Title: First-principles Derivation of the AdS/CFT Y- and T-systems

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Abstract: I will present a first-principles derivation of the AdS5/CFT4 T-system up to first non-trivial order in the large 't Hooft coupling expansion. The proof relies on the computation of quantum effects in the fusion of some special line operators, namely the transfer matrices. This computation is done in the pure spinor formalism for the superstring in AdS5xS5. I will also discuss the generalization of this computation to other integrable 2D CFTs that define string theory in AdS backgrounds.

First-principles derivation of the AdS/CFT Y-systems

IGST 2011

Raphael Benichou
VUB, Brussels

Setting up the stage

Type IIB string theory in $AdS_5 \times S^5$ ⟷ (AdS/CFT) ⟷ $\mathcal{N}=4$ $SU(N)$ SYM

Large N limit: Integrable structures appear.

This talk is about the spectrum problem.

Energy of string states ⟷ Conformal dimensions

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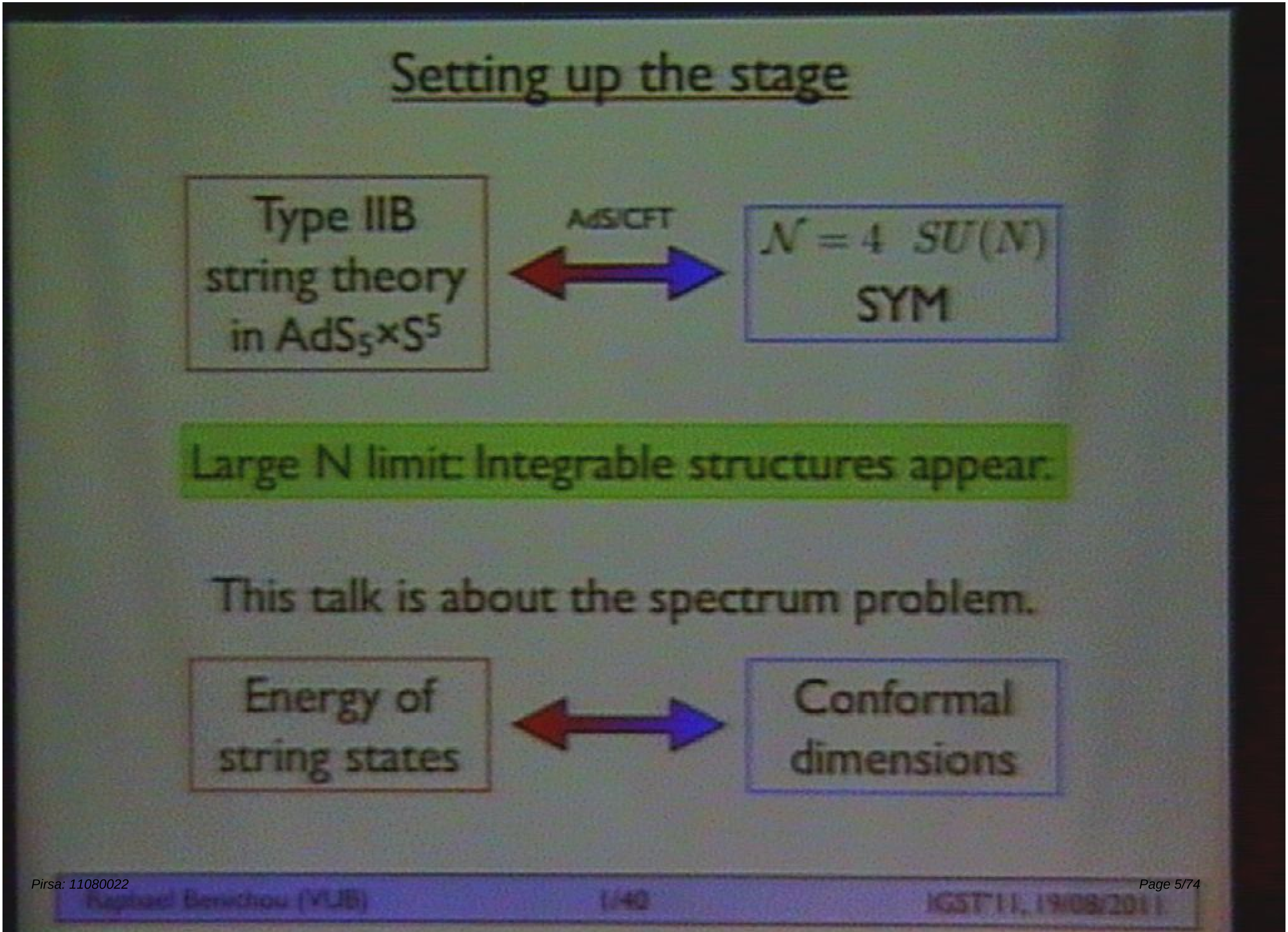

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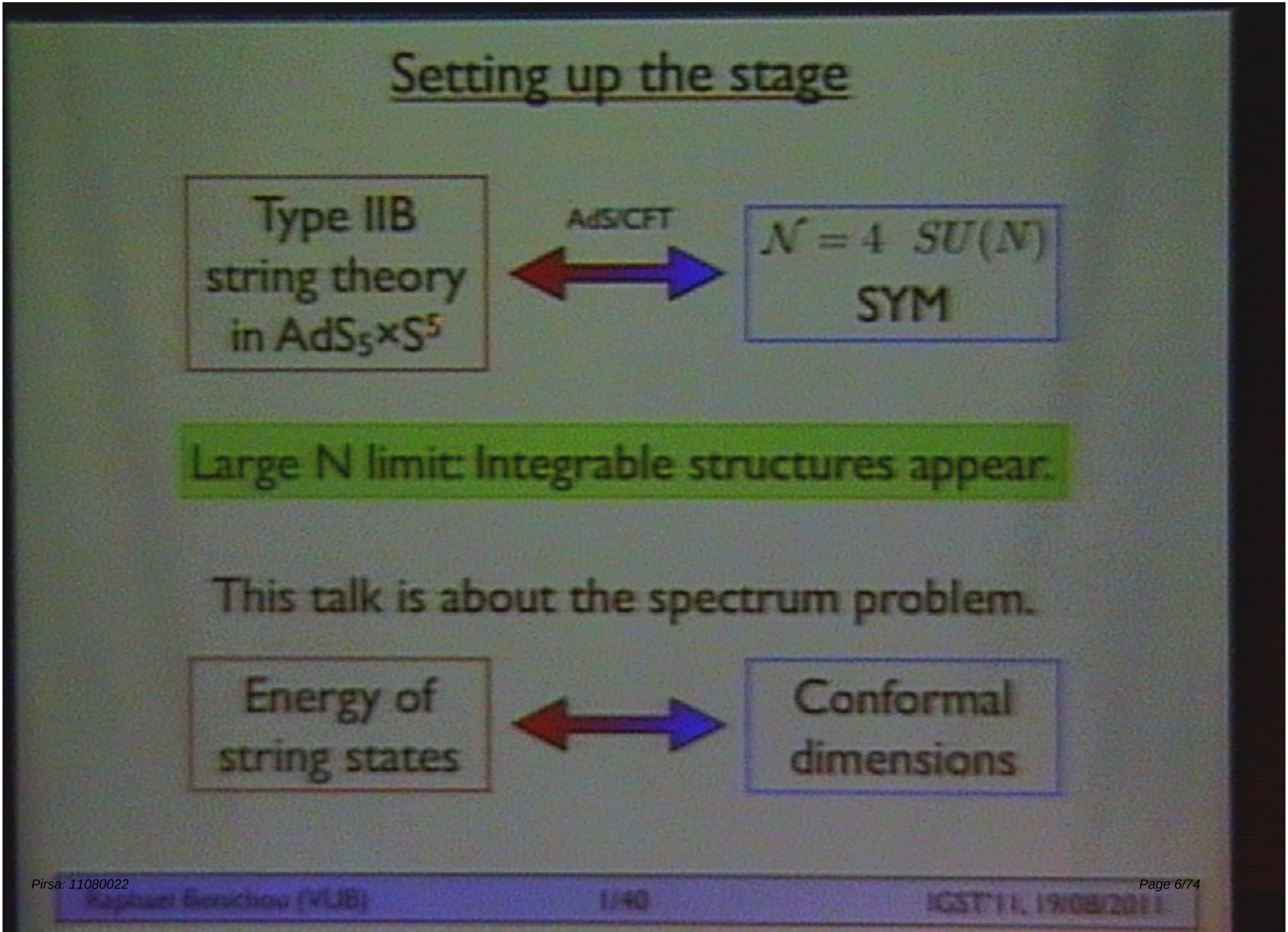
Raphael Benichou (VUB)
1/40
IGST'11, 19/08/2011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Energy of string states ⟷ Conformal dimensions

The Y- and T-systems

- A system of equations has been proposed to solve the spectrum problem:

Gromov, Kazakov & Vieira, 2009

Y-system

⇕

$$T_{a,s}(u+1)T_{a,s}(u-1) = T_{a+1,s}(u+1)T_{a-1,s}(u-1) + T_{a,s+1}(u-1)T_{a,s-1}(u+1)$$

T-system, or Hirota equation

- Each string state corresponds to a solution of the Y-system with specific analytic properties.

- The energy of a string state can be computed easily from the Y-functions.

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T-system, or Hirota equation

- Each string state corresponds to a solution of the Y-system with specific analytic properties.

- The energy of a string state can be computed easily from the Y-functions.

Good reasons to appreciate the Y-system

- It is compatible with the Asymptotic Bethe Ansatz

Gromov, Kazakov & Vieira, 2009a

- It reproduces the spectrum of the quasi-classical string a large 't Hooft coupling.

Gromov, 2009

Gromov, Kazakov & Tsuboi, 2010

- It gave correct predictions for the dimension of the Konishi operator at large and small 't Hooft coupling.

Gromov, Kazakov & Vieira, 2009c

Arutyunov, Frolov & Suzuki, 2010

Now it would be nice to prove the validity of the Y-system.

Thermodynamic Bethe Ansatz

Zamolodchikov, 1990

- The Y-system can be derived using the Thermodynamic Bethe Ansatz approach. Gromov, Kazakov, Kozak & Vieira, 2009 | Bombardelli, Fioravanti & Tateo, 2009 | Arutyunov & Frolov, 2009

- This derivation relies on some crucial assumptions:
  - Quantum integrability
  - String hypothesis
  - Analytic continuation for the excited states

In this talk we present another approach:

☺ First-principles ☹ Perturbative

closer in spirit to the work of Bazhanov, Lukyanov & Zamolodchikov, 1994

Plan

1. The strategy of the proof
2. The worldsheet theory
3. Line operators and UV divergences
4. Fusion of line operators
5. Derivation of the T-system
6. Generalizations and conclusions

1. The strategy of the proof

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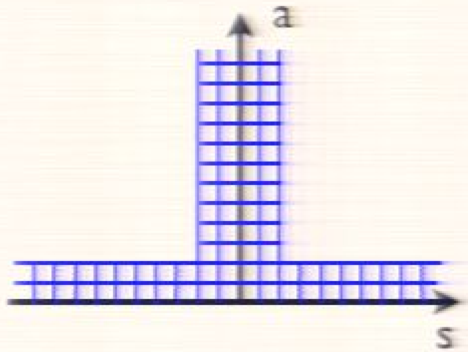
T-system: generalities

$$\mathcal{T}_{a,s}(u+1)\mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1)\mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1)\mathcal{T}_{a,s-1}(u+1)$$

- The integrer indices (a,s) label representations of PSU(2,2|4). They take value in a T-shaped lattice.



- The T-functions are presumably related to the transfer matrices of the underlying theory.

see e.g. Gromov, Kazakov & Tsuboi, 2010

The classical limit of the T-system

- The classical transfer matrix is a super-character:

$$\mathcal{T}_R(u) = STr\,\boxed{P\ \exp\left(-\oint A_R(u)\right)} \longrightarrow \boxed{\in PSU(2,2|4)}$$

- Characters of PSU(2,2|4) satisfy:

$$\boxed{\chi(a,s)\ \chi(a,s)\ =\ \chi(a+1,s)\ \chi(a+1,s)\ +\ \chi(a,s+1)\ \chi(a,s-1)}$$

↑ $u \gg 1$ ~ Classical limit

$$\boxed{\mathcal{T}_{a,s}(u+1)\mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1)\mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1)\mathcal{T}_{a,s-1}(u+1)}$$

- The shifts of the spectral parameter presumably come from some kind of quantum effects.

The starting point of the derivation

$\mathcal{T}$'s = Transfer matrices

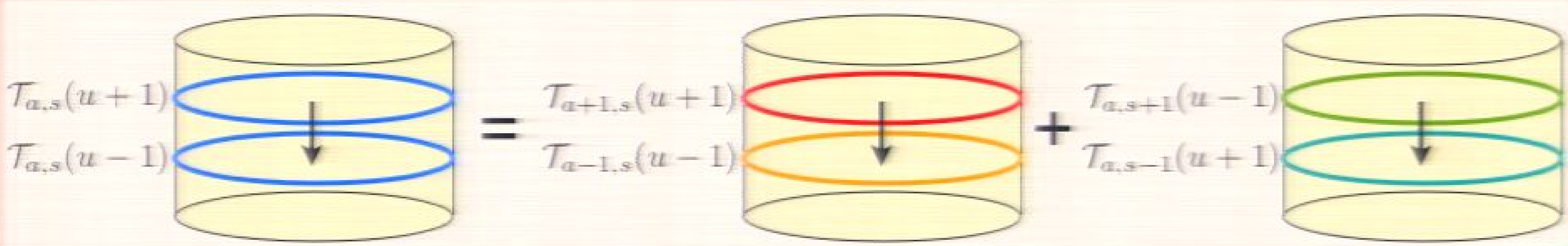
→ The T-system is promoted to an operator identity.

Product of $\mathcal{T}$'s = Fusion of line operators

→ The shifts come from quantum effects associated with fusion.

$$\mathcal{T}_{a,s}(u+1) \triangleright \mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1) \triangleright \mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1) \triangleright \mathcal{T}_{a,s-1}(u+1)$$

$$\Updownarrow$$



We will demonstrate that this picture is correct at first order in the large 't Hooft coupling expansion.

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2. The worldsheet theory

3. Line operators and UV divergences

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6. Generalizations and conclusions

The pure spinor string on $AdS_5 \times S^5$

The worldsheet theory is a sigma-model on $\frac{PSU(2,2|4)}{SO(5)\times SO(4,1)}$ coupled to ghosts.

Berkovits, 2000

The action is:

$$S = \frac{R^2}{4\pi} STr \int d^2w \left(J_2 \bar{J}_2 + \frac{3}{2} J_3 \bar{J}_1 + \frac{1}{2} \bar{J}_3 J_1 \right)$$

$$+ \frac{R^2}{2\pi} STr \int d^2w \left(N \bar{J}_0 + \hat{N} J_0 - N \hat{N} + w \bar{\partial} \lambda + \hat{w} \partial \hat{\lambda} \right)$$

The J_i's are the $\mathbb{Z}_4$ components of the Maurer-Cartan current:

$$g \in PSU(2,2|4): \; g^{-1} dg = J_0 + J_1 + J_2 + J_3$$

Pure spinor ghosts and their conjugate momenta (λ, w) $(\hat{\lambda}, \hat{w})$ → $N = -\{w, \lambda\}$, $N = -\{\hat{w}, \hat{\lambda}\}$ Pure spinor Lorentz currents

Classical integrability

The pure spinor string on $AdS_5 \times S^5$ is classically integrable.
There exists a one-parameter family of flat connections:

$$A(y) = (J_0 + yJ_1 + y^2 J_2 + y^3 J_3 + (y^4 - 1)N)dz \\ + (\bar{J}_0 + y^{-3}\bar{J}_1 + y^{-2}\bar{J}_2 + y^{-1}\bar{J}_3 + (y^{-4} - 1)\hat{N})d\bar{z}$$

Vallilo, 2003

$$\forall y, \quad dA(y) + A(y) \wedge A(y) = 0$$

Bena, Polchinski & Roiban, 2003

Consequently the monodromy matrix Ω codes an infinite number of conserved charges.

$$\Omega(y) = P \exp\left(-\oint A(y)\right)$$

Current-current OPEs

- The structure of the current-current OPEs is the following:

$$J(z)J(0) = (\text{2nd}-\text{order pole})Id + (\text{1st}-\text{order pole})J(0) + \ldots$$

Computed at tree-level by: Bianchi & Kluson, 2006; Puletti, 2006; Mikhailov & Schafer-Nameki, 2007

Computed at one-loop by: Bedoya, Marchioro, Nedel & Vallilo, 2010

- Perturbation theory in R^{-2} is easily implemented:

The coefficients of all poles are of order R^{-2}

↓

Computation at p-th order in R^{-2}
$\Leftrightarrow$ Perform p OPEs.

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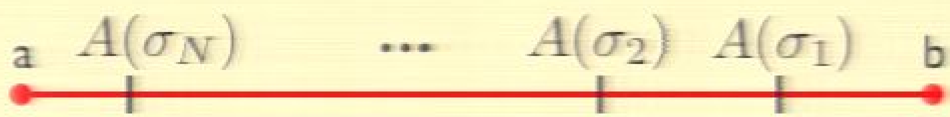
UV divergences in line operators

We expand the line operators:

$$W^{b,a} = P\exp\left(-\int_a^b A\right) = \sum_{N=0}^{\infty} W_N^{b,a}$$

with: $W_N^{b,a}$:

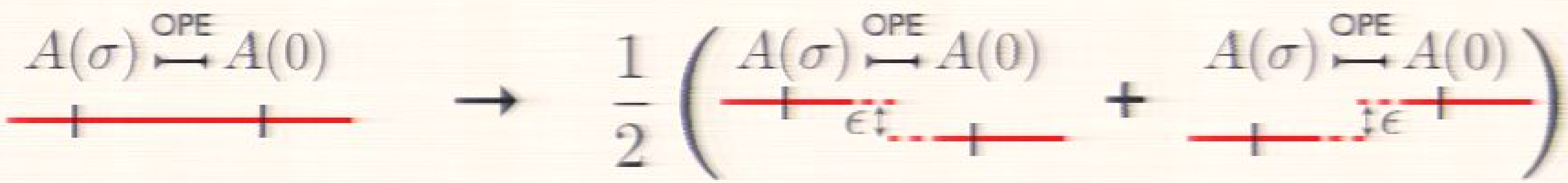


Collisions of integrated operators lead to divergences.

$\Rightarrow$ We need to regularize and potentially renormalize the line operators.

Regularization of divergences

We use a "principal value" regularization scheme:



For instance for a simple pole:

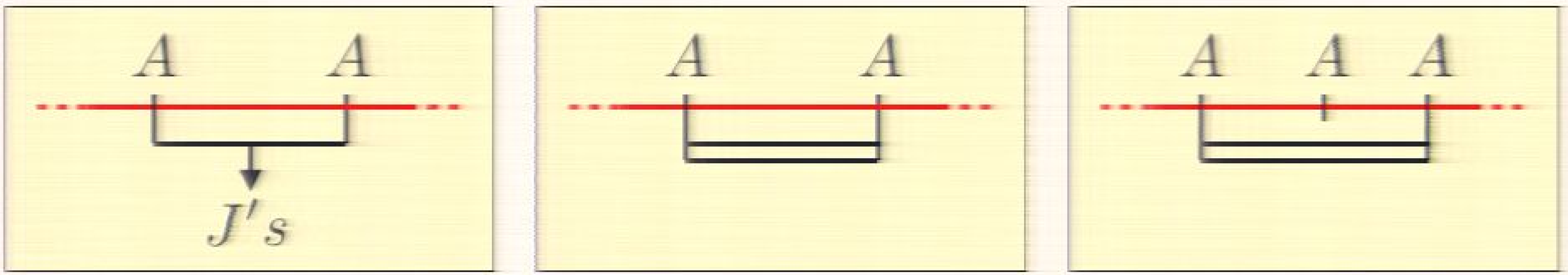
$$\frac{1}{\sigma} \longrightarrow \frac{1}{2}\left(\frac{1}{\sigma + i\epsilon} + \frac{1}{\sigma - i\epsilon} \right)$$

$$= \frac{\sigma}{\sigma^2 + \epsilon^2} \equiv P.V.\frac{1}{\sigma}$$

Line operator: Divergences at first order

There are three sources of divergences:

1st-order poles: 2nd-order poles:



The sum of these three terms cancel, but there are less of these.
We end up with a logarithmic divergence:

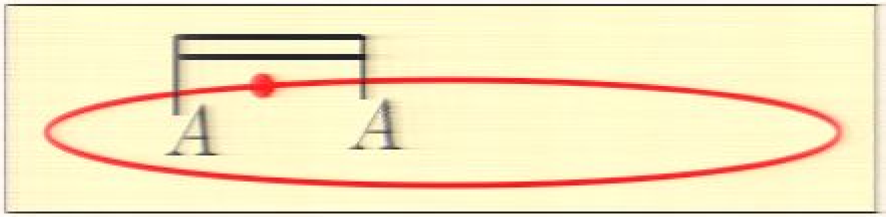
$$\log \epsilon \sum_{i=0}^{3} \# \left(W \, t^{a_i} t_{a_i} + t^{a_i} t_{a_i} W \right)$$

Line operator

Generator of the algebra of grade i.

Divergences in the Monodromy matrix

There is a new source of divergences in the monodromy matrix:



It contributes to the logarithmic divergences:

$$\log \epsilon \sum_{i=0}^{3} \# \left(\Omega\, t^{a_i} t_{a_i} + t^{a_i} t_{a_i} \Omega - 2\, t^{a_i} \Omega\, t_{a_i} \right)$$

We deduce that:

The transfer matrix is free of divergences up to first order in perturbation theory.

Mikhailov & Schafer-Nameki, 2007a

The vanishing of the dual Coxeter number of PSU(2,2|4) is crucial.

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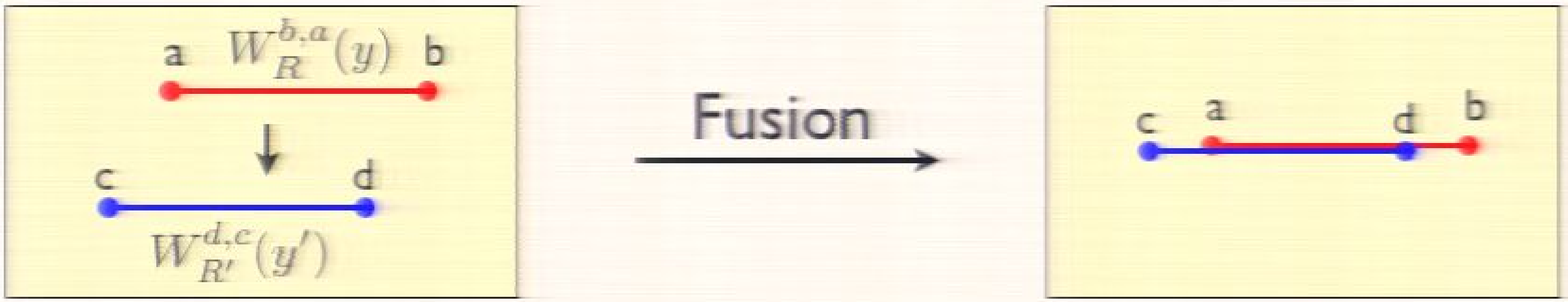
Fusion of line operators



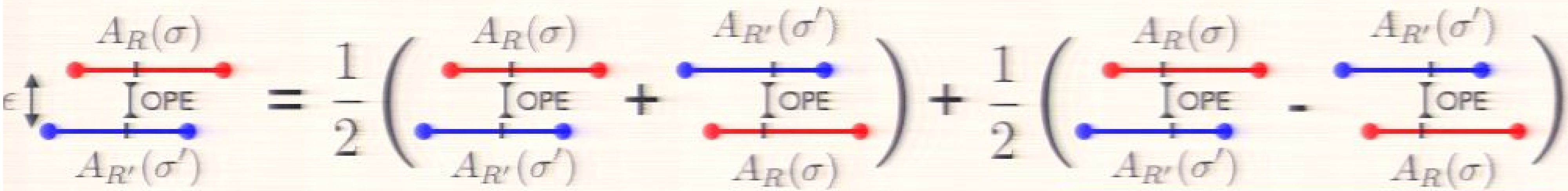
We denote the fusion as: $W_R^{b,a}(y) \triangleright W_{R'}^{d,c}(y')$

- The classical process is simple.
- Collisions of integrated connections induce quantum corrections that we are going to compute.

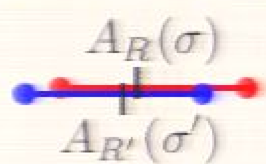
Disentangling the OPEs

Mikhailov & Schafer-Nameki, 2007b

We write the OPE between two connections as:



Regularized OPE in the double-line operator



Quantum correction associated with fusion.

For instance for a simple pole:

$$\frac{1}{\sigma + i\epsilon - \sigma'} = \frac{1}{2}\left(\frac{1}{\sigma + i\epsilon - \sigma'} + \frac{1}{\sigma - i\epsilon - \sigma'}\right) + \frac{1}{2}\left(\frac{1}{\sigma + i\epsilon - \sigma'} - \frac{1}{\sigma - i\epsilon - \sigma'}\right)$$

$$P.V.\ \frac{1}{\sigma - \sigma'}$$

$$-i\pi\delta_\epsilon(\sigma - \sigma')$$

Commutator of connections

- To compute the quantum corrections in the process of fusion, the relevant OPE is:

$$\lim_{\epsilon \to 0^+} (1 - P.V.) A_R(y;\sigma + i\epsilon) A_{R'}(y';\sigma') = \frac{1}{2}[A_R(y;\sigma), A_{R'}(y';\sigma')]$$

- From the current-current OPEs, we obtain:

$$[A_R(y;\sigma), A_{R'}(y';\sigma')] = 2s\delta'(\sigma - \sigma') + [A_R(y;\sigma), r + s]\,\delta(\sigma - \sigma') + [A_{R'}(y';\sigma'), r - s]\,\delta(\sigma - \sigma')$$

We recognize a (r,s) Maillet system with:

Maillet, 1985

Maillet, 1986

$$r, s \sim \sum_{i=0}^{3} \#\, t^{a_i,R} \otimes t^{R'}_{a_i}$$

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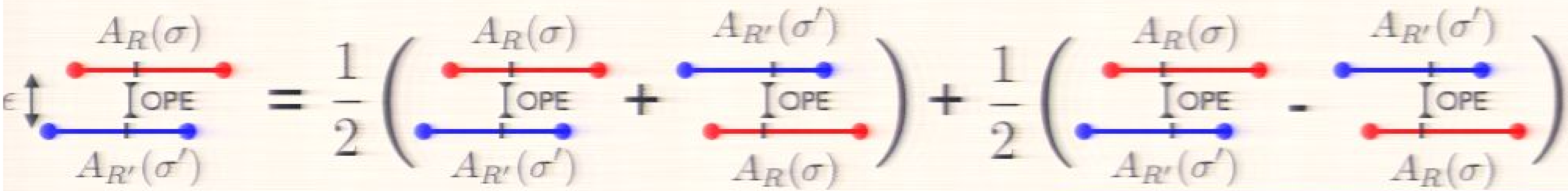

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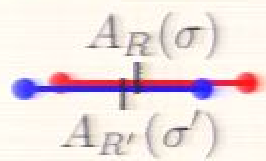

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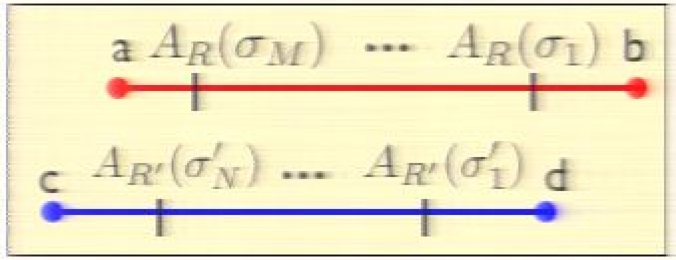
Fusion at first order

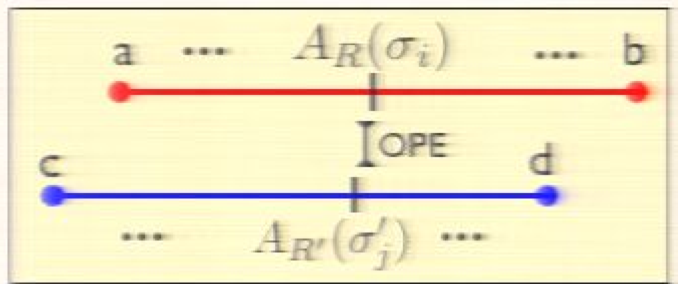
We start from the line operators:

$$\sum_{M,N=0}^{\infty}$$



We perform one OPE between two connections sitting on different contours:

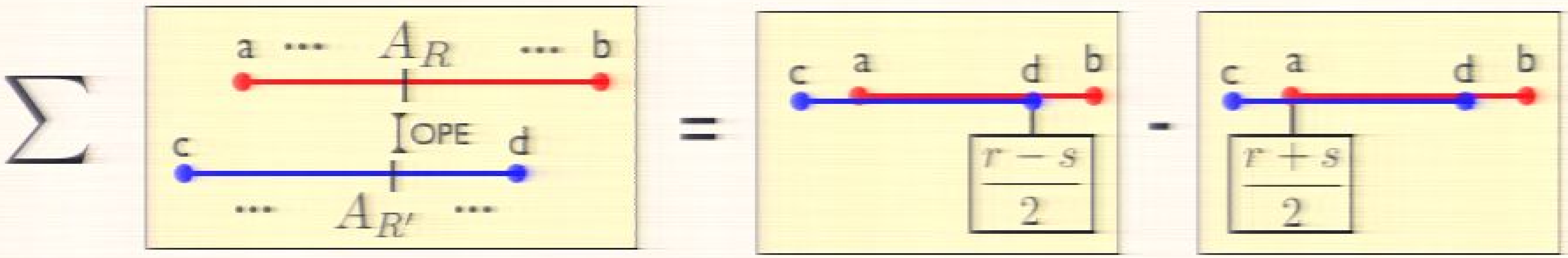
$$\sum_{M,N=0}^{\infty} \sum_{i=1}^{M} \sum_{j=1}^{N}$$



The first-order corrections only contribute to the commutator of the line operators.

Fusion at first order

With some efforts we can sum all terms to get:



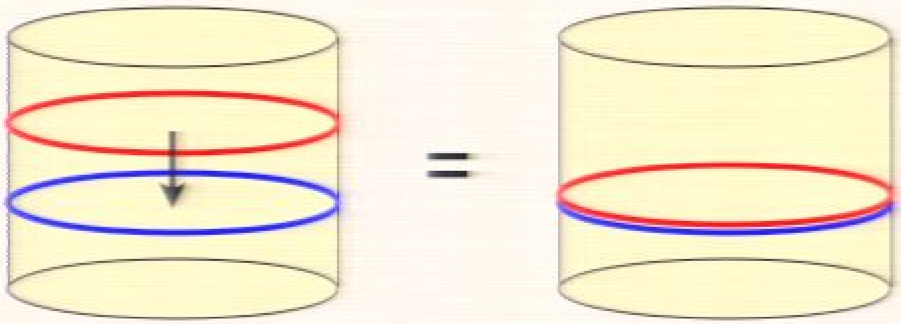
This agrees with the commutator of transition matrices derived in the Hamiltonian formalism.

Maillet, 1986

For line operators with coincinding endpoints, amiguities appear in the Hamiltonian formalism. There is no such problem in the OPE formalism.

Fusion of transfer matrices at first order

The fusion of transfer matrices is trivial at first order:



$$+ \mathcal{O}(R^{-4})$$

In particular the transfer matrices commute:

$$[\mathcal{T}_R(x), \mathcal{T}_{R'}(x')] = 0 + \mathcal{O}(R^{-4})$$

→ To get leading quantum correction to the fusion of transfer matrices, we have to go to second order.

Fusion at second order

To get a second-order correction, we can:

- Take one OPE and include the R^{-4} corrections to the current-current OPEs.

Need to be computed

One OPE $\Rightarrow$ contribute to the commutator.

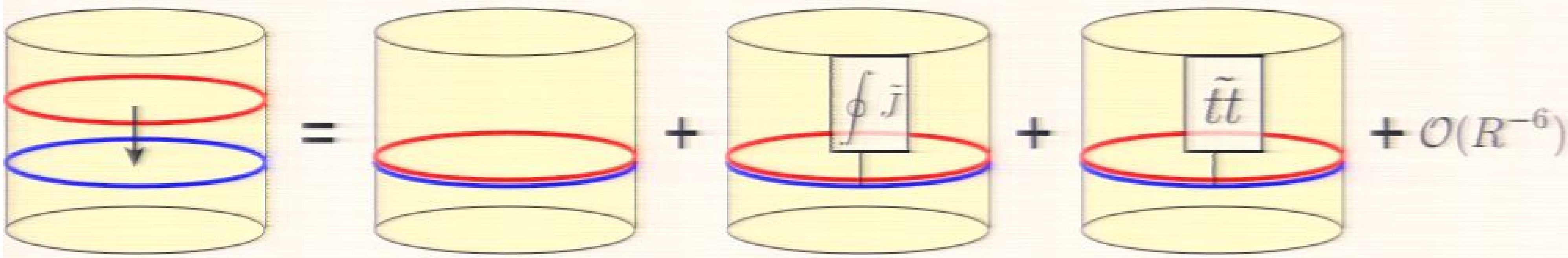
- Take two OPEs using the current-current OPEs at order R^{-2}

Two OPEs $\Rightarrow$ contribute to the symmetric fusion product.

This is what we will compute

Symmetric fusion of transfer matrices

We obtain:



Additional operator integrated on the contour

$$\tilde{J} \sim \tilde{J}^a \times f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b$$

Constant matrix inserted between the integrated connections

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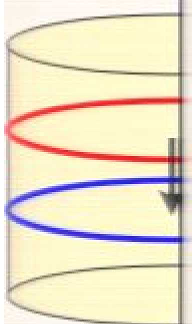

$$\tilde J = (i\pi R^{-2})^2 \sum_{m,n,p,q,r} f_{C_p}{}^{B_n A_m} f_{E_r}{}^{C_p D_q} \{t^R_{D_q}, t^R_{A_m}] t^{R'}_{B_n}$$
$$\times \Big(J_r^{E_r}\big(\tilde{D}'^{p}_{mn} F_q C^r_{pq} - \tilde{D}'^{p}_{mn} \bar F_q C^r_{p\bar q} - \tilde{D}'^{\bar p}_{mn} F_q C^r_{\bar p q} - \tilde{D}'^{\bar p}_{mn} \bar F_q C^r_{\bar p \bar q}$$
$$+ \frac{1}{2} F_r (\tilde{D}'^{s}_{mn} F_p C_{sp} + \tilde{D}'^{s}_{pn} F_m C_{sm} - \tilde{D}'^{\bar s}_{mn} \bar F_p C_{\bar s \bar p} - \tilde{D}'^{\bar s}_{pn} \bar F_m C_{\bar s \bar m})\big)$$
$$+ \bar J_r^{E_r}\big(\tilde{D}'^{p}_{mn} F_q C^{\bar r}_{pq} + \tilde{D}'^{p}_{mn} \bar F_q C^{\bar r}_{p\bar q} + \tilde{D}'^{\bar p}_{mn} F_q C^{\bar r}_{\bar p q} - \tilde{D}'^{\bar p}_{mn} \bar F_q C^{\bar r}_{\bar p \bar q}$$
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$$+ f_{C_p}{}^{B_n A_m} f_{E_r}{}^{D_q C_p} t^R_{A_m} \{t^{R'}_{B_n}, t^{R'}_{D_q}]$$
$$\times \Big(J_r^{E_r}\big(-\tilde{D}^{p}_{mn} F_q C^r_{pq} + \tilde{D}^{p}_{mn} \bar F_q C^r_{p\bar q} + \tilde{D}^{\bar p}_{mn} F_q C^r_{\bar p q} + \tilde{D}^{\bar p}_{mn} \bar F_q C^r_{\bar p \bar q}$$
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$\mathcal{O}(R^{-6})$

A

tween

ns

$\tilde J$

$d t_a$

Derivation of the T-system I

The goal is to show that:

$$\mathcal{T}_{a,s}(u+1) \triangleright \mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1) \triangleright \mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1) \triangleright \mathcal{T}_{a,s-1}(u+1)$$

- We consider:

$$\mathcal{T}_{a,s}(y+\delta) \triangleright \mathcal{T}_{a,s}(y-\delta) - \mathcal{T}_{a+1,s}(y+\delta) \triangleright \mathcal{T}_{a-1,s}(y-\delta) - \mathcal{T}_{a,s+1}(y-\delta) \triangleright \mathcal{T}_{a,s-1}(y+\delta)$$

- We look for δ such that the previous quantity vanishes. We assume: $\delta = \mathcal{O}(R^{-2})$
- We perform an expansion in powers of R^{-2}

$$\sum_{R,R'} \mathcal{T}_R(y+\delta) \triangleright \mathcal{T}_{R'}(y-\delta)$$
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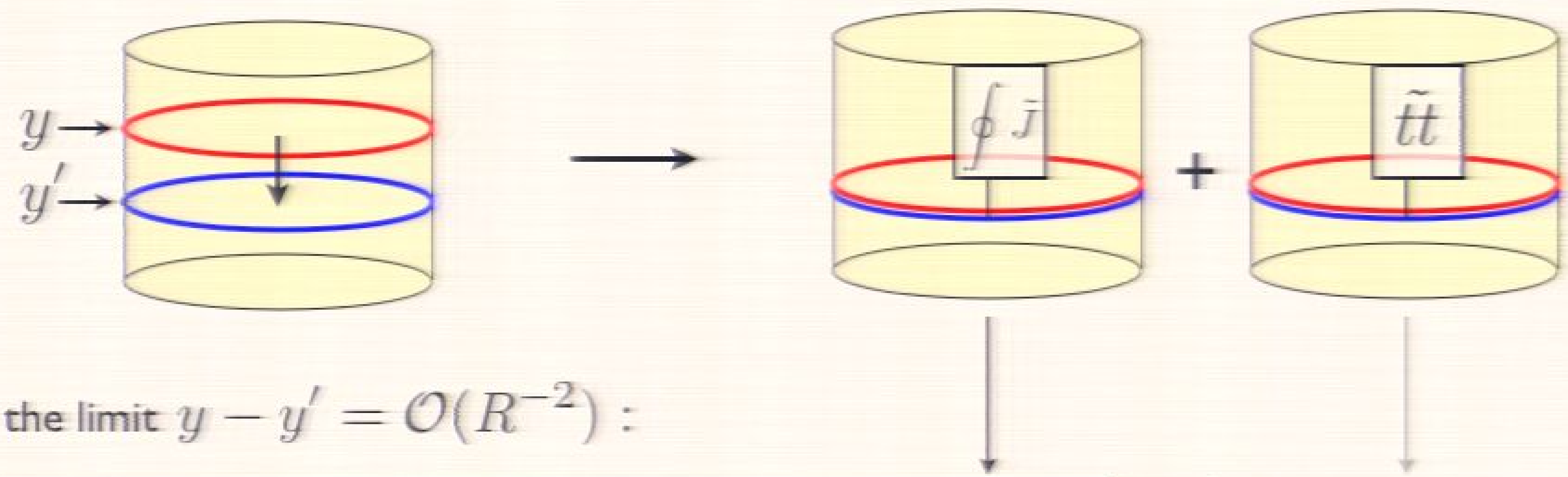
Derivation of the T-system II

- Previously we computed the leading quantum correction:



- In the limit $y - y' = \mathcal{O}(R^{-2})$:

$$\tilde{J} \sim \tilde{J}^a \times f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b$$

$$\tilde{J}^a = \frac{R^{-4}}{y - y'} \frac{\pi^2}{16} y^2 (y^2 + y^{-2})^4 \partial_y A^a(y) + \mathcal{O}(R^{-4})$$

$\frac{R^{-4}}{y-y'} \sim R^{-2}$

$$\tilde{t}t = \mathcal{O}(R^{-4})$$

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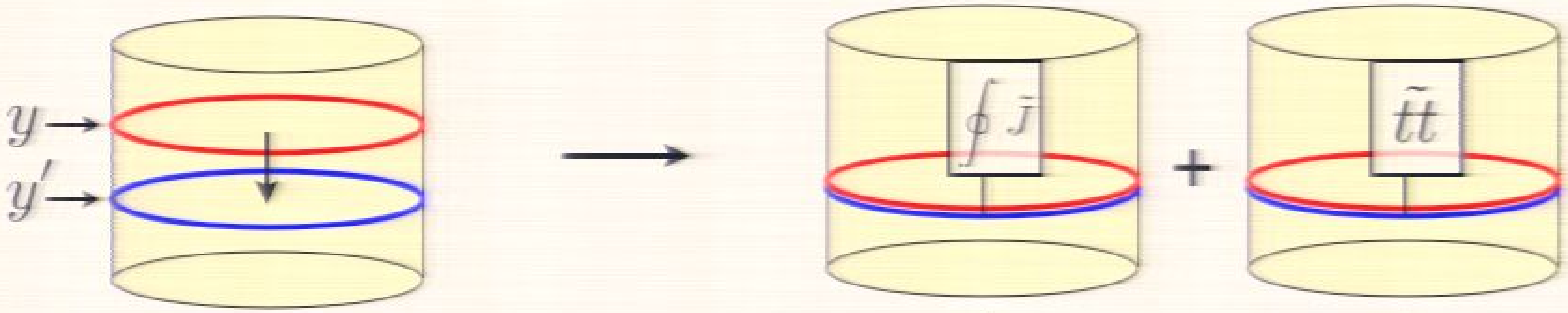


- In the limit $y - y' = \mathcal{O}(R^{-2})$:

$$\tilde{J} \sim \tilde{j}^a \times f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b$$

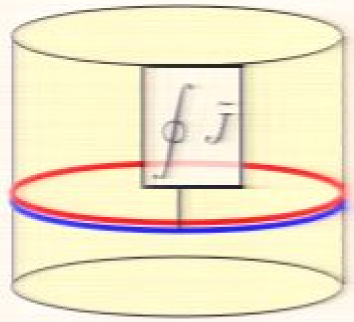
$$\tilde{j}^a = \frac{R^{-4}}{y - y'} \frac{\pi^2}{16} y^2 (y^2 + y^{-2})^4 \partial_y A^a(y) + \mathcal{O}(R^{-4})$$

$\sim R^{-2}$

$$\tilde{t}t = \mathcal{O}(R^{-4})$$

Derivation of the T-system III

We consider: $\sum_{R,R'}$



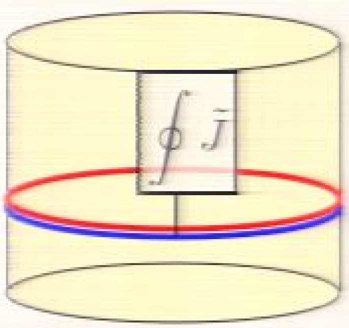
$$\tilde{J} = \frac{R^{-4}}{y-y'}\frac{\pi^2}{16}y^2(y^2+y^{-2})^4\partial_y A^a(y)\left(f_a{}^{bc}f_b{}^{de}\; t_c^R t_e^R \otimes t_d^{R'} + f_a{}^{bc}f_c{}^{de}\; t_e^R \otimes t_d^R t_b^{R'}\right) + \mathcal{O}(R^{-4})$$

For the particular combination of representations appearing in the T-system, we can use character identities from Kazakov & Vieira, 2007

$$t_a^R \otimes 1 - 1 \otimes t_a^{R'}$$

The leading quantum corrections from fusion simplify to:

$\sum_{R,R'}$



$$= -\frac{R^{-4}}{\delta}\frac{\pi^2}{16}y^2(y^2+y^{-2})^4\sum_{R,R'}\left(\partial_y \mathcal{T}_R(y)\mathcal{T}_{R'}(y) - \mathcal{T}_R(y)\partial_y\mathcal{T}_{R'}(y)\right) + \mathcal{O}(R^{-4})$$

Derivation of the T-system IV

We obtain eventually:

$$\sum_{R,R'} \mathcal{T}_R(y+\delta) \triangleright \mathcal{T}_{R'}(y-\delta) = \sum_{R,R'} \mathcal{T}_R(y)\mathcal{T}_{R'}(y)$$

Character identity $\Rightarrow \varnothing$

$$+ \left(\delta - \frac{R^{-4}}{\delta}\frac{\pi^2}{16} y^2 (y^2+y^{-2})^4 \right) \sum_{R,R'} \left(\partial_y \mathcal{T}_R(y)\mathcal{T}_{R'}(y) - \mathcal{T}_R(y)\partial_y\mathcal{T}_{R'}(y)\right) + \mathcal{O}(R^{-4})$$

From the derivative expansion (δ)

From the quantum effects in fusion ($\frac{R^{-4}}{\delta}\frac{\pi^2}{16} y^2 (y^2+y^{-2})^4$)

If we choose: $\delta = R^{-2}\frac{\pi}{4} y (y^2+y^{-2})^2$

We obtain: $\sum_{R,R'} \mathcal{T}_R(y+\delta) \triangleright \mathcal{T}_{R'}(y-\delta) = 0 + \mathcal{O}(R^{-4})$

The fact that all divergences vanish in the transfer matrices is important. Else the different terms in the T-system would be renormalized differently.

Derivation of the T-system V

We have shown: $\sum_{R,R'} \mathcal{T}_R(y+\delta) \triangleright \mathcal{T}_{R'}(y-\delta) = 0 + \mathcal{O}(R^{-4})$ for $\delta = R^{-2}\frac{\pi}{4}y(y^2+y^{-2})^2$

We perform a change of variables such that the T-system takes its usual form:

$$u = \frac{R^2}{\pi}\frac{1}{1-y^4}$$

$$\Rightarrow \mathcal{T}_{a,s}(u+1) \triangleright \mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1) \triangleright \mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1) \triangleright \mathcal{T}_{a,s-1}(u+1) + \ldots$$

We have derived from first principles the T-system up to first non-trivial order in perturbation theory.

1. The strategy of the proof
2. The worldsheet theory
3. Line operators and UV divergences
4. Fusion of line operators
5. Derivation of the T-system

6. Generalizations and conclusions

Summary of the results

- We studied the fusion of line operators in the pure spinor string on $AdS_5 \times S^5$ up to second order in perturbation theory.

- We deduced a perturbative proof of the T-system as an operator identity.

" The shifts come from fusion "

Fusion vs TBA

☺ No hypothesis	☺ All states	← Fusion wins
? Energy(T's)	? Analytic properties	
☹ Perturbative		← TBA wins

At that point, the two approaches are complementary.

First generalization

- The same computation can be done in the sigma model on the supergroups $PSl(n|n)$.

R.B., 2010

- In that case the current-current OPEs are known to all orders.

Ashok, R.B. & Troost, 2009

Bershadsky, Zhukov & Vaintrob, 1999

→ No divergence in the transfer matrix up to second order.

→ Transfer matrices commute up to second order.

Superstrings in $AdS_3 \times S^3$

- Strings in $AdS_3 \times S^3$ with RR and/or NS fluxes can be described in the hybrid formalism.

Hybrid string on $AdS_3 \times S^3$	⇔	Sigma model on $PSU(1,1|2)$ + ghosts	Berkovits, Vafa & Witten, 1999

- This theory admits a consistent expansion in the ghosts.

⇒ **String theory in $AdS_3 \times S^3$ realizes the T-system**

- Up to first order in the large radius expansion.
- At zeroth-order in the ghosts expansion.

Further generalizations

- Two features are necessary and (almost) sufficient for the computation to work:

The commutator of the flat connection can be written as a (r,s) system.

The transfer matrix is free of divergences.

↑

This strongly relies on the vanishing of the dual Coxeter number.

Bachas & Gaberdiel, 2004

- There are good chances that it works for string theory on:

$AdS_4 \times CP^3$ $\quad$ $AdS_2 \times S^2$ $\quad$ $AdS_3 \times S^3 \times S^3$ ... $\quad$ Zarembo, 2010

Thank you.

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From the quantum effects in fusion

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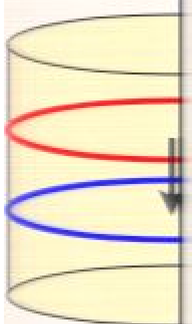
We obtain:



$$
\begin{aligned}
\tilde{J} = (i\pi R^{-2})^2 \sum_{m,n,p,q,r} & f_{C_p}{}^{B_n A_m} f_{E_r}{}^{C_p D_q} \{t^R_{D_q}, t^R_{A_m}\} t^{R'}_{B_n} \\
& \times \big(J_r^{E_r}(\tilde{D}'^{p}_{mn} F_q C^r_{pq} - \tilde{D}'^{p}_{mn} \bar{F}_q C^r_{p\bar{q}} - \tilde{D}'^{\bar{p}}_{mn} F_q C^r_{\bar{p}q} - \tilde{D}'^{\bar{p}}_{mn} \bar{F}_q C^r_{\bar{p}\bar{q}} \\
& \quad + \frac{1}{2} F_r (\tilde{D}'^{s}_{mn} F_p C_{sp} + \tilde{D}'^{s}_{pn} F_m C_{sm} - \tilde{D}'^{\bar{s}}_{mn} \bar{F}_p C_{\bar{s}\bar{p}} - \tilde{D}'^{\bar{s}}_{pn} \bar{F}_m C_{\bar{s}\bar{m}})) \\
& \quad + \bar{J}_r^{E_r}(\tilde{D}'^{p}_{mn} F_q C^{\bar{r}}_{pq} + \tilde{D}'^{p}_{mn} \bar{F}_q C^{\bar{r}}_{p\bar{q}} + \tilde{D}'^{\bar{p}}_{mn} F_q C^{\bar{r}}_{\bar{p}q} - \tilde{D}'^{\bar{p}}_{mn} \bar{F}_q C^{\bar{r}}_{\bar{p}\bar{q}} \\
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& + f_{C_p}{}^{B_n A_m} f_{E_r}{}^{D_q C_p} t^R_{A_m} \{t^{R'}_{B_n}, t^{R'}_{D_q}\} \\
& \times \big(J_r^{E_r}(-\tilde{D}^{p}_{mn} F_q C^r_{pq} + \tilde{D}^{p}_{mn} \bar{F}_q C^r_{p\bar{q}} + \tilde{D}^{\bar{p}}_{mn} F_q C^r_{\bar{p}q} + \tilde{D}^{\bar{p}}_{mn} \bar{F}_q C^r_{\bar{p}\bar{q}} \\
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\end{aligned}
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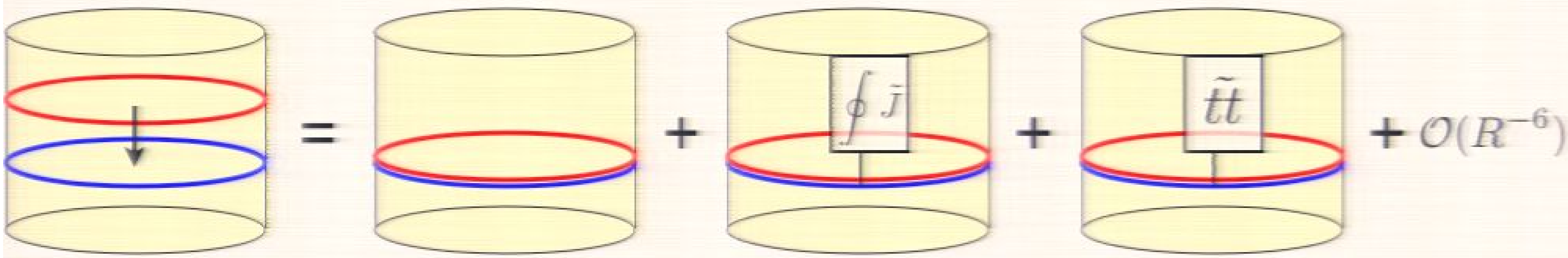
Symmetric fusion of transfer matrices

We obtain:



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$$\tilde{J} \sim \tilde{J}^a \times f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b$$

Constant matrix inserted between the integrated connections

$$\tilde{t}t \sim f^{abc} f_{cb}{}^{d} \times t_d t_a$$

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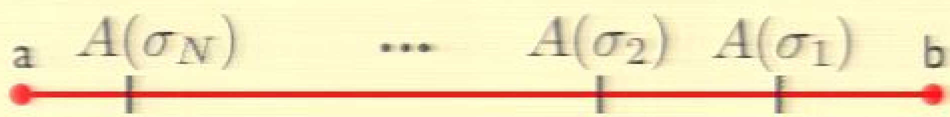
UV divergences in line operators

We expand the line operators:

$$W^{b,a} = P\exp\left(-\int_a^b A\right) = \sum_{N=0}^{\infty} W_N^{b,a}$$

with: $W_N^{b,a}$:



Collisions of integrated operators lead to divergences.

$\Rightarrow$ We need to regularize and potentially renormalize the line operators.