Title: Baxter Q-Operators for Integrable Spin Chains

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URL: http://pirsa.org/11080028

Abstract:

Baxter Q-operators for integrable spin chains

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Exact Results in Gauge/Gravity Dualities
Perimeter Institute.

09.08.2011

arXiv:1005.3261 V. Bazhanov, TL, C. Meneghelli, M. Staudacher

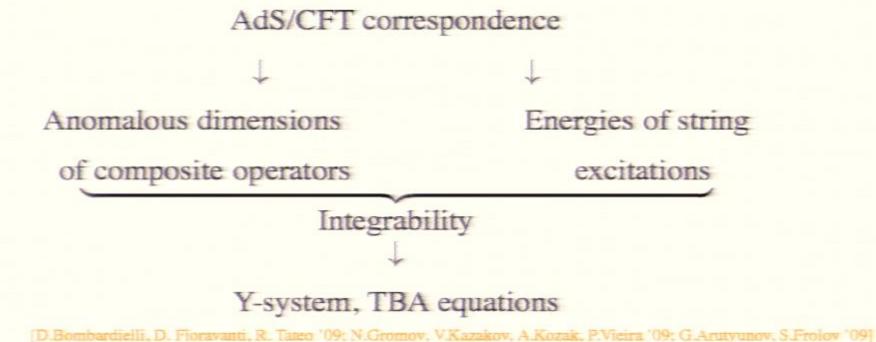
arXiv:1010.3699 V. Bazhanov, R. Frassek, TL, C. Meneghelli, M. Staudacher

arXiv:1012.6021 R. Frassek, TL, C. Meneghelli, M. Staudacher

work in progress R. Frassek, TL, C. Meneghelli, M. Staudacher, Y. Xu

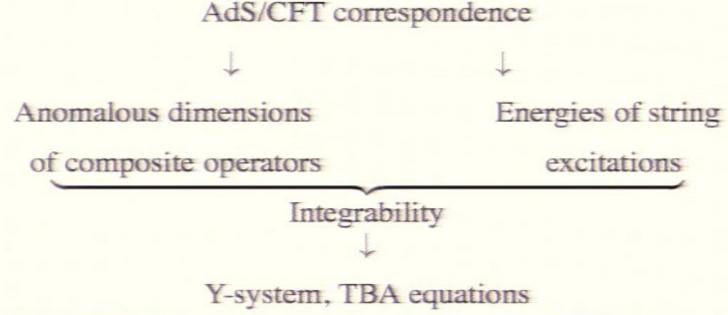
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Motivation



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Motivation



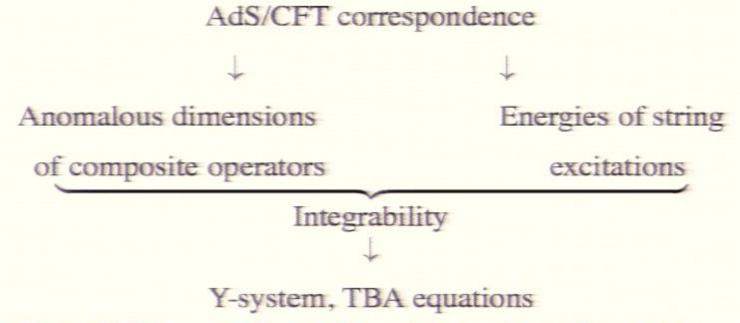
[D.Bombardielli, D. Fioravanti, R. Tateo '09; N.Gromov, V.Kazakov, A.Kozak, P.Vieira '09; G.Arutyunov, S.Frolov '09]

- Still many questions and unsolved problems
- We need to find analytic properties of Y-functions

Let us focus on some simpler integrable models

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Motivation



[D.Bombardielli, D. Fioravanti, R. Tateo '09; N.Gromov, V.Kazakov, A.Kozak, P.Vieira '09; G.Arutyunov, S.Frolov '09]

- Still many questions and unsolved problems
- We need to find analytic properties of Y-functions
- "In integrable models it is frequently much easier to guess the exact

Pirsa: 11080028 Ition than to prove and understand it!"

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Let us focus on some simpler integrable models

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- Let us focus on some simpler integrable models
- Problem of finding one-loop anomalous dimensions is mapped to a diagonalization of the psu(2, 2|4)-invariant spin chain

 Minahan, Zarembo;

Beisert, Staudacher

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 Minahan, Zarembo;

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• More general: diagonalizing integrable spin chains with a given representation of $gl(n_1, n_2|m)$ in the quantum space

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- In order to do it we will construct so called Baxter Q-operators all
 Y-functions are functions of eigenvalues of Q-operators

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$$V^{\otimes L}$$
 = $V^{\otimes L}$ + periodic boundary conditions

Hamiltonian

$$H: V^L \to V^L$$

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- Other approaches:
 - Derkachov, Korchemsky, Manashov, . . .
 - Kazakov, Laurent, Tsuboi, Vieira

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Hamiltonian

$$H: V^L \to V^L$$

Nearest neighbor spin chain

$$H = \sum_{i=1}^{L} \mathcal{H}_{i,i+1}, \qquad \mathcal{H}_{i,i+1} : V^2 \to V^2$$

- We want to diagonalize it
 - directly

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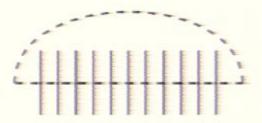
Nearest neighbor spin chain

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 - with use of Bethe equations efficient method to find them is the

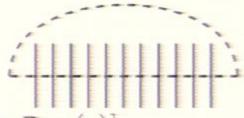
Quantum Inverse Scattering Method

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• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

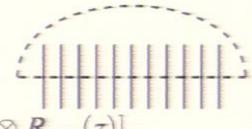
 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

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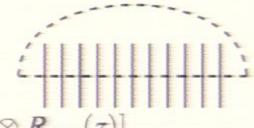
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Transfer matrices form a family of commuting operators

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Pirsa: 11080028 Inster matrices diagonalized with use of the Algebraic Bethe Ange 21/315 Z

Bethe equations (for the fundamental rep of $\mathfrak{sl}(n)$)

$$\begin{pmatrix}
\frac{z_{1,k} + \frac{1}{2}}{z_{1,k} - \frac{1}{2}}
\end{pmatrix}^{L} = \prod_{j \neq k} \frac{z_{1,k} - z_{1,j} + 1}{z_{1,k} - z_{1,j} - 1} \prod_{j} \frac{z_{1,k} - z_{2,j} + \frac{1}{2}}{z_{1,k} - z_{2,j} + \frac{1}{2}}$$

$$1 = \prod_{j} \frac{z_{2,k} - z_{1,j} - \frac{1}{2}}{z_{2,k} - z_{1,j} + \frac{1}{2}} \prod_{j \neq k} \frac{z_{2,k} - z_{2,j} + 1}{z_{2,k} - z_{2,j} - 1} \prod_{j} \frac{z_{2,k} - z_{3,j} - \frac{1}{2}}{z_{2,k} - z_{3,j} + \frac{1}{2}}$$

$$1 = \prod_{j} \frac{z_{n-1,k} - z_{n-2,j} - \frac{1}{2}}{z_{n-1,k} - z_{n-1,j} + 1} \prod_{j \neq k} \frac{z_{n-1,k} - z_{n-1,j} + 1}{z_{n-1,k} - z_{n-1,j} - 1}$$

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 $E = \sum_{i=1}^{n} \frac{1}{\frac{1}{4} - z_{1-k}^2}$

ABA works best if we have a reference state

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- For XYZ model Bethe equations were found by Baxter with use of so called Q-operators – operators which eigenvalues are polynomials with roots being Bethe roots

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Baxter Q-operators – the most powerful tool in integrable systems

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Only recently fully understood for XXX_{1/2} Heisenberg spin chain

[V. Bazhanov, TL, C. Menegheili, M. Staudacher]

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We want to construct Baxter operators in the QISM framework

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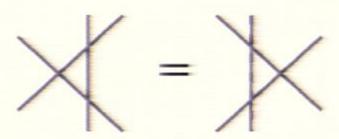
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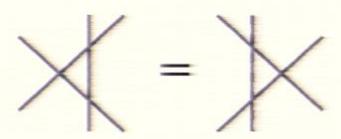
• We can find R-matrices from the Yang-Baxter equation



$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

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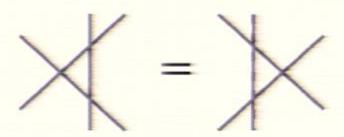


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 YBE is a defining relation for Yangian – every solution of the YBE gives a representation of Yangian

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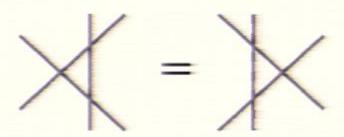
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- Example $\square \otimes \square \otimes \square$

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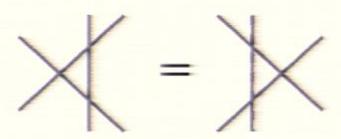
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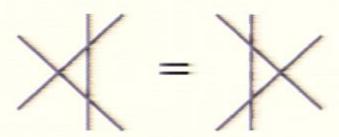
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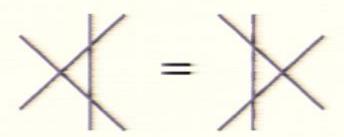
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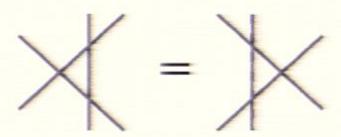
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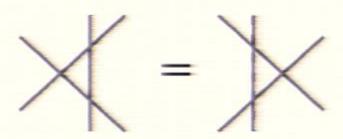
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• Take $\square \otimes \square \otimes W$ (W – any space) and try to find all linear solutions of

$$R_{\square,\square}(x-y)R_{\square,W}(x)R_{\square,W}(y) = R_{\square,W}(y)R_{\square,W}(x)R_{\square,\square}(x-y)$$

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Pirsa: 11080028 Page 42/31:

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Pirsa: 11080028 Page 61/31

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Pirsa: 11080028 Page 62/31

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$$Q_p(z) = \operatorname{Tr}_{aux} \left[\mathbb{D} R_p(z) \otimes R_p(z) \otimes \ldots \otimes R_p(z) \right]$$

Pirsa: 11080028 Page 63/315

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Pirsa: 11080028 Page 64/315

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Pirsa: 11080028 Page 65/315

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Pirsa: 11080028 Page 66/315

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Pirsa: 11080028 Page 67/315

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Pirsa: 11080028 Page 68/315

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Pirsa: 11080028 Page 69/315

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Pirsa: 11080028 Page 70/315

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Pirsa: 11080028 Page 73/315

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Pirsa: 11080028 Page 74/315

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Pirsa: 11080028 Page 75/315

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Pirsa: 11080028 Page 76/315

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Pirsa: 11080028 Page 77/315

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Pirsa: 11080028 Page 78/315

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Pirsa: 11080028

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Pirsa: 11080028 Page 80/315

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Pirsa: 11080028 Page 81/315

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• Belong to the same family of commuting operators: $[Q_p(z), T(z')] = 0$

Pirsa: 11080028 Page 82/315

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Pirsa: 11080028 Page 83/315

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Pirsa: 11080028 Page 84/315

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Pirsa: 11080028 Page 85/315

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Pirsa: 11080028 Page 90/315

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Pirsa: 11080028 Page 92/315

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Pirsa: 11080028 Page 94/315

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$$Q_p(z) = \operatorname{Tr}_{aux} \left[\mathbb{D} R_p(z) \otimes R_p(z) \otimes \ldots \otimes R_p(z) \right]$$

• Belong to the same family of commuting operators: $[Q_p(z), T(z')] = 0$

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• Take $\square \otimes \square \otimes W$ (W – any space) and try to find all linear solutions of

$$R_{\square,\square}(x-y)R_{\square,W}(x)R_{\square,W}(y) = R_{\square,W}(y)R_{\square,W}(x)R_{\square,\square}(x-y)$$

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Pirsa: 11080028 Page 96/315

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Pirsa: 11080028 Page 97/315

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Pirsa: 11080028 Page 98/315

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Pirsa: 11080028 Page 99/315

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Pirsa: 11080028

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Pirsa: 11080028 Page 101/315

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Pirsa: 11080028 Page 102/315

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Pirsa: 11080028

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Pirsa: 11080028 Page 104/315

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Pirsa: 11080028 Page 105/315

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Pirsa: 11080028 Page 106/315

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Pirsa: 11080028

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Pirsa: 11080028 Page 108/315

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Pirsa: 11080028 Page 110/315

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Pirsa: 11080028 Page 111/315

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- For p = n we have previous solutions transfer matrices
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$$Q_p(z) = \operatorname{Tr}_{aux} \left[\mathbb{D} R_p(z) \otimes R_p(z) \otimes \ldots \otimes R_p(z) \right]$$

- Belong to the same family of commuting operators: $[Q_p(z), T(z')] = 0$
- Even more Q-operators $-\binom{n}{p}$ different ways of fixing $\mathfrak{gl}(p)$ in $\mathfrak{gl}(n)$

Page 215/315

• Take $\square \otimes \square \otimes W$ (W – any space) and try to find all linear solutions of

$$R_{\square,\square}(x-y)R_{\square,W}(x)R_{\square,W}(y) = R_{\square,W}(y)R_{\square,W}(x)R_{\square,\square}(x-y)$$

- W can be the representation space of the tensor product of any gl(p)
 (p ≤ n) representation and an oscillator algebra representation
- For p = n we have previous solutions transfer matrices
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Page 216/315

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Page 217/315

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Page 218/315

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Page 219/315

• Take $\square \otimes \square \otimes W$ (W – any space) and try to find all linear solutions of

$$R_{\square,\square}(x-y)R_{\square,W}(x)R_{\square,W}(y) = R_{\square,W}(y)R_{\square,W}(x)R_{\square,\square}(x-y)$$

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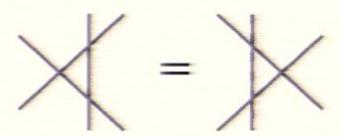
$$Q_p(z) = \operatorname{Tr}_{aux} \left[\mathbb{D} R_p(z) \otimes R_p(z) \otimes \ldots \otimes R_p(z) \right]$$

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- Even more Q-operators $-\binom{n}{p}$ different ways of fixing $\mathfrak{gl}(p)$ in $\mathfrak{gl}(n)$

Pirsa: 11080028

Yang-Baxter equation

• We can find R-matrices from the Yang-Baxter equation

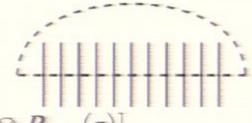


$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

 YBE is a defining relation for Yangian – every solution of the YBE gives a representation of Yangian

Pirsa: 11080028

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

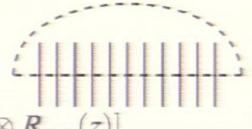
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 222/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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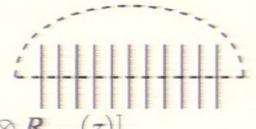
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 223/215 TZ

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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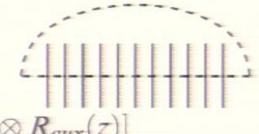
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 224/315 Z

Transfer matrices (D is for convergence)



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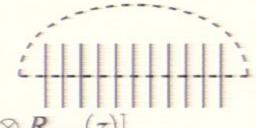
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 225/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

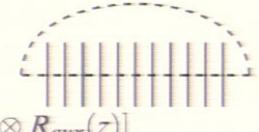
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 229/315 Z

• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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Transfer matrices form a family of commuting operators

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Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 227/315 Z

• Transfer matrices (D is for convergence)



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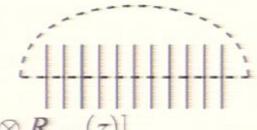
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 228/215 Z

■ Transfer matrices (D is for convergence)



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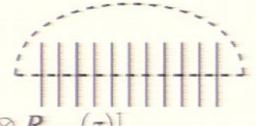
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

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Pirsa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 229/315 Z

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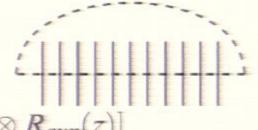
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 230/215 Z

• Transfer matrices (D is for convergence)



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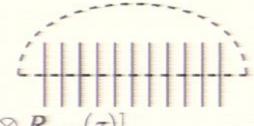
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 231/215 TZ

Transfer matrices (D is for convergence)



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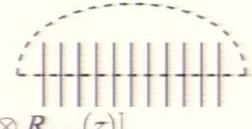
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 232/215 Z

• Transfer matrices (D is for convergence)



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 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

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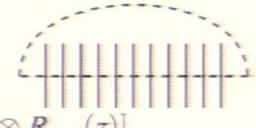
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 239/215 Z

Transfer matrices (D is for convergence)



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 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

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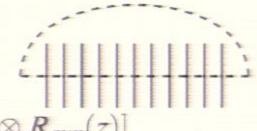
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 234/315 Z

• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 235/215 Z

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 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

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Transfer matrices form a family of commuting operators

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Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 239/215 Z

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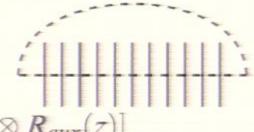
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 237/315 Z

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$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Inster matrices diagonalized with use of the Algebraic Bethe Page 238/215 TZ

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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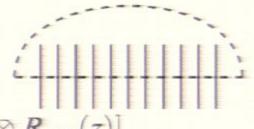
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 239/215 Z

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$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

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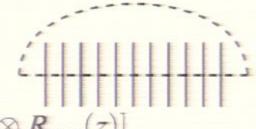
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pirsa: 11080028 INSTET matrices diagonalized with use of the Algebraic Bethe Page 240/315 Z

• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

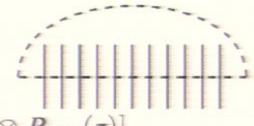
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Inster matrices diagonalized with use of the Algebraic Bethe Page 241/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pirsa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 242/215 TZ

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 243/215 Z

• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

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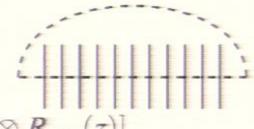
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 244/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

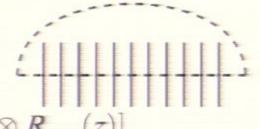
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 INSTET matrices diagonalized with use of the Algebraic Bethe Page 245/315 Z

• Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

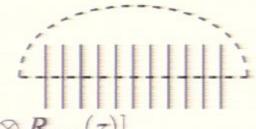
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 240/315 Z

• Transfer matrices (D is for convergence)



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 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

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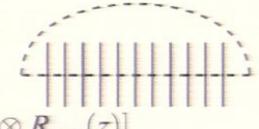
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 247/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

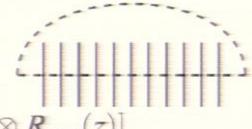
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 248/315 Z

Transfer matrices (D is for convergence)



$$T_{aux}(z) = \operatorname{Tr}_{aux}\left[\mathbb{D} R_{aux}(z) \otimes R_{aux}(z) \otimes \ldots \otimes R_{aux}(z)\right]$$

 From the transfer matrix with auxiliary=quantum we can extract local conserved charges

$$U = T(z_{\star})$$

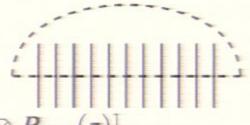
$$H = \frac{d}{dz} \log T(z) \big|_{z=z_{\star}}$$

Transfer matrices form a family of commuting operators

$$[T(z), T(z')] = [T(z), H] = 0$$

Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 249/315 Z

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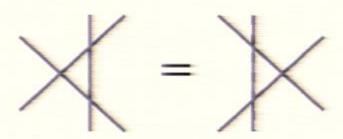
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Pisa: 11080028 Insfer matrices diagonalized with use of the Algebraic Bethe Page 250/315 Z

Yang-Baxter equation

• We can find R-matrices from the Yang-Baxter equation



$$R_{12}(x-y)R_{13}(x)R_{23}(y) = R_{23}(y)R_{13}(x)R_{12}(x-y)$$

- YBE is a defining relation for Yangian every solution of the YBE gives a representation of Yangian
- Example $\square \otimes \square \otimes \square$

$$R_{12}(z) = zI + P_{12}$$

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• Take $\square \otimes \square \otimes W$ (W – any space) and try to find all linear solutions of

$$R_{\square,\square}(x-y)R_{\square,W}(x)R_{\square,W}(y) = R_{\square,W}(y)R_{\square,W}(x)R_{\square,\square}(x-y)$$

- W can be the representation space of the tensor product of any gl(p)
 (p ≤ n) representation and an oscillator algebra representation
- For p = n we have previous solutions transfer matrices
- For the singlet in $\mathfrak{gl}(p)$ we define $(R_{\square,W_p} \equiv R_p)$

$$Q_p(z) = \operatorname{Tr}_{aux} \left[\mathbb{D} R_p(z) \otimes R_p(z) \otimes \ldots \otimes R_p(z) \right]$$

- Belong to the same family of commuting operators: $[Q_p(z), T(z')] = 0$
- Even more Q-operators $-\binom{n}{p}$ different ways of fixing $\mathfrak{gl}(p)$ in $\mathfrak{gl}(n)$

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• Solve YBE for $\square \otimes \square \otimes \Lambda$ – leads to $R_{\square,\Lambda}$ (quadratic)

$$R_{\Box,\Lambda}(z) = z e_{ii} + e_{ij} \otimes J_{ji}$$

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 - a priori, complicated functions of Casimir operators

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given by

$$c_i = \sum_{k=1}^{p} \prod_{j \neq k} \left(1 - \frac{1}{A_k - A_j} \right) A_k^i$$

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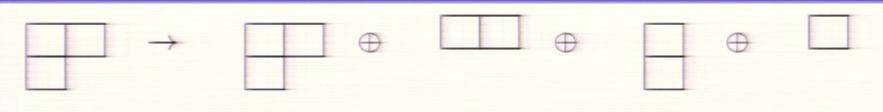
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An example: $\mathfrak{gl}(2)$ in $\mathfrak{gl}(3)$



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0

3

0

1

 \oplus

2

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• Basis of spin $s = -\frac{1}{2}$ representation: $|M\rangle = (\bar{a})^M |0\rangle, a|0\rangle = 0$

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$$Q_{0}(z) = 1$$

$$Q_{1}(z,M) = \frac{(M+z-1/2)! \,_{2}F_{1}\left(-M,-M;-M-z+1/2;\frac{1}{1-e^{-i\phi}}\right)}{(z-1/2)!}$$

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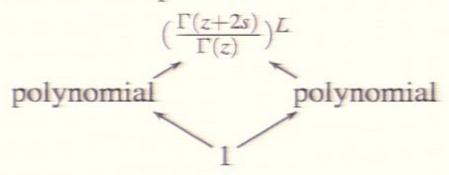
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- \circ Q_2 is a very complicated meromorphic function with an infinite number

Analytic properties of Q-functions

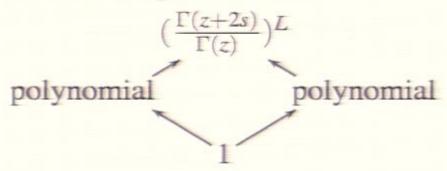
• Hasse diagram for s > 0 – compact case



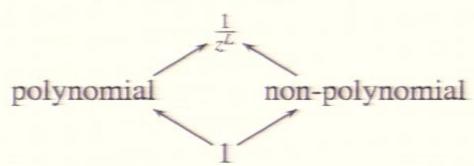
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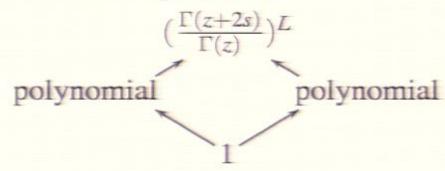


• Hasse diagram for s = -1/2 – non-compact case

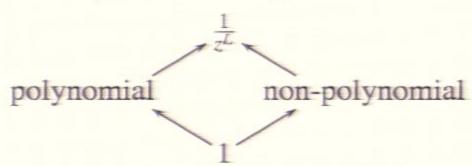


Analytic properties of Q-functions

• Hasse diagram for s > 0 – compact case



• Hasse diagram for s = -1/2 – non-compact case



 For su(2, 2|4) we have 2⁸ different Q-operators with various analytic properties - some of them are polynomials and some are very

Pirsa: 11080028 Dicated meromorphic functions

Anybody seen the Hamiltonian?

• We know Bethe equations for all representations for all (super)algebras:

"The nested Bethe ansatz for 'all' closed spin chains" – Belliard, Ragoucy (2008)

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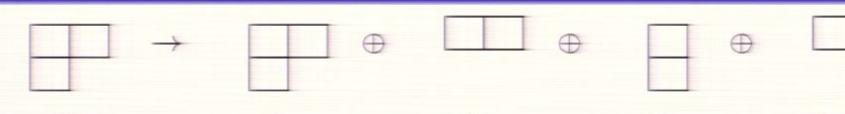
$$c_k = S_{i_2}^{i_1} S_{i_3}^{i_2} \dots S_{i_1}^{i_k}$$

given by

$$c_i = \sum_{k=1}^p \prod_{i \neq k} \left(1 - \frac{1}{A_k - A_j} \right) A_k^i$$

• however, they are simply shifted gl(p) weights of the decomposition of the gl(n) representation with respect to the gl(p) subalgebra!

An example: $\mathfrak{gl}(2)$ in $\mathfrak{gl}(3)$



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0

3

1

1

0

2

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• Solve YBE for $\square \otimes \square \otimes \Lambda$ – leads to $R_{\square,\Lambda}$ (quadratic)

$$R_{\Box,\Lambda}(z) = z e_{ii} + e_{ij} \otimes J_{ji}$$

• Solve YBE for $\square \otimes \Lambda \otimes W_p$ – leads to R_{Λ,W_p} (linear)

$$R_{\Lambda,W_p}(z) = e^{a_i^j S_j^i} \prod_{k=1}^p rac{\Gamma(z+A_k)}{\Gamma(z+p-k)} e^{-\overline{a}_j^i S_i^j}, \qquad i=1,\ldots,p \ j=p+1,\ldots,n$$

- Works for:
 - compact representations,
 - non-compact representations,
 - conjecture: works also for non-hws representations
- What are the operators A_k ?

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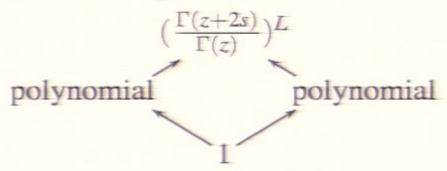
Example: spin $s = -\frac{1}{2}$ spin chain

• Basis of spin $s = -\frac{1}{2}$ representation: $|M\rangle = (\bar{a})^M |0\rangle, a|0\rangle = 0$

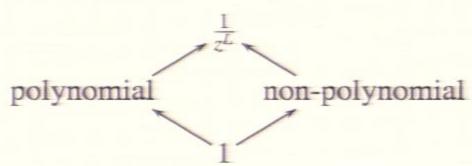
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Analytic properties of Q-functions

• Hasse diagram for s > 0 – compact case



• Hasse diagram for s = -1/2 – non-compact case



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Anybody seen the Hamiltonian?

• We know Bethe equations for all representations for all (super)algebras:

"The nested Bethe ansatz for 'all' closed spin chains" - Belliard, Ragoucy (2008)

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 Similar situation in all loop N = 4 SYM – it was conjectured that the solution is given by the Y-system/TBA equations....
 However, what is the Hamiltonian?

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• How to generalize the following formula from $\mathfrak{gl}(2)$ to $\mathfrak{gl}(n)$?

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Conjecture:

$$R_{\Lambda,\Lambda}(z) = \prod_{k=1}^{n} \frac{\Gamma(z + A_k)}{\Gamma(z + n - k)}$$

where A_k are shifted weights of the decomposition of the tensor product

Pirsa: 11080028. WO copies of the representation Λ

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Pirsa: 11080028 Page 303/315

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Pirsa: 11080028 Page 304/315

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- The resulting Hamiltonian takes the form

$$\mathcal{H} = \sum_{k=1}^{n} \psi(z_{\star} + A_{k}) + \operatorname{const}(n, z_{\star})$$

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We found a universal (we believe) method for the construction of Baxter
 Q-operators through the QISM

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Page 308/315 Pirsa: 11080028

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 - understand analytic properties of all-loop Q-, T-, Y-functions

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Thank you!

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No Signal

VGA-1

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No Signal VGA-1

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