Title: A Relativistic Relative of GS Superstrings on AdS5 x S5

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Abstract: The motion of superstrings on symmetric space target spaces is classically equivalent, via the Pohlmeyer reduction, to a family of 2-d relativistic integrable field theories known as semisymmetric space sine-Gordon (SSSSG) theories. In this talk I will review recent progress in constructing the relativistic S-matrix corresponding to the quantum solution of the AdS5 x S5 SSSSG theory.

A relativistic relative of GS supertrings on $AdS_5 \times S^5$
The Semi-Symmetric Space sine-Gordon theory

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Based on arXiv:1107.0628, 1104.2429, ...

Outline

1. The SSSG and SSSSG theories

2. The $AdS_5 \times S^5$ SSSSG theory

3. The relativistic S-matrix

4. Bound states: bootstrap

5. Open questions

The SSSG and SSSSG theories

Pohlmeyer, Eichenherr, Forger, D'Auria, Regge, ...'76-81

☞ | Symmetric space sine-Gordon (SSSG) theories |

Tseytlin'03
Mikhailov'05

- Relativistic integrable theories in $1+1$ dimensions that are classically equivalent, via the Pohlmeyer reduction, to the non-relativistic (gauged fixed) world-sheet theories of strings on symmetric space spacetimes like $\mathbb{R}_t \times S^n$, $\mathbb{R}_t \times \mathbb{C}P^n$, $AdS_n \times S^1_\vartheta$, AdS_n, ...

Hofman-Maldacena'06
Chen-N.Dorey-Okamura'06

- Admit soliton solutions that, for $\mathbb{R}_t \times S^n$ or $\mathbb{R}_t \times \mathbb{C}P^n$, are the images of the string giant magnons

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Serganova'83
Zarembo'10

$\longmapsto$ $\mathbb{Z}_4$ automorphism: $\sigma^4 = 1, \quad \sigma(G) = G$

$\longmapsto$ Lie superalgebra decomposition: $\mathfrak{f} = \underbrace{\mathfrak{f}_0}_{\text{even}} \oplus \underbrace{\mathfrak{f}_1}_{\text{odd}} \oplus \underbrace{\mathfrak{f}_2}_{\text{even}} \oplus \underbrace{\mathfrak{f}_3}_{\text{odd}}$

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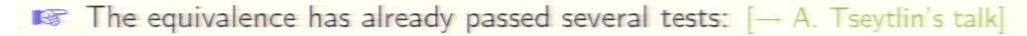
Roiban-Tseytlin'09

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- Focus not so much on the Lagrangian and perturbation theory but rather on the solitons: perturbative fields re-appear!
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The (relativistic) SSSSG Lagrangian

$$\mathscr{L} = \mathscr{L}_{\text{gWZW}}[G/H] - \frac{k}{\pi}\text{STr}\left(\Lambda\gamma^{-1}\Lambda\gamma\right)$$

$$+ \frac{k}{2\pi}\text{STr}\left(\psi_+[\Lambda, D_-\psi_+] - \psi_-[\Lambda, D_+\psi_-] - 2\psi_+\gamma^{-1}\psi_-\gamma\right)$$

- $\gamma \in G = e^{\mathfrak{f}_0}$ and $A_\pm \longrightarrow$ bosonic fields
  $\psi_+ \in \mathfrak{f}_1, \psi_- \in \mathfrak{f}_3 \longrightarrow$ fermionic fields
- The potential is fixed by $\Lambda \in \mathfrak{f}_2$ (constant)

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Equations of motion

- Zero-curvature conditions $\boxed{\left[\mathcal{L}_\mu(z), \mathcal{L}_\nu(z)\right] = 0}$

$$\mathcal{L}_+(z) = \partial_+ + \gamma^{-1}\partial_+\gamma + \gamma^{-1}A_+\gamma + z\psi_+ - z^2\Lambda,$$
$$\mathcal{L}_-(z) = \partial_- + A_- + z^{-1}\gamma^{-1}\psi_-\gamma - z^{-2}\gamma^{-1}\Lambda\gamma$$

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Goykhman-Ivanov'11
Hollowood-JLM'11

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Solitons

■ Continuous spectrum of relativistic non-abelian Q-ball kinks with bosonic and fermionic degrees of freedom

labelled by $\varphi \in (0, \pi/2) \longrightarrow$ $\boxed{m_\varphi = \dfrac{2k}{\pi} \sin \varphi}$

Non-trivial moduli space $\boxed{\mathfrak{M} = \dfrac{SU(2|2)}{U(2|1)} \times \dfrac{SU(2|2)}{U(2|1)}}$

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The extended $sl(2|2)$ superalgebra

- Generators

$$\text{Even} \rightarrow sl(2)^2: \quad \mathfrak{R}^a{}_b, \quad \mathfrak{L}^\alpha{}_\beta$$

$$\text{Odd} \rightarrow \text{SUSY:} \quad \mathfrak{Q}^\alpha{}_b, \quad \mathfrak{S}^a{}_\beta \qquad \text{Centres:} \quad \boxed{\mathfrak{C}, \; \mathfrak{P}, \; \mathfrak{K}}$$

- $\boxed{psl(2|2) \ltimes \mathbb{R}^3}$ algebra

$$[\mathfrak{R}^a{}_b, \mathfrak{R}^c{}_d] = \delta^c_b \mathfrak{R}^a{}_d - \delta^a_d \mathfrak{R}^c{}_b \qquad\qquad [\mathfrak{L}^\alpha{}_\beta, \mathfrak{L}^\gamma{}_\delta] = \delta^\gamma_\beta \mathfrak{L}^\alpha{}_\delta - \delta^\alpha_\delta \mathfrak{L}^\gamma{}_\beta$$

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$$\boxed{\begin{aligned} \{\mathfrak{Q}^\alpha{}_b, \mathfrak{S}^c{}_\delta\} &= \delta^c_b \mathfrak{L}^\alpha{}_\delta + \delta^\alpha_\delta \mathfrak{R}^c{}_b + \delta^c_b \delta^\alpha_\delta \, \mathfrak{C} \\ \{\mathfrak{Q}^\alpha{}_b, \mathfrak{Q}^\gamma{}_d\} &= \varepsilon^{\alpha\gamma}\varepsilon_{bd}\, \mathfrak{P} \qquad \{\mathfrak{S}^a{}_\beta, \mathfrak{S}^c{}_\delta\} = \varepsilon^{ac}\varepsilon_{\beta\delta}\, \mathfrak{K} \end{aligned}}$$

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Defining representation(s)

☞ $\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3$ has a family of 4-dimensional representations labelled by four parameters a, b, c, d constrained by $\boxed{ad - bc = 1}$

- Odd generators

$$\mathfrak{Q}^1{}_1 = a\,\mathbf{e}_{31} + b\,\mathbf{e}_{24}, \qquad \mathfrak{Q}^1{}_2 = a\,\mathbf{e}_{32} - b\,\mathbf{e}_{14},$$

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- $\mathbb{Z}$-gradation of $\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3$

$$s(\mathfrak{R}^a{}_b) = s(\mathfrak{L}^\alpha{}_\beta) = 0, \qquad s(\mathfrak{Q}^\alpha{}_b) = 1, \qquad s(\mathfrak{S}^a{}_\beta) = -1,$$

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$$\boxed{\mathfrak{sl}(2|2)^{(\sigma)} = \mathcal{L}(\mathfrak{sl}(2|2), \sigma) \oplus \mathbb{C}\,\mathfrak{C}} \qquad \boxed{\mathfrak{P} \equiv z^2 \otimes \mathbf{1}, \quad \mathfrak{K} \equiv z^{-2} \otimes \mathbf{1}}$$

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☞ $\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3$ is a finite subalgebra of the centrally extended twisted affine algebra

$$\boxed{\mathfrak{sl}(2|2)^{(\sigma)} = \mathcal{L}(\mathfrak{sl}(2|2), \sigma) \oplus \mathbb{C}\,\mathfrak{C}} \qquad \boxed{\mathfrak{P} \equiv z^2 \otimes \mathbf{1}, \quad \mathfrak{K} \equiv z^{-2} \otimes \mathbf{1}}$$

Z-gradation and $\mathbb{Z}_4$ automorphism

- Z-gradation of $\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3$

$$s(\mathfrak{R}^a{}_b) = s(\mathfrak{L}^\alpha{}_\beta) = 0, \qquad s(\mathfrak{Q}^\alpha{}_b) = 1, \qquad s(\mathfrak{S}^a{}_\beta) = -1,$$

$$s(\mathfrak{C}) = 0, \qquad\qquad\qquad s(\mathfrak{P}) = 2, \qquad\quad s(\mathfrak{K}) = -2,$$

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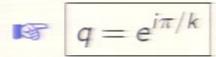# Quantum deformation $U_q(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^3)$

- Chevalley generators

$$[\mathcal{E}_1, \mathcal{F}_1] = [\mathfrak{H}_1]_q, \qquad \{\mathcal{E}_2, \mathcal{F}_2\} = -[\mathfrak{H}_2]_q, \qquad [\mathcal{E}_3, \mathcal{F}_3] = -[\mathfrak{H}_3]_q$$

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+ deformed Serre relations

- Cartan matrix: $\quad A_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \qquad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$

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- Chevalley generators in the fundamental representation(s)

$$\mathfrak{E}_1 = q^{1/2} \, \mathfrak{R}^2{}_1, \qquad \mathfrak{E}_2 = \mathfrak{Q}^2{}_2, \qquad \mathfrak{E}_3 = q^{-1/2} \, \mathfrak{L}^1{}_2$$

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☞ Centres

$$P = ab, \quad K = cd, \quad ad = [C + 1/2]_q, \quad bc = [C - 1/2]_q$$

$$\Rightarrow \quad \boxed{[C]_q^2 - PK = [1/2]_q^2} \quad \equiv \quad \text{deformed shortening condition}$$

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- Beisert-Koroteev parameterization of $\{a, b, c, d\}$

$$a = \sqrt{g}\gamma, \qquad\qquad b = \frac{\sqrt{g}\alpha}{\gamma}\left(1 - q^{2C-1}\frac{x^+}{x^-}\right),$$

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subject to

$$\frac{x^+}{q} + \frac{q}{x^+} - qx^- - \frac{1}{qx^-} + ig(q - q^{-1})\left(\frac{x^+}{qx^-} - \frac{qx^-}{x^+}\right) = \frac{i}{g}$$

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$$\boxed{q \to 1} \quad (\text{or } k \to \infty) \quad \Rightarrow \quad \boxed{\frac{x^+}{x^-} = e^{ip}}$$

$p \equiv$ world-sheet momentum $\qquad g \equiv$ string tension

$$[C]_q^2 - PK = [1/2]_q^2 \quad \longrightarrow \quad \epsilon(p) = \sqrt{1 + 4g^2 \sin^2(p/2)}$$

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- Involves a new abelian generator $\mathfrak{U}$

$$\Delta(\mathfrak{E}_j) = \mathfrak{E}_j \otimes 1 + q^{-\mathfrak{H}_j}\,\mathfrak{U}^{s_j} \otimes \mathfrak{E}_j, \quad \Delta(\mathfrak{F}_j) = \mathfrak{F}_j \otimes q^{\mathfrak{H}_j} + \mathfrak{U}^{-s_j} \otimes \mathfrak{F}_j$$

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Fundamental (relativistic) S-matrix

Hoare–Tseytlin'11
Hoare–Hollowood–JLM'11

- $\tilde{S}_{11}(\theta_{12}) : V_1(\theta_1) \otimes V_1(\theta_2) \longrightarrow V_1(\theta_2) \otimes V_1(\theta_1)$ $\boxed{\theta_{12} = \theta_1 - \theta_2}$

$$\boxed{\tilde{S}_{11}(\theta) = Y(\theta)\,Y(i\pi - \theta)\,\check{R}(e^{\theta})}$$

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- $\check{R}$-matrix: basis $\{|\phi^{a}\rangle, |\psi^{\alpha}\rangle\}$, $x = e^{\theta}$

$$\check{R}(x)|\phi^{a}\phi^{a}\rangle = A|\phi^{a}\phi^{a}\rangle, \qquad \check{R}(x)|\psi^{\alpha}\psi^{\alpha}\rangle = D|\psi^{\alpha}\psi^{\alpha}\rangle,$$

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$$\check{R}(x)|\phi^{a}\psi^{\alpha}\rangle = G|\psi^{\alpha}\phi^{a}\rangle + H|\phi^{a}\psi^{\alpha}\rangle, \qquad \check{R}(x)|\psi^{\alpha}\phi^{a}\rangle = K|\psi^{\alpha}\phi^{a}\rangle + L|\phi^{a}\psi^{\alpha}\rangle,$$

$$A = \frac{(qx-1)(x+1)}{q^{1/2}x}, \qquad\qquad D = \frac{(q-x)(x+1)}{q^{1/2}x},$$

$$B = \frac{q^{3}-(q^{3}-2q^{2}+2q-1)x-x^{2}}{q^{3/2}x}, \qquad E = \frac{q^{3}x^{2}-(q^{3}-2q^{2}+2q-1)x-1}{q^{3/2}x},$$

$$C = F = \frac{i(q-1)(q^{2}+1)(x-1)}{q^{3/2}x^{1/2}}, \qquad G = L = x - x^{-1},$$

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$$A = \frac{(qx-1)(x+1)}{q^{1/2}x}, \qquad\qquad D = \frac{(q-x)(x+1)}{q^{1/2}x},$$

$$B = \frac{q^3 - (q^3 - 2q^2 + 2q - 1)x - x^2}{q^{3/2}x}, \qquad E = \frac{q^3x^2 - (q^3 - 2q^2 + 2q - 1)x - 1}{q^{3/2}x},$$

$$C = F = \frac{i(q-1)(q^2+1)(x-1)}{q^{3/2}x^{1/2}}, \qquad G = L = x - x^{-1},$$

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★ $\tilde{S}_{11}(\theta)$ has $U_q\left(\mathfrak{psl}(2|2) \ltimes \mathbb{R}^2\right)$ symmetry

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$Y(\theta)$ is fixed by the unitarity condition $\tilde{S}_{11}(\theta)\tilde{S}_{11}(-\theta) = 1 \otimes 1$

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If a bound state corresponding to a multiplet V_c is produced in the collision of V_a and V_b in the direct channel, then

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$$\tilde{S}_{11}(\theta_{12}) : \overbrace{V_{\langle 0,0\rangle}(\theta_1) \otimes V_{\langle 0,0\rangle}(\theta_2)}^{V_{\{0,0\}}} \longrightarrow V_{\langle 0,0\rangle}(\theta_2) \otimes V_{\langle 0,0\rangle}(\theta_1)$$

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☞ Res $S_{ab}(iu_{ab}^c)$ is a weighted sum of "projectors"

$$V_c = \oplus_j V_c^{(j)}, \quad V_c^{(j)} = \text{irreducible representations of } U_q\left(\mathfrak{sl}(2)^{\times 2}\right)$$

$$S_{ab}(\theta) \sim \frac{i}{\theta - iu_{ab}^c} \sum_j \rho_j \, \mathbb{P}_{ab}^{c.j}, \qquad \mathbb{P}_{ab}^{c.j} = \mathcal{P}_{c.j}^{ba} \, \mathcal{P}_{ab}^{c.j}$$

The magnon and the soliton representations

- Magnon representation:

$$\boxed{q \to 1} \quad (\text{or } k \to \infty) \Rightarrow \boxed{\frac{x^+}{x^-} = e^{ip}}$$

$p \equiv$ world-sheet momentum $\qquad g \equiv$ string tension

$$[C]_q^2 - PK = [1/2]_q^2 \longrightarrow \epsilon(p) = \sqrt{1 + 4g^2 \sin^2(p/2)}$$

Hoare-Hollowood-JLM'11

☞ Soliton representation:

$$\boxed{g \to \infty} \Rightarrow \boxed{\frac{x^+}{x^-} = q, \quad C = 0}$$

$$\Rightarrow [C]_q^2 - PK = [1/2]_q^2 \quad \text{becomes a relativistic mass-shell condition}$$

★ $m_a = \mu \sin\left(\frac{\pi a}{2k}\right)$ can be understood as the BPS condition for the short representation of $\boxed{U_q(\mathfrak{psu}(2|2) \ltimes \mathbb{R}^2)}$ of dimension $4a \times 4a$ with $\boxed{q = e^{i\pi/k}}$

☣ The semiclassical spectrum consists of a tower of massive states in short symmetric representations, just like the magnon bound states of the superstring world sheet theory.......

but in the SSSG theory the tower is truncated by k, the level of the WZW term.....

Solitons

■ Continuous spectrum of relativistic non-abelian Q-ball kinks with bosonic and fermionic degrees of freedom

labelled by $\varphi \in (0, \pi/2)$ $\longrightarrow$ $\boxed{ m_\varphi = \dfrac{2k}{\pi} \sin \varphi }$

Non-trivial moduli space $\boxed{ \mathfrak{M} = \dfrac{SU(2|2)}{U(2|1)} \times \dfrac{SU(2|2)}{U(2|1)} }$

Hidden symmetries

- Infinite (classical) symmetry algebra

$$\hat{\mathfrak{f}}^{\perp} = \mathrm{Ker}(\mathrm{ad}\Lambda) \cap \mathcal{L}(\mathfrak{psu}(2,2|4),\sigma)$$

Hollowood–JLM'11
Hoare–Tseytlin'11

☞ The elements of grade ± 1 (or Lorentz spin $\pm 1/2$) generate SUSY transformations whose closure is the (exotic) $\mathcal{N} = (8|8)$ superalgebra

$$\boxed{\mathfrak{s} = (\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)) \ltimes (\mathbb{R} \oplus \mathbb{R})} = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_{+1} \oplus \mathfrak{s}_{+2}$$

$\longmapsto$ $\mathfrak{s}_{\pm 2}$: central elements corresponding to the components of p_μ
$\longmapsto$ $\mathfrak{s}_{\pm 1}$: generators of SUSY transformations
$\longmapsto$ $\mathfrak{s}_0$: generators of global gauge transformations
$\qquad \equiv$ non-abelian R-symmetry group $SU(2)^{\times 4}$

☞ $\mathfrak{s}$ is a finite subalgebra of $\boxed{\mathcal{L}(\mathfrak{p}(\mathfrak{su}(2|2) \oplus \mathfrak{su}(2|2)),\sigma) \subset \hat{\mathfrak{f}}^{\perp}}$ and the derivation $\boxed{z d/dz}$ is the generator of Lorentz boosts

The (relativistic) SSSSG Lagrangian

$$\mathcal{L} = \mathcal{L}_{\text{gWZW}}[G/H] - \frac{k}{\pi}\text{STr}\left(\Lambda\gamma^{-1}\Lambda\gamma\right)$$

$$+ \frac{k}{2\pi}\text{STr}\left(\psi_+[\Lambda, D_-\psi_+] - \psi_-[\Lambda, D_+\psi_-] - 2\psi_+\gamma^{-1}\psi_-\gamma\right)$$

- $\gamma \in G = e^{\mathfrak{f}_0}$ and $A_\pm \longrightarrow$ bosonic fields
  $\psi_+ \in \mathfrak{f}_1, \psi_- \in \mathfrak{f}_3 \longrightarrow$ fermionic fields
- The potential is fixed by $\Lambda \in \mathfrak{f}_2$ (constant)

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