Title: Twistor Methods in N=4 SYM

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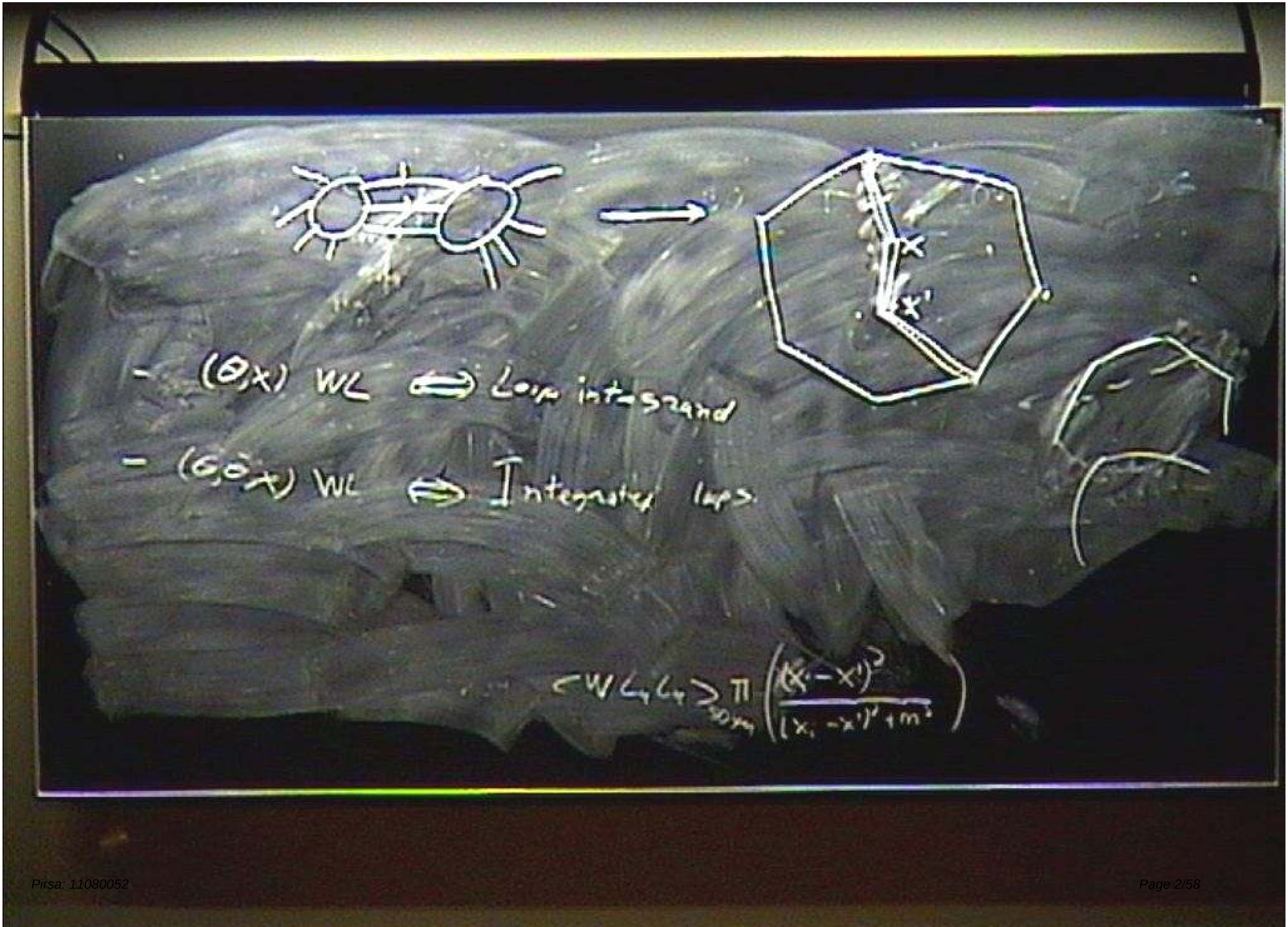
Abstract: I review how N=4 SYM can be reformulated as a theory on twistor space, and explain various calculations that have been performed there. In particular, twistors turn out to be a powerful tool for investigating the duality between scattering amplitudes and null polygonal Wilson Loops in the planar limit. The BCFW recursion relations are interpreted as the loop equations for a supersymmetric generalization of the Wilson Loop.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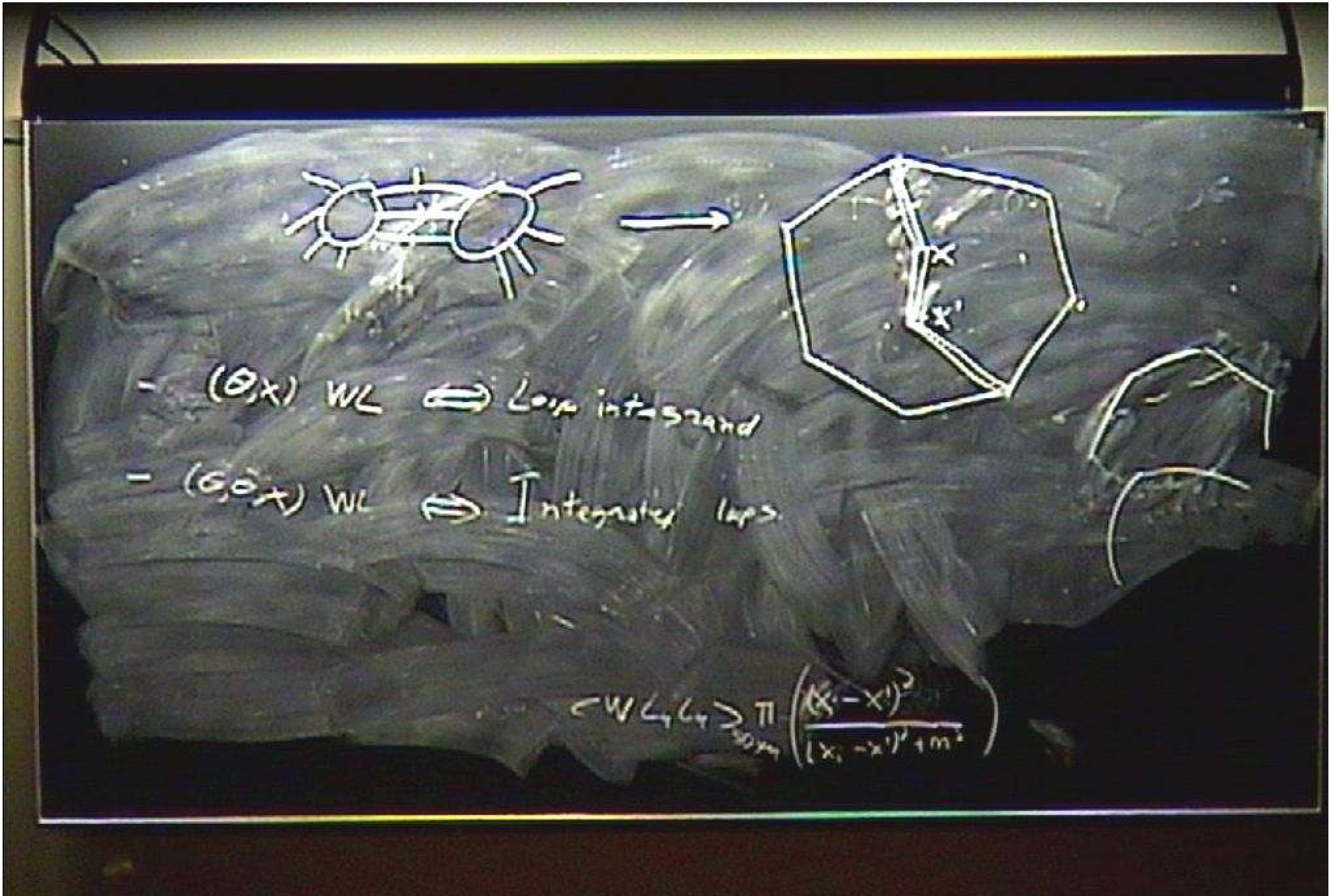
x
x'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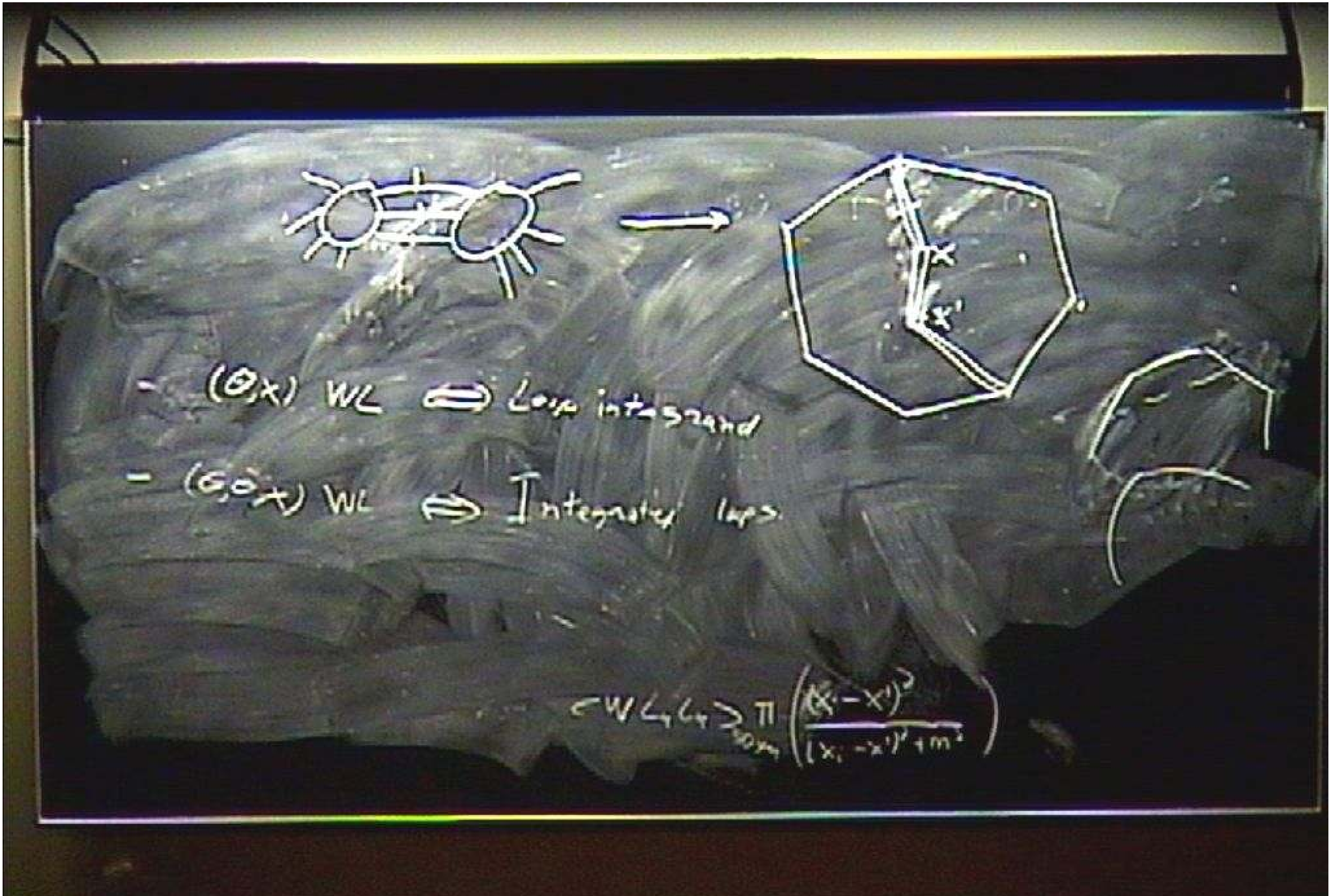
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Introduction

Twistor methods have proved useful in a variety of calculations in 4d gauge theories.

There are two main reasons for this:

- **Twistor space carries a natural action of the space-time conformal group**
- **The twistor data for scattering processes / null polygonal Wilson Loops is unconstrained**

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- **The twistor data for scattering processes / null polygonal Wilson Loops is unconstrained**

In this talk, I will review how we can take advantage of these features by reformulating $\mathcal{N}=4$ SYM in twistor space, particularly in the context of the duality between scattering amplitudes and null polygonal Wilson Loops.

- **Working in twistor space immediately allows us to generalize this duality beyond MHV amplitudes**
- **It also provides insight into why the duality holds at all**

Basic twistor geometry

Twistor space is a copy of $\mathbb{CP}^3$ with homogeneous coordinates $Z^a = (\lambda_\alpha, \mu^{\dot\alpha})$

The (complexified) space-time conformal group $\mathrm{SL}(4,\mathbb{C})$ acts via the generators $J^a{}_b = Z^a \dfrac{\partial}{\partial Z^b}$

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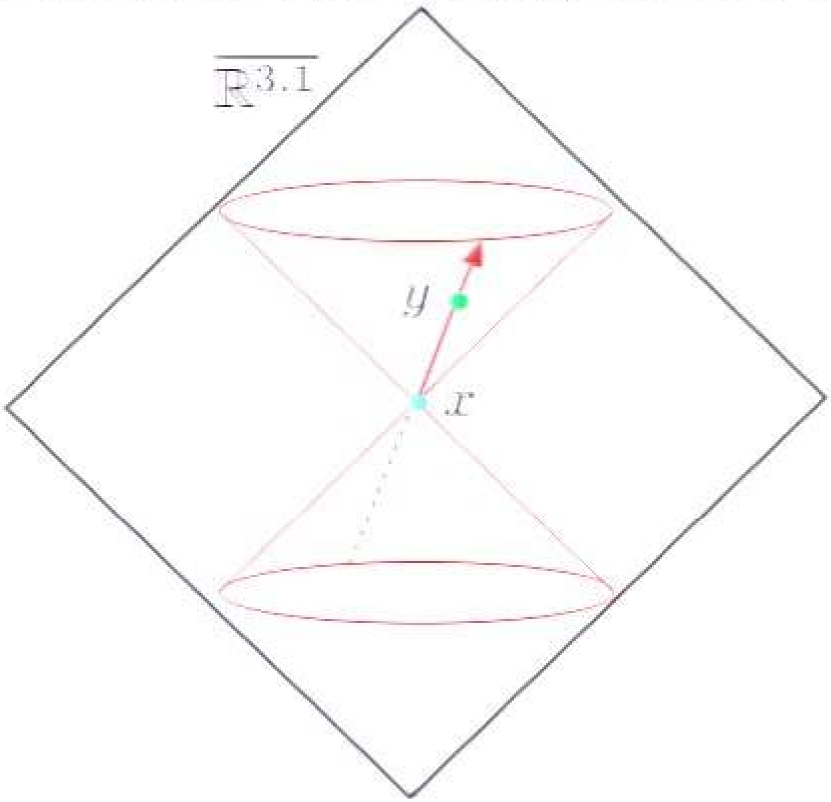

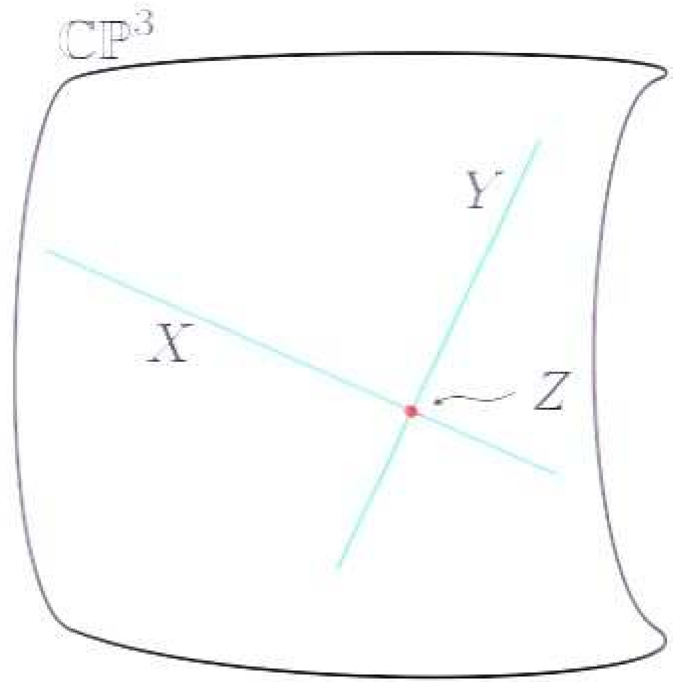

- As Z varies over the Riemann sphere X in twistor space, the rays sweep out the null cone centered on x in space-time

- If two twistor lines intersect, their corresponding space-time points are null-separated

The Penrose transform

Massless free fields of all helicities have a beautiful description on twistor space:

$$H^1(\mathbb{CP}^{3'}, \mathcal{O}(2h-2)) \quad \cong \quad \left\{ \begin{array}{l} \text{Analytic sol}^{\text{n}}\text{s of wave eqn for} \\ \text{massless free field, helicity } h \end{array} \right\}$$

described locally by **arbitrary** holomorphic function of homogeneity $2h-2$

$$\phi(x) = \oint \langle \lambda \, \mathrm{d}\lambda \rangle \, \Phi_{-2}(Z)|_{\mu = \mathrm{i}x\lambda} \qquad \Phi \in H^1(\mathbb{CP}^{3'}, \mathcal{O}(-2))$$

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In a Dolbeault representation, this cohomology group arises as the field equation of twistor action

$$S = \int \mathrm{D}^3 Z \wedge \Phi \bar\partial \Phi \qquad \text{where} \quad \Phi \in \Omega^{0,1}(\mathbb{CP}^3, \mathcal{O}(-2)) \ \text{off-shell.}$$

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Non-linear extensions: Penrose-Ward construction considers **holomorphic bundles** rather than just functions, leading to **self-dual Yang-Mills**. This is the basis of the ADHM construction of instantons.

$\mathcal{N} = 4$ SYM on twistor space

$\mathcal{N} = 4$ SYM can be described by the twistor space action

$$S = \int \mathrm{D}^{3|4}Z \wedge \mathrm{Tr}\left(\mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right) + \mathrm{g}^2 \int \mathrm{d}^{4|8}x \; \log\det\left(\bar{\partial} + \mathcal{A}\right)_X$$

where $\mathcal{A} \in \Omega^{(0,1)}(\mathbb{CP}^{3|4}, \mathrm{End}E)$ is a connection (0,1)-form superfield

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- Expanding in powers of the field, the terms proportional to the coupling are

$$\int \mathrm{d}^{4|8}x \, \log\det\left(\bar{\partial}+\mathcal{A}\right)_X = \sum_{n=2}^{\infty}\frac{1}{n}\int \mathrm{d}^{4|8}x \, \mathrm{Tr}\,\underbrace{\left(\bar{\partial}^{-1}\mathcal{A}\,\bar{\partial}^{-1}\mathcal{A}\cdots\bar{\partial}^{-1}\mathcal{A}\right)}_{n \text{ terms}}$$

giving an infinite sum of MHV vertices.

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$\mathbb{CP}^1 \subset \mathbb{CP}^{3|4}$

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- By choosing the gauge $\left(\bar\partial^{\dagger}\mathcal{A}\right)_X=0$, we can reduce the twistor action to the space-time SYM action

$$S=\int \mathrm{Tr}\left(G^-\wedge F_A+\text{susy}\right)+2\mathrm{g}^2\int \mathrm{Tr}\left(G^-\wedge G^-+\text{susy}\right)$$

$$=\frac{1}{4\mathrm{g}^2}\int \mathrm{Tr}\left(F_A\wedge *F_A\right)\ +\ \text{susy}$$ (after integrating out G^- using its algebraic eom).

Scattering Amplitudes

Projective delta-functions & the twistor propagator

To use this action, we first need the propagator. This is one of a family of projective delta-functions:

$\bar{\delta}^{3|4}(Z_1, Z_2) \equiv \int \frac{\mathrm{d}s}{s}\, \bar{\delta}^{4|4}(Z_1 + sZ_2)$ imposing coincidence of $Z_1, Z_2 \in \mathbb{CP}^{3|4}$

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for any $z \in \mathbb{C}$ we have

$$\bar{\delta}(z) = \frac{1}{2\pi \mathrm{i}} \bar{\partial}\left(\frac{1}{z}\right) = \mathrm{d}\bar{z}\, \delta(\mathrm{Re} z)\delta(\mathrm{Im} z)$$

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$$\bar{\delta}^{2|4}(Z_1, Z_2, Z_3) \equiv \int \frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}s_2}{s_2}\, \bar{\delta}^{4|4}(Z_1 + s_1 Z_2 + s_2 Z_3)$$ imposing collinearity

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$$\vdots$$

$$\bar\delta^{0|4}(Z_1, Z_2, Z_3, Z_4, Z_5) \equiv \int \bigwedge_{i=1}^{4} \frac{\mathrm{d}s_i}{s_i}\,\bar\delta^{4|4}(Z_1 + s_1Z_2 + s_2Z_3 + s_3Z_4 + s_4Z_5)$$

$$= \frac{\delta^{0|4}(\chi_1(2345) + \text{cyclic})}{(1234)(2345)(3451)(4512)(5123)}$$

$$\equiv [1,2,3,4,5] \quad (= R_{5;13})$$ (the basic superconformal invariant)

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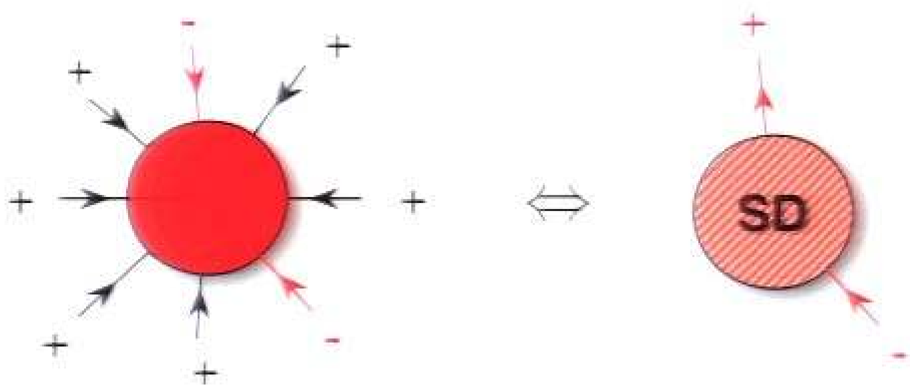
For $r \leq 3$ these obey $\bar\partial\left(\bar\delta^{r|4}(Z_1, \dots, Z_{5-r})\right) = 2\pi\mathrm{i}\sum_i (-1)^{i-1}\bar\delta^{r-1|4}(Z_1, \dots, \hat{Z}_i, \dots, Z_{5-r})$

so in the axial gauge $\bar Z_*^a \mathcal{A}_a = 0$, the holomorphic Chern-Simons propagator is

$\langle \mathcal{A}(Z)\mathcal{A}(Z')\rangle = \bar\delta^{2|4}(Z, Z_*, Z')$ since $\bar\partial\left(\bar\delta^{2|4}(Z, Z_*, Z')\right) = \bar\delta^{3|4}(Z, Z') - \text{spurious}$

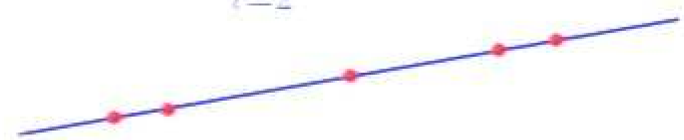
CSW diagrams are Feynman diagrams

MHV tree amplitudes can be interpreted as the scattering of a particle traveling on a self-dual background



In twistor space they are just the statement that the amplitude is supported on a line [illegible]

$$A_{\text{MHV}} = \prod_{i=2}^{n-1} \bar{\delta}^{2|4}(Z_1, Z_i, Z_{i-1})$$



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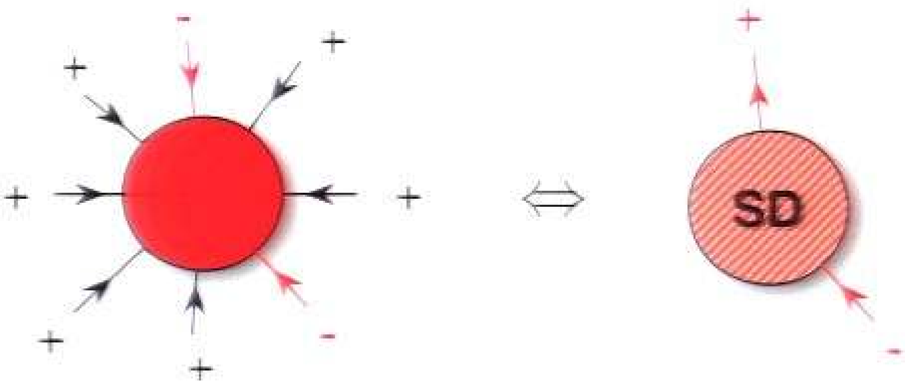

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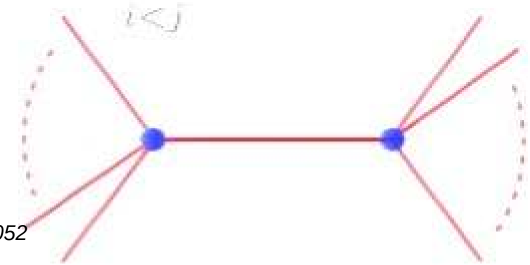
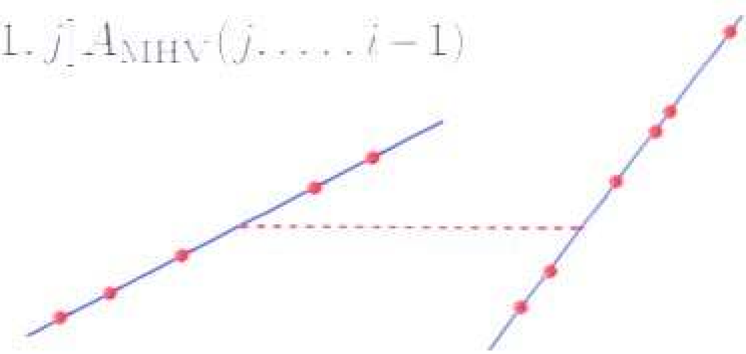
Beyond MHV we join vertices with propagators as usual. For example

$$A_{\mathrm{NMHV}} = \sum_{i<j} \int \mathrm{D}^{3|4}Z\,\mathrm{D}^{3|4}Z'\, A_{\mathrm{MHV}}(i,\ldots,j-1,Z)\,\bar{\delta}^{2|4}(Z,Z_*,Z')\,A_{\mathrm{MHV}}(Z',j,\ldots,i-1)$$

$$= \sum_{i<j} A_{\mathrm{MHV}}(i,\ldots,j-1)\,[*,i-1,i,j-1,j]\,A_{\mathrm{MHV}}(j,\ldots,i-1)$$



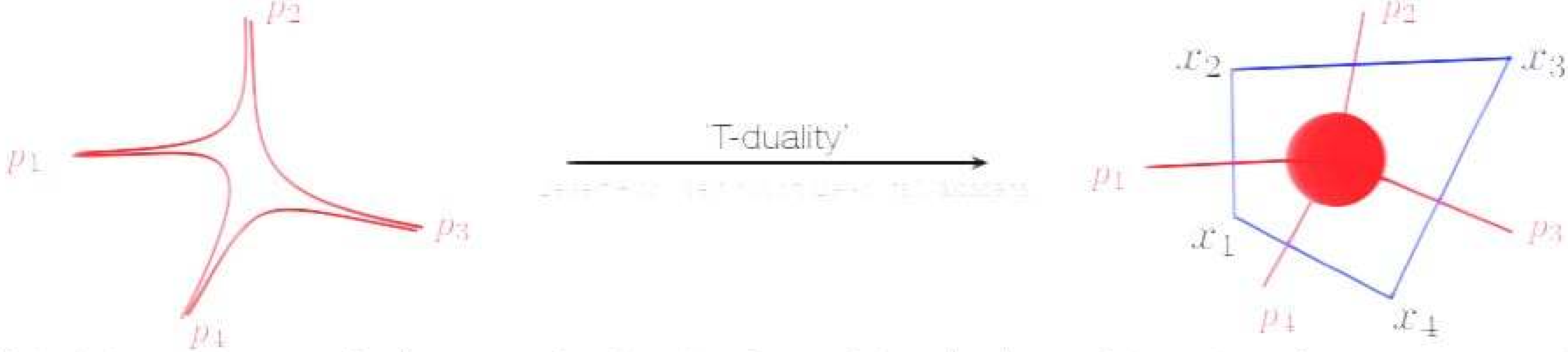
$\Longleftrightarrow$



Null Polygonal Wilson Loops

Null polygonal Wilson Loops in space-time

In 2007, Alday & Maldacena studied the strong-coupling limit of 4-particle scattering using AdS/CFT.



Calculation reduces to finding area of minimal surface with null polygonal boundary. Answer agrees with expectation from Bern-Dixon-Smirnov ansatz for all-orders scattering amplitude.

Inspired by this, Drummond, Henn, Korchemsky & Sokatchev studied NPWLs at weak coupling, again finding agreement with the MHV scattering amplitude at 1-loop. Results subsequently extended to all 1-loop MHV amplitudes, and 2-loop MHV amplitudes by many authors, and to n particles at strong coupling.

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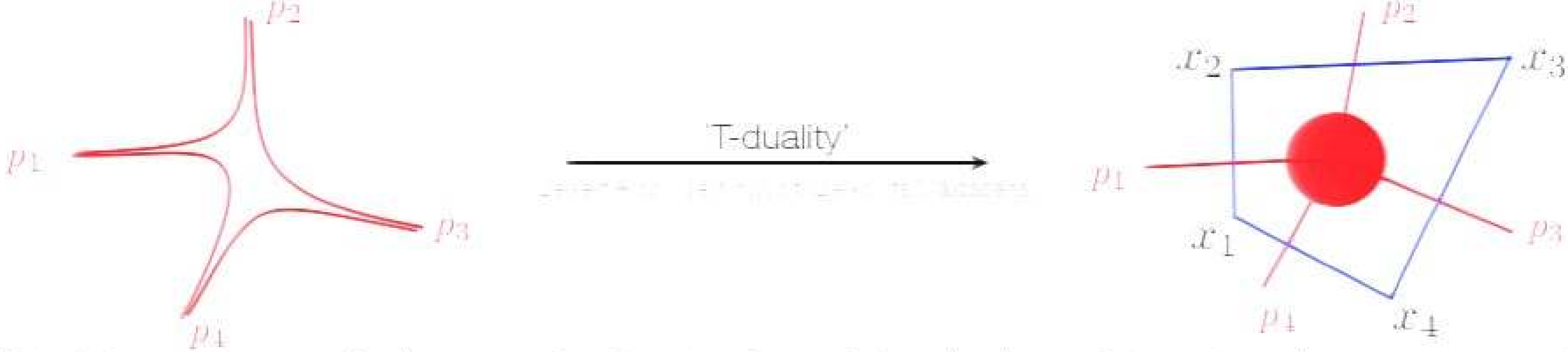

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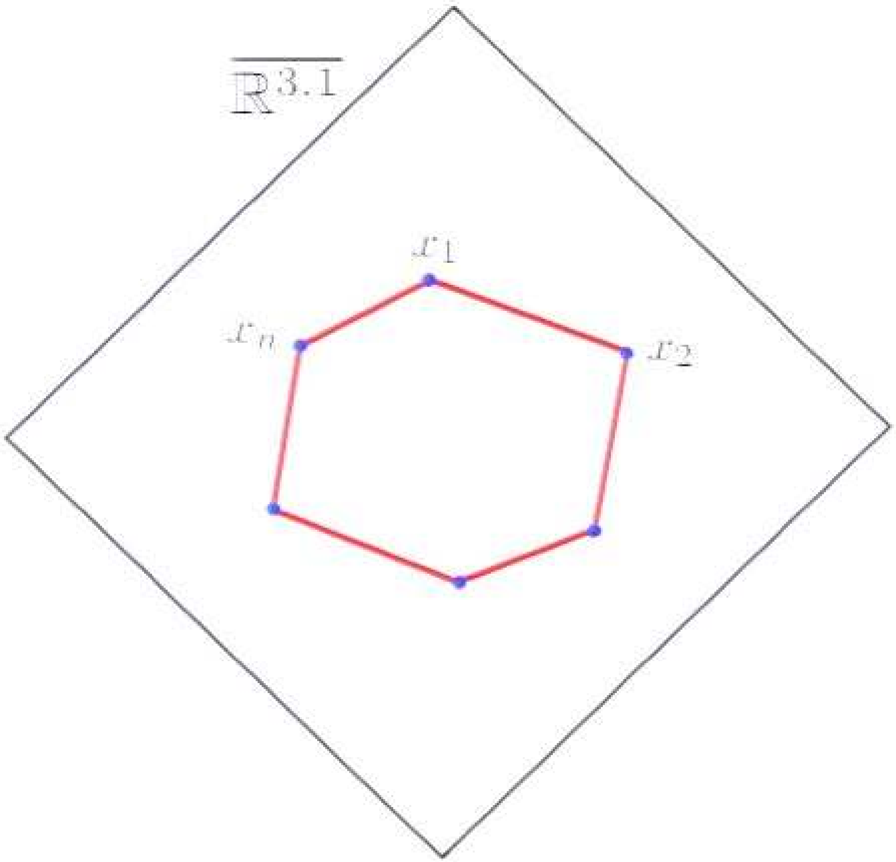
These case-by-case calculations raise two important questions:

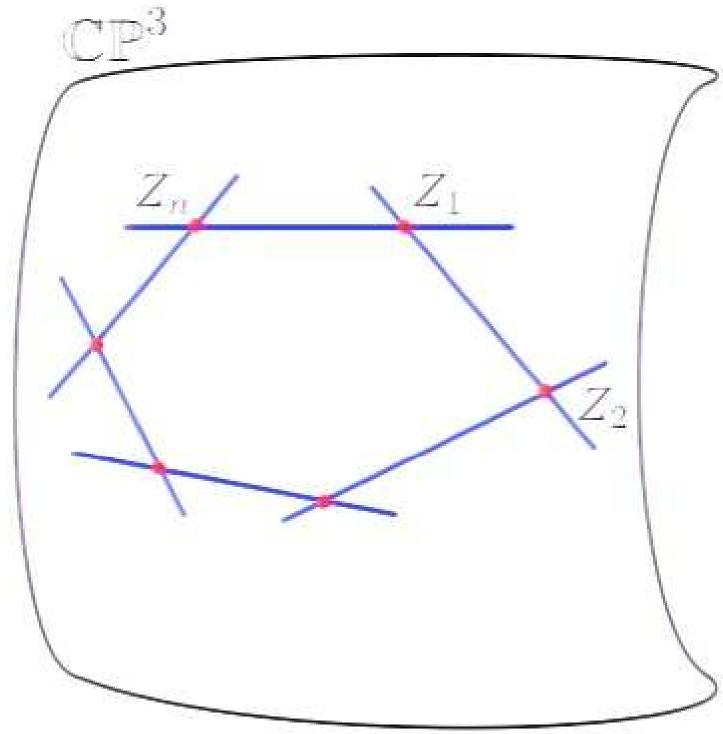
1) Can one extend the duality beyond the MHV sector?

2. Why does the duality work at all / Does it continue to hold to all orders?

Null polygonal Wilson Loops in twistor space

- The null edges of the polygon correspond to points Z_i in twistor space, while the vertices $\{x_1, x_2, \ldots, x_n\}$ of the space-time polygon correspond to the lines $\{\ Z_n Z_1\ ,\ Z_1 Z_2\ ,\ \ldots\ ,\ Z_{n-1} Z_n\ \}$ in twistor space.





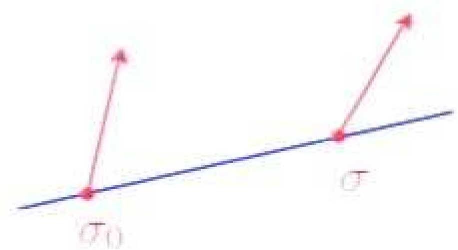
- The twistor data is **unconstrained**: given *arbitrary* Z_i , the twistor lines intersect by construction, so the corresponding space-time vertices are inevitably null separated.

Holomorphic frames

Just like a real parallel propagator, there is a unique ***holomorphic frame*** $\mathrm{U}(\sigma,\sigma_0)$ that obeys

$(\bar{\partial} + \mathcal{A})_X \mathrm{U}(\sigma,\sigma_0) = 0$ with boundary condition $\mathrm{U}(\sigma_0,\sigma_0) = 1$

$\mathrm{U}(\sigma_1,\sigma_2)\mathrm{U}(\sigma_2,\sigma_3) = \mathrm{U}(\sigma_1,\sigma_3)$	***concatenation***
$\mathrm{U}(\sigma_2,\sigma_1) = \mathrm{U}(\sigma_1,\sigma_2)^{-1}$	***inverse***
$\mathrm{U}(\sigma_1,\sigma_2) \to g^{-1}(\sigma_1)\mathrm{U}(\sigma_1,\sigma_0)g(\sigma_0)$	***gauge transformation***



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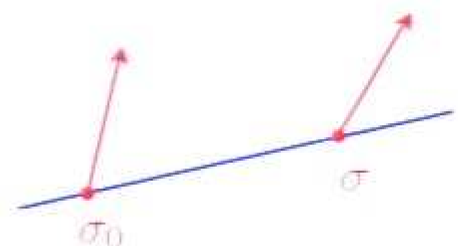

We can (formally) solve for $U(\sigma, \sigma_0)$ as

$$U(\sigma, \sigma_0) = \mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)$$

where ω is the Green's function for the $\bar\partial$-operator on X

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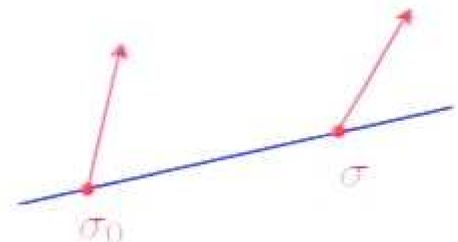

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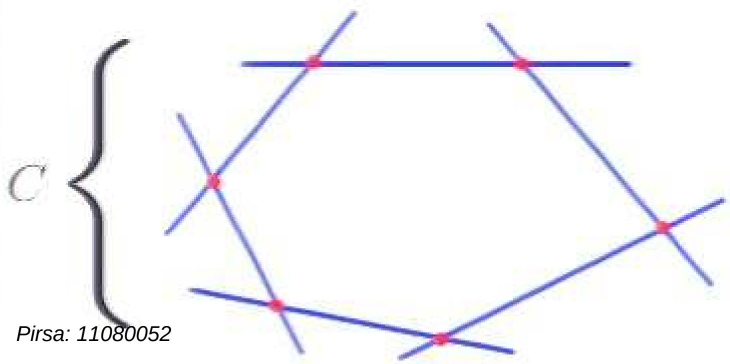

$$\langle \mathrm{W}[C] \rangle = \frac{1}{N}\left\langle \mathrm{Tr}\,\mathrm{P}\exp\left(-\int_C \omega \wedge \mathcal{A}\right)\right\rangle_{\mathcal{N}=4\ \mathrm{SYM}}$$

- Computes trace of holonomy of (super-)connection around C.
- Is the twistor field expression for space-time Wilson loop.
- Expectation value gives all **complete planar S-matrix**.

- Trees and Loops.
- $(L\ \text{loops})_{\text{amp}} = (L\ \text{loops} + k)_{\text{WL}}$ ← $= 4k$
- $A_n(p,\eta) = \dfrac{\delta^4(\Sigma p)\,\delta^{0|8}(\Sigma \lambda\eta)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\, W_n\left(x, \dfrac{\theta}{\alpha'^{1/2}}\right)$, $\epsilon = \dfrac{g^2 N}{4\pi}$.
- $\left\{ q_{\alpha A} + \alpha'^{1/2} \dfrac{\partial}{\partial \theta^{\alpha A}} \right\} W_n\left(x, \dfrac{\theta}{\alpha'^{1/2}}\right) = 0$
- Trees are dual to SDYM WL ⇒ $q=0$ ⟺ Yangian Symmetry
- $(x_i - x_{i+1})^2 = 0$
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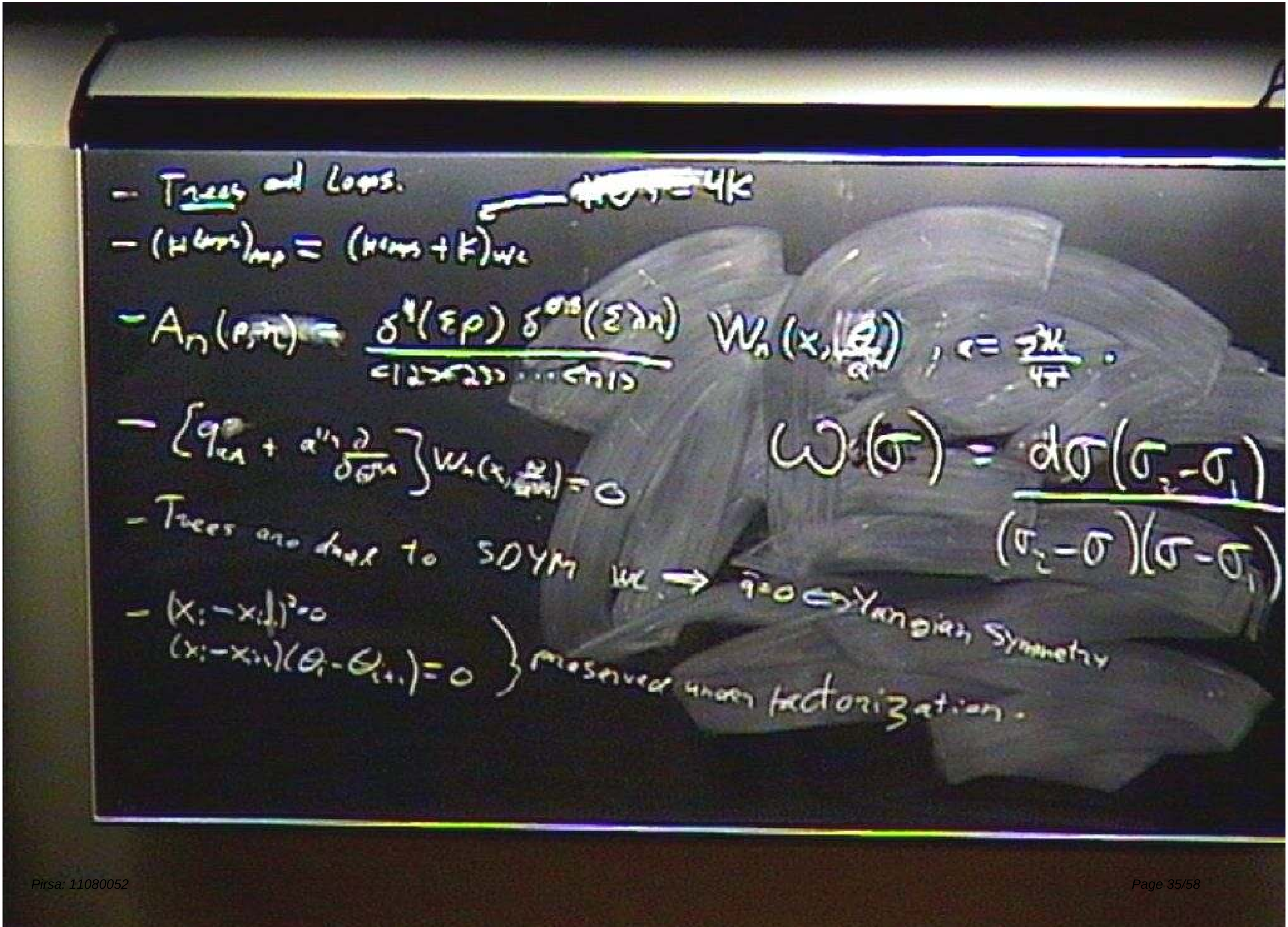
$$\omega(\sigma) = \frac{d\sigma\,(\sigma_2-\sigma_1)}{(\sigma_2-\sigma)(\sigma-\sigma_1)}$$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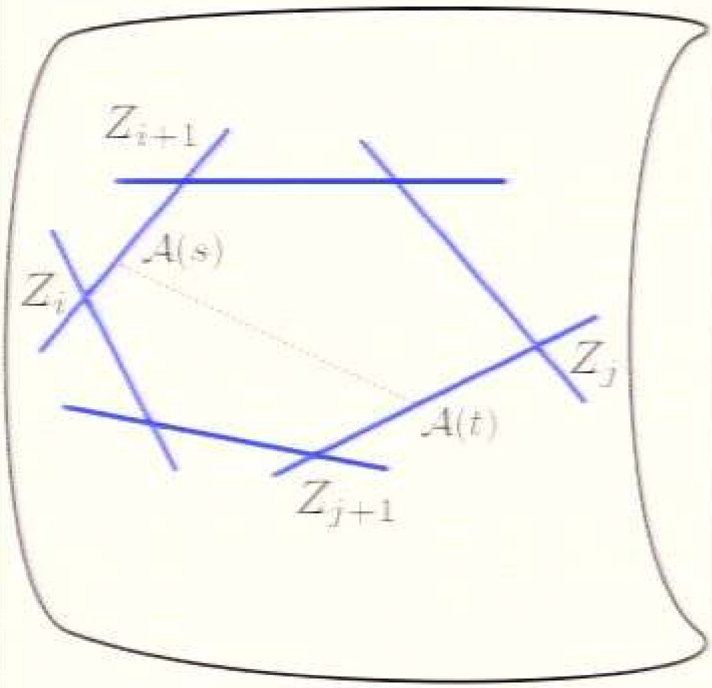
Explicit examples - tree amplitudes

The coupling constant appears only in front of the MHV vertices, so to lowest order (g^0) the Wilson Loop correlator is computed purely using holomorphic Chern-Simons theory.



- At lowest order in the field, the Wilson Loop correlator becomes

$$\langle \mathrm{W}[C] \rangle = 1 + \frac{1}{2} \sum_{i,j} \int \frac{\mathrm{d}s}{s} \frac{\mathrm{d}t}{t} \langle \mathcal{A}(s) \mathcal{A}(t) \rangle + \cdots$$

$$= 1 + \frac{1}{2} \sum_{i,j} [*, i, i+1, j, j+1] + \cdots$$

giving the NMHV tree amplitude (divided by an overall factor of the MHV tree).

- Diagrams between adjacent edges require care[Belitsky, Korchemsky, Sokatchev]. They may be handled by *framing* the Wilson Loop, whereupon they vanish in limit the framing is taken to zero.

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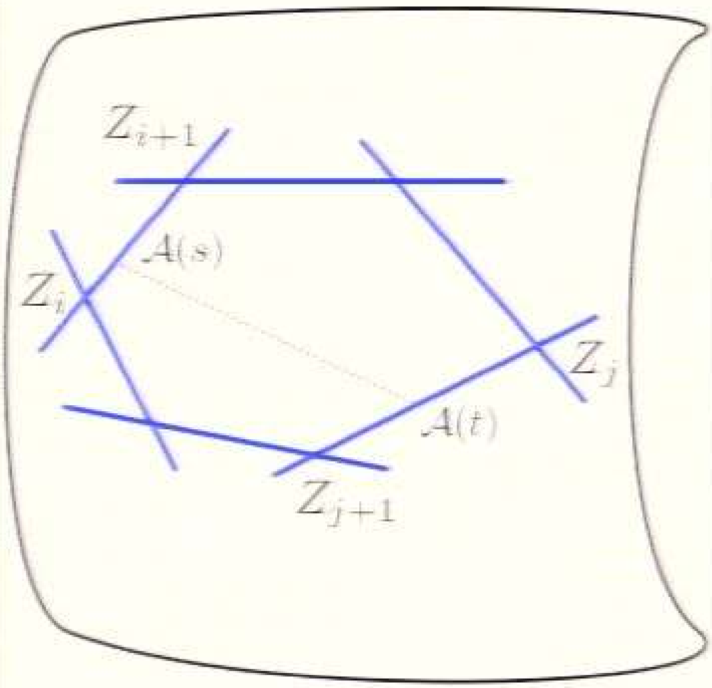

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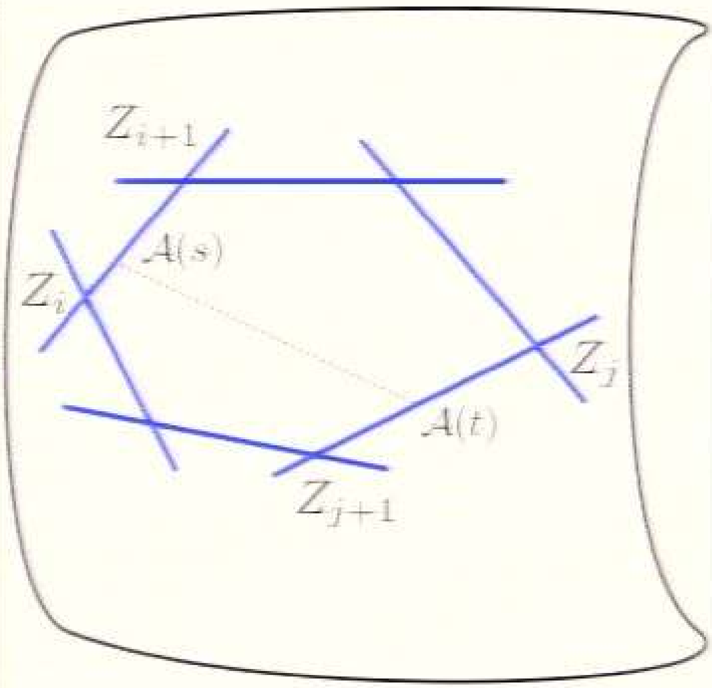

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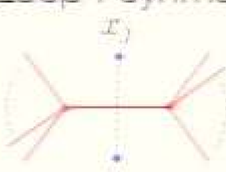
In axial gauge, Wilson Loop Feynman diagrams are the planar duals of MHV diagrams for the amplitude.

$$\langle \mathrm{WL} \rangle = 1 + \sum_{i<j-1}$$



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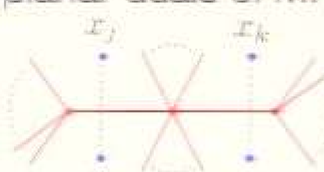
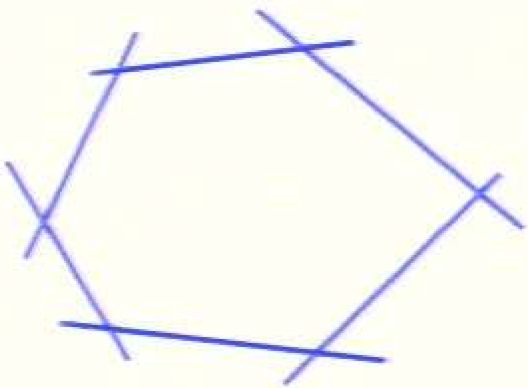


$$+ \sum_{i<j-1<l-2}$$

$$+ \cdots + \mathcal{O}(g^2)$$

Explicit examples - the loop integrand

All dependence on the coupling constant comes from insertions of $g^2 \int d^{4|8}x \log\det(\bar{\partial} + \mathcal{A})_X$ from the action, representing an infinite series of MHV vertices



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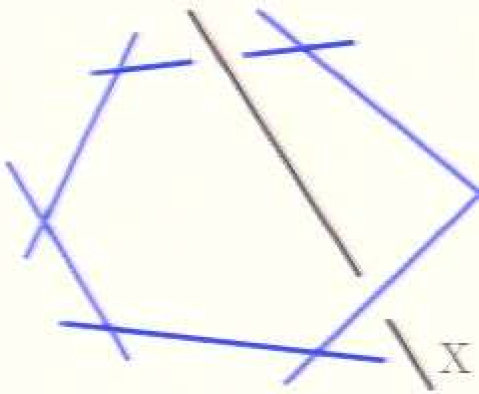

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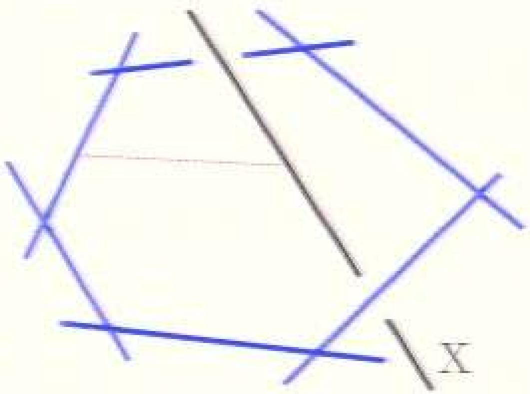

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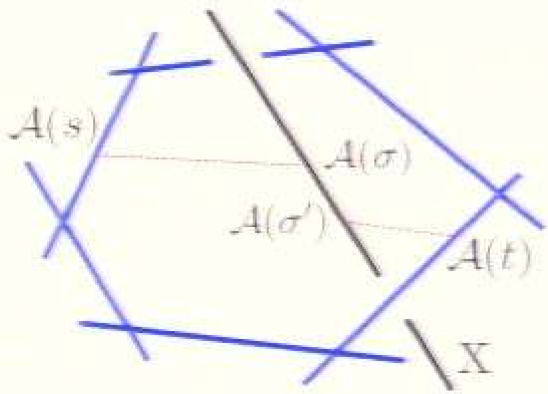

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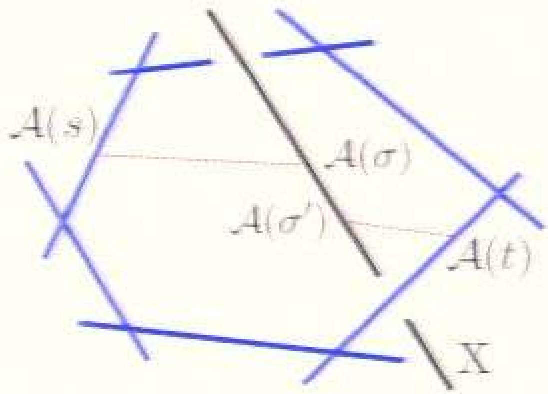

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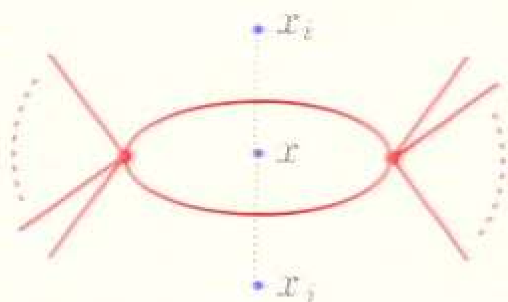

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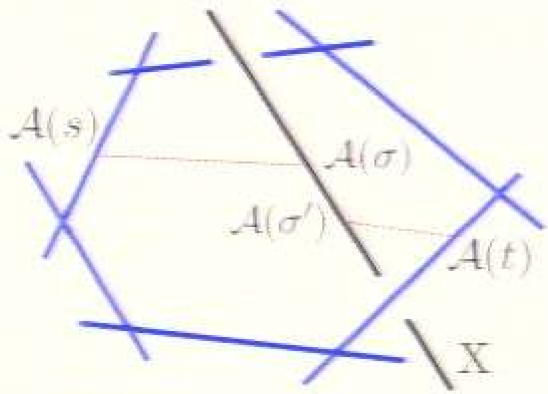

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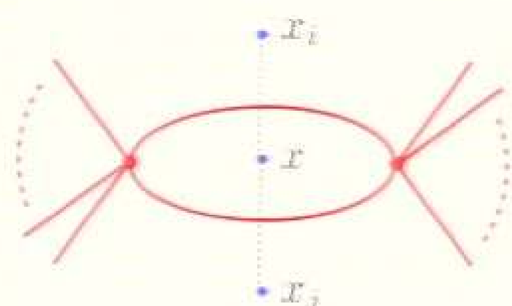


We explicitly checked all NMHV and N²MHV trees, MHV and NMHV 1-loop and MHV 2-loop integrands

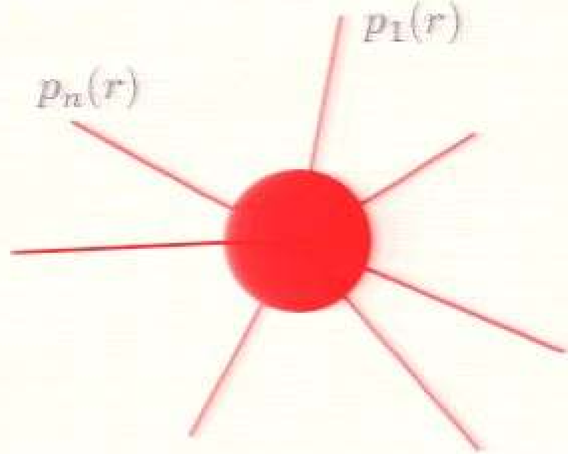
BCFW recursion and Wilson Loops

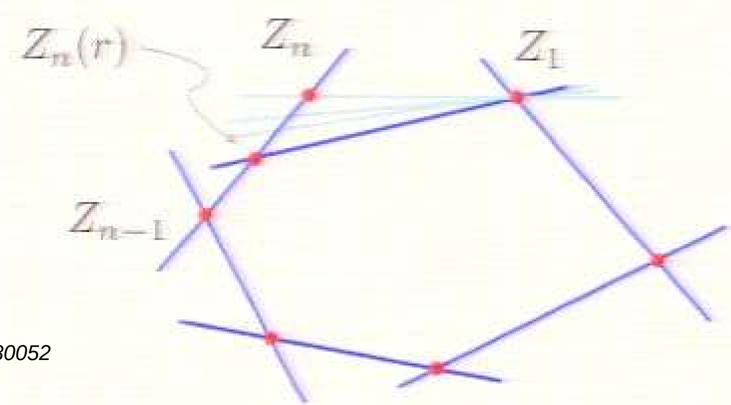
These examples provide some evidence that the duality between **Scattering Amplitudes and Wilson Loops** continues to hold for **the full superamplitude** and to **all orders in the loop expansion**.

To understand *why* it holds, we showed that the twistor Wilson Loop obeys the extension of the BCFW recursion relation for the all-loop integrand.

- In scattering amplitudes, BCFW recursion starts by deforming the external momenta $p_i \to p_i(r)$ subject to the **constraints**

$$\sum_i p_i(r) = 0 \qquad p_i^2(r) = 0$$





- These constraints mean that the deformed space-time Wilson Loop remains a closed polygon with null (but generically complex) edges.

- In twistor space, **there are no constraints!** We just vary the locations of the vertices Z_i arbitrarily.

Varying the Wilson Loop

Recall that under a smooth change in the curve, a real Wilson Loop obeys

$$\delta W[\gamma] = -\int_\gamma \mathrm{d}x^\mu \wedge \delta x^\nu \,\mathrm{Tr}\left[F_{\mu\nu}(x)\,\mathrm{P}\exp\left(-\int_x^x A\right)\right]$$

saying (*e.g.*) that the Wilson Loop is unchanged if γ varies only in a region where the field-strength vanishes.

$$\bar{\delta}\mathrm{W}[C] = -\int_C \omega \wedge \mathrm{d}\bar{Z}^{\bar{a}} \wedge \bar{\delta}\bar{Z}^{\bar{b}}\,\mathrm{Tr}\left[\mathcal{F}_{\bar{a}\bar{b}}(Z)\,\mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)\right]$$

$\mathcal{F}^{0,2} = 0$ $\mathrm{W}[C]$

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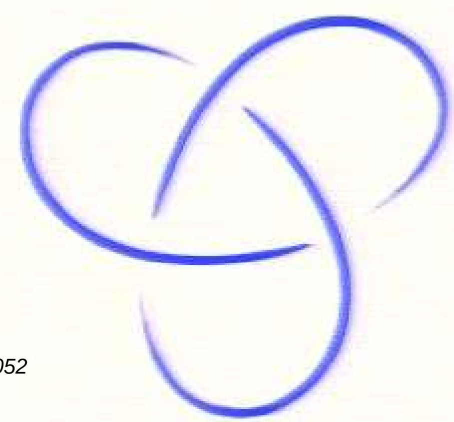
The Migdal-Makeenko loop equations tell us how the correlation function, rather than the operator, behaves.



- In real Chern-Simons, $\langle \mathrm{W}[\gamma] \rangle$ computes knot invariants such as the HOMFLY polynomial[Witten].
- Migdal-Makeenko equations give (poor man's) derivation of the **skein relations** - *i.e.* recursion relations for the knot polynomial[Cotta-Rasmusino, Guadagnini, Martellini, Mintchev].
- From the point of view of Wilson Loops, tree-level BCFW recursion is a holomorphic analogue of these skein relations, while the amplitude itself is a **holomorphic linking invariant** of the twistor curve

BCFW recursion as the loop equations

Beginning in the pure holomorphic Chern-Simons theory, integration by parts in the path integral gives

$$\langle \bar\delta \mathrm{W}[C(r)] \rangle = -\int_C \omega \wedge \left\langle \mathrm{Tr}\, \mathcal{F}^{(0,2)}(Z)\, \mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)\right\rangle_{\mathrm{hCS}}$$
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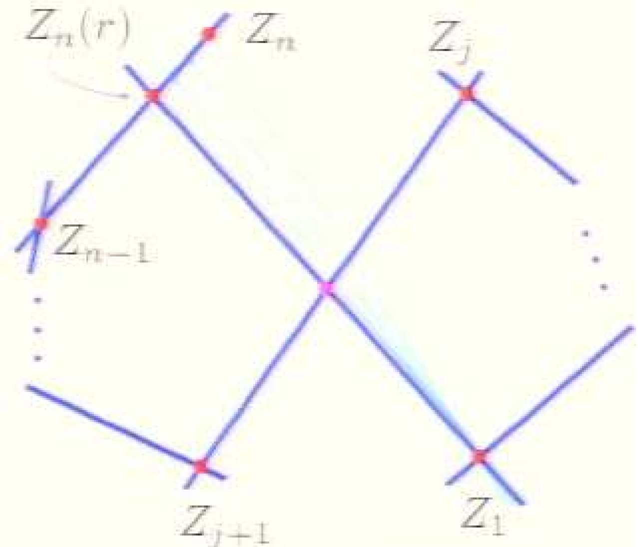
$$= \int_{C\times C} \omega \wedge \omega' \wedge \bar\delta^{3|4}(Z, Z')\, \langle \mathrm{W}[C']\, \mathrm{W}[C''] \rangle_{\mathrm{hCS}}$$



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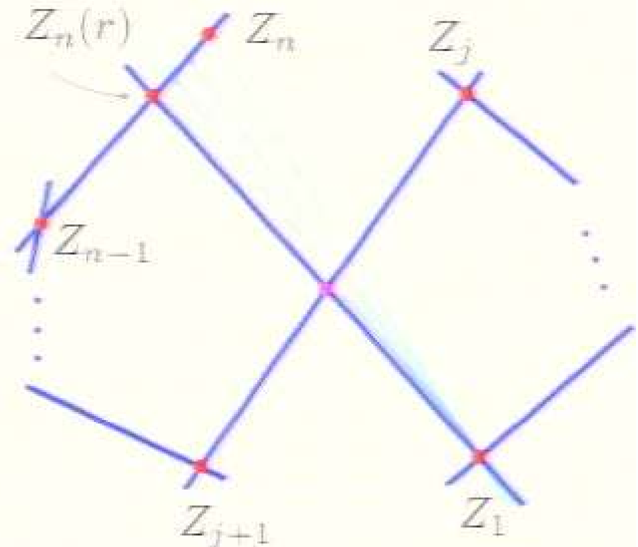
for $U(N)$ theory

ensures only get contribution if $C(r)$ self-intersects as we deform



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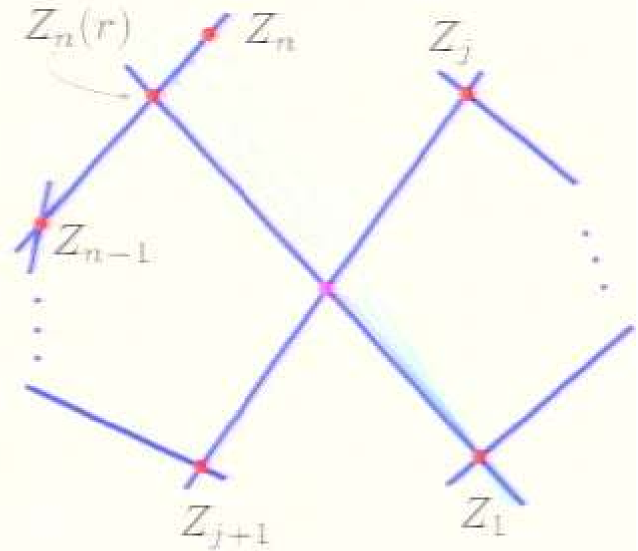

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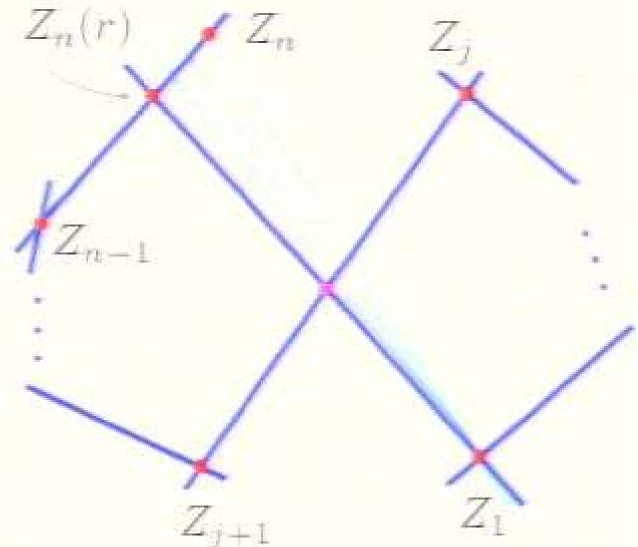
&= -\int_C \omega \wedge \left\langle \mathrm{Tr}\,\frac{\delta S_{\mathrm{hCS}}}{\delta \mathcal{A}(Z)}\,\mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)\right\rangle_{\mathrm{hCS}} \\
&= \int_{C\times C} \omega \wedge \omega' \wedge \bar{\delta}^{3|4}(Z,Z')\,\langle \mathrm{W}[C']\,\mathrm{W}[C''] \rangle_{\mathrm{hCS}} \\
&= \int_{C\times C} \omega \wedge \omega' \wedge \bar{\delta}^{3|4}(Z,Z')\,\langle \mathrm{W}[C'] \rangle\,\langle \mathrm{W}[C''] \rangle
\end{aligned}$$



When the line $(n_r 1)$ intersects $(j\, j+1)$ we have $(x_1(r) - x_j)^2 = 0$, corresponding to a factorization channel of the amplitude.

BCFW recursion as the loop equations

Beginning in the pure holomorphic Chern-Simons theory, integration by parts in the path integral gives

$$\langle \bar{\delta} \mathrm{W}[C(r)] \rangle = -\int_C \omega \wedge \left\langle \mathrm{Tr}\, \mathcal{F}^{(0,2)}(Z)\, \mathrm{P}\exp\left(-\int \omega \wedge \mathcal{A}\right)\right\rangle_{\mathrm{hCS}}$$
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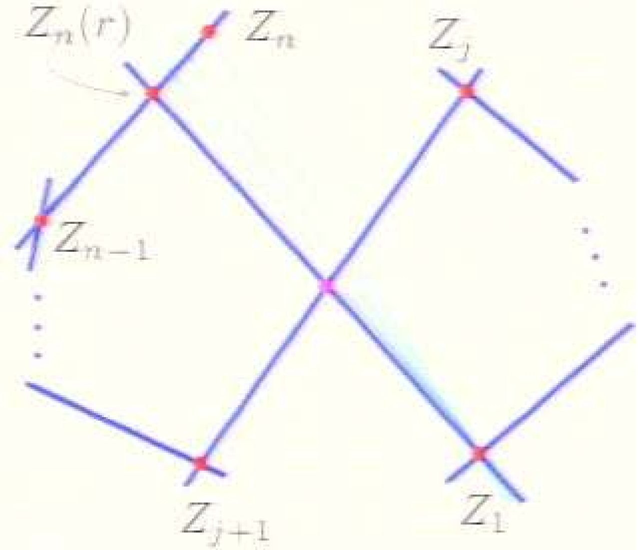

When the line $(n_r 1)$ intersects $(j\, j+1)$ we have $(x_1(r) - x_j)^2 = 0$, corresponding to a factorization channel of the amplitude.

$$\langle \mathrm{W}[1,\ldots,n] \rangle = \langle \mathrm{W}[1,\ldots,n-1] \rangle + \sum_{j=2}^{n-2} [n-1, n, 1, j, j+1]\, \langle \mathrm{W}[1,\ldots,j,Z_I] \rangle \langle \mathrm{W}[Z_I, j+1, \ldots, n_r] \rangle$$

i.e., the **tree-level** BCFW recursion relation in momentum twistor space, summed over MHV degree.

BCFW recursion as the loop equations

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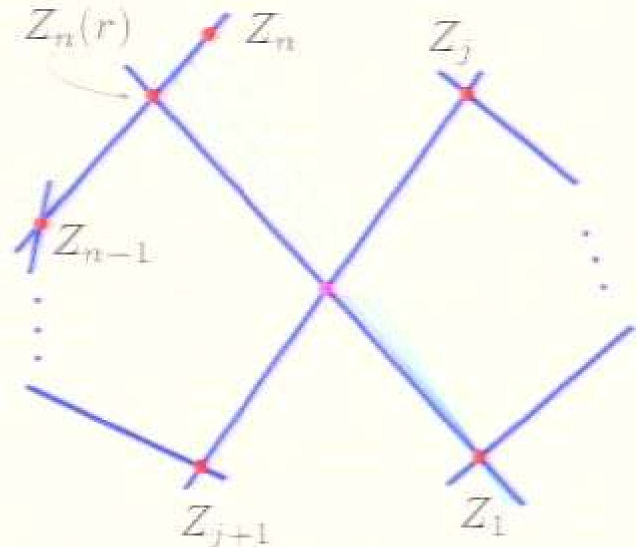
$$\begin{aligned}\langle \bar{\delta}\mathrm{W}[C(r)]\rangle &= -\int_C \omega\wedge\left\langle \mathrm{Tr}\,\mathcal{F}^{(0,2)}(Z)\,\mathrm{P}\exp\left(-\int\omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}}\\ &= -\int_C \omega\wedge\left\langle \mathrm{Tr}\,\frac{\delta S_{\mathrm{hCS}}}{\delta\mathcal{A}(Z)}\,\mathrm{P}\exp\left(-\int\omega\wedge\mathcal{A}\right)\right\rangle_{\mathrm{hCS}}\\ &= \int_{C\times C}\omega\wedge\omega'\wedge\bar{\delta}^{3|4}(Z,Z')\,\langle \mathrm{W}[C']\,\mathrm{W}[C'']\rangle_{\mathrm{hCS}}\\ &= \int_{C\times C}\omega\wedge\omega'\wedge\bar{\delta}^{3|4}(Z,Z')\,\langle \mathrm{W}[C']\rangle\,\langle \mathrm{W}[C'']\rangle\end{aligned}$$



When the line $(n_r 1)$ intersects $(j\,j{+}1)$ we have $(x_1(r)-x_j)^2=0$, corresponding to a factorization channel of the amplitude.

$$\langle \mathrm{W}[1,\ldots,n]\rangle = \langle \mathrm{W}[1,\ldots,n{-}1]\rangle + \sum_{j=2}^{n-2}[n{-}1,n,1,j,j{+}1]\,\langle \mathrm{W}[1,\ldots,j,Z_I]\rangle\,\langle \mathrm{W}[Z_I,j{+}1,\ldots,n_r]\rangle$$

i.e., the **tree-level** BCFW recursion relation in momentum twistor space, summed over MHV degree.

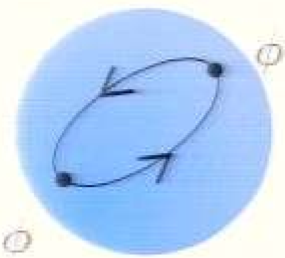
Repeating this derivation for the full action (including the "log det" term) gives a new contribution to the loop equations that leads[Bullimore, DS] to the **all-loop extension** of BCFW[Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka].

Correlation functions of local operators

Local operators in twistor space

Local operators on space-time are non-local on twistor space:

$$\text{e.g.}\quad \mathrm{Tr}\,\phi^2(x) = \int_{X\times X} \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\mathrm{Tr}\left(\Phi(\sigma)\mathrm{U}(\sigma,\sigma')\Phi(\sigma')\mathrm{U}(\sigma',\sigma)\right)$$



Once again, using the twistor superfield immediately yields the (chiral part) of the corresponding supermultiplet

$$\text{e.g.}\quad \mathcal{O}_{\mathrm{K}}(x,\theta,\bar\theta=0) = \int_{X\times X} \mathrm{d}\sigma\,\mathrm{d}\sigma'\,\mathrm{Tr}\left(\frac{\partial^2\mathcal{A}}{\partial\psi^a\partial\psi^b}\mathrm{U}(\sigma,\sigma')\frac{\partial^2\mathcal{A}}{\partial\psi^c\partial\psi^d}\mathrm{U}(\sigma',\sigma)\right)\epsilon^{abcd}$$

There is also [Alday, Eden, Heslop, Korchemsky, Maldacena, Sokatchev] a correspondence between scattering amplitudes and correlation functions of local operators in the null-separated limit.

Using twistors we [Adamo, Bullimore, Mason, DS] extended this to the full superamplitude / correlator of chiral supermultiplets at the level of the integrand.

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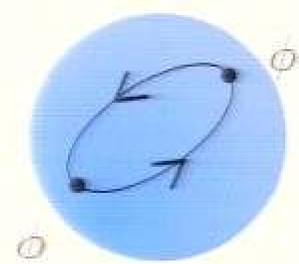

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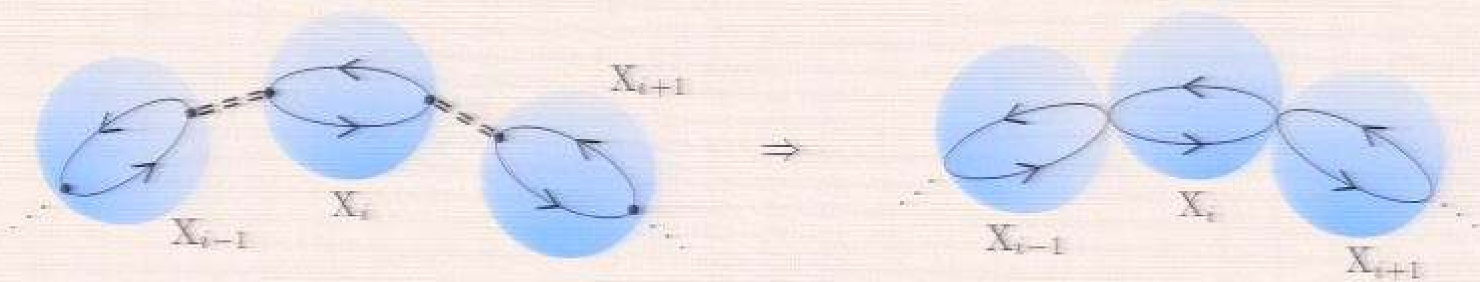
There is also[Alday, Eden, Heslop, Korchemsky, Maldacena, Sokatchev] a correspondence between scattering amplitudes and correlation functions of local operators in the null-separated limit.

Correlators on the null cone and null Wilson Lines are closely related in any QFT. In twistor space, in the limit that the twistor lines intersect, factoring out a singular piece from the integrand leaves us with a Wilson Loop in the **adjoint**. In the planar limit this becomes the **square** of the scattering amplitude.



Using twistors we[Adamo, Bullimore, Mason, DS] extended this to the full superamplitude / correlator of chiral supermultiplets at the level of the integrand.

Conclusions

We constructed a natural holomorphic & supersymmetric Wilson Loop in twistor space and saw that its correlator using the twistor action for $\mathcal{N}=4$ SYM computes the ratio of the all-loop S-matrix to the MHV tree.

- **Perturbative computation in axial gauge $\Leftrightarrow$ MHV / CSW diagrams.**
- **Migdal-Makeenko Loop equations $\Leftrightarrow$ BCFW recursion for the all-loop integrand.**

Despite much progress over the past few years, understanding QFT from the point of view of twistors is still very much in its infancy.

Nonetheless, I hope I've convinced you that twistors provide both insightful perspectives and useful tools for studying some interesting problems in QFT.