Title: Critical Values of the Yang-Yang Functional in the Quantum Sine-Gordon Model

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Abstract:

Critical values of the Yang-Yang functional in the quantum sine-Gordon model

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Outline

- Intro
- Fendley-Saleur-Zamolodchikov relations
- On-shell action for the ShG equation
- Generalized FSZ relations
- Yang-Yang function in the sine-Gordon model
- Conclusion

aces of integrability in field theory

Classical PDEs: Lax representation, Inverse Scattering Method ...

1+1 QFT: Bethe Ansatz, CFT, Factorized S-matrix ...

Faces of integrability in field theory

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↕

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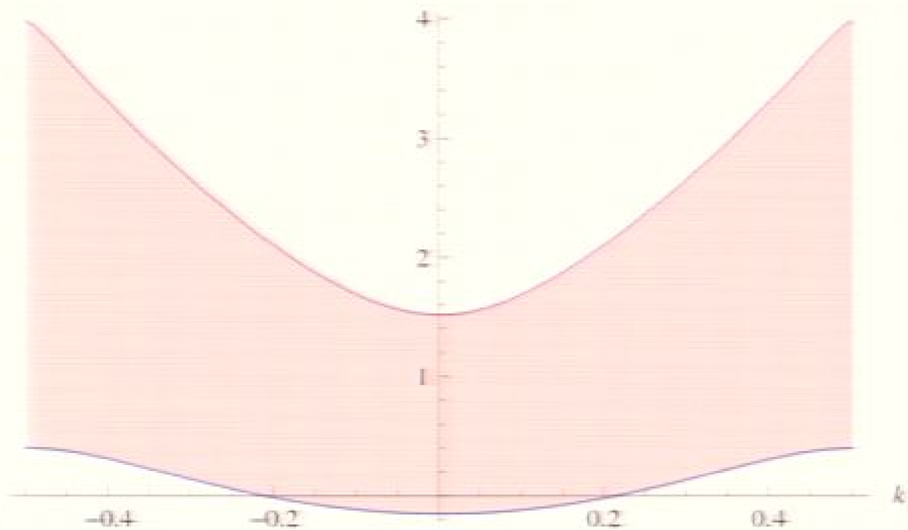
Particle in the cosine potential

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial\phi^2} - \Lambda\ \cos(\phi) - E \right] \Psi = 0$$

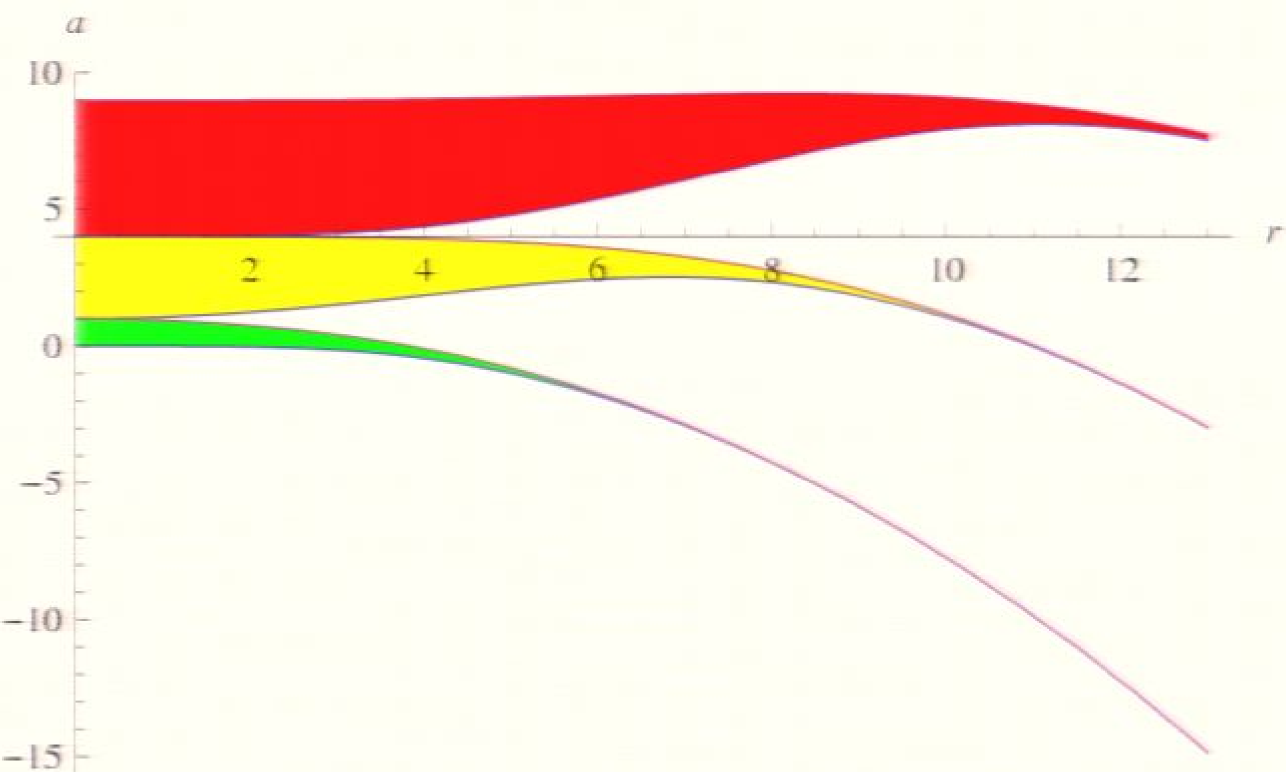
$$\Psi_k(\phi + 2\pi) = e^{2\pi i k}\, \Psi_k(\phi)$$

Ground state energy E_k (the first conducting band)



Stability regions for the Matheiu equation

$$\Psi'' + \left(a - \frac{r^2}{8}\cos(2x)\right)\,\Psi = 0 \qquad \left(\,E = 8\,a\,,\quad \Lambda = \left(\frac{r}{8}\right)^2\,\right)$$



Singularities of the Painleve III equation

$$q = q(t) \quad : \quad q'' + \frac{q'}{t} - \frac{(q')^2}{q^2} = \frac{1}{2}\left(q^2 - 1\right)$$

- Movable (depend on the initial data) double poles. The solution with a double pole at $t = r$ have the Laurent series expansion

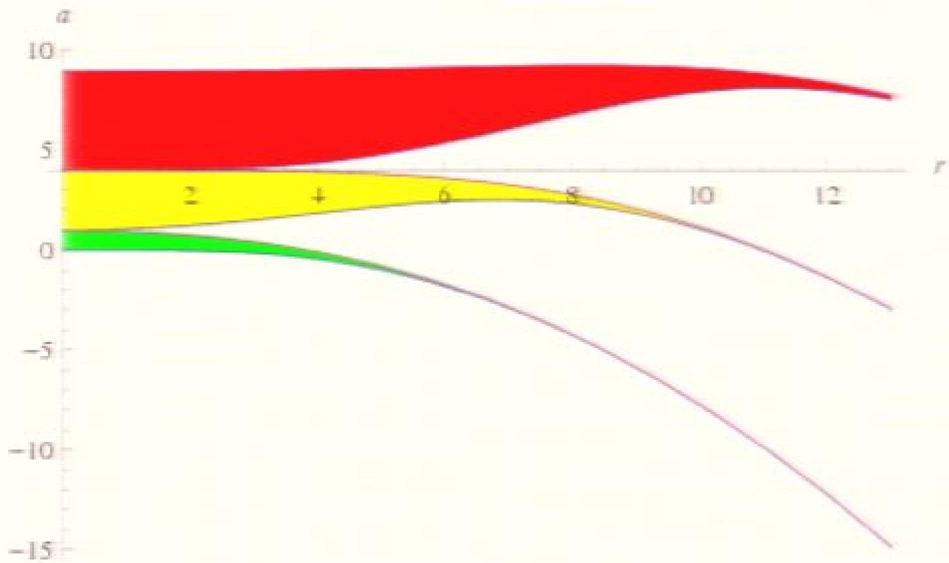
$$q(t) = \frac{4}{(t-r)^2} - \frac{4}{r(t-r)} + \frac{13-16a}{3r^2} + O(t-r)$$

  where a is some number.

- The point $t = 0$
  - Accumulation point for second order poles
  - $q(t) = \kappa^2\, t^{4\nu-2} + o(t^{4\nu-2})$ as $t \to 0$

- The point $t = \infty$ (Accumulation point for second order poles and zeroes, or $q(t)|_{t\to+\infty} \to 1$)

Relation beween Mathieu and Painleve III



The union of colored regions is an admissible domain for the parameters (r, a) of the solutions

$$q(t) = \frac{4}{(t-r)^2} - \frac{4}{r(t-r)} + \frac{13-16a}{3r^2} + O(t-r)$$

of the Painlevé III equation which develop the asymptotic behavior

$$q(t) = \kappa^2\ t^{4\nu-2} + o(t^{4\nu-2}) \quad \text{as} \quad t \to 0$$

The solutions $q(t)$ corresponding to the green region are free of poles and zeroes at the interval $t \in (0, r)$. For the yellow region $q(t)$ has one pole and one zero at $t \in (0, r)$ etc.

Quantum sine-Gordon in finite volume

$$\mathcal{L} = \frac{1}{\beta_{\text{sg}}^2}\left(\frac{1}{2}(\partial_\mu \phi)^2 + \Lambda \ \cos(\phi)\right)$$

Periodic boundary: $\phi(x+R) = \phi(x)$

The space of states $\mathcal{H}$ splits into orthogonal subspaces $\mathcal{H}_k$, characterized by the quasi-momentum k

$$\phi \to \phi + 2\pi : \qquad |\,\Psi_k\,\rangle \ \to \ \mathrm{e}^{2\pi \mathrm{i}\, k}\,|\,\Psi_k\,\rangle$$

The ground state of the finite-size system in the sector $\mathcal{H}_k$:

$$|\,\Psi_k^{(\text{vac})}\,\rangle \quad : \quad E_k$$

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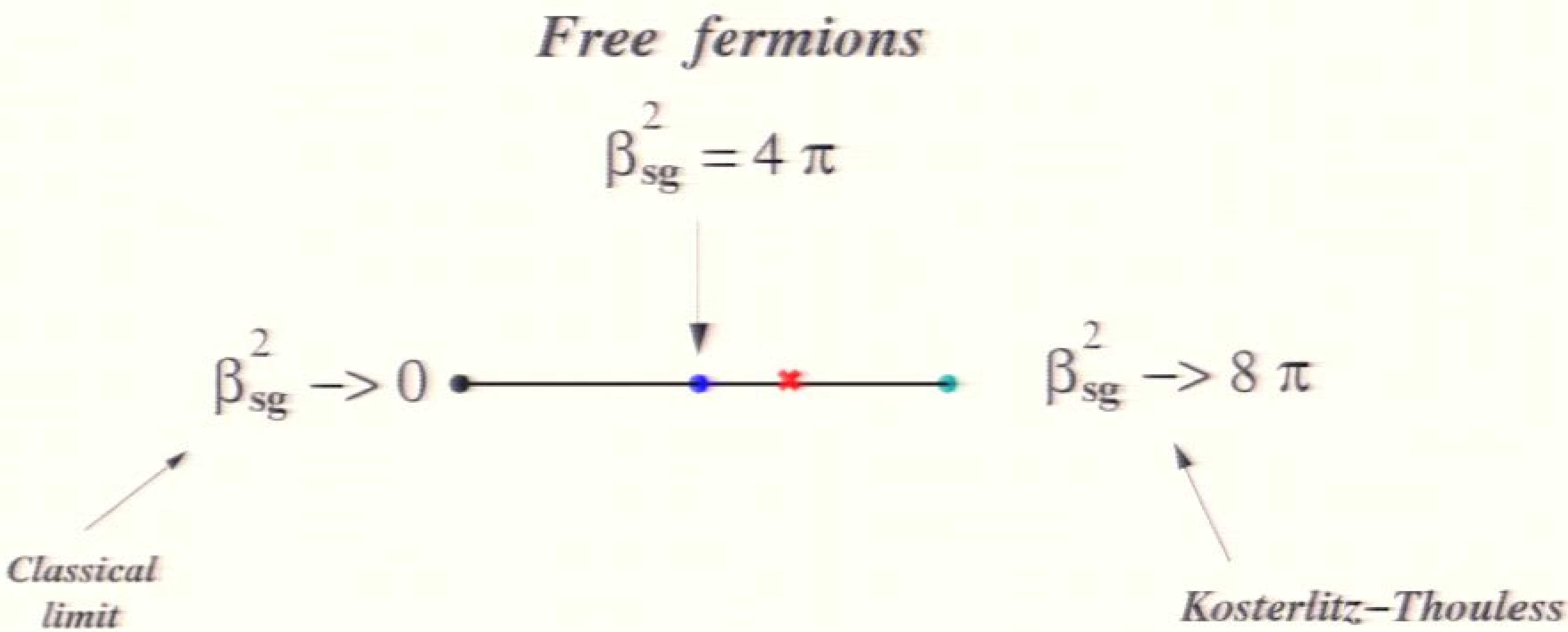
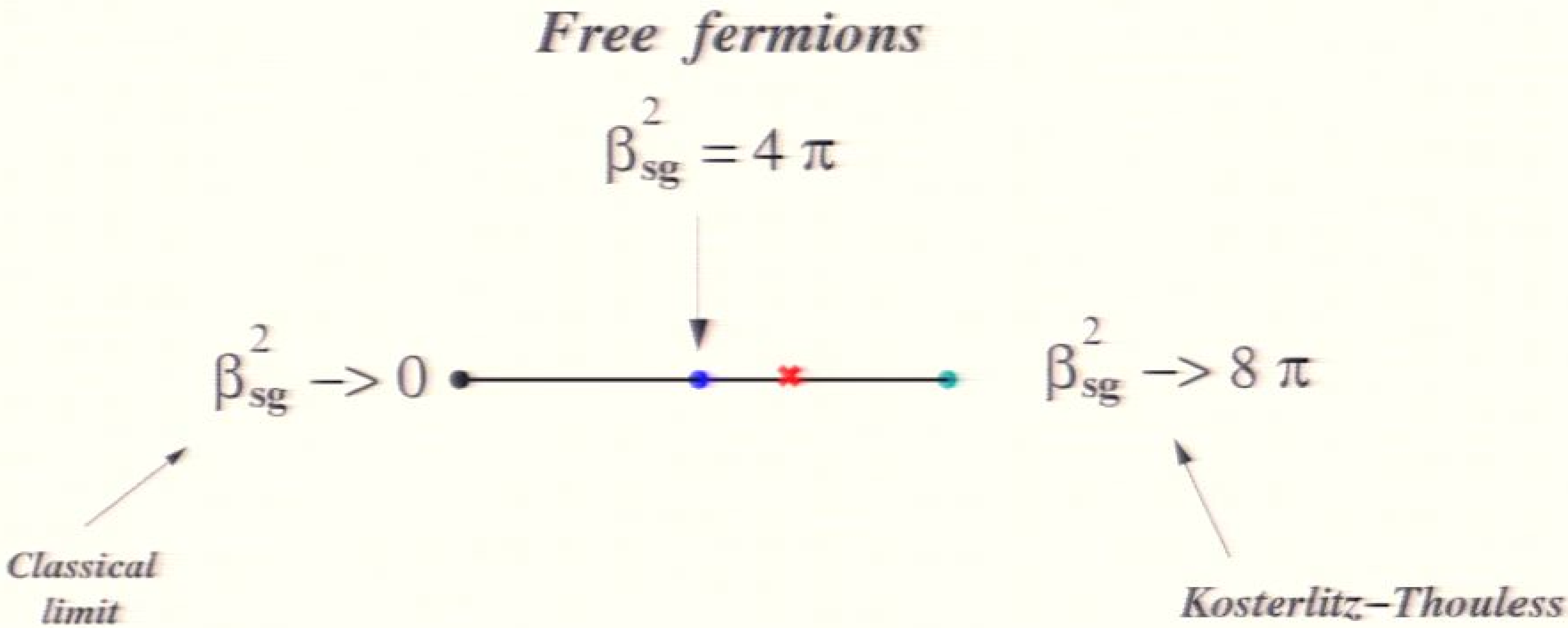
Scaling function: $RE_k = F\Big(r,\, \beta_{\text{sg}}^2,\, k\Big)$ $\qquad r = MR$

M-soliton mass (inverse correlation length)


Free fermions
$\beta_{sg}^2 = 4\pi$
$\beta_{sg}^2 \to 0$
$\beta_{sg}^2 \to 8\pi$
Classical limit
Kosterlitz–Thouless

Renormalized coupling : $\xi = \frac{\beta_{sg}^2}{8\pi-\beta_{sg}^2}$



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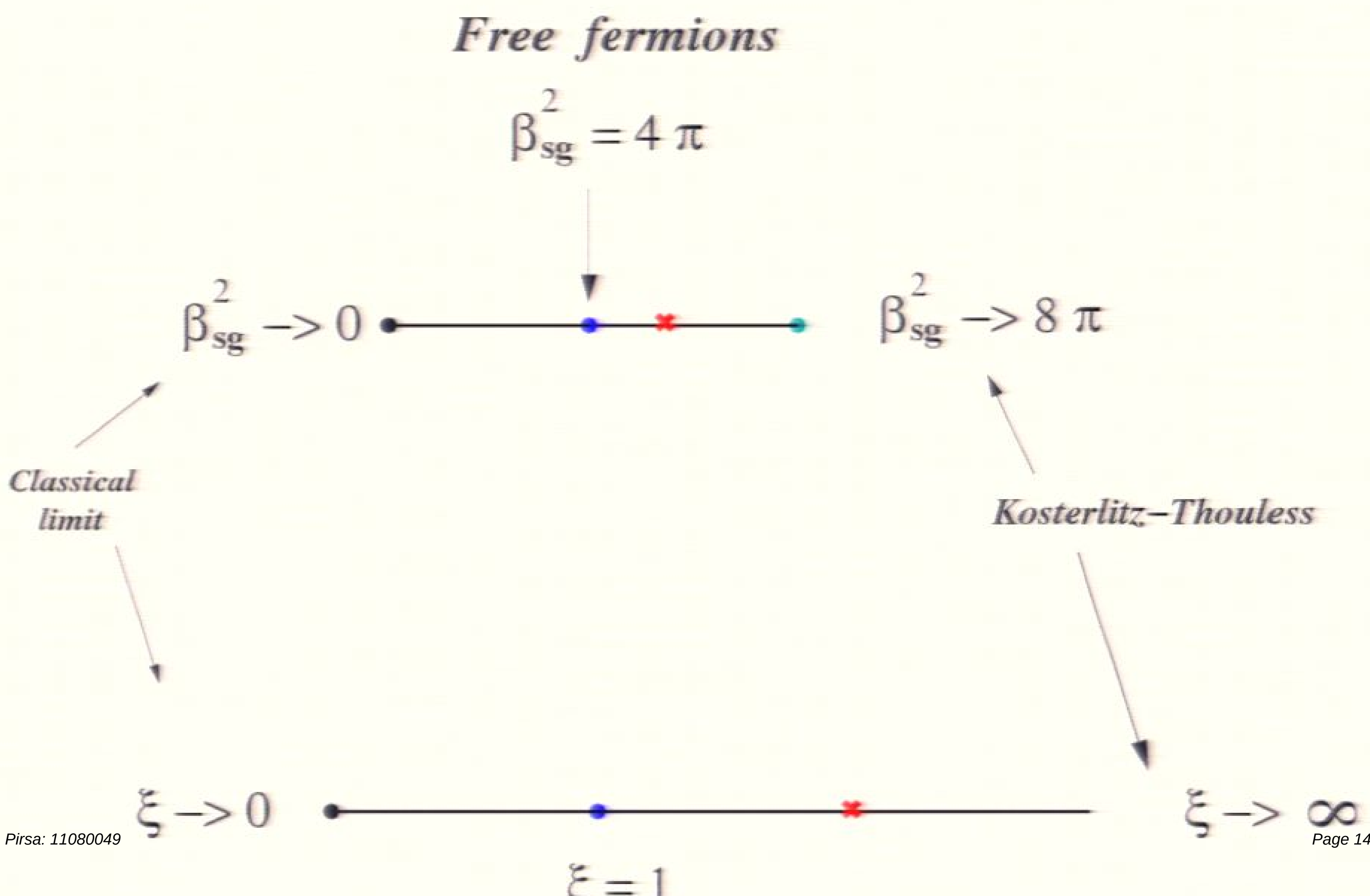

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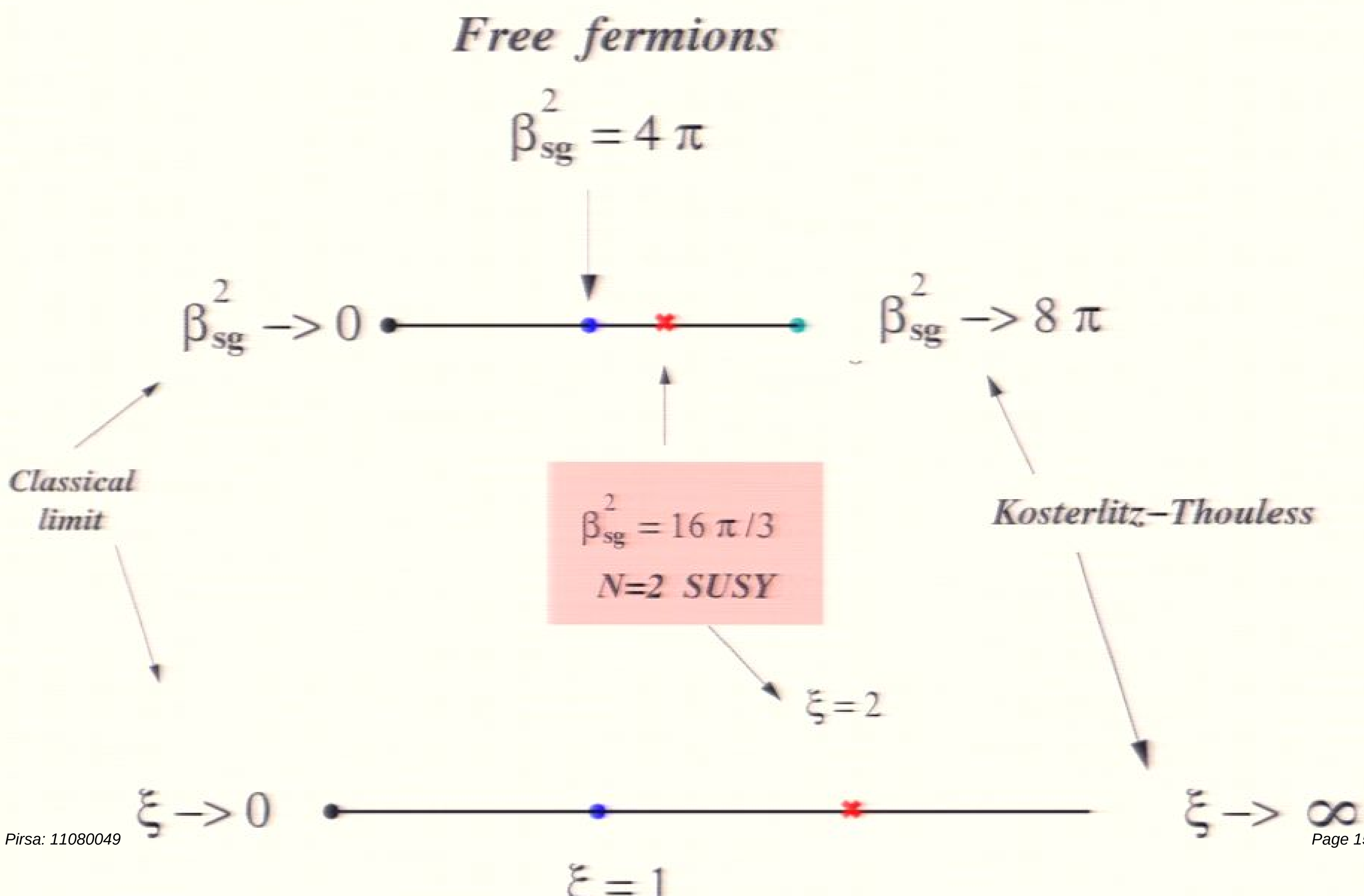

- Self-avoiding polymer problem **Saleur (1991)**

$\xi = 2$ sine-Gordon $\rightarrow \mathcal{N} = 2$ SUSY is spontaneously broken, except the subspaces $\mathcal{H}_k$ corresponding to $k = \pm\frac{1}{4}$.

- General approach for $D = 2,\ \mathcal{N} = 2$ supersymmetric QFT

Cecotti, Vafa (1991)

Cecotti, Fendley, Intriligator, Vafa (1992)

Fendley, Saleur (1992)
Al. Zamolodchikov (1994) relations

FS relation
$$\frac{R}{\pi}\left(\frac{\partial E_k}{\partial k}\right)_{\substack{\xi=2\\ k=\pm 1/4}} = \mp 4r\,\frac{\mathrm{d}U(r)}{\mathrm{d}r}$$

(t)- the Painleve III transcendent

$(t) = \mathrm{e}^{2U(t)})$

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$U(t)$- the Painleve III transcendent $\frac{1}{t}\frac{\mathrm{d}}{\mathrm{dt}}\left(t\frac{\mathrm{dU}}{\mathrm{dt}}\right)=\frac{1}{2}\,\sinh(2U)$

$(t) = e^{2U(t)})$

$$U(t) \to \begin{cases} -\frac{1}{3}\,\log(t) + O(1) & \text{as} \quad t\to 0 \\ 0 & \text{as} \quad t\to\infty \end{cases}$$

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Z relation $$\frac{R}{\pi}\left(\frac{\partial E_k}{\partial \xi}\right)_{\substack{\xi=2\\k=\pm1/4}} = -\frac{r^2}{8}+\frac{1}{2}\int_r^\infty \mathrm{dt}\ \mathrm{t}\ \sinh^2\mathrm{U(t)}$$

Modified Sinh-Gordon (MShG)

$$\partial_z \partial_{\bar{z}} \eta + \mathrm{e}^{2\eta} - p(z)\, p(\bar{z})\ \mathrm{e}^{-2\eta} = 0$$

- Constant mean curvature surfaces in $\mathbb{A}dS_3$
- Strong coupling scattering amplitudes in the $4D$, $\mathcal{N} = 4$ SUSY Yang-Mills theory.

$$p(z) = z^{n-2} + m_{n-4}\, z^{n-4} + \ldots m_0$$

Alday, Maldacena (2007, 2009)

- BPS states of $4D$ $\mathcal{N} = 2$ theories and wall crossing Gaiotto, Moore, Neitzke (2008, 2009)

MShG on the cone

Zamolodchikov, SL (2010)

$$\partial_z \partial_{\bar{z}} \eta + \mathrm{e}^{2\eta} - p(z)\, p(\bar{z})\ \mathrm{e}^{-2\eta} = 0$$

$p(z) = z^{2\alpha} - s^{2\alpha}$, η respects the symmetry $z \to \mathrm{e}^{\frac{i\pi}{\alpha}} z\,,\, \bar{z} \to \mathrm{e}^{-\frac{i\pi}{\alpha}} \bar{z}$

η continuous at all finite nonzero z

η grows slower then exponential at $|z| \to \infty$

$\eta \to 2l \log|z| + O(1) \qquad |z| \to 0 \qquad -\frac{1}{2} < l < \frac{1}{2}$

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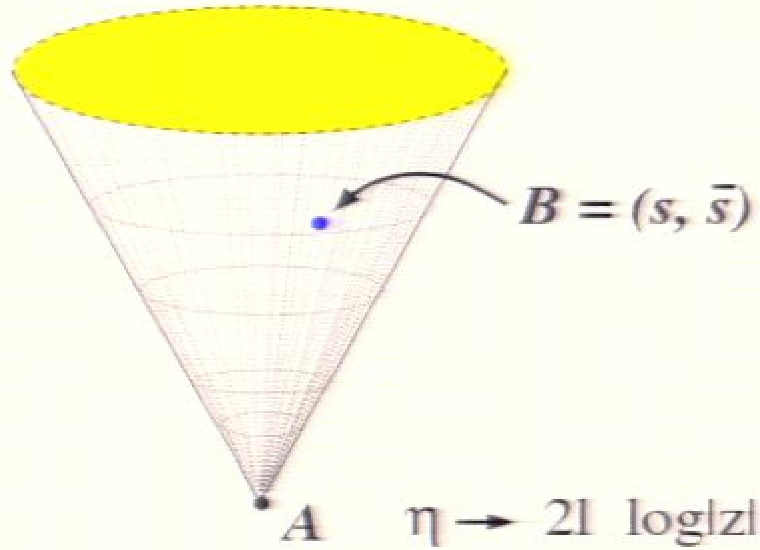

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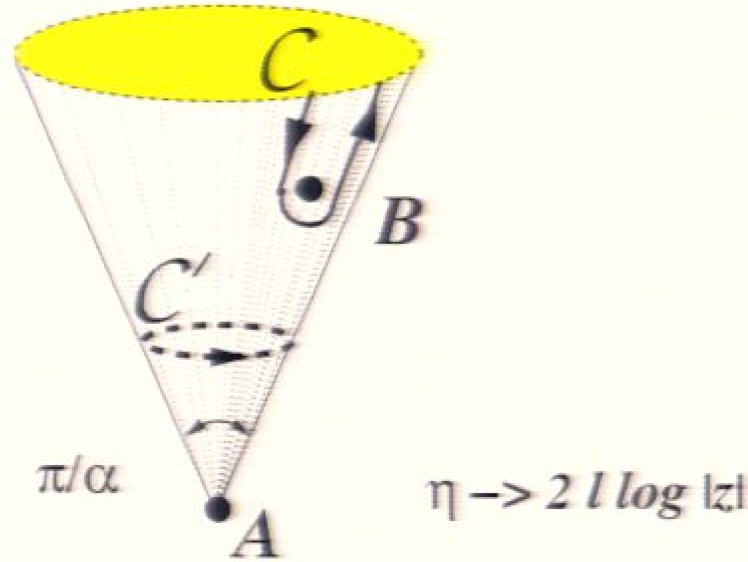
MShG v.s. Quantum Sine-Gordon

MShG Integrals of motions

$$I_{2n-1} = \int_C \left(\mathrm{d}z\, T_{2n} + \mathrm{d}\bar{z}\, \Theta_{2n-2} \right)$$

$$\bar{I}_{2n-1} = \int_C \left(\mathrm{d}\bar{z}\, \bar{T}_{2n} + \mathrm{d}z\, \Theta_{2n-2} \right)$$



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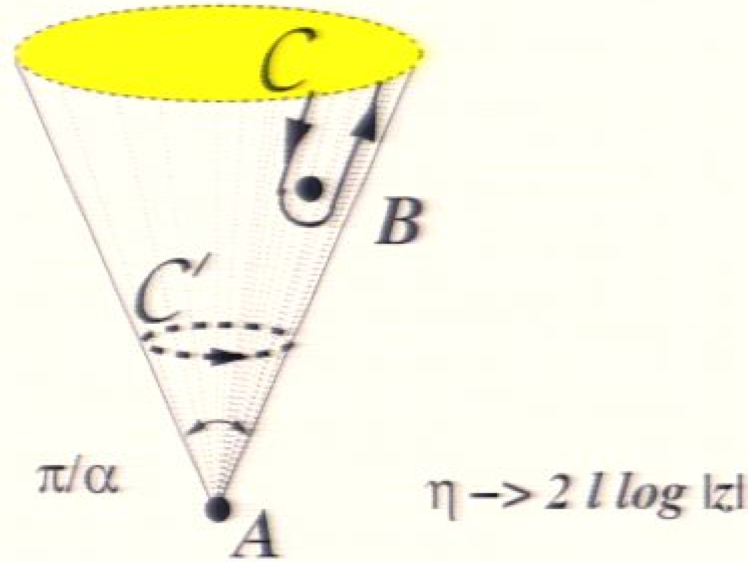

I_{2n-1}, $\bar{I}_{2n-1}$ coincide with the k-vacuum eigenvalues of local
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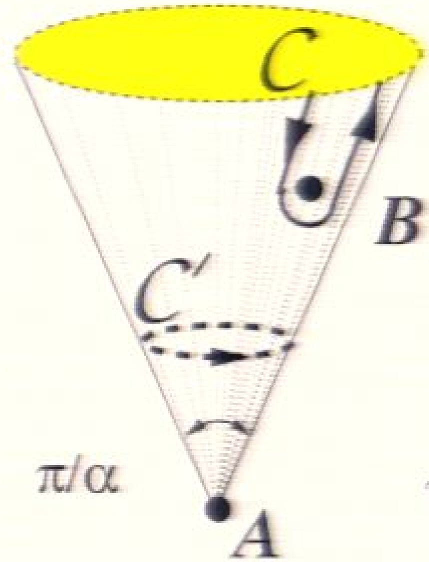

$\eta \to 2\, l \log |z|$

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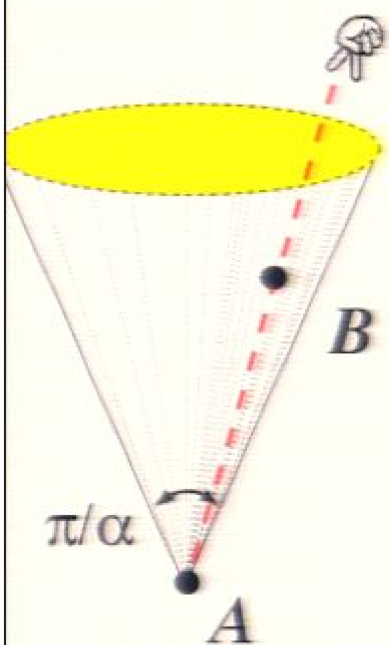
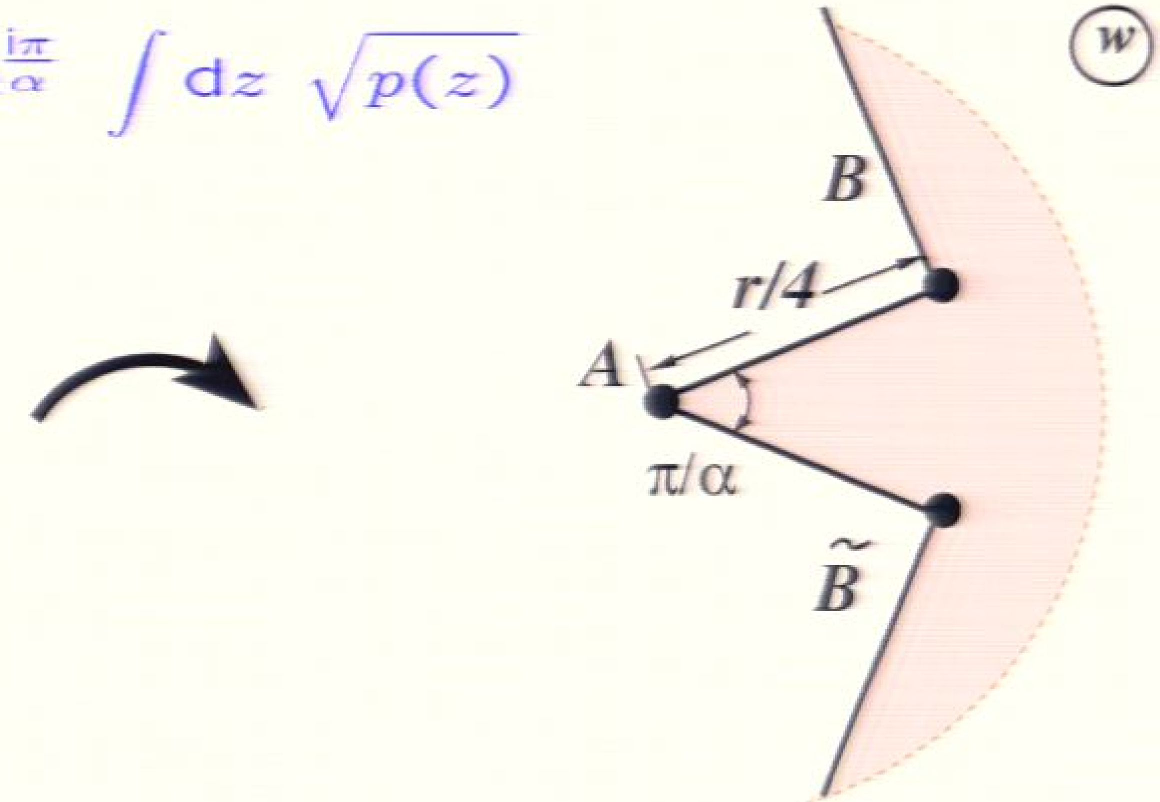
I_{2n-1}, $\bar{I}_{2n-1}$ coincide with the k-vacuum eigenvalues of local M in the quantum sine-Gordon model

$$\alpha = \xi^{-1} \qquad l = 2\,|k| - \frac{1}{2} \qquad s = \left[\frac{r\, \Gamma(\frac{3}{2} + \frac{\xi}{2})}{2\sqrt{\pi}\,\Gamma(1 + \frac{\xi}{2})} \right]^{\frac{\xi}{1+\xi}}$$

From MShG to ShG

$$w = \mathrm{i}\ e^{\frac{\mathrm{i}\pi}{\alpha}} \int dz\ \sqrt{p(z)}$$





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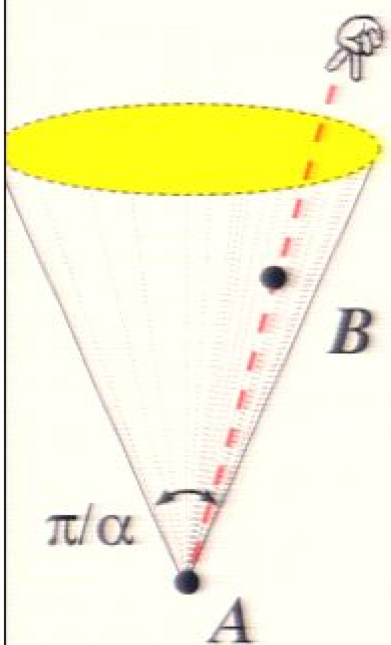

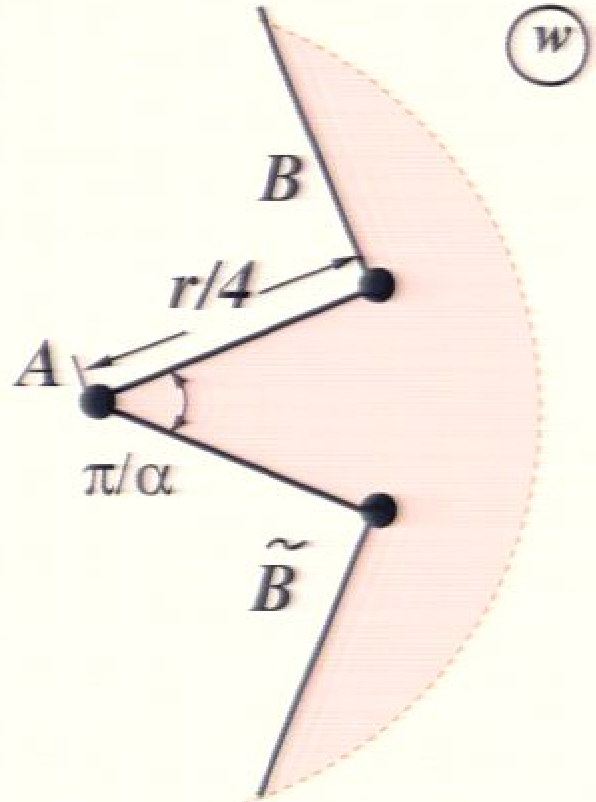

$$\hat{\eta}(w, \bar{w}) = \eta - \frac{1}{4}\ \log(p\bar{p})$$

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + p(z)\,\bar{p}(\bar{z})\ \mathrm{e}^{-2\eta} = 0 \quad \rightarrow \quad \partial_w \partial_{\bar{w}} \hat{\eta} - \mathrm{e}^{2\hat{\eta}} + \mathrm{e}^{-2\hat{\eta}} = 0$$

On-shell action for the ShG equation

$\delta\mathcal{A} = 0$: ShG equation + Boundary conditions

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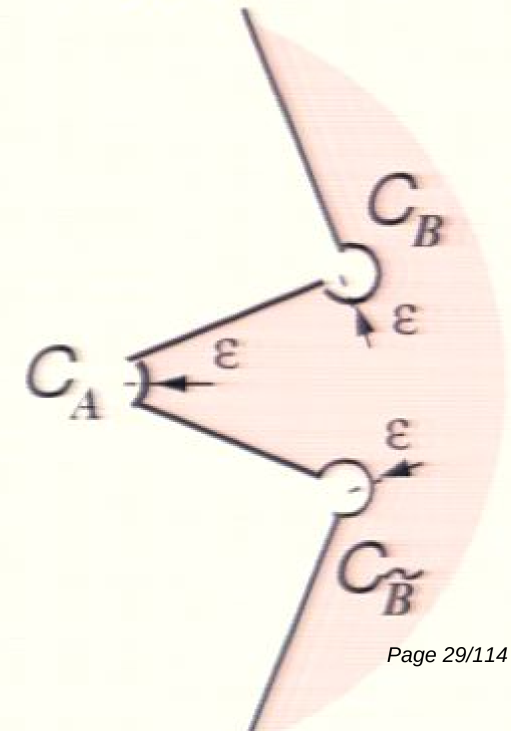
$$\mathcal{A}[\hat{\eta}] = \lim_{\epsilon\to 0}\Big[\int_{D_\epsilon}\frac{\mathrm{d}w\wedge\mathrm{d}\bar{w}}{2\pi\mathrm{i}}\Big(\partial_w\hat{\eta}\partial_{\bar{w}}\hat{\eta} + 4\,\sinh^2(\hat{\eta})\Big) + \frac{l}{\pi\epsilon}\int_{C_A}\mathrm{d}\ell\ \hat{\eta}$$
$$-\frac{1}{6\pi\epsilon}\int_{C_B}\mathrm{d}\ell\ \hat{\eta} - \frac{1}{6\pi\epsilon}\int_{C_{\tilde{B}}}\mathrm{d}\ell\ \hat{\eta} - \frac{l^2}{\alpha}\ \log(\epsilon) - \frac{1}{12}\ \log(\epsilon)\Big]$$



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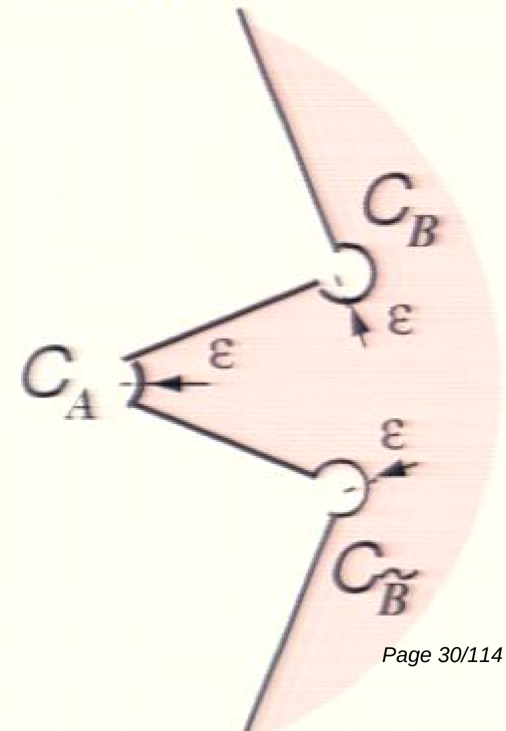
$\mathcal{A}^* = \mathcal{A}^*(r,\alpha,l)$ - on-shell action



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$$- \frac{1}{6\pi\epsilon} \int_{C_B} \mathrm{d}\ell \; \hat{\eta} - \frac{1}{6\pi\epsilon} \int_{C_{\tilde{B}}} \mathrm{d}\ell \; \hat{\eta} - \frac{l^2}{\alpha} \log(\epsilon) - \frac{1}{12} \log(\epsilon) \Big]$$

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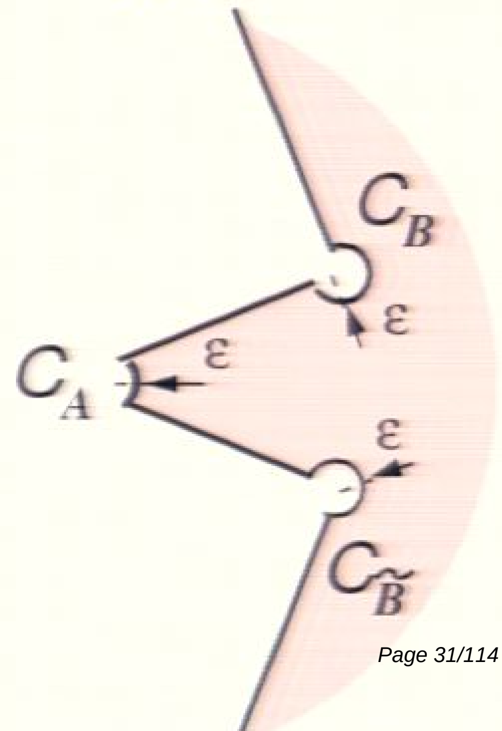

Variations of $\mathcal{A}^*(r, \alpha, l)$

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$$\mathrm{m}_{\to 0}\ \delta_l \left[\int_{D_\epsilon} \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{2\pi \mathrm{i}} \left(\partial_w \widehat{\eta} \partial_{\bar{w}} \widehat{\eta} + 4\ \sinh^2(\widehat{\eta}) \right) + \frac{l}{\pi\epsilon} \int_{C_A} \mathrm{d}\ell\ \widehat{\eta} \right.$$

$$\left. \frac{1}{6\pi\epsilon} \int_{C_B} \mathrm{d}\ell\ \widehat{\eta} - \frac{1}{6\pi\epsilon} \int_{C_{\bar{B}}} \mathrm{d}\ell\ \widehat{\eta} - \frac{l^2}{\alpha}\ \log(\epsilon) - \frac{1}{12}\ \log(\epsilon) \right]$$

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$$\left(\frac{\partial \mathcal{A}^*}{\partial l}\right)_{r,\alpha} = \frac{1}{\alpha}\ \hat{\eta}_A$$

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$$\frac{1}{6\pi\epsilon}\int_{C_B} \mathrm{d}\ell\ \hat{\eta} - \frac{1}{6\pi\epsilon}\int_{C_{\bar{B}}} \mathrm{d}\ell\ \hat{\eta} - \frac{l^2}{\alpha}\ \log(\epsilon) - \frac{1}{12}\ \log(\epsilon)\Big]$$

$$\left(\frac{\partial\mathcal{A}^*}{\partial l}\right)_{r,\alpha} = \frac{1}{\alpha}\ \hat{\eta}_A$$

$$\hat{\eta}_A = \lim_{|w-w_A|\to 0} \Big(\hat{\eta}(w,\bar{w}) - 2l\ \log|w-w_A| \Big)$$

Regularized value of the solution $\hat{\eta}$ at the apex.

$$ {}_{\alpha}\mathcal{A}^* = \delta\left(\frac{\pi}{\alpha}\right) \ \lim_{\epsilon \to 0} \left[-\int_{\partial D_\epsilon} \frac{d\ell}{\pi} \ \epsilon^{\mu\nu} x^{\mu} n^{\sigma} \ T_{\nu\sigma} + \frac{l}{\pi} \ \widehat{\eta}_A + \frac{l^2}{\pi} \ \log(\epsilon) \right] $$

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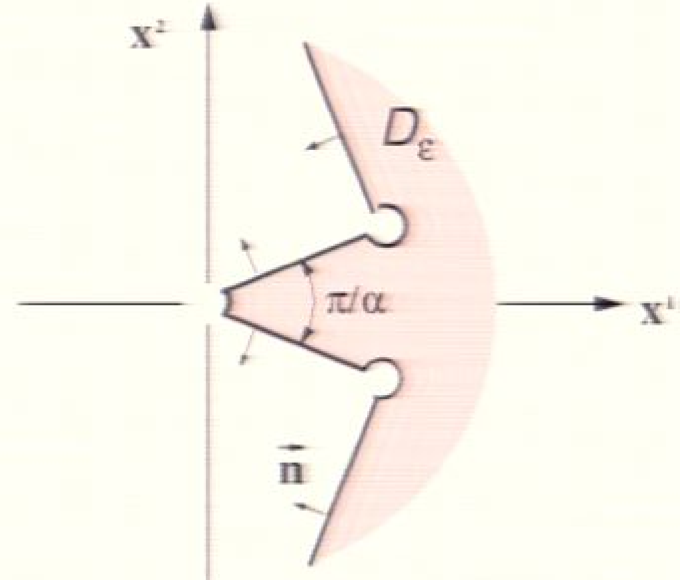

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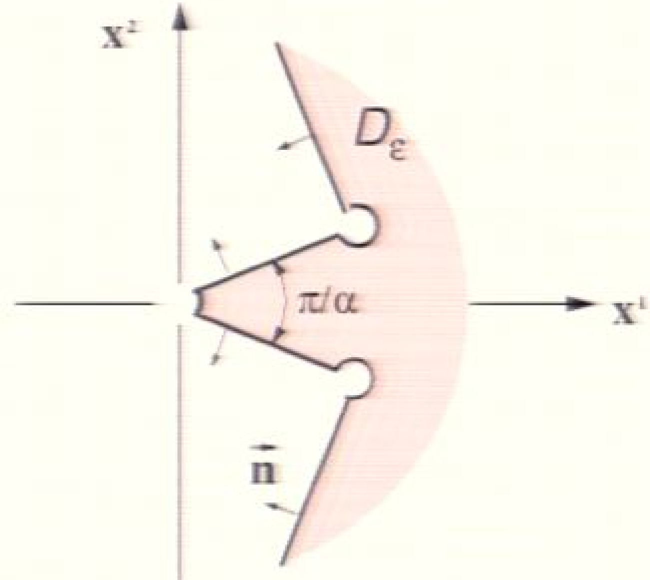

$$_{\mu\nu} = -\frac{1}{4}\ \partial_\mu\hat{\eta}\partial_\nu\hat{\eta} + \delta_{\mu\nu}\ \left[\frac{1}{8}\ \left(\partial_\sigma\hat{\eta}\right)^2 + 2\sinh^2(\hat{\eta})\right]$$

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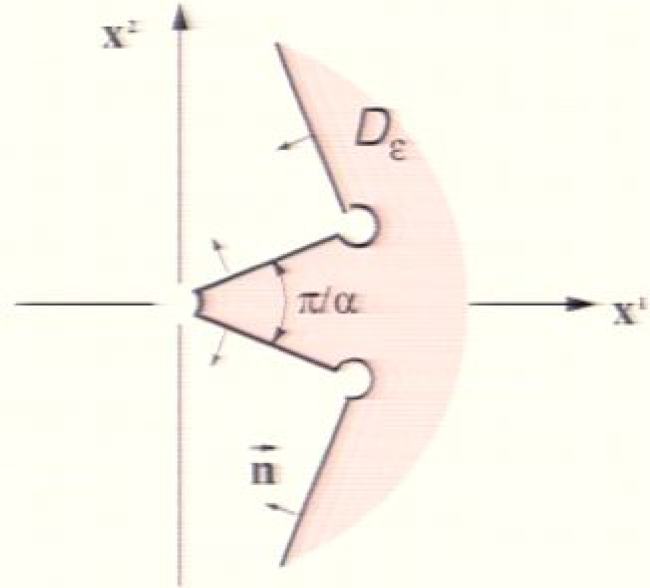

$$ {}_{\mu\nu} = -\frac{1}{4}\ \partial_\mu\hat{\eta}\partial_\nu\hat{\eta} + \delta_{\mu\nu}\ \left[\frac{1}{8}\ \left(\partial_\sigma\hat{\eta}\right)^2 + 2\sinh^2(\hat{\eta})\right] $$

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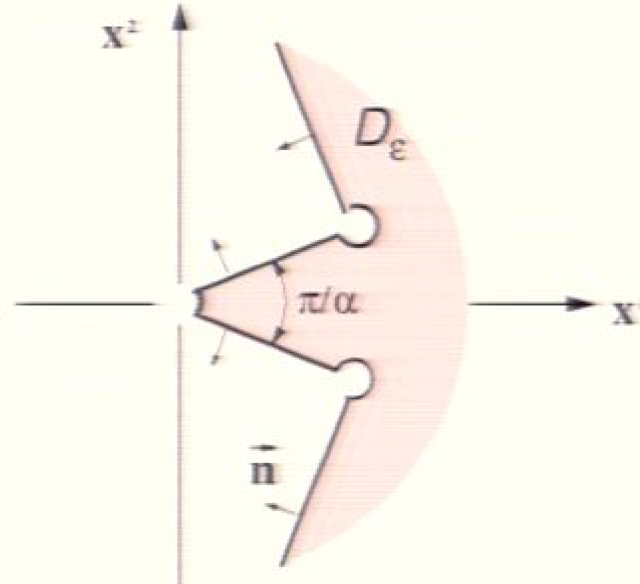
$$ {}_{\alpha}\mathcal{A}^* = \delta\left(\frac{\pi}{\alpha}\right) \ \lim_{\epsilon\to 0}\left[-\int_{\partial D_\epsilon} \frac{d\ell}{\pi} \ \epsilon^{\mu\nu} x^\mu n^\sigma \ T_{\nu\sigma} + \frac{l}{\pi}\ \widehat{\eta}_A + \frac{l^2}{\pi} \ \log(\epsilon) \right] $$



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$$ \alpha^2\left(\frac{\partial\mathcal{A}^*}{\partial\alpha}\right)_{r,l} = -\frac{1}{2}\ \Phi_A - l\ \widehat{\eta}_A $$

$$ \Phi_A = \lim_{|w-w_A|\to 0}\left(\Phi(w,\bar{w}) + 2l^2\ \log|w-w_A|\right) $$

Dilations $\frac{\delta r}{r} = \frac{\delta \epsilon}{\epsilon} = \lambda \ll 1$

$$\delta_r \mathcal{A}^* = \frac{\delta r}{r}\left[\lim_{\epsilon \to 0} \int_{D_\epsilon} \frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{\pi \mathrm{i}}\, \Theta - \left(\frac{l^2}{\alpha} + \frac{1}{12}\right)\right]$$

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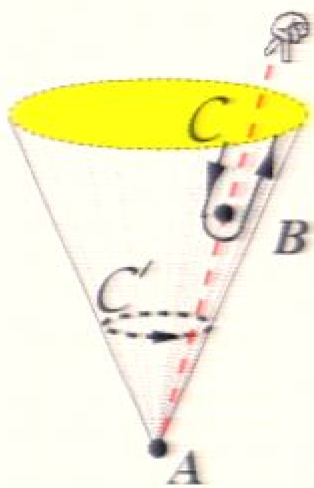
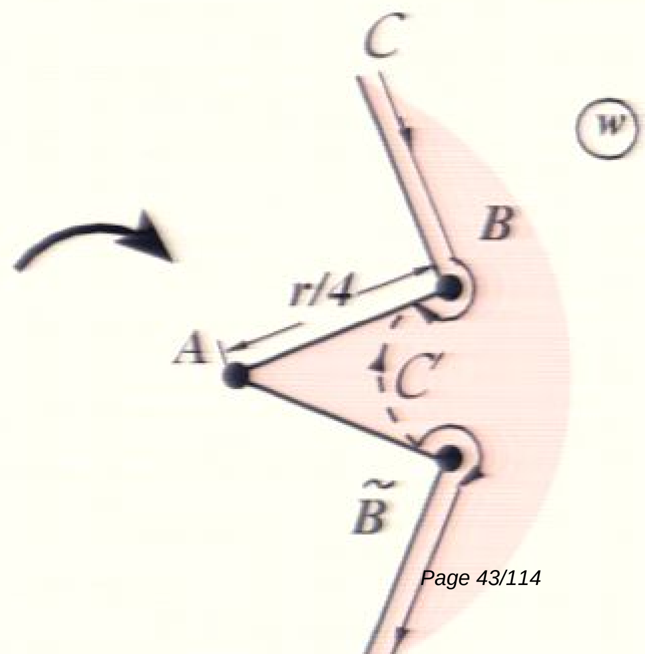
$$\left(\frac{\partial \mathcal{A}^*}{\partial r}\right)_{\alpha, l} = -\frac{1}{8\pi}\left(I_1 + \bar{I}_1\right)$$





$$I_1 = \int_C \left(\mathrm{d}w\, T + \mathrm{d}\bar{w}\, \Theta\right)$$

$$\bar{I}_1 = \int_C \left(\mathrm{d}\bar{w}\, \bar{T} + \mathrm{d}w\, \Theta\right)$$

$$\left(\frac{\partial \mathcal{A}^*}{\partial l}\right)_{r,\alpha} = \frac{1}{\alpha}\,\hat{\eta}_A$$

$$\alpha^2\left(\frac{\partial \mathcal{A}^*}{\partial \alpha}\right)_{r,l} = -\tfrac{1}{2}\,\Phi_A - l\,\hat{\eta}_A$$

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⋆ The compatibility conditions

$$\alpha\left(\frac{\partial F}{\partial l}\right)_{r,\alpha} = -r\left(\frac{\partial \hat{\eta}_A}{\partial r}\right)_{\alpha,l}$$

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$e_\infty = \lim_{R\to\infty}\frac{E_k}{R} = -\frac{M^2}{4}\ \tan\left(\frac{\pi\xi}{2}\right)$

Destri, de Vega (1991)

Generalized FSZ

$$\frac{R}{\pi\xi}\left(\frac{\partial E_k}{\partial k}\right)_{r,\xi} = -r\left(\frac{\partial \widehat{\eta}_A}{\partial r}\right)_{\alpha,l}$$

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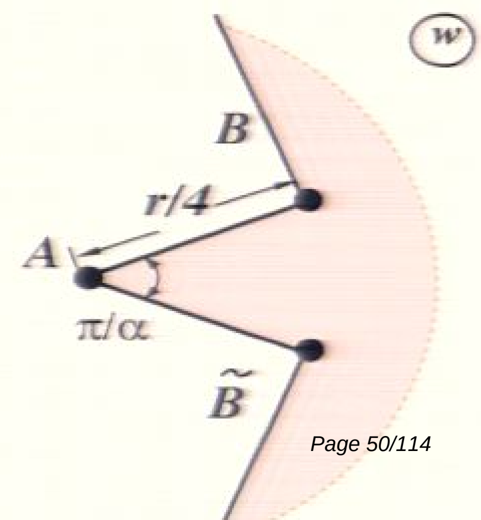

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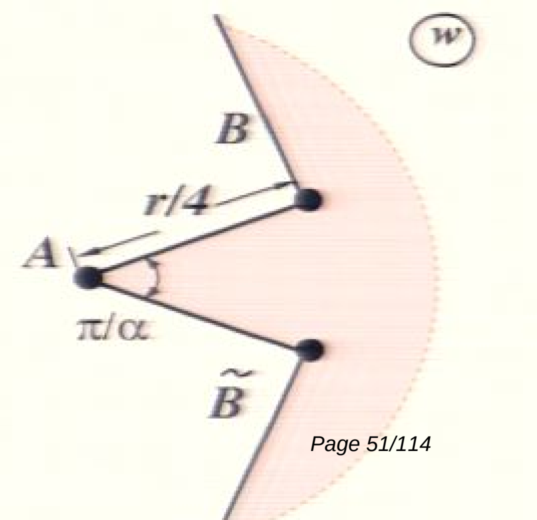

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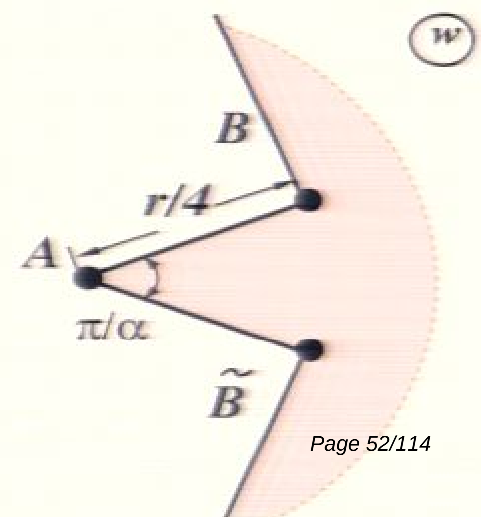

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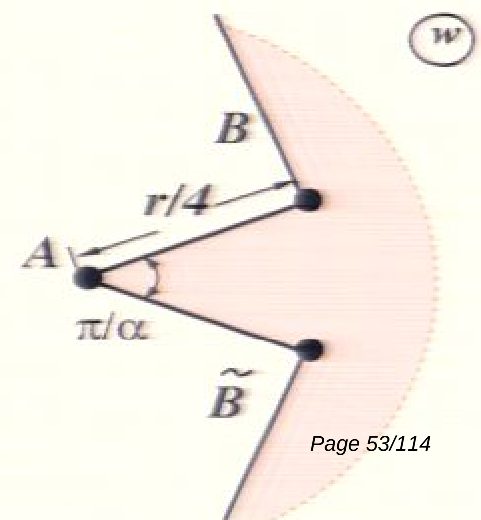

$$\hat{\eta}_A = U(r) \qquad r\,\frac{d\Phi_A}{dr} = -\int_r^\infty dt\ t\ \sinh^2 U(t)$$

Normalized on-shell action $\mathfrak{Y} = \mathcal{A}^* - \mathcal{A}^*_\infty$

$$\mathfrak{Y} = \int_R^\infty \frac{\mathrm{d}R}{\pi}\,(\,E_k - e_\infty\,R\,)$$

$$\lim_{r\to\infty} \mathfrak{Y} = 0$$

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Small R limit

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Small R limit

$$\lim_{R\to 0} R E_k = -\frac{\pi}{6} c_{\mathrm{eff}} \qquad c_{\mathrm{eff}} = 1 - \frac{24\xi k^2}{1+\xi}$$

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$$\mathfrak{Y} = \frac{1}{6}\,c_{\mathrm{eff}}\;\log(MR) + \mathfrak{Y}_0 - \frac{(MR)^2}{8\pi}\;\tan\left(\frac{\pi\xi}{2}\right) - \int_0^R \frac{\mathrm{d}R}{\pi}\left(E_k + \frac{\pi c_{\mathrm{eff}}}{6R}\right)$$

$$R \to 0$$

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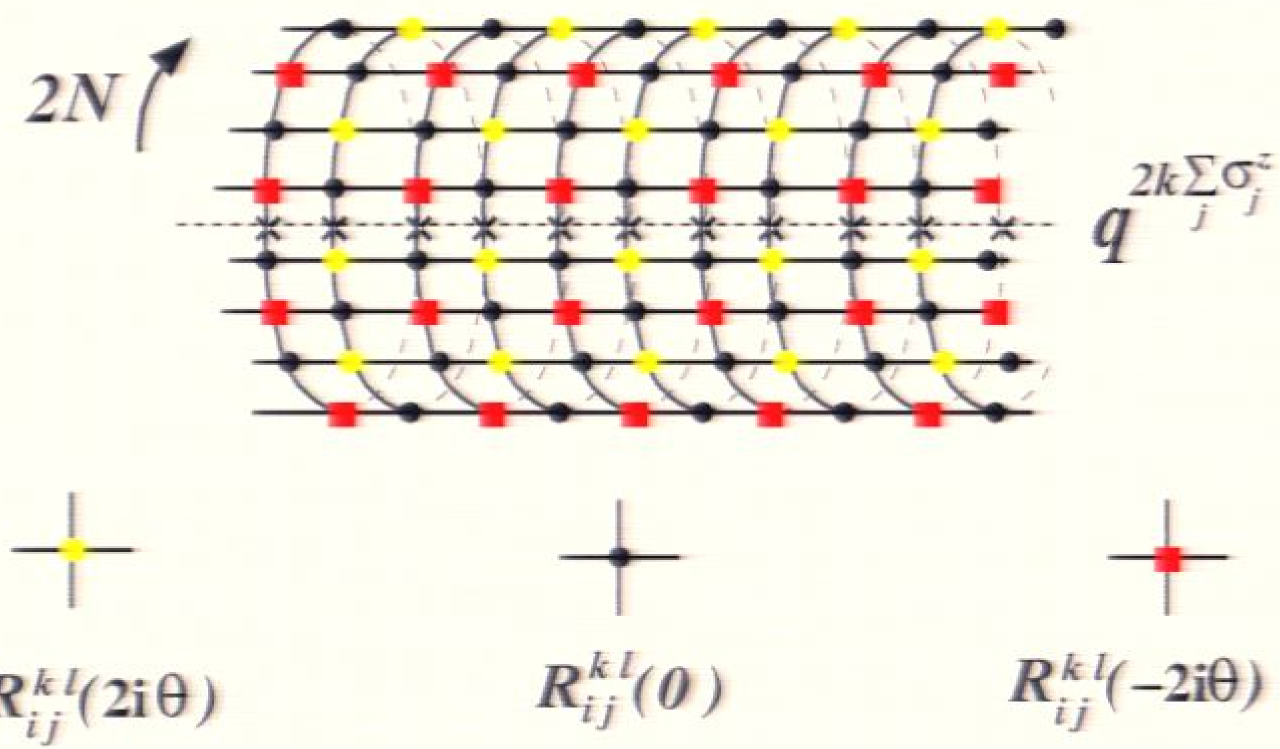
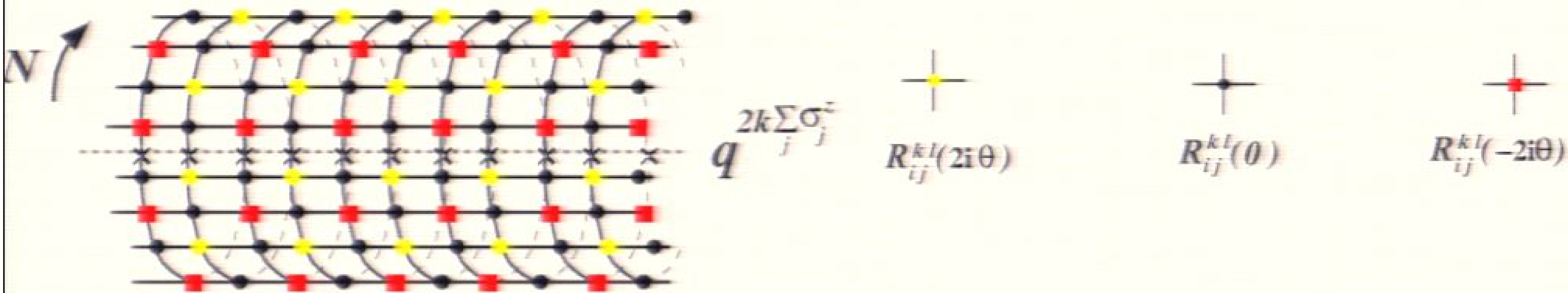
$$\mathfrak{Y}_0 \;=\; \tfrac{1}{12}\, \log\left(4\,\xi^{\xi}\,(1+\xi)^{-1-\xi}\right) - \tfrac{1}{6}\, c_{\mathrm{eff}}\ \log\left(\frac{2\sqrt{\pi}\,\Gamma(\frac{\xi}{2})}{\Gamma(\frac{3}{2}+\frac{\xi}{2})}\right)$$
$$-\ \int_0^{\infty} \frac{\mathrm{d}x}{x}\ \left(\frac{\sinh(x)\cosh(4\xi k x)}{2x\sinh(\xi x)\sinh\left(x(1+\xi)\right)} - \frac{1}{2\xi(1+\xi)\,x^2} + \frac{c_{\mathrm{eff}}}{6}\,\mathrm{e}^{-2x}\right)$$

YY-function for the inhomogeneous 6-vertex model (Destri, de Vega 1989)



Partition function $Z_N = \mathrm{Tr}\left[q^{k\sum_j \sigma_j^z} \tau^N \right]$ of the inhomogeneous 6-vertex model on an infinite cylinder. Here τ is the monodromy matrix along the infinite direction and $q = e^{\frac{i\pi\xi}{1+\xi}}$. $R_{ik}^{kl}(\lambda)$ are conventional Boltzmann weights for the 6-vertex model

Bethe Ansatz equations



$$s(x) = \sinh\left(\frac{x}{1+\xi}\right)$$

$$\left[\frac{s(\theta_j + \Theta + \frac{i\pi}{2})\, s(\theta_j - \Theta + \frac{i\pi}{2})}{s(\theta_j + \Theta - \frac{i\pi}{2})\, s(\theta_j - \Theta - \frac{i\pi}{2})}\right]^N = -e^{\frac{4i\pi\xi k}{1+\xi}} \prod_n \frac{s(\theta_j - \theta_n + i\pi)}{s(\theta_j - \theta_n - i\pi)}$$

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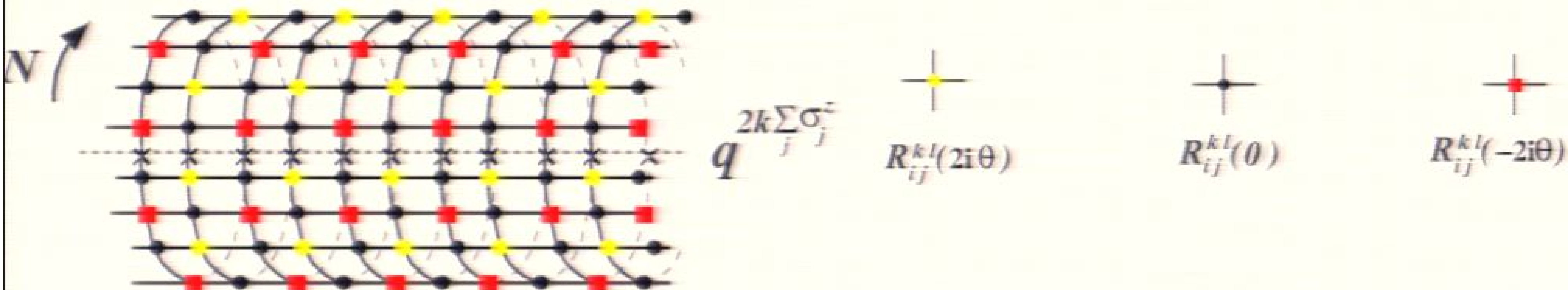

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The energy $E^{(N)}$ and momentum $P^{(N)}$ of the BA state

$$\exp\left(-i\,\frac{E^{(N)} \pm P^{(N)}}{2N}\right) = \prod_j \frac{s(\frac{i\pi}{2} + \Theta \pm \theta_j)}{s(\frac{i\pi}{2} - \Theta \mp \theta_j)}$$

Yang-Yang functional (1966)

The vacuum BA equations can be bring to the form of extremum condition

$$\frac{\partial Y^{(N)}}{\partial \theta_j} = 0 \qquad j = -\tfrac{N}{2}, -\tfrac{N}{2}+1, \ldots \tfrac{N}{2}-2, \tfrac{N}{2}-1$$

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$$V(\theta) = -\frac{N}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \frac{\sinh(\frac{\pi\omega\xi}{2}) \cos(\omega\Theta)}{\sinh(\frac{\pi\omega(1+\xi)}{2})} e^{i\omega\theta}$$

$$U(\theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \frac{\sinh(\frac{\pi\omega\xi}{2}) \cosh(\frac{\pi\omega}{2})}{\sinh(\frac{\pi\omega(1+\xi)}{2})} e^{i\omega\theta}$$

Mechanical analogy

$$Y^{(N)} = 2 \sum_{j} \Big(V(\theta_j) - 2gk\,\theta_j \Big) + \sum_{j,n} U(\theta_j - \theta_n) \qquad g = \tfrac{\xi}{1+\xi}$$

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- 2-body potential: 1D repulsive Coulomb potential slightly modified at short distances

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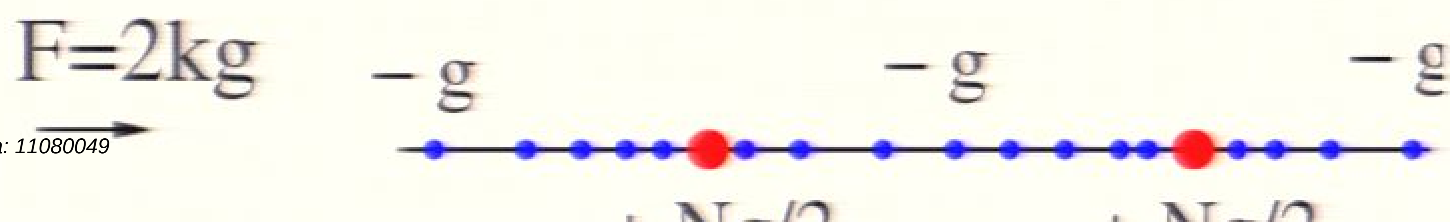

BA roots for the vacuum state

$$\theta^{(N)}_{-\frac{N}{2}} < \theta^{(N)}_{-\frac{N}{2}+1} < \ldots < \theta^{(N)}_{\frac{N}{2}-2} < \theta^{(N)}_{\frac{N}{2}-1}$$

ritical value of YY-functional

$$Y^{(N)} = Y^{(N)}(\Theta, \xi, k)$$

$$E^{(N)} = \left(\frac{\partial Y^{(N)}}{\partial \Theta}\right)_{N,\xi,k}$$

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t large N and finite Θ

$$\rho^{(N)}(\theta_{n+\frac{1}{2}}) = \frac{1}{N(\theta_{n+1} - \theta_n)}$$

well approximated by the continuous density

$$\rho(\theta) = \frac{1}{2\pi}\left[\frac{1}{\cosh(\theta - \Theta)} + \frac{1}{\cosh(\theta + \Theta)}\right]$$

BA roots for the vacuum state

$$\theta^{(N)}_{-\frac{N}{2}} < \theta^{(N)}_{-\frac{N}{2}+1} < \ldots < \theta^{(N)}_{\frac{N}{2}-2} < \theta^{(N)}_{\frac{N}{2}-1}$$

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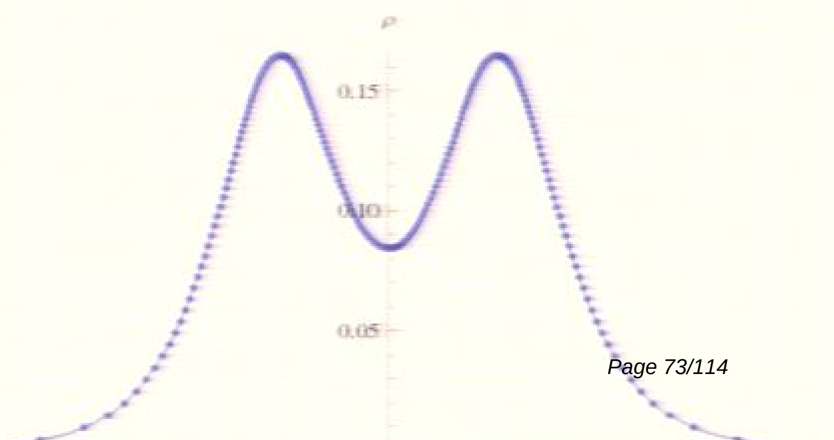

$$|\Theta| \to +\infty$$

$$\left[\frac{s(\theta_j + \Theta + \frac{i\pi}{2})\, s(\theta_j - \Theta + \frac{i\pi}{2})}{s(\theta_j + \Theta - \frac{i\pi}{2})\, s(\theta_j - \Theta - \frac{i\pi}{2})}\right]^N = -e^{\frac{4i\pi\xi k}{1+\xi}} \prod_n \frac{s(\theta_j - \theta_n + i\pi)}{s(\theta_j - \theta_n - i\pi)}$$

$$s(x) = \sinh\left(\frac{x}{1+\xi}\right)$$

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"monopole-monopole" interaction of the clasters

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Intrinsic potential energy of the claster

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$$E^{(N)}(\Theta) - \frac{\xi N^2}{1+\xi} = o(1)$$

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$N, \Theta \to +\infty$

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Normalized on-shell action

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Normalized on-shell action $\mathfrak{Y} = \mathcal{A}^* - \mathcal{A}^*_\infty$

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Normalized on-shell action $\quad \mathfrak{Y} = \mathcal{A}^* - \mathcal{A}^*_\infty$

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$$\mathfrak{Y} = \tfrac{1}{6}\, c_{\text{eff}}\ \log(MR) + \mathfrak{Y}_0 - \frac{(MR)^2}{8\pi}\ \tan\left(\frac{\pi\xi}{2}\right) - \int_0^R \frac{\mathrm{d}R}{\pi}\left(E_k + \frac{\pi c_{\text{eff}}}{6R}\right)$$

Conclusion: Faces of integrability

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Classical Sinh-Gordon PDE: On-shell Action

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:lassical Sinh-Gordon PDE: On-shell Action

;ine-Gordon QFT: Critical values of YY-functional

Conclusion: Faces of integrability

Classical Sinh-Gordon PDE: On-shell Action

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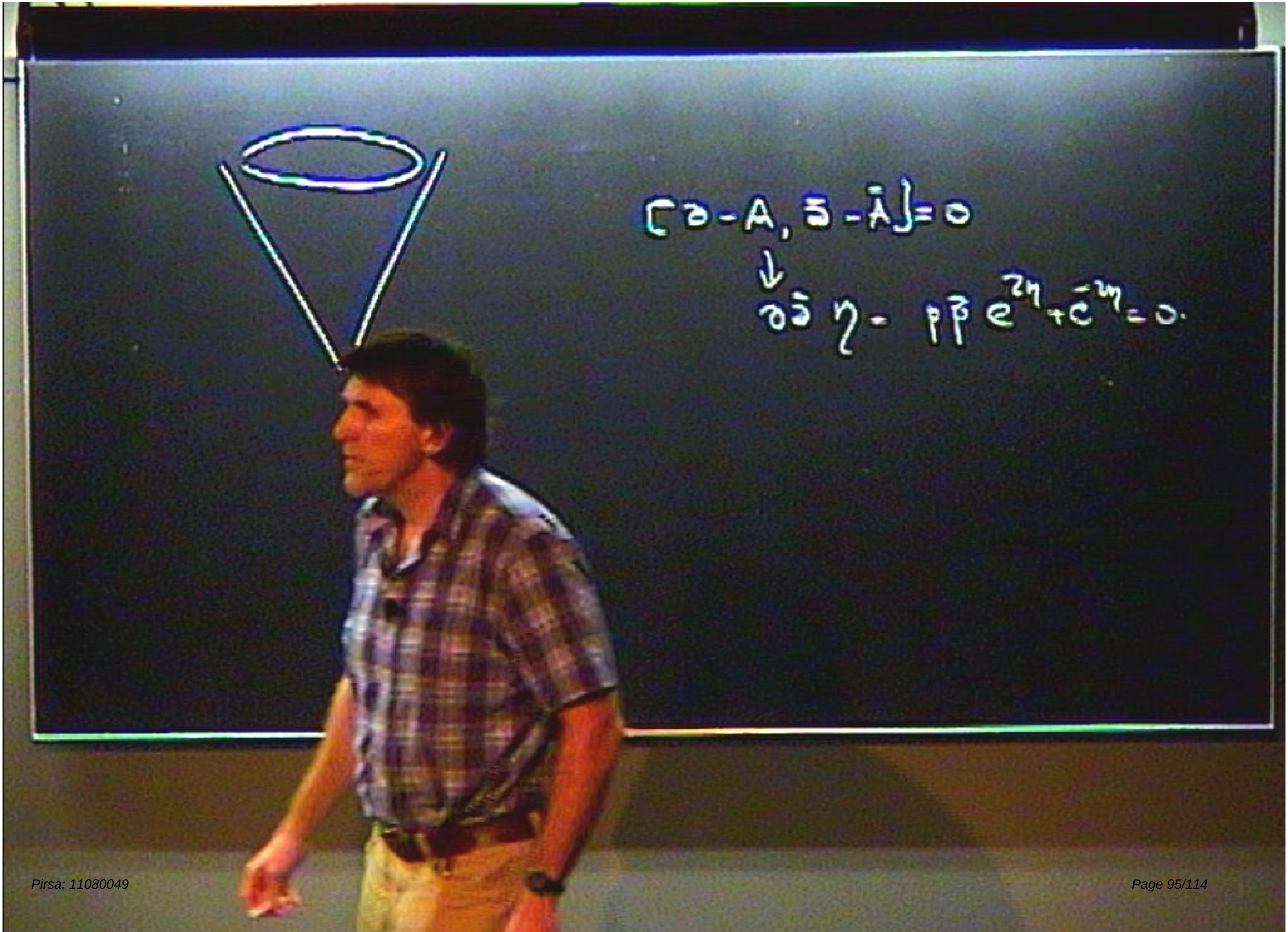
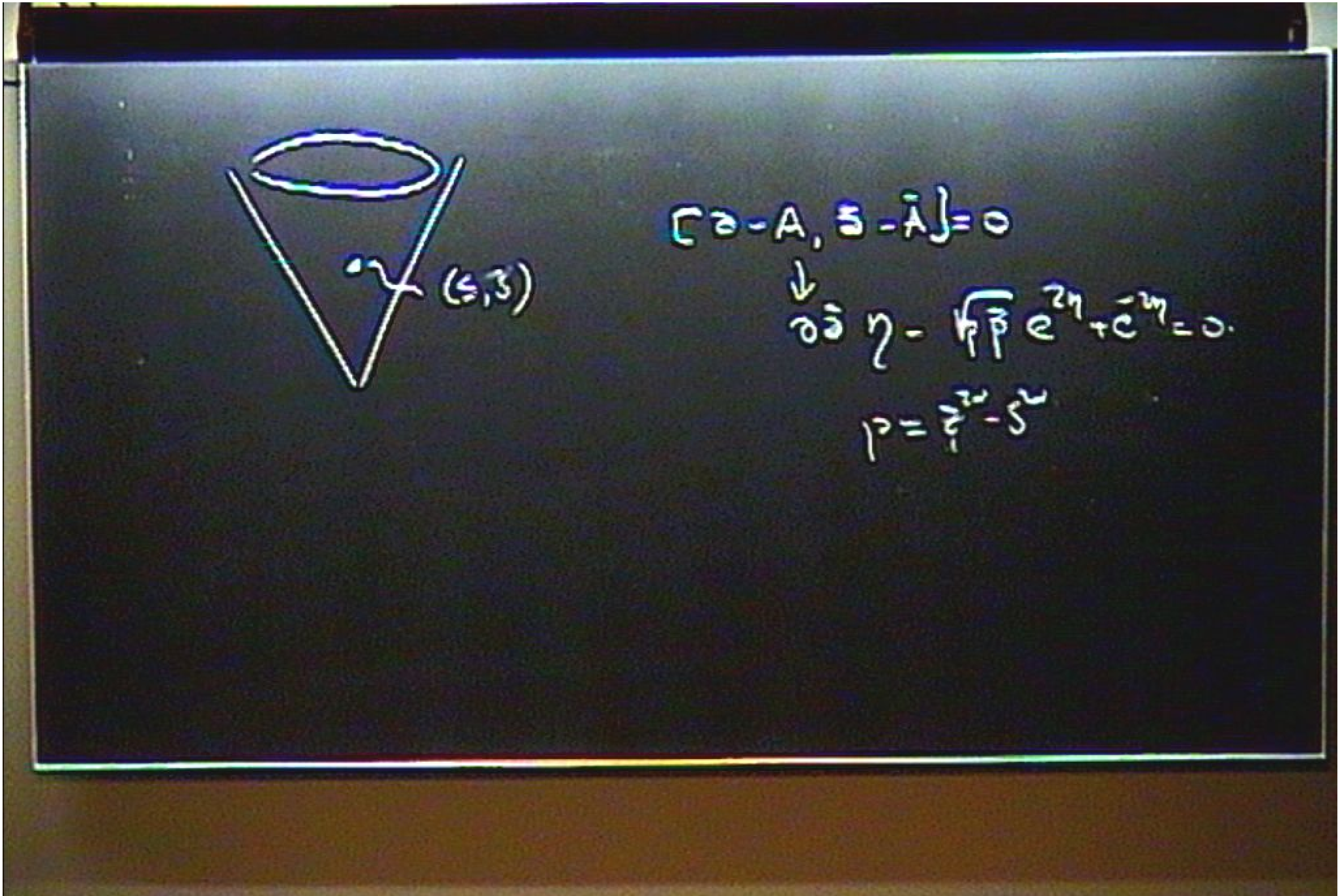
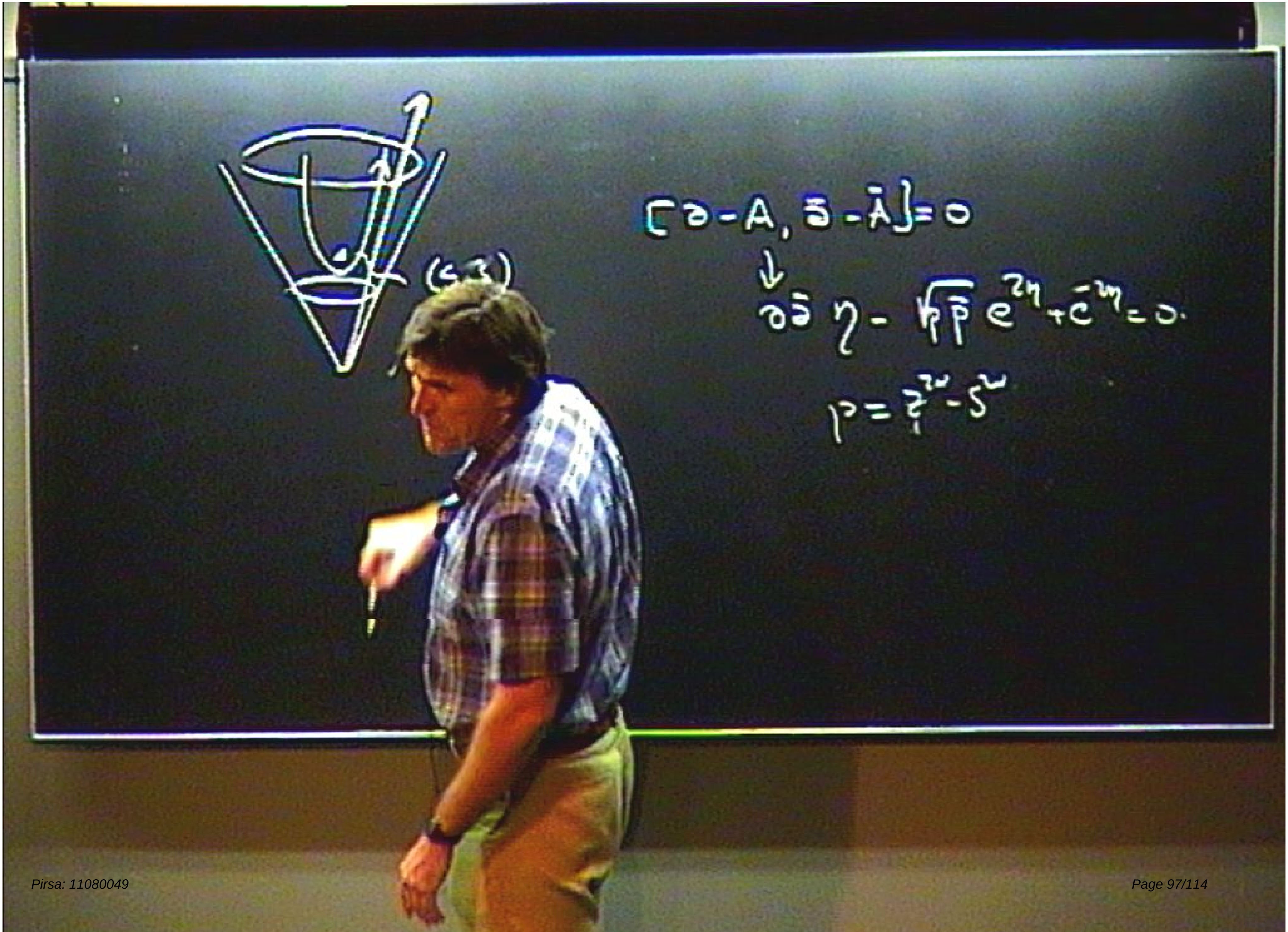
Sine-Gordon QFT: Critical values of YY-functional


[∂-A, ∂̄-Ā]=0
∂∂̄ η - p p̄ e^{2η} + c̄ e^{-2η} = 0.


(s, z̄)
[∂ − A, ∂̄ − Ā] = 0
∂∂̄ η − √(p p̄) e^{2η} + e^{−2η} = 0.
p = z^{2ω} − s^ω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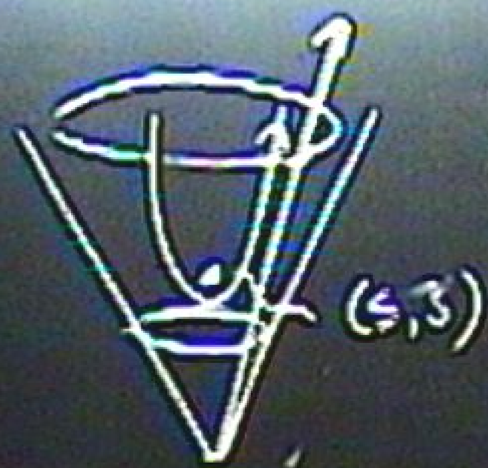
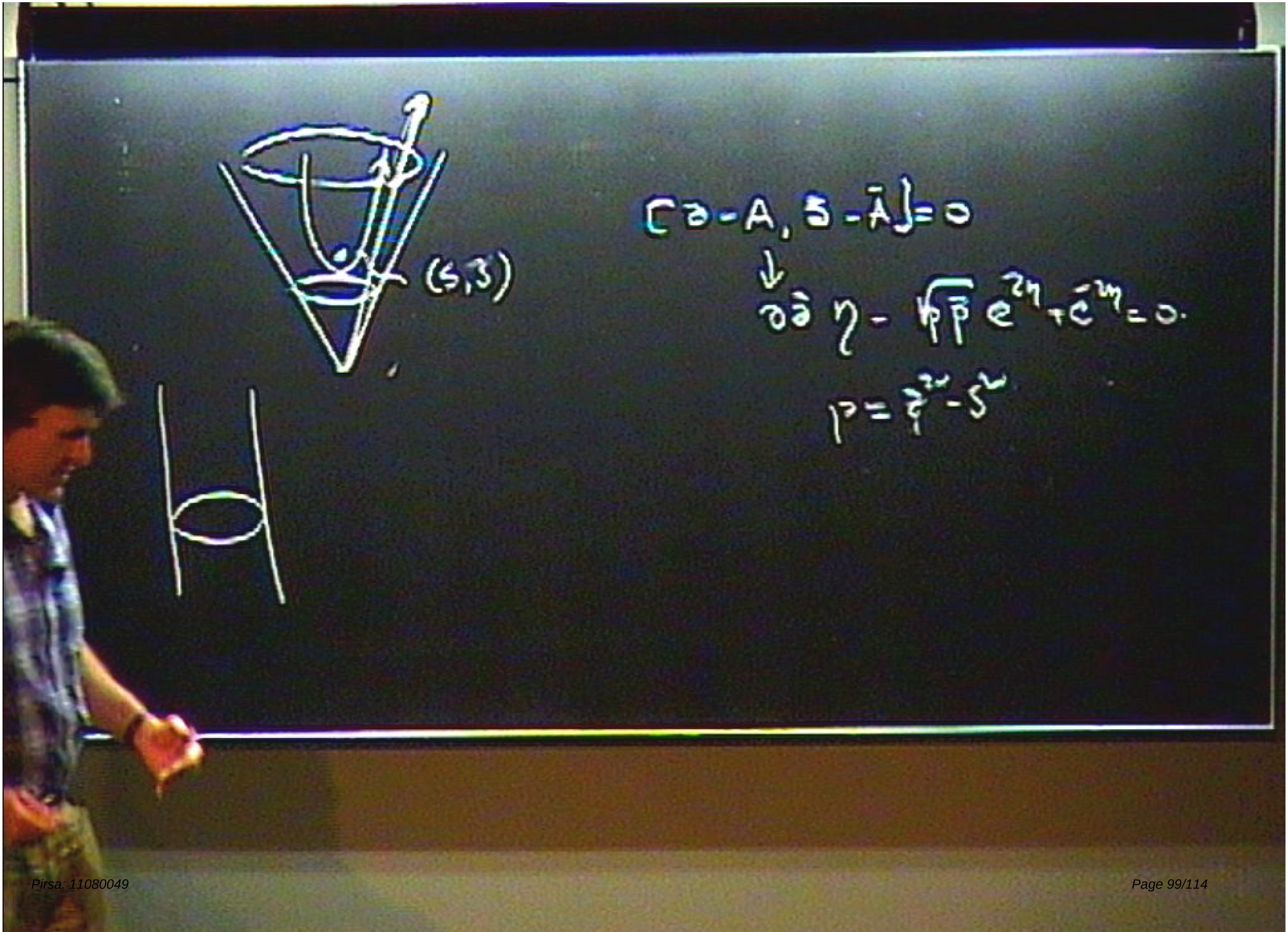


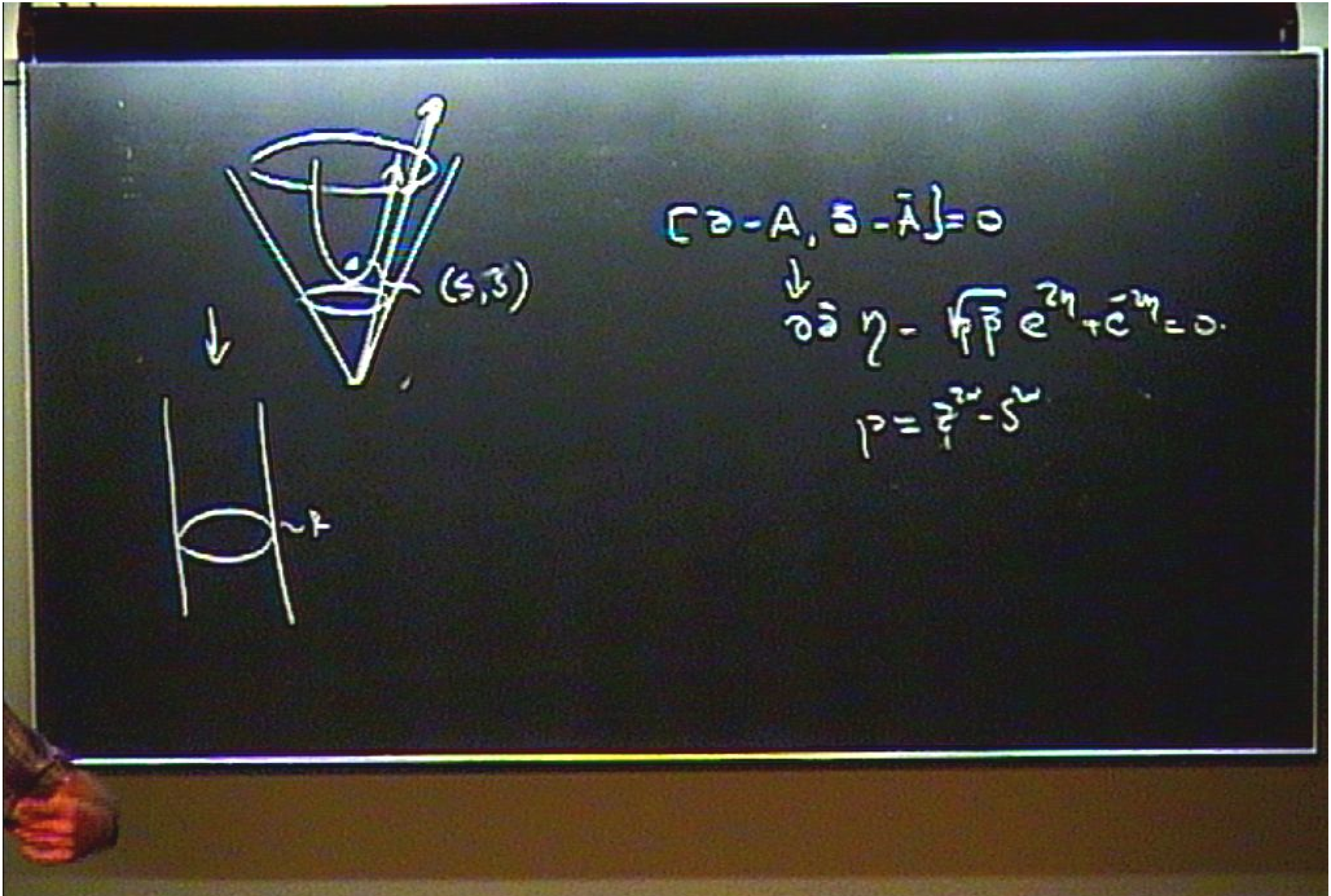
$$[\partial - A, \bar\partial - \bar A] = 0$$

$$\downarrow$$

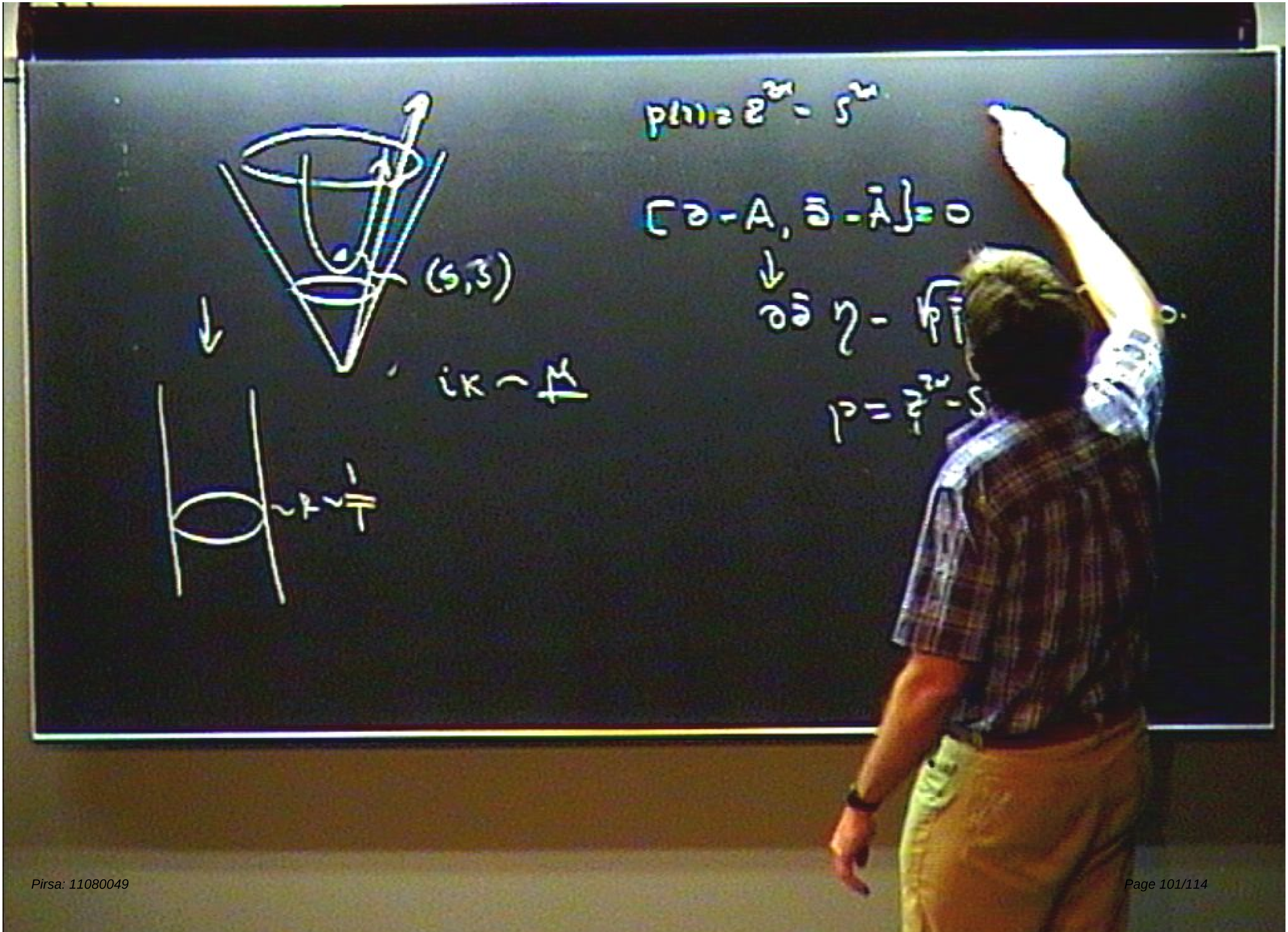
$$\partial\bar\partial\,\eta - \sqrt{p\bar p}\,e^{2\eta} + e^{-2\eta} = 0.$$

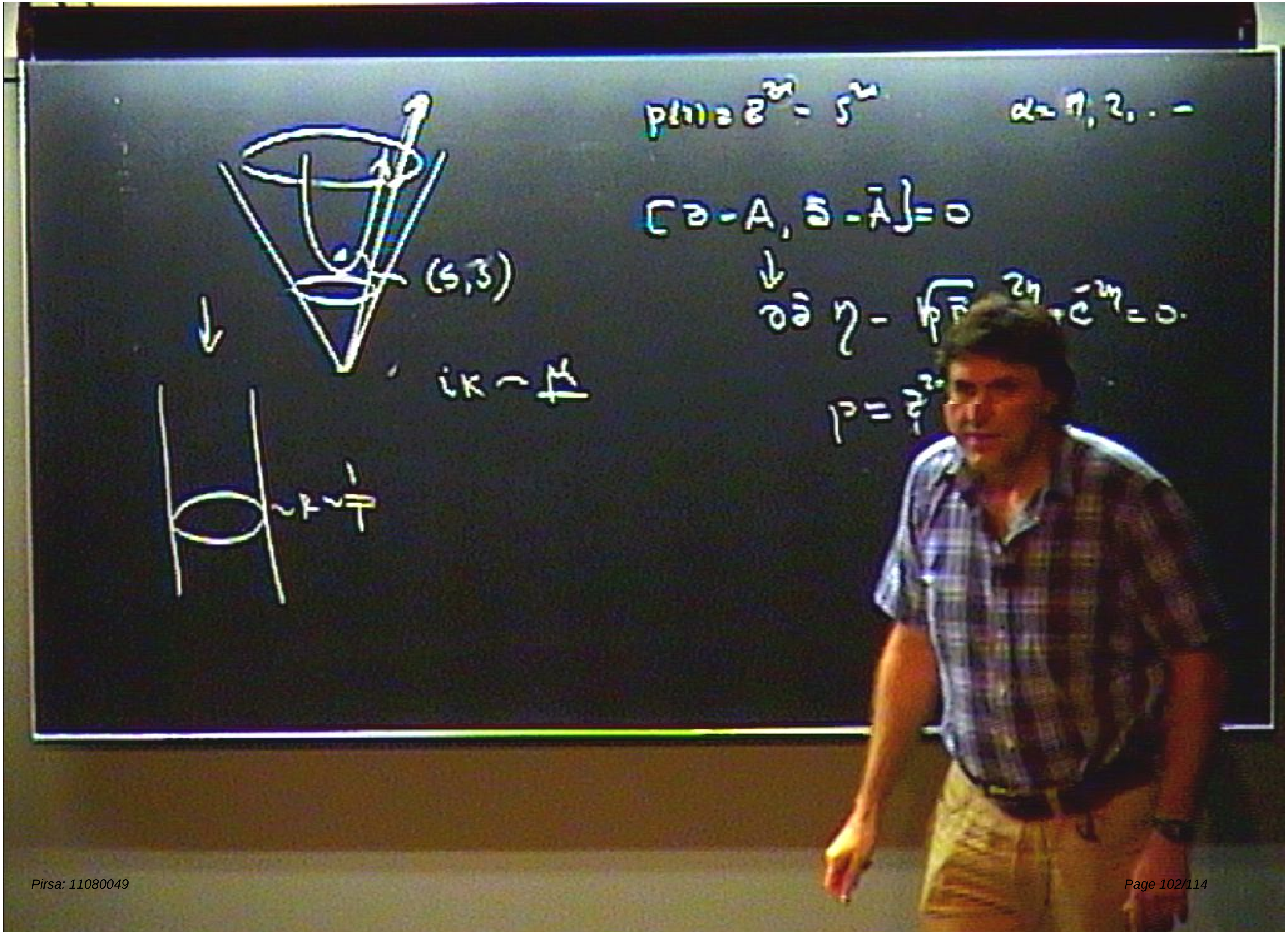
$$p = z^{2\omega} - s^{\omega}$$


(s,3)
[∂ - A, ∂̄ - Ā] = 0
∂∂̄ η - √(p p̄) e^{2η} + e^{-2η} = 0.
p = ξ² - s²



(ς,ζ)
[∂ − A, ∂̄ − Ā] = 0
∂∂̄ η − √(p p̄) e^{2η} + e^{−2η} = 0.
p = ζ^{2w} − ς^{2w}


(s,3)
[∂ - A, ∂̄ - Ā] = 0
ik ~ μ


(s,3)
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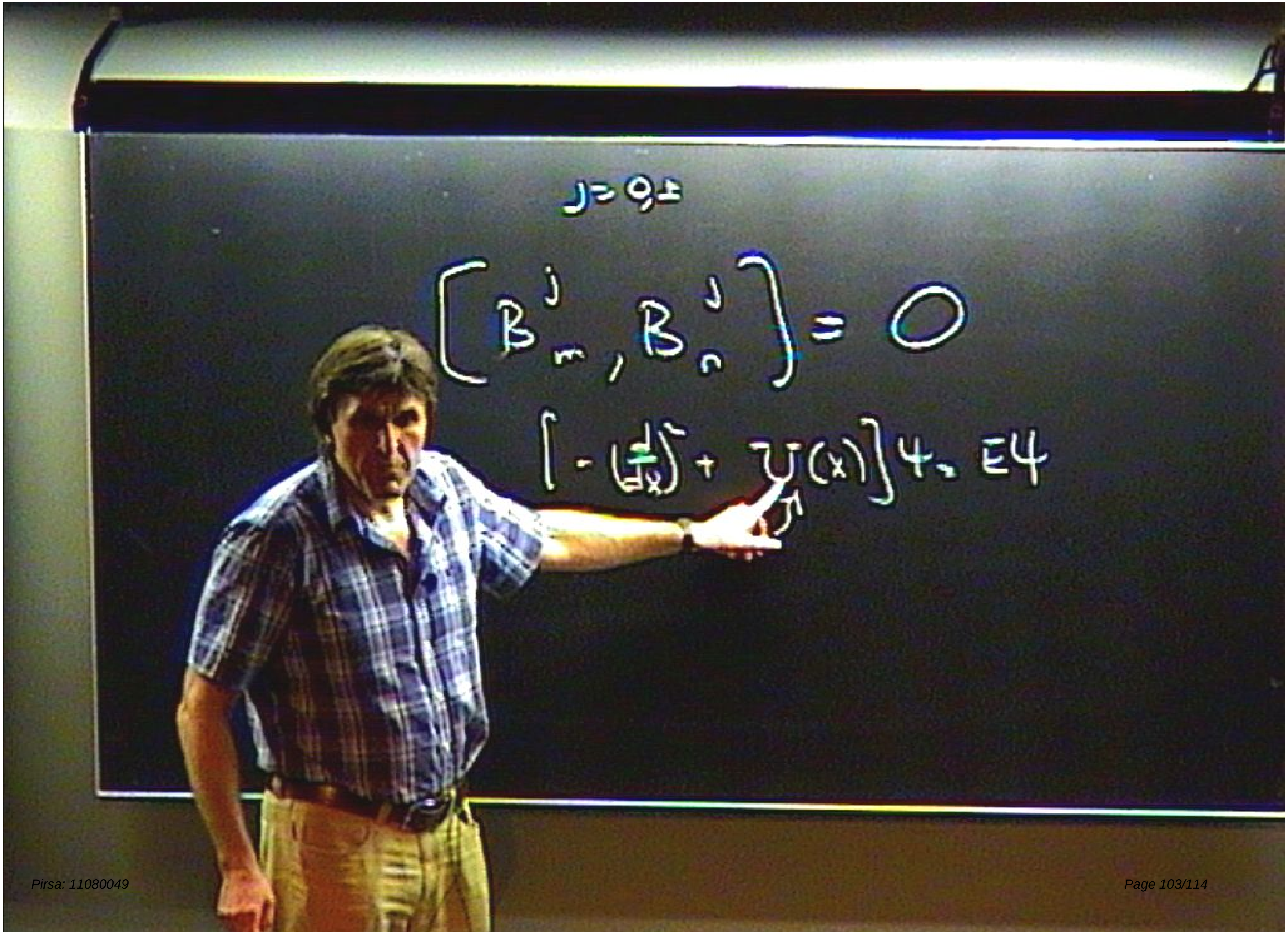
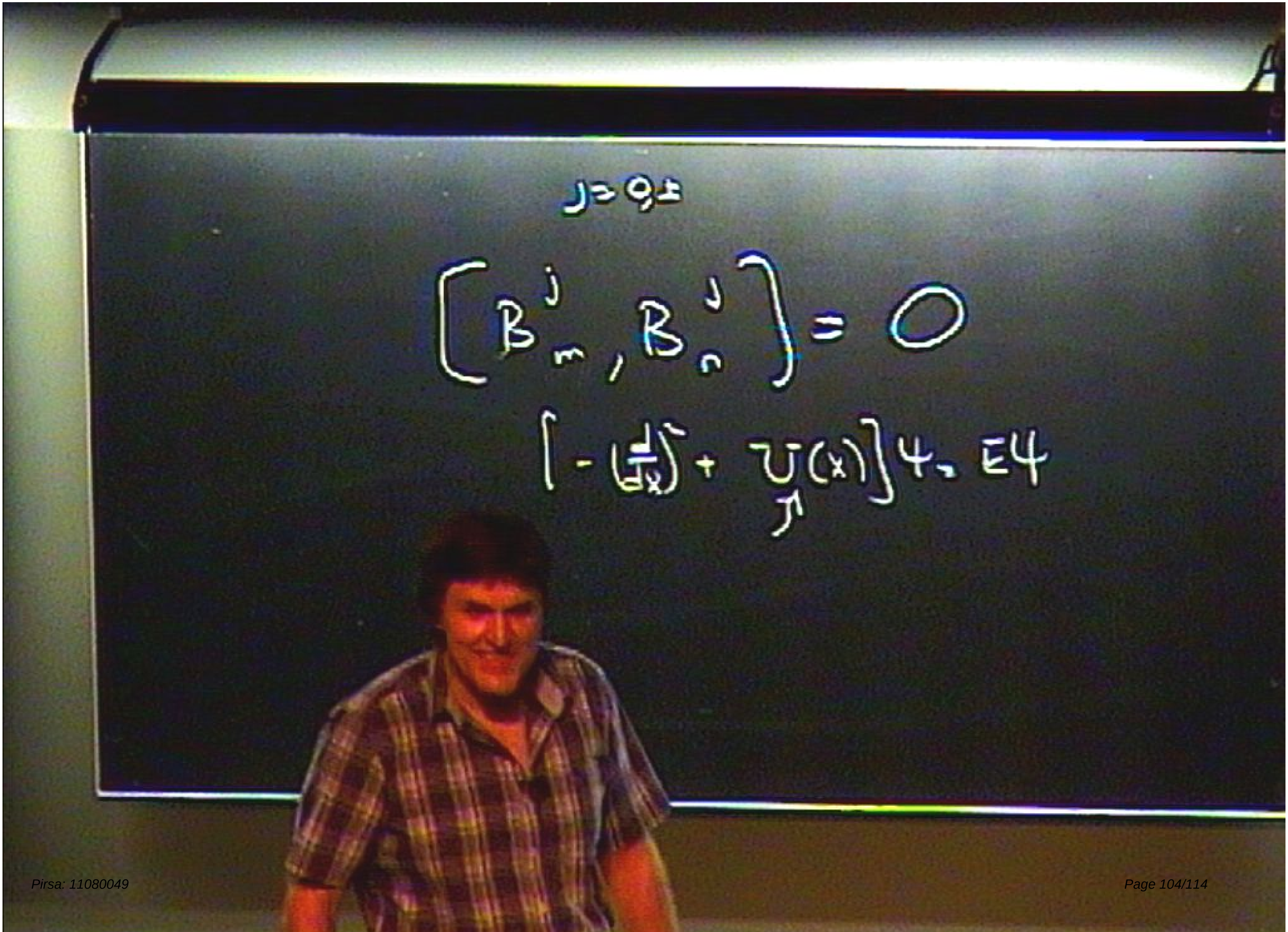

j = 0, 1
[B^j_m , B^j_n] = 0
[- (d/dx)^2 + V(x)] ψ = Eψ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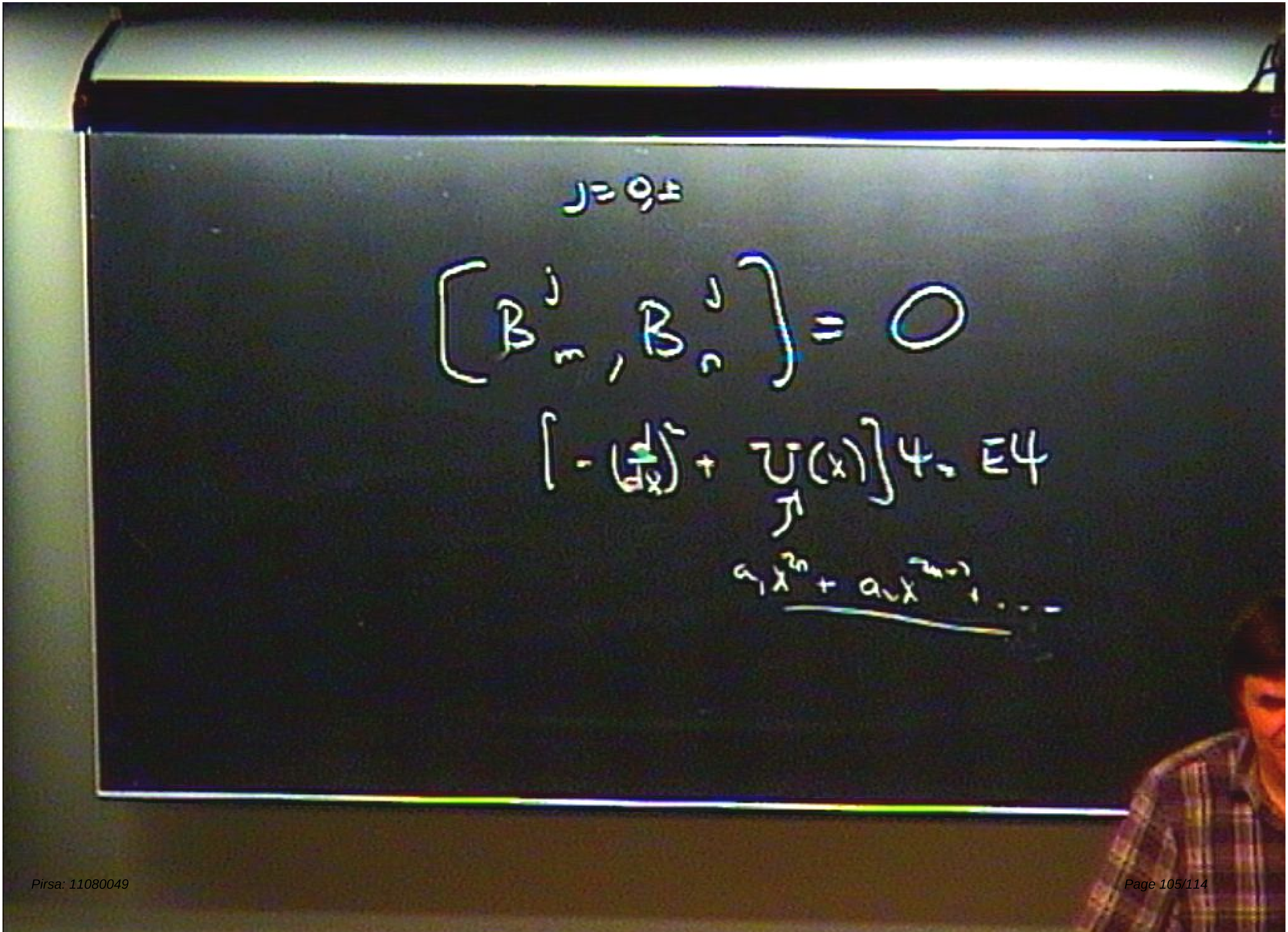
j = 0, ±
[B^j_m, B^j_n] = 0
[-(d/dx)^2 + V(x)]ψ = Eψ
a_1 x^2n + a_2 x^2n-1 + ...

Conclusion: Faces of integrability

Classical Sinh-Gordon PDE: On-shell Action

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Sine-Gordon QFT: Critical values of YY-functional

$$\lim_{\substack{N,\Theta\to+\infty\\ r-\text{fixed}}} Y_{\text{int}}^{(N)}(\Theta) = -\int_0^r \frac{\mathrm{d}r}{\pi r}\,\left(R E_k + \frac{\pi c_{\text{eff}}}{6} \right)$$

$$\lim_{\substack{\Theta\to+\infty\\ -\text{fixed}}} Y_{\text{int}}^{(N)}(\Theta) = \mathfrak{Y} - \tfrac{1}{6}\, c_{\text{eff}}\ \log(r) + \frac{r^2}{8\pi}\ \tan\left(\frac{\pi\xi}{2}\right) - \mathfrak{Y}_0$$

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$N, \Theta \to +\infty$

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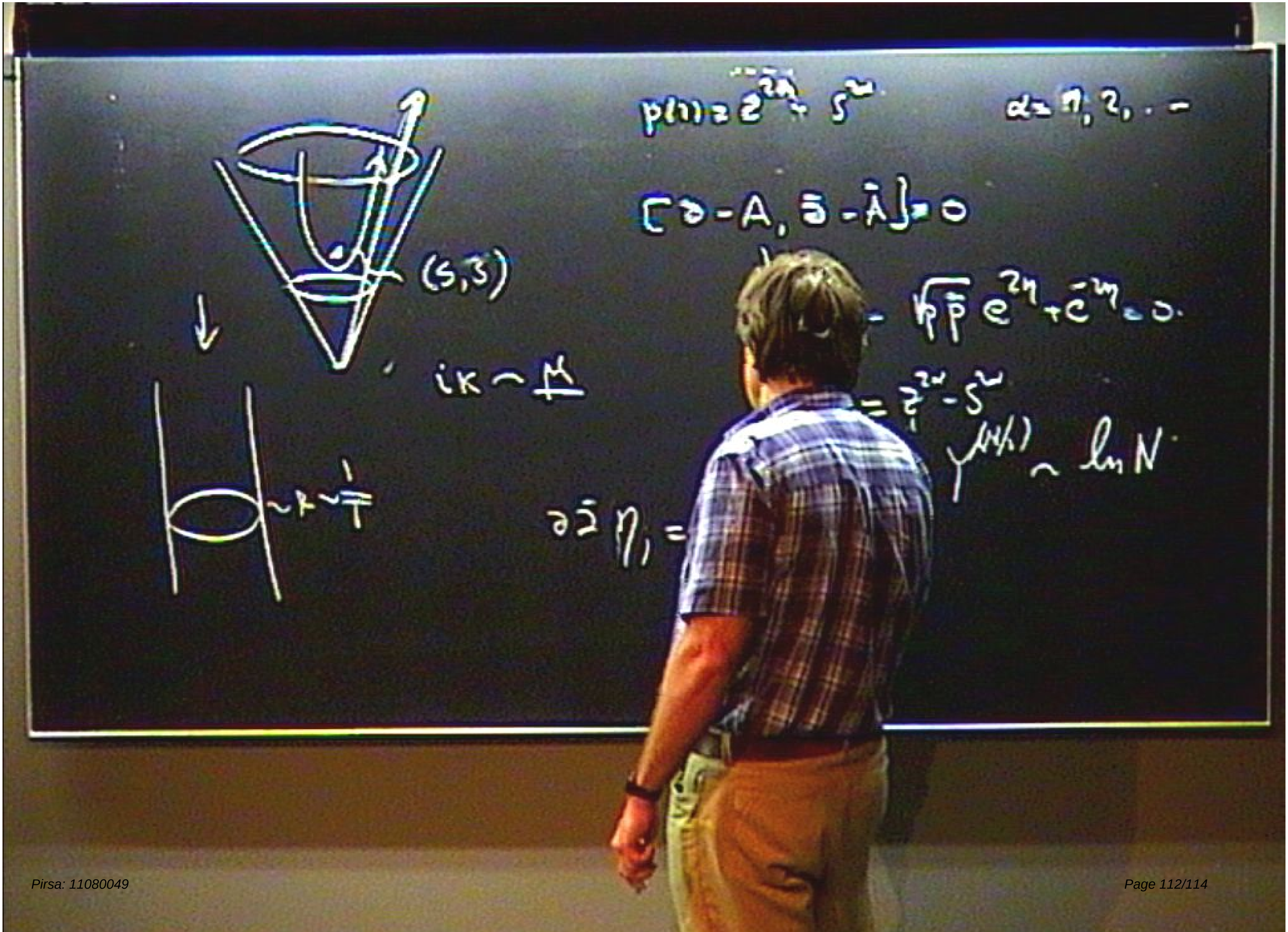
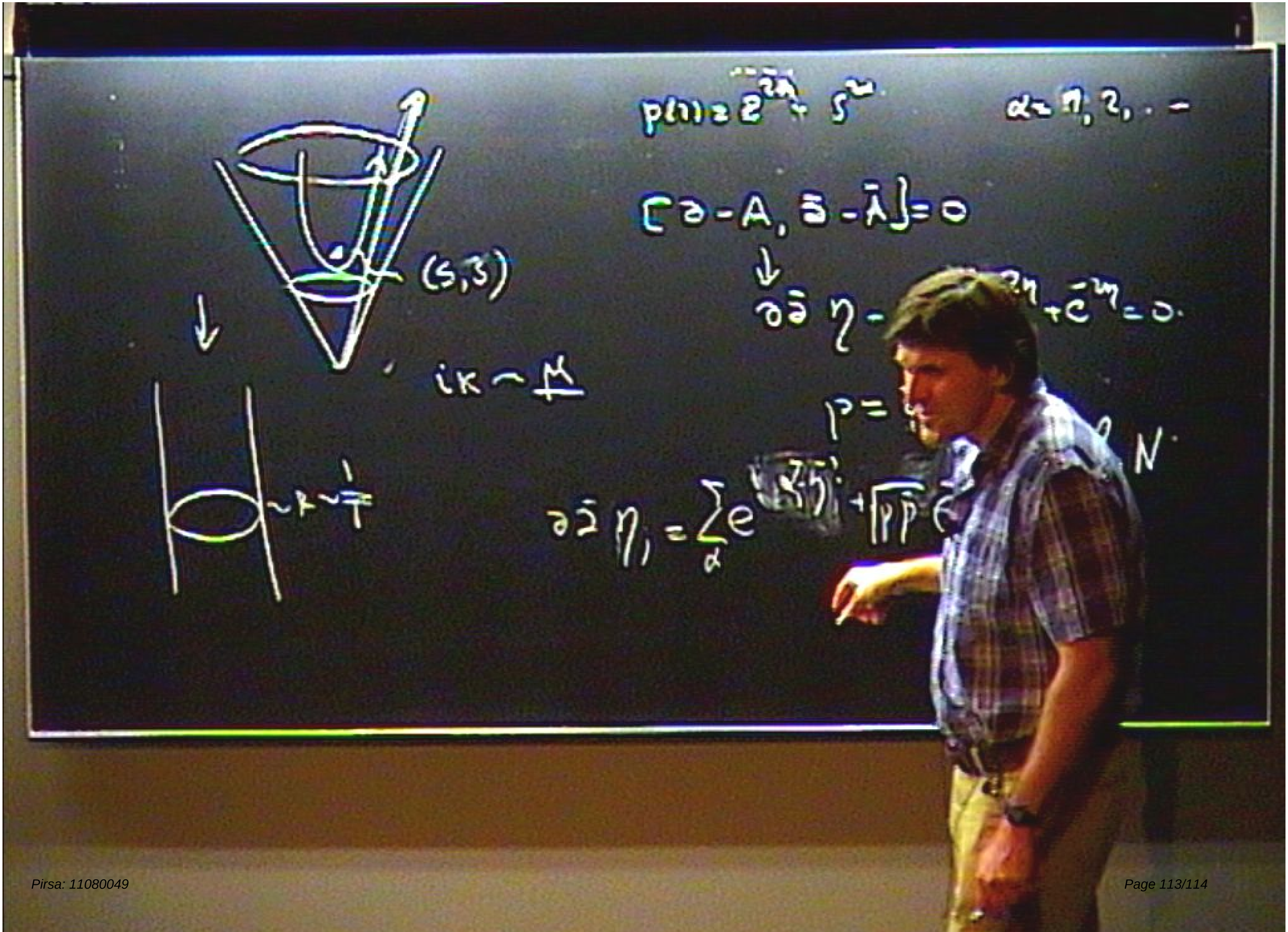
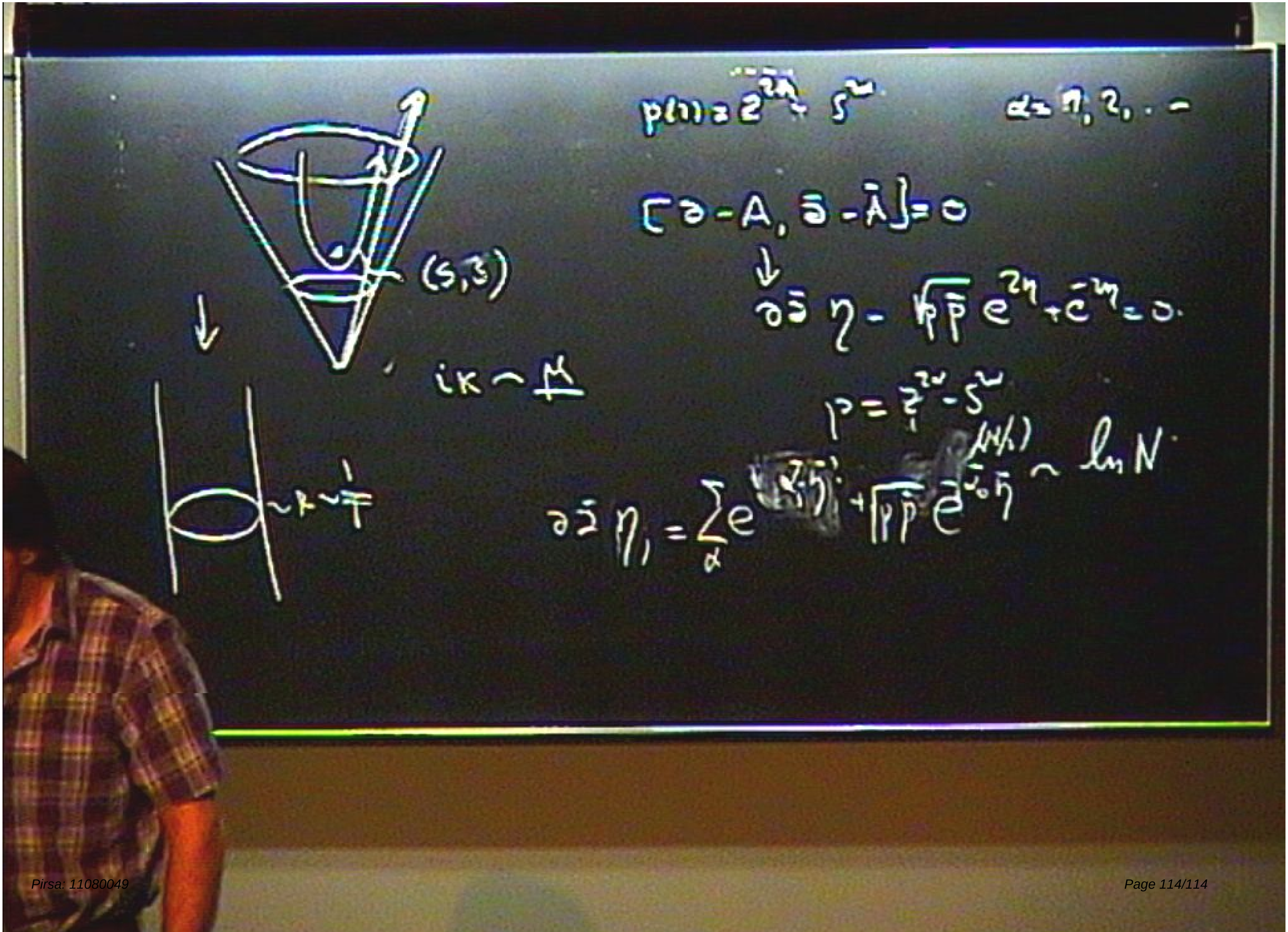

(5,3)
ln N




(5,3)
[∂ - A, ∂̄ - Ā] = 0
~ ln N