Title: From Weak to Strong Coupling with GKP

Date: Aug 18, 2011 03:20 PM

URL: http://pirsa.org/11080054

Abstract:

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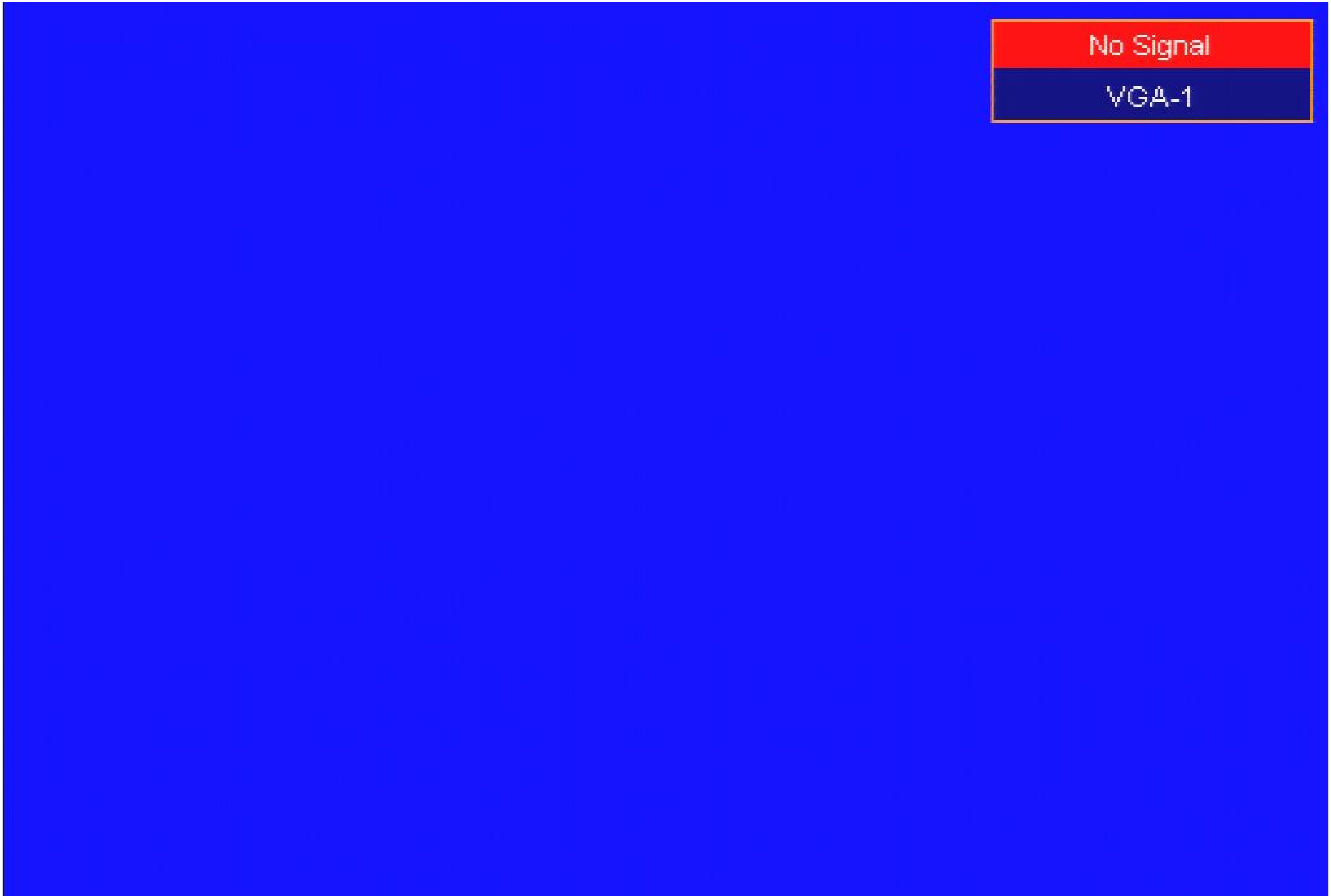
Benjamin Basso

Princeton Center for Theoretical Science
Princeton University

August 18, 2011
Perimeter Institute

arXiv:1010.5237
arXiv:1108.0999 with A. Belitsky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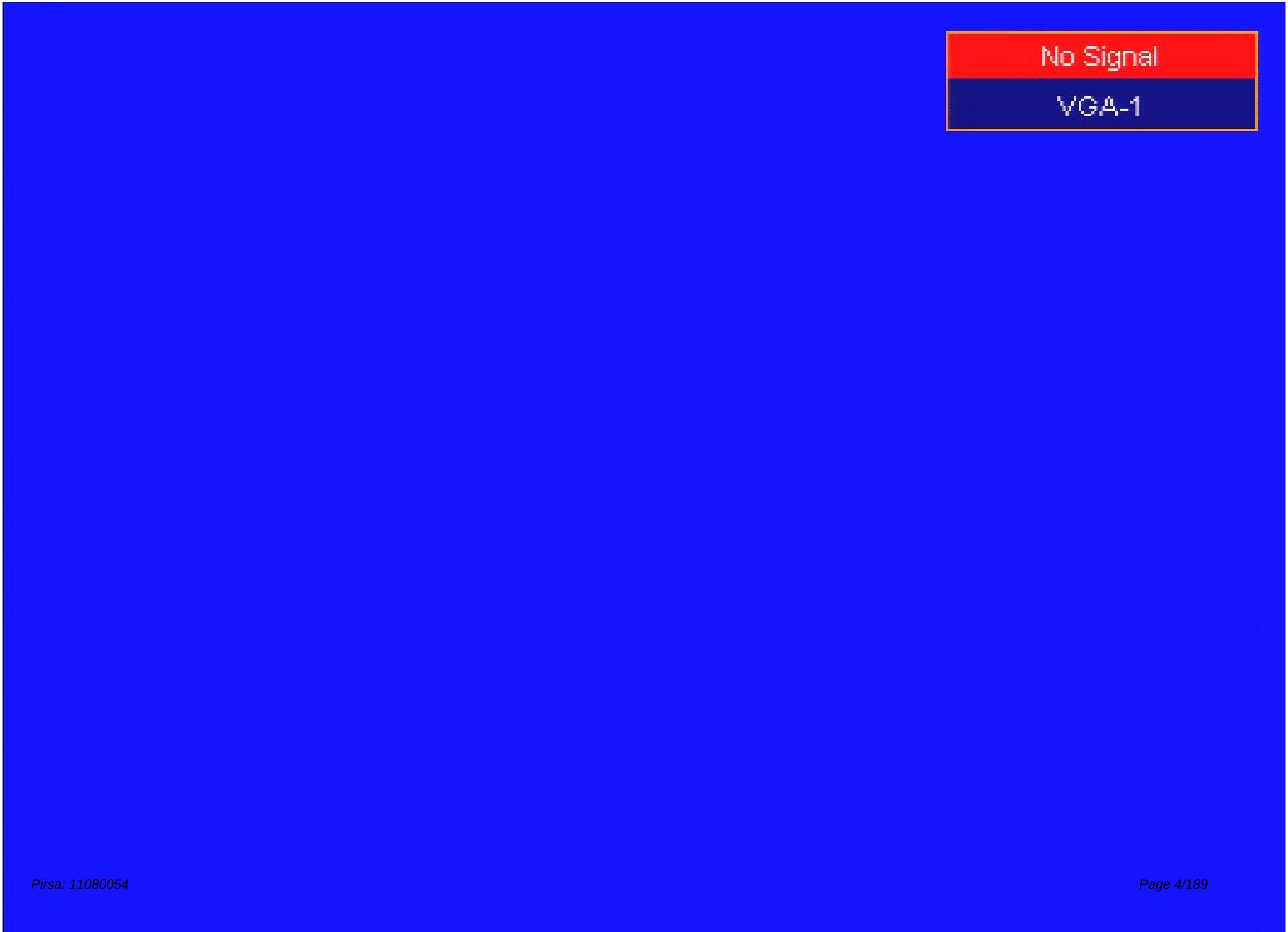
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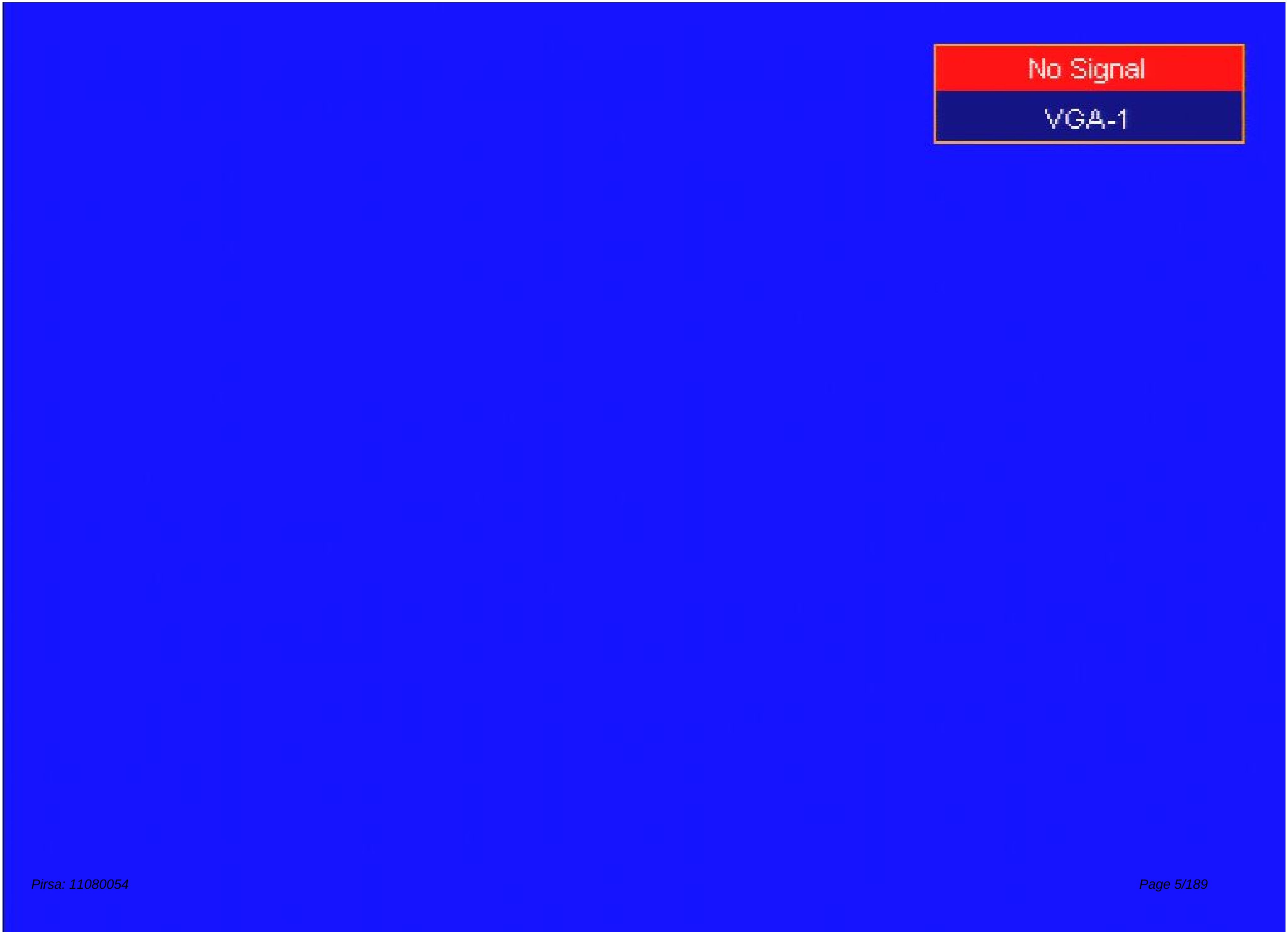
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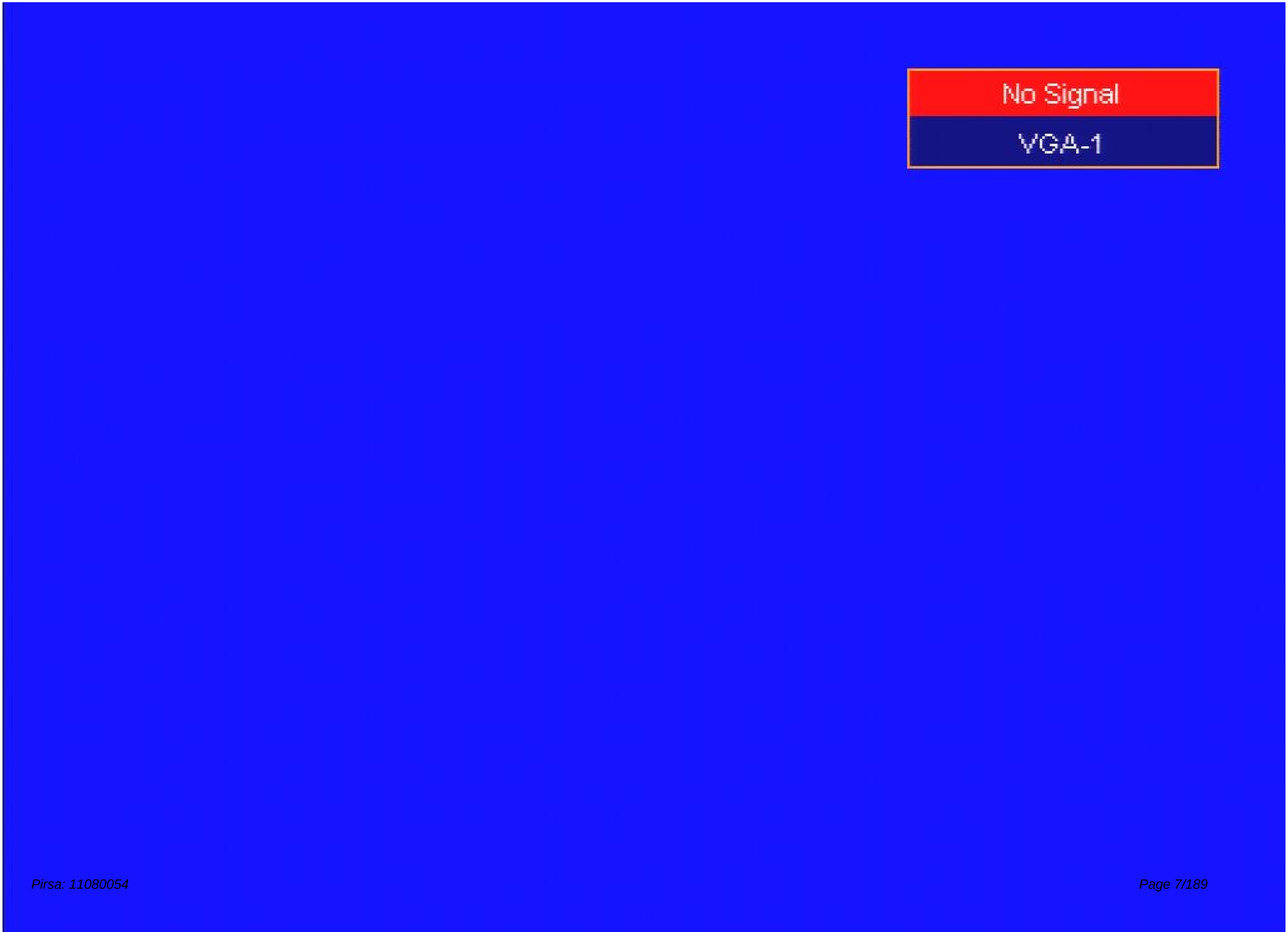
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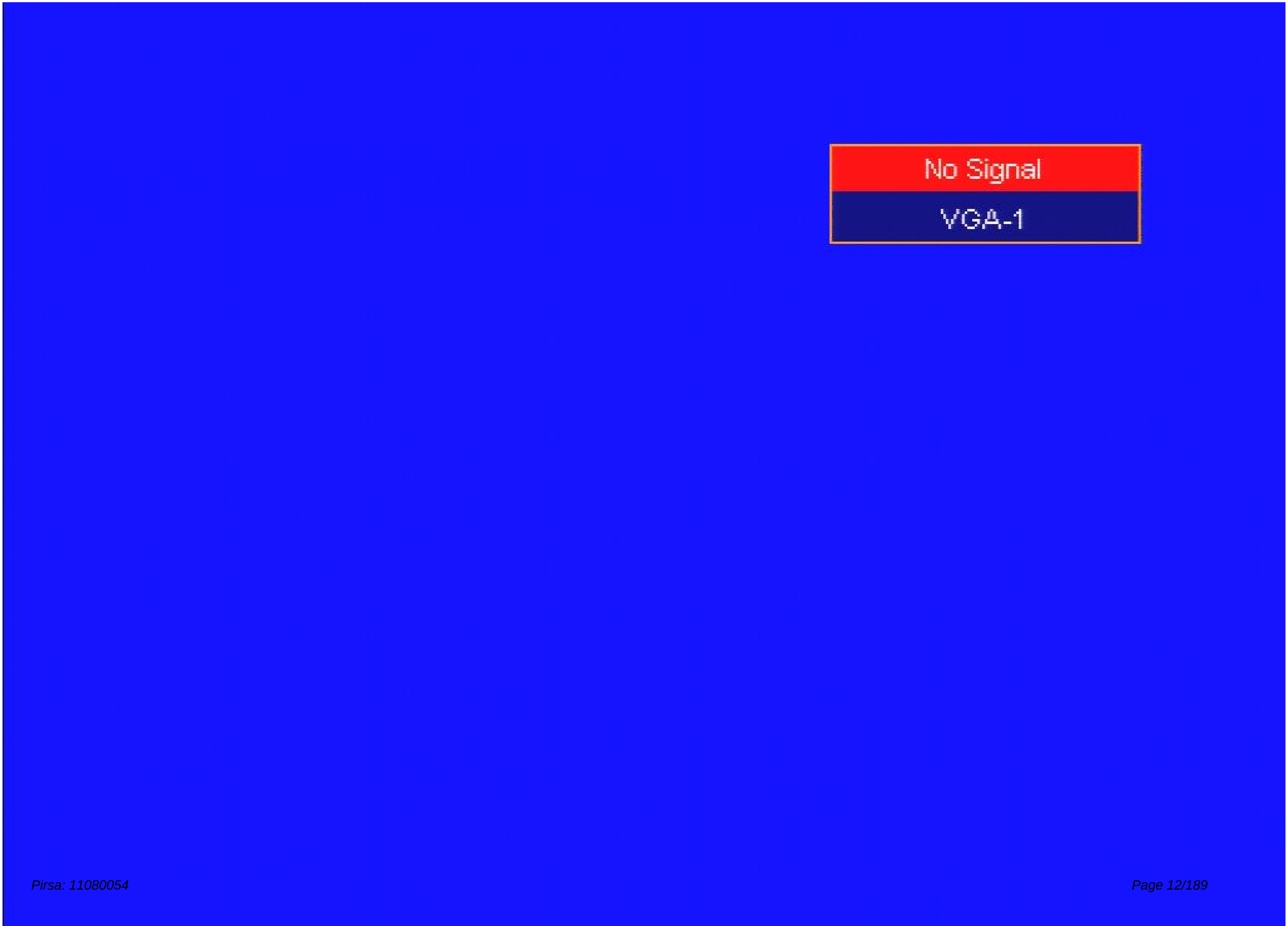
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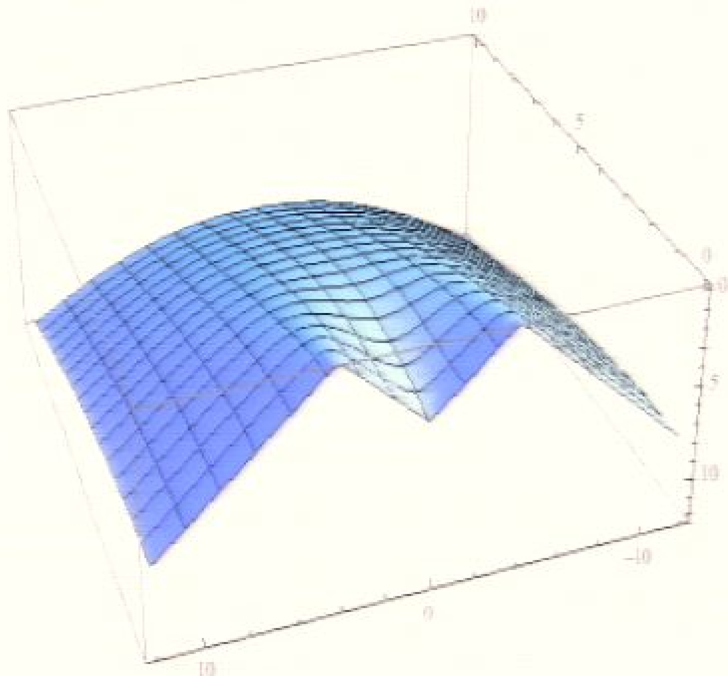
Four cusps solution = GKP string



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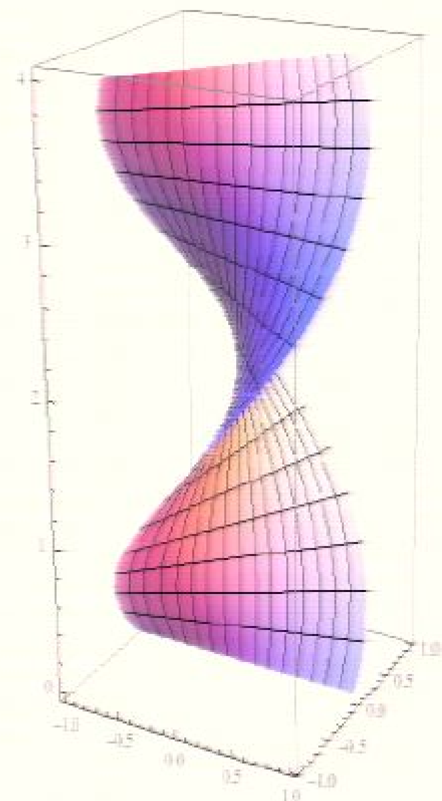

... a bridge to the spectral problem

Outline

- The string theory point of view: excitations around long rotating GKP string
- The gauge theory perspective: excitations around large spin twist-two operator
- Interpolation via Bethe ansatz equations.... all-loop dispersion relations
- Application to twist-two anomalous dimension
  - Mirror kinematics
  - Double Wick rotation symmetry
  - Lüscher formula

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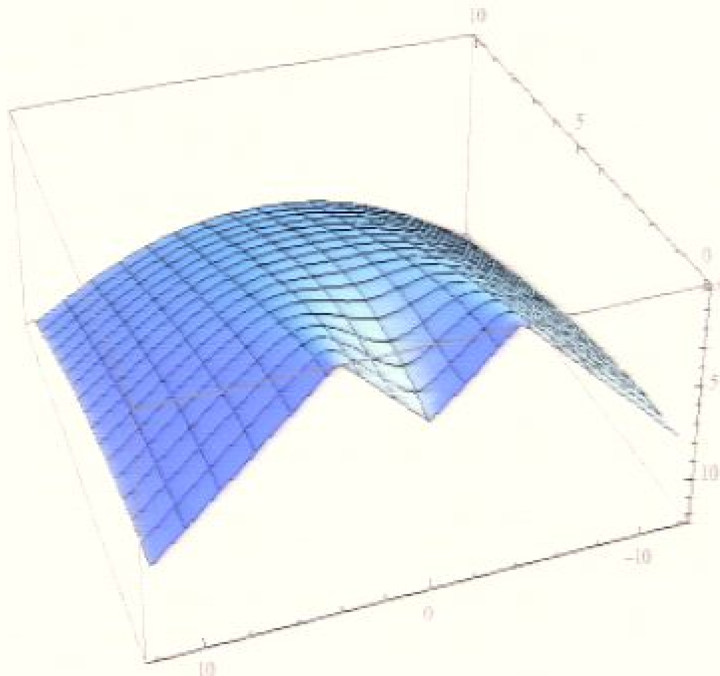

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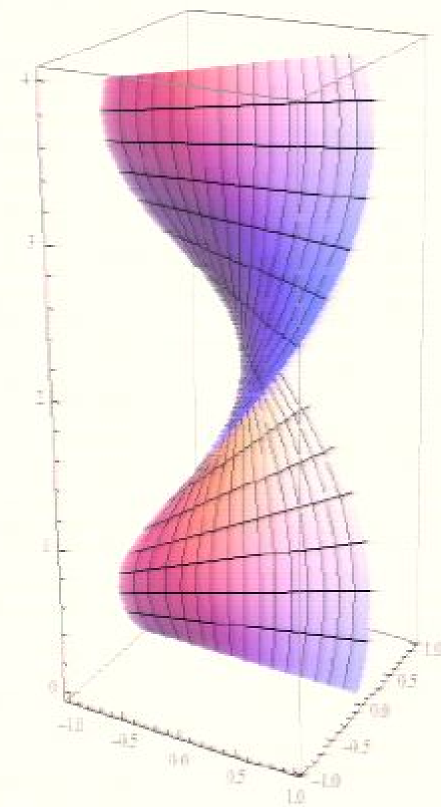

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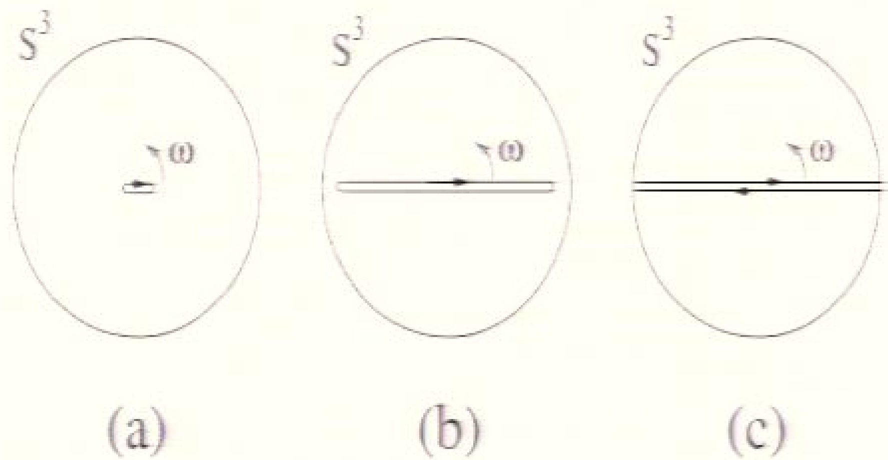
Folded string rotating in $AdS_3 \subset AdS_5$ with spin S [Gubser,Klebanov,Polyakov'02]



- (a) Short string : $S \sim 0$ — length $\sim S^{1/2} \sim 0$
- (c) Long string : $S \sim \infty$ — length $= 2 \log S \gg 1$ + worldsheet homogeneous

Energy of long GKP string : (string tension) $2g = \sqrt{\lambda}/2\pi \gg 1$

$$E \equiv \Delta - S = 4g \log S + O(\log^0 S) — 2\Gamma_{\text{cusp}}(g) \log S + \ldots$$

Spectrum of excitations from string theory I

Quadratic fluctuations (relativistic spectrum) [Frolov,Tseytlin'02]

- 5 massless bosons for $\perp$ fluctuations in S^5
- 2 bosons with mass $\sqrt{2}$ for $\perp$ fluctuations in AdS_5/AdS_3
- 1 boson with mass 2 for $\perp$ fluctuation in AdS_3
- 8 fermions with mass 1

Symmetries [Alday,Maldacena'07]

- All supersymmetries are broken (complicated background)
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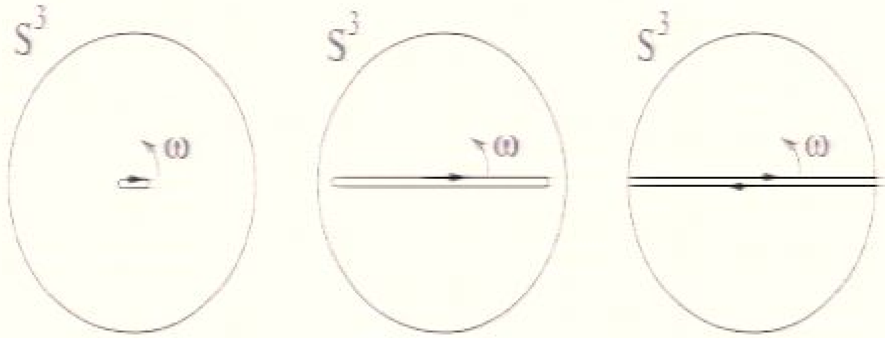

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Higher-loop correction?... recently: one-loop correction [Giombi,Ricci,Roiban,Tseytlin'10]

$$E(p) = \sqrt{p^2 + m(g)^2}\left[1 - c\,\frac{p^3}{g} + O(1/g^2)\right]$$

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- Spectrum undisturbed otherwise.... though evidence for possible bound states [Zarembo (unpublished)]

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... identified with the help of the finite gap method

... related to spiky strings [Kruczenski'04]

Asymptotics: universal behavior at large momentum

$$E \sim 2g\log p \longrightarrow \Gamma_{\rm cusp}(g)\log p$$

Nota bene: scaling $E \sim \log p$ similar to the one observed for magnon dispersion relation in mirror kinematics

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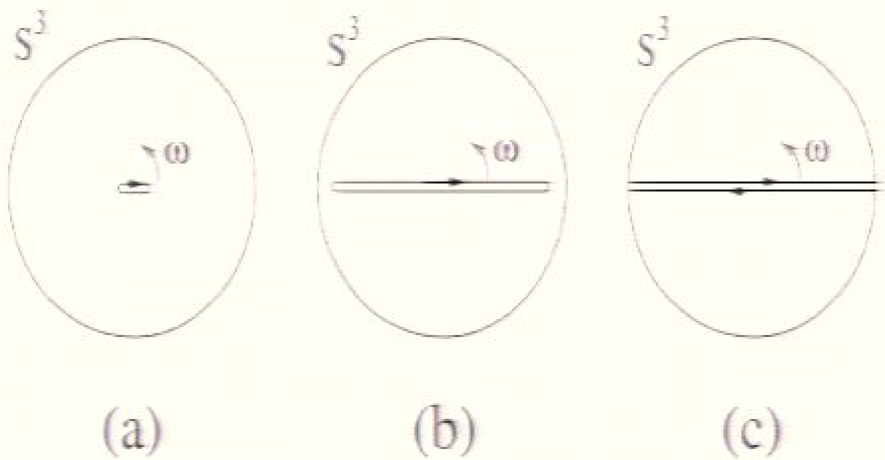

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Gauge theory picture

Vacuum

Long GKP string = large spin twist-two operator $\sim \mathrm{tr}\, ZD_+^S Z$

Vacuum energy: $E_{\mathrm{vacuum}} = \Delta - S = 2\Gamma_{\mathrm{cusp}}(g) \log S + \dots$

Excitation

Insertion of operators in the background of covariant derivatives

One-particle state = length-three operator

$$\mathrm{tr}\, ZD_+^{k_1} \Phi D_+^{k_2} Z$$

$S \sim k_1 + k_1 \gg 1$

Gauge theory spectrum

Fundamental (lightest) excitations

Analogy with BMN operators

— fundamental excitations : twist $= (\Delta - S)_{\text{tree-level}} = 1$ operators

Energy: $\Delta - S = E_{\text{vacuum}} + E_{\Phi}(p)$

Mass: $E_{\Phi}(p=0) =$ twist $+O(g^2)$

Flavors

- 6 scalars Z
- 8 fermions Ψ
- 2 field strength components $F = F_{+\perp} \sim \partial_+ A_\perp$ — gauge field excitation

Bound states...

... of gauge fields: embedded as length = 1, high-twist operators

$$D_\perp^{\ell-1} F_{+\perp} \qquad \text{twist} = \ell$$

with $\ell = 1, 2, 3, \ldots$

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Spectrum of masses

Masses	Weak coupling	Strong coupling
Scalar	1	$\sim e^{-\pi g}$
Fermion	1	1
Gauge field	1	$\sqrt{2}$
?	2	2

$$\text{`?'} \quad = \quad D_{-}.F_{+-}.\ \text{or two-fermion state ?}$$

Interpolation?

Spectral problem

Large spin operators with $L-2$ insertions:

$$\mathcal{O} = \operatorname{tr} Z D_+^{k_1} \Phi \dots D_+^{k_{L-2}} \Phi D_+^{k_{L-1}} Z$$

Elementary excitations: (twist-1 partons)

$$\Phi \;=\; Z \text{ (scalars)}, \quad \Psi \text{ (fermions)}, \quad F_{+\perp} \text{ (gauge fields)}$$

Mixing problem — Spectrum of scaling dimensions at large spin $S \sim \sum_i k_i \gg 1$?

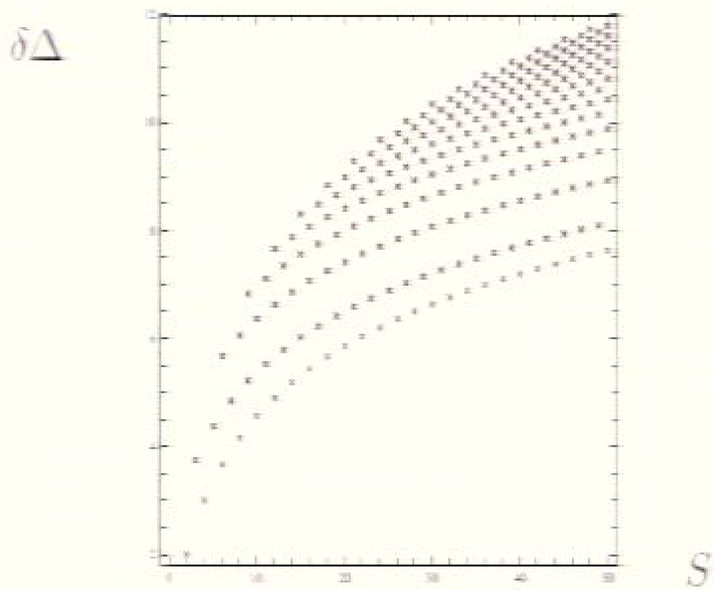
Solution:

$$E_{\text{eigenstate}} = \Delta - S = E_{\text{twist-two}} + \sum_{i \in \text{excitations}} E_i(p_i) + \dots$$

Spectrum of high-spin operators

Illustration:

One-loop spectrum of anomalous dimensions of twist-3 operators $\mathcal{O} = \mathrm{tr}\, Z D_+^{k_1} Z D_+^{k_2} Z$



Goal: parameterize anomalous dimensions trajectories close to the minimal one...

$$\delta\Delta - \delta\Delta_{\text{twist-two}} = E(p) \qquad p \sim 1/\log S$$

and extract the dispersion relation (here for a scalar Z)

Tool : Integrability

Kinematics

Operators

$$\mathcal{O}_{\{k_m\}} = \mathrm{tr}\, D_+^{k_1} Z ... D_+^{k_L} Z$$

- $\mathrm{tr}\, Z...Z...Z$ — vacuum state of the spin chain
- $\mathrm{tr}\, Z...D_+Z...Z$ — one-particle state of the spin chain (magnon)

Quantum numbers

- Twist L — spin chain length
- Lorentz spin $S = k_1 + ... + k_L$ — number of excitations (magnons) over the vacuum

Dynamics

Callan-Symanzik equation

$$\mu \frac{\partial}{\partial \mu} \mathcal{O}_{\{k_m\}} = -\delta D \cdot \mathcal{O}_{\{k_m\}}$$

- δD — Hamiltonian of the spin chain
- Spectrum of anomalous dimensions $\delta\Delta$ — spectrum of energies of the spin chain

One-loop example

Mapping with $\mathfrak{sl}(2)$ integrable Heisenberg spin chains

[Lipatov'97],[Braun,Belitsky,Derkachov,Korchemsky,Manashov'98]
[Minahan,Zarembo'02],[Beisert,Staudacher'03]

Kinematics : spin-chain Hilbert space $\mathcal{H} = V_s^{\otimes L}$

- scalar : conformal spin $s = 1/2$
- fermion : conformal spin $s = 1$
- gauge fields : conformal spin $s = 3/2$

Dynamics : δD = Hamiltonian of XXX_s $\mathfrak{sl}(2)$ Heisenberg spin chain

- System with L degrees of freedom... and L commuting conserved charges

  Liouville definition of a completely integrable system

- The complete family of conserved charges can be diagonalized simultaneously with δD by means of the algebraic Bethe ansatz

Bethe ansatz solution

Solution to mixing problem (here for scalar)

- Bethe ansatz equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

- S magnons — S rapidities u_k
- One-loop anomalous dimension

$$\delta\Delta = 2g^2 \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4} + O(g^4)$$

Large spin limit $S \gg 1$

Continuum limit of the Bethe ansatz equations

[Korchemsky'95],[Belitsky,Gorsky,Korchemsky'06]
[Freyhult,Rej,Staudacher'07]

- Continuous distribution of Bethe roots described by density $\rho(u)$
- Bethe ansatz equations turn into integral equation for $\rho(u)$

$$2\pi\rho(u) = \frac{4L}{1+4u^2} - \sum_{l=1}^{L-2} \frac{2}{(u-\hat{u}_l)^2+1} + 2\int dv\, \frac{\rho(v)}{(u-v)^2+1}$$

- What are the $\hat{u}_l$'s?... they are the holes' rapidities

Particle/hole transformation : trading dynamics of magnons D_+ on spin chain of Z's for dynamics of Z's through the background of D_+'s

One hole — One Z insertion

- Solve integral equation for given set of holes rapidities and compute energy

— vacuum energy + energy of excitation carrying hole rapidity

Bethe ansatz solution

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$$\delta\Delta = 2g^2 \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4} + O(g^4)$$

Large spin limit $S \gg 1$

Continuum limit of the Bethe ansatz equations

[Korchemsky'95],[Belitsky,Gorsky,Korchemsky'06]
[Freyhult,Rej,Staudacher'07]

- Continuous distribution of Bethe roots described by density $\rho(u)$
- Bethe ansatz equations turn into integral equation for $\rho(u)$

$$2\pi\rho(u) = \frac{4L}{1+4u^2} - \sum_{l=1}^{L-2} \frac{2}{(u-\hat{u}_l)^2+1} + 2\int dv\, \frac{\rho(v)}{(u-v)^2+1}$$

- What are the $\hat{u}_l$'s?... they are the holes' rapidities

Particle/hole transformation : trading dynamics of magnons D_+ on spin chain of Z's for dynamics of Z's through the background of D_+'s

One hole — One Z insertion

- Solve integral equation for given set of holes rapidities and compute energy

— vacuum energy + energy of excitation carrying hole rapidity

Bethe ansatz solution

Solution to mixing problem (here for scalar)

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$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

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One-loop example

Mapping with $\mathfrak{sl}(2)$ integrable Heisenberg spin chains

[Lipatov'97],[Braun,Belitsky,Derkachov,Korchemsky,Manashov'98]
[Minahan,Zarembo'02],[Beisert,Staudacher'03]

Kinematics : spin-chain Hilbert space $\mathcal{H} = V_s^{\otimes L}$

- scalar : conformal spin $s = 1/2$
- fermion : conformal spin $s = 1$
- gauge fields : conformal spin $s = 3/2$

Dynamics : δD = Hamiltonian of XXX_s $\mathfrak{sl}(2)$ Heisenberg spin chain

- System with L degrees of freedom... and L commuting conserved charges

  Liouville definition of a completely integrable system

- The complete family of conserved charges can be diagonalized simultaneously with δD by means of the algebraic Bethe ansatz

Large spin limit $S \gg 1$

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Changing flavor: one-loop case

Equations for $\mathfrak{sl}(2)$ spin chain with conformal spin s

$$\left(\frac{u_k + is}{u_k - is}\right)^L = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

Anomalous dimensions :

$$\delta\Delta = 2g^2 \sum_{j=1}^{S} \frac{2s}{u_j^2 + s^2} + 4g^2 L(\psi(2s) - \psi(1))$$

Excitation around large spin twist-two anomalous dimension

Large spin limit = continuum distribution of roots u_k

Excitations = holes in the distribution of roots

How many of them? $L - 2$ = number of parton insertions 'on top of' twist-two operator

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Vacuum energy: (twist-two scaling dimension) [Korchemsky'89],[Korchemsky,Marchesini'92]

$$E_{\text{vacuum}} = \Delta - S\big|_{\text{twist-two}} = 2\Gamma_{\text{cusp}}(g)\log S + O(\log^0 S)$$

with $\Gamma_{\text{cusp}}(g) = 4g^2 + O(g^4)$

Dispersion relation: (for a hole with rapidity u) [Korchemsky'95], [Belitsky,Gorsky,Korchemsky'06] [Freyhult,Rej,Staudacher'07]

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$$E(u) = \Delta - S\big|_{\text{above vacuum}} = 1 + 2g^2\left(\psi(s+iu) + \psi(s-iu) - 2\psi(1)\right) + O(g^4)$$

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Particular points

Masses: (i.e.. energy at zero momentum $p = 0$)

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Agreement with expected asymptotic behavior for highly-excited trajectory :

[Belitsky,Gorsky,Korchemsky '03'06]

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$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp\left(2i\theta(u_k, u_j)\right)$$

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First step: Classify large-spin solutions of ABA equations

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- bound states

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Inside the dashed square

twist-one excitations

Ouside the dashed square

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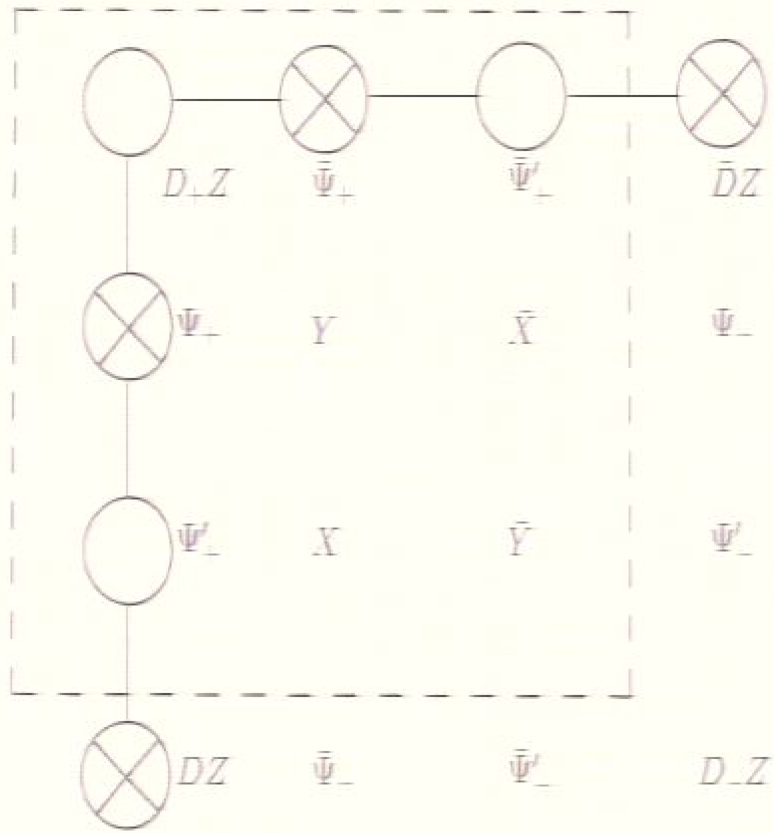

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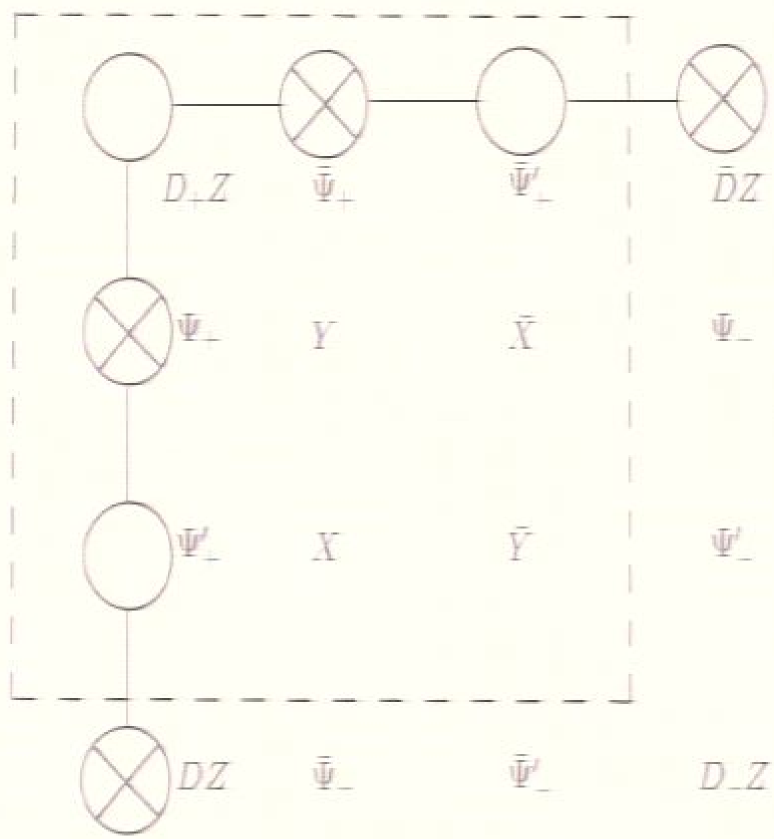

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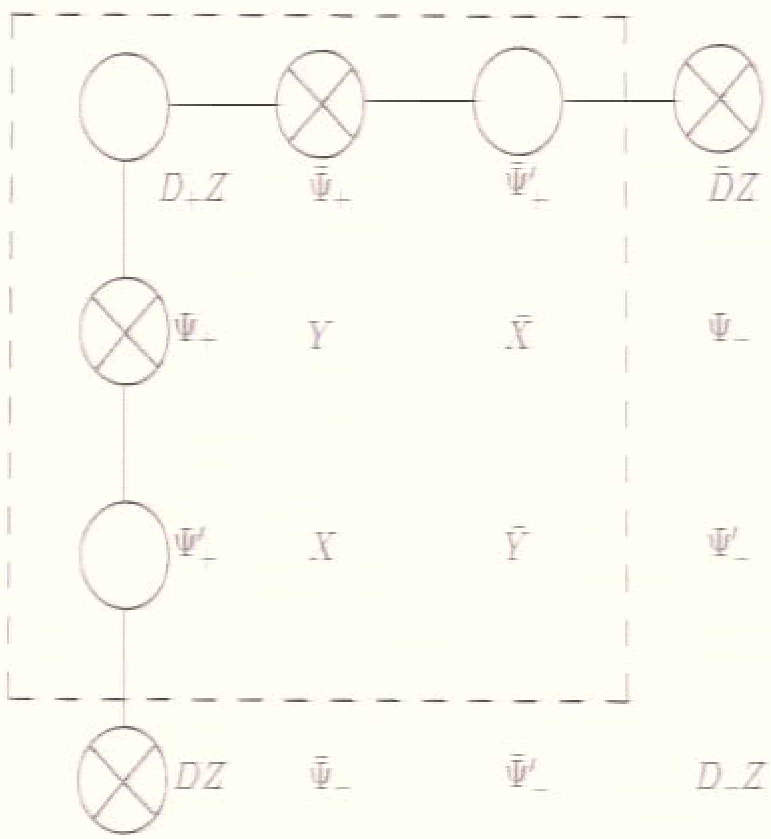

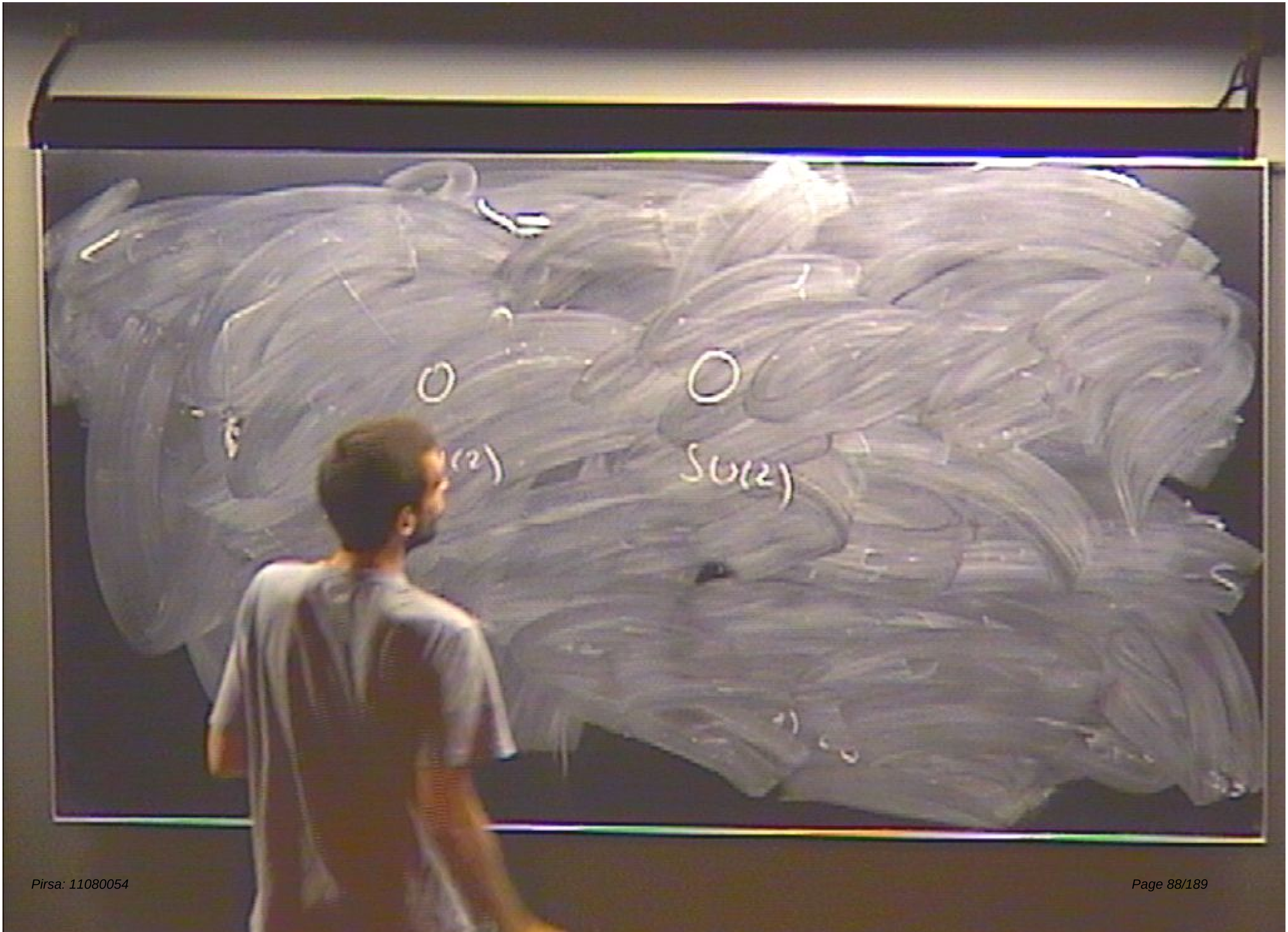
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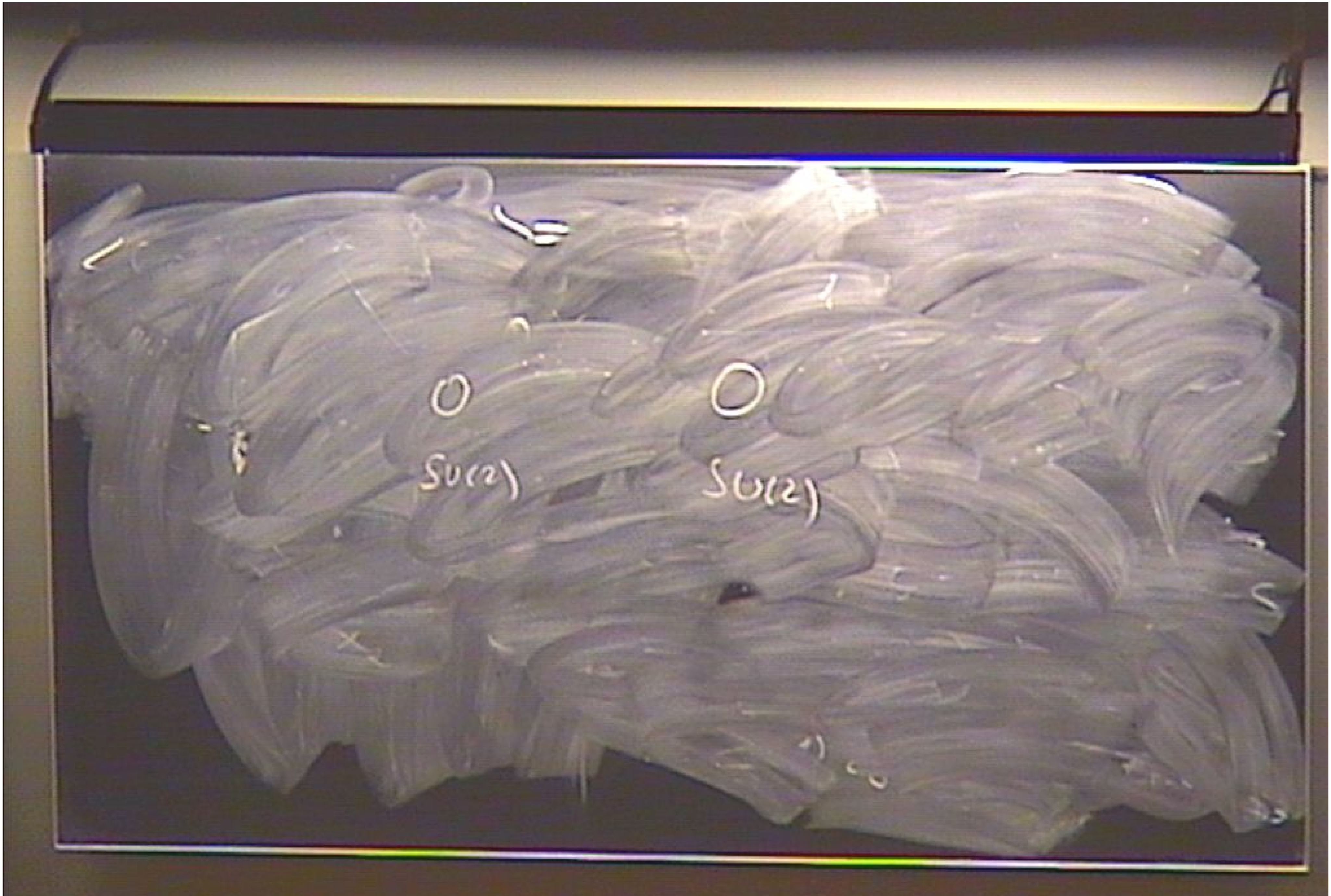
O
SU(2)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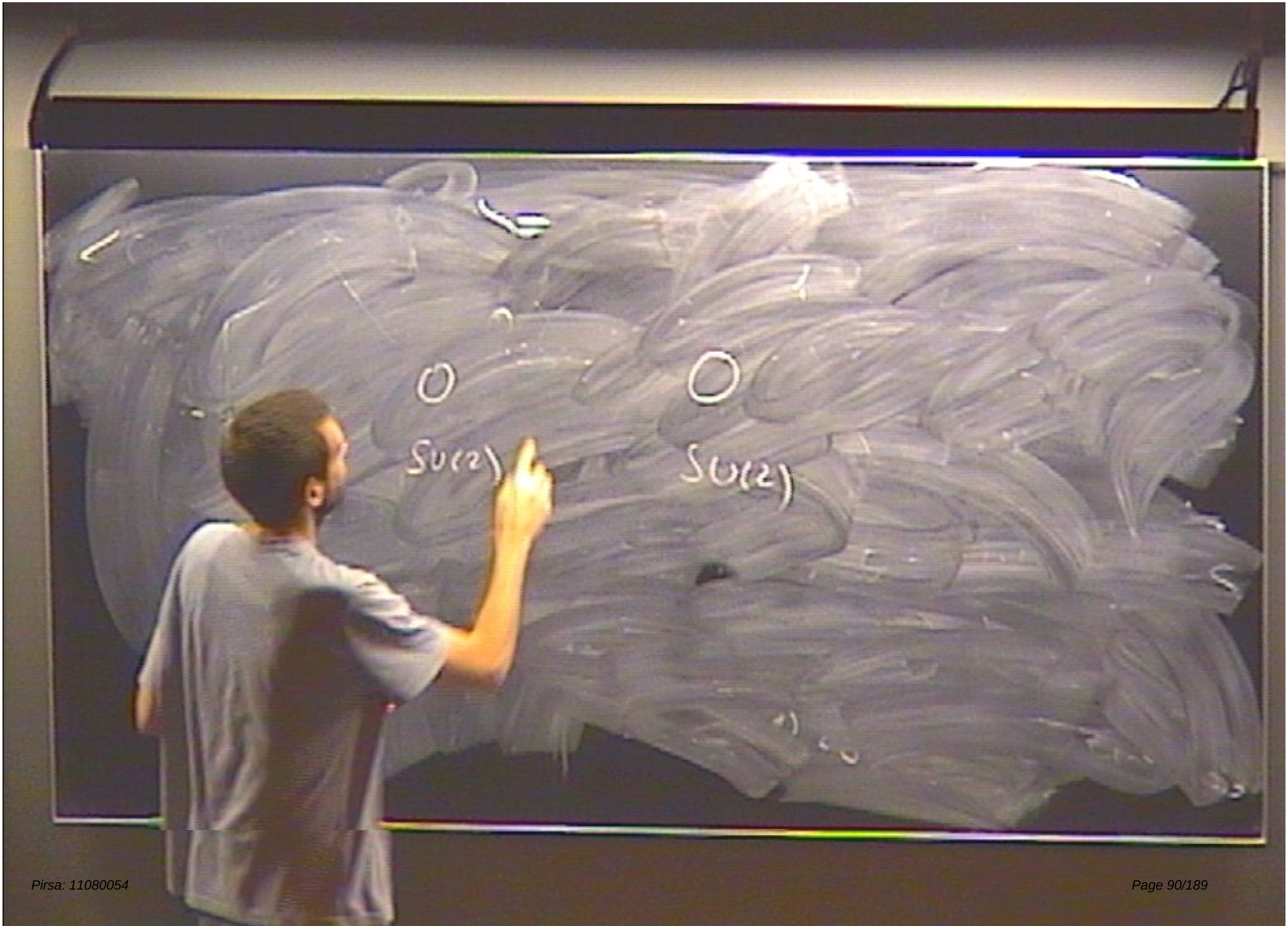
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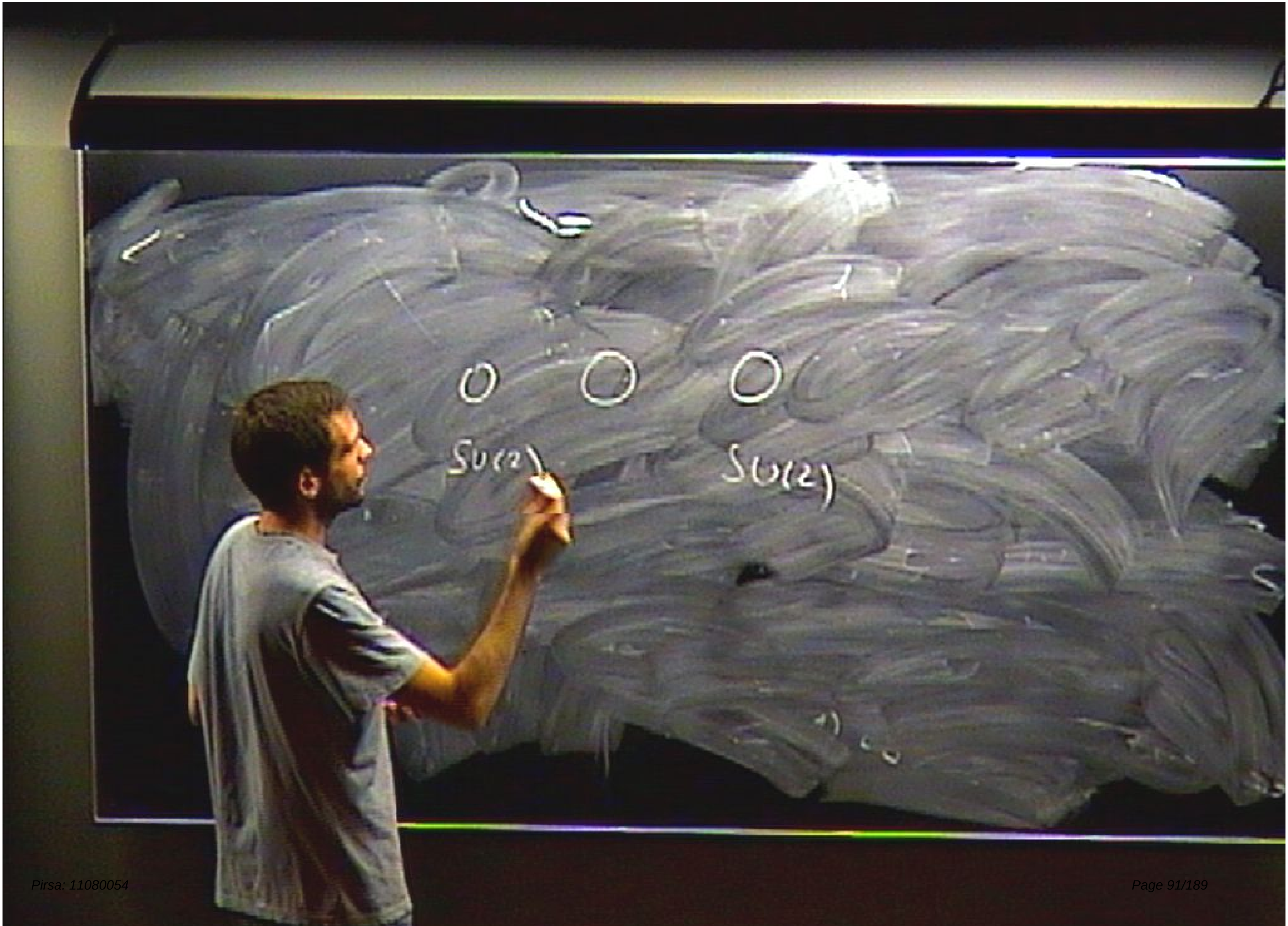
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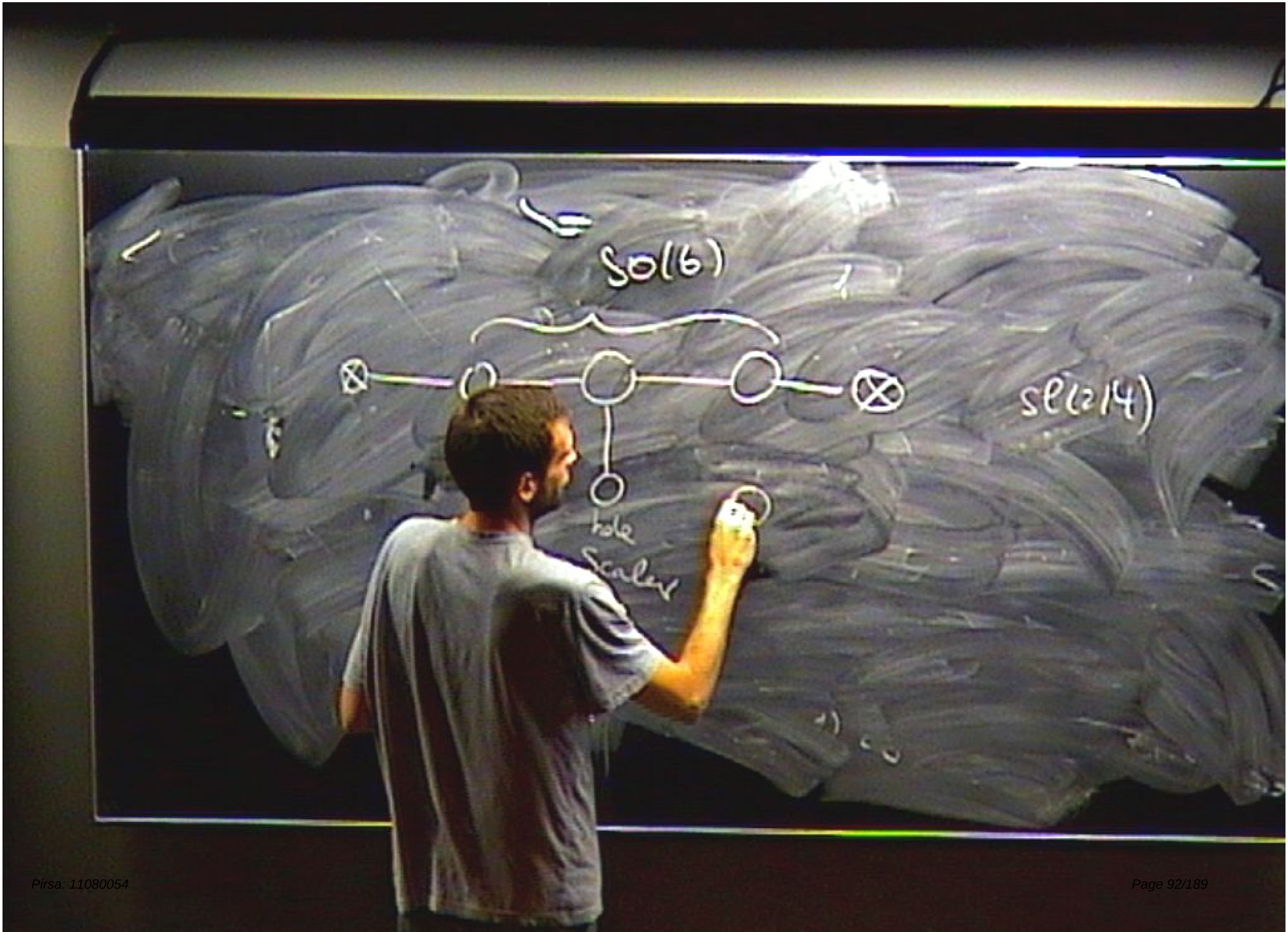
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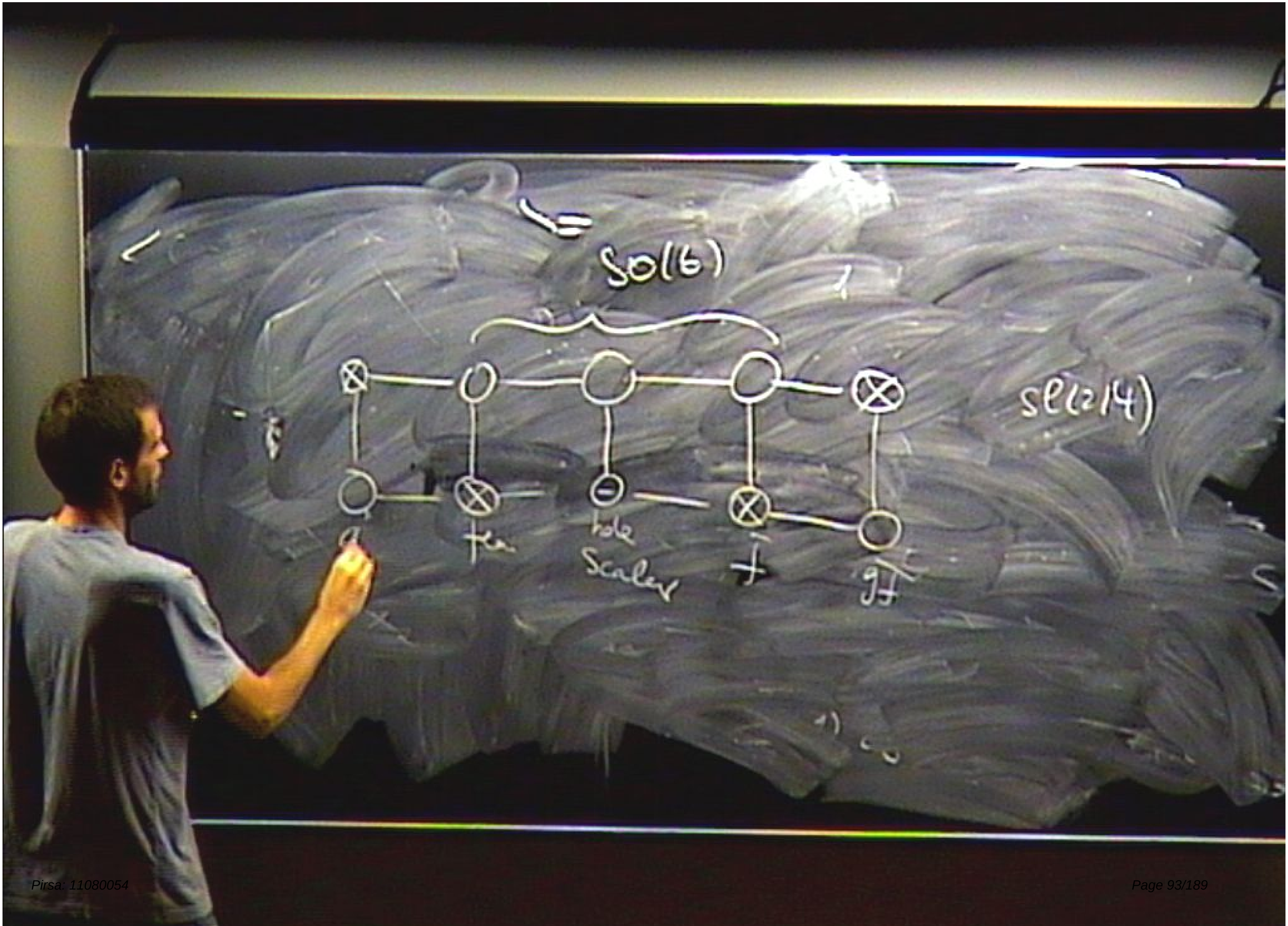
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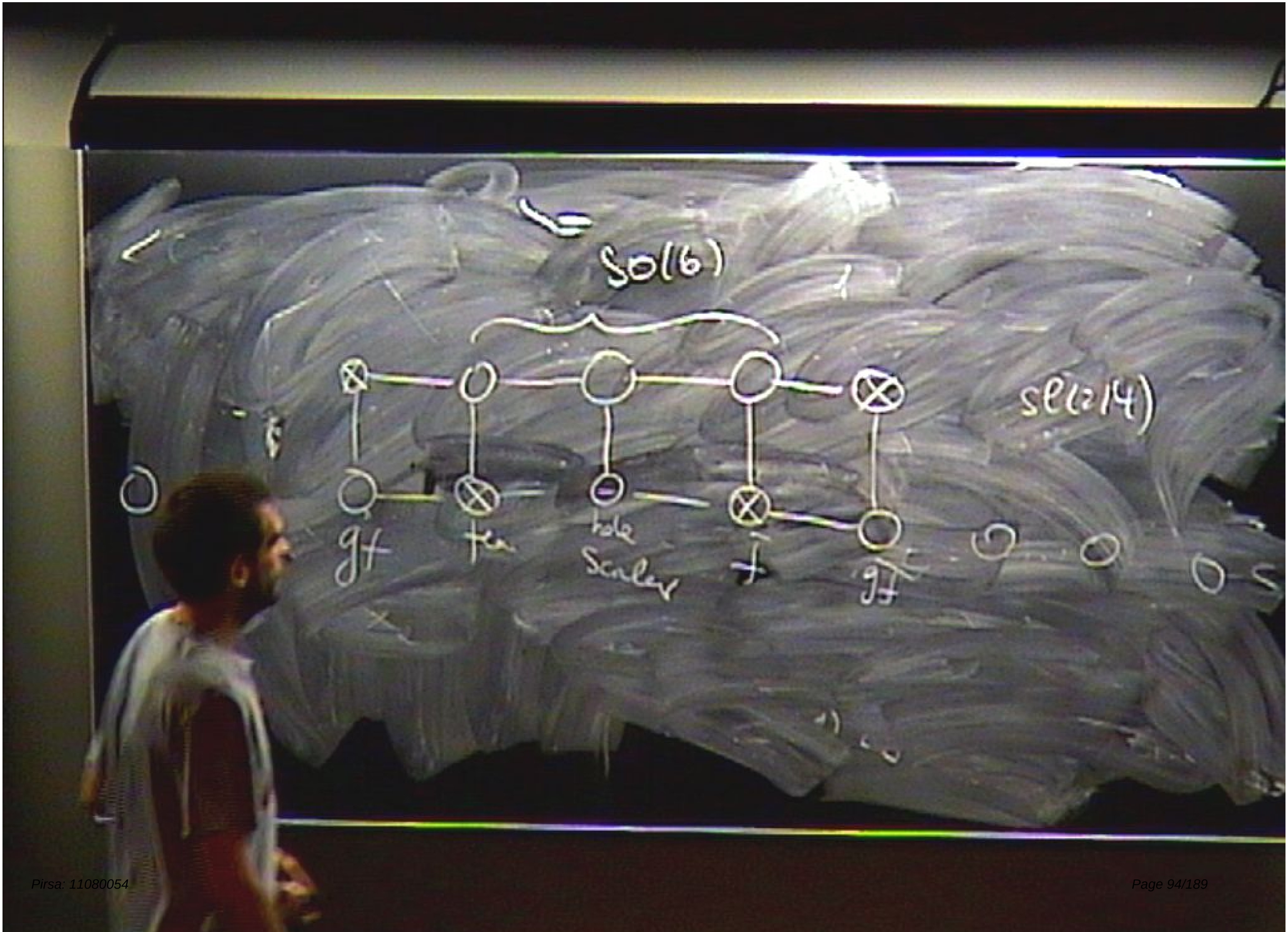
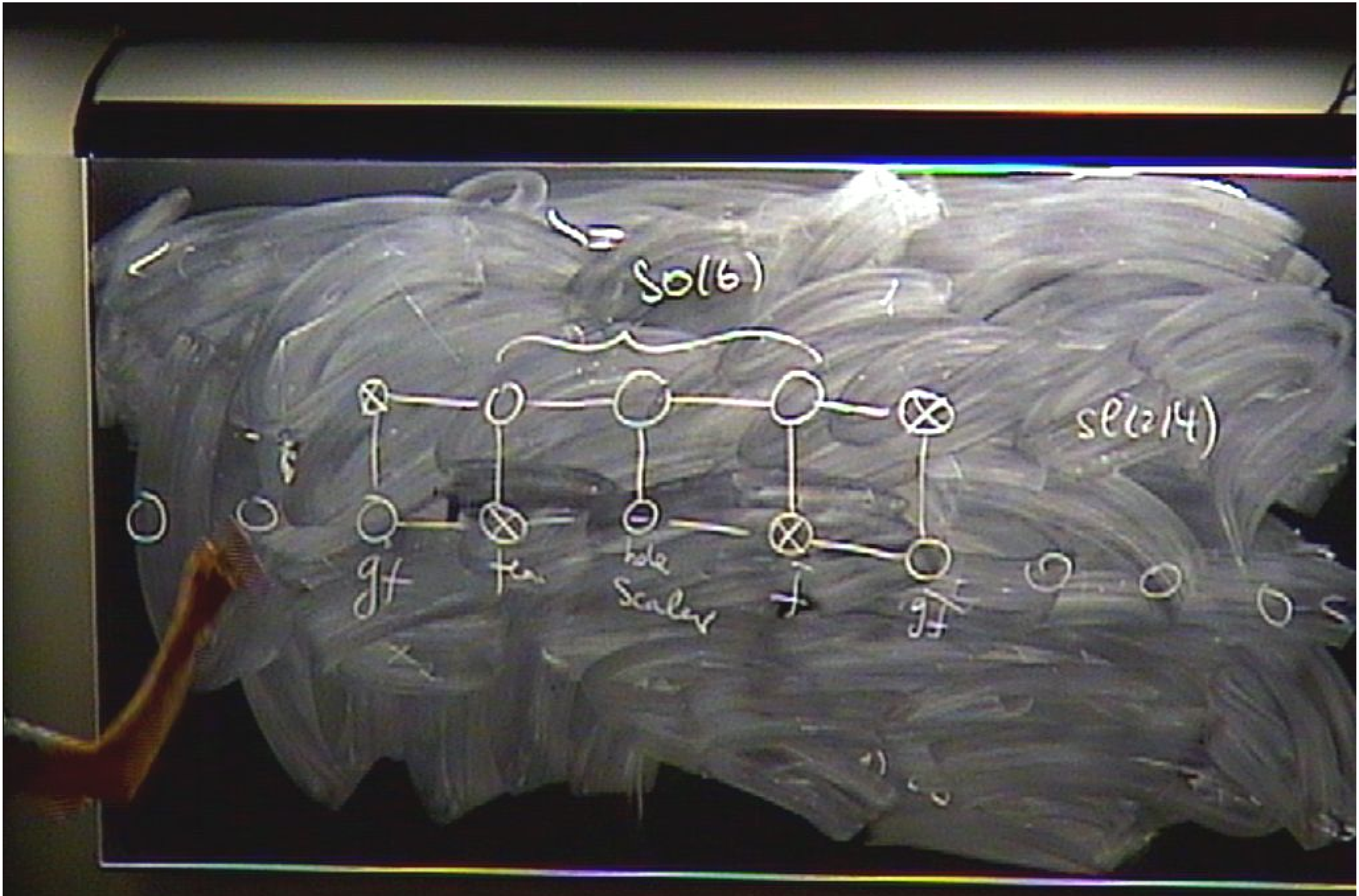


SO(6)
sl(2|4)
gf
fer
hole
Scalar
f̄
gf̄


SO(6)
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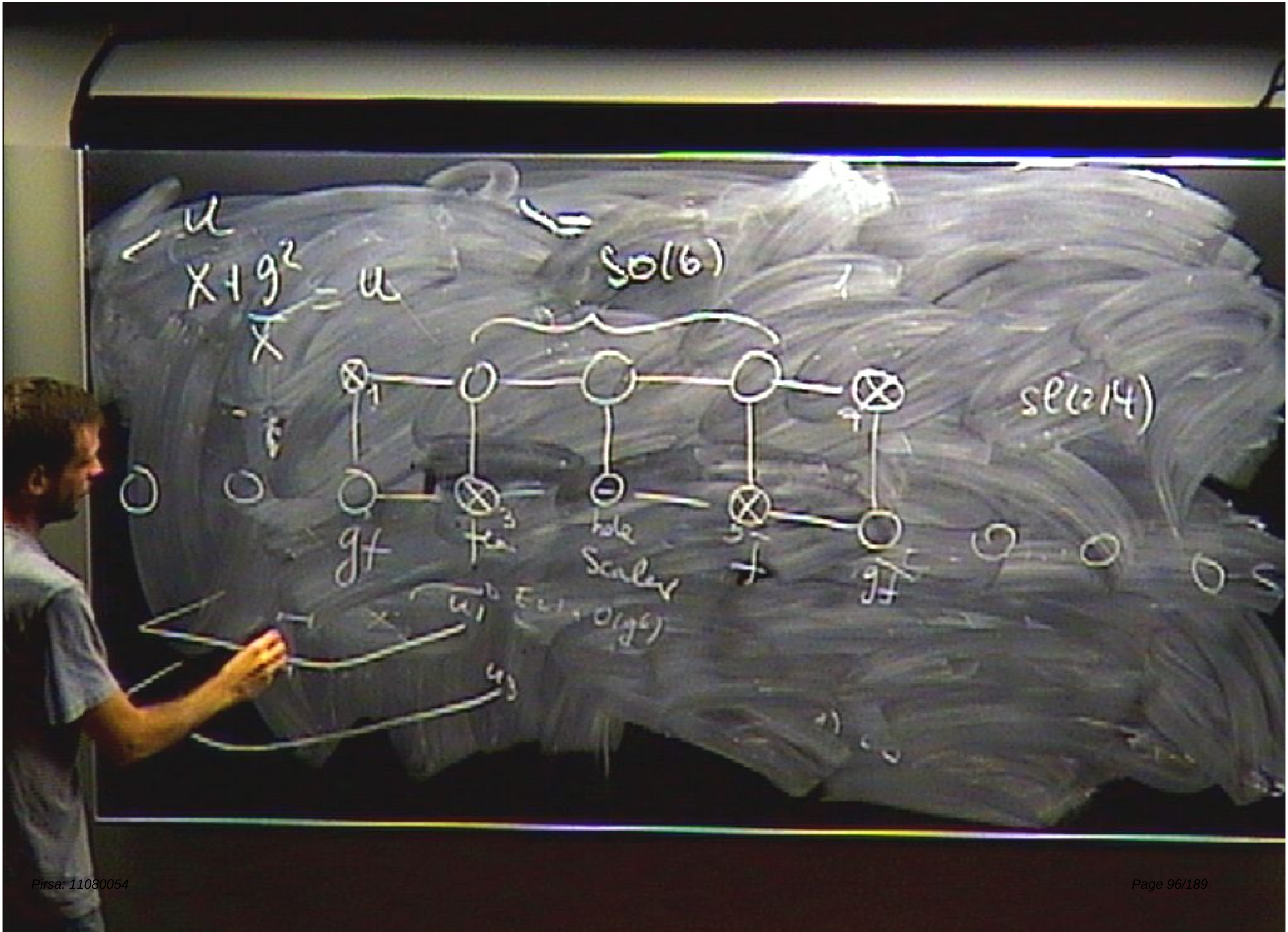


SO(6)
sl(2|4)

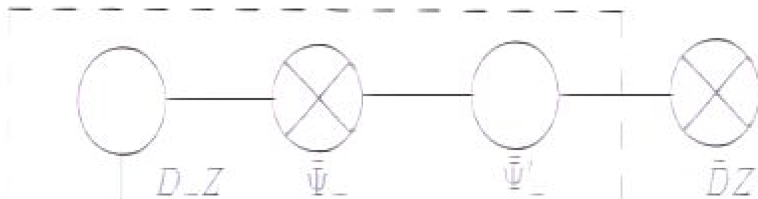
All-loop analysis

First step: Classify large-spin solutions of ABA equations

Nesting equations — table of BMN excitations

Inside the dashed square

twist-one excitations



Outside the dashed square

- bound states

Some help: existing results from spectroscopy of large-spin scaling dimensions

[Beisert,Bianchi,Morales,Samtleben'04],[Freyhult,Rej,Zieme'09]

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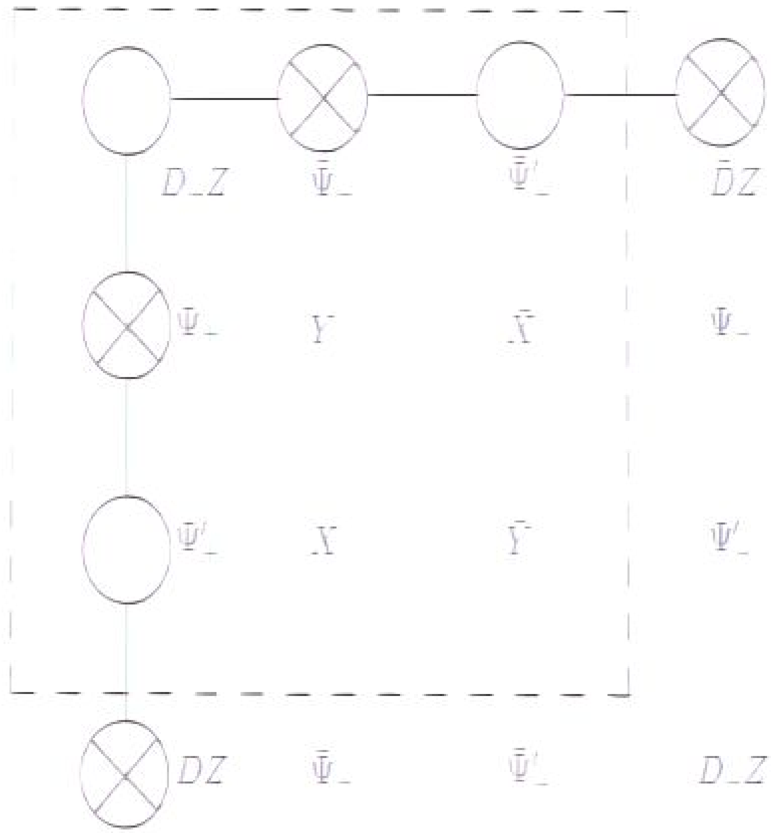

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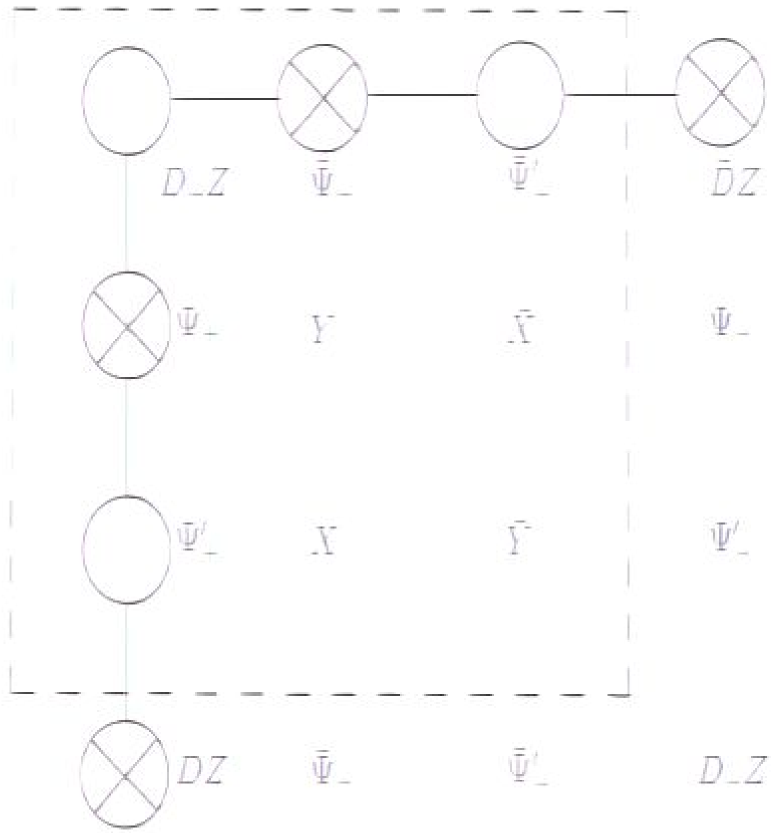

All-loop analysis

Second step: Derive integral equation for large-spin distribution density of roots

$$\gamma_n^\star + \int_0^\infty \frac{dt}{t} J_n(2gt) \frac{\gamma_-^\star(2gt) - (-1)^n \gamma_-^\star(2gt)}{e^t - 1} = \kappa_n^\star$$

Generalization of the system of equations for vacuum solution $(\star = \text{ø})$

$$\kappa_n^{\text{ø}} = 2g\,\delta_{n,1}$$

Equivalent to solving the Beisert-Eden-Staudacher equation for vacuum distribution of roots

Asymptotic Bethe Ansatz equations for (planar) dilatation operator (all loops)

- Proposal for the $\mathfrak{sl}(2)$ sector [Beisert.Staudacher'05],[Beisert'05]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp\left(2i\theta(u_k, u_j)\right)$$

with the deformed spectral parameter $u_k \pm i/2 = x_k^\pm - g^2/x_k^\pm$ [Beisert.Dippel.Staudacher'05]

and with the dressing phase $\theta(u_k, u_j)$ $\left(= O\left(g^6\right)\right)$ [Beisert.Eden.Staudacher'06] [Beisert.Hernández.López'06]

- All-loop 'asymptotic' anomalous dimensions

$$\delta\Delta = 2g^2 \sum_{j=1}^{S} \left[\frac{i}{x_j^+} - \frac{i}{x_j^-}\right]$$

Wrapping effect — avoided at large spin....

evidence from explicit computations at weak coupling
[Bajnok.Janik.Lukowski'08],[Beccaria.Forini.Lukowski.Zieme'09]

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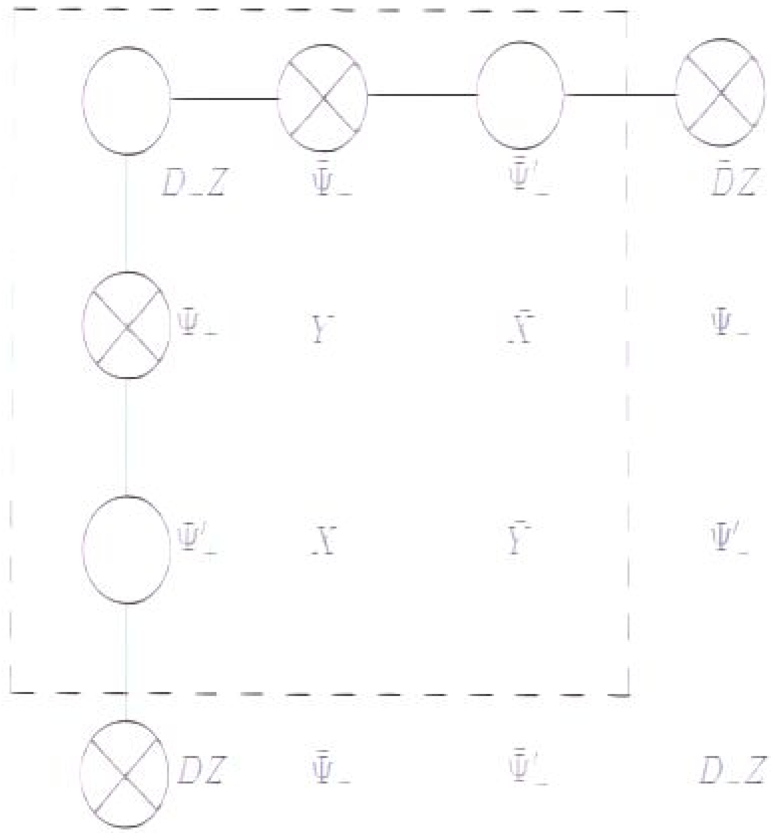

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Generalization of the system of equations for vacuum solution $(\star = \emptyset)$

$$\kappa_n^\emptyset = 2g\, \delta_{n,1}$$

Equivalent to solving the Beisert-Eden-Staudacher equation for vacuum distribution of roots

All-loop analysis

Last step:

$$\gamma_n^\star + \int_0^\infty \frac{dt}{t} J_n(2gt) \frac{\gamma_-^\star(2gt) - (-1)^n \gamma_-^\star(2gt)}{e^t - 1} = \kappa_n^\star$$

Apply methodology to deal with this type of problem...

[Kotikov.Lipatov'06],[Benna.Benvenuti.Klebanov.Scardicchio'07],
[Alday.Arutyunov.Benna.Eden.Klebanov'07],[Kostov.Serban.Volin'07],[Beccaria.DeAngelis.Forini'07]
[B.,Korchemsky.Kostanski'07],[Kostov.Serban.Volin'08],[B.,Korchemsky'08'09]

... and in particular a trick for evaluating the energy $\sim \gamma_1^\star$ in terms of sol. to BES equation

All-loop dispersion relation

Example for scalar

$$E_{\mathrm{h}}(u) = 1 + \int_0^{\infty} \frac{dt}{t} \frac{e^{t/2}\cos(ut) - J_0(2gt)}{e^t - 1} \gamma^{\circ}(-2gt)$$

$$p_{\mathrm{h}}(u) = 2u - \int_0^{\infty} \frac{dt}{t} \frac{e^{t/2}\sin(ut)}{e^t - 1} \gamma^{\circ}(2gt)$$

The function $\gamma^{\circ}(t)$ solves the BES equation...

...known explicitely at both weak/strong coupling

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Weak coupling expansion

Example: parametric representation for a scalar

- Energy:

[Belitsky.Pasechnik.Korchemsky'06]

$$E_{\rm h}(u) = 1 + \frac{1}{2}\Gamma_{\rm cusp}(g)\left(\psi(1/2 - iu) + \psi(1/2 - iu) - 2\psi(1)\right)$$

$$- g^4\left(\psi_2(1/2 + iu) + \psi_2(1/2 - iu) - 12\zeta_3\right) + O(g^6)$$

- Momentum:

$$p_{\rm h}(u) = 2u - \frac{\pi}{2}\Gamma_{\rm cusp}(g)\tanh(\pi u) + 2\pi^3 g^4 \frac{\tanh(\pi u)}{\cosh^2(\pi u)} - O(g^6)$$

with

$$\Gamma_{\rm cusp}(g) = 4g^2 - \frac{4\pi^2 g^4}{3} + O(g^6)$$

All-loop dispersion relation

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Particular points (scalar)

Mass:

Weak coupling: mass starts at 1 and decreases as coupling increases

Strong coupling: mass becomes exponentially small [Fioravanti,Grinza,Rossi'08],[B.,Korchemsky'08]]

$$m_{\text{scalar}} = k\, g^{1/4}\, e^{-\pi g}\,(1 + O(1/g))$$

Expression for constant k agrees with string theory prediction [Alday,Maldacena'07],[B.,Korchemsky'08]]

Large rapidity:

Dispersion relation expressible in terms of vacuum data

$$E = \Gamma_{\text{cusp}}(g) \log p + O(1)$$

Agreement with string theory computation [Dorey,Losi'10]

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Strong coupling regimes (scalar)

- Non-perturbative regime: $E \sim p \sim m \sim \mathrm{e}^{-\pi g}$

$$E = \sqrt{p^2 + m^2}\left[1 - c(g)p^2 + O(m^4, m^2p^2, p^4)\right]$$

- Perturbative regime: $E \sim p \sim 1$

$$E = p\left[1 - c\frac{p^2}{g} + O(1/g^2)\right]$$

- Near-flat space regime : $E \sim p \sim g^{1/4}$

$$\tilde{E} = \tilde{p} + O(1/\sqrt{g})$$

- Giant hole regime: $E \sim p \sim g$

Matching with result obtained on the string theory side using algebraic curve

[Dorey,Losi'10]

Comparison with string theory

Perturbative regime: $E \sim p \sim 1$

$$E = \sqrt{p^2 + m^2(g)}\left[1 - c\,\frac{p^2}{g} + O(1/g^2)\right]$$

Agreement with string theory computations for most of the excitations...

[Giombi,Ricci,Roiban,Tseytlin '10]

- gauge field: $m(g) = \sqrt{2}\left[1 - 1/(8g) + O(1/g^2)\right]$ $\qquad c = \frac{1}{8}$
- fermion: $m(g) = 1$ $\qquad c = \frac{1}{4}$

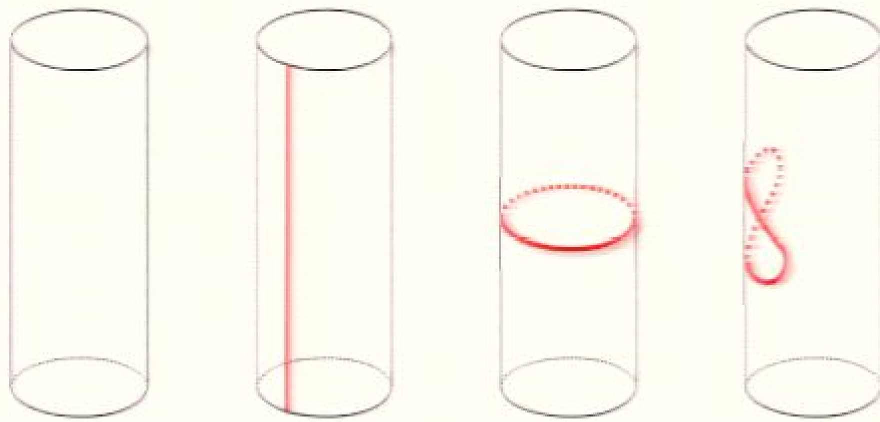
... except for scalar

$$c_{\text{ABA}} = \frac{\Gamma(\frac{1}{4})^4}{8(12)^{5/4}\pi^2} \neq \frac{7}{24\pi} = c_{\text{string}}$$

Possible explanation:

- ABA computes dispersion relation for asymptotic scalar
- String theory computes dispersion relation for Goldstone mode in S^5 : infrared divergences — not asymptotic

Twist-two energy as vacuum energy



(a) (b) (c) (d)

- (a) Twist two = vacuum for a theory on cylinder of length $R \sim 2 \log S \gg 1$
- (b) Twist three = one-particle state
- (c) Finite-size correction due to exchange of virtual excitation
- (d) Bulk contribution reflecting that the background is corrected

Vacuum

Ansatz

$$E = E^{\text{bulk}} + E^{\text{FS}}$$

- $E^{\text{bulk}} \sim 2\log S \sim$ length
- First finite-size correction as a Lüscher formula

$$E^{\text{FS}} = -\sum_{\star} \frac{n_\star}{2\pi} \int dp\, Y_\star^{\text{mirror}}(p)$$

with $\star$ = scalar, fermion, gauge field, ...

Ingredient : amplitude for propagation of excitation $\star$ in mirror kinematics

$$Y_\star^{\text{mirror}}(p) \sim \exp\left(-RE_\star^{\text{mirror}}(p)\right)$$

What are they?

- Remember that for BMN vacuum $R \sim L$ (spin-chain length) and [Bajnok,Janik'08] [Ambjorn,Janik,Kristjansen'05]

$$E_Q^{\text{mirror}}(p) = 2\text{arcsinh}\frac{\sqrt{Q^2+p^2}}{4g} \sim -2\log g \qquad \text{for a bound state of } Q \text{ magnons}$$

- Here $R \sim 2\log S$ but what are $E_\star^{\text{mirror}}(p)$?

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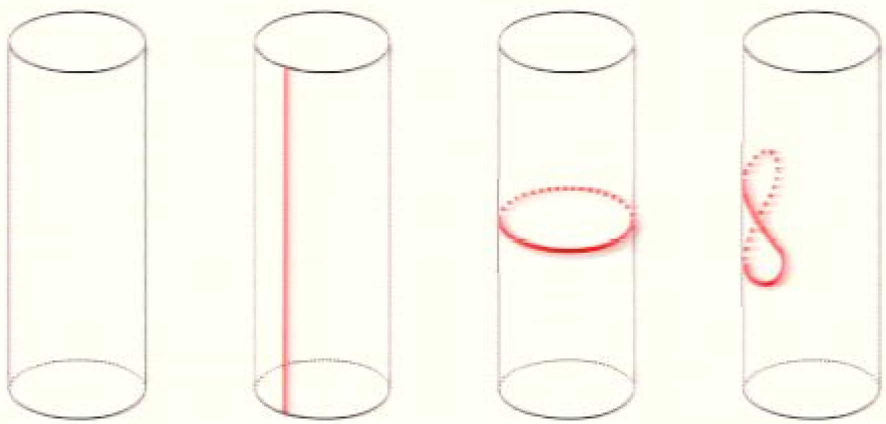

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Double Wick rotation

Program:

- Continuation to imaginary values

$$E = ip^{\text{mirror}} \qquad p = iE^{\text{mirror}}$$

- Parametrically equivalent to a transformation $u \to u'$ such that

$$p^{\text{mirror}}(u) \equiv -iE(u') \qquad E^{\text{mirror}}(u) \equiv -ip(u')$$

  achieve real values simultaneously

- After eliminating u one gets the dispersion relation in the mirror theory

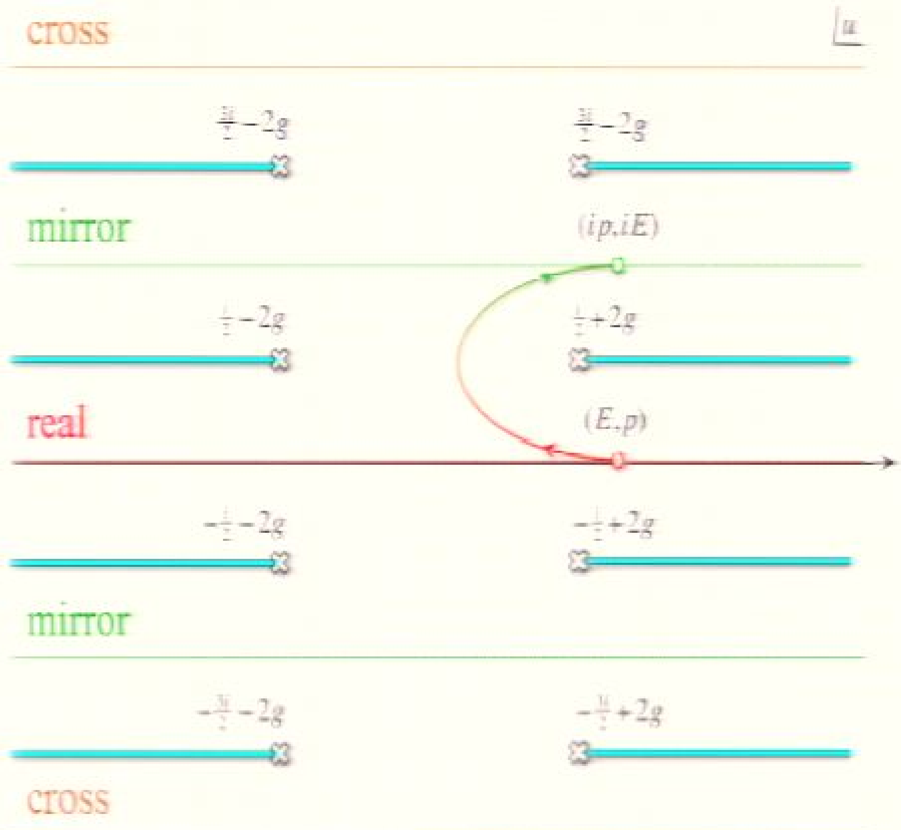
Outcome: Double Wick rotation symmetry [Alday,Gaiotto,Maldacena,Sever,Vieira'10]

$$p^{\text{mirror}}(u) = p(u) \qquad E^{\text{mirror}}(u) = E(u)$$

or equivalently, invariance of the dispersion relation

$$E^{\text{mirror}}(p) = E(p)$$

Rapidity plane for a scalar

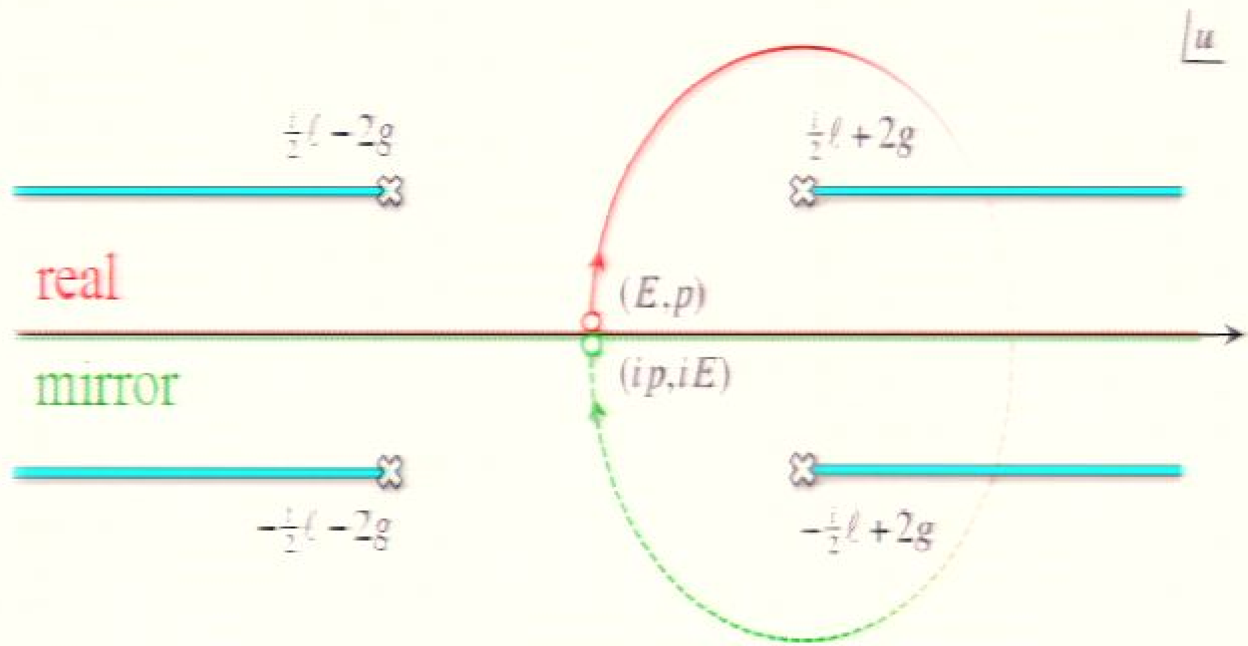


Similar to relativistic dispersion relation

$$E = m\cosh\theta \qquad p = m\sinh\theta$$

with $\theta = \pi u/2$... if not for the presence of cuts

Rapidity plane for gauge field and bound states



Special point fixed by symmetry [Alday,Maldacena'07]

$$E = \ell \qquad p = \pm i\ell$$

with ℓ the number of gluons in the bound state

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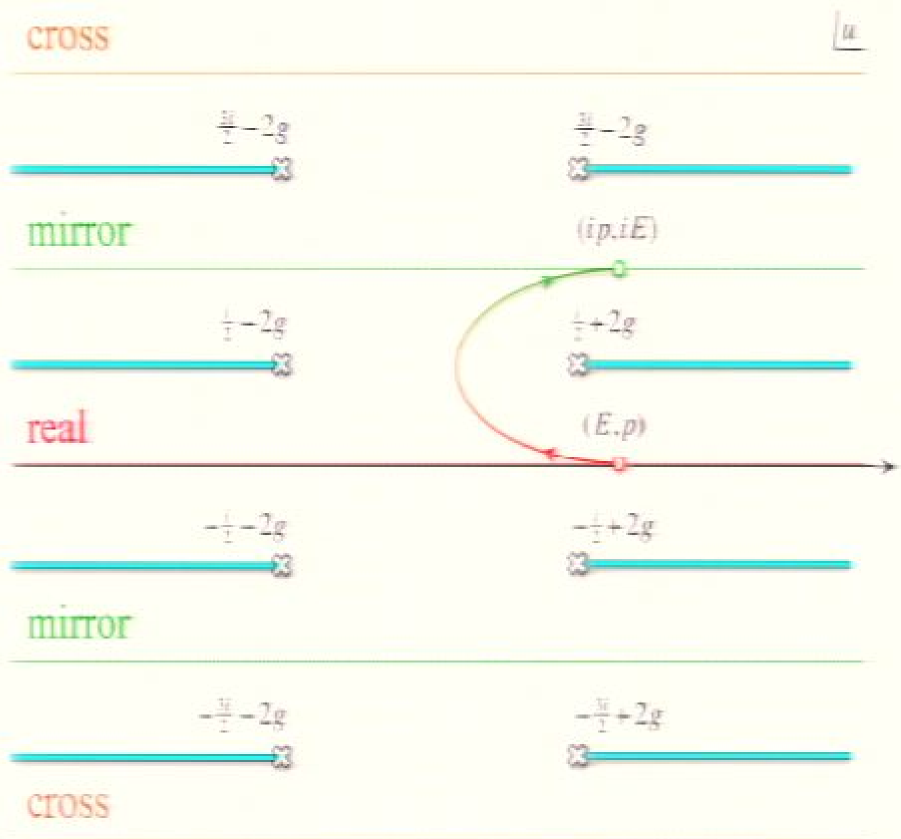

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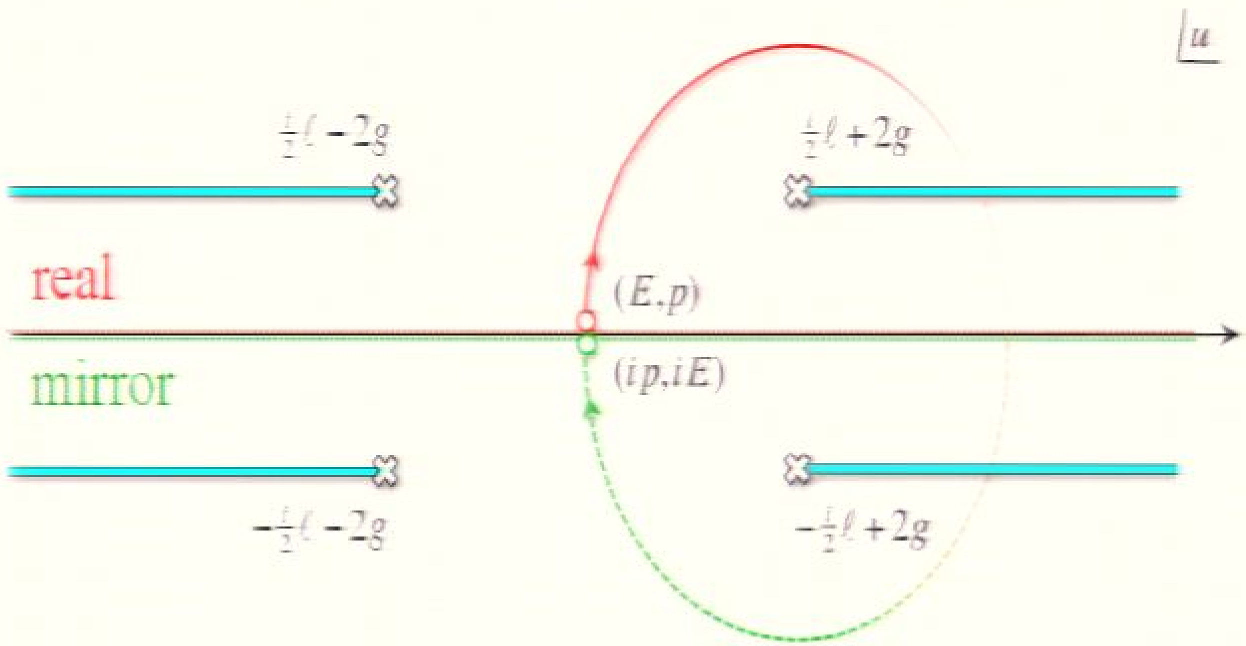

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Y functions IA

- Look at a generic state of $L-2$ scalars, ie. $\in \mathfrak{sl}(2)$ sector $\sim \operatorname{tr} D_+^S Z^L$
- Derive effective Bethe ansatz equations for the hole rapidities δ_k

$$1 + Y_{\rm h}(\delta_k) = 0$$

- One-loop expression [Belitsky,Gorsky,Korchemsky'06]

$$Y_{\rm h}(u) = e^{i\vartheta}\, e^{2ip_{\rm h}(u)\log S}\, \frac{\Gamma(\frac{1}{2}-iu)^2}{\Gamma(\frac{1}{2}+iu)^2} \prod_{j=1}^{L-2} S_{\rm hh}(u,u_j)$$

with $p_{\rm h}(u) = 2u + O(g^2)$ and

$$S_{\rm hh}(u,\delta_j) = \frac{\Gamma(iu-i\delta_j)}{\Gamma(i\delta_j-iu)} \frac{\Gamma(\frac{1}{2}-iu)}{\Gamma(\frac{1}{2}+iu)} \frac{\Gamma(\frac{1}{2}+i\delta_j)}{\Gamma(\frac{1}{2}-i\delta_j)} + O(g^2)$$

- Generalization

$$Y_{\rm h}(u) = e^{i\vartheta}\, e^{iP_{\rm h}(u)} \prod_{j=1}^{L-2} S_{\rm hh}(u,u_j)$$

S matrix recently analyzed at strong coupling in giant hole regime [Dorey,Zhao'11]

Y functions IB

Restriction to vacuum

$$L = 2 \qquad e^{i\vartheta} = (-1)^S$$

Y function in vacuum

$$Y_{\rm h}(u) = (-1)^S \, e^{iP_{\rm h}(u)}$$

with

$$P_{\rm h}(u) = 2p_{\rm h}(u) \log \bar{S} + 2\delta p_{\rm h}(u)$$

Important: Both the momentum $p_{\rm h}(u)$ and its anomalous sibling $\delta p_{\rm h}(u)$ can be found exactly in the coupling from BES and [Freyhult,Zieme'09],[Fioravanti,Grinza,Rossi'09] equations

Example at strong coupling and (very) low momentum ($\theta = \pi u/2 \ll \pi g$)

$$Y_{\rm h}(u) = (-1)^S \, e^{imR\sinh\theta}$$

with effective length

$$R = 2\log\left(\frac{8\pi S}{\sqrt{\lambda}}\right) + O(1/\sqrt{\lambda})$$

and mass

$$m \sim e^{-\sqrt{\lambda}/4}$$

and $\sqrt{\lambda} \equiv 4\pi g \gg 1$

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with effective length

$$R = 2 \log\left(\frac{8\pi S}{\sqrt{\lambda}}\right) + O(1/\sqrt{\lambda})$$

and mass

$$m \sim e^{-\sqrt{\lambda}/4}$$

and $\sqrt{\lambda} \equiv 4\pi g \gg 1$

Y functions II

Mirror kinematics

- Continuation $u \to u'$

$$Y_{\rm h}^{\rm mirror}(u) \equiv Y_{\rm h}(u')$$

- General structure

$$Y_{\rm h}^{\rm mirror}(u) = (-1)^S \, e^{-\mathcal{E}_{\rm h}^{\rm mirror}(u)}$$

where by definition

$$\mathcal{E}_{\rm h}^{\rm mirror}(u) \equiv -iP_{\rm h}(u')$$

Main properties

$$\mathcal{E}_{\rm h}^{\rm mirror}(u) = 2E_{\rm h}(u)\log\bar{S} + 2\delta E_{\rm h}(u)$$

yielding

$$Y_{\rm h}^{\rm mirror}(u) \sim \exp\left(-E_{\rm h}(u)R\right)$$

with length $R \sim 2\log S$

Generalization and overal estimate

$$Y_{\star}^{\rm mirror}(u) \sim 1/S^{2m_\star} \qquad \text{with } m_\star \text{ the mass of } \star$$

Finite-size corrections

$$E^{\text{FS}} = -6\int \frac{dp}{2\pi} Y_{\text{h}}^{\text{mirror}}(p) - 2\int \frac{dp}{2\pi} Y_{\text{gf}}^{\text{mirror}}(p) - 8\int \frac{dp}{2\pi} Y_{\text{f}}^{\text{mirror}}(p)$$

with exchange of twist-one excitations and expand at weak coupling

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$$\mathcal{P}_2 = 0$$

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and observe cancellation between bosonic and fermionic contributions at two loops

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Y functions IB

Restriction to vacuum

$$L = 2 \qquad e^{i\vartheta} = (-1)^S$$

Y function in vacuum

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Y functions IA

- Look at a generic state of $L-2$ scalars, ie. $\in \mathfrak{sl}(2)$ sector $\sim \mathrm{tr}\, D_+^S Z^L$
- Derive effective Bethe ansatz equations for the hole rapidities δ_k

$$1 + Y_{\mathrm{h}}(\delta_k) = 0$$

- One-loop expression [Belitsky,Gorsky,Korchemsky'06]

$$Y_{\mathrm{h}}(u) = \mathrm{e}^{i\vartheta}\, \mathrm{e}^{2ip_{\mathrm{h}}(u)\log S}\, \frac{\Gamma(\frac{1}{2}-iu)^2}{\Gamma(\frac{1}{2}+iu)^2} \prod_{j=1}^{L-2} S_{\mathrm{hh}}(u,u_j)$$

with $p_{\mathrm{h}}(u) = 2u + O(g^2)$ and

$$S_{\mathrm{hh}}(u,\delta_j) = \frac{\Gamma(iu-i\delta_j)}{\Gamma(i\delta_j-iu)} \frac{\Gamma(\frac{1}{2}-iu)}{\Gamma(\frac{1}{2}+iu)} \frac{\Gamma(\frac{1}{2}+i\delta_j)}{\Gamma(\frac{1}{2}-i\delta_j)} + O(g^2)$$

- Generalization

$$Y_{\mathrm{h}}(u) = \mathrm{e}^{i\vartheta}\, \mathrm{e}^{iP_{\mathrm{h}}(u)} \prod_{j=1}^{L-2} S_{\mathrm{hh}}(u,u_j)$$

S matrix recently analyzed at strong coupling in giant hole regime [Dorey,Zhao'11]

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Clash

$$E^{\rm FS} = O(g^6) \qquad \neq \qquad E^{\rm wrap} = O(g^8)$$

Empirical resolution

- Go back to ABA energy

$$E^{\rm ABA} = E^{\rm reg} + E^{\rm alt}$$

  where $E^{\rm alt} \sim (-1)^S / S^2$

- Observe that $E^{\rm alt}$ has large spin expansion similar to $E^{\rm FS}$ (at least for $\sim 1/S^2$ corrections in weak coupling expansion)

- Exact the ABA prediction by the substitution

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- In short

$$E = E^{\rm bulk} + E^{\rm FS}$$

  with (at least within our current accuracy at large spin)

$$E^{\rm bulk} = E^{\rm reg}$$

If true $\longrightarrow$ $E^{\rm FS} = E^{\rm alt} + E^{\rm wrap}$

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$$E^{\rm alt} = \frac{1}{\pi} \oint dp_{\rm h}(u - i/2) Y_{\rm h}(u - i/2)$$

from long-range Baxter equation

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Comparison

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Strong coupling analysis

Relevant sector : scalars because lightest excitations with mass $m \sim \exp(-\pi g)$

$$E^{\text{alt}} \sim -(-1)^S \frac{2m}{\pi} K_1(mR)$$

compared

$$E^{\text{FS}} \sim -(-1)^S \frac{6m}{\pi} K_1(mR)$$

with length $R \sim 2 \log S$

Somehow ABA detects only 2 scalar excitations maybe because the BMN vacuum breaks $O(6)$ symmetry down to $O(4)$

$$Z, \bar{Z} \qquad \text{and} \qquad Y, \bar{Y}, X, \bar{X} \qquad \text{treated differently}$$

Substitution

$$E^{\text{alt}} \to E^{\text{FS}}$$

restores the proper counting of fluctuations and includes fermions and gauge fields

Comparison with string theory

Our result is non-perturbative... perturbative string theory gives at one loop

[Schafer-Nameki,Zamaklar,Zarembo'05]
[Beccaria,Dunne,Forini,Pawellek,Tseytlin'10],[Giombi,Ricci,Roiban,Tseytlin'10]
[Gromov,Serban,Shenderovich,Volin'11]

$$E^{\mathrm{FS}} = -\frac{5\pi}{12\log S} = -\frac{5\pi}{6R}$$

Change of scale

$$1/S^2 - 1/S^{2m} \sim e^{-mR} - 1/\log S \sim 1/R$$

Resolution

$$E^{\mathrm{FS}} = -\frac{\pi c(mR)}{6R}$$

where $c(mR)$ is the effective central charge of the O(6) sigma model

- Lüscher formula is good in the massive regime $mR \gg 1$
- String Casimir energy is found in the UV regime $mR \ll 1$
- Inbetween one needs the full-fledged TBA equations for vacuum energy of O(6) sigma model [Balog,Hegedus'01]

Summary

- Solution to spectrum of excitations over the GKP string at any coupling
- Exact representation for dispersion relations
- Uncover analytical structure of dispersion relations
- Application to large-spin correction to twist-two anomalous dimension

Outlook

- Dynamics.... S matrix...
- Missing boson... Stability...
- Lüscher formula for higher twist operators...
  - Twist three = one-particle state... finite-size correction to mass gap...
  - Large-twist limit and generalized scaling function... finite-size correction to finite-density gas
- Generalization to ABJM theory
  - Lightest modes = twist = 1/2 partons in $4 + \bar{4}$
  - Finite-size correction to twist-one operator $\sim (-1)^S/S$
- Application to deformation of light-like Wilson loops and scattering amplitudes

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Y functions IA

- Look at a generic state of $L-2$ scalars, ie. $\in \mathfrak{sl}(2)$ sector $\sim \operatorname{tr} D_+^S Z^L$
- Derive effective Bethe ansatz equations for the hole rapidities δ_k

$$1 + Y_{\mathrm{h}}(\delta_k) = 0$$

- One-loop expression [Belitsky,Gorsky,Korchemsky'06]

$$Y_{\mathrm{h}}(u) = \mathrm{e}^{i\vartheta}\, \mathrm{e}^{2ip_{\mathrm{h}}(u)\log S}\, \frac{\Gamma(\frac{1}{2} - iu)^2}{\Gamma(\frac{1}{2} + iu)^2} \prod_{j=1}^{L-2} S_{\mathrm{hh}}(u, u_j)$$

with $p_{\mathrm{h}}(u) = 2u + O(g^2)$ and

$$S_{\mathrm{hh}}(u, \delta_j) = \frac{\Gamma(iu - i\delta_j)}{\Gamma(i\delta_j - iu)} \frac{\Gamma(\frac{1}{2} - iu)}{\Gamma(\frac{1}{2} + iu)} \frac{\Gamma(\frac{1}{2} + i\delta_j)}{\Gamma(\frac{1}{2} - i\delta_j)} + O(g^2)$$

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S matrix recently analyzed at strong coupling in giant hole regime [Dorey,Zhao'11]

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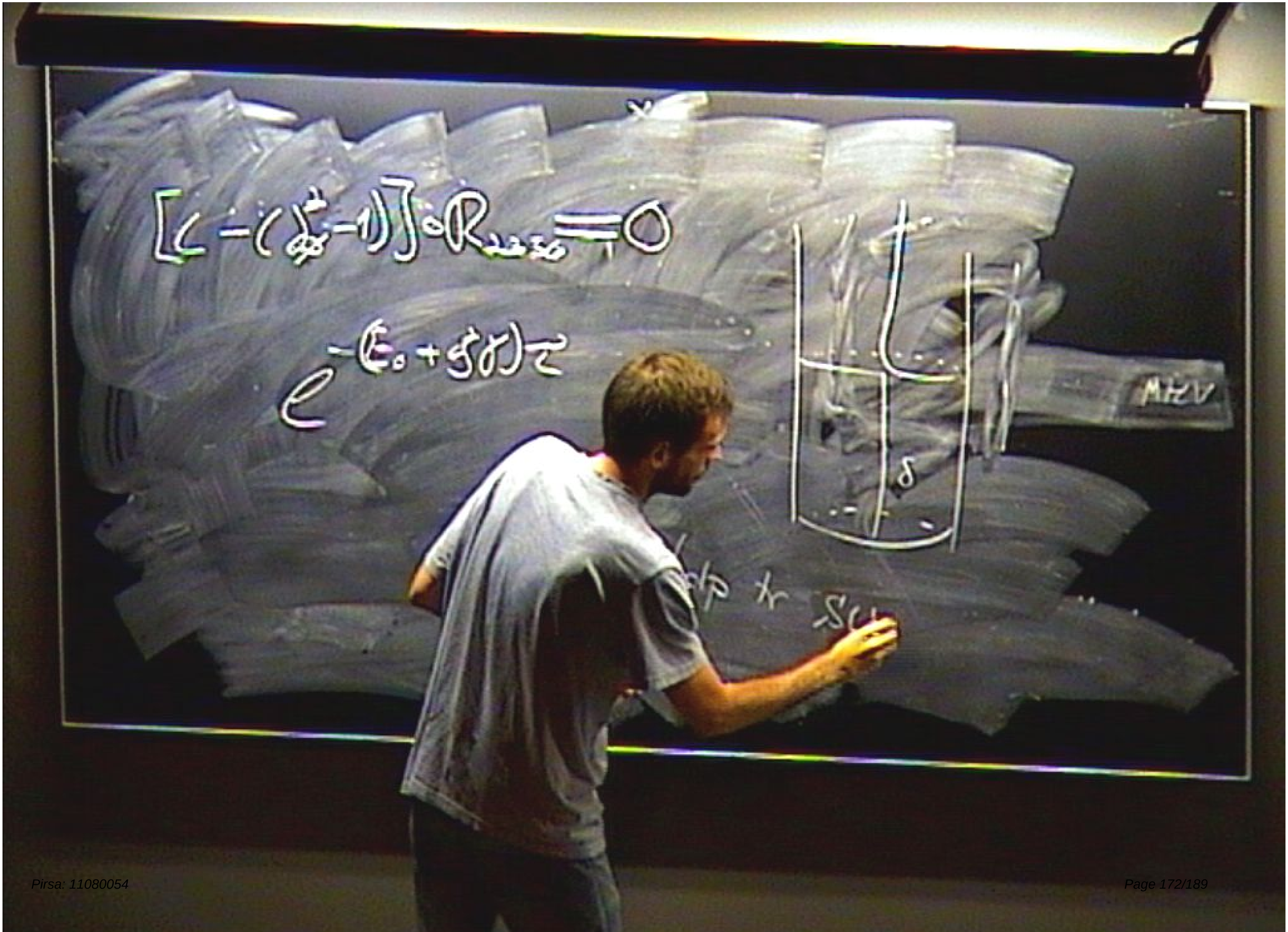
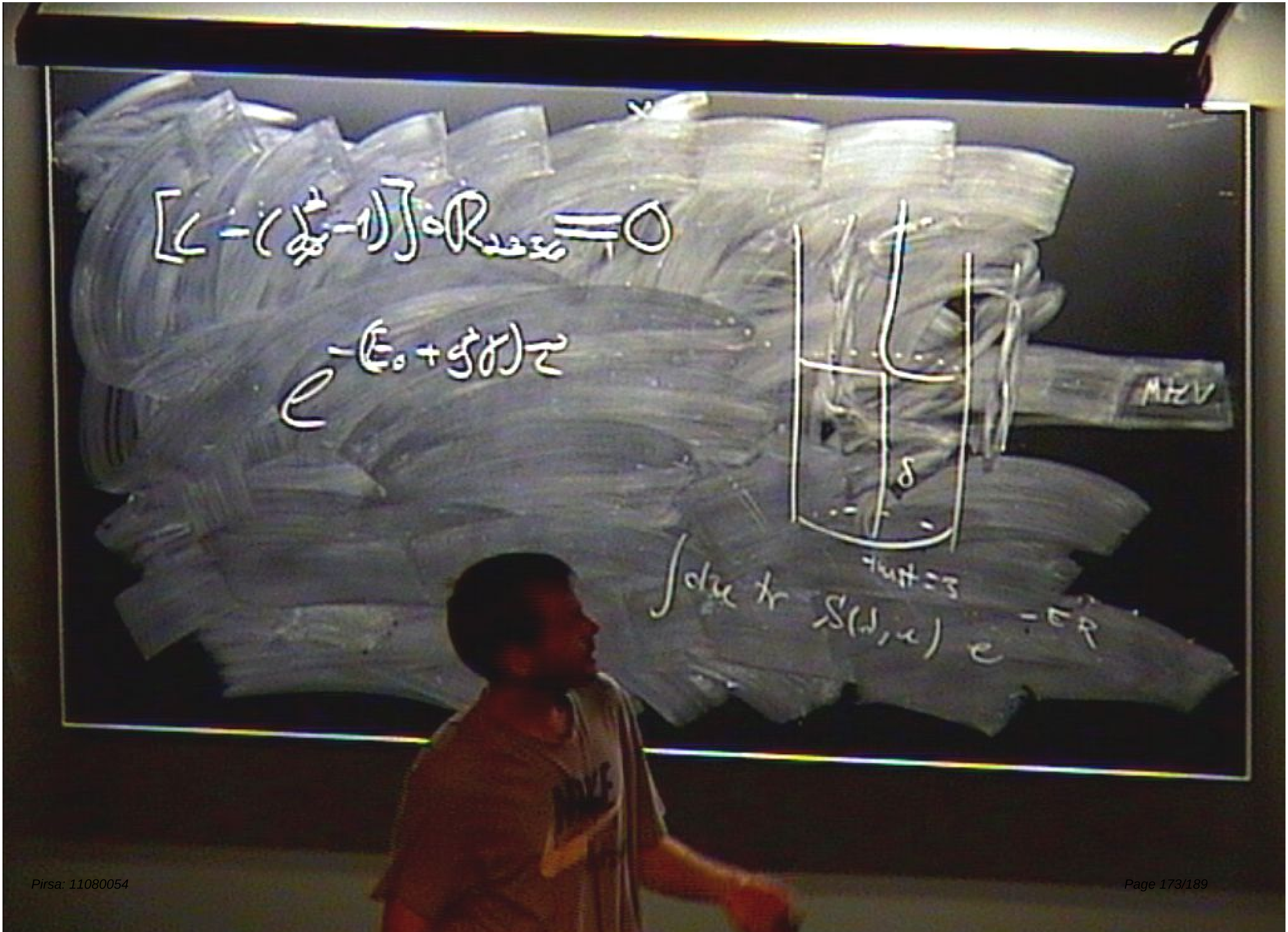

MHV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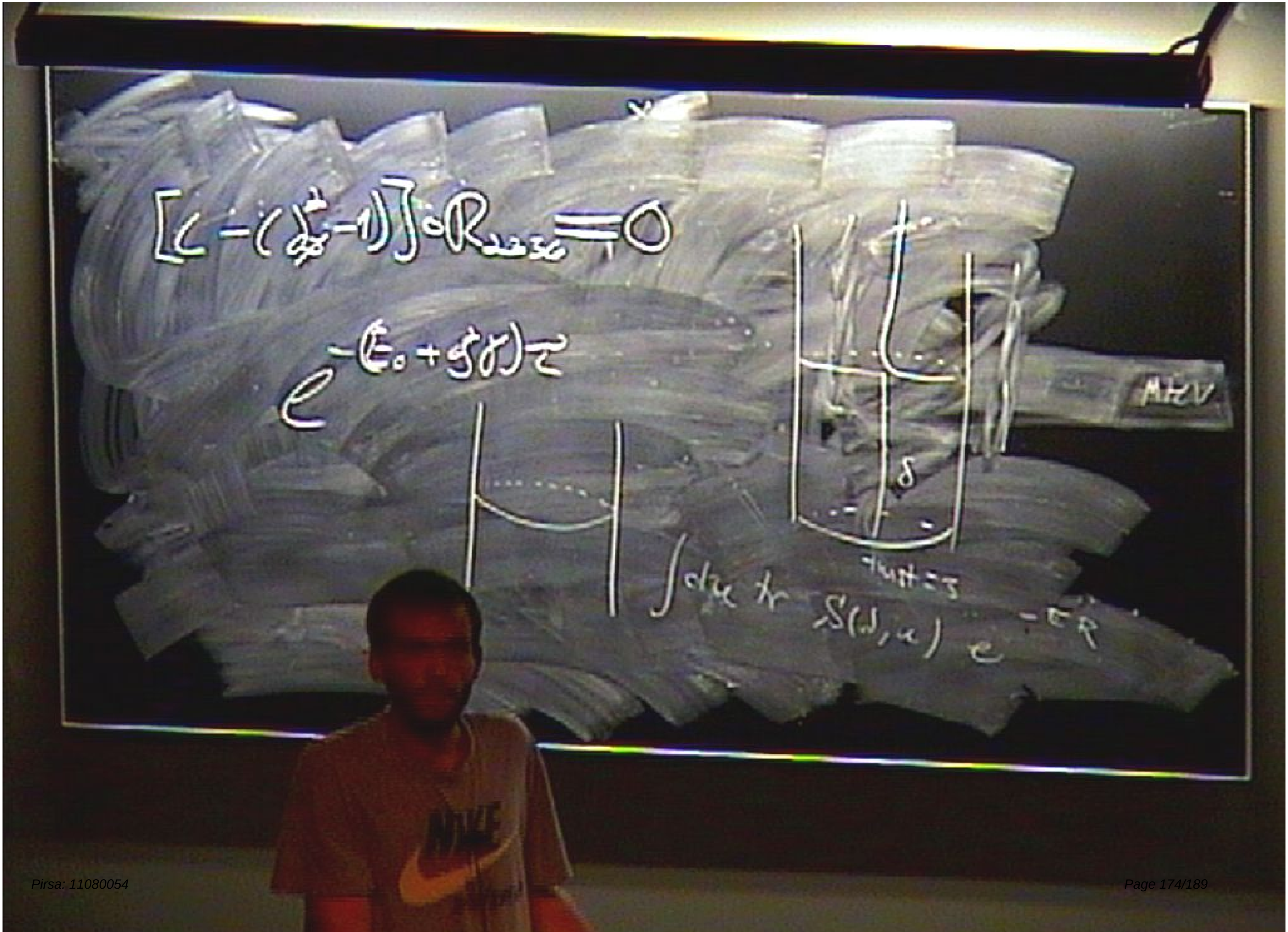
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Outlook

- Dynamics.... S matrix...
- Missing boson... Stability...
- Lüscher formula for higher twist operators...
  - Twist three = one-particle state... finite-size correction to mass gap...
  - Large-twist limit and generalized scaling function... finite-size correction to finite-density gas
- Generalization to ABJM theory
  - Lightest modes = twist = 1/2 partons in $4+\bar{4}$
  - Finite-size correction to twist-one operator $\sim (-1)^S/S$
- Application to deformation of light-like Wilson loops and scattering amplitudes

Comparison with string theory

Our result is non-perturbative... perturbative string theory gives at one loop

[Schafer-Nameki,Zamaklar,Zarembo'05]
[Beccaria,Dunne,Forini,Pawellek,Tseytlin'10],[Giombi,Ricci,Roiban,Tseytlin'10]
[Gromov,Serban,Shenderovich,Volin'11]

$$E^{\rm FS} = -\frac{5\pi}{12\log S} = -\frac{5\pi}{6R}$$

Change of scale

$$1/S^2 - 1/S^{2m} \sim e^{-mR} - 1/\log S \sim 1/R$$

Resolution

$$E^{\rm FS} = -\frac{\pi c(mR)}{6R}$$

where $c(mR)$ is the effective central charge of the O(6) sigma model

- Lüscher formula is good in the massive regime $mR \gg 1$
- String Casimir energy is found in the UV regime $mR \ll 1$
- Inbetween one needs the full-fledged TBA equations for vacuum energy of O(6) sigma model [Balog,Hegedus'01]

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Clash

$$E^{\rm FS} = O(g^6) \qquad \neq \qquad E^{\rm wrap} = O(g^8)$$

Empirical resolution

- Go back to ABA energy
$$E^{\rm ABA} = E^{\rm reg} + E^{\rm alt}$$
where $E^{\rm alt} \sim (-1)^S/S^2$
- Observe that $E^{\rm alt}$ has large spin expansion similar to $E^{\rm FS}$ (at least for $\sim 1/S^2$ corrections in weak coupling expansion)
- Exact the ABA prediction by the substitution
$$E^{\rm alt} \to E^{\rm FS}$$
- In short
$$E = E^{\rm bulk} + E^{\rm FS}$$
with (at least within our current accuracy at large spin)
$$E^{\rm bulk} = E^{\rm reg}$$

If true $\longrightarrow$ $E^{\rm FS} = E^{\rm alt} + E^{\rm wrap}$

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- Compute integrals at weak coupling

$$E^{\text{FS}} = \mathbf{0}\, g^4 + O(g^6)$$

and observe cancellation between bosonic and fermionic contributions at two loops

Similar to near-collinear expansion for amplitudes from OPE

[Alday,Gaiotto,Maldacena,Sever,Vieira'10]

... but the situation is better since we know corrections to the measures to all loops

Finite-size corrections

$$E^{\rm FS} = -6\int \frac{dp}{2\pi}\, Y_{\rm h}^{\rm mirror}(p) - 2\int \frac{dp}{2\pi}\, Y_{\rm gf}^{\rm mirror}(p) - 8\int \frac{dp}{2\pi}\, Y_{\rm f}^{\rm mirror}(p)$$

with exchange of twist-one excitations and expand at weak coupling

$$E^{\rm FS} = \frac{(-1)^S}{S^2}\sum_{n\geq 2} \mathcal{P}_n g^{2n}$$

with polynomials in $\log \bar{S} = \log S + \gamma_E$

$$\mathcal{P}_2 = 0$$

$$\mathcal{P}_3 = -\frac{16\pi^2}{3}\log \bar{S}$$

$$\mathcal{P}_4 = -\left(256\zeta_3 + \frac{64\pi^2}{3}\right)\log^2 \bar{S} + \frac{112\pi^4}{15}\log \bar{S} + 16\pi^2\zeta_3$$

$$\mathcal{P}_5 = -\left(\frac{448\pi^4}{45} + 1024\zeta_3 - \frac{256\pi^2}{3}\right)\log^3 \bar{S}$$

$$+ \left(2560\zeta_5 + \frac{896\pi^2\zeta_3}{3} + \frac{1664\pi^4}{45} + 256\zeta_3 + \frac{128\pi^2}{3}\right)\log^2 \bar{S}$$

$$+ \left(1536\zeta_3^2 - \frac{8992\pi^6}{945} + 128\pi^2\zeta_3\right)\log \bar{S} - \left(\frac{320\pi^2\zeta_5}{3} + \frac{896\pi^4\zeta_3}{45}\right)$$

Y functions II

Mirror kinematics

- Continuation $u \to u'$

$$Y_{\mathrm{h}}^{\mathrm{mirror}}(u) \equiv Y_{\mathrm{h}}(u')$$

- General structure

with

$$P_{\mathrm{h}}(u) = 2p_{\mathrm{h}}(u) \log \bar{S} + 2\delta p_{\mathrm{h}}(u)$$

Important: Both the momentum $p_{\mathrm{h}}(u)$ and its anomalous sibling $\delta p_{\mathrm{h}}(u)$ can be found exactly in the coupling from BES and [Freyhult,Zieme'09],[Fioravanti,Grinza,Rossi'09] equations

Example at strong coupling and (very) low momentum ($\theta = \pi u/2 \ll \pi g$)

$$Y_{\mathrm{h}}(u) = (-1)^{S}\, \mathrm{e}^{imR\sinh\theta}$$

with effective length

$$R = 2\log\left(\frac{8\pi S}{\sqrt{\lambda}}\right) + O(1/\sqrt{\lambda})$$

and mass

$$m \sim \mathrm{e}^{-\sqrt{\lambda}/4}$$

and $\sqrt{\lambda} \equiv 4\pi g \gg 1$

Y functions IB

Restriction to vacuum

$$L \to 2 \qquad e^{i\vartheta} = (-1)^S$$

Y function in vacuum

$$Y_{\rm h}(u) = (-1)^S \, e^{iP_{\rm h}(u)}$$

with

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Y functions IA

- Look at a generic state of $L-2$ scalars, ie. $\in \mathfrak{sl}(2)$ sector $\sim \operatorname{tr} D_+^S Z^L$
- Derive effective Bethe ansatz equations for the hole rapidities δ_k

$$1 + Y_{\mathrm{h}}(\delta_k) = 0$$

- One-loop expression [Belitsky,Gorsky,Korchemsky'06]

$$Y_{\mathrm{h}}(u) = \mathrm{e}^{i\vartheta}\, \mathrm{e}^{2ip_{\mathrm{h}}(u)\log S}\, \frac{\Gamma(\frac{1}{2}-iu)^2}{\Gamma(\frac{1}{2}+iu)^2} \prod_{j=1}^{L-2} S_{\mathrm{hh}}(u,u_j)$$

with $p_{\mathrm{h}}(u) = 2u + O(g^2)$ and

$$S_{\mathrm{hh}}(u,\delta_j) = \frac{\Gamma(iu-i\delta_j)}{\Gamma(i\delta_j-iu)} \frac{\Gamma(\frac{1}{2}-iu)}{\Gamma(\frac{1}{2}+iu)} \frac{\Gamma(\frac{1}{2}+i\delta_j)}{\Gamma(\frac{1}{2}-i\delta_j)} + O(g^2)$$

- Generalization

$$Y_{\mathrm{h}}(u) = \mathrm{e}^{i\vartheta}\, \mathrm{e}^{iP_{\mathrm{h}}(u)} \prod_{j=1}^{L-2} S_{\mathrm{hh}}(u,u_j)$$

S matrix recently analyzed at strong coupling in giant hole regime [Dorey,Zhao'11]

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