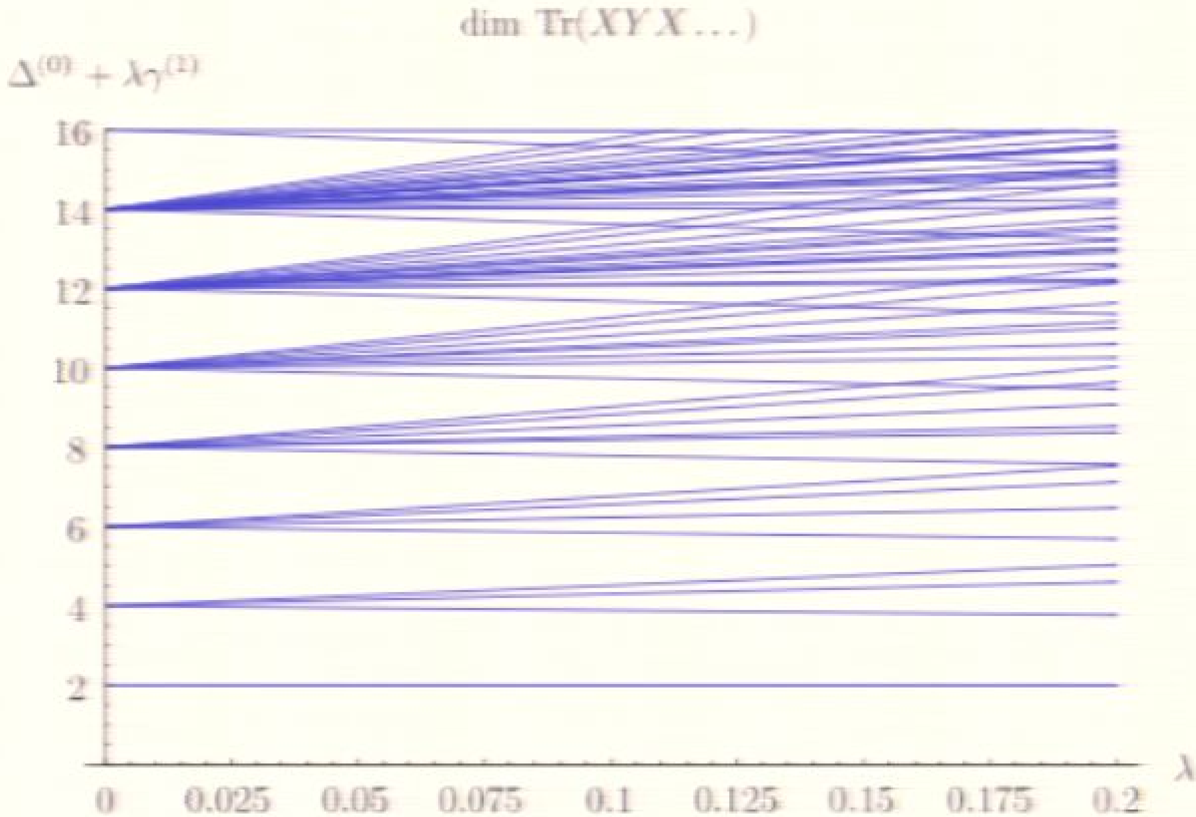
$$\gamma = \left(2+\frac{1}{2}\sqrt{10+6\cos 0}-\frac{1}{2}\sqrt{10+6\cos\pi}\right)\lambda = 3\lambda$$

- Even parity

$$\underbrace{|0\rangle, \quad b_{\frac{\pi}{2}}^\dagger b_{\frac{3\pi}{2}}^\dagger|0\rangle}_{\sum k=0} \qquad \longleftrightarrow \qquad \mathrm{Tr}(X^2+Y^2), \quad \mathrm{Tr}(XY)$$

$$\gamma = \left(2\pm\frac{1}{2}\sqrt{10+6\cos\frac{\pi}{2}}\pm\frac{1}{2}\sqrt{10+6\cos\frac{3\pi}{2}}\right)\lambda = (2\pm\sqrt{10})\lambda$$

Asymptotics



$$\gamma_\pm \;=\; \lambda L \pm \frac{\lambda L}{4\pi}\int_0^{2\pi} dk\sqrt{10+6\cos k} \qquad \text{(Large } L\text{)}$$

$$\approx\; (1 \pm 1.54)\lambda L$$

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Outlook

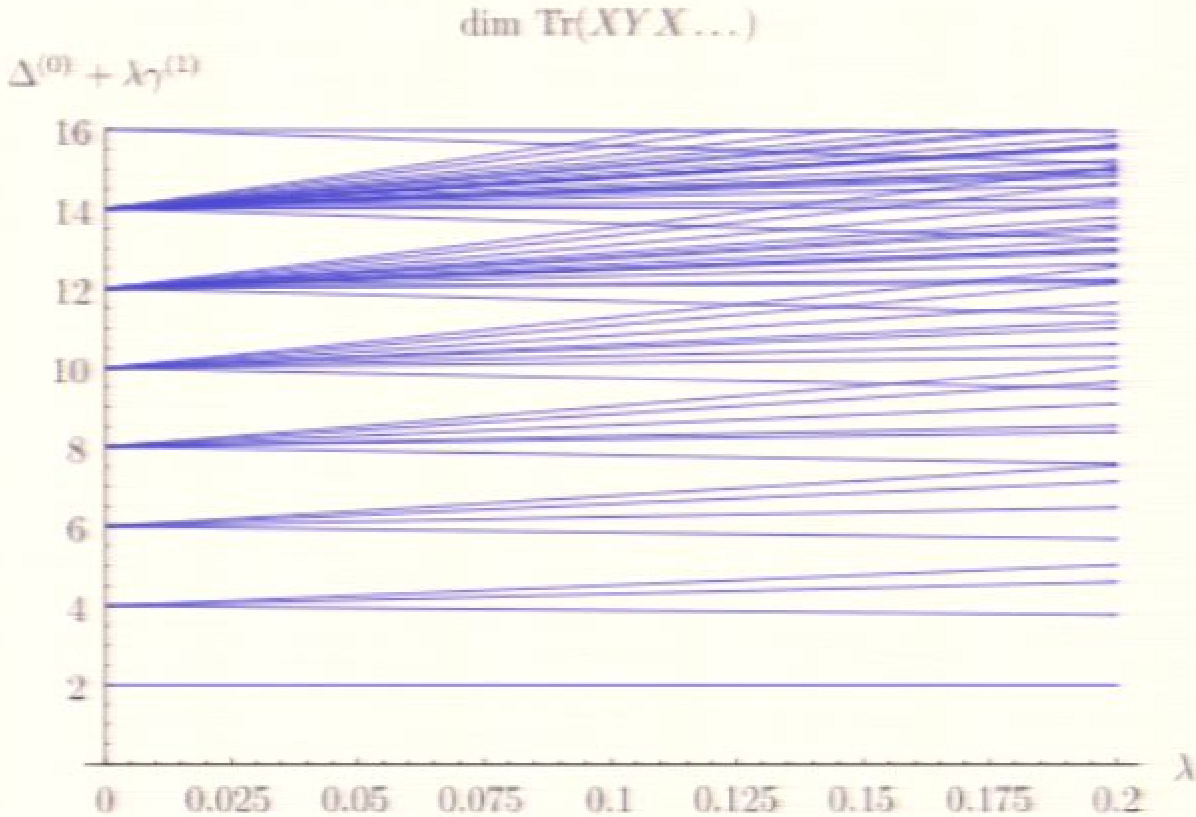
New features (cf. $\mathcal{N}=4$ SYM)

- $|0\rangle$ is not BPS.
- fundamental excitations are non-local "dimer solitons".

Possible directions

- Other operators at 1-loop? At 2-loops, mixing with fermions+gauge fields, but still nearest neighbor in X, Y. Nontrivial S-matrix for $b^\dagger, b$?
- Magnetic dual? Building blocks aren't so simple: $q(MM^\dagger)^k q^\dagger$, $\tilde{q}^\dagger(M^\dagger M)^k \tilde{q}$, $q(MM^\dagger)^k M\tilde{q}$, $\tilde{q}^\dagger(M^\dagger M)^k M^\dagger q^\dagger$. Need a good way to organize DOF.
- Speculation: TFIM self-dual around $h=1$ (naively $\frac{N_f}{N_c}=1$). Could this be relevant in SQCD?

Asymptotics



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