Title: Hidden symmetry of correlation functions and amplitudes in N=4 SYM

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Abstract: We study the four-point correlation function of stress-tensor supermultiplets in N=4 SYM using the method of Lagrangian insertions. We argue that, as a corollary of N=4 superconformal symmetry, the resulting all-loop integrand possesses an unexpected complete symmetry under the exchange of the four external and all the internal (integration) points. This alone allows us to predict the integrand of the three-loop correlation function up to four undetermined constants. Further, exploiting the conjectured amplitude/correlation function duality, we are able to fully determine the three-loop integrand in the planar limit. We perform an independent check of this result by verifying that it is consistent with the operator product expansion, in particular that it correctly reproduces the three-loop anomalous dimension of the Konishi operator. As a byproduct of our study, we also obtain the three-point function of two half-BPS operators and one Konishi operator at three-loop level. We use the same technique to work out a compact form for the four-loop four-point integrand and discuss the generalisation to higher loops.

Hidden symmetry of correlation functions and amplitudes in $\mathcal{N}=4$ SYM

Gregory Korchemsky
IPhT, Saclay

Work in collaboration with

Burkhard Eden, Paul Heslop, Emery Sokatchev

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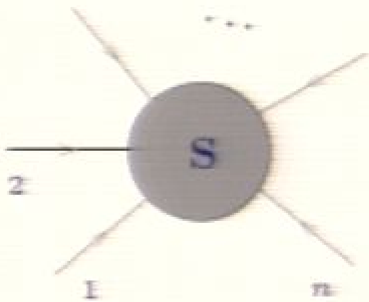
Gluon amplitudes in $\mathcal{N}=4$ SYM

- On-shell matrix elements of $S-$matrix:

$\mathcal{A}_n$ =



- Quantum numbers of scattered gluons:

Color:	$a_i = 1, \ldots, N_c^2 - 1$
Light-like momenta:	$(p_i^\mu)^2 = 0$
Polarization state (helicity):	$h_i = \pm 1$

- Color-ordered planar gluon amplitudes:

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\ldots T^{a_n}\right] A_n^{h_1,h_2,\ldots,h_n}(p_1,p_2,\ldots,p_n) + \text{[Bose symmetry]}$$

- Four-gluon amplitude in $\mathcal{N}=4$ SYM at weak coupling $a = g^2 N_c/(8\pi^2)$

$$A_4^{++--}/A_4^{(\mathrm{tree})} = 1 + a\, st I^{(1)}(s,t) + O(a^2)\,,$$

Scalar box in the dimensional regularization (for IR divergences) with $D = 4 - 2\epsilon$

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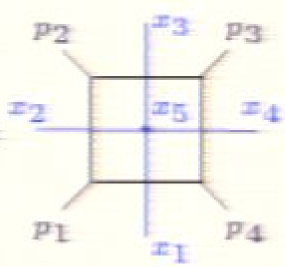

$$\sim \int \frac{d^D x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}\,, \qquad (x_{12}^2 = x_{23}^2 = x_{34}^2 = x_{41}^2 = 0)$$

Dual variables $p_i = x_i - x_{i+1}$ with $p_i^2 = x_{i,i+1}^2 = 0$

Correlation functions

- Protected superconformal operators made from six real scalars Φ^I

$$\mathcal{O}(x) = \mathrm{Tr}(ZZ)\,, \qquad \bar{\mathcal{O}}(x) = \mathrm{Tr}(\bar{Z}\bar{Z})\,, \qquad Z = \Phi^1 + i\Phi^2$$

All-loop scaling dimension = tree level dimension

Two- and three-point correlation functions do not receive quantum corrections

- Simplest correlation function

$$G_4 = \langle \mathcal{O}(x_1)\bar{\mathcal{O}}(x_2)\mathcal{O}(x_3)\bar{\mathcal{O}}(x_4)\rangle = G_4^{(0)}\left[1 + 2a\, x_{13}^2 x_{24}^2 g(1,2,3,4) + O(a^2)\right]$$

One-loop 'cross' integral

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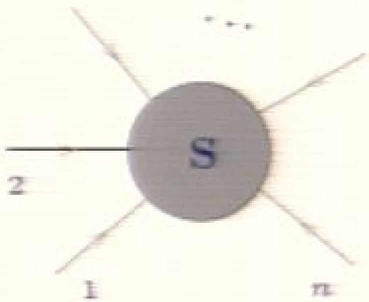
$A_n =$ 

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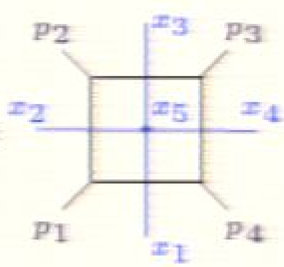
$I^{(1)}(s,t) =$ 

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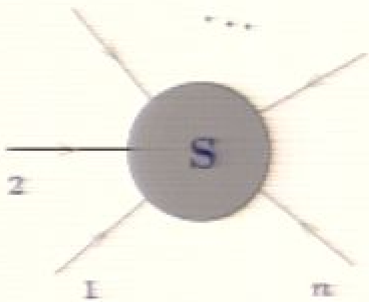
$A_n =$ 

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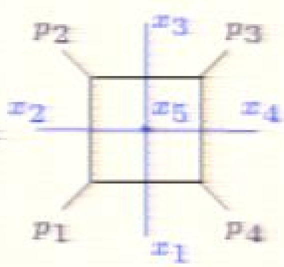
$I^{(1)}(s,t) =$ 

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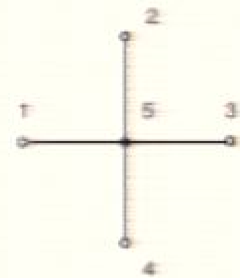
Hint for a new symmetry

Examine one-loop correction to the correlator

$$G_4^{(1)}/G_4^{(0)} \sim \int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \quad =$$



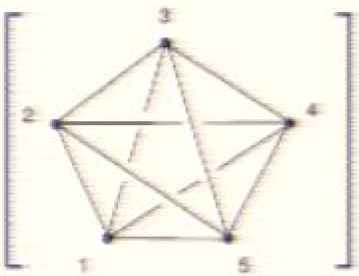
The corresponding integrand for $G_4/G_4^{(0)}$

$$[G_4^{(1)}/G_4^{(0)}]_{\text{Integrand}} \sim \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

The r.h.s. has S_4 permutation symmetry w.r.t. exchange of the external points $1,2,3,4$

Equivalent form of writing

$$G_4^{(1)}/G_4^{(0)}]_{\text{Integrand}} \sim x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times \left[\prod_{i<j} \frac{1}{x_{ij}^2}\right] = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \times$$



The second factor in the r.h.s. has the complete S_5 permutation symmetry!

Looks like triviality ... but

- The integrand of *all-loop* correlator has the complete permutation symmetry exchanging the external $1,2,3,4$ and internal (Integration) points
- Combined with the amplitude/correlator duality, it allows us to reconstruct the integrand of the four-point correlation function to *all loops*

$\mathcal{N}=4$ stress-tensor supermultiplet

Object of study: four-point correlator of half-BPS operators made of the six scalars,

$$\mathcal{O}^{IJ}_{\mathbf{20'}} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \frac{1}{6}\delta^{IJ}\,\mathrm{tr}\left(\Phi^K \Phi^K\right)$$

To simplify $SO(6)$ structure, project indices with 6-dim (complex) null vector, $y^2 \equiv y_I y_I = 0$

$$\mathcal{O}(x,y) = y_I\, y_J\, \mathcal{O}^{IJ}_{\mathbf{20'}}(x) = y_I\, y_J\ \mathrm{tr}\left[\Phi^I \Phi^J(x)\right]$$

The lowest-weight state of the $\mathcal{N}=4$ stress-tensor (chiral) supermultiplet

$$\mathcal{T}(x,\rho,y) = \exp\left(\rho^a_\alpha Q^\alpha_a\right)\mathcal{O}(x,y) = \mathcal{O}(x,y) + \ldots + (\rho)^4 \mathcal{L}_{\mathcal{N}=4}(x)$$

- Top component is the (on-shell) chiral Lagrangian of $\mathcal{N}=4$ SYM
- Auxiliary $y-$variables covariantly break the R symmetry $SU(4) \mapsto SU(2)\times SU(2)' \times U(1)$
- $\mathcal{T}(x,\rho,y)$ depends on half of the odd variables $\theta^A_\alpha \to \rho^a_\alpha$ with $A=(a,a')$
- Is annihilated by the other half of $\mathcal{N}=4$ supercharges $Q_{a'}$

The on-shell action of the $\mathcal{N}=4$ theory

$$S_{\mathcal{N}=4} = \int d^4x \int d^4\rho\, \mathcal{T}(x,\rho,y)$$

Method of Lagrangian insertions

Four-point correlator (with $a = g^2 N_c/(4\pi)^2$)

$$G_4 = \langle \mathcal{O}(x_1, y_1) \dots \mathcal{O}(x_4, y_4) \rangle = \sum_{l=0}^{\infty} a^l \, G_4^{(\ell)}(1,2,3,4)$$

The tree-level expression (with $y_{12}^2 = (y_1 - y_2)^2$)

$$G_4^{(0)}(1,2,3,4) \quad = \quad \frac{N_c^4}{4\,(4\pi^2)^4} \frac{y_{12}^4}{x_{12}^4} \frac{y_{34}^4}{x_{34}^4} + \frac{N_c^2}{(4\pi^2)^4} \; \frac{y_{12}^2}{x_{12}^2} \frac{y_{23}^2}{x_{23}^2} \frac{y_{34}^2}{x_{34}^2} \frac{y_{41}^2}{x_{41}^2} + \text{(2 permutations)}$$

Compute loop corrections using the method of Lagrangian insertions:

$$a\frac{\partial}{\partial a} G_4 = a \frac{\partial}{\partial a} \int D\Phi \, e^{-\frac{1}{a} S_{\mathcal{N}=4}[\Phi]} \, \mathcal{O}(x_1, y_1) \dots \mathcal{O}(x_4, y_4)$$
$$= \int d^4 x_5 \, \langle \mathcal{O}(x_1, y_1) \dots \mathcal{O}(x_4, y_4) \mathcal{L}_{\mathcal{N}=4}(x_5) \rangle$$

The 1-loop correction to G_4 is determined by the *tree-level* 5-point correlation function with insertions of the $\mathcal{N} = 4$ SYM action

The ℓ–loop correction to $G_4 \Rightarrow$ (integrated) tree-level correlation function with ℓ insertions of $\mathcal{L}_{\mathcal{N}=4}$

$$G_4^{(\ell)}(1,2,3,4) = \int d^4 x_5 \dots \int d^4 x_{4+\ell} \, \langle \mathcal{O}(x_1, y_1) \dots \mathcal{O}(x_4, y_4) \mathcal{L}(x_5) \dots \mathcal{L}(x_{4+\ell}) \rangle^{(0)}$$

The correlators in both sides are particular components of *supercorrelator* $G_n = \langle \mathcal{T}(1) \dots \mathcal{T}(n) \rangle$

Method of Lagrangian insertions II

Supercorrelators (have the exact $Q-$ and $\bar{S}-$superconformal symmetry)

$$G_4^{(\ell)} = \langle \mathcal{T}(1)...\mathcal{T}(4)\rangle^{(\ell)} = \langle \mathcal{O}(x_1,y_1)...\mathcal{O}(x_4,y_4)\rangle^{(\ell)}$$

$$G_{4+\ell}^{(0)} = \langle \mathcal{T}(1)...\mathcal{T}(4+\ell)\rangle^{(0)} = \langle \mathcal{O}(x_1,y_1)...\mathcal{O}(x_4,y_4)\mathcal{L}(x_5)...\mathcal{L}(x_{4+l})\rangle^{(0)}(\rho_5)^4...(\rho_{4+l})^4 + ...$$

The $\ell-$loop correction to 4-point correlator $\Longrightarrow$ the tree-level $(4+\ell)-$point correlator

$$G_4^{(\ell)}(1,2,3,4) = \int d^4x_5...d^4x_{4+\ell}\left(\int d^4\rho_5 \dots d^4\rho_{4+\ell}\, G_{4+\ell}^{(0)}(1,\dots,4+\ell)\right)$$

The *integrand* of the loop corrections to the four-point correlation function

$$\left[G_4^{(\ell)}(1,2,3,4)\right]_{\text{Integrand}} = \int d^4\rho_5 \dots d^4\rho_{4+\ell}\, G_{4+\ell}^{(0)}(1,...,4,5...,4+\ell)$$

Ultimate goal is determine the integrands of $G_4^{(\ell)}$ without any Feynman graph calculations!

This is possible because

- ✓ $G_{4+\ell}^{(0)}$ reveals a new permutation $S_{4+\ell}$ symmetry involving all the $(4+\ell)$ points
- ✓ The correlation function/amplitude duality leads to powerful restriction on the form of $G_{4+\ell}^{(0)}$
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Integrands to one- and two-loops

The known results for 5- and 6-point tree-level correlators of Grassmann degree 4 and 8:

$$G_5^{(0)}(1,2,3,4,5) \quad = \frac{N_c^2}{(4\pi^2)^5} \times \mathcal{I}_5 \times \frac{1}{\prod_{1\leq i<j\leq 5} x_{ij}^2} + O(\rho^0)$$

$$G_6^{(0)}(1,2,3,4,5,6) = \frac{N_c^2}{(4\,\pi^2)^6} \times \mathcal{I}_6 \times \frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2} x^2_{\sigma_3\sigma_4} x^2_{\sigma_5\sigma_6}}{\prod_{1\leq i<j\leq 6} x_{ij}^2} + O(\rho^4)$$

The sum runs over all permutations $(\sigma_1,\ldots,\sigma_6)$ of the indices $(1,2,\ldots,6)$

The dependence on the Grassmann variables is hidden in $\mathcal{I}_n$ (with $n=5,6$)

- $\mathcal{I}_n$ is a nilpotent (chiral) superconformally covariant function of n superspace points (x_i,ρ_i,y_i)
- $\mathcal{I}_n$ is a homogenous polynomial in the Grassmann variables $\rho_1,\ldots,\rho_n$ of degree $4(n-4)$

$$\mathcal{I}_n = (x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2) \times R(1,2,3,4) \times (\rho_5)^4 \ldots (\rho_n)^4 + \ldots .$$

R is a rational S_4-symmetric function of the bosonic coordinates of $1,2,3,4$:

$$R(1,2,3,4) = \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} + \text{(2 permutations)}$$

- The complete expression for $\mathcal{I}_n(1,\ldots,n)$ is invariant under exchange of any two points
- These properties fix $\mathcal{I}_n$ up to an arbitrary conformally covariant factor depending on x_i only

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$$\mathcal{I}_n = (x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2) \times R(1,2,3,4) \times (\rho_5)^4 \dots (\rho_n)^4 + \dots .$$

R is a rational S_4−symmetric function of the bosonic coordinates of $1,2,3,4$:

$$R(1,2,3,4) = \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2}(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2) + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} + \text{(2 permutations)}$$

- The complete expression for $\mathcal{I}_n(1,\dots,n)$ is invariant under exchange of any two points
- These properties fix $\mathcal{I}_n$ up to an arbitrary conformally covariant factor depending on x_i only

All-loop integrand

General expression for $G^{(0)}_{4+\ell}$ of Grassmann degree 4ℓ

$$G^{(0)}_{4+\ell}(1,\dots,4+\ell) = \frac{N_c^2}{(4\pi^2)^4} \times \mathcal{I}_{4+\ell} \times f^{(\ell)}(x_1,\dots,x_{4+\ell}) + O(\rho^{4(\ell-1)})$$

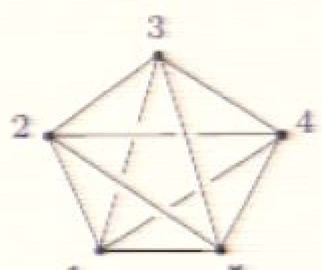
The full crossing symmetry of the correlator + the permutation symmetry of $\mathcal{I}_{4+\ell}$ lead to:

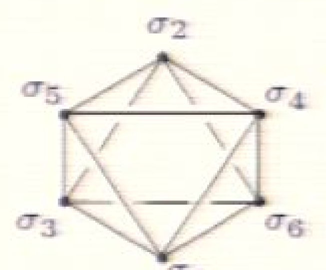
The function $f^{(\ell)}$ must be completely symmetric under the exchange of any of $4+\ell$ points:

$$f^{(\ell)}(\dots,x_i,\dots,x_j,\dots) = f^{(\ell)}(\dots,x_j,\dots,x_i,\dots)$$

- This symmetry is specific to the $\mathcal{N}=4$ theory (is not present in $\mathcal{N}=2$ SYM)
- Exchanges external $(1,2,3,4)$ and internal $(5,\dots,4+\ell)$ points and puts the two sets of points on an equal footing
- Explicit expessions for $\ell=1$ and $\ell=2$

$$f^{(1)}(x_1,\dots,x_5) = \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}\,, \qquad f^{(2)}(x_1,\dots,x_6) = \frac{1}{48}\sum_{\sigma\in S_6} \frac{x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 4+l} x_{ij}^2}$$





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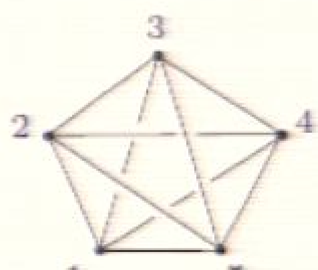
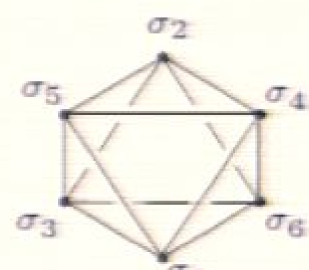




All-loop integrand II

- Loop corrections to the 4-point correlator

$$G_4^{(\ell)}(1,2,3,4) = \frac{N_c^2}{(4\pi^2)^4} \times R(1,2,3,4) \times F^{(\ell)},$$

with the function $F^{(\ell)}$ given by the integral

$$F^{(\ell)}(x_1,x_2,x_3,x_4) = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 \dots d^4x_{4+\ell}\, f^{(\ell)}(x_1,\dots,x_{4+\ell}).$$

The integration breaks $S_{4+\ell}$ symmetry of $f^{(\ell)}$ down to S_4 symmetry for the function $F^{(\ell)}$.

- General form of $f^{(\ell)}$ for arbitrary ℓ:

$$f^{(\ell)}(x_1 \dots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1,\dots x_{4+\ell})}{\prod_{1\le i<j\le 4+\ell} x_{ij}^2},$$

Can be deduced from the operator product expansion (OPE) analysis of the tree-level correlator

The polynomial $P^{(\ell)}$ should satisfy the conditions:

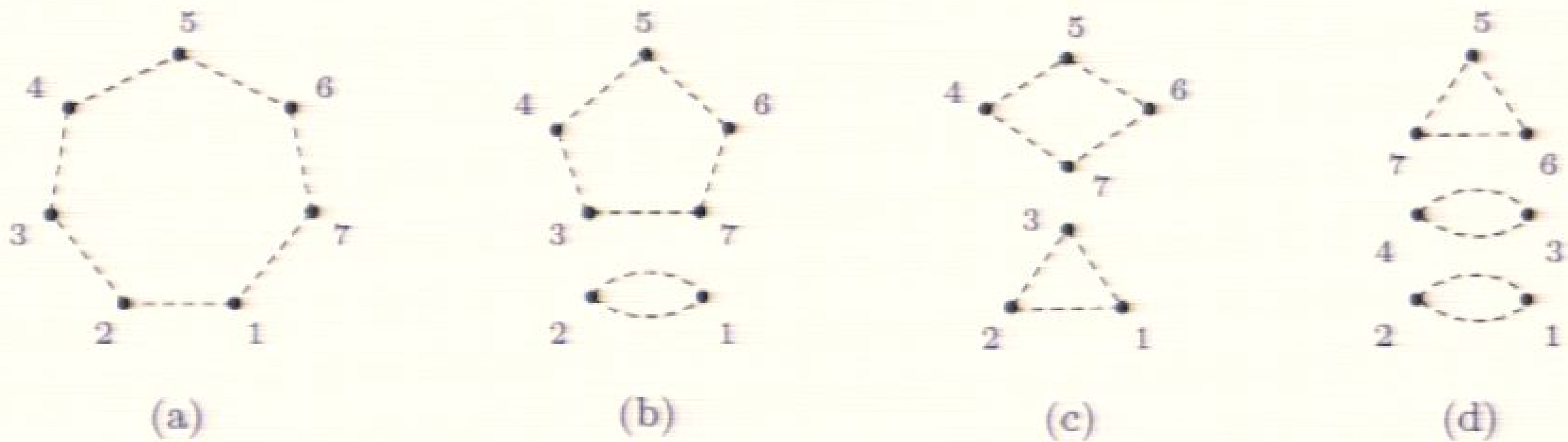
- be invariant under $S_{4+\ell}$ permutations of $x_1, \dots, x_{4+\ell}$
- have a uniform conformal weight $-(\ell-1)$ at each point, both external and internal.

Graph theory solution:

$$P^{(\ell)} \mapsto \textit{\textbf{Multi-graph with}}\ (4+\ell)\ \textit{\textbf{vertices of degree}}\ (\ell-1)$$

Three loops

$P^{(3)} \mapsto$ ***Multi-graph with* 7 *vertices of degree* 2**



There are only 4 independent possibilities for $P^{(3)}$:

(a) heptagon: $x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{71}^2 \;+\; S_7$ permutations

(b) 2-gon × pentagon: $(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \;+\; S_7$ permutations

(c) triangle × square: $(x_{12}^2 x_{23}^2 x_{31}^2)(x_{45}^2 x_{56}^2 x_{67}^2 x_{74}^2) \;+\; S_7$ permutations

(d) 2-gon × 2-gon × triangle: $(x_{12}^4)(x_{34}^4)(x_{56}^2 x_{67}^2 x_{75}^2) \;+\; S_7$ permutations .

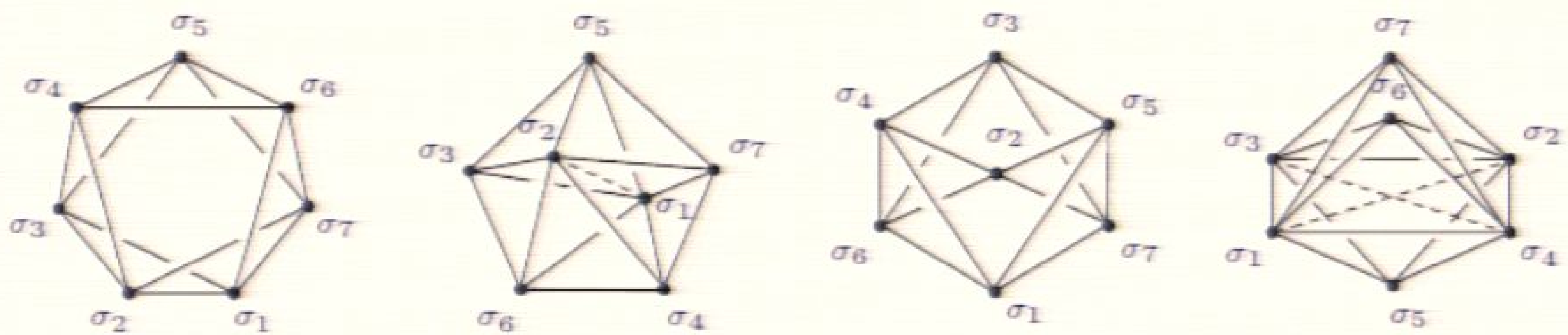
$P^{(3)}$ = **linear combination of four terms with *arbitrary* coefficients**

Three loops II

From polynomial to integrand

$$f^{(3)}(x_1,\dots,x_7) = \frac{P^{(3)}(x_1,\dots x_7)}{\prod_{1\le i<j\le 7} x_{ij}^2}$$

Four different contributions corresponding to four graphs for $f^{(3)}$:



(a) (b) (c) (d)

Solid lines denote $1/x^2_{\sigma_i\sigma_j}$, dashed lines stand for $x^2_{\sigma_i\sigma_j}$

$f^{(3)}$ = **linear combination of four diagrams with *arbitrary* coefficients**

This holds in $\mathcal{N}=4$ SYM for *arbitrary* gauge group $SU(N_c)$

To fix the value of four coefficients we have to impose additional conditions

In planar $\mathcal{N}=4$ SYM they follow from the correlation function/amplitude duality

Correlation function/Amplitude duality

$$\lim_{x^2_{i,i+1}\to 0}(G_4(x_i)/G_4^{(0)}(x_i)) = (A_4(p_i)/A_4^{(0)}(p_i))^2\,, \qquad p_i = x_i - x_{i+1}$$

s understood at the level of the *integrands* (and not in terms of the divergent *integrals*)!

$$\lim_{x^2_{i,i+1}\to 0}\left(1 + 2x_{13}^2 x_{24}^2 \sum_{\ell\geq 1} a^\ell F^{(\ell)}\right) = \left(1 + \sum_{\ell\geq 1} a^\ell M^{(\ell)}\right)^2 .$$

r equivalently

$$\lim_{x^2_{i,i+1}\to 0}(x_{13}^2 x_{24}^2 F^{(1)}) = M^{(1)}\,,$$

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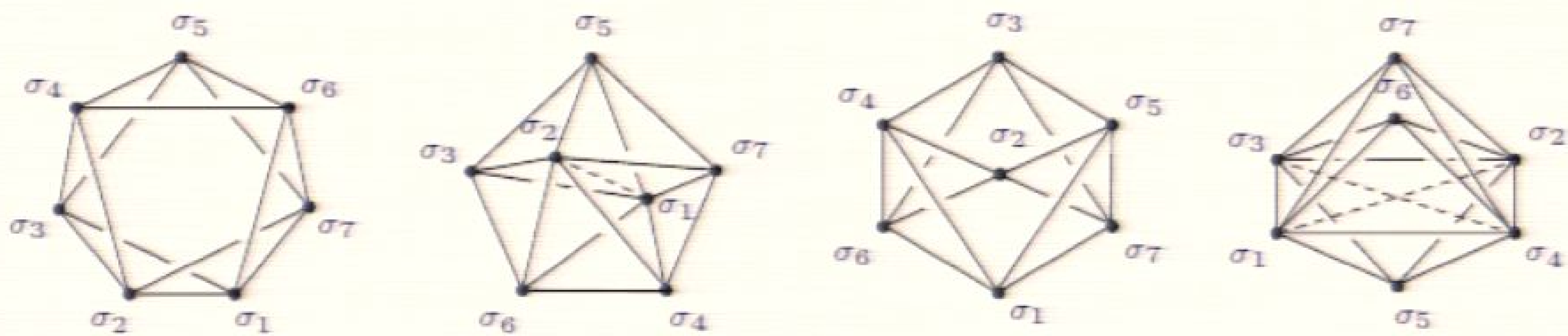

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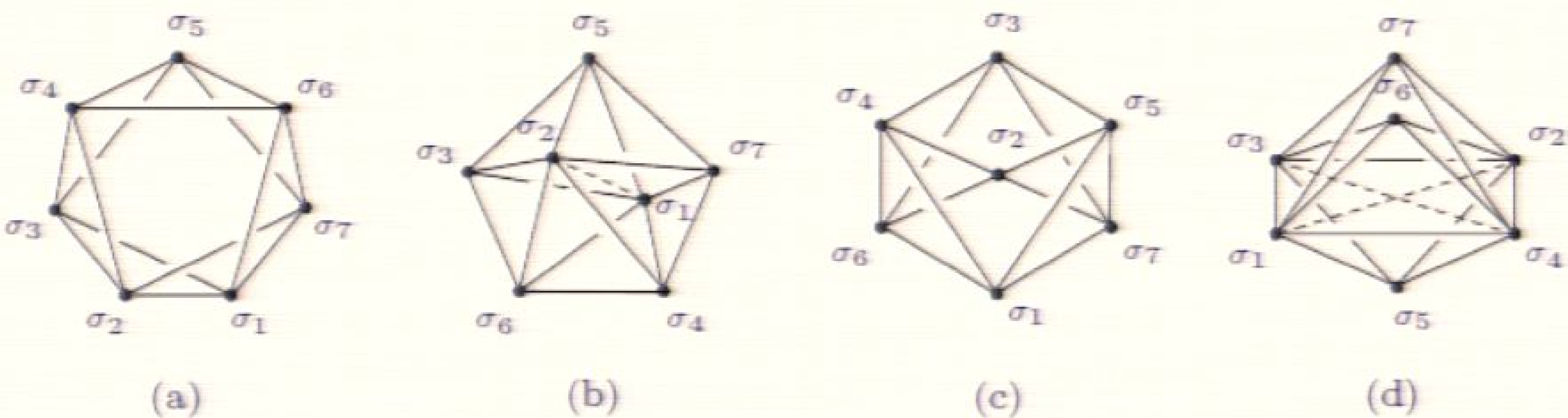

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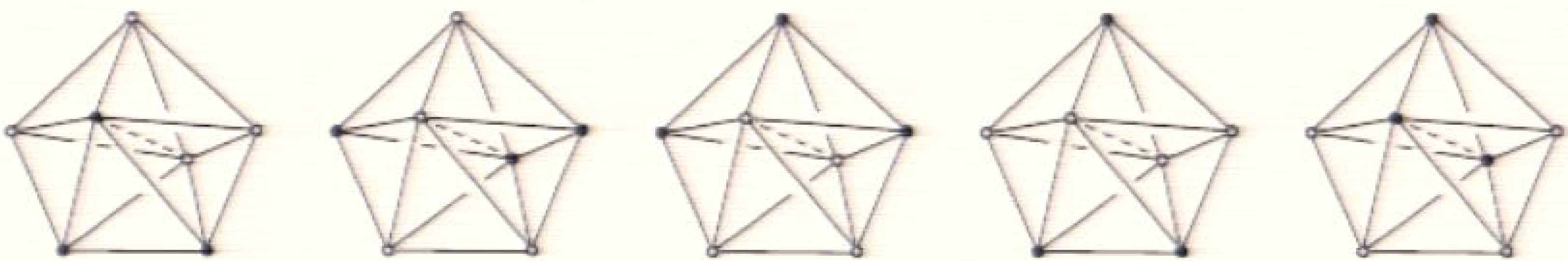
The duality leads to the selection rule for the topologies contributing to the correlator!

Correlation function/Amplitude duality II

At three loops, the amplitude/correlator duality selects only one topology

$$f^{(3)}(x_1,\ldots,x_7)=\frac{\frac{1}{20}\sum_{\sigma\in S_7}x^4_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_4\sigma_5}x^2_{\sigma_5\sigma_6}x^2_{\sigma_6\sigma_7}x^2_{\sigma_7\sigma_3}}{\prod_{1\le i<j\le 7}x^2_{ij}}$$

Diagrammatic representation for $f^{(3)}$



(a) (b) (c) (d) (e)

White nodes are the external points $x_1,\ldots,x_4$ and black nodes are the internal points x_5,x_6,x_7.

3-loop integrand for 4-point correlation function

$$[F^{(3)}]_{\text{integrand}}=x^2_{12}x^2_{13}x^2_{14}x^2_{23}x^2_{24}x^2_{34}\times f^{(3)}(x_1,\ldots,x_7)\,,$$

Diagrams for $F^{(3)}$ look simpler - the factors of x^2_{ij} remove some of the solid lines in the above diagrams

Four-point correlator at three loops

Our result for 4-point correlation function in planar $\mathcal{N}=4$ SYM

$$G_4(1,2,3,4) = G_4^{(0)} + \frac{N_c^2}{(4\pi^2)^4}\, R(1,2,3,4)\, \left[aF^{(1)} + a^2F^{(2)} + a^3F^{(3)} + O(a^4)\right]$$

$F^{(\ell)}$ are given by the sum of scalar Feynman integrals

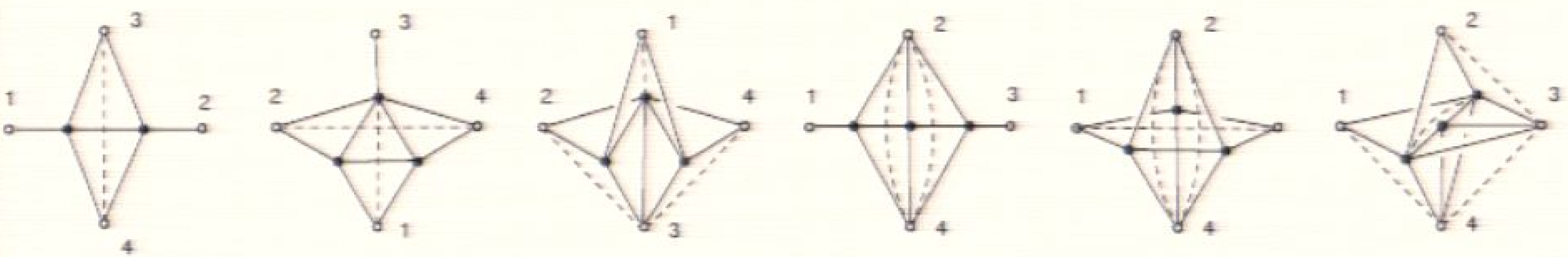
$$F^{(1)} = g(1,2,3,4)\,,$$

$$F^{(2)} = \left[h(1,2;3,4) + 5 \text{ perms}\right] + \tfrac{1}{2}\left[x_{12}^2 x_{34}^2 (g(1,2,3,4))^2 + 3 \text{ perms}\right],$$

$$F^{(3)} = \left[T(1,3;2,4) + 11 \text{ perms}\right] + \left[E(2;1,3;4) + 11 \text{ perms}\right] + \left[L(1,3;2,4) + 5 \text{ perms}\right]$$

$$+ \left[(g \times h)(1,3;2,4) + 5 \text{ perms}\right] + \tfrac{1}{2}\left[H(1,3;2,4) + 11 \text{ perms}\right],$$

2- and 3-loop topologies:



$h(1,2;3,4)$ $\quad T(1,3;2,4)$ $\quad E(1;2,4;3)$ $\quad L(1,3;2,4)$ $\quad g\times h(1,3;2,4)$ $\quad H(1,2;3,4)$

Verifying the amplitude/correlation duality

The proposed 3-loop $G_4(1,2,3,4)$ should reproduce the known result for the 3-loop amplitude A_4

Dramatic simplification in the light-like limit $x^2_{i,i+1} \to 0$:

$$\lim F^{(1)} = g(1,2,3,4)\,,$$

$$\lim F^{(2)} = h(1,3;2,4) + h(2,4;1,3) + \frac{1}{2} x^2_{13} x^2_{24} \left[g(1,2,3,4)\right]^2 ,$$

$$\lim F^{(3)} = T(1,3;2,4) + T(1,3;4,2) + T(2,4;1,3) + T(2,4;3,1) + L(1,3;2,4) + L(2,4;1,3) + (g\times h)(1,3;2,4) + (g\times h)(2,4;1,3)\,.$$

The integrals are divergent but we only need their integrands (= well-defined rational functions)

Compare with the analogous 3-loop expressions for 4-gluon amplitude

$$M^{(1)} = x^2_{13} x^2_{24}\, g(1,2,3,4)\,,$$

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Precise agreement with the amplitude/correlator duality

$$\lim(x^2_{13} x^2_{24} F^{(2)}) = M^{(2)} + \tfrac{1}{2}(M^{(1)})^2 , \qquad \lim\,(x^2_{13} x^2_{24} F^{(3)}) = M^{(3)} + M^{(1)} M^{(2)}$$

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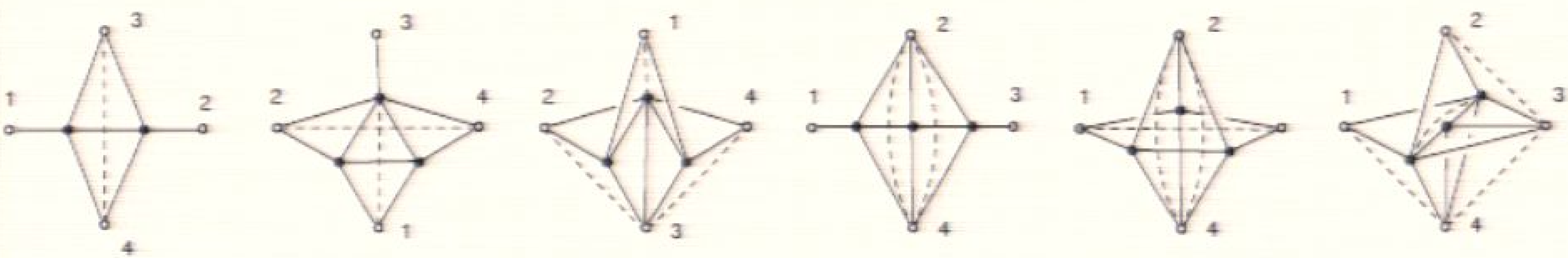

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The OPE test for 3-loop correlator

Examine $G_4(1,2,3,4)$ in the short-distance limit $x_1 \to x_2$, $x_3 \to x_4$ and apply the OPE

$$\mathcal{O}(x_1, y_1)\, \mathcal{O}(x_2, y_2) = c_{\mathcal{I}} \frac{y_{12}^4}{x_{12}^4}\, \mathcal{I} + c_{\mathcal{K}}(a) \frac{y_{12}^4}{(x_{12}^2)^{1-\frac{\gamma_{\mathcal{K}}}{2}}}\, \mathcal{K}(x_2) + c_{\mathcal{O}} \frac{y_{12}^2}{x_{12}^2} y_{1I} y_{2J}\, \mathcal{O}^{IJ}_{\mathbf{20'}}(x_2) + \dots ,$$

Konishi operator $\mathcal{K} = \mathrm{tr}[\Phi^I \Phi^I]$ is normalized as

$$\langle \mathcal{K}(x_2)\, \mathcal{K}(x_4) \rangle = \frac{d_{\mathcal{K}}}{(x_{24}^2)^{2+\gamma_{\mathcal{K}}(a)}} , \qquad c_{\mathcal{K}}^2(a) d_{\mathcal{K}} = \left(\frac{1}{3} + \sum_{\ell \geq 1} \alpha_\ell\, a^\ell \right) \frac{N_c^2}{(4\pi^2)^4}$$

The OPE prediction ($u = x_{12}^2 x_{34}^2 / (x_{13}^2 x_{24}^2)$ is a conformal cross-ratio)

$$G_4 = \frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4}\, c_{\mathcal{I}}^2 + \frac{y_{12}^2 y_{34}^2 (y_{13}^2 y_{24}^2 + y_{14}^2 y_{23}^2)}{x_{12}^2 x_{34}^2 x_{24}^4} \frac{c_{\mathcal{O}}^2 c_{\mathcal{I}}}{2} + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2 x_{24}^4} \left[d_{\mathcal{K}} c_{\mathcal{K}}^2(a)\, u^{\gamma_{\mathcal{K}}(a)/2} - \frac{1}{6} c_{\mathcal{O}}^2 c_{\mathcal{I}} \right] + \dots ,$$

Matching with the asymptotics of the obtained 3-loop result (for $u \to 0$)

$$x_{24}^4 \sum_{\ell \geq 1} a^\ell F^{(\ell)}(x) \sim \left(c_{\mathcal{K}}^2(a) d_{\mathcal{K}}\, u^{\gamma_{\mathcal{K}}(a)/2} - \frac{1}{6} c_{\mathcal{O}}^2 c_{\mathcal{I}} \right) + \dots ,$$

The OPE test for 3-loop correlator II

OPE prediction to three loops:

$$x_{24}^4 F^{(3)} = \tfrac{1}{288}\gamma_1^3 (\ln u)^3 + \left(\tfrac{1}{24}\gamma_1\gamma_2 + \tfrac{1}{16}\gamma_1^2\alpha_1\right)(\ln u)^2 + \left(\tfrac{1}{12}\gamma_3 + \tfrac{1}{4}\gamma_2\alpha_1 + \tfrac{1}{4}\gamma_1\alpha_2\right)\ln u + \tfrac{1}{2}\alpha_3 + O(u)$$

Our result for 3-loop correlator (for $u \to 0$)

$$x_{24}^4 F^{(3)} = \frac{3}{32}(\ln u)^3 - \frac{15}{16}(\ln u)^2 + \left(\frac{9}{8}\zeta(3) + \frac{61}{16}\right)\ln u - \left(\frac{25}{8}\zeta(5) + \zeta(3) + \frac{21}{8}\right) + O(u)$$

✓ Correct structure of $\ln u$ terms

✓ Reproduce the known result for 3-loop Konishi anomalous dimension $\gamma_1 = 3,\ \gamma_2 = -3,\ \gamma_3 = \frac{21}{4}$

✓ Predict 3-loop normalization coefficients

$$\alpha_1 = -1, \quad \alpha_2 = \frac{3}{2}\zeta(3) + \frac{7}{2}, \quad \alpha_3 = -\left(\frac{25}{4}\zeta(5) + 2\zeta(3) + \frac{21}{4}\right)$$

✓ An interesting byproduct: 3-point correlation function

$$\langle \mathcal{O}(x_1, y_1)\mathcal{O}(x_2, y_2)\mathcal{K}(x_3)\rangle = \frac{C(a)\,(Y_1 \cdot Y_2)^2}{(x_{12}^2)^{1-\gamma_{\mathcal{K}}/2}(x_{13}^2)^{1+\gamma_{\mathcal{K}}/2}(x_{23}^2)^{1+\gamma_{\mathcal{K}}/2}}$$

3-loop prediction for the normalization constant $C(a)$

$$C(a) = \left(1 + 3\sum \alpha_\ell\, a^\ell\right)^{1/2} \frac{N_c^2}{(4\pi^2)^3}$$

The OPE test for 3-loop correlator

Examine $G_4(1,2,3,4)$ in the short-distance limit $x_1 \to x_2$, $x_3 \to x_4$ and apply the OPE

$$\mathcal{O}(x_1,y_1)\,\mathcal{O}(x_2,y_2) = c_{\mathcal{I}}\frac{y_{12}^4}{x_{12}^4}\,\mathcal{I} + c_{\mathcal{K}}(a)\frac{y_{12}^4}{(x_{12}^2)^{1-\frac{\gamma_{\mathcal{K}}}{2}}}\,\mathcal{K}(x_2) + c_{\mathcal{O}}\frac{y_{12}^2}{x_{12}^2}y_{1I}y_{2J}\,\mathcal{O}^{IJ}_{\mathbf{20'}}(x_2) + \dots\,,$$

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Conclusions

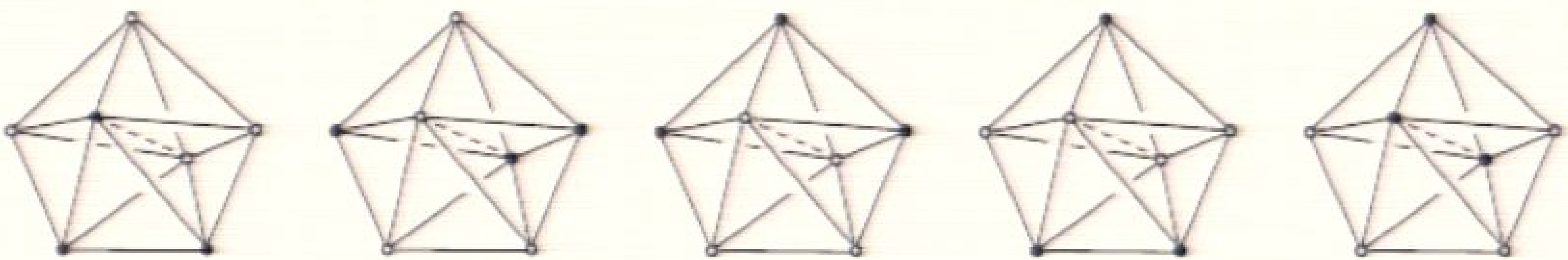
- The all-loop integrand of 4-point correlator $\langle \mathcal{T}(1) \dots \mathcal{T}(4) \rangle$ possesses an unexpected complete symmetry under the exchange of the four external and all internal (integration) points
- This alone allows us to predict the integrand of the three-loop correlation function up to four undetermined constants (for arbitrary gauge group!)
- The conjectured amplitude/correlation function duality fixes the constants and fully determines the three-loop correlator in the planar limit
- The obtained result for the three-loop correlator is consistent with the operator product expansion
- As a byproduct, it predicts the three-point function of two half-BPS operators and one Konishi operator at three-loop level
- Straightforward generalization to higher loops, towards all-loop result for the 4-point correlator

Correlation function/Amplitude duality II

At three loops, the amplitude/correlator duality selects only one topology

$$f^{(3)}(x_1,\ldots,x_7) = \frac{\frac{1}{20}\sum_{\sigma\in S_7} x^4_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_4\sigma_5}x^2_{\sigma_5\sigma_6}x^2_{\sigma_6\sigma_7}x^2_{\sigma_7\sigma_3}}{\prod_{1\le i<j\le 7} x^2_{ij}}$$

Diagrammatic representation for $f^{(3)}$



(a) (b) (c) (d) (e)

White nodes are the external points $x_1,\ldots,x_4$ and black nodes are the internal points x_5, x_6, x_7.

3-loop integrand for 4-point correlation function

$$[F^{(3)}]_{\text{integrand}} = x^2_{12}x^2_{13}x^2_{14}x^2_{23}x^2_{24}x^2_{34} \times f^{(3)}(x_1,\ldots,x_7)\,,$$

Diagrams for $F^{(3)}$ look simpler - the factors of x^2_{ij} remove some of the solid lines in the above diagrams

All-loop integrand

General expression for $G^{(0)}_{4+\ell}$ of Grassmann degree 4ℓ

$$G^{(0)}_{4+\ell}(1,\ldots,4+\ell) = \frac{N_c^2}{(4\pi^2)^4} \times \mathcal{I}_{4+\ell} \times f^{(\ell)}(x_1,\ldots,x_{4+\ell}) + O(\rho^{4(\ell-1)})$$

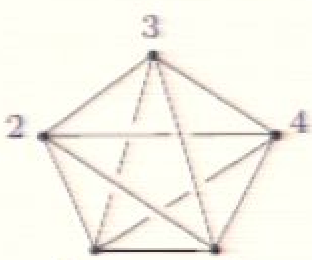
The full crossing symmetry of the correlator + the permutation symmetry of $\mathcal{I}_{4+\ell}$ lead to:

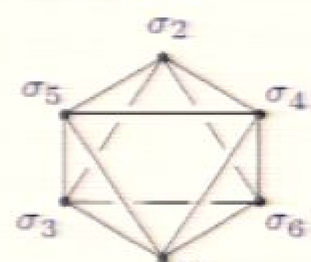
The function $f^{(\ell)}$ must be completely symmetric under the exchange of any of $4+\ell$ points:

$$f^{(\ell)}(\ldots,x_i,\ldots,x_j,\ldots) = f^{(\ell)}(\ldots,x_j,\ldots,x_i,\ldots)$$

- This symmetry is specific to the $\mathcal{N}=4$ theory (is not present in $\mathcal{N}=2$ SYM)
- Exchanges external $(1,2,3,4)$ and internal $(5,\ldots,4+\ell)$ points and puts the two sets of points on an equal footing
- Explicit expessions for $\ell=1$ and $\ell=2$

$$f^{(1)}(x_1,\ldots,x_5) = \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}, \qquad f^{(2)}(x_1,\ldots,x_6) = \frac{1}{48}\sum_{\sigma\in S_6} \frac{x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 4+l} x_{ij}^2}$$





Integrands to one- and two-loops

The known results for 5- and 6-point tree-level correlators of Grassmann degree 4 and 8:

$$G_5^{(0)}(1,2,3,4,5) \quad = \frac{N_c^2}{(4\pi^2)^5} \times \mathcal{I}_5 \times \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2} + O(\rho^0)$$

$$G_6^{(0)}(1,2,3,4,5,6) = \frac{N_c^2}{(4\pi^2)^6} \times \mathcal{I}_6 \times \frac{\frac{1}{48}\sum_{\sigma\in S_6} x^2_{\sigma_1\sigma_2}x^2_{\sigma_3\sigma_4}x^2_{\sigma_5\sigma_6}}{\prod_{1\le i<j\le 6} x_{ij}^2} + O(\rho^4)$$

The sum runs over all permutations $(\sigma_1,\dots,\sigma_6)$ of the indices $(1,2,\dots,6)$

The dependence on the Grassmann variables is hidden in $\mathcal{I}_n$ (with $n=5,6$)

- $\mathcal{I}_n$ is a nilpotent (chiral) superconformally covariant function of n superspace points (x_i,ρ_i,y_i)
- $\mathcal{I}_n$ is a homogenous polynomial in the Grassmann variables $\rho_1,\dots,\rho_n$ of degree $4(n-4)$

$$\mathcal{I}_n = (x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2) \times R(1,2,3,4) \times (\rho_5)^4 \dots (\rho_n)^4 + \dots .$$

R is a rational S_4-symmetric function of the bosonic coordinates of $1,2,3,4$:

$$R(1,2,3,4) = \frac{y_{12}^2y_{23}^2y_{34}^2y_{14}^2}{x_{12}^2x_{23}^2x_{34}^2x_{14}^2}(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) + \frac{y_{12}^4y_{34}^4}{x_{12}^2x_{34}^2} + \text{(2 permutations)}$$

- The complete expression for $\mathcal{I}_n(1,\dots,n)$ is invariant under exchange of any two points
- These properties fix $\mathcal{I}_n$ up to an arbitrary conformally covariant factor depending on x_i only

Three loops

$$P^{(3)} \mapsto \textbf{\textit{Multi-graph with }} 7 \textbf{\textit{ vertices of degree }} 2$$

$$G_4^{[illegible]}(1,2,3,4) = \frac{[illegible]}{(4\pi^2)^4} \times R(1,2,3,4) \times F^{[illegible]},$$

with the function $F^{(\ell)}$ given by the integral

$$F^{(\ell)}(x_1, x_2, x_3, x_4) = x_{12}^2 x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^2 \int d^4x_5 \ldots d^4x_{4+\ell}\, f^{(\ell)}(x_1, \ldots, x_{4+\ell}) \,.$$

The integration breaks $S_{4+\ell}$ symmetry of $f^{(\ell)}$ down to S_4 symmetry for the function $F^{(\ell)}$.

General form of $f^{(\ell)}$ for arbitrary ℓ:

$$f^{(\ell)}(x_1 \ldots, x_{4+\ell}) = \frac{P^{(\ell)}(x_1, \ldots x_{4+\ell})}{\prod_{1 \le i < j \le 4+\ell} x_{ij}^2} \,,$$

Can be deduced from the operator product expansion (OPE) analysis of the tree-level correlator

The polynomial $P^{(\ell)}$ should satisfy the conditions:

- be invariant under $S_{4+\ell}$ permutations of $x_1, ..., x_{4+\ell}$
- have a uniform conformal weight $-(\ell - 1)$ at each point, both external and internal.

Graph theory solution:

$$P^{(\ell)} \mapsto \textbf{\textit{Multi-graph with }} (4+\ell) \textbf{\textit{ vertices of degree }} (\ell - 1)$$

Four-point correlator at three loops

Our result for 4-point correlation function in planar $\mathcal{N}=4$ SYM

$$G_4(1,2,3,4) = G_4^{(0)} + \frac{N_c^2}{(4\pi^2)^4}\, R(1,2,3,4)\ \left[aF^{(1)} + a^2F^{(2)} + a^3F^{(3)} + O(a^4)\right]$$

$F^{(\ell)}$ are given by the sum of scalar Feynman integrals

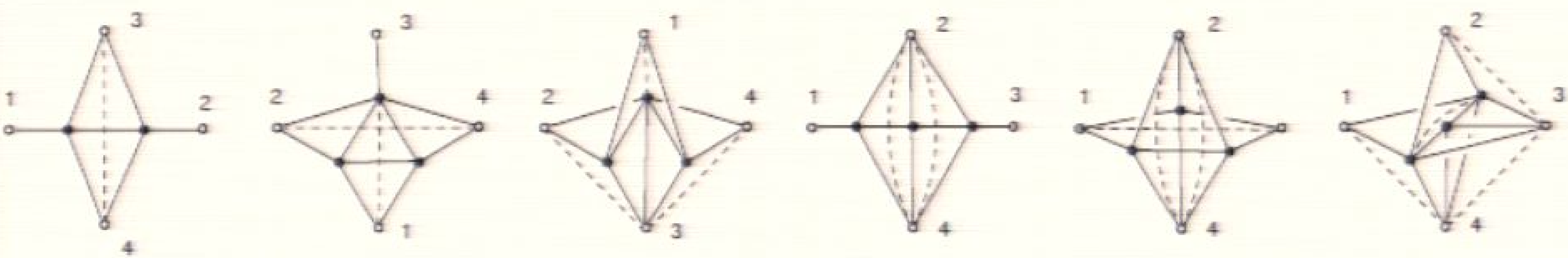
$$F^{(1)} = g(1,2,3,4)\,,$$

$$F^{(2)} = \left[h(1,2;3,4) + 5\text{ perms}\right] + \tfrac{1}{2}\left[x_{12}^2x_{34}^2(g(1,2,3,4))^2 + 3\text{ perms}\right],$$

$$F^{(3)} = \left[T(1,3;2,4) + 11\text{ perms}\right] + \left[E(2;1,3;4) + 11\text{ perms}\right] + \left[L(1,3;2,4) + 5\text{ perms}\right]$$

$$+ \left[(g\times h)(1,3;2,4) + 5\text{ perms}\right] + \tfrac{1}{2}\left[H(1,3;2,4) + 11\text{ perms}\right],$$

2- and 3-loop topologies:



$h(1,2;3,4)$ $T(1,3;2,4)$ $E(1;2,4;3)$ $L(1,3;2,4)$ $g\times h(1,3;2,4)$ $H(1,2;3,4)$