Title: Euclidean Wilson Loops and Riemann Theta Functions

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Abstract: For N=4 super Yang-Mills theory, in the large-N limit and at strong coupling, Wilson loops can be computed using the AdS/CFT correspondence. In the case of flat Euclidean loops the dual computation consists in finding minimal area surfaces in Euclidean AdS3 space. In such case very few solutions were known. In this talk I will describe an infinite parameter family of minimal area surfaces that can be described analytically using Riemann Theta functions. Furthermore, for each Wilson loop a one parameter family of deformations that preserve the area can be exhibited explicitly. The area is given by a one dimensional integral over the world-sheet boundary.

- **Theta functions solve e.o.m.**
  - Theta functions associated w/ Riemann surfaces
  - Main Properties and some interesting facts.
  - Formula for the renormalized area.
- **Closed Wilson loops for g=3 (and g=1)**
  - Particular solutions, plots, etc.
- **Conclusions**

AdS/CFT correspondence (Maldacena, GKP, Witten)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N}=4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ, Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

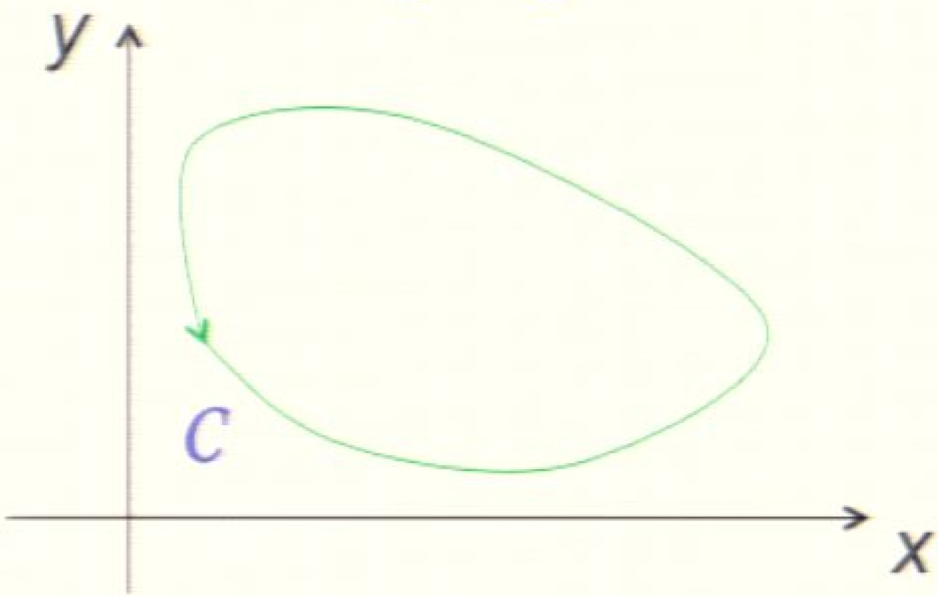
$$g_s = g_{YM}^2; \qquad R/l_s = (g_{YM}^2 N)^{1/4}$$

$N \to \infty, \lambda = g_{YM}^2 N$ fixed ➡

λ large → string th.
λ small → field th.

Wilson loops

Basic operators in gauge theories.



$$W = \frac{1}{N} \mathrm{Tr}\, \hat{P} \exp\left\{ i \oint_C \left(A_\mu \frac{dx^\mu}{ds} + \theta_0^I \Phi_I \left| \frac{dx^\mu}{ds} \right| \right) ds \right\}$$

Simplest example: single, flat, smooth, space-like curve (with constant scalar).

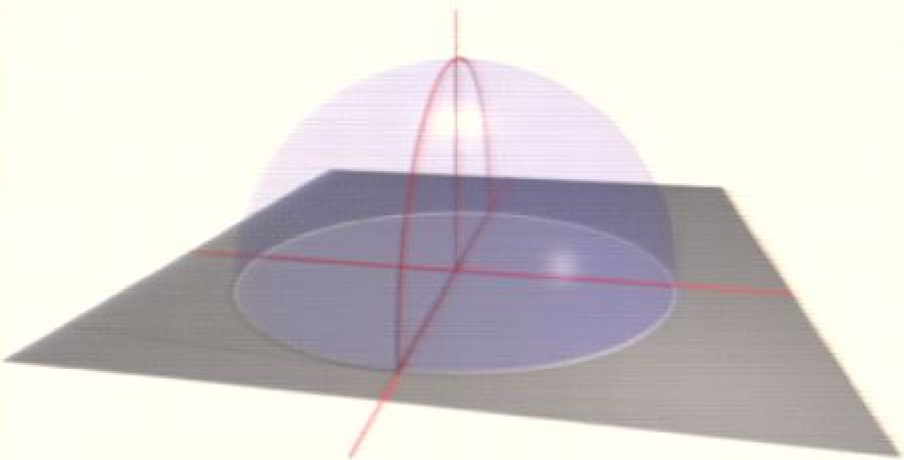
Wilson loops in the AdS/CFT correspondence

(Maldacena, Rey, Yee)

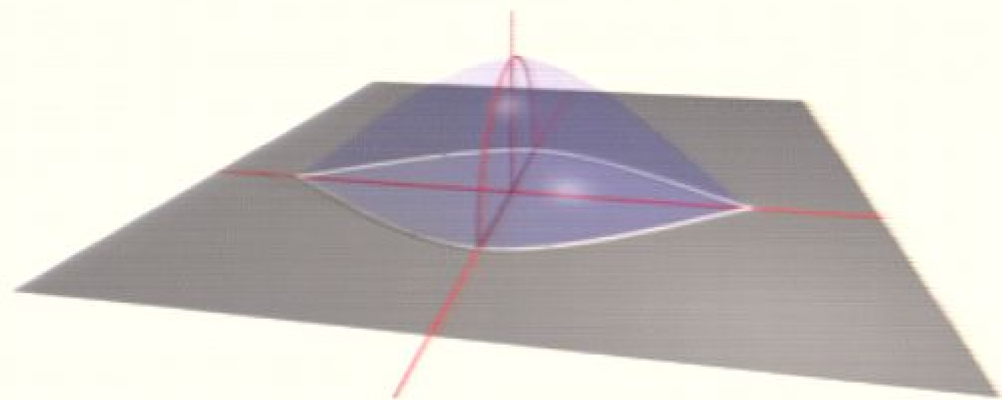
Euclidean, Wilson loops with constant scalar =
Minimal area surfaces in Euclidean AdS_3

Closed curves:

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$



circular



lens-shaped

Berenstein Corrado Fischler Maldacena
Gross Ooguri, Erickson Semenoff Zarembo
Drukker Gross, Pestun

Drukker Giombi Ricci Trancanelli

Wilson loops

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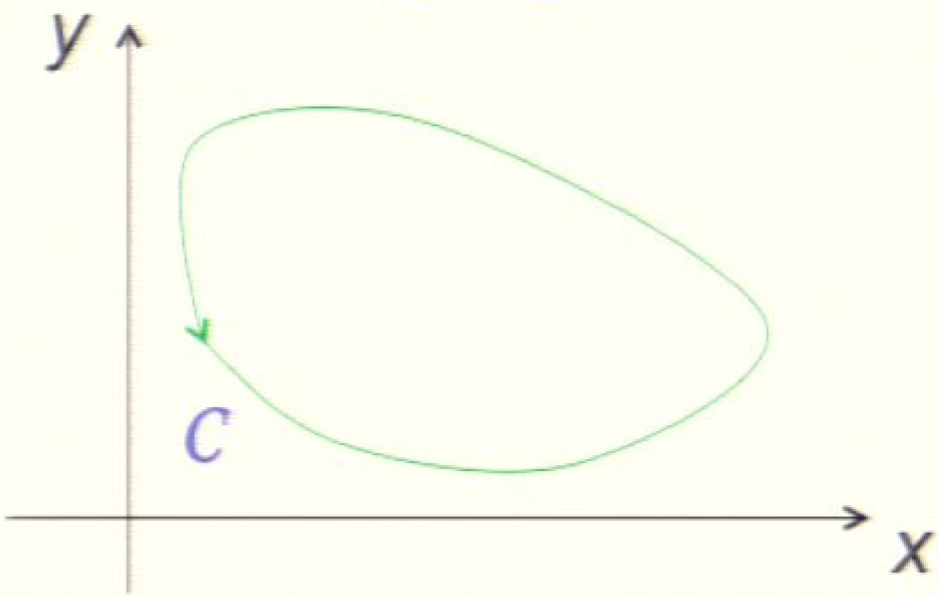

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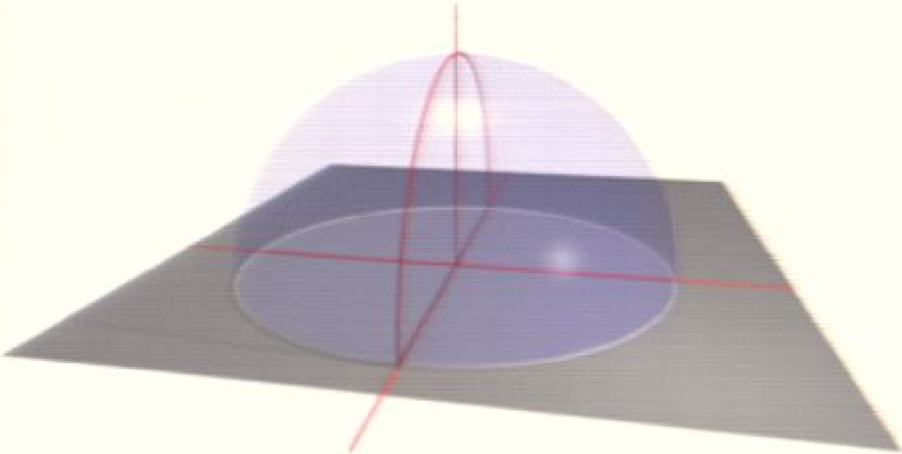

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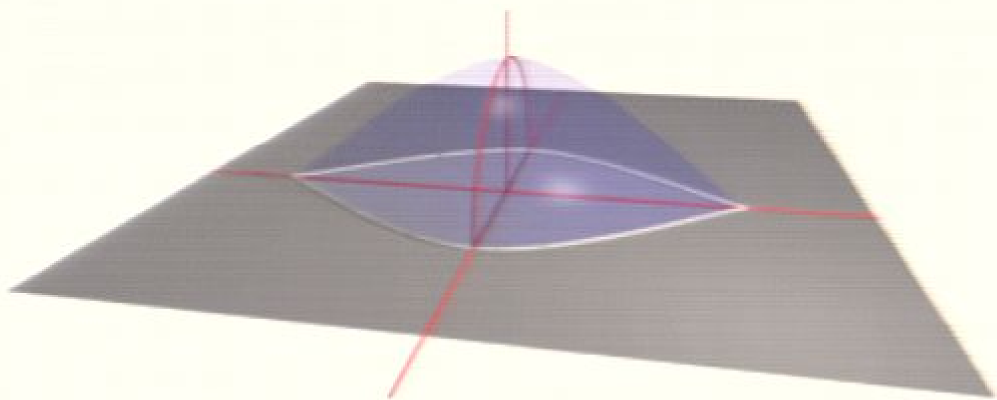

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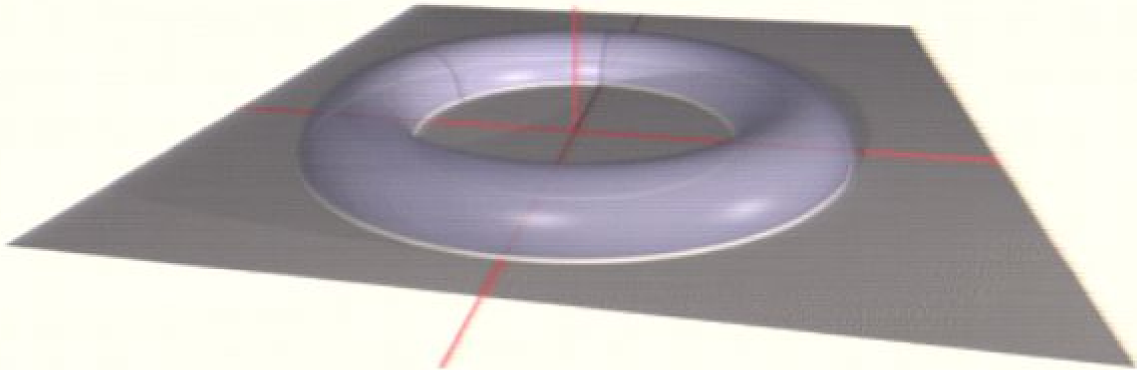
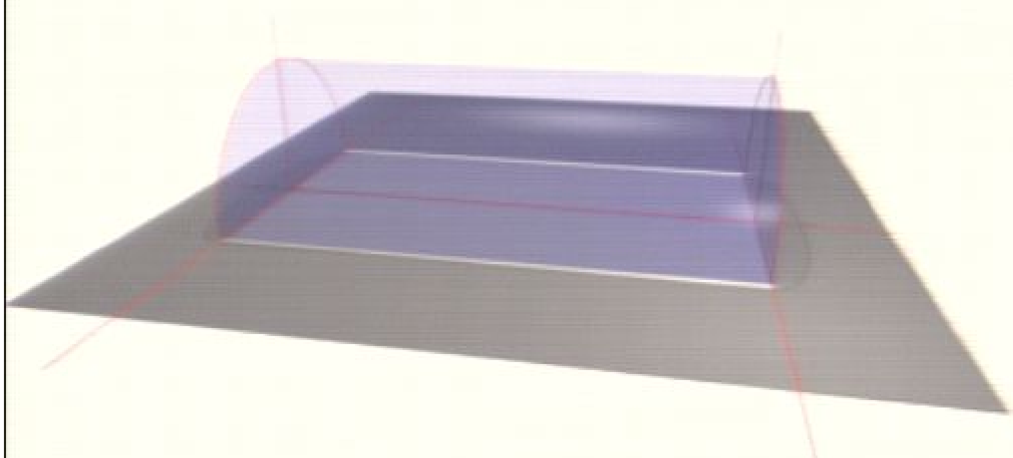
Multiple curves:

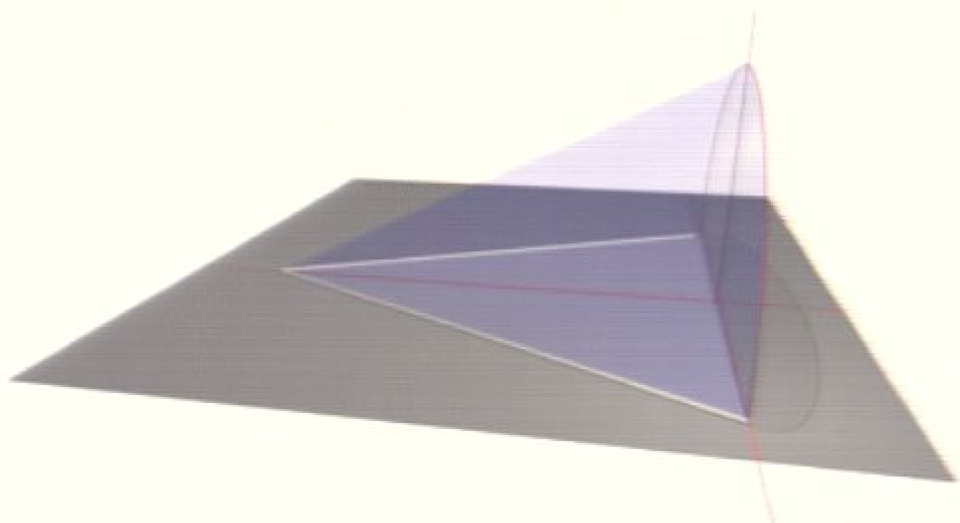


Drukker Fiol

concentric circles

Euclidean, open Wilson loops:





Maldacena, Rey Yee parallel lines

Drukker Gross Ooguri cusp

Many interesting and important results for Wilson loops with non-constant scalar and for Minkowski Wilson loops (lots of recent activity related to light-like cusps and their relation to scattering amplitudes).

Result: more generic examples for Euclidean Wilson loops can be found using Riemann theta functions.

Babich, Bobenko. **(our case)**
Kazakov, Marshakov, Minahan, Zarembo **(sphere)**
Dorey, Vicedo. **(Minkowski space-time)**
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In Poincare coordinates the minimal area surfaces are:

$$Z = \left| \frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})} \right| \frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)|\,|e^{\mu z+\nu\bar{z}}|^2}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2 + |\theta(\zeta-\int_{p_1}^{p_4})|^2} .$$

$$X + iY = e^{2\bar{\mu}\bar{z}+2\bar{\nu}z} \frac{\theta(\zeta-\int_{p_1}^{p_4})\overline{\theta(\zeta+\int_{p_1}^{p_4})} - \hat{\theta}(\zeta-\int_{p_1}^{p_4})\overline{\hat{\theta}(\zeta+\int_{p_1}^{p_4}}}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2 + |\theta(\zeta-\int_{p_1}^{p_4})|^2}$$

which we will now describe in detail.

Minimal Area surfaces in AdS_3

Equations of motion and Pohlmeyer reduction

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 = 1 \quad X + iY = \frac{X_1 + iX_2}{X_0 - X_3}, \quad Z = \frac{1}{X_0 - X_3}$$

$$z = \sigma + i\tau, \ \bar{z} = \sigma - i\tau$$

$$\begin{aligned} S &= \frac{1}{2}\int \left(\partial X_\mu \bar{\partial} X^\mu - \Lambda(X_\mu X^\mu - 1)\right) \, d\sigma \, d\tau \\ &= \frac{1}{2}\int \frac{1}{Z^2}\left(\partial_a X \partial^a X + \partial_a Y \partial^a Y + \partial_a Z \partial^a Z\right) \, d\sigma \, d\tau \end{aligned}$$

$$\partial\bar{\partial} X_\mu = \Lambda X_\mu \qquad \Lambda = -\partial X_\mu \bar{\partial} X^\mu$$

$$\partial X_\mu \partial X^\mu = 0 = \bar{\partial} X_\mu \bar{\partial} X^\mu$$

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We can also use:

$$\mathbb{X} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} = X_0 + X_i\sigma^i$$

$$\mathbb{X}^\dagger = \mathbb{X}. \quad \det \mathbb{X} = 1. \quad \partial\bar{\partial}\mathbb{X} = \Lambda\mathbb{X}. \quad \det(\partial\mathbb{X}) = 0 = \det(\bar{\partial}\mathbb{X})$$

X hermitian can be solved by:

$$\mathbb{X} = \mathbb{A}\mathbb{A}^\dagger, \qquad \det \mathbb{A} = 1, \qquad \mathbb{A} \in SL(2,\mathbb{C})$$

Global and gauge symmetries:

$$\mathbb{X} \to U\mathbb{X}U^\dagger, \quad \mathbb{A} \to U\mathbb{A}, \quad U \in SL(2,\mathbb{C})$$

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The currents:

$$J = \mathbb{A}^{-1}\partial\mathbb{A}, \quad \bar{J} = \mathbb{A}^{-1}\bar{\partial}\mathbb{A}$$

$$\mathcal{A} = \frac{1}{2}(\bar{J} + J^\dagger), \quad \mathcal{B} = \frac{1}{2}(J - \bar{J}^\dagger)$$

satisfy:

$$\begin{aligned}
\mathrm{Tr}\mathcal{A} = \mathrm{Tr}\mathcal{B} &= 0\,, \\
\det\mathcal{A} &= 0\,, \\
\partial\mathcal{A} + [\mathcal{B},\mathcal{A}] &= 0\,, \\
\bar{\partial}\mathcal{B} + \partial\mathcal{B}^\dagger &= [\mathcal{B}^\dagger,\mathcal{B}] + [\mathcal{A}^\dagger,\mathcal{A}].
\end{aligned}$$

$$\mathcal{A} \to \mathcal{U}^\dagger\mathcal{A}\mathcal{U}, \quad \mathcal{B} \to \mathcal{U}^\dagger\mathcal{B}\mathcal{U} + \mathcal{U}^\dagger\partial\mathcal{U}, \quad \mathcal{U}(z,\bar{z}) \in SU(2$$

Up to a gauge transformation (rotation) $\mathcal{A}$ is given by:

$$\mathcal{A} = \frac{1}{2} e^{\alpha(z,\bar{z})}(\sigma_1 + i\sigma_2) = e^{\alpha(z,\bar{z})}\sigma_+$$

$\mathrm{Tr}\mathcal{A} = 0$
$\det \mathcal{A} = 0$
gauge

Then: $\mathcal{B} \;=\; -\frac{1}{2}\partial\alpha\sigma_z + f(z)e^{-\alpha}\sigma_+$

$$\mathcal{A} \;=\; \bar{\lambda} e^{\alpha}\sigma_+ \;, \qquad |\lambda| = 1$$

$$\mathcal{B} \;=\; -\frac{1}{2}\partial\alpha\sigma_z + e^{-\alpha}\sigma_+ \;,$$

$$\partial\bar{\partial}\alpha \;=\; 2\cosh(2\alpha) \;,$$

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Summary

Solve $\partial\bar{\partial}\alpha = 2\cosh 2\alpha$

plug it in $\mathcal{A}$, $\mathcal{B}$ giving:

$$J = \begin{pmatrix} -\frac{1}{2}\partial\alpha & e^{-\alpha} \\ \lambda e^{\alpha} & \frac{1}{2}\partial\alpha \end{pmatrix}, \qquad \bar{J} = \begin{pmatrix} \frac{1}{2}\bar{\partial}\alpha & \bar{\lambda} e^{\alpha} \\ -e^{-\alpha} & -\frac{1}{2}\bar{\partial}\alpha \end{pmatrix}$$

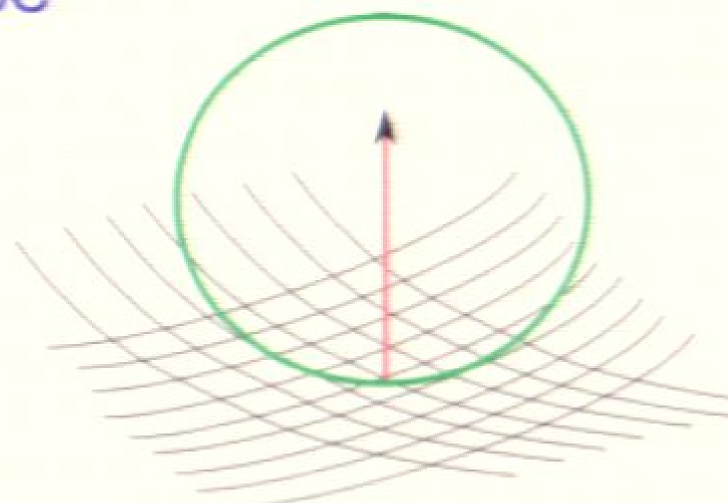
Solve:

$$\begin{aligned} \partial\mathbb{A} &= \mathbb{A}J\,, \\ \bar{\partial}\mathbb{A} &= \mathbb{A}\bar{J}. \end{aligned} \quad \Longrightarrow \quad \mathbb{X} = \mathbb{A}\mathbb{A}^{\dagger}$$

Babich, Bobenko: Solve eqns. using Theta functions

Motivation: Willmore tori in flat space

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$



Surface: $\kappa_1 = \frac{1}{R_1},\ \kappa_2 = \frac{1}{R_2},$ $R_{1,2}$ max. and min. R

Gauss curvature: $K = \kappa_1 \kappa_2$

Mean curvature: $H = \frac{1}{2}(\kappa_1 + \kappa_2)$

Willmore functional: $\mathcal{W} = \frac{1}{4}\int (\kappa_1 - \kappa_2)^2 \, d\mathcal{A} = \int H^2 - \int K$

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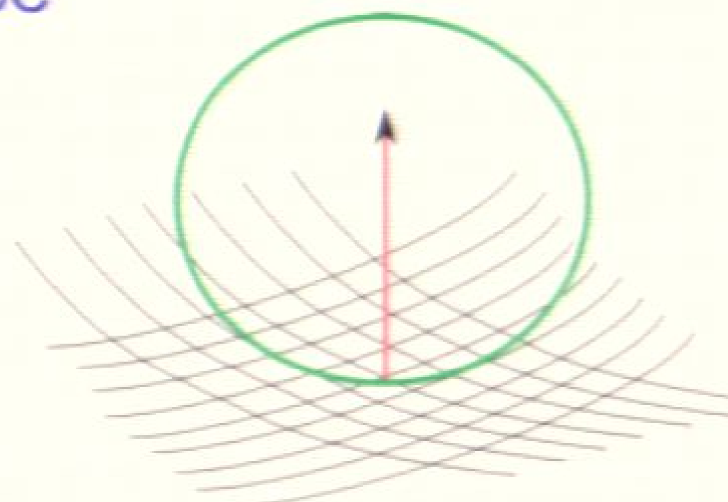

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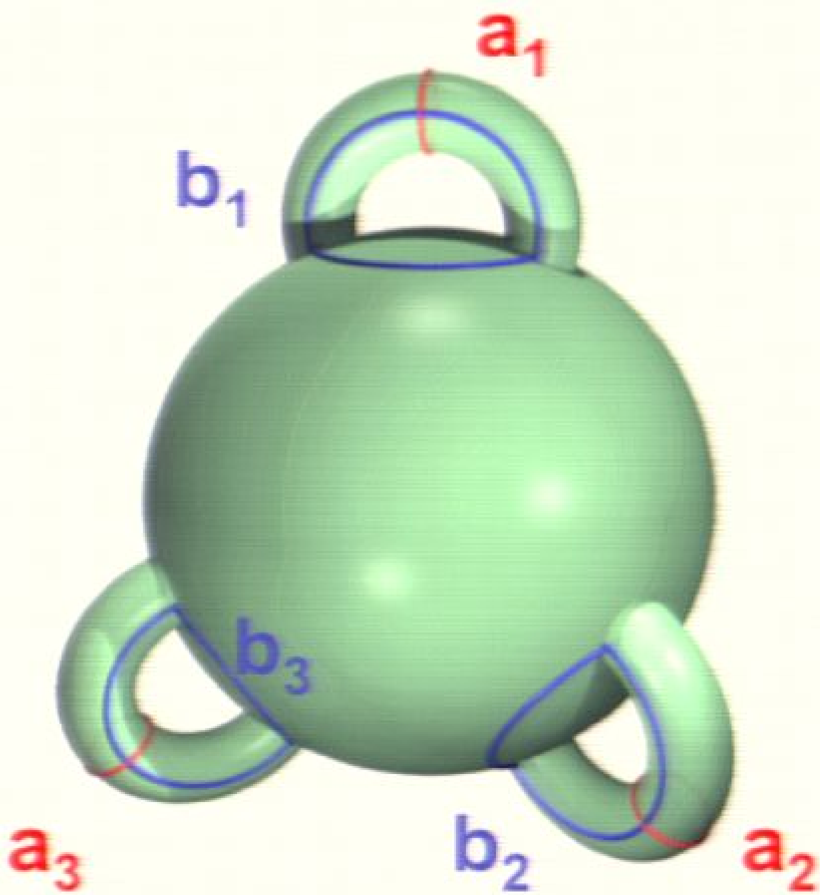
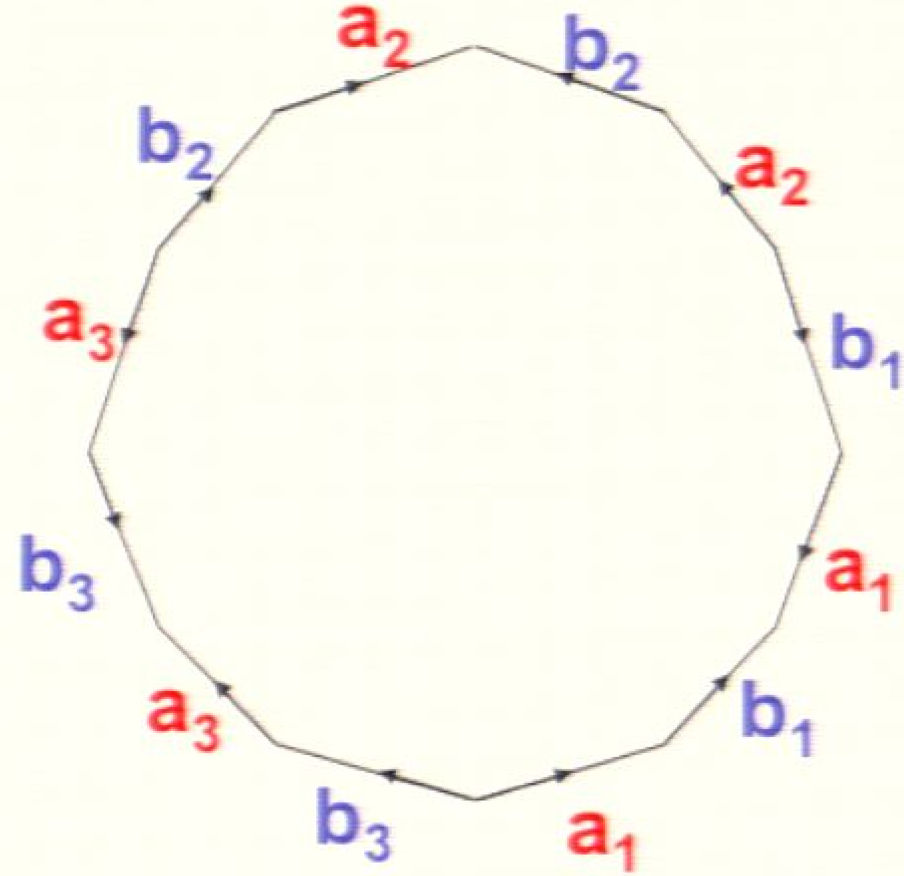
Theta functions associated with (hyperelliptic) Riemann surfaces

Riemann surface:





hyperelliptic: $(\mu, \lambda), \quad \mu^2 = \lambda \prod_{i=1}^{2g} (\lambda - \lambda_i)$

Holomorphic differentials and period matrix:

$$\omega_{i=1\ldots g} \qquad \oint_{a_i} \omega_j = \delta_{ij}$$

$$\Pi_{ij} = \oint_{b_i} \omega_j$$

Theta functions:

$$\theta(\zeta) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i(\frac{1}{2}n^t \Pi n + n^t \zeta)}$$

Theta functions with characteristics:

$$\hat{\theta}(\zeta) = \theta\left[\begin{array}{c}\Delta_1\\ \Delta_2\end{array}\right](\zeta) = \exp\left\{2\pi i\left[\frac{1}{8}\Delta_1^t\Pi\Delta_1 + \frac{1}{2}\Delta_1^t\zeta + \frac{1}{4}\Delta_1^t\Delta_2\right]\right\}\theta\left(\zeta + \frac{1}{2}\Delta_2 + \frac{1}{2}\Pi\Delta_1\right.$$

Simple properties:

Symmetry:

$$\theta(-\zeta) = \theta(\zeta)$$

Periodicity

$$\theta\left(\zeta + \Delta_2 + \Pi\Delta_1\right) = e^{-2\pi i\left[\Delta_1^t\zeta + \frac{1}{2}\Delta_1^t\Pi\Delta_1\right]}\theta(\zeta$$

Antisymmetry

$$\hat{\theta}(-\zeta) = e^{i\pi\Delta_1^t\Delta_2}\hat{\theta}(\zeta) = -\hat{\theta}(\zeta) \Rightarrow \hat{\theta}(0) = 0$$

$\Delta_1, \Delta_2 \in \mathbb{Z}^g$ and $\Delta_1^t\Delta_2$ is an odd integer

Holomorphic differentials and period matrix:

$$\omega_{i=1\ldots g} \qquad \oint_{a_i} \omega_j = \delta_{ij}$$

$$\Pi_{ij} = \oint_{b_i} \omega_j$$

Theta functions:

$$\theta(\zeta) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2} n^t \Pi n + n^t \zeta\right)}$$

Theta functions with characteristics:

$$\hat\theta(\zeta) = \theta\left[\begin{array}{c}\Delta_1\\ \Delta_2\end{array}\right](\zeta) = \exp\left\{2\pi i\left[\frac{1}{8}\Delta_1^t\Pi\Delta_1 + \frac{1}{2}\Delta_1^t\zeta + \frac{1}{4}\Delta_1^t\Delta_2\right]\right\}\theta\left(\zeta + \frac{1}{2}\Delta_2 + \frac{1}{2}\Pi\Delta_1\right.$$

Simple properties:

Symmetry:

$$\theta(-\zeta) = \theta(\zeta)$$

Periodicity

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Special functions

Algebraic problems:

Roots of polynomial in terms of coefficients.

Square root: quadratic equations (compass and straight edge or ruler)

$\sin\alpha \longrightarrow \sin(\alpha/2)$ ~~$[\sin\alpha \longrightarrow \sin(\alpha/3)]$~~

Exponential and log: generic roots, allows solutions of cubic and quartic eqns.

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Differential Equations

sin, cos, exp: harmonic oscillator (Klein-Gordon).

theta functions: sine-Gordon, sinh-Gordon, <u>cosh-Gordon</u>.

Trisecant identity:

$$\theta(\zeta)\,\theta\left(\zeta+\int_{p_2}^{p_1}\omega+\int_{p_3}^{p_4}\omega\right)=\gamma_{1234}\,\theta\Big(\zeta+\int_{p_2}^{p_1}\omega\Big)\,\theta\Big(\zeta+\int_{p_3}^{p_4}\omega\Big)+\gamma_{1324}\,\theta\Big(\zeta+\int_{p_3}^{p_1}\omega\Big)\,\theta\Big(\zeta+\int_{p_2}^{p_4}\omega$$

$$\gamma_{ijkl}=\frac{\theta(a+\int_{p_k}^{p_i}\omega)\,\theta(a+\int_{p_l}^{p_j}\omega)}{\theta(a+\int_{p_l}^{p_i}\omega)\,\theta(a+\int_{p_k}^{p_j}\omega)}$$

Derivatives:

$$D_{p_1} F(\zeta) = \omega_j(p_1)\nabla_j F(\zeta$$

$$D_{p_1}\ln\left[\frac{\theta(\zeta)}{\theta(\zeta+\int_{p_3}^{p_4})}\right] = -D_{p_1}\ln\left[\frac{\theta(a+\int_{p_3}^{p_1})}{\theta(a+\int_{p_4}^{p_1})}\right]$$

$$-\frac{D_{p_1}\theta(a)\,\theta\left(a+\int_{p_4}^{p_3}\right)}{\theta\left(a+\int_{p_4}^{p_1}\right)\theta\left(a+\int_{p_1}^{p_3}\right)}\frac{\theta\left(\zeta+\int_{p_3}^{p_1}\right)\theta\left(\zeta+\int_{p_1}^{p_4}\right)}{\theta(\zeta)\,\theta\left(\zeta+\int_{p_3}^{p_4}\right)}$$

$$D_{p_3p_1}\ln\theta(\zeta) = D_{p_3p_1}\ln\theta\left(a+\int_{p_3}^{p_1}\right) - \frac{D_{p_1}\theta(a)D_{p_3}\theta(a)}{\theta\left(a+\int_{p_3}^{p_1}\right)\theta\left(a+\int_{p_1}^{p_3}\right)}\frac{\theta\left(\zeta+\int_{p_3}^{p_1}\right)\theta\left(\zeta+\int_{p_1}^{p_3}\right)}{\theta^2(\zeta)}$$

cosh-Gordon: $\partial\bar\partial\alpha = 2\cosh 2\alpha = e^{2\alpha}+e^{-2\alpha}$

$$e^{2\alpha} = -e^{-2\pi i\Delta_1^t\zeta - \frac{i\pi}{2}\Delta_1^t\Pi\Delta_1}\frac{\theta^2(\zeta)}{\theta^2(\zeta+\int_{p_1}^{p_3})} = \frac{\theta^2(\zeta)}{\hat\theta^2(\zeta)}$$

Theta functions solve e.o.m.

Hyperelliptic Riemann surface $\mu(\lambda) = \sqrt{\lambda \prod_{j=1}^{2g} (\lambda - \lambda_j)}$

$$\mathbb{A} = \begin{pmatrix} \psi_1 & \psi_2 \\ \tilde{\psi}_1 & \tilde{\psi}_2 \end{pmatrix}$$

$$\psi_1 = \sqrt{-\lambda}\, \frac{\hat{\theta}(\zeta + \int_{p_1}^{p_4})}{\hat{\theta}(\zeta)} e^{-\frac{1}{2}\alpha} e^{\mu z + \nu \bar{z}}$$

$$\psi_2 = \frac{\theta(\zeta + \int_{p_1}^{p_4})}{\theta(\zeta)} e^{\frac{1}{2}\alpha} e^{\mu z + \nu \bar{z}} \,, \qquad \zeta = 2\omega(p_1)\bar{z} + 2\omega(p_3) z$$

$$\sqrt{-\lambda} = 2 \frac{D_{p_3}\hat{\theta}(0)\hat{\theta}(\int_{p_1}^{p_4})}{\theta(\int_{p_1}^{p_4})\theta(0)} \qquad \mu = -2D_{p_3} \ln \theta(\int_{p_1}^{p_4}), \qquad \nu = -2D_{p_1} \ln \hat{\theta}(\int_{p_1}^{p_4}$$

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Finally we write the solution in Poincare coordinates:

$$Z = \left| \frac{\hat{\theta}(2\int_{p_1}^{p_4})}{\hat{\theta}(\int_{p_1}^{p_4})\theta(\int_{p_1}^{p_4})} \right| \frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)|\,|e^{\mu z+\nu\bar{z}}|^2}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2}\,.$$

$$X+iY = e^{2\bar{\mu}\bar{z}+2\bar{\nu}z}\,\frac{\theta(\zeta-\int_{p_1}^{p_4})\overline{\theta(\zeta+\int_{p_1}^{p_4})}-\hat{\theta}(\zeta-\int_{p_1}^{p_4})\overline{\hat{\theta}(\zeta+\int_{p_1}^{p_4}}}{|\hat{\theta}(\zeta-\int_{p_1}^{p_4})|^2+|\theta(\zeta-\int_{p_1}^{p_4})|^2}$$

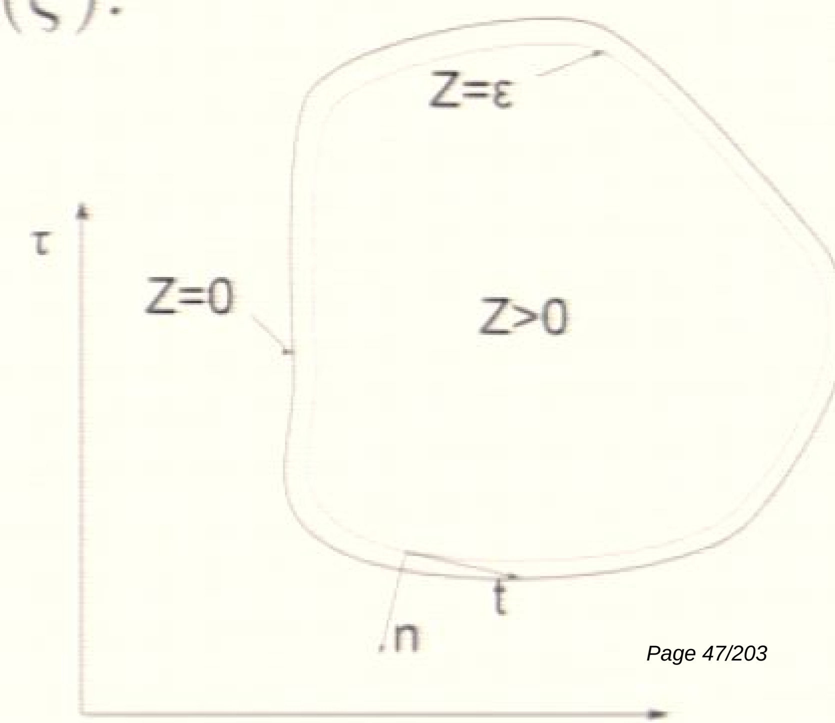
$$Z=0 \quad \Leftrightarrow \quad \theta(\zeta)=0 \text{ or } \boxed{\hat{\theta}(\zeta)=0}$$

Renormalized area:

$$A = 2\int \partial X_\mu \bar{\partial} X^\mu d\sigma d\tau = 2\int \Lambda d\sigma d\tau = 4\int e^{2\alpha} d\sigma d\tau$$

$$\begin{aligned} e^{2\alpha} &= 4\left\{ D_{p_1 p_3} \ln\theta(0) - D_{p_1 p_3} \ln\hat{\theta}(\zeta) \right\} \\ &= 4D_{p_1 p_3} \ln\theta(0) - \partial\bar{\partial} \ln\hat{\theta}(\zeta). \end{aligned}$$

$$A = \frac{L}{\epsilon} + A_f$$



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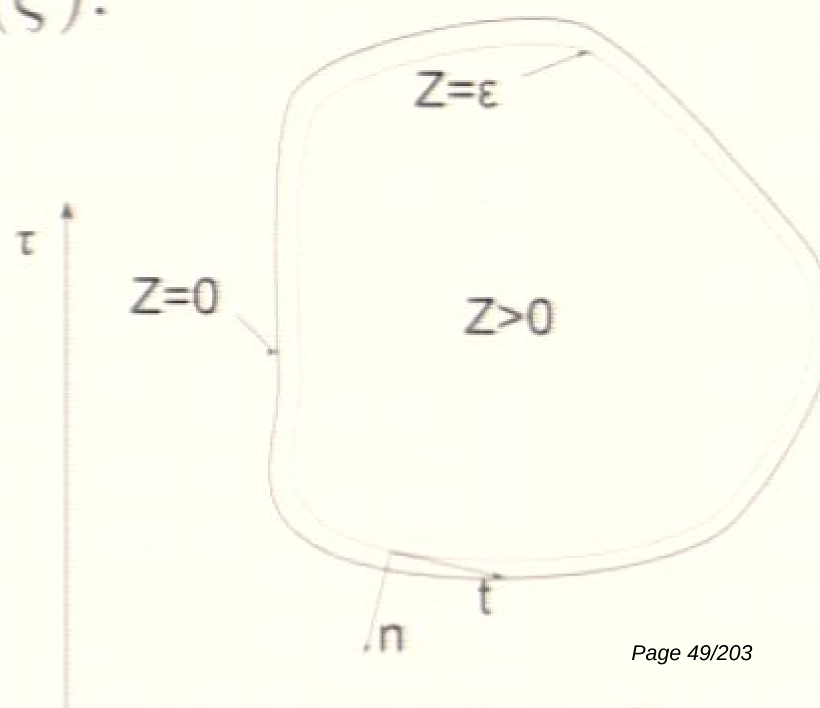


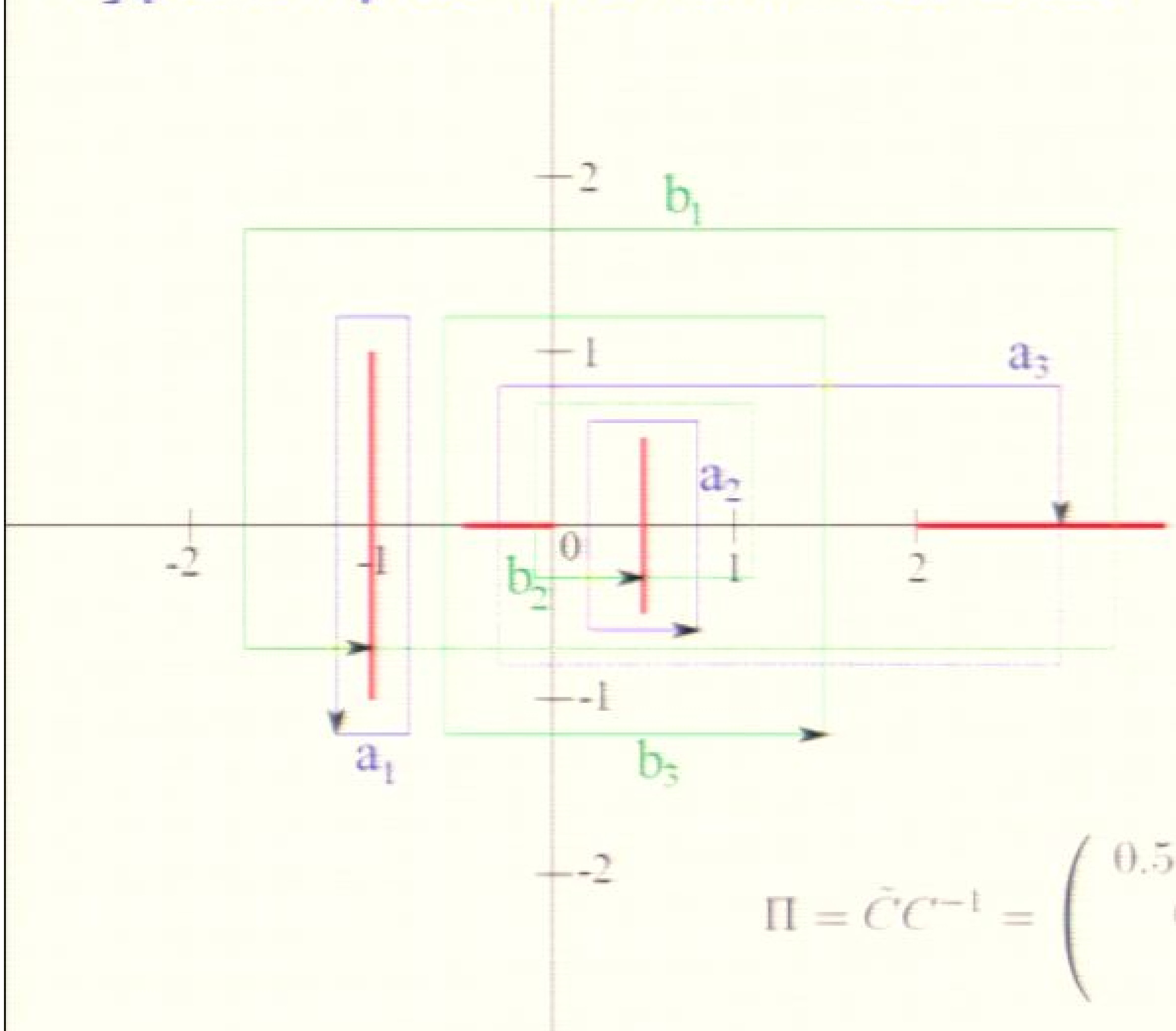
Subtracting the divergence gives:

$$\begin{aligned} A_{\mathrm{f}} &= 16 D_{p_1 p_3} \ln \theta(0) \int d\sigma d\tau - \frac{1}{2} \oint \frac{\nabla^2 \theta(\zeta)}{|\nabla \hat{\theta}(\zeta)|} \, d\ell \\ &= 16 D_{p_1 p_3} \ln \theta(0) \int d\sigma d\tau - 2 \oint \frac{D_{p_1 p_3} \hat{\theta}(\zeta)}{|D_{p_1} \hat{\theta}(\zeta)|} \, d\ell \\ &= 8 D_{p_1 p_3} \ln \theta(0) \oint (\sigma d\tau - \tau d\sigma) - 2 \oint \frac{D_{p_1 p_3} \hat{\theta}(\zeta)}{|D_{p_1} \hat{\theta}(\zeta)|} \, d\ell \end{aligned}$$

$$\langle W \rangle = e^{-\frac{\sqrt{\lambda_t}}{2\pi} A_f}$$

Example of closed Wilson loop for g=3

Hyperelliptic Riemann surface



$$\nu_k = \frac{\lambda^{k-1}}{\mu} d\lambda, \quad k = 1 \ldots 3$$

$$C_{ij} = \oint_{a_i} \nu_j, \quad \tilde{C}_{ij} = \oint_{b_i} \nu_j$$

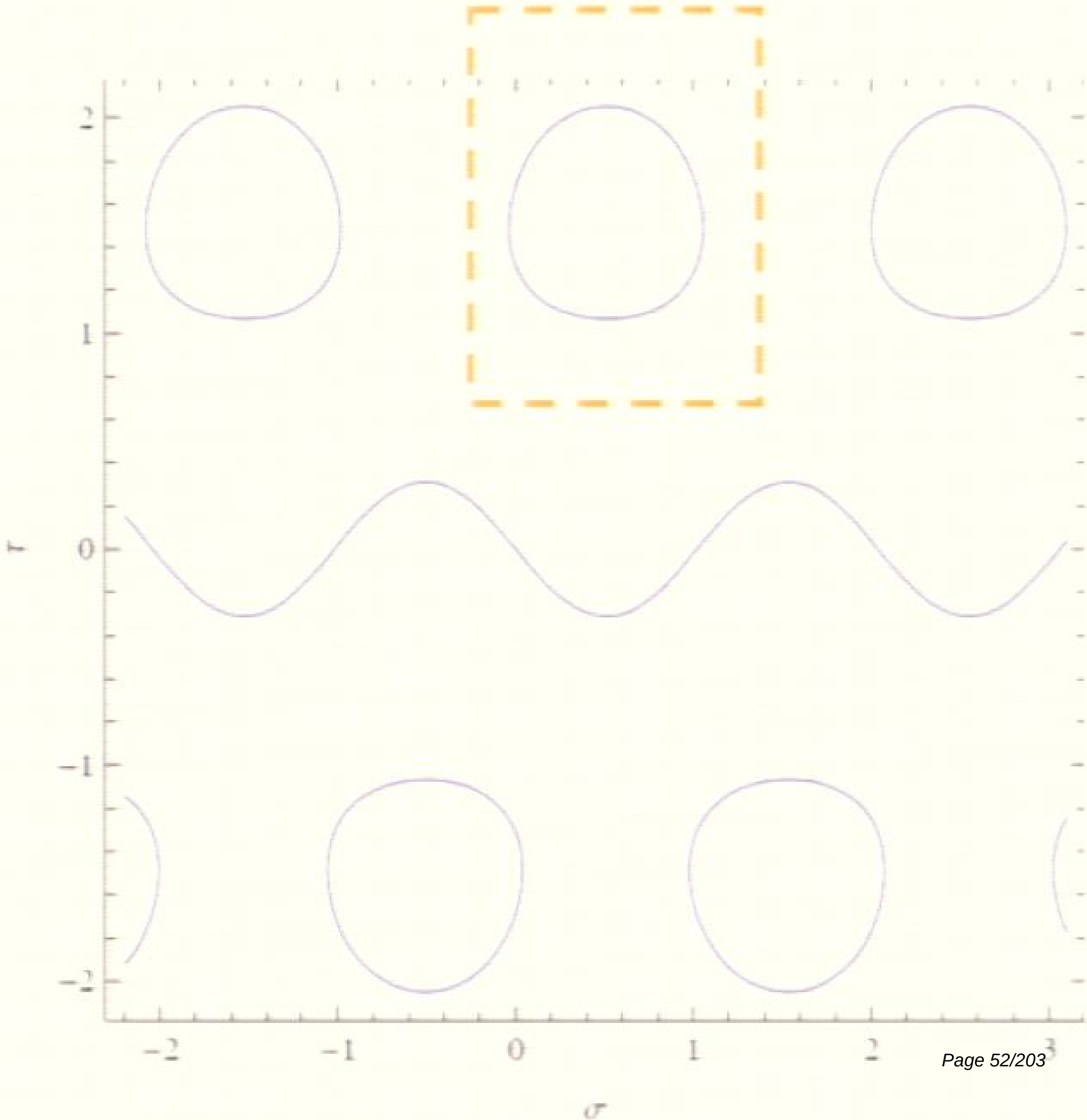
$$\omega_i = \nu_j \left(C^{-1}\right)_{ji} ,$$

$$\Pi = \tilde{C} C^{-1} = \begin{pmatrix} 0.5 + 0.64972i & 0.14972i & -0.5 \\ 0.14972i & -0.5 + 0.64972i & 0.5 \\ -0.5 & 0.5 & 0.639631 \end{pmatrix}$$

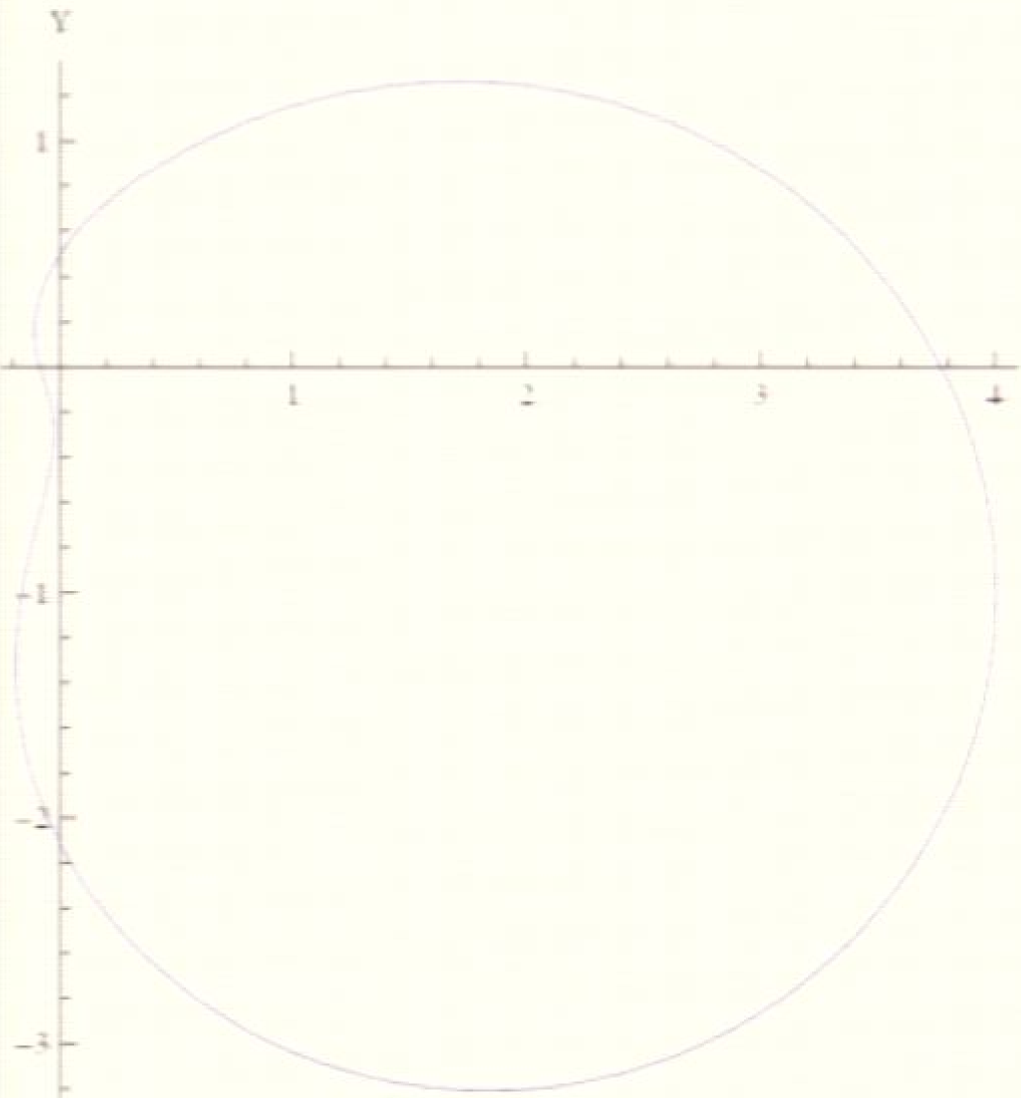
$$\mu = i\sqrt{-i(\lambda + 1 - i)}\sqrt{-i(\lambda + 1 + i)}\sqrt{-i(\lambda - \frac{1+i}{2})}\sqrt{-i(\lambda - \frac{1-i}{2})}\sqrt{2-\lambda}\sqrt{\lambda}\sqrt{\lambda +$$

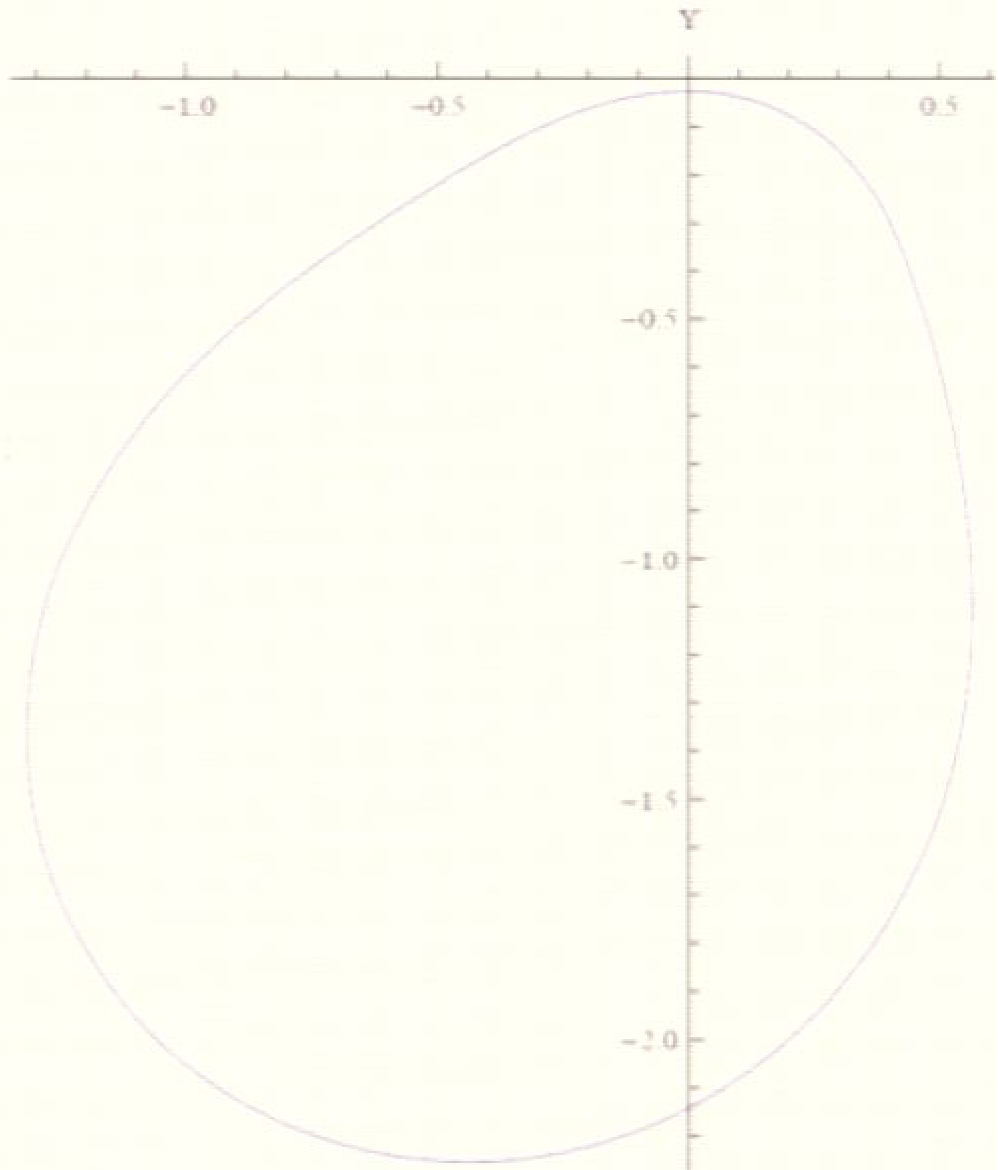
Zeros

$$\hat{\theta}(\zeta) = 0$$



Shape of Wilson loop:

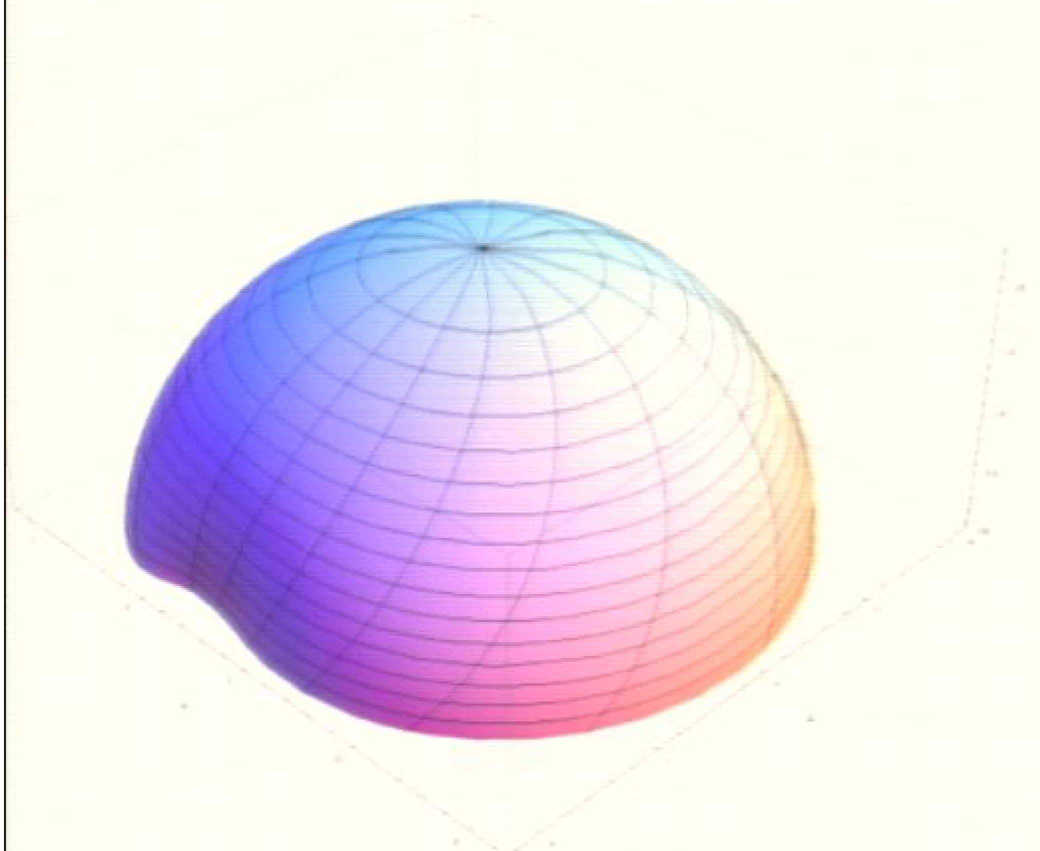




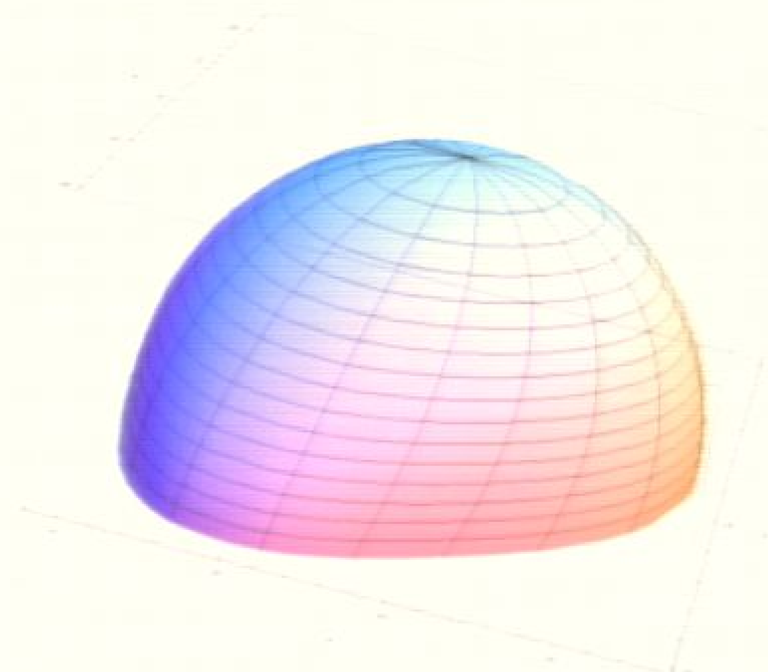
$$\lambda = i$$

$$\lambda = -\frac{1+i}{\sqrt{2}}$$

Shape of dual surface:



$$\lambda = i$$



$$\lambda = -\frac{1+i}{\sqrt{2}}$$

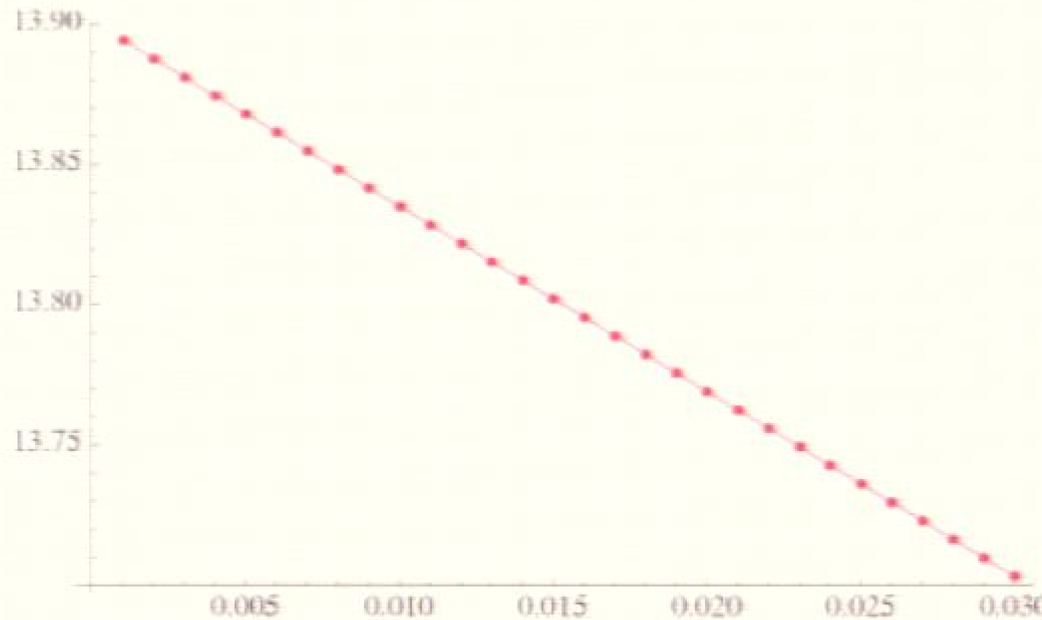
Computation of area:

Using previous formula

$$L_1 = 13.901, \quad L_2 = 6.449$$
$$A_f = -6.598 \quad \text{for both.}$$

Direct computation:

$$\varepsilon A = L + A_f \varepsilon$$



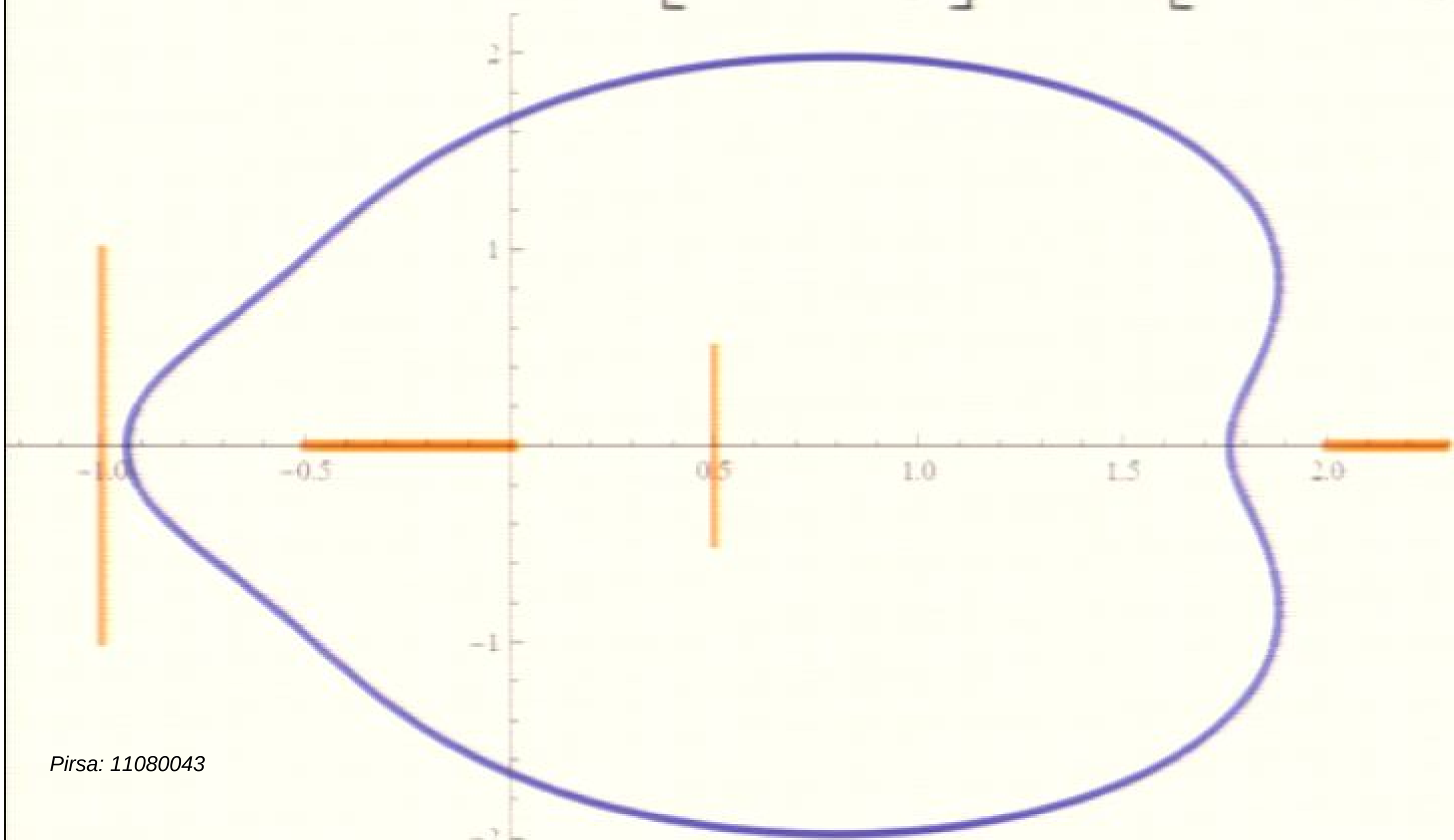
Circular Wilson loops , maximal area for fixed length.

Map from Wilson loop into the Riemann surface

Zeros determine shape of the WL. $\hat{\theta}(\zeta) = 0$

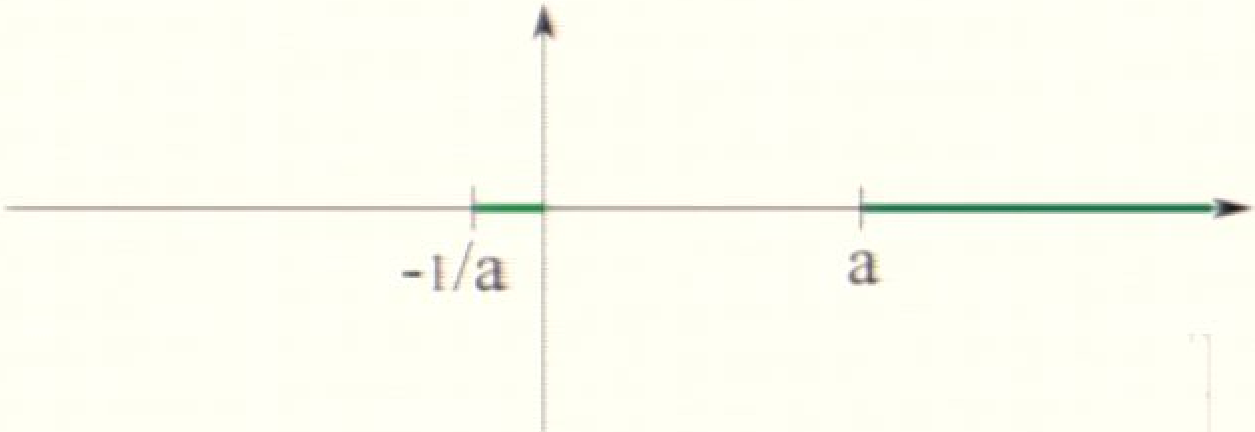
z can be written as:

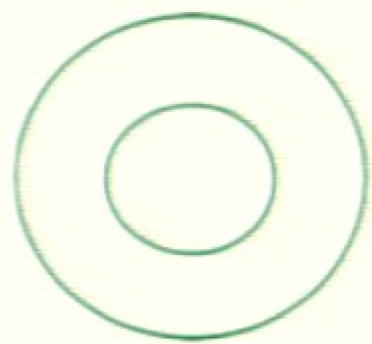
$$2\left(\omega(\infty)z + \omega(0)\bar{z}\right) = \frac{1}{2}\begin{bmatrix} 2n_1 \\ 2n_2 \\ 1+2n_3 \end{bmatrix} + \frac{1}{2}\Pi\begin{bmatrix} 1+2m_1 \\ 1+2m_2 \\ 1+2m_3 \end{bmatrix} - \phi(\lambda_1) + \phi\left(-\frac{1}{\bar{\lambda}_1}\right)$$

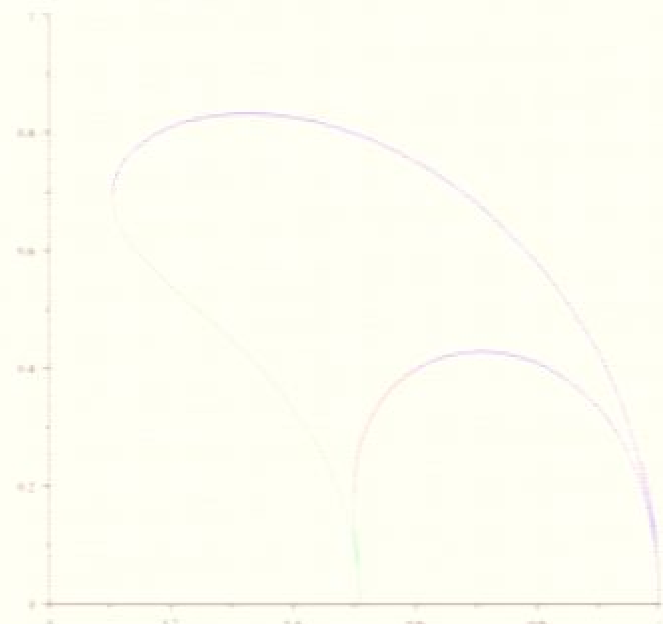


$$\phi(\lambda)_j = \int_0^\lambda \omega_j$$

Simpler case g=1

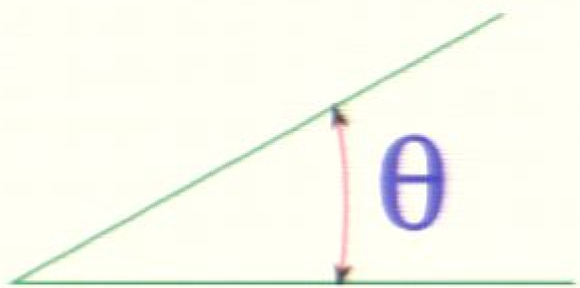






$a>1,\ \lambda=1$

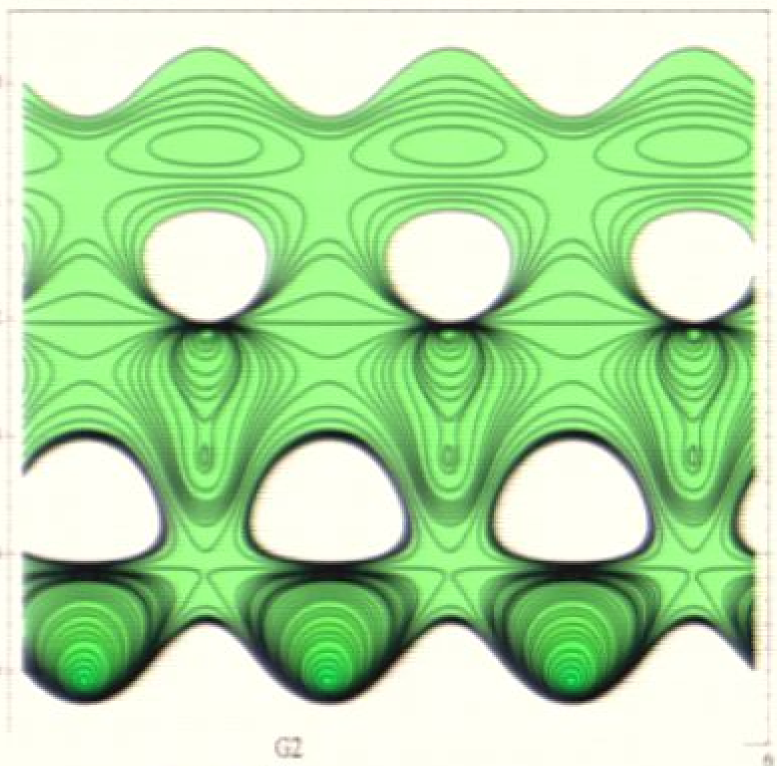
$a<1,\ \lambda=-1$

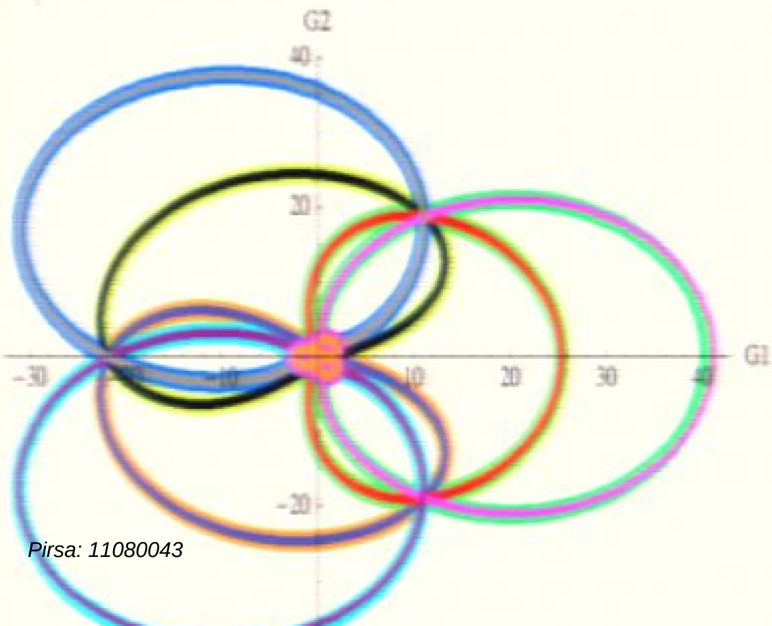


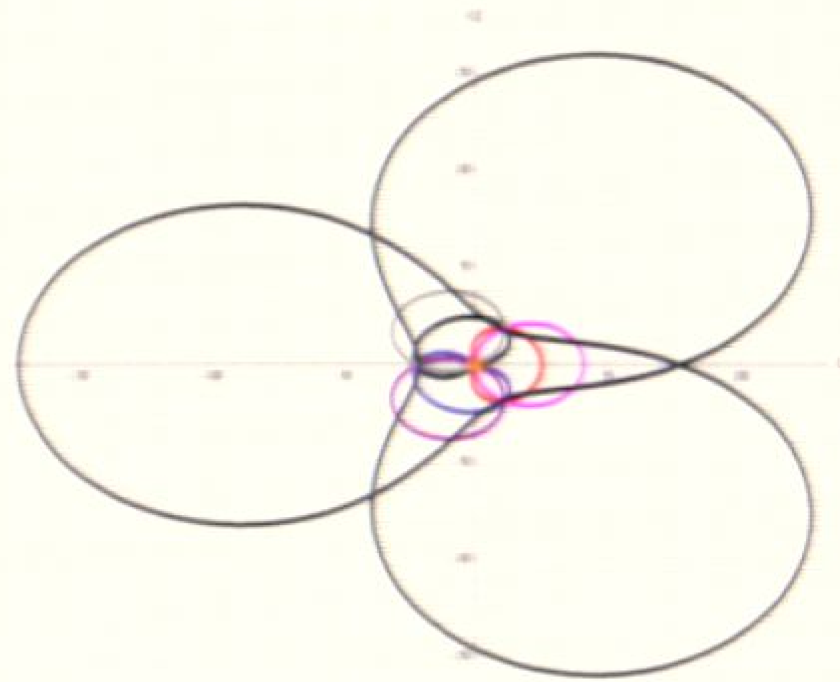
$a \to 1,\ \ \theta \to 0$

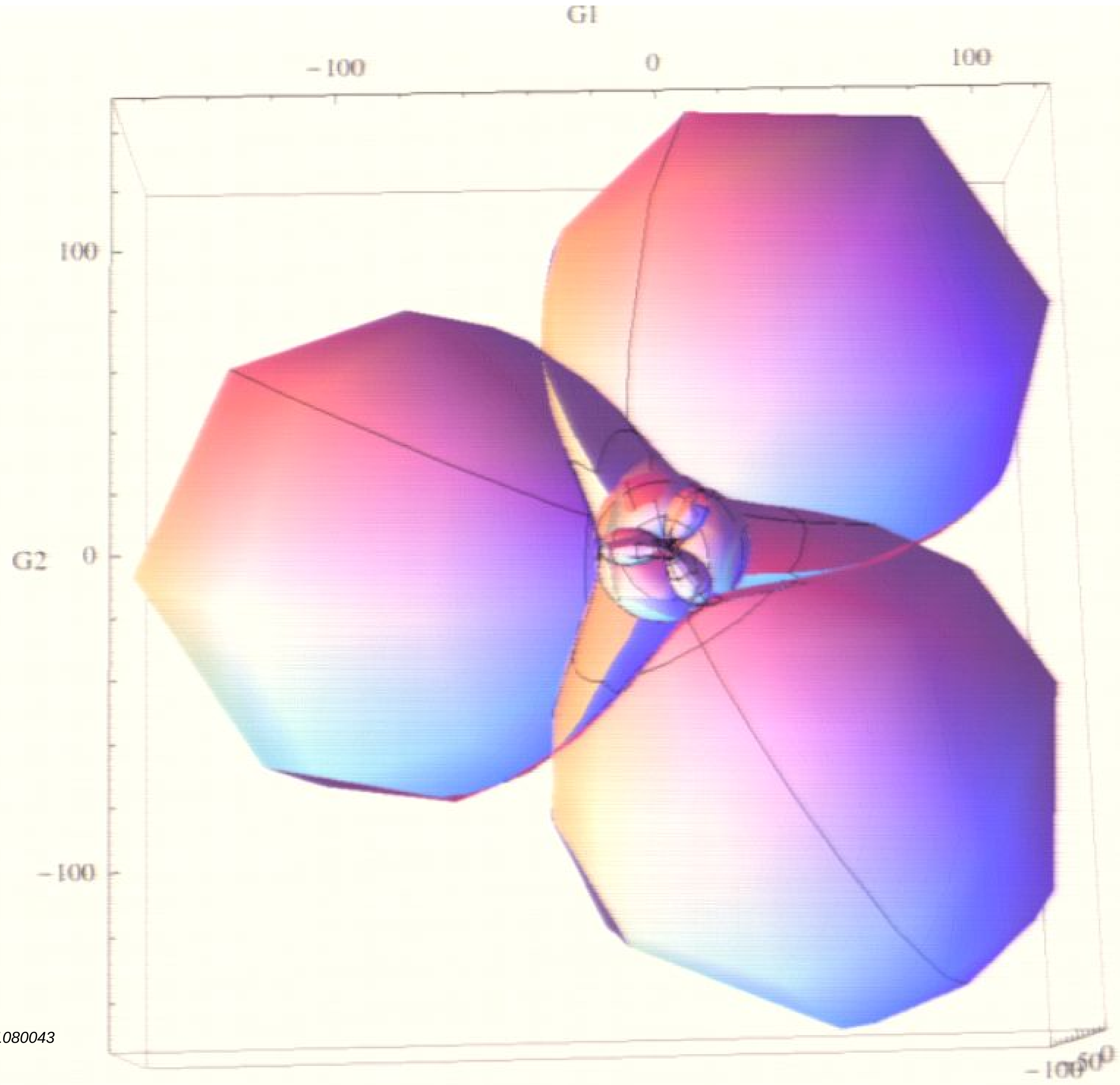
$a \to 0,\ \ \theta \to \pi$

Multiple contours

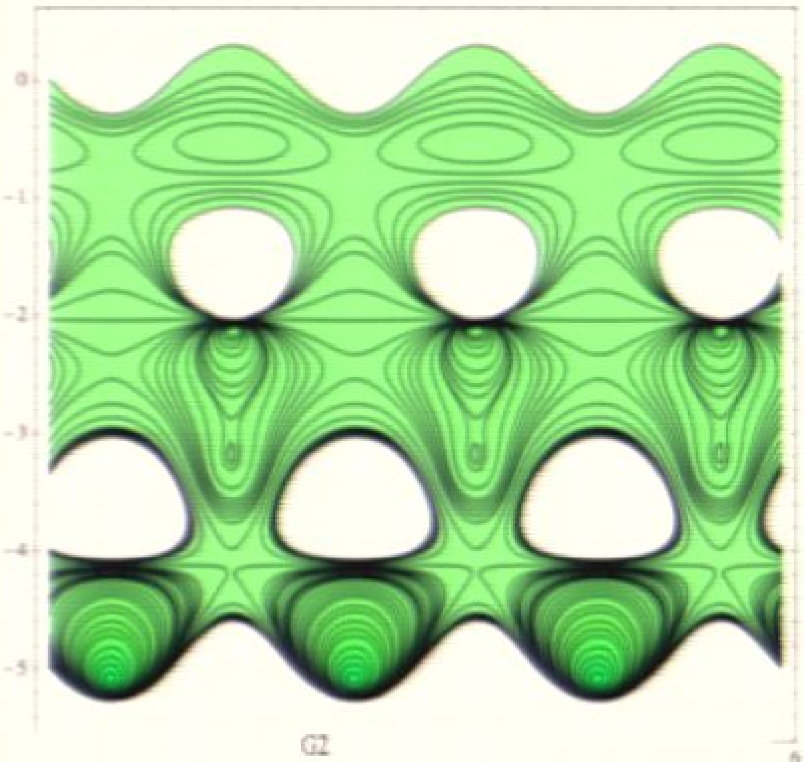


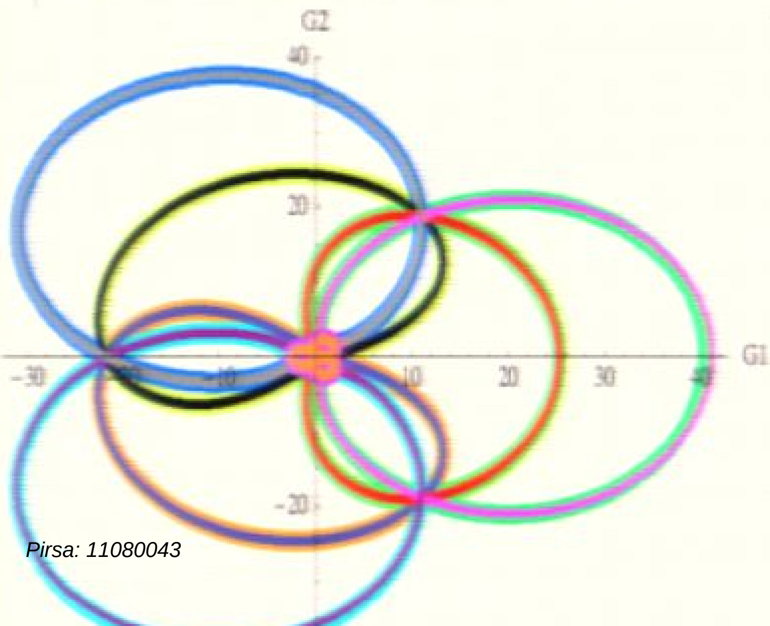


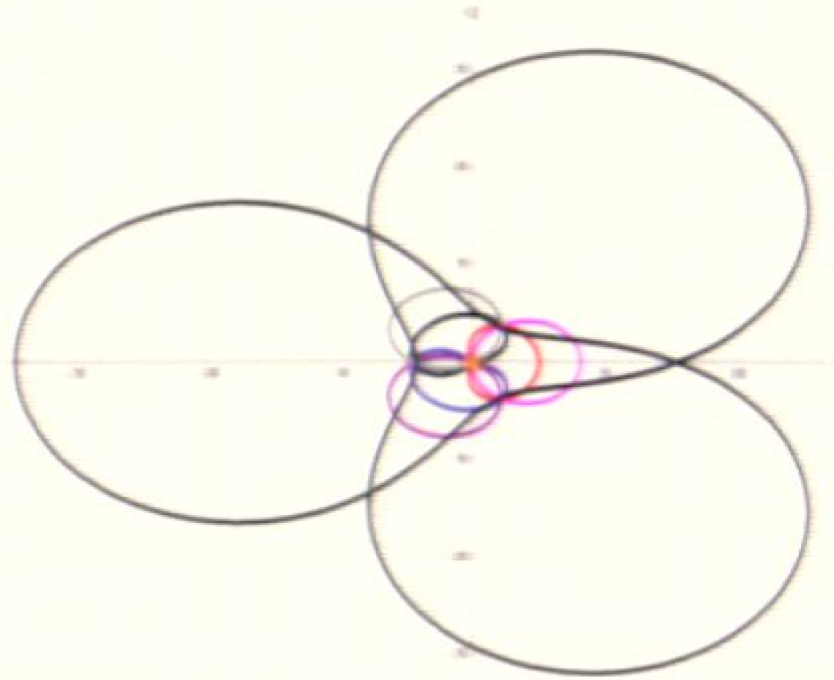


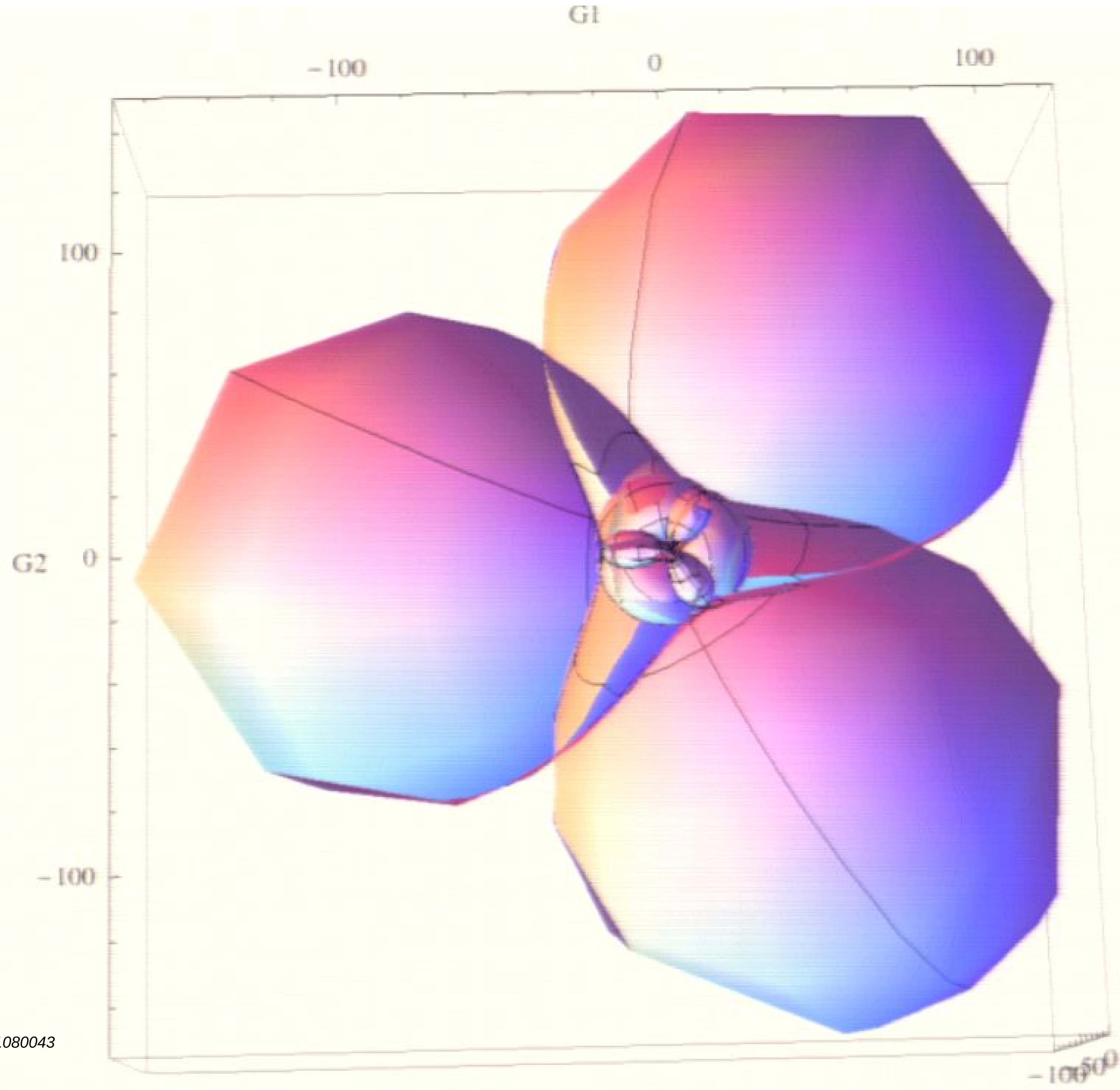

G1
−100
0
100
100
G2 0
−100
−100
50

Multiple contours








G1
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Conclusions

We argue that there is an infinite parameter family of closed Wilson loops whose dual surfaces can be found analytically. The world-sheet has the topology of a disk and the renormalized area is found as a finite one dimensional contour integral over the world-sheet boundary.

We showed specific examples for g=3 including multiple contours. The case g=1 is also interesting.

Integrability properties of Euclidean Wilson loops deserve further study.

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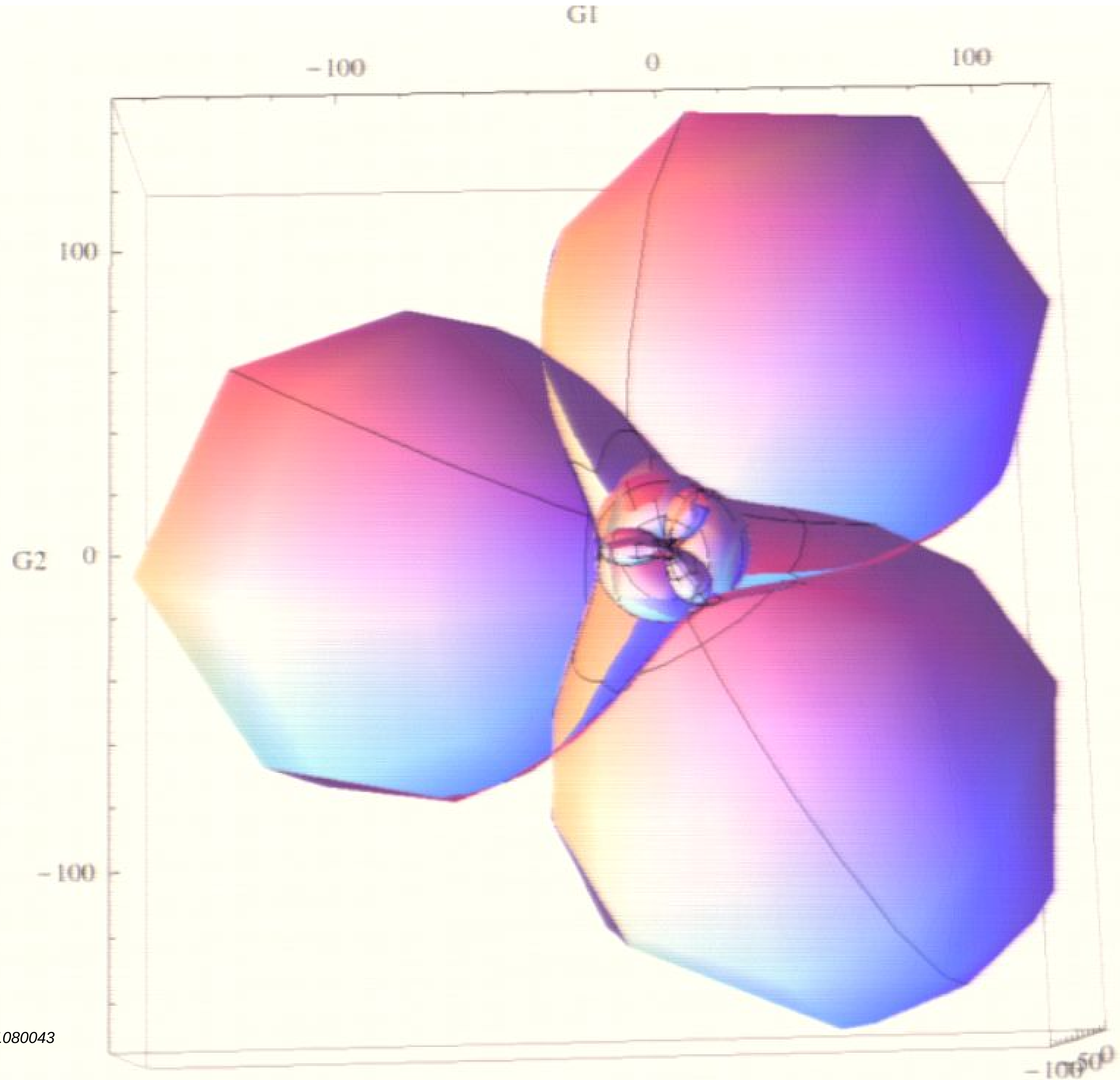
G1
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100
100
G2
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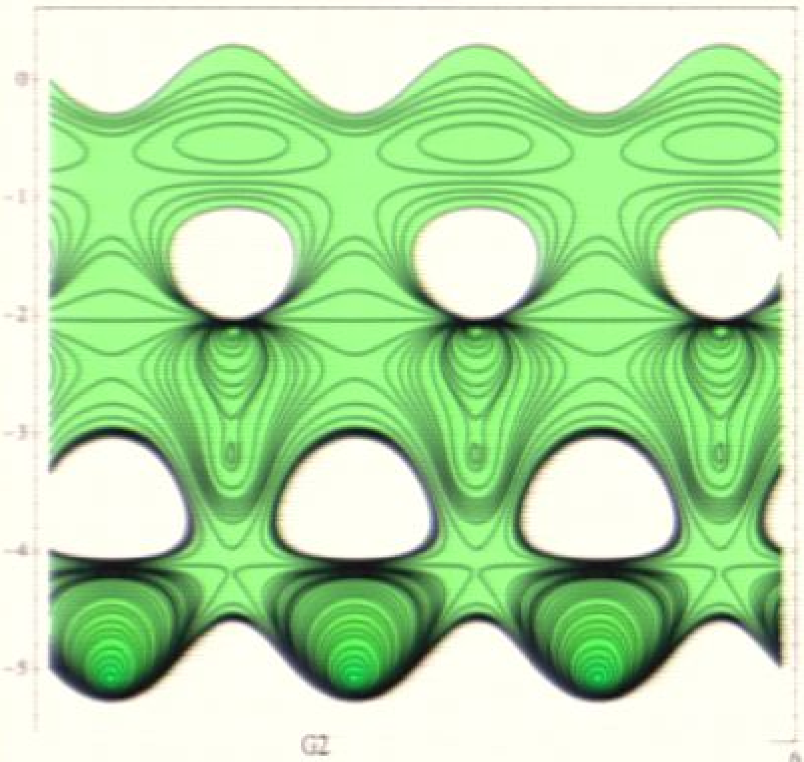
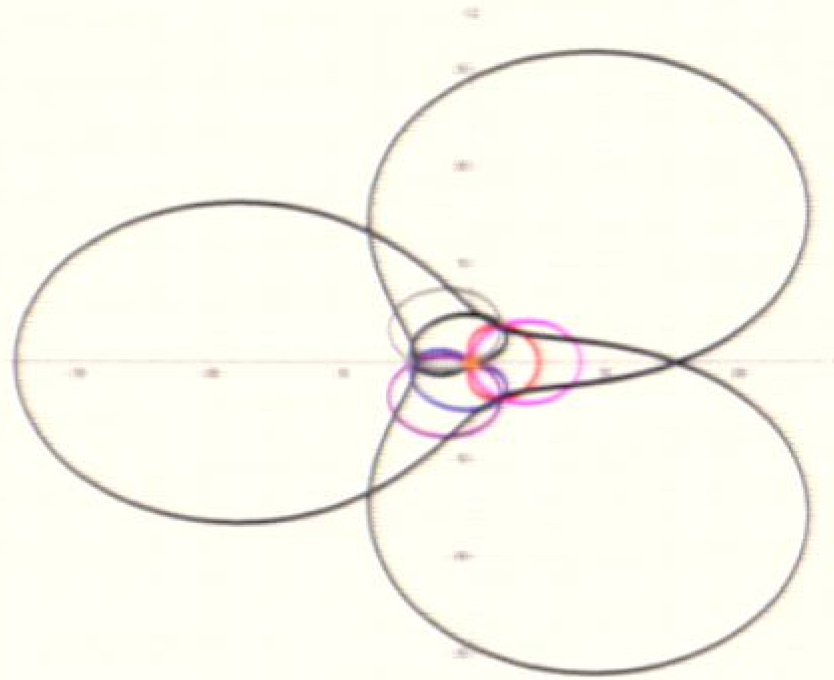
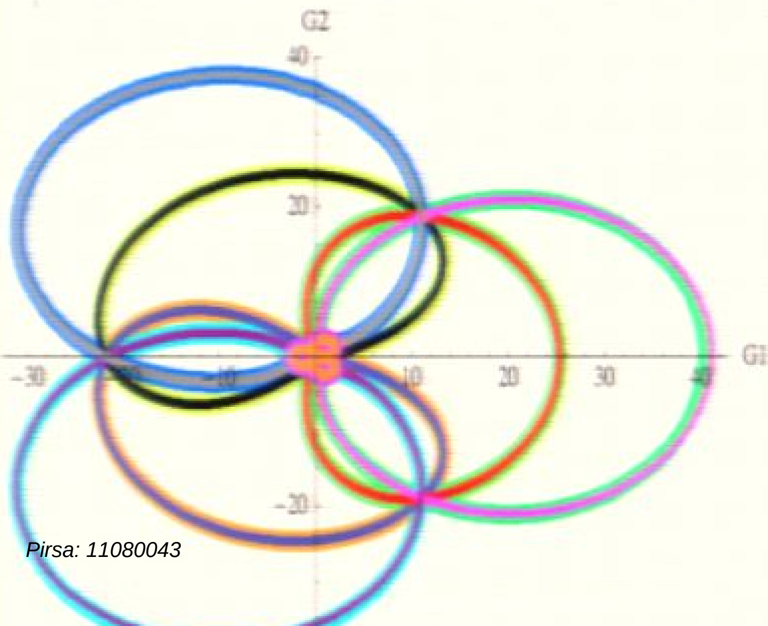
−100
50

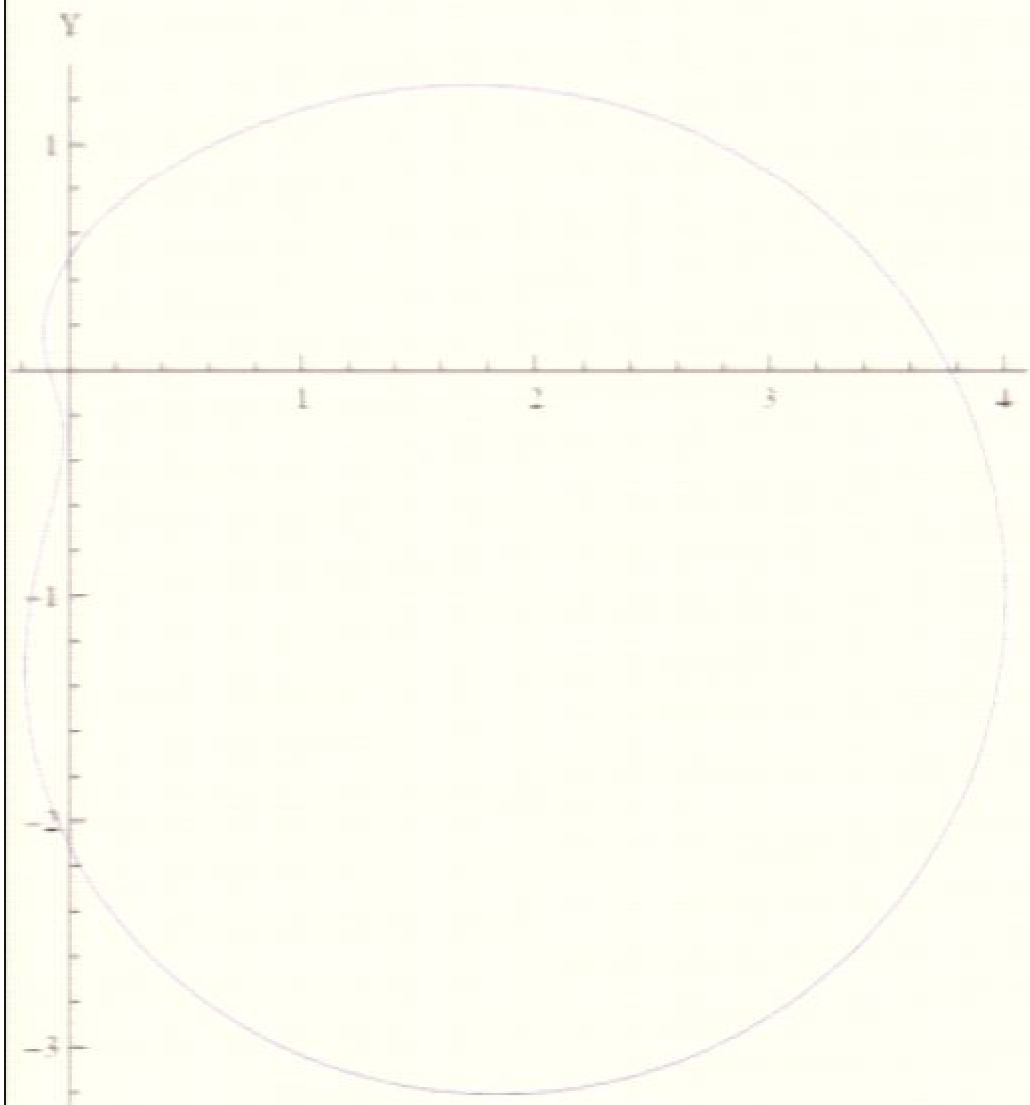
Multiple contours

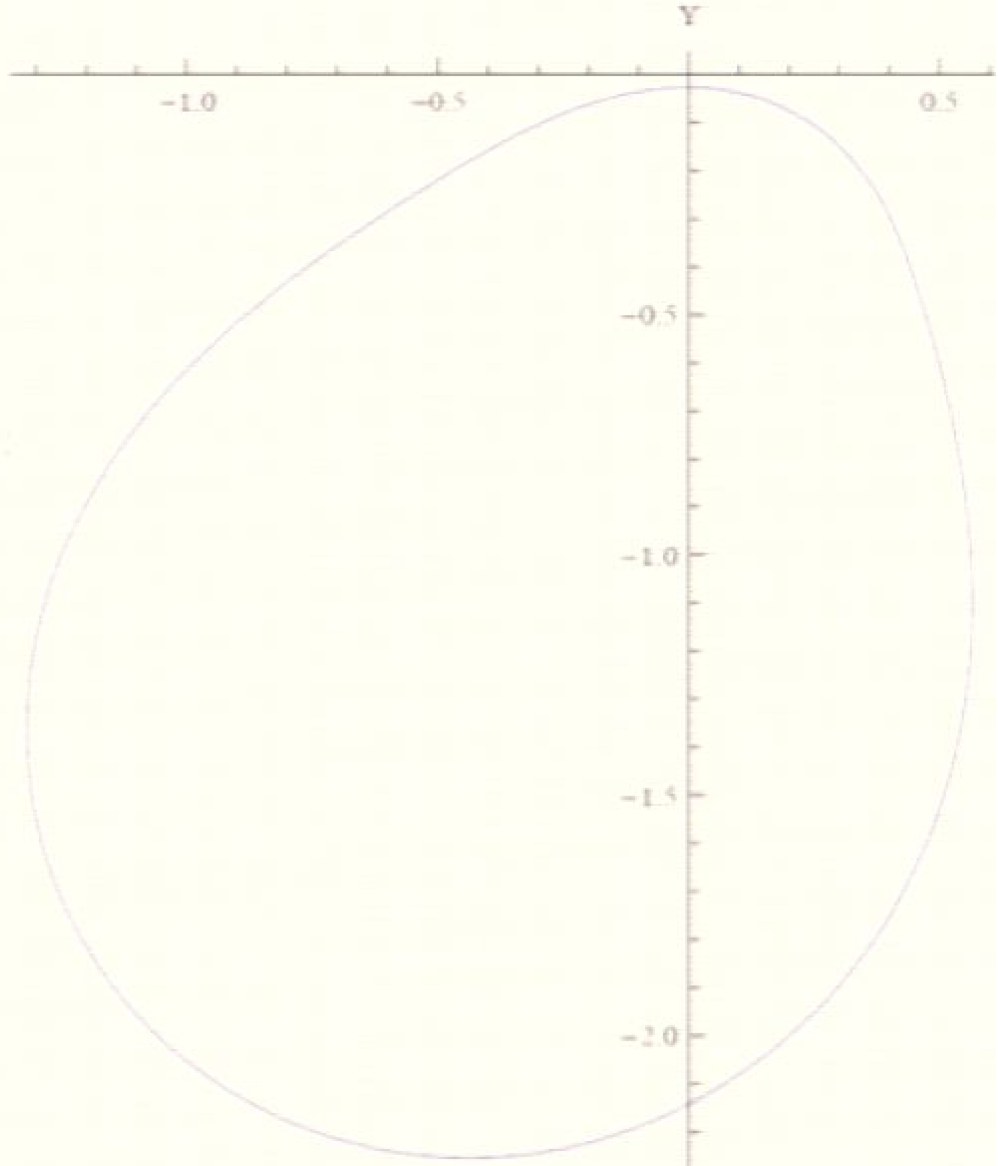






Shape of Wilson loop:



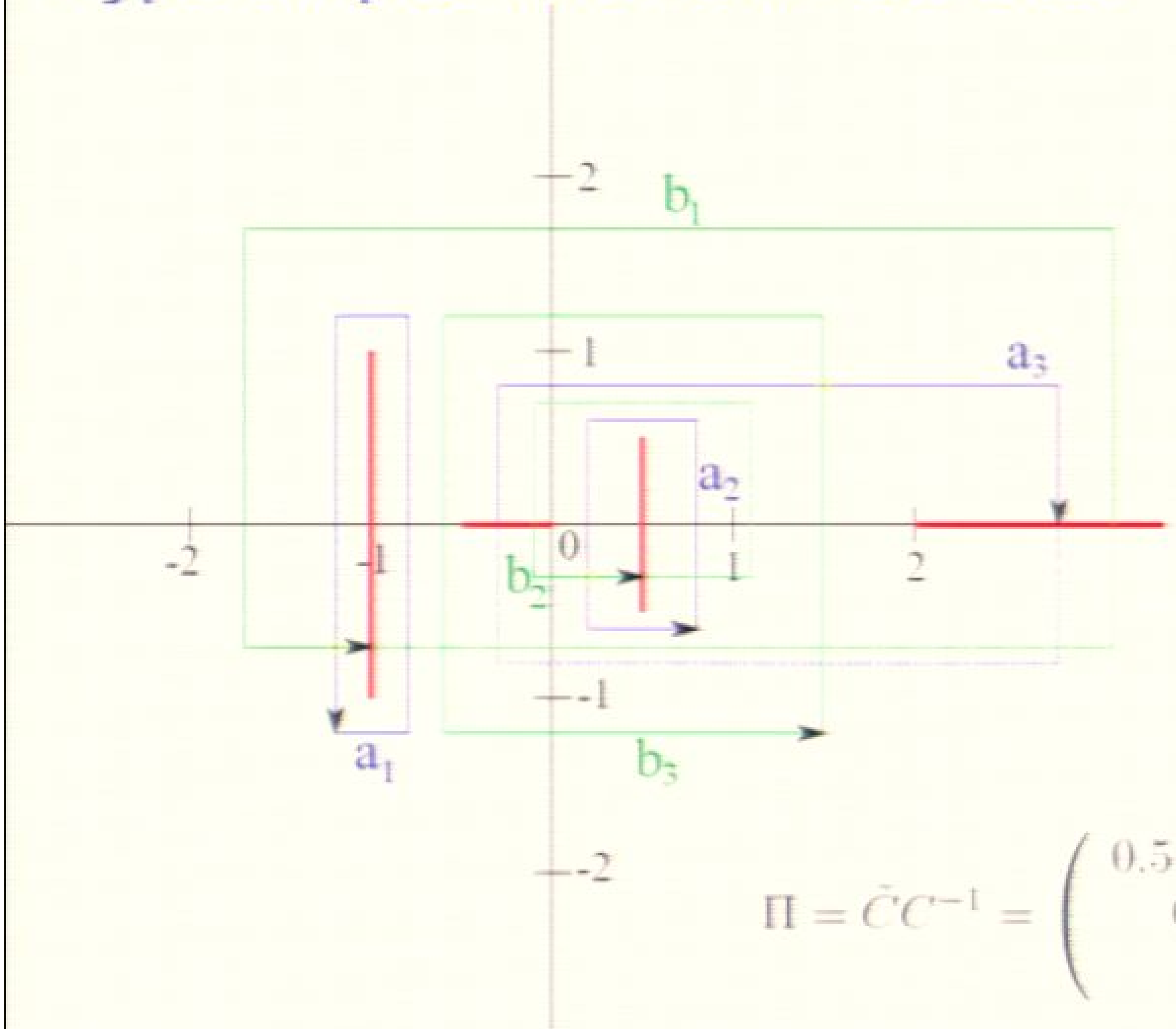


$\lambda = i$

$\lambda = -\frac{1+i}{\sqrt{2}}$

Example of closed Wilson loop for g=3

Hyperelliptic Riemann surface



$$\nu_k = \frac{\lambda^{k-1}}{\mu} d\lambda, \quad k = 1 \dots 3$$

$$C_{ij} = \oint_{a_i} \nu_j, \quad \tilde{C}_{ij} = \oint_{b_i} \nu_j$$

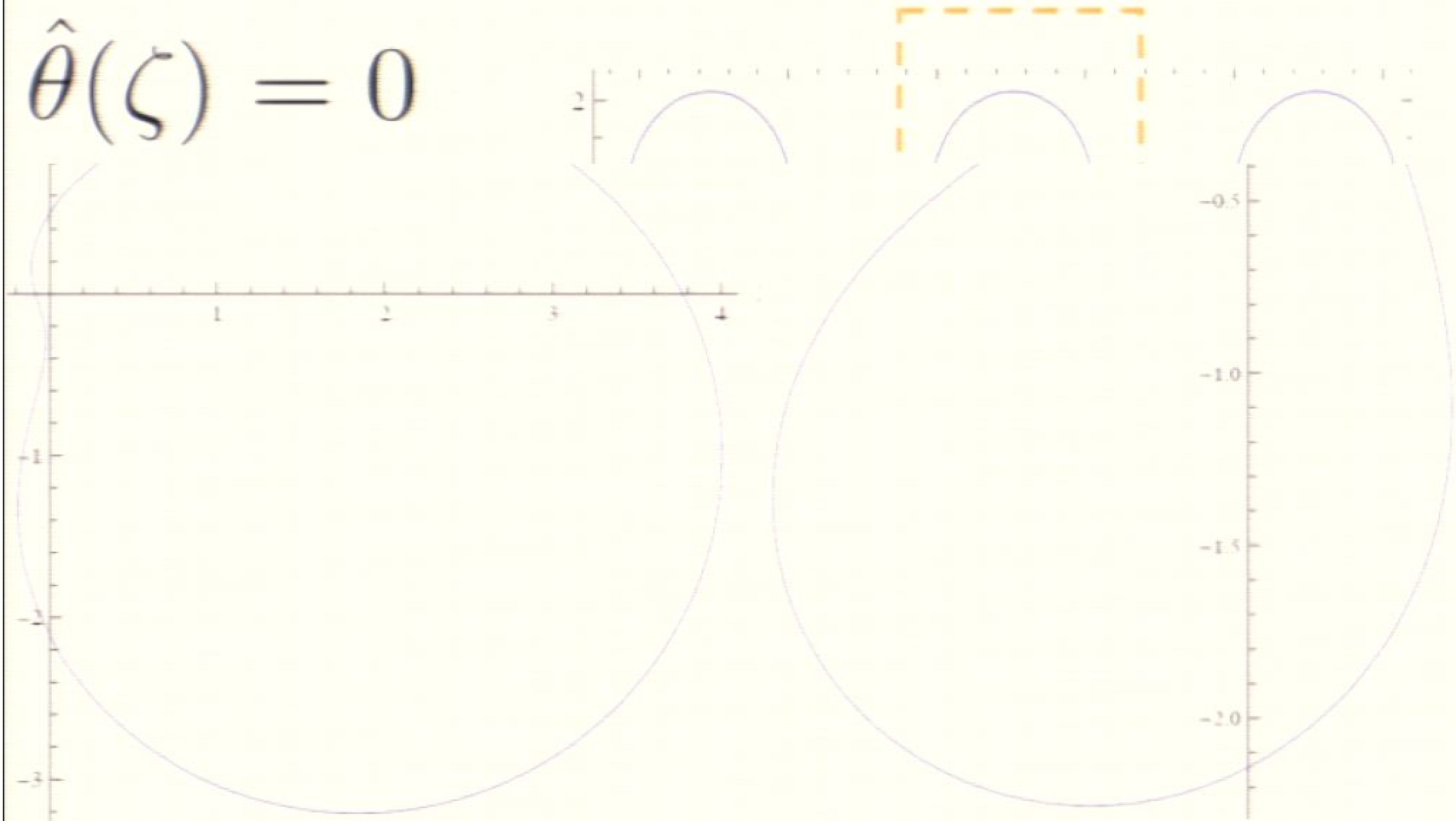
$$\omega_i = \nu_j \left(C^{-1}\right)_{ji} ,$$

$$\Pi = \tilde{C} C^{-1} = \begin{pmatrix} 0.5 + 0.64972i & 0.14972i & -0.5 \\ 0.14972i & -0.5 + 0.64972i & 0.5 \\ -0.5 & 0.5 & 0.639631 \end{pmatrix}$$

$$\mu = i\sqrt{-i(\lambda + 1 - i)}\sqrt{-i(\lambda + 1 + i)}\sqrt{-i(\lambda - \frac{1+i}{2})}\sqrt{-i(\lambda - \frac{1-i}{2})}\sqrt{2 - \lambda}\sqrt{\lambda}\sqrt{\lambda +$$

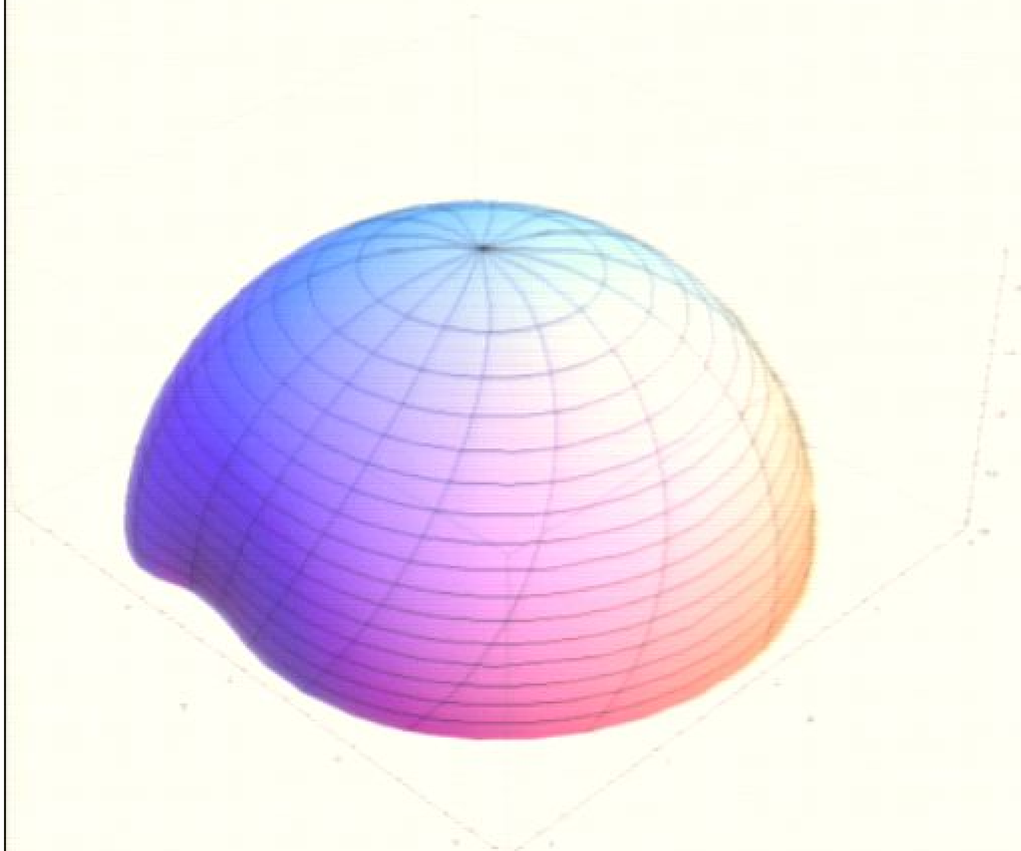
Zeros

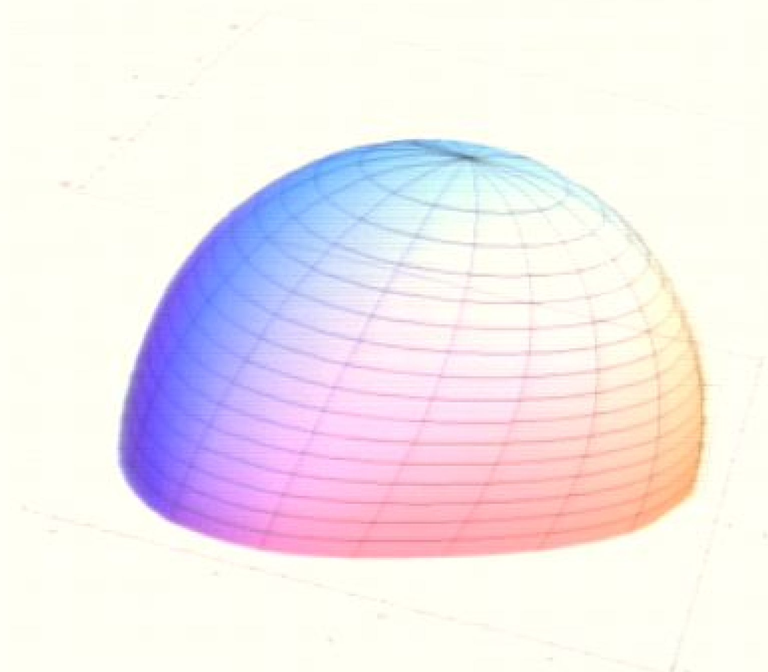
$$\hat{\theta}(\zeta) = 0$$



$$\lambda = i \qquad\qquad \lambda = -\frac{1+i}{\sqrt{2}}$$

Shape of dual surface:

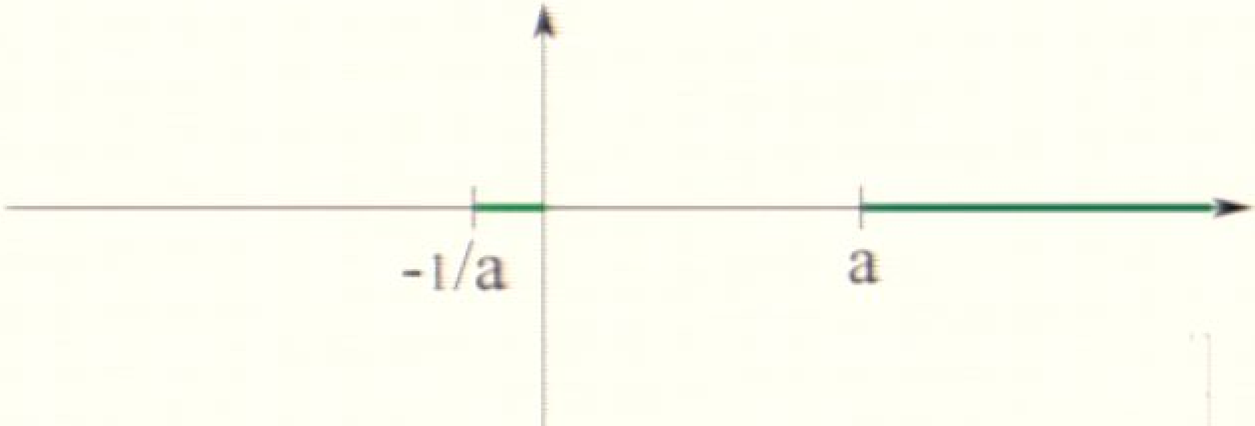


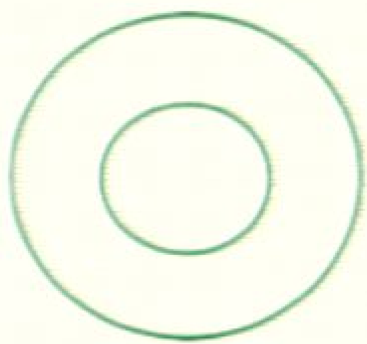


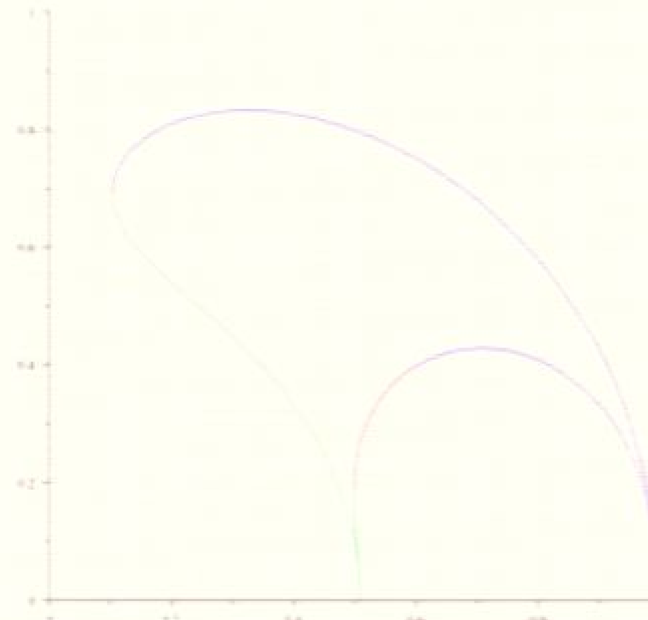
$$\lambda = i$$

$$\lambda = -\frac{1+i}{\sqrt{2}}$$

Simpler case g=1

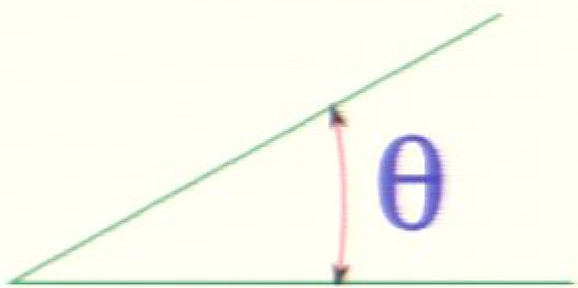






$a>1,\ \lambda=1$

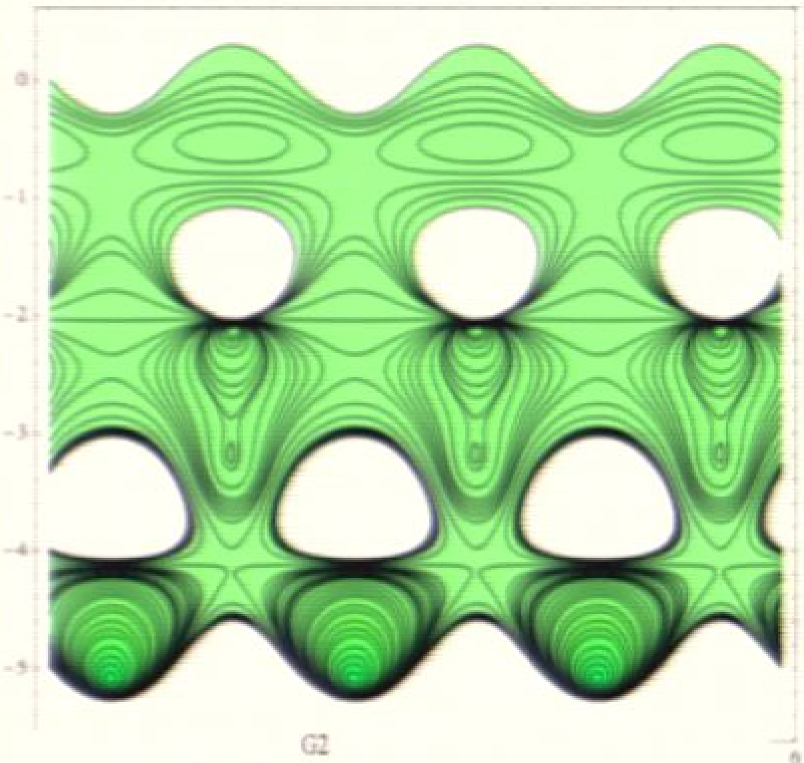
$a<1,\ \lambda=-1$

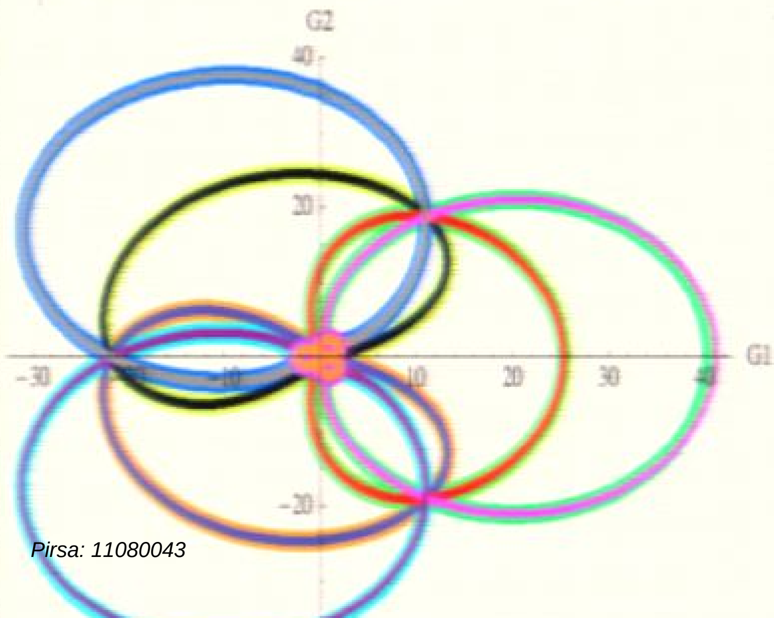


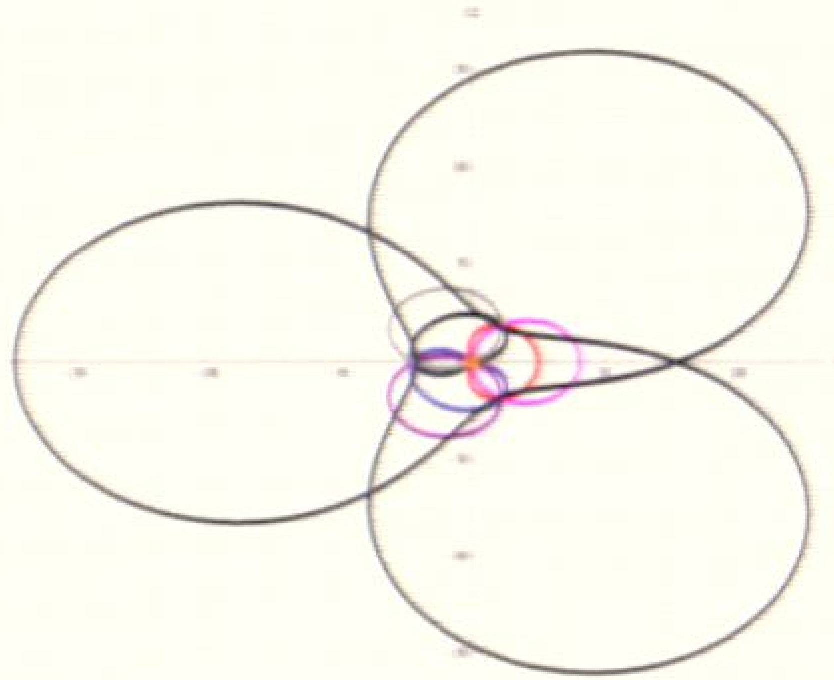
$a \to 1\ ,\ \ \theta \to 0$

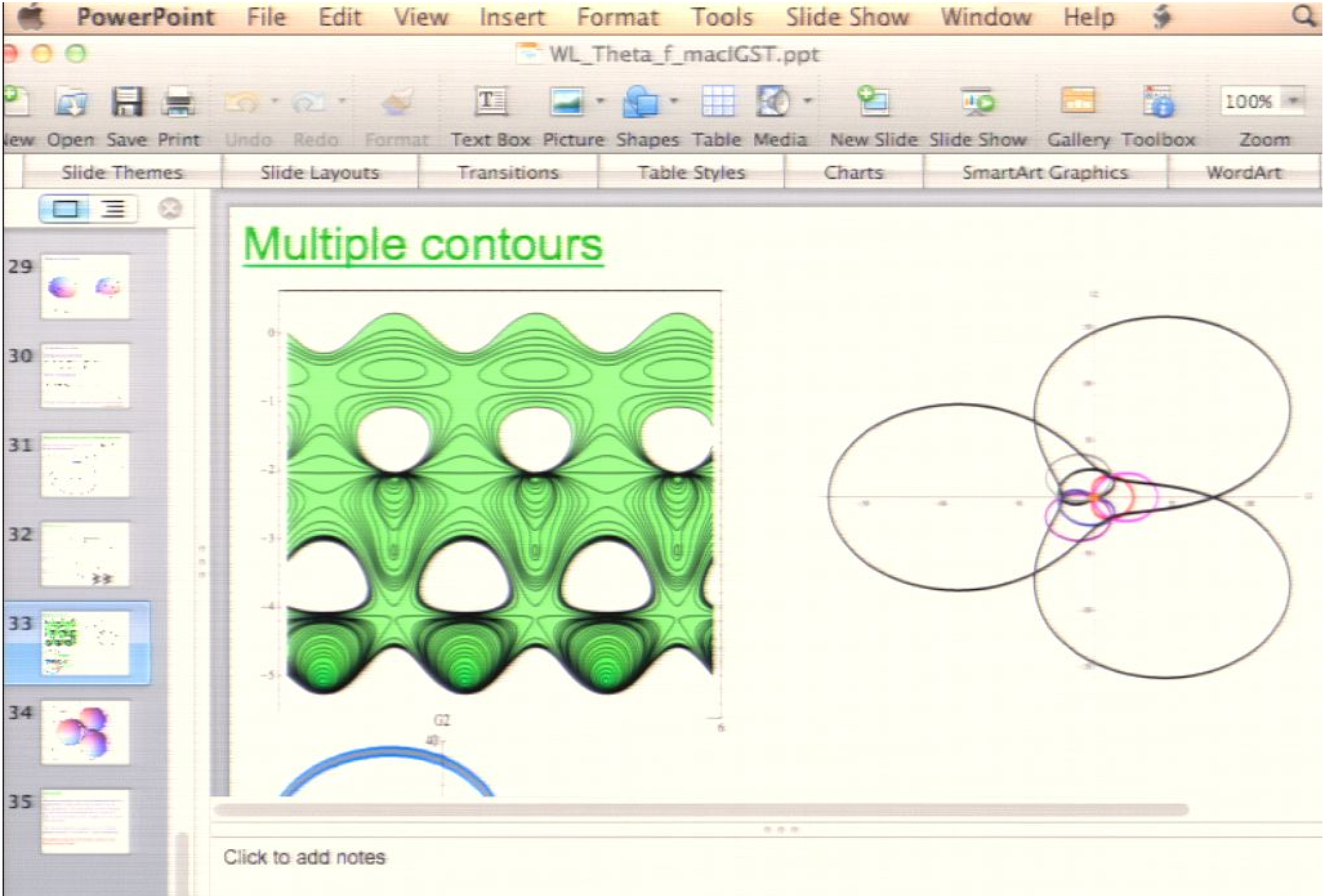
$a \to 0\ ,\ \ \theta \to \pi$

Multiple contours





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New Open Save Print Undo Redo Format Text Box Picture Shapes Table Media New Slide Slide Show Gallery Toolbox Zoom
100%
Slide Themes Slide Layouts Transitions Table Styles Charts SmartArt Graphics WordArt
29
30
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Multiple contours
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-1
-2
-3
-4
-5
6
G2
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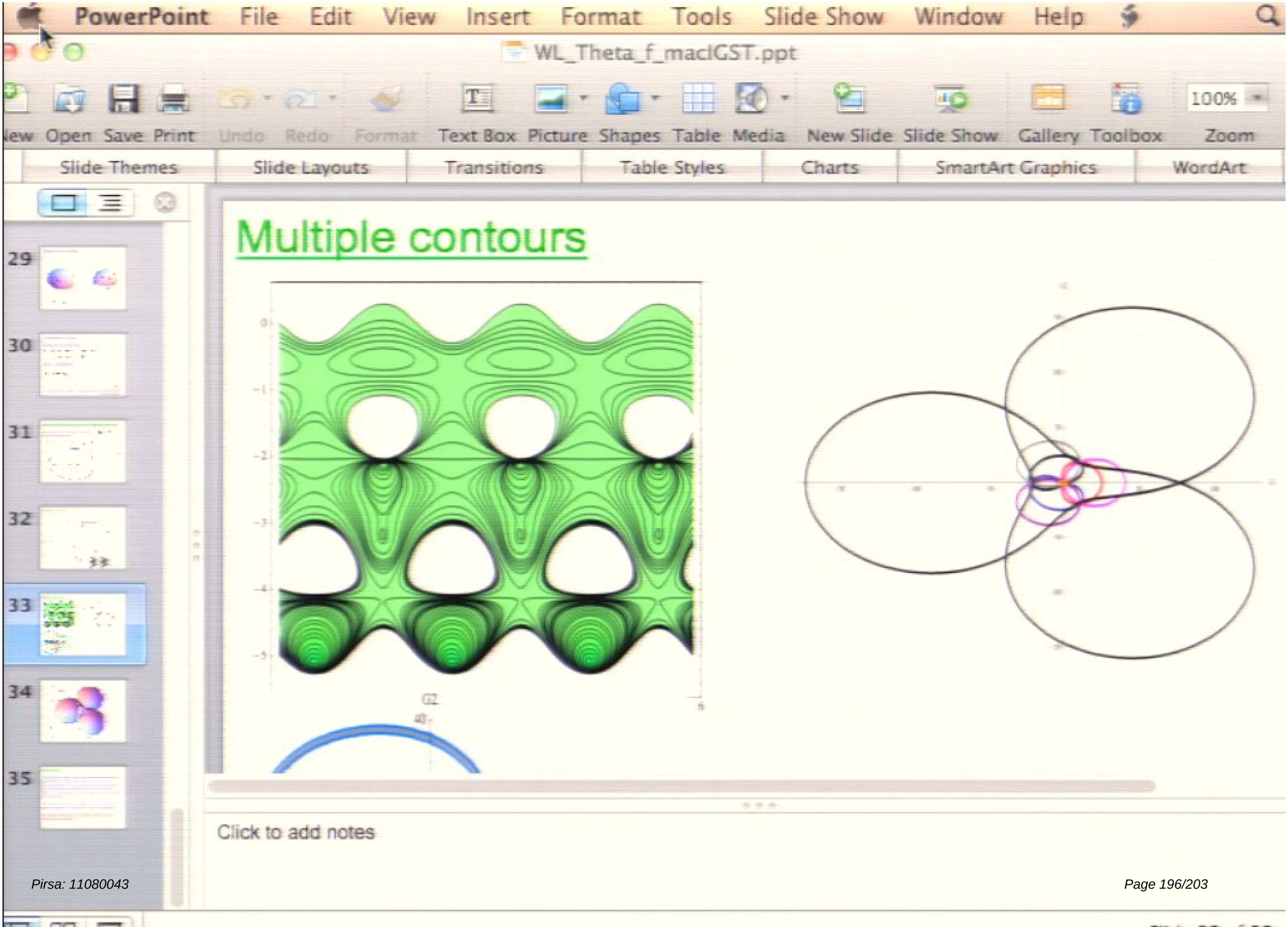
Click to add notes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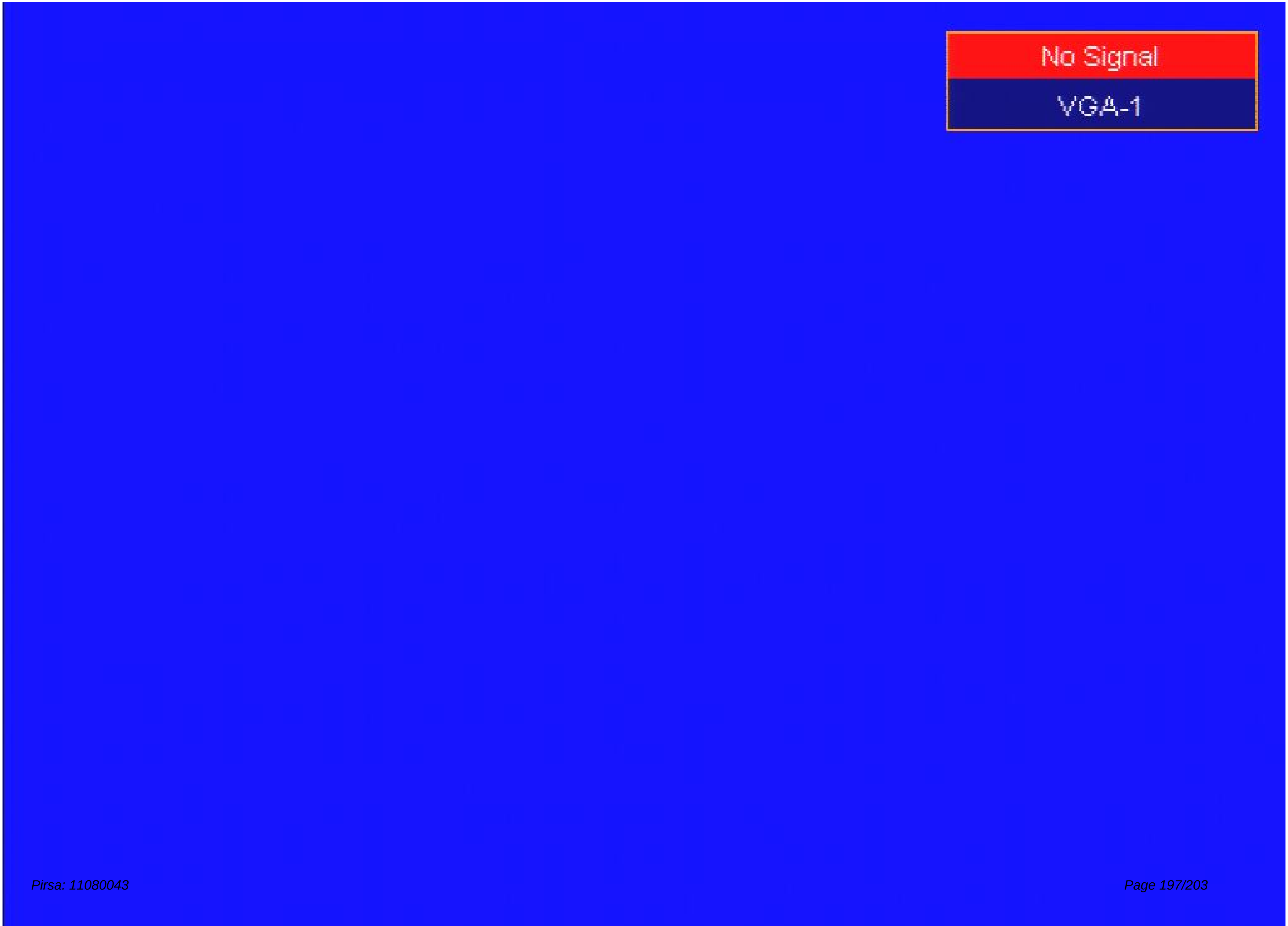
Click to add notes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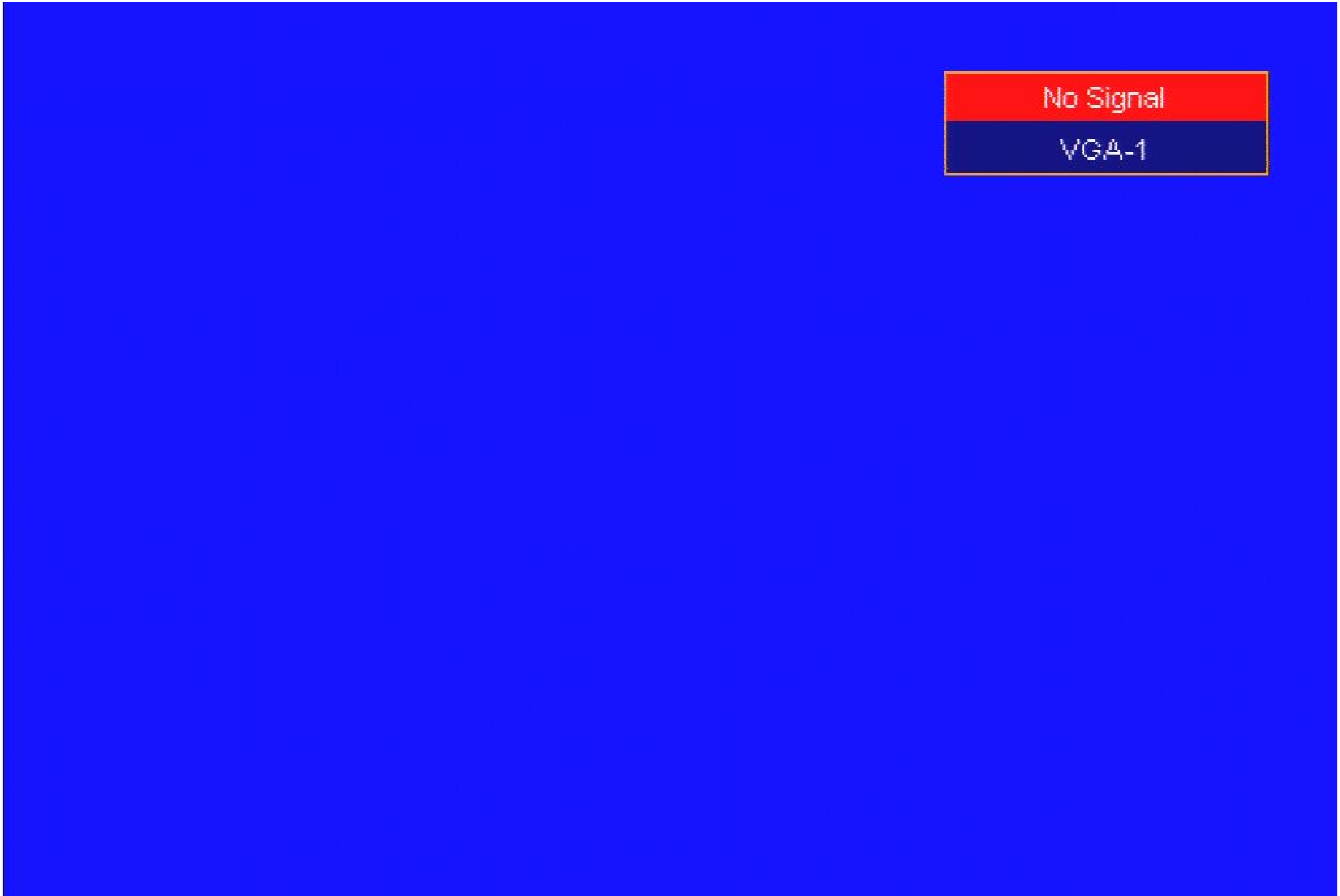
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Announcements on Gong Show

- It will be interesting to have talks of more students and postdocs.
- We plan to have an informal series of talks on Tuesday and/or Thursday, 17:00-18:00 (provisional).
- This is NOT official part* of the Conference IGST2011. Still we can use nice theater and projector facilities.
- If you want to join the show, please gather in front of the blackboard after Martin's talk THIS evening.

(* Spontaneously organized by participants)