Title: Edge Modes, Zero Modes and Conserved Charges in Parafermion Chains

Date: Aug 16, 2011 02:30 PM

URL: http://pirsa.org/11080023

Abstract:

Edge modes, zero modes and conserved charges in parafermion chains

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As wise men said...

It has become quite commonplace for concepts to move up and back between statistical physics and field theory.

This paper is concerned with elaborating an example from statistical physics which might perhaps illuminate in a simple context some ideas which have been employed in particle physics. In particular, we study some fields which appear (at least) superficially similar to those describing (fractionally) charged particles and topological excitations like 't Hooft monopoles...

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Nuclear Physics B170 [FS1] (1980) 1–15

DISORDER VARIABLES AND PARA-FERMIONS IN TWO-DIMENSIONAL STATISTICAL MECHANICS

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Received 4 March 1980

It is shown that "clock" type models in two-dimensional statistical mechanics possess order and disorder variables σ_n and χ_m with n and m falling in the range $1, 2, \ldots, p$. These variables respectively describe abelian analogs to charged fields and the fields of 't Hooft monopoles with charges $q = n/p$ and topological quantum number m. They are related to one another by a dual symmetry. Products of these operators generate, via a short-distance expansion, para-fermion operators in which rotational symmetry and the internal symmetry group are tied together. The clock models in two dimensions are shown to be an ideal laboratory where these ideas have a very simple realization.

1. Introduction

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- Correlators in these CFTs are used to construct the Read-Rezayi wavefunctions for the fractional quantum Hall effect.

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A familiar example is Chern-Simons theory.

Lattice models are fundamental to both condensed matter physics and to integrable systems.

Maybe it would be a good idea to go back and see if the original lattice parafermions of Fradkin and Kadanoff have something to do with topological order...

I'll describe interacting lattice models with edge modes that are not perturbations of free fermions. This is progress toward a $\mathbb{Z}_N$-invariant interacting generalization of the Kitaev honeycomb model.

Outline

- Edge/zero modes in the Majorana chain
- Edge/zero modes in the 3-state (chiral) Potts chain using parafermions

  an unusual form of integrability
- Coupling chains to make 2d $\mathbb{Z}_n$ gauge theories

  generalizing the Kitaev honeycomb model

How to fermionize the quantum Ising chain

$$H = \sum_j \left[f\sigma_j^x + J\sigma_j^z \sigma_{j+1}^z \right]$$

flip term　　　interaction

Critical point is when $J = f$, ordered phase is $J > f$.

$\mathbb{Z}_2$ symmetry operator is flipping all spins:

$$\prod_j \sigma_j^x$$

Jordan-Wigner transformation in terms of Majorana fermions

$$\psi_j = \sigma_j^z \prod_{i<j} \sigma_i^x , \qquad \chi_j = \sigma_j^y \prod_{i<j} \sigma_i^x$$

string

$$\{\psi_i, \psi_j\} = \{\chi_i, \chi_j\} = 2\delta_{ij}, \qquad \{\psi_i, \chi_j\} = 0$$

$\mathbb{Z}_2$ symmetry measures even or odd number of fermions:

$$(-1)^F = \prod_j \sigma_j^x = \prod_j (i\psi_j \chi_j)$$

The Hamiltonian in terms of fermions

- with free boundary conditions:

$$H = if \sum_{j=1}^{N} \psi_j \chi_j + iJ \sum_{j=1}^{N-1} \chi_j \psi_{j+1}$$

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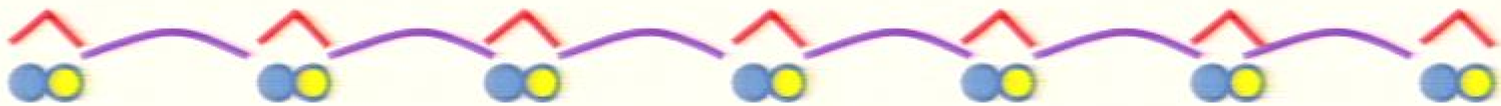

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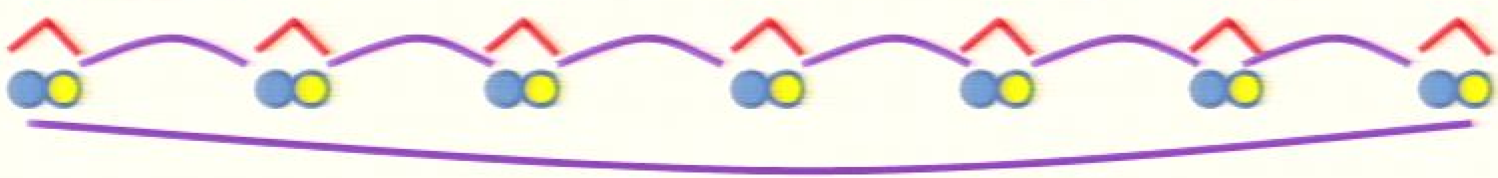

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A catch: when written in terms of spins, this is twisted by $-(-1)^F$

Extreme limits:

- J=0 (disordered in spin language):

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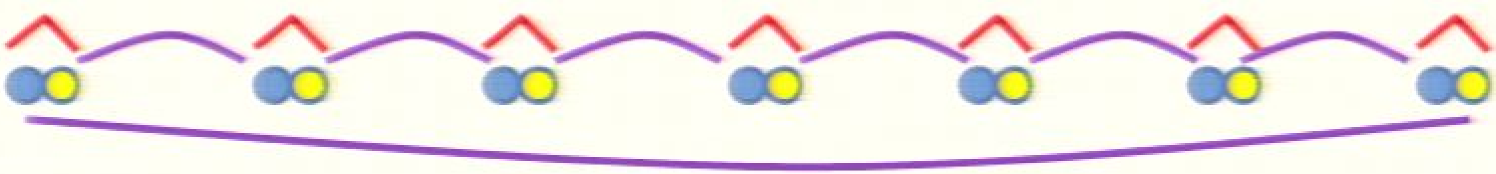

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- f=0 (ordered in spin language):

The fermions on the edges, ψ_1 and χ_N, do not appear in H when f=0. They commute with H!

Gapless edge modes = topological order

- When f=0, the operators χ_N and ψ_1 map one ground state to the other – they form an exact zero mode.

- The topological order persists for all $f < J$, even though for finite N, the two states split in energy.

- Can identify topological order (or lack thereof) for Hamiltonians of arbitrary fermion bilinears.

Kitaev

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Heuristic way: with translation invariance, k=0 and k=π fermion operators are raising/lowering operators:

$$\left[H, \sum_j (\psi_j \pm i\chi_j)(\pm 1)^j\right] = (\Delta E) \sum_j (\psi_j \pm i\chi_j)(\pm 1)^j$$

with $\Delta E = \pm f \pm J$.

One of these is the "zero" mode – at the critical point it gives an exact degeneracy between the two sectors

The 3-state (chiral) Potts model

The quantum chain version of the 3-state Potts model:

$$H = -\sum_j \Big[f(\tau_j + \tau_j^\dagger) + J(\sigma_j^\dagger \sigma_{j+1} + \mathrm{h.c.}) \Big]$$

flip term

potential

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\boxed{\tau\sigma = e^{2\pi i/3}\sigma\tau}$$

Define parafermions just like fermions:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain,

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Instead of anticommutators, for $i < j$ and $\alpha = \chi$ or ψ,

$$\alpha_i \alpha_j = e^{2\pi i/3} \alpha_j \alpha_i$$

The Hamiltonian in terms of parafermions:

$$= f(\psi_j^\dagger \chi_j + \chi_j^\dagger \psi_j) \qquad\qquad = J(\psi_{j+1}^\dagger \chi_j + \chi_j^\dagger \psi_{j+1})$$

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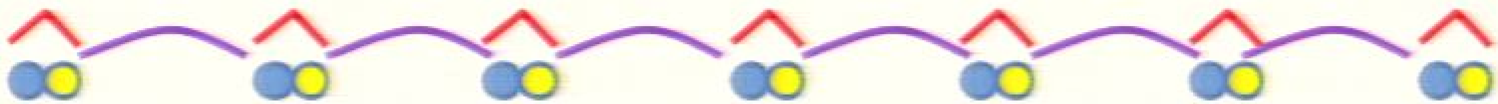

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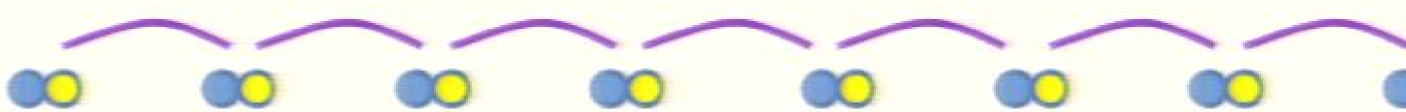
However, when f=0, there are edge modes!



Does the zero mode remain for $J > f > 0$?

Take periodic boundary conditions on parafermions.
Can we find a Ψ so that $[H, \Psi] = (\Delta E)\,\Psi$?

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The answer is yes only if the couplings obey an interesting constraint!

Generalize to the chiral Potts model:

$$= f(e^{i\phi}\psi_j^\dagger \chi_j + e^{-i\phi}\chi_j^\dagger \psi_j) \qquad = J(e^{i\theta}\psi_{j+1}^\dagger \chi_j + e^{-i\theta}\chi_j^\dagger \psi_{j+1})$$

Then there is an exact "zero" mode Y if the couplings obey:

$$f\cos(3\phi) = J\cos(3\theta)$$

This is strong evidence that non-abelian topological order exists for all $f < J$ in this interacting system.

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The "superintegrable" line $\theta=\phi=\pi/6$ (halfway between ferro and antiferromagnet) is very special.

Here the "zero" mode occurs for any value of f and J.

The integrable chiral Potts model is quite peculiar. The Boltzmann weights of the 2d classical analog are parametrized by higher genus Riemann surfaces instead of theta functions. They satisfy a generalized Yang-Baxter equation with no difference property.

Along the superintegrable line the model a direct way of finding the infinite number of conserved charges is to use the Onsager algebra.

Dolan and Grady; von Gehlen and Rittenberg; Davies

This algebra can be rewritten in a more intuitive fashion.

Work in a basis where τ is diagonal and σ is not, and then rewrite in terms of the usual spin-1 matrices, e.g.

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = S^+ + (S^-)^2$$

Then split the Hamiltonian into terms that conserve the U(1) symmetry generated by S^z and those violating it by +3 or -3:

$$\sum_j (\sigma_j^\dagger \sigma_{j+1} e^{i\pi/6} + h.c.) \equiv B_1^0 + B_1^+ + B_1^-$$

terms such as $S_i^+ (S^+)_{i+1}^2 e^{i\pi/6}$

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$$[B_1^+,\ B_n^-] = B_{n+1}^0 + B_{n-1}^0 \ ,$$

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$j = q\pm$

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Using this makes it easy to find the infinite number of conserved charges commuting with the Hamiltonian

$$H = \alpha B_0^0 + \beta B_1^0 + \gamma(B_1^+ + B_1^-)$$

More interesting stuff happens. The $\gamma = 0$ case can be solved via the standard Bethe ansatz, with the Bethe equations those of the XXZ chain at a special point.

Hidden susy!?! Presumably related to the hidden susy in XXZ/XYZ, which changes the number of sites by 1.

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Topological order in 2d

The Ising/Majorana chain has an elegant generalization to 2d via the Kitaev honeycomb model.

This is a spin model that can be mapped to free fermions coupled to a background gauge field.

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I'll describe the analog for parafermions.

View the 2d model as coupled 1d chains

The quantum YZ chain

 $= J_z \, \sigma_i^z \sigma_{i+1}^z$

 $= J_y \sigma_i^y \sigma_{i+1}^y$

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The quantum YZ chain

 $= i J_z \chi_i \psi_{i+1}$

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The quantum YZ chain

$$= i J_z \chi_i \psi_{i+1}$$

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is comprised of two commuting Hamiltonians:

+ 

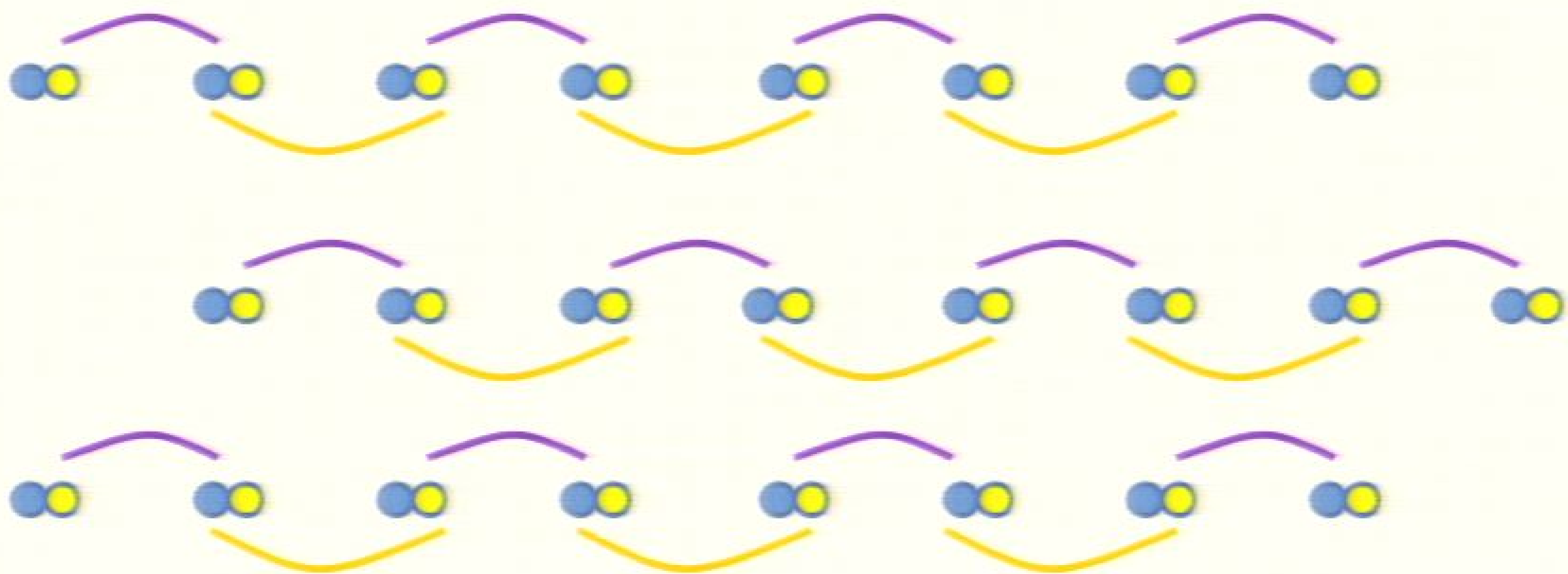
Consider one of these Hamiltonians:

$$H^{(1)} = \sum_j \left[\sigma^z_{2j-1}\sigma^z_{2j} + \sigma^y_{2j}\sigma^y_{2j+1}\right] = i\sum_j \left[\chi_{2j-1}\psi_{2j} + \psi_{2j}\chi_{2j+1}\right.$$

Just like the edge modes, the fermions ψ_{2j-1} and χ_{2j} do not appear!

They commute with each individual term in $H^{(1)}$.

Now couple chains together into a honeycomb lattice:



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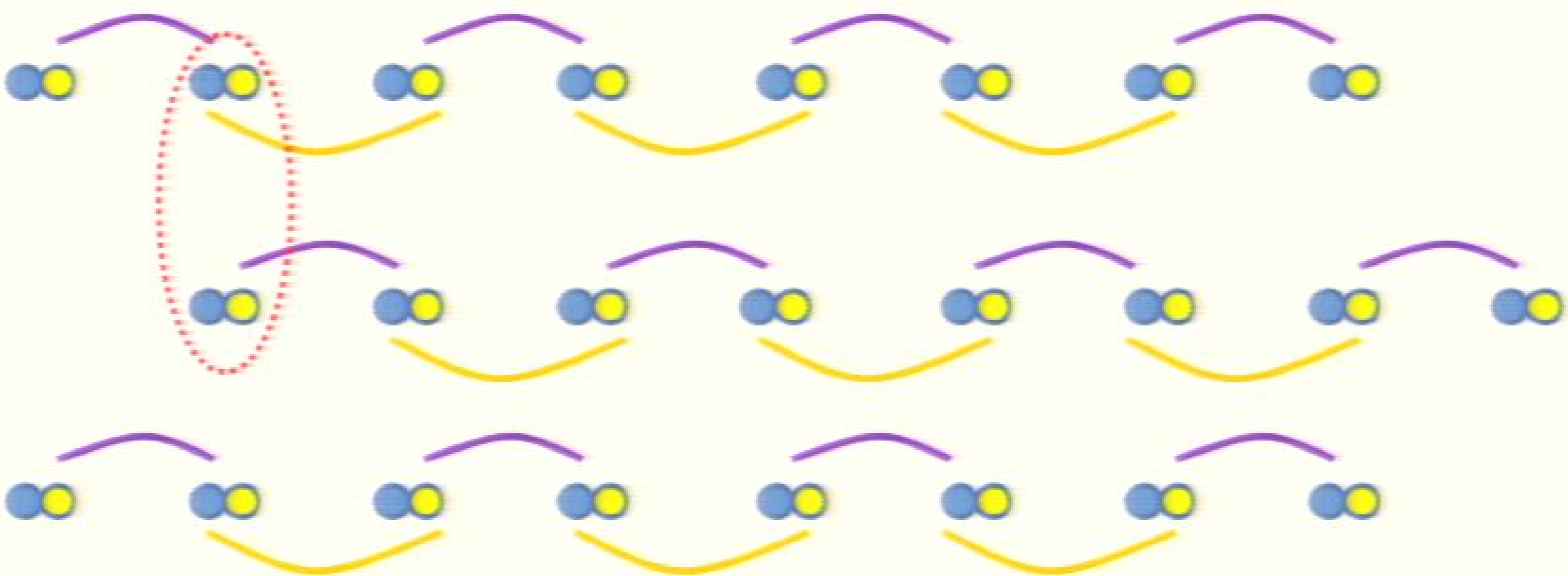

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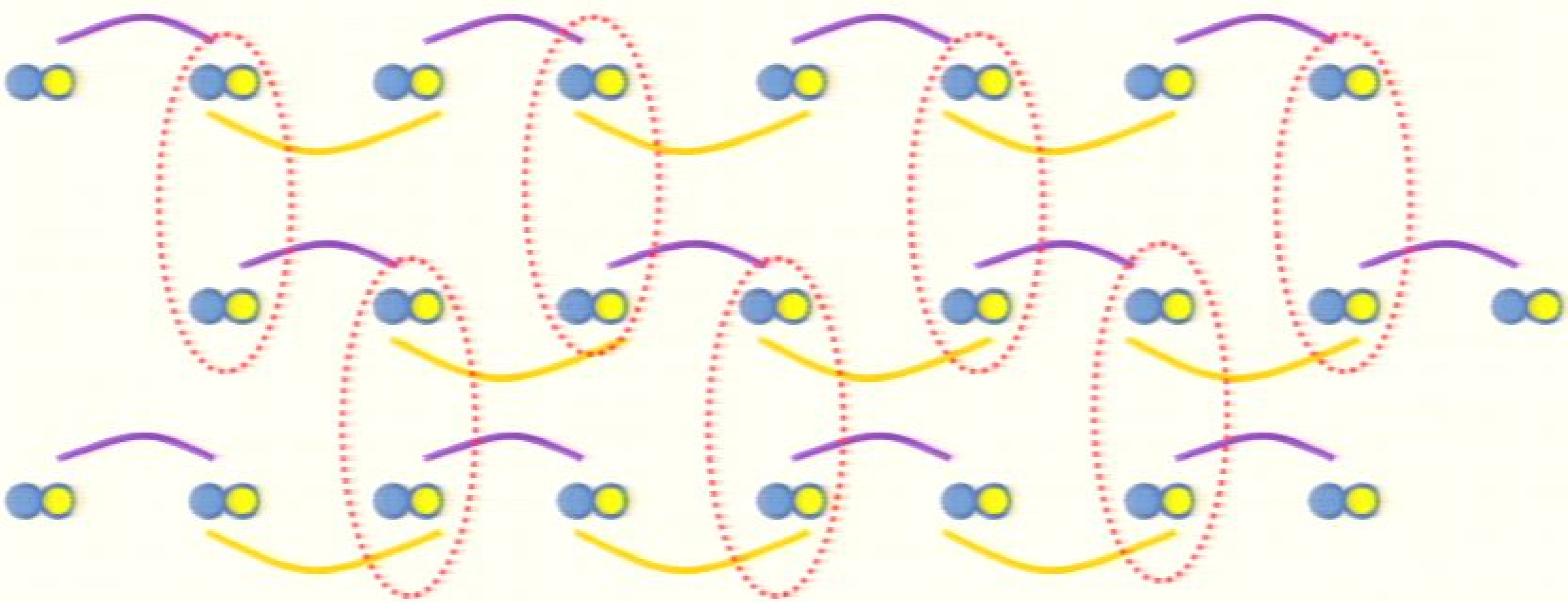

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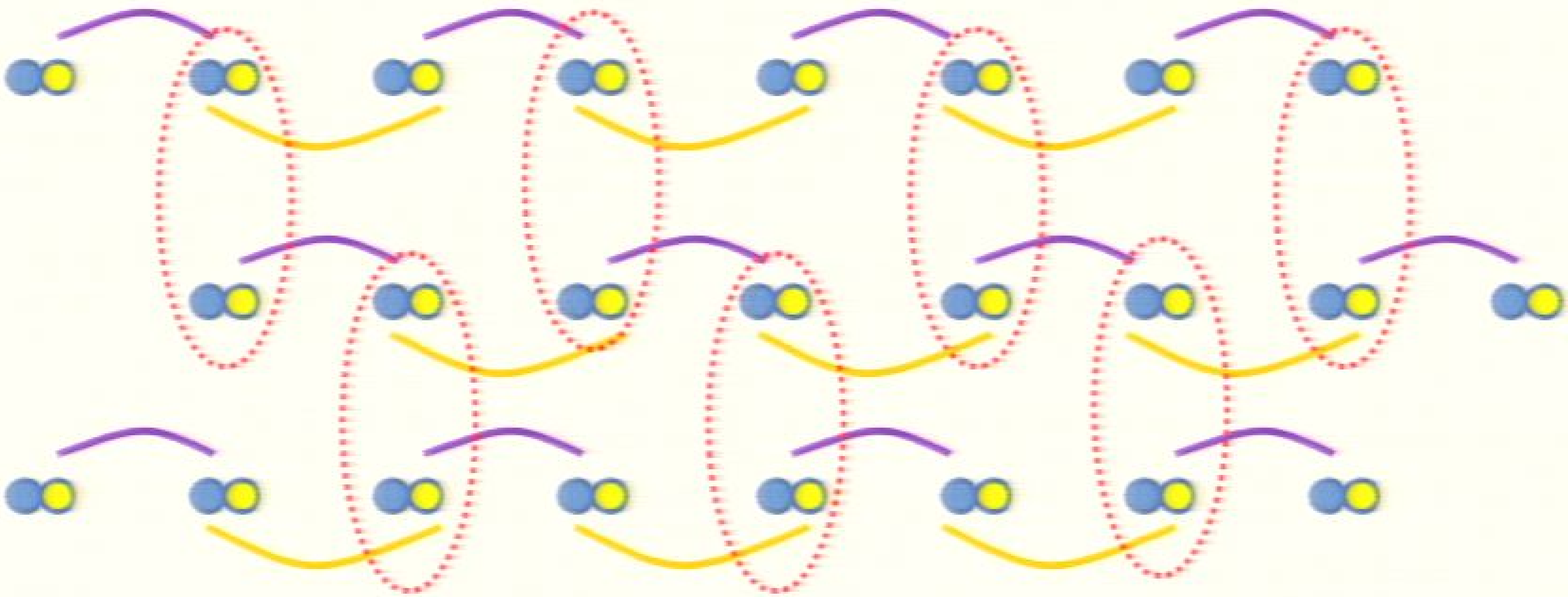

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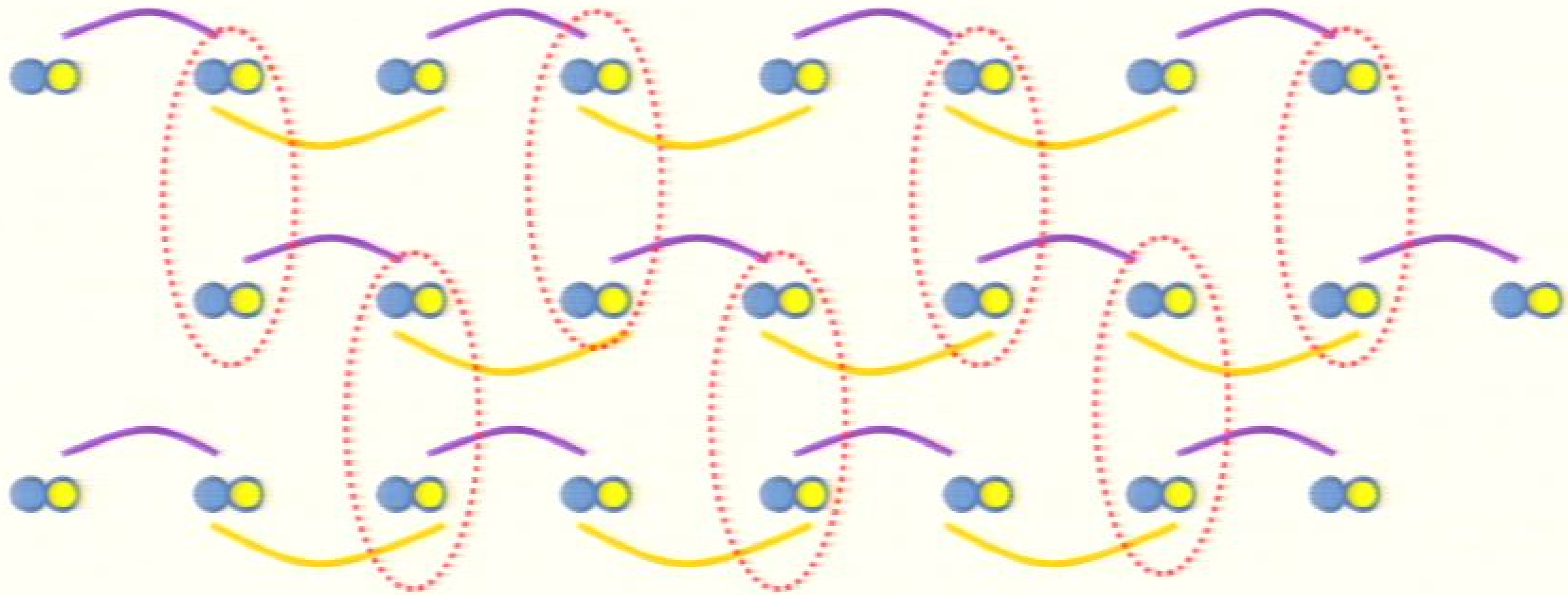
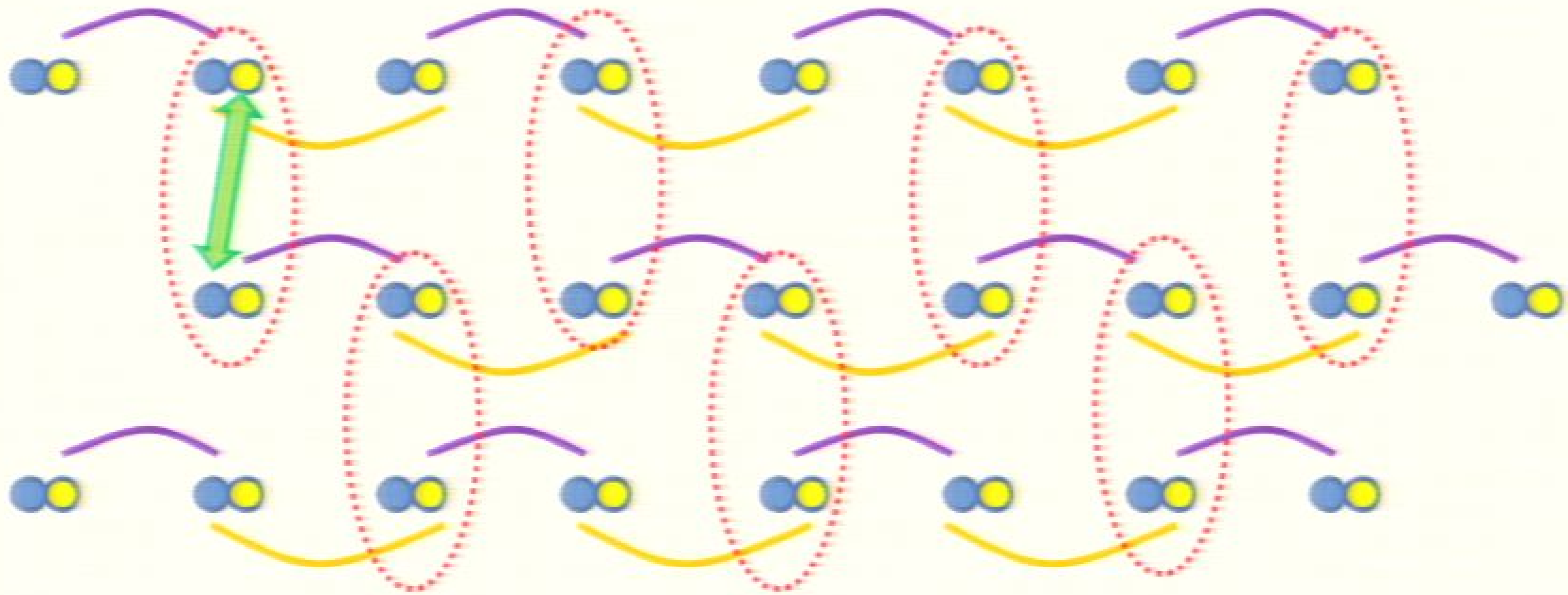
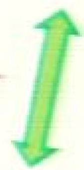
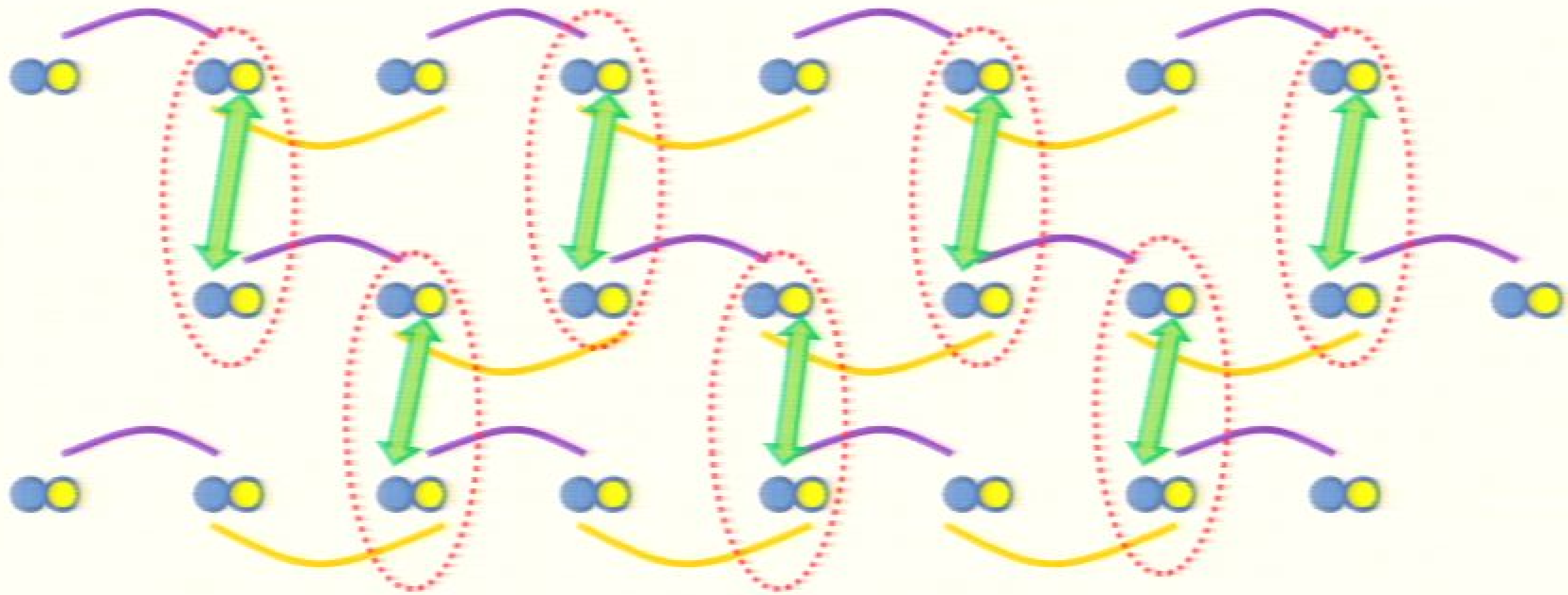
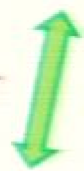
$$H = \sum \sigma^x\sigma^x + \sum \sigma^y\sigma^y + \sum \sigma^z\sigma^z$$

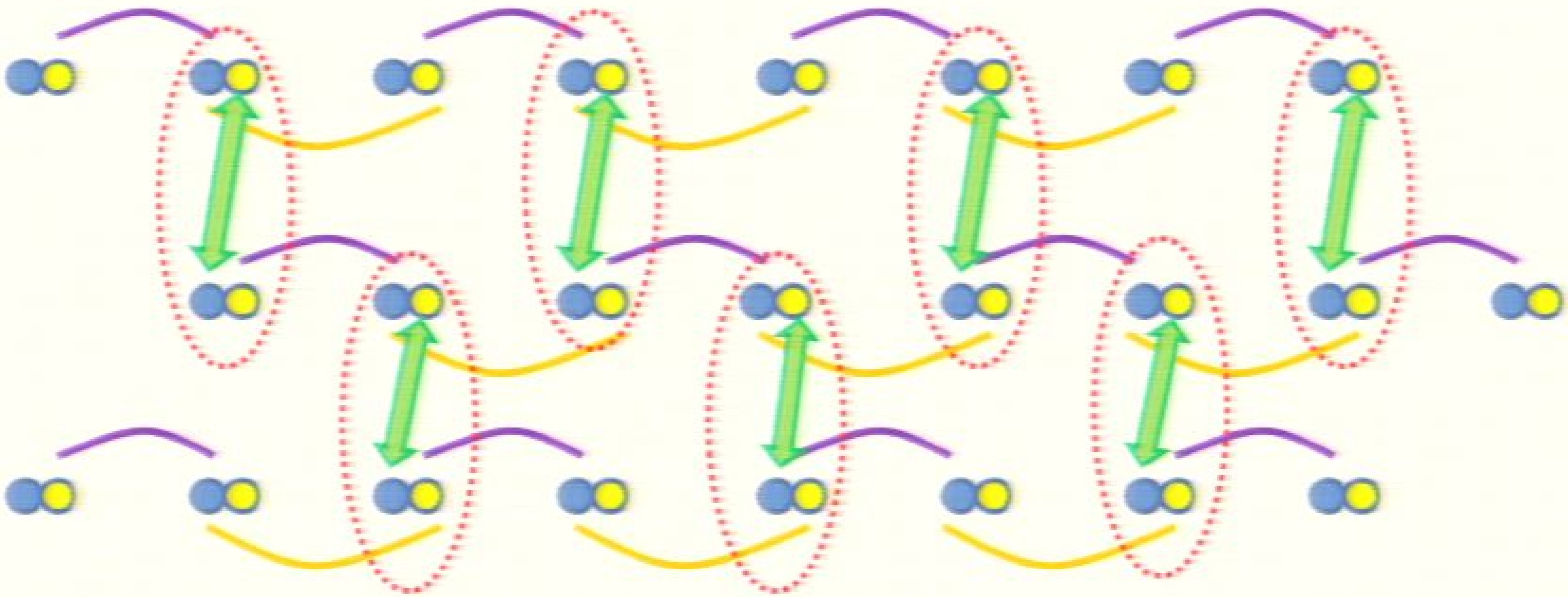




Each fermion bilinear  commutes with each term in H.



Each fermion bilinear  commutes with each term in H.



Each fermion bilinear commutes with each term in H.

The $\mathbb{Z}_2$ gauge flux is the product $= \sigma^z \sigma^x \sigma^y \sigma^z \sigma^y \sigma^x$ around a hexagon.

The flux through each plaquette can be chosen individually, and is not dynamical.

Thus the Kitaev honeycomb model is simply free fermions coupled to a background $\mathbb{Z}_2$ gauge field.

A magnetic field destroys the solvability, but causes non-abelian topological order.

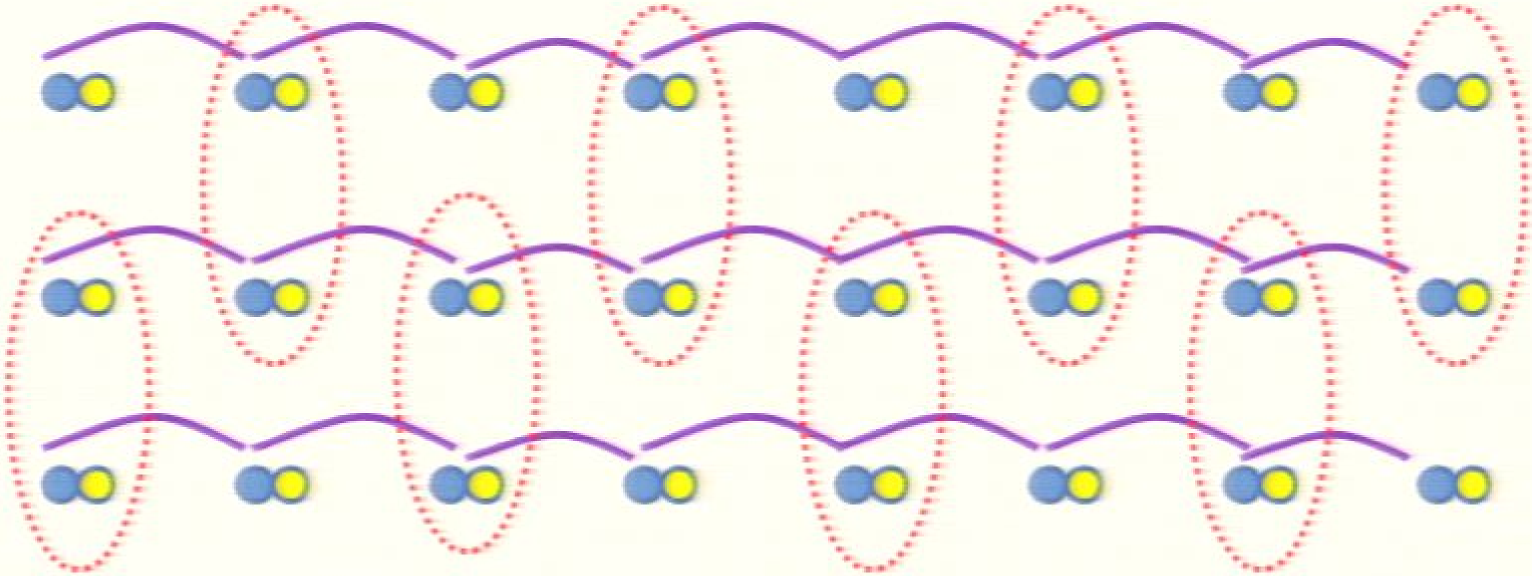
On the Fisher lattice, non-abelian topological order occurs without the magnetic field.

Yao and Kivelson

So what about parafermions?

The same trick yields a "YZ" Hamiltonian that doesn't involve half the parafermions:

$$= (\tau_i \sigma_i)^\dagger \tau_{i+1} + \mathrm{h.c.}$$

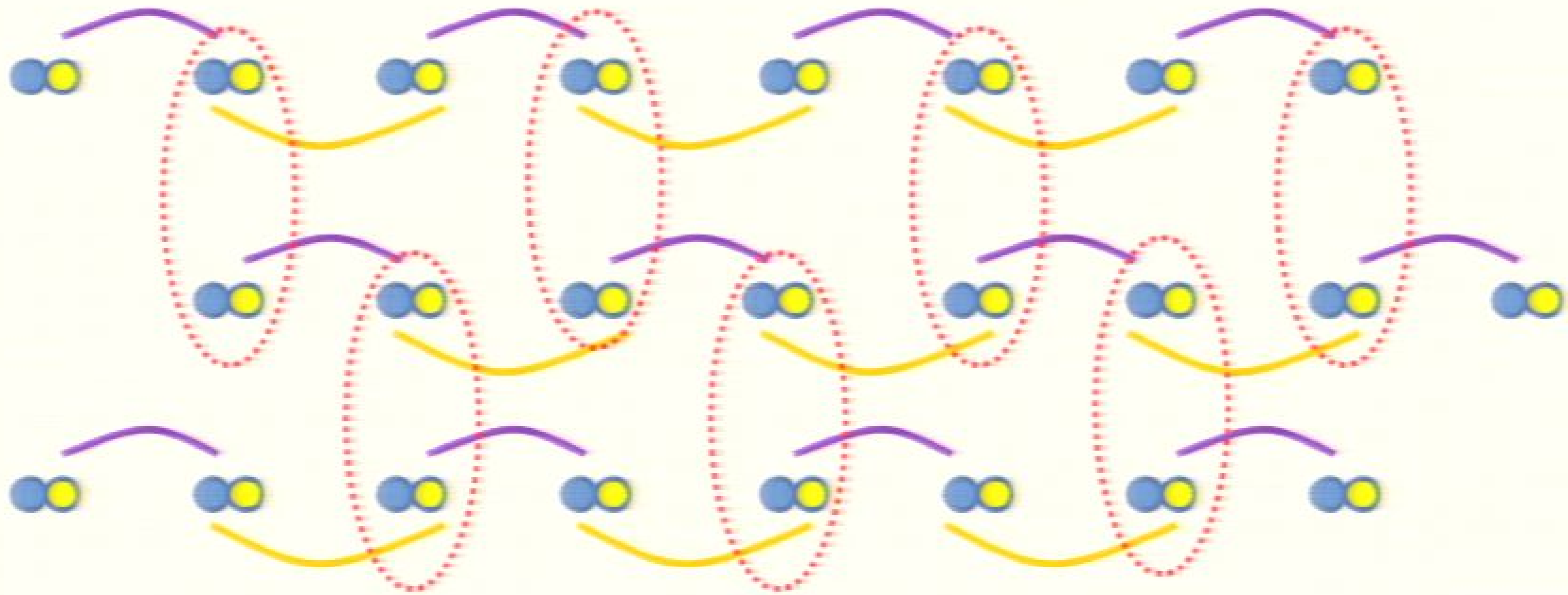


 $= \sigma_k^\dagger \sigma_{k+1} + \text{h.c.}$

The $\mathbb{Z}_3$ gauge flux is = $\begin{matrix} \tau^\dagger \ \sigma \ (\tau\sigma) \\ \\ \tau^\dagger \ \sigma \ (\tau\sigma) \end{matrix}$ around a hexagon.

Questions

- The handwaving arguments for topological order work for parafermions. Presumably non-abelian?
- Is there a formula for the parafermions generalizing the Pfaffian/Chern number for fermions?
- If so, will this result go "up" to statistical mechanics?
- Is there a connection to 2+1d integrable models?
- Should work for all $\mathbb{Z}_N$, what about $U(1)$?
- What's with the Onsager algebra?



Consider one of these Hamiltonians:

$$H^{(1)} = \sum_j \left[\sigma^z_{2j-1}\sigma^z_{2j} + \sigma^y_{2j}\sigma^y_{2j+1}\right] = i\sum_j \left[\chi_{2j-1}\psi_{2j} + \psi_{2j}\chi_{2j+1}\right.$$

Just like the edge modes, the fermions ψ_{2j-1}
and χ_{2j} do not appear!

They commute with each individual term in $H^{(1)}$.

Using this makes it easy to find the infinite number of conserved charges commuting with the Hamiltonian

$$H = \alpha B_0^0 + \beta B_1^0 + \gamma(B_1^+ + B_1^-)$$

More interesting stuff happens. The $\gamma = 0$ case can be solved via the standard Bethe ansatz, with the Bethe equations those of the XXZ chain at a special point.

Hidden susy!?! Presumably related to the hidden susy in XXZ/XYZ, which changes the number of sites by 1.

Fendley, Nienhuis and Schoutens; Hagendorf and Fendley

Label the rest of the Hamiltonian as

$$\sum_j (\tau_j e^{i\pi/6} + \tau_j^\dagger e^{-i\pi/6}) \equiv B_0^0 \;, \qquad B_0^+ = B_0^- = 0$$

The remaining elements of the Onsager algebra are defined via the commutators

$$[B_1^+,\ B_n^-] = B_{n+1}^0 + B_{n-1}^0 \;,$$

$$\pm[B_1^0,\ B_n^\pm] = B_{n+1}^\pm + B_{n-1}^\pm$$

Then the Onsager algebra is remarkably beautiful:

$$[B_m^+,\ B_n^-] = B_{n+m}^0 + B_{n-m}^0$$

$$\pm[B_m^\pm,\ B_n^0] = B_{n+m}^\pm + B_{n-m}^\pm$$