Title: Aspects of the Generalized Unitarity Method

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Abstract: The generalized unitarity method reconstructs loop-level scattering amplitudes from tree-level scattering amplitudes. We review some of the steps of this method which are necessary for applications in any quantum field theory, discuss several points of view and emphasize some of the aspects that can be easily automated.

We mainly use N=4 super-Yang-Mills theory to illustrate the discussion and comment on the necessary changes for applications to super-Yang-Mills theories with reduced supersymmetry.

Aspects of generalized unitarity

- tree-level comments
- generalized unitarity: generalities, sums, regularization, type of cuts, etc
- Supersums; example, general props.
- numerics vs. analytics
- general strategy for supersums.
- ....

- <u>Trees</u> t^{abc}

$A^{(0)}$

• **Trees** f^{abc} $SU(N)$

$$A^{(L)} = N^L \sum_{S_n/Z_n} \mathrm{Tr}(T^{a_1} \cdots T^{a_n})\, A^{(L)}(k_{a_1}, \cdots k_{a_n}; \varepsilon)$$

$$+ \sum \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\quad A^{(L)}(\qquad)$$

$$+ \sum \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\, A^{(L)}(\qquad)$$

• Trees f^{abc} $SU(N)$

$$A^{(L)} = N^L \sum_{S_n/Z_n} \mathrm{Tr}(T^{a_1}\cdots T^{a_n})\, A^{(L)}(k_{\sigma_1}, \cdots k_{\sigma_n}, N)$$

$$+ \sum \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\, A^{(L)}(\quad)$$

$$+ \sum \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\, \mathrm{Tr}(\quad)\, A^{(L)}(\quad)$$

$$0 = A^{(0)}(k_1 k_2 k_3 k_4) + A^{(0)}(k_1 k_3 k_4 k_2) + A^{(0)}(k_1 k_4 k_2 k_3)$$

• Trees t^{ree} $SU(N)$

$$A^{(L)} = N \sum_{S_n/Z_n} Tr(T^{a_1}\cdots T^{a_n})\, A^{(L)}(k_{\sigma_1}, \cdots k_{\sigma_n})$$

$$+ \sum Tr(\quad)\, Tr(\quad)\; A^{(L)}(\qquad)$$

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$$\sum_{C_{2\dots n}} A(1, \sigma_2 \dots \sigma_n) = 0$$ $U(1)$ decoupling id

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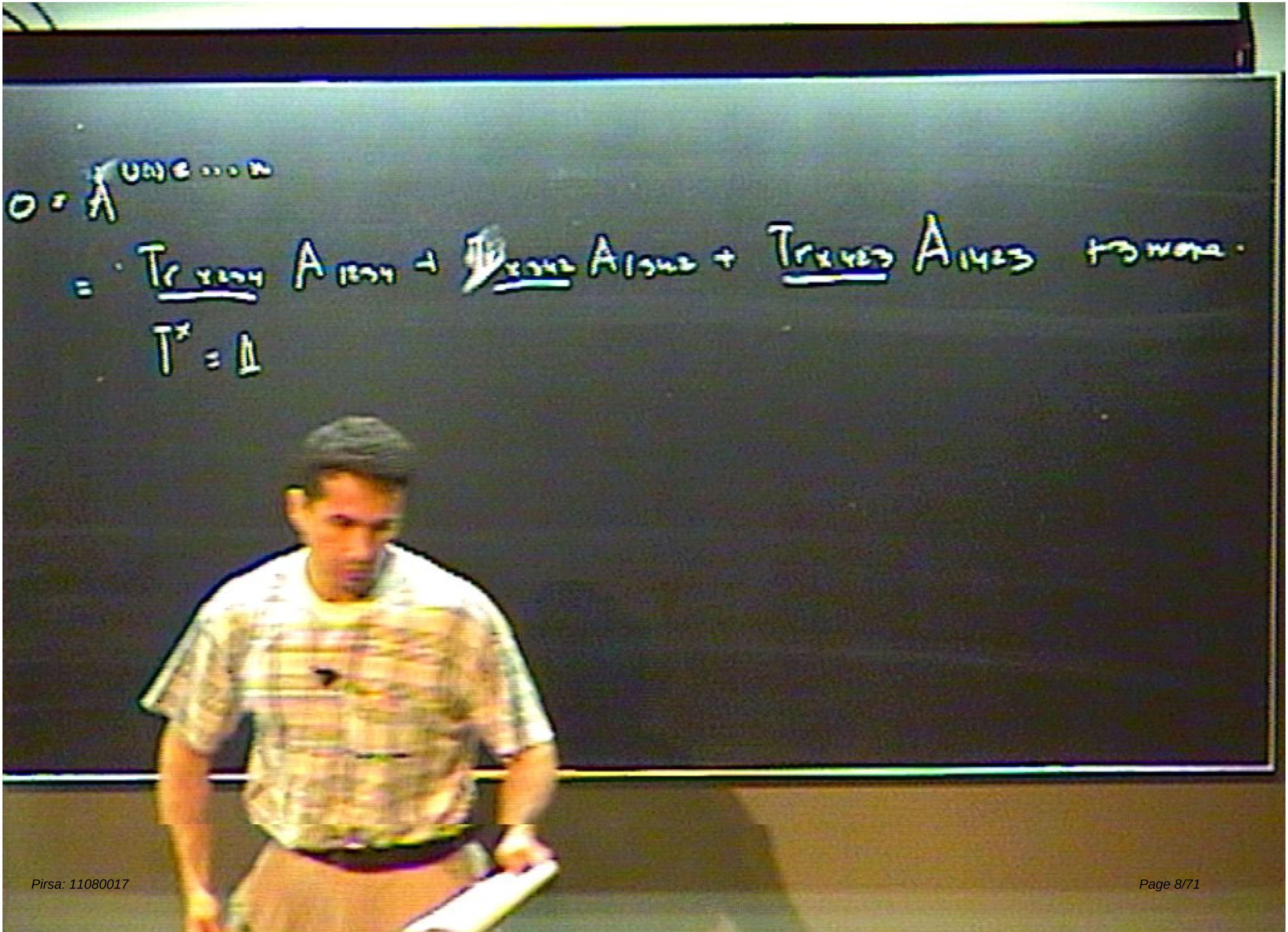
$$\sum_{C_{2\cdots n}} A(1, \sigma_2 \ldots \sigma_n) = 0 \qquad U(1) \text{ decoupling id}$$



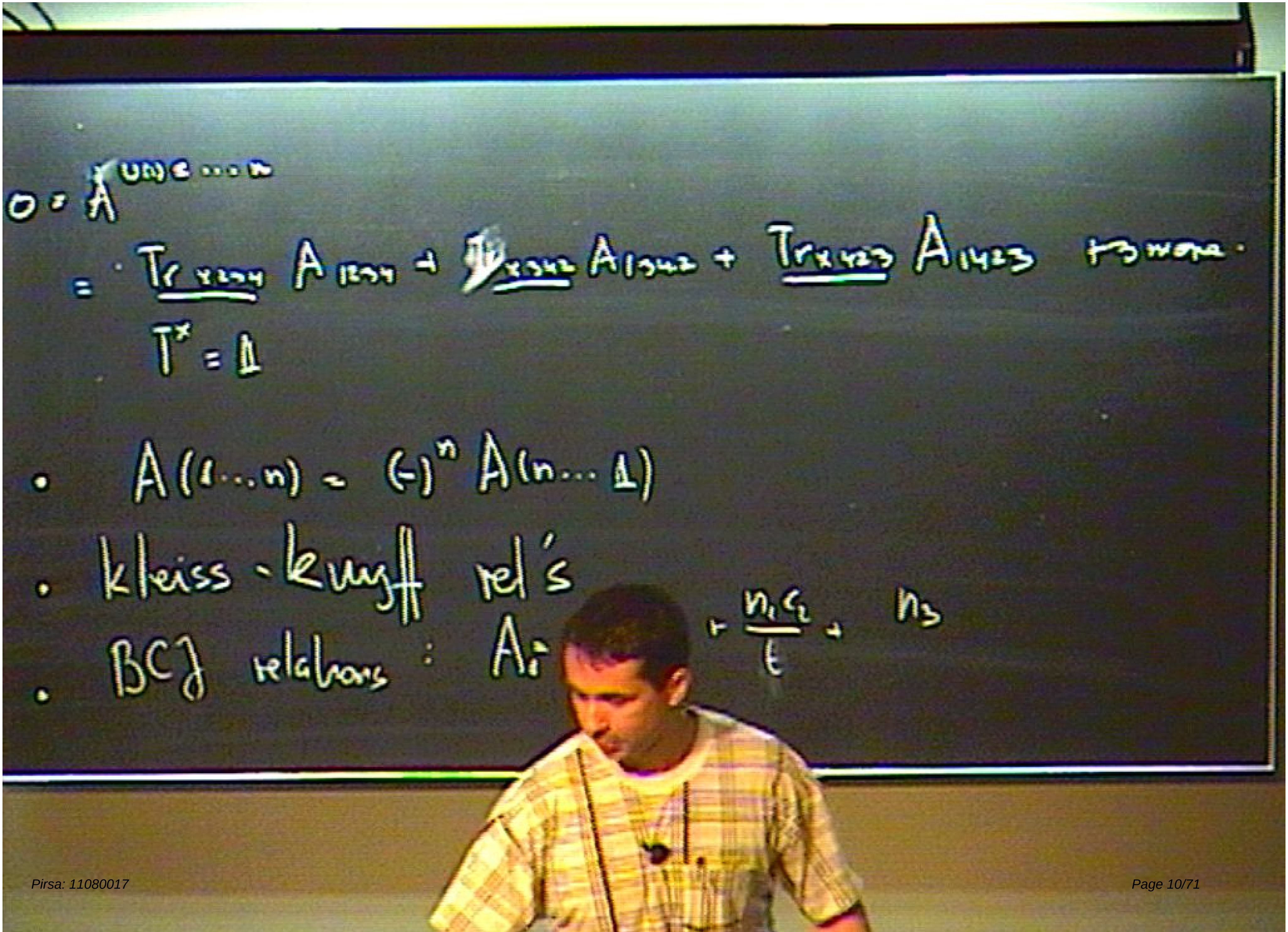
0 = A
= Tr A + Tr A + Tr A + 3 more.
T = 1

$$0 = A^{U(N)\,3\ldots n}$$

$$= \underline{\mathrm{Tr}_{x234}}\, A_{1234} + \underline{\mathrm{Tr}_{x342}}\, A_{1342} + \underline{\mathrm{Tr}_{x423}}\, A_{1423} \quad + 3 \text{ more.}$$

$$T^x = \mathbb{1}$$

- $$A(1 \ldots n) = (-)^n A(n \ldots 1)$$

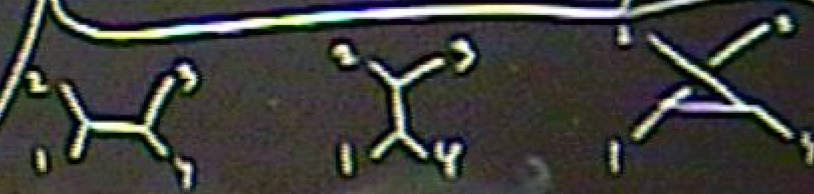

0 = A^{U(N) ... n}
= Tr_{x234} A_{1234} + Tr_{x342} A_{1342} + Tr_{x423} A_{1423} +3 more.
T^x = 1
• A(1...n) = (-)^n A(n...1)
• kleiss - kuijf rel's
• BCJ relations : A_i n_1c_2/t + n_3

$$0 = A^{U(1)\,2\ldots n}$$

$$= \mathrm{Tr}_{x234}\, A_{1234} + \mathrm{Tr}_{x342}\, A_{1342} + \mathrm{Tr}_{x423}\, A_{1423} \quad + 3 \text{ more}.$$

$$T^x = \mathbb{1}$$

$$\boxed{\begin{aligned} n_1 + n_2 + n_3 &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned}}$$

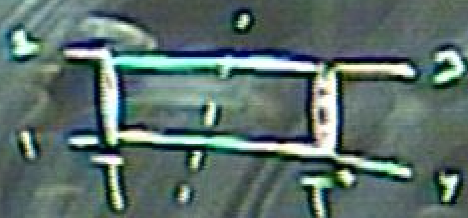


- $A(1 \ldots n) = (-)^n A(n \ldots 1)$
- Kleiss-Kuijf rel's
- BCJ relations: $A_4 = \dfrac{n_1 c_1}{s} + \dfrac{n_2 c_2}{t} + \dfrac{n_3 c_3}{u}$

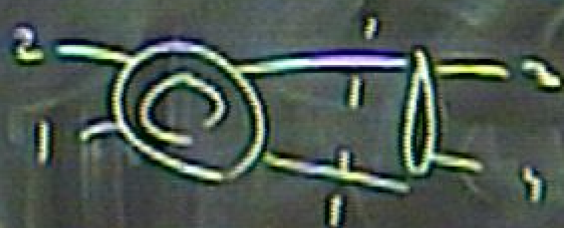
• $\mathbb{1} = S S^{\dagger} \Rightarrow 2\,\mathrm{Im}\,T = T\,T^{\dagger}$

$S = \mathbb{1} + iT$ $\qquad g^{L} \qquad g^{n}\, g^{L-n}$

1 loop :



2 loops :



• Tree

• $\mathbb{1} = S S^\dagger \Rightarrow 2\,\mathrm{Im}\,T = T\,T^\dagger$

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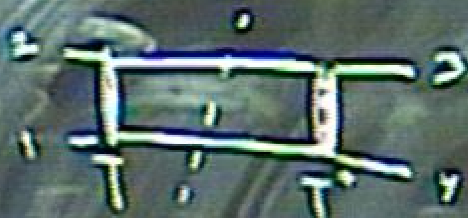
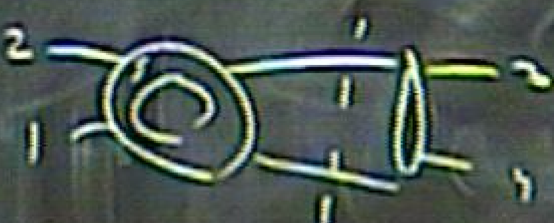
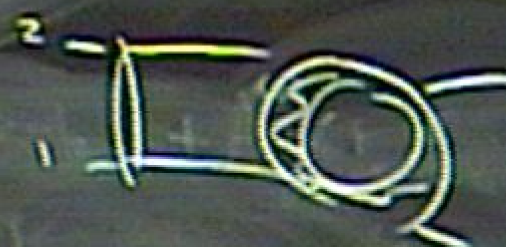
1 loop :



2 loops :



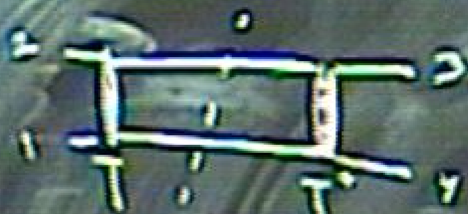


• Tw

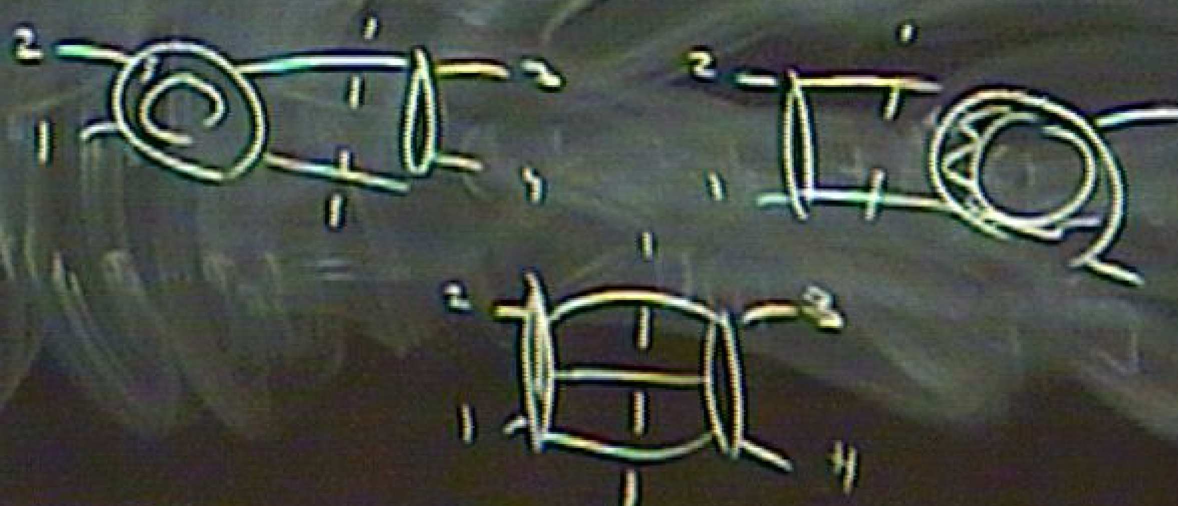
• $1 = S S^\dagger \Rightarrow 2\,\mathrm{Im}\,T = T\,T^\dagger$

$S = 1 + iT$ $\qquad g^L \qquad g^n\,g^{L-n}$

1 loop :



2 loops :



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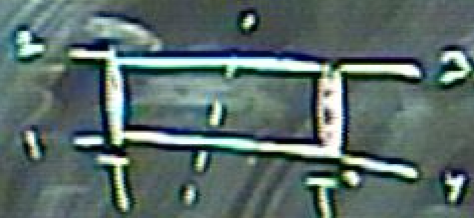
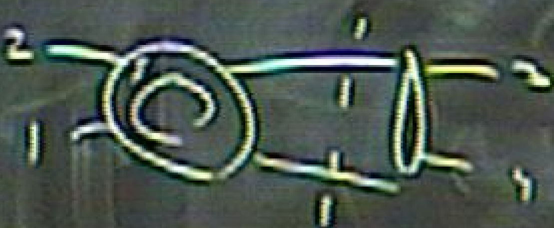
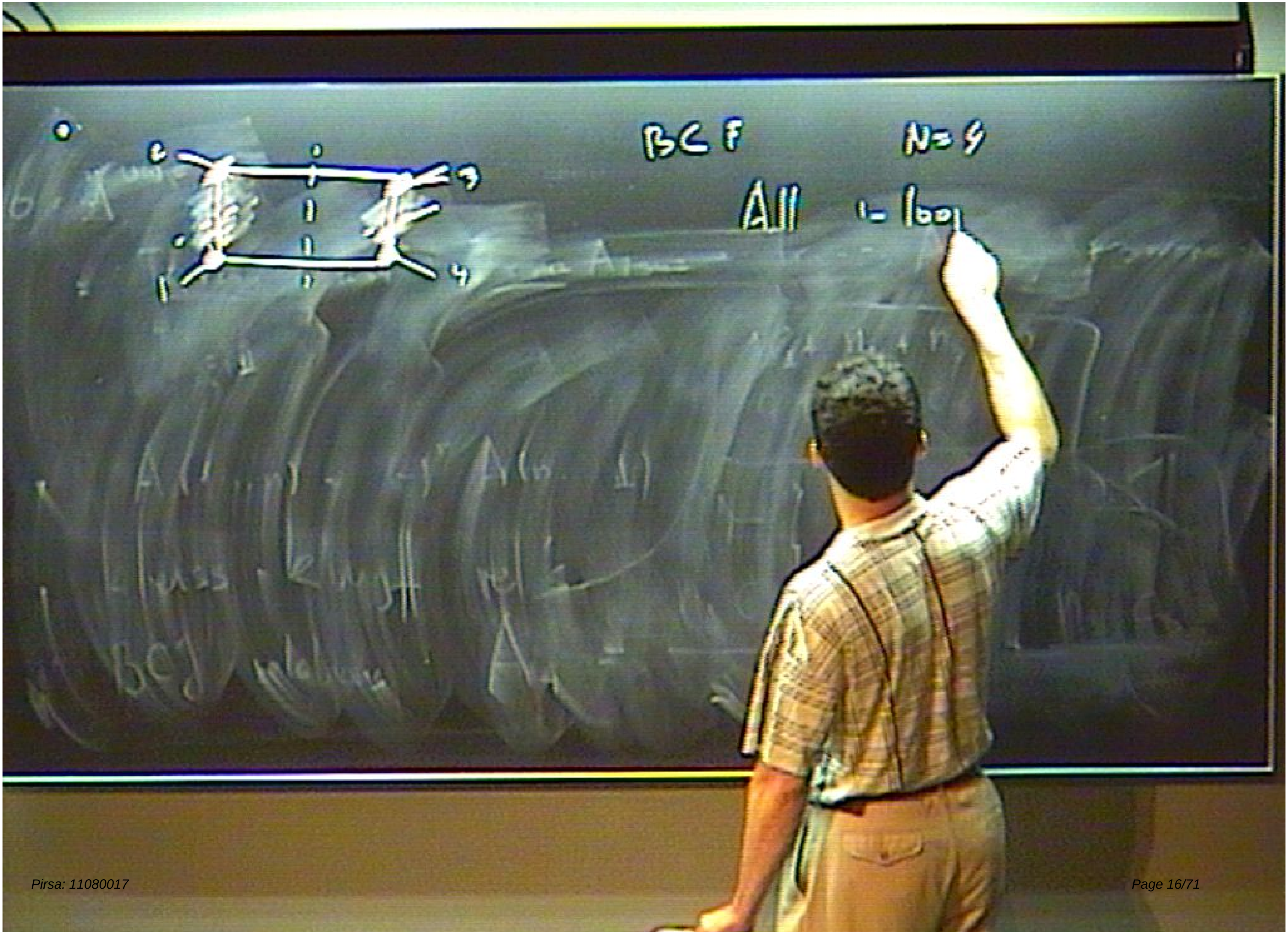
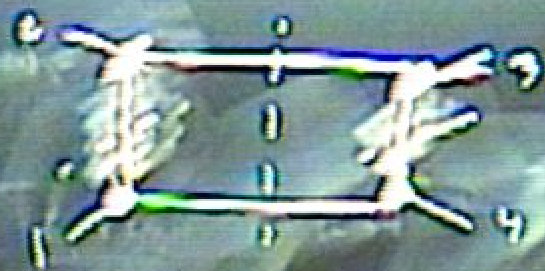


2 loops :

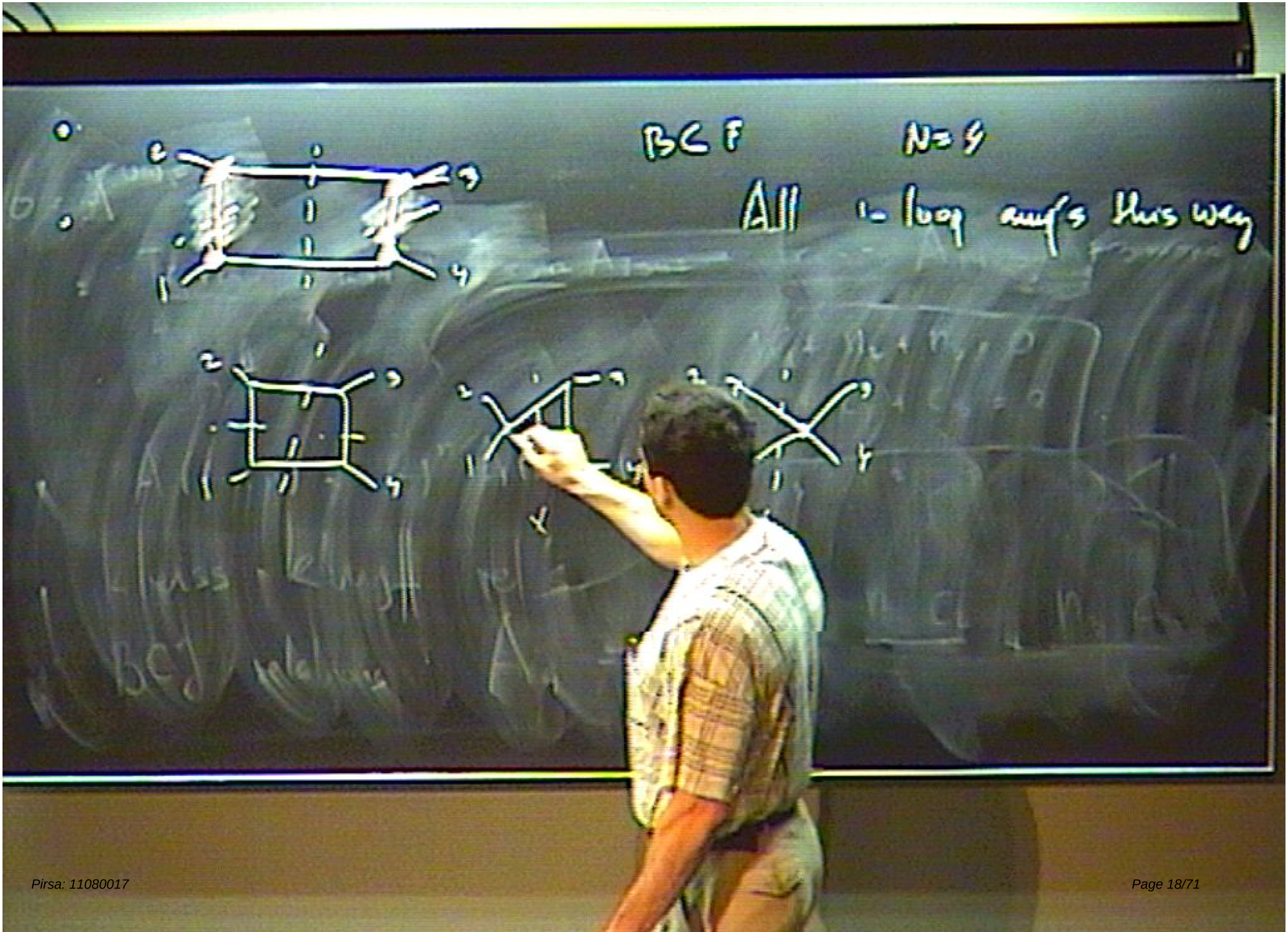


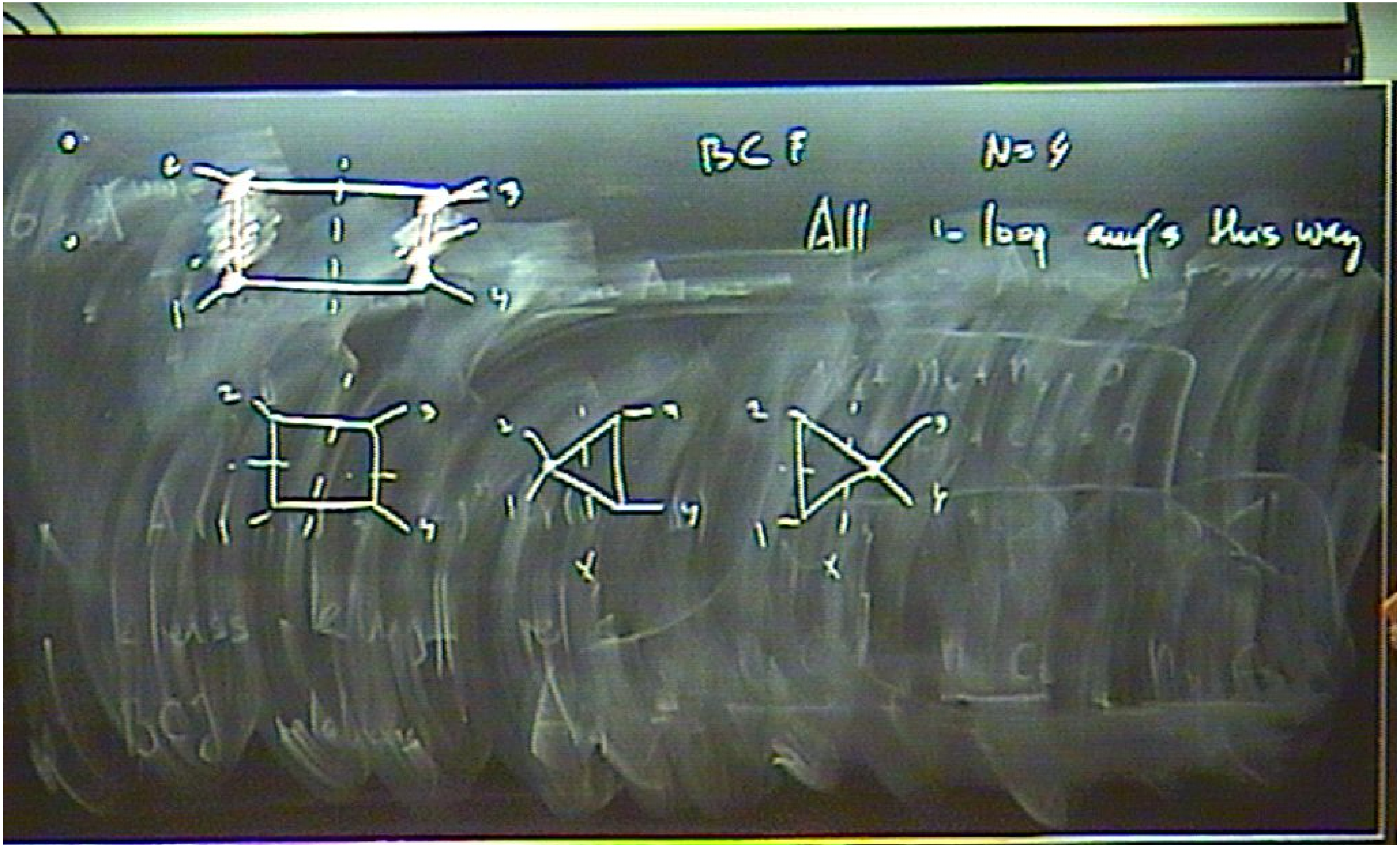

BCF
N= 4
All
2
3
1
4



BCF $\mathcal{N}=4$

All 1-loop amp's this way


BCF
N=4
All 1-loop amp's this way


BCF
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All 1-loop amp's this way

• Im

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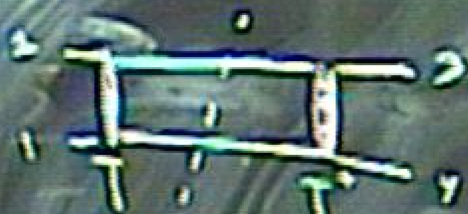
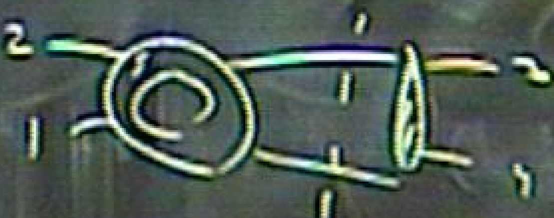
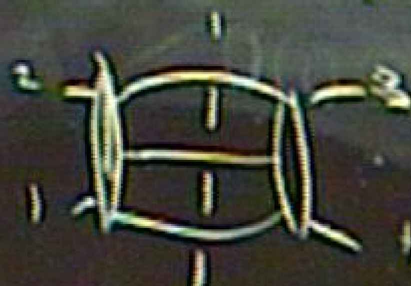
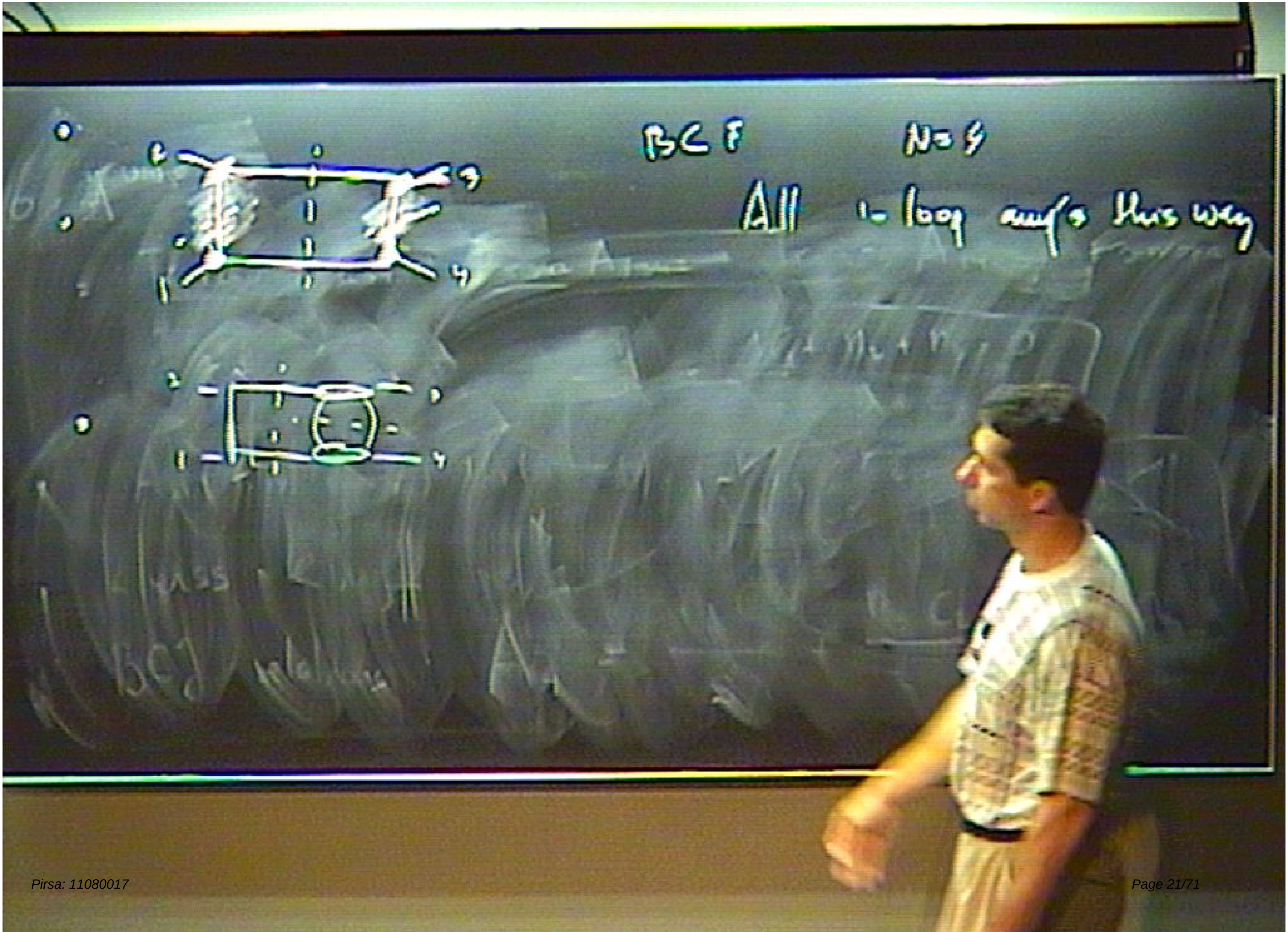
1 loop :

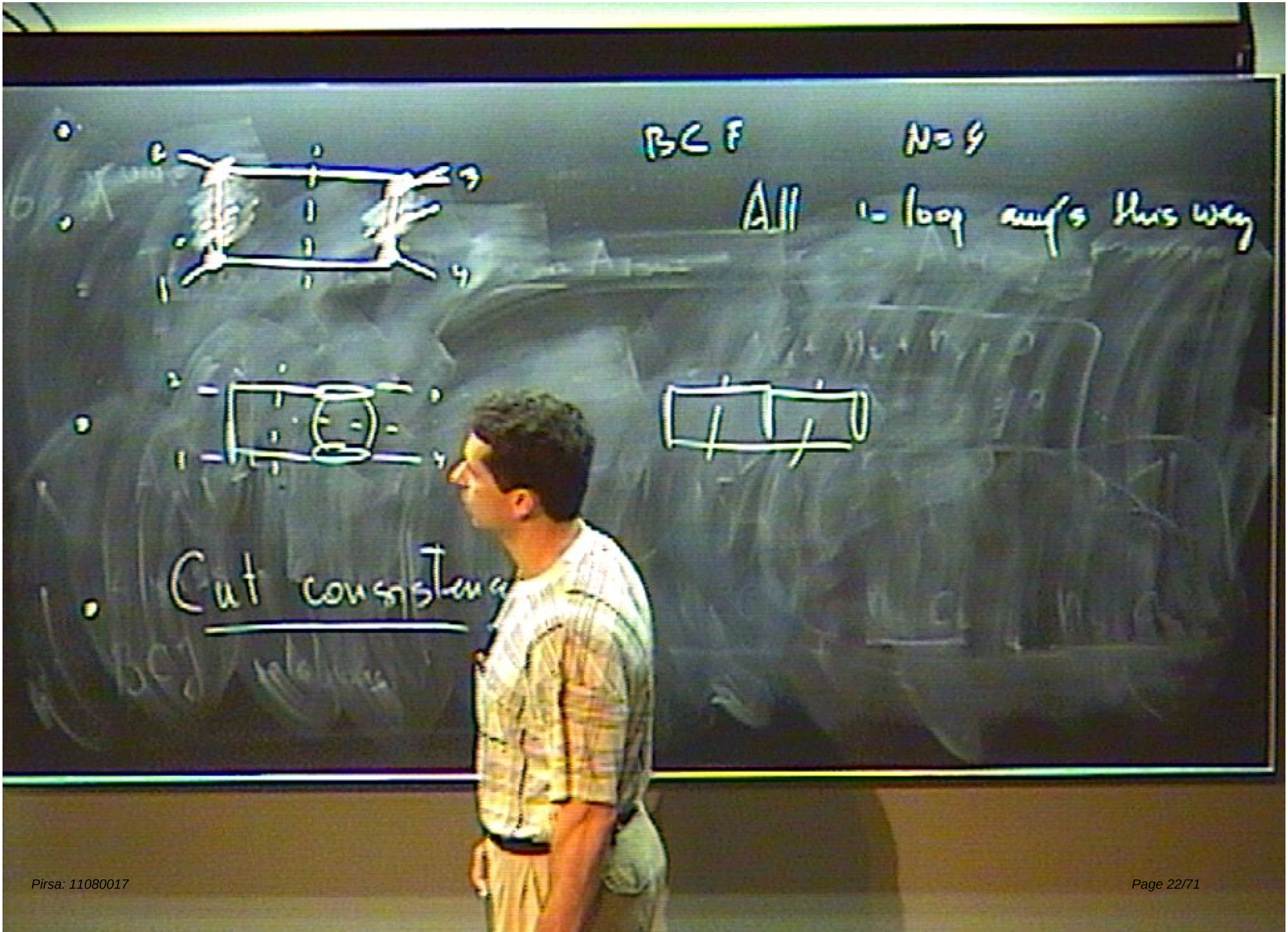


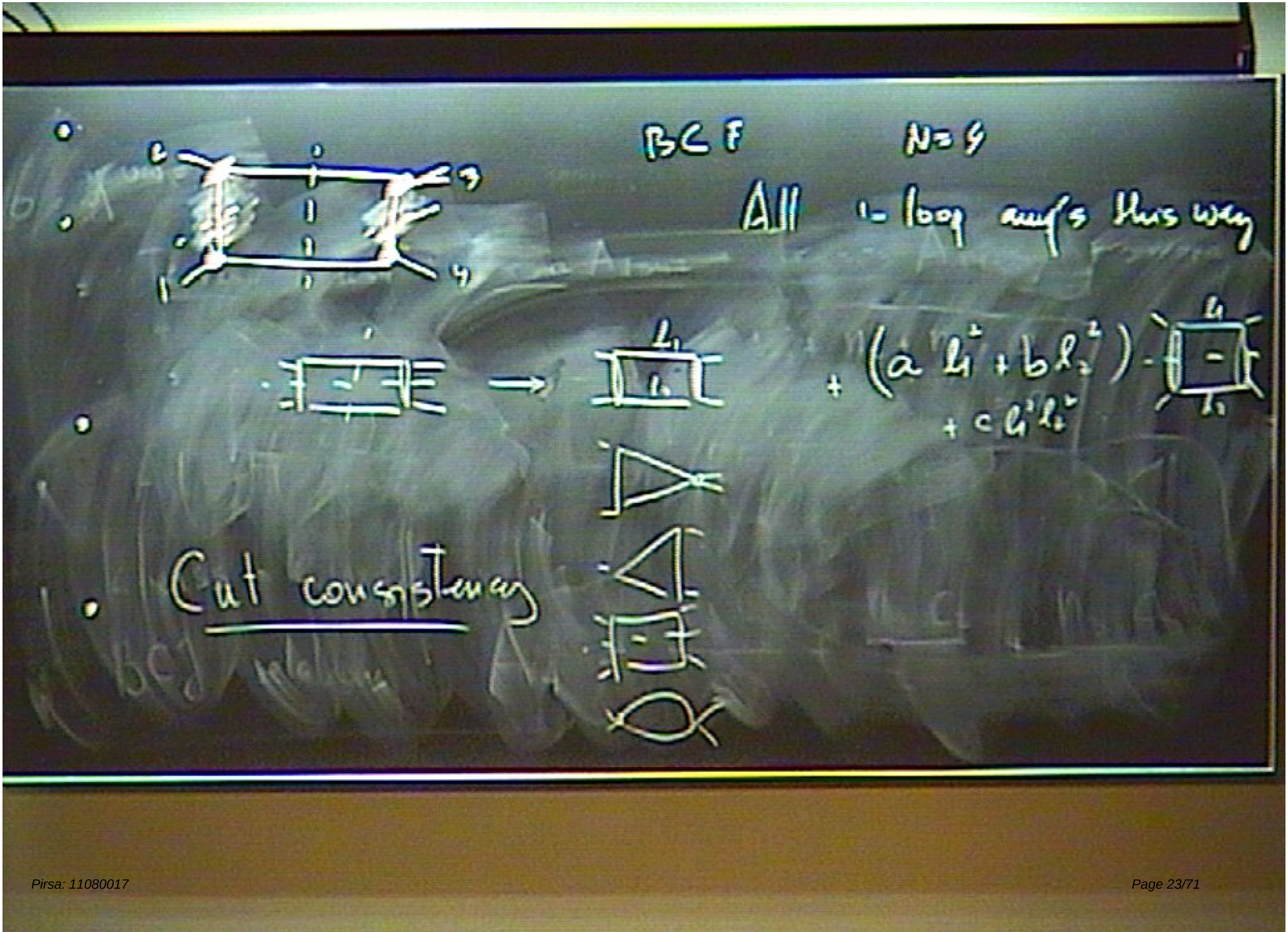
2 loops :

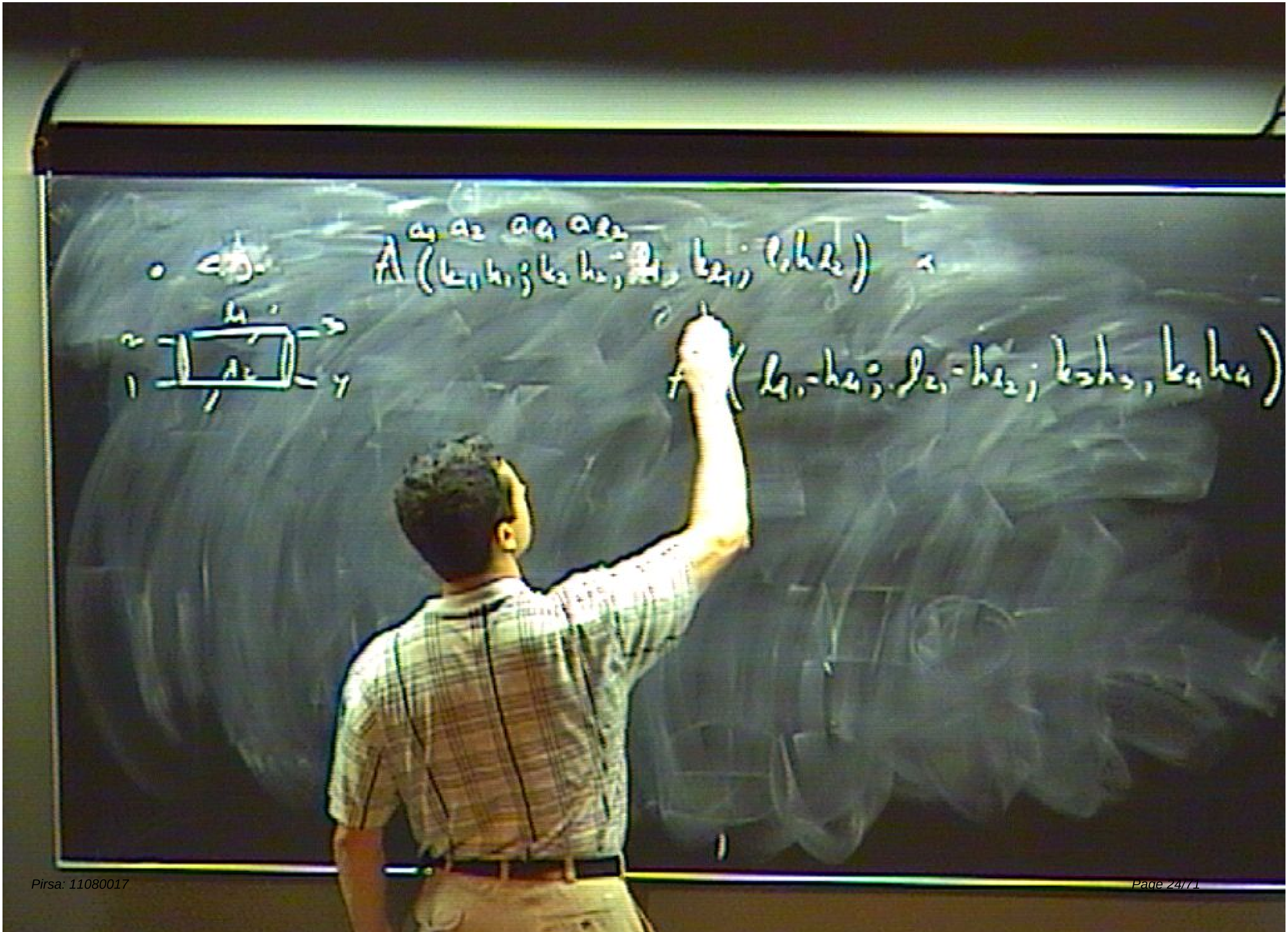





BCF
N=4
All 1-loop amp's this way


BCF
N=4
All 1-loop amp's this way
Cut consisten


BCF
N=4
All 1-loop amp's this way
Cut consistency


e.g.

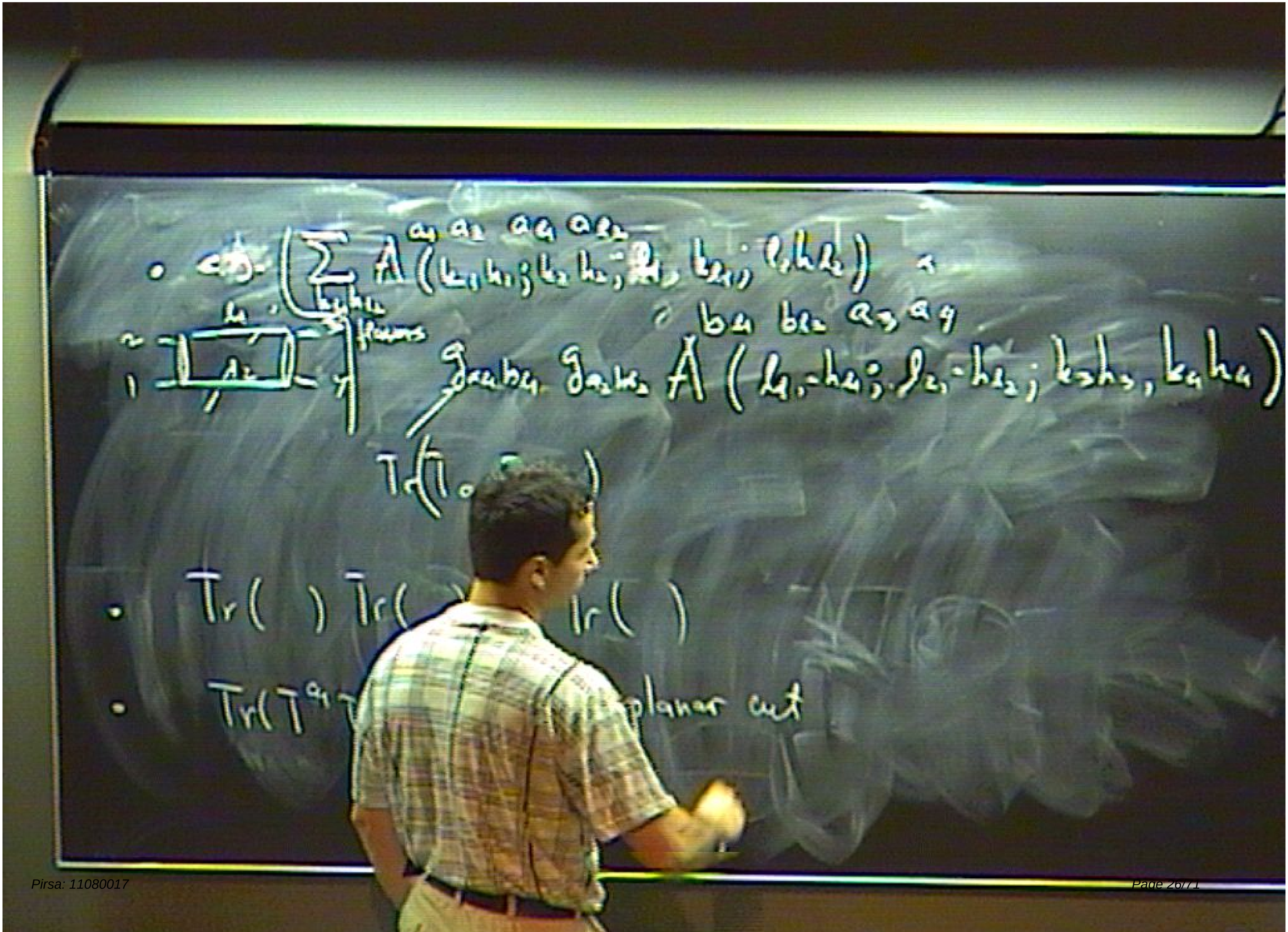
• e.g. $\left(\sum_{h_{\ell_1} h_{\ell_2}} A^{a_1 a_2 a_{\ell_1} a_{\ell_2}} (k_1 h_1 ; k_2 h_2 ; \ell_1, h_{\ell_1} ; \ell_2 h_{\ell_2}) \right) \times$

helicities

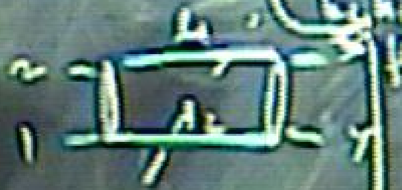
$$b_{\ell_1} \, b_{\ell_2} \, a_3 \, a_4$$

$$g_{a_{\ell_1} b_{\ell_1}} \, g_{a_{\ell_2} b_{\ell_2}} \, A \left(\ell_1, -h_{\ell_1} ; \ell_2, -h_{\ell_2} ; k_3 h_3 , k_4 h_4 \right)$$

$\mathrm{Tr}\left(T_{a_{\ell_1}} T_{b_{\ell_1}} \right)$


planar cut

• e.g. $\left(\sum_{h_{\ell_1} h_{\ell_2}} A^{a_1 a_2 a_{\ell_1} a_{\ell_2}}(k_1 h_1; k_2 h_2; \ell_1, h_{\ell_1}; \ell_2 h_{\ell_2})\right) \times$

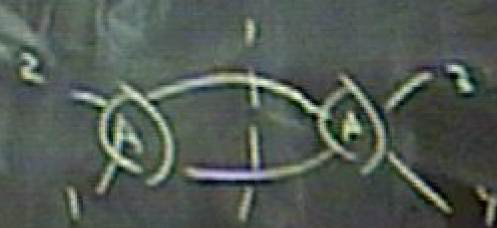


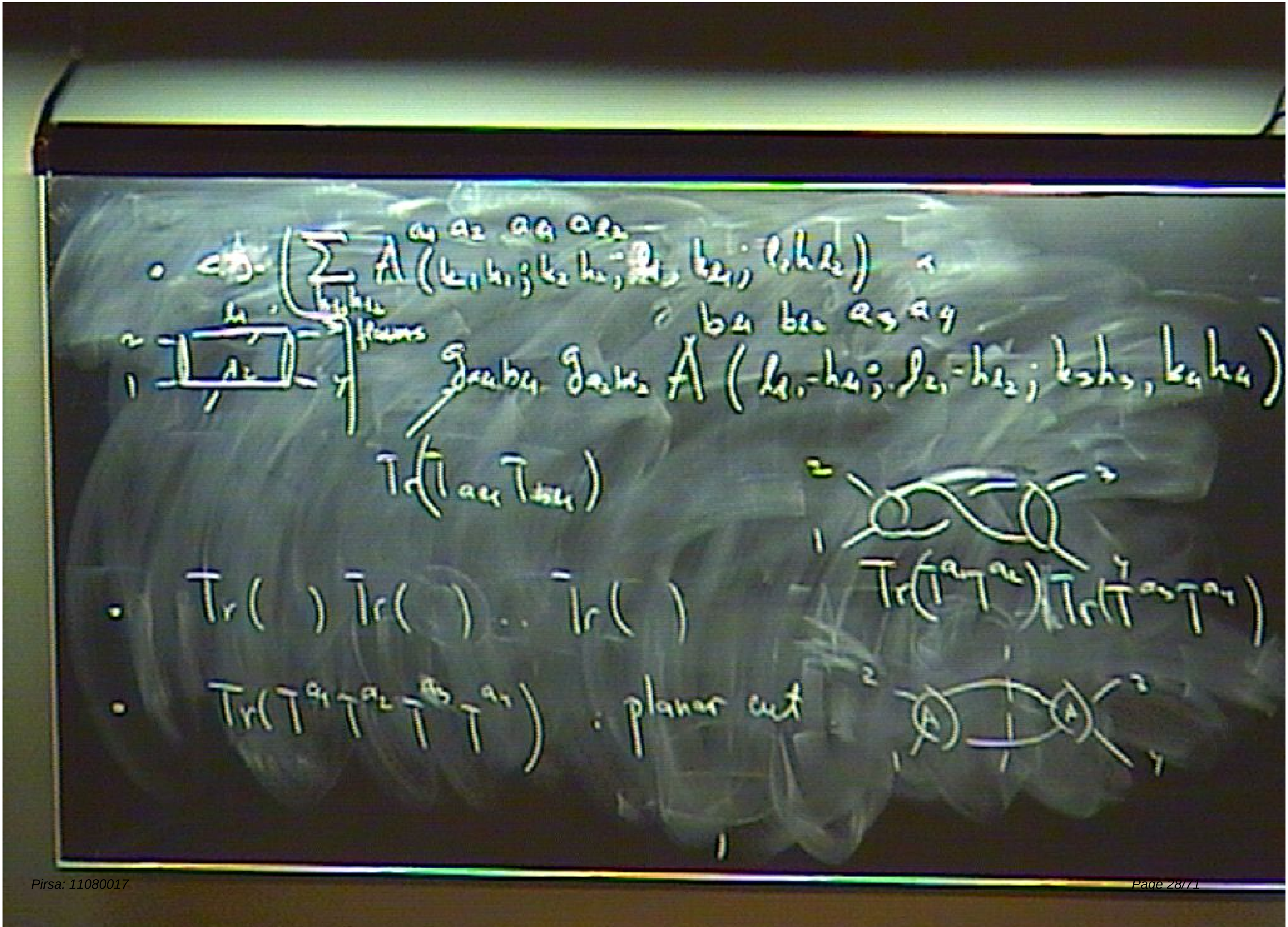
states $\quad g_{a_{\ell_1} b_{\ell_1}} \, g_{a_{\ell_2} b_{\ell_2}} \, A^{b_{\ell_1} b_{\ell_2} a_3 a_4}(\ell_1, -h_{\ell_1}; \ell_2, -h_{\ell_2}; k_3 h_3, k_4 h_4)$

$\mathrm{Tr}(T_{a_{\ell_1}} T_{b_{\ell_1}})$

• $\mathrm{Tr}(\ \ ) \, \mathrm{Tr}(\ \ ) \ldots \mathrm{Tr}(\ \ )$

• $\mathrm{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4})$: planar cut




e.g.
fermions
$\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})$
$\mathrm{Tr}(\ )\ \mathrm{Tr}(\ )\ \ldots\ \mathrm{Tr}(\ )$
$\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$: planar cut

• e.g. $\left(\sum_{h_1 h_4} A^{a_1 a_2 a_{l_1} a_{l_2}}(k_1 h_1; k_2 h_2; l_1, h_{l_1}; l_2 h_{l_2})\right) \times$

flavors

$b_{l_1}\, b_{l_2}\, a_3\, a_4$

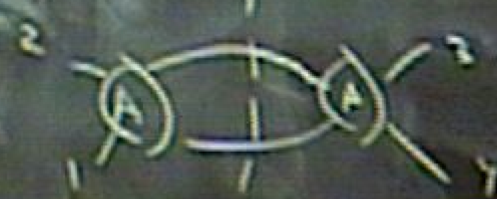
$g_{a_{l_1} b_{l_1}}\, g_{a_{l_2} b_{l_2}}\, A(l_1, -h_{l_1}; l_2, -h_{l_2}; k_3 h_3, k_4 h_4)$

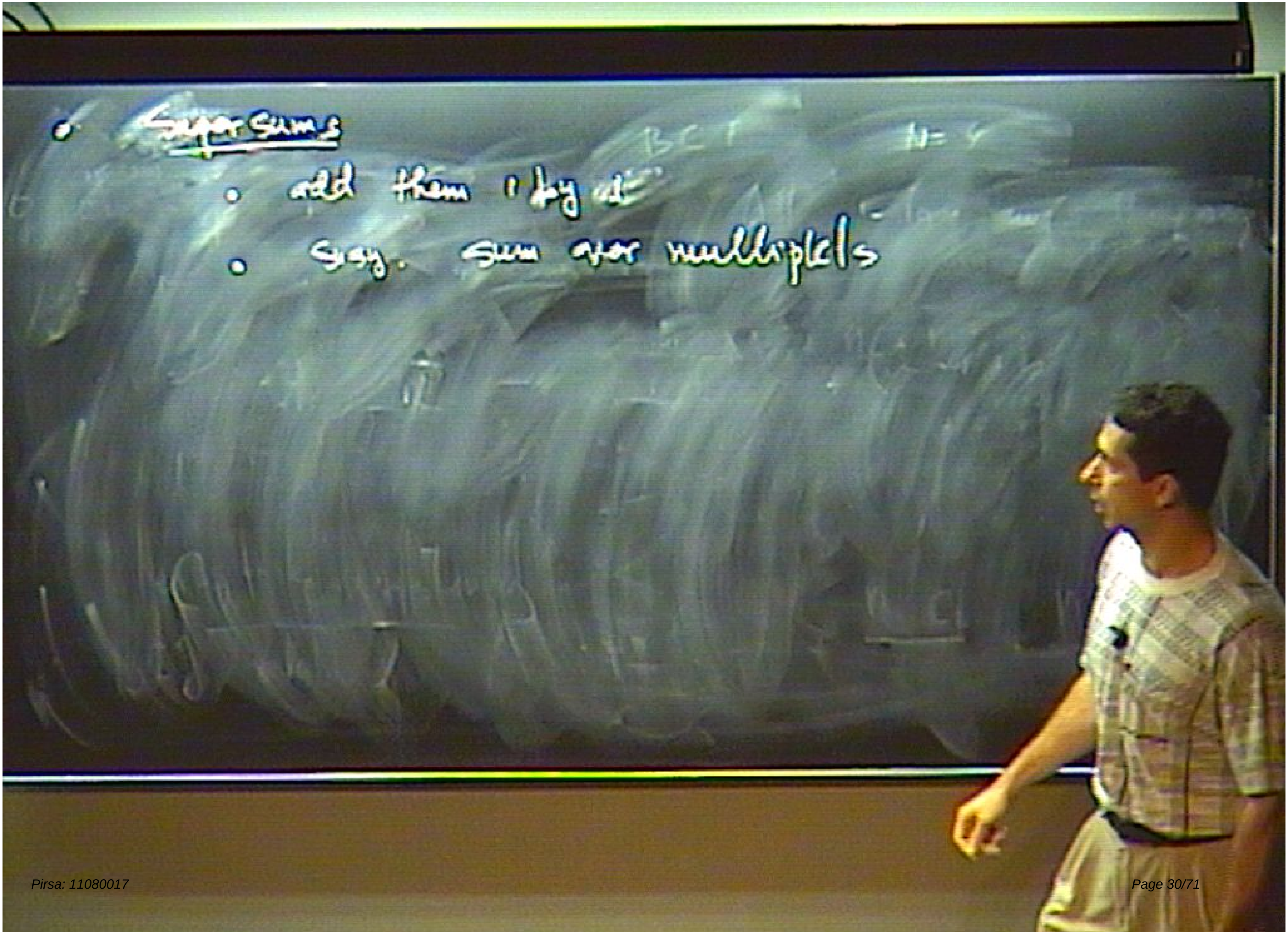
$\mathrm{Tr}(T_{a_{l_1}} T_{b_{l_1}})$

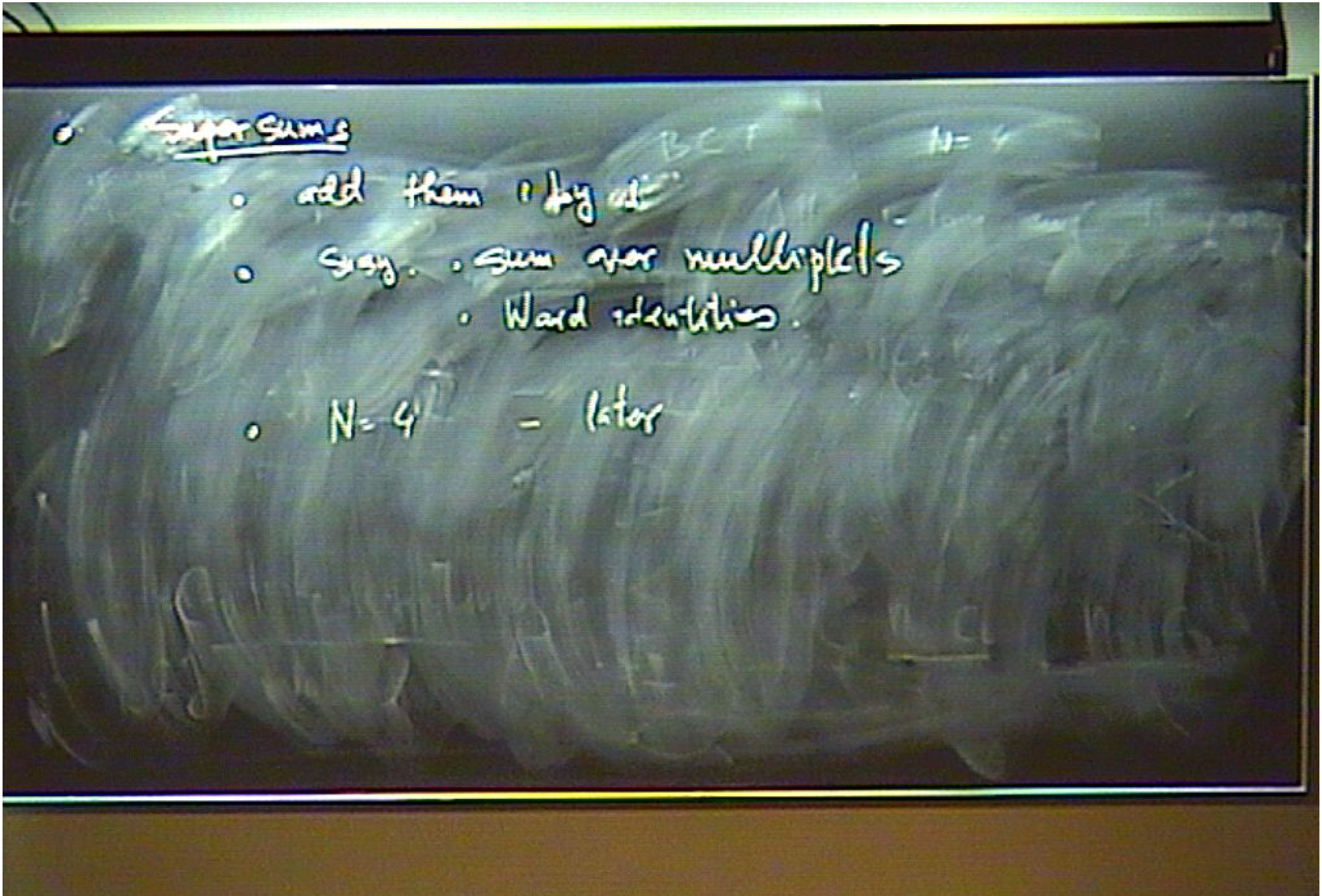
• $\mathrm{Tr}(\quad)\,\mathrm{Tr}(\quad)\ldots\mathrm{Tr}(\quad)$

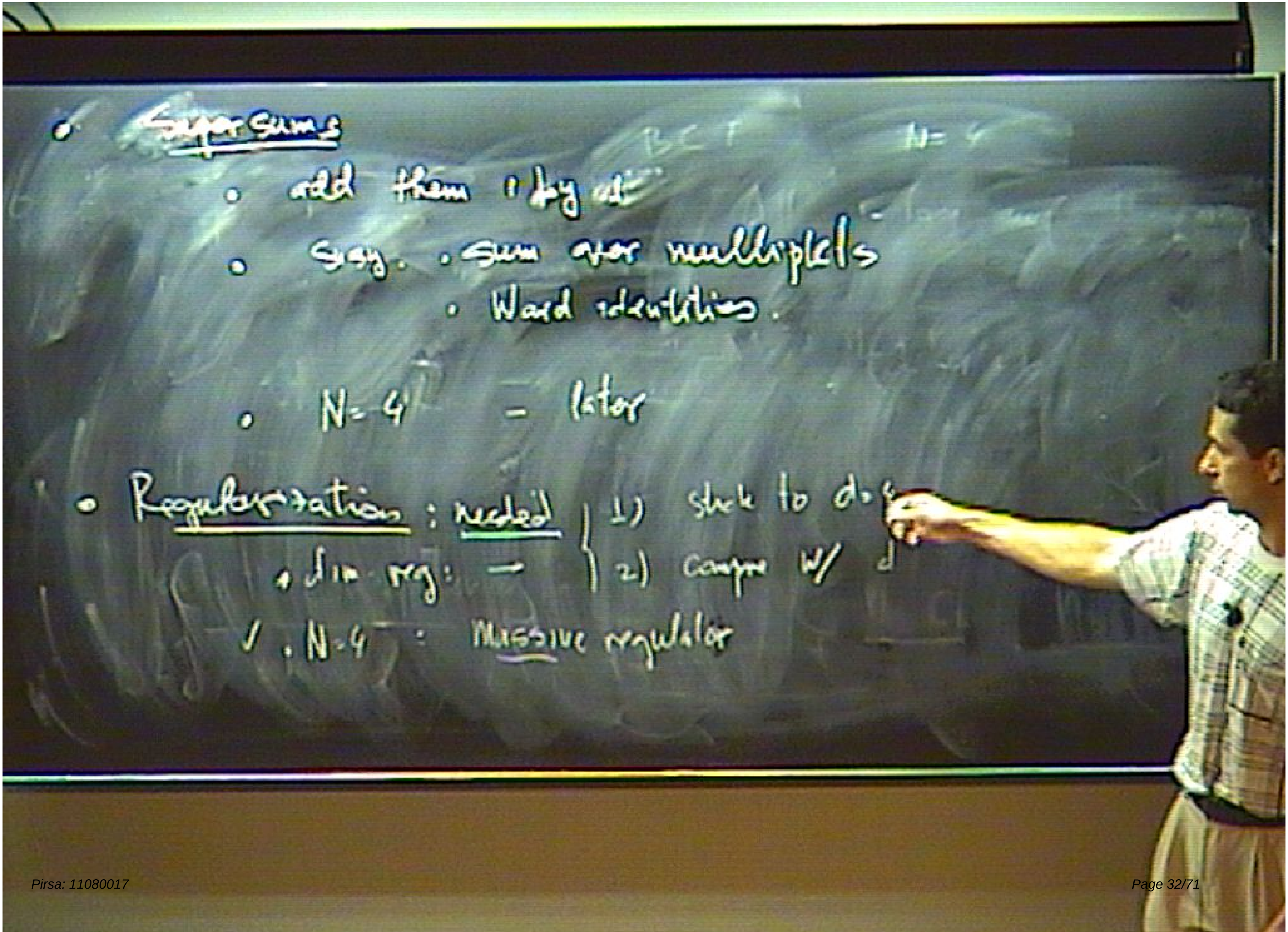
$\mathrm{Tr}(T^{a_1}T^{a_2})\,\mathrm{Tr}(T^{a_3}T^{a_4})$

• $\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$: planar cut




Supersums
sum over multiplets


Supersums
add them
susy. · sum over multiplets
· Ward identities.
N=4 – later



Supersums
add them
susy. sum over multiplets
Ward identities.
N=4 — later
Regularization : needed
1) stick to
2) compare w/
dim. reg. : —
N=4 : Massive regulator

- Supersums
  - add them by d
  - susy. • sum over multiplets
    - Ward identities.
  - N=4 — later
- Regularization : needed
  - 1) stick to d=4
  - 2) compare w/ d
  - dim. reg: —
  - ✓ N=4 : massive regulator

- Cuts

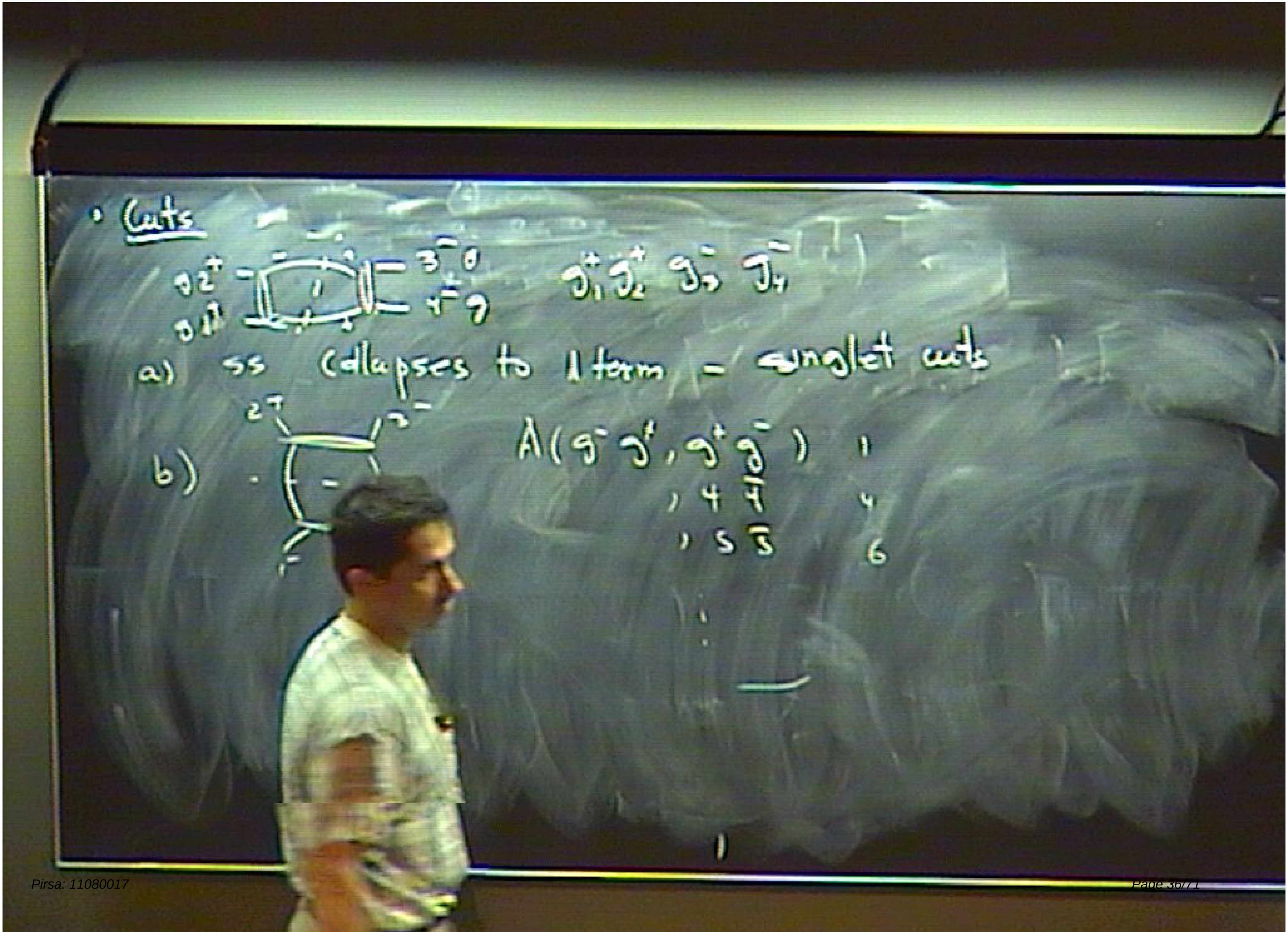
$g_1^+ \, g_2^+ \, g_3^- \, g_4^-$

a) ss collapses to 1 term

- **Cuts**

$g_1^+ g_2^+ g_3^- g_4^-$

a) ss collapses to 1 term

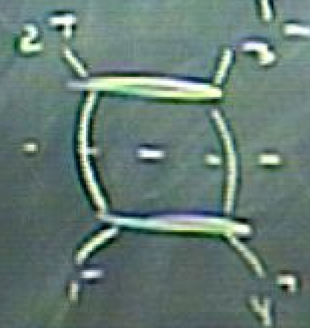

Cuts
$g_1^+ g_2^+ g_3^- g_4^-$
a) ss collapses to 1 term – singlet cuts
b)
$A(g^- g^+, g^+ g^-)$ 1
, 4 $\bar{4}$ 4
, 5 $\bar{5}$ 6

• Cuts

$g_1^+ \, g_2^+ \, g_3^- \, g_4^-$

a) ss collapses to 1 term – singlet cuts

b)



$A(g^- g^+, g^+ g^-)$ 1

$, \; 4 \; \bar{4}$ 4 – non-singlet

$, \; 5 \; \bar{5}$ 6   cuts

⋮

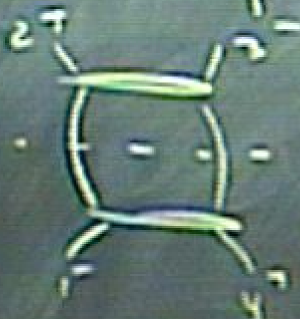
c) Super amp's.

• **Cuts**

$g_1^+ \, g_2^+ \, g_3^- \, g_4^-$

a) ss collapses to 1 term – singlet cuts

b)

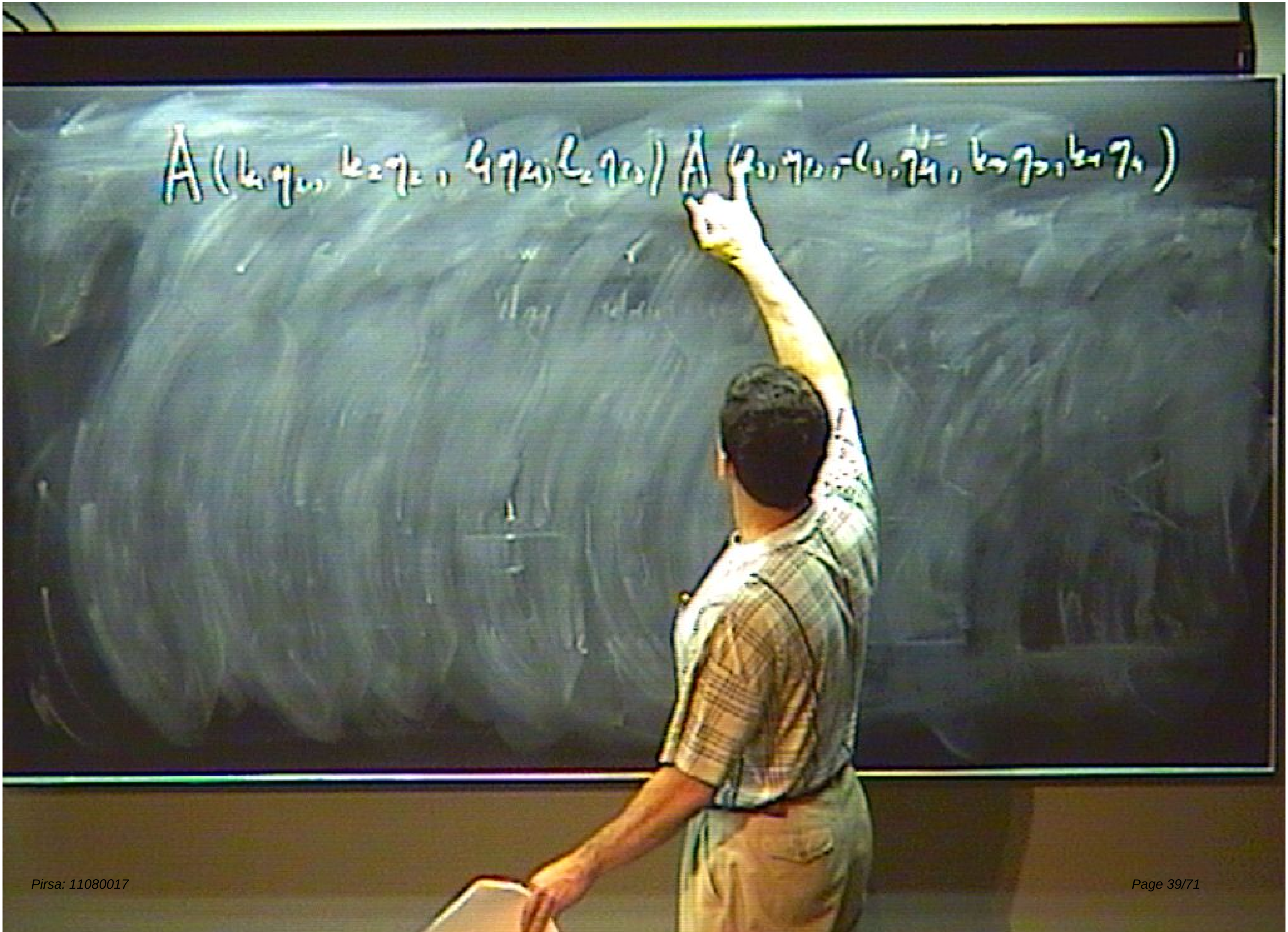


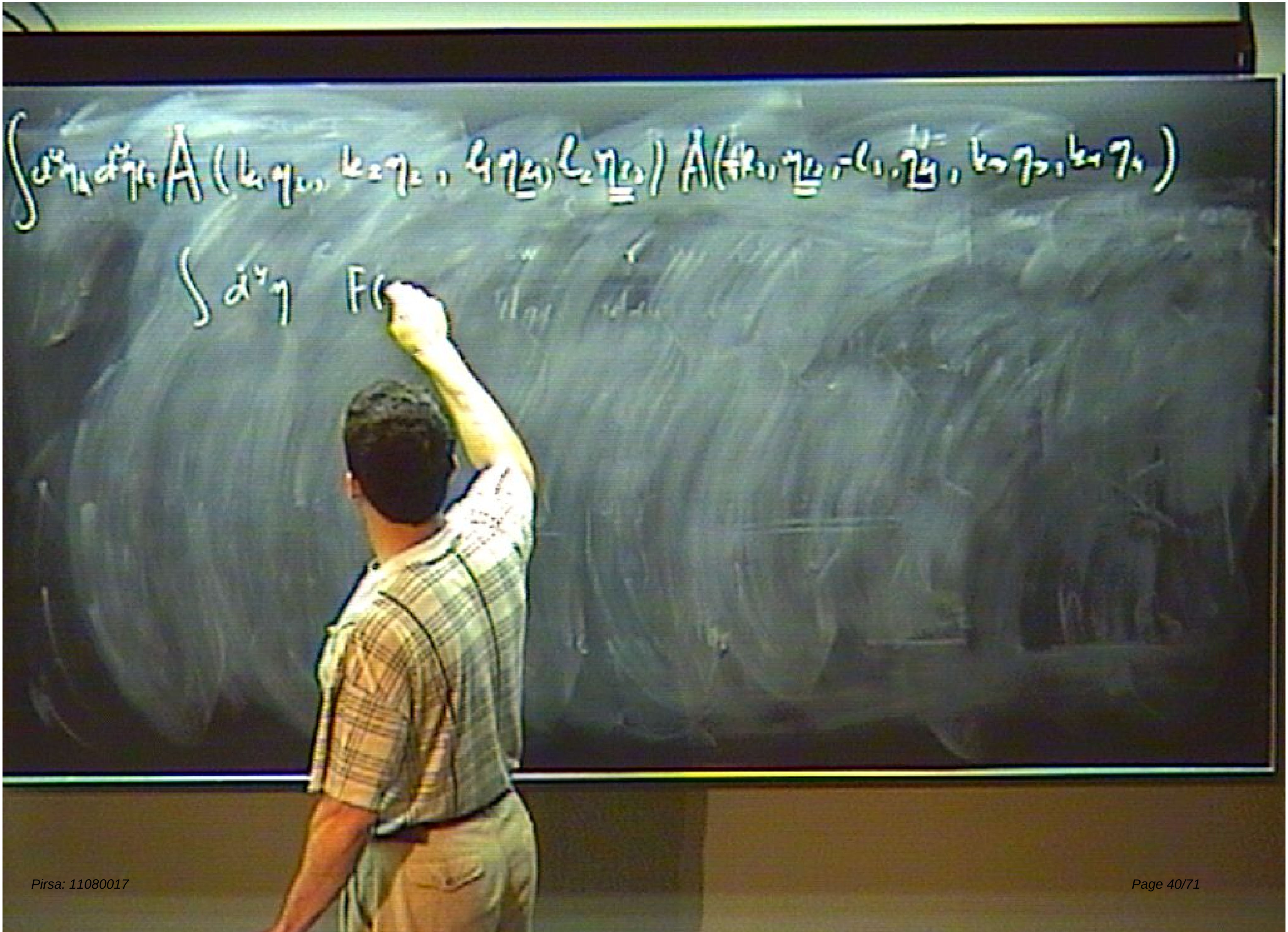
$A(g^- g^+, g^+ g^-)$ 1

$, 4 \, \bar{4}$ 4 – non-singlet cuts

$, 5 \, \bar{5}$ 6

c) Super amp's.




$\int d^4\eta \quad F($

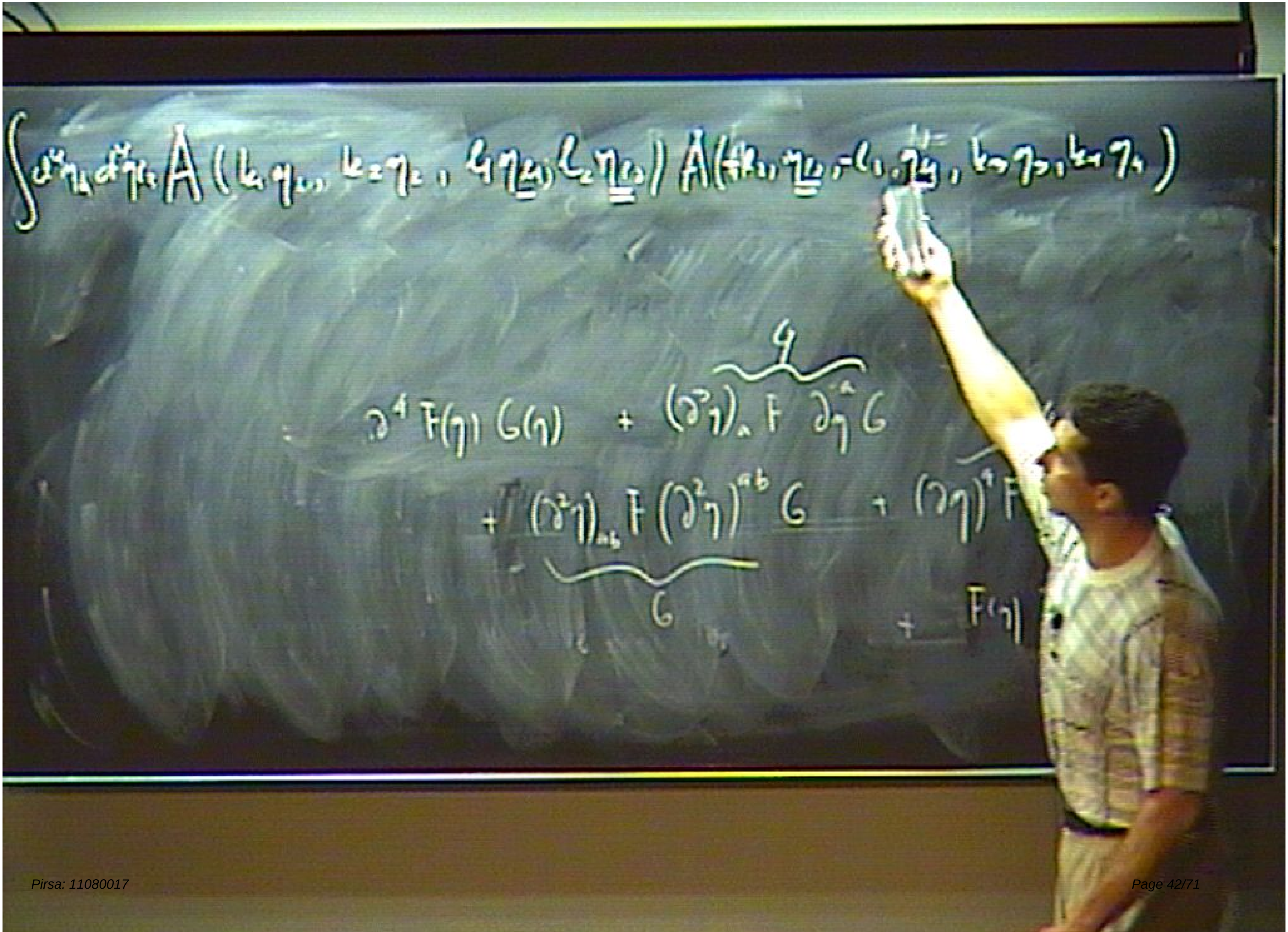
$$\int d^4\eta_{\ell_1} d^4\eta_{\ell_2}\, A\left(k_1\eta_1, k_2\eta_2, \ell_1\eta_{\ell_1}, \ell_2\eta_{\ell_2}\right) A\left(-\ell_2, \eta_{\ell_2}, -\ell_1, \eta_{\ell_1}, k_3\eta_3, k_4\eta_4\right)$$

$$\int d^4\eta \; F(\eta)\, G(\eta) =$$

$$= \partial^4 F(\eta)\, G(\eta) + \overbrace{(\partial^3\eta)_a F\, \partial_\eta^a G}^{4}$$

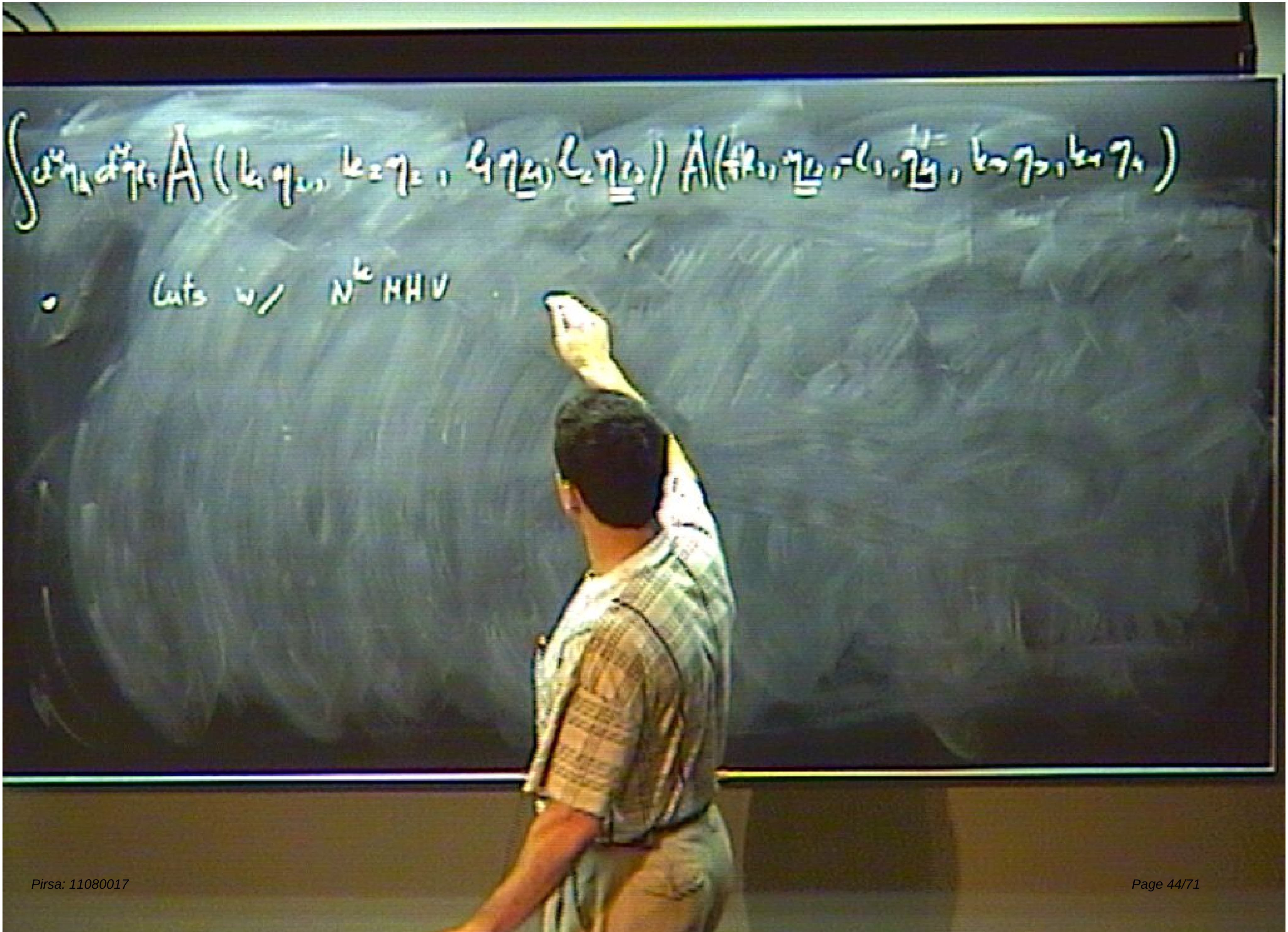
$$+ \underbrace{(\partial^2\eta)_{ab} F\, (\partial^2\eta)^{ab} G}_{6} + \overbrace{(\partial\eta)^a F\, (\partial^3\eta)_a G}^{4}$$

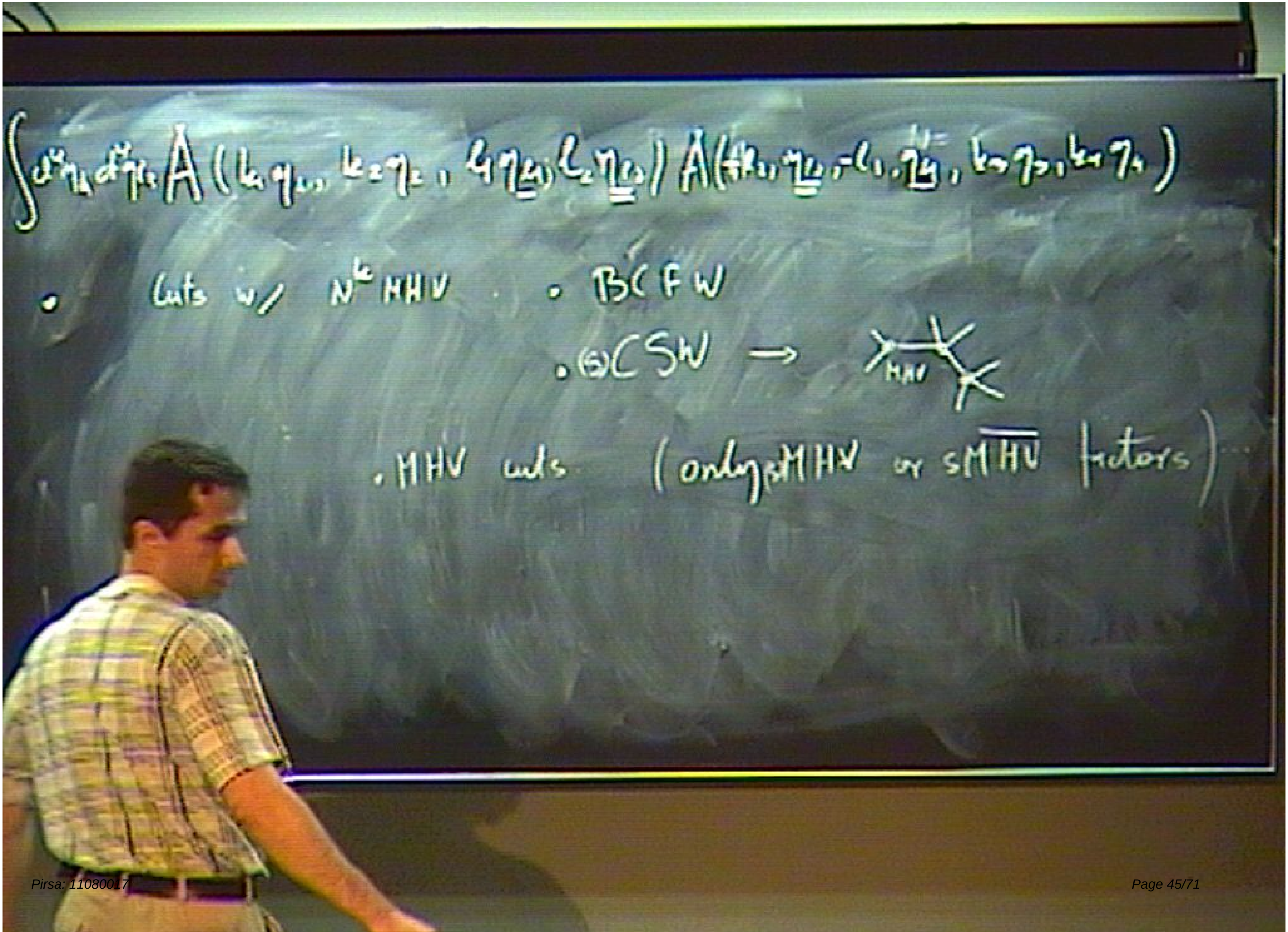
$$+ F(\eta)\, \partial_\eta^4 G$$



$\partial^4 F(\eta)\, G(\eta) + (\partial^3\eta)_a F\, \partial_\eta^a G$
$+ (\partial^2\eta)_{ab} F\, (\partial^2_\eta)^{ab} G + (\partial\eta)^4 F$
$+ F(\eta)$

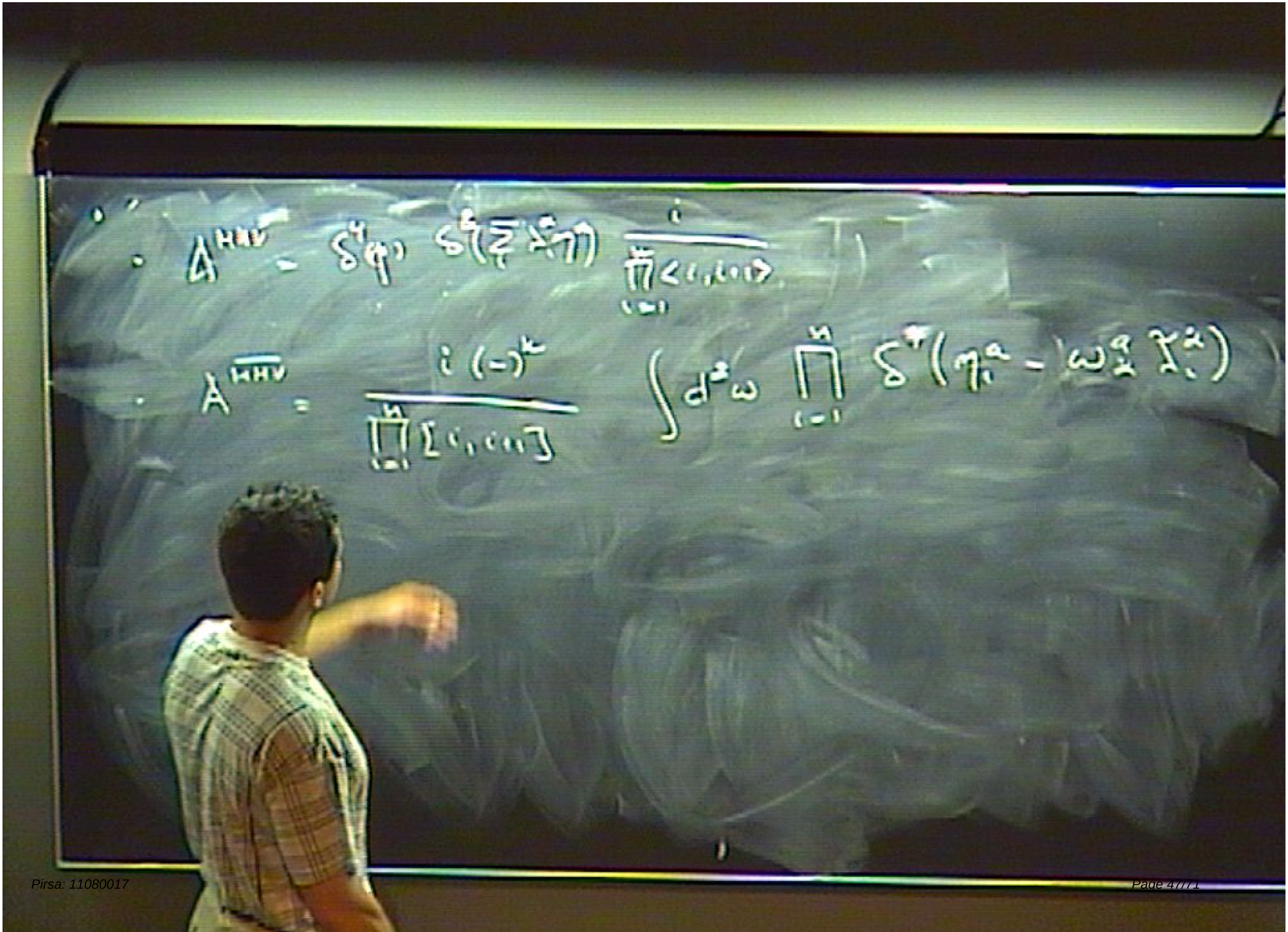
$$\int d^4\eta_{l_1}\, d^4\eta_{l_2}\, \mathring{A}\left(k_1\eta_1,\ k_2\eta_2,\ l_1\eta_{l_1},\ l_2\eta_{l_2}\right) \mathring{A}\left(-l_2,\ \eta_{l_2},\ -l_1,\ \eta_{l_1},\ k_3\eta_3,\ k_4\eta_4\right)$$

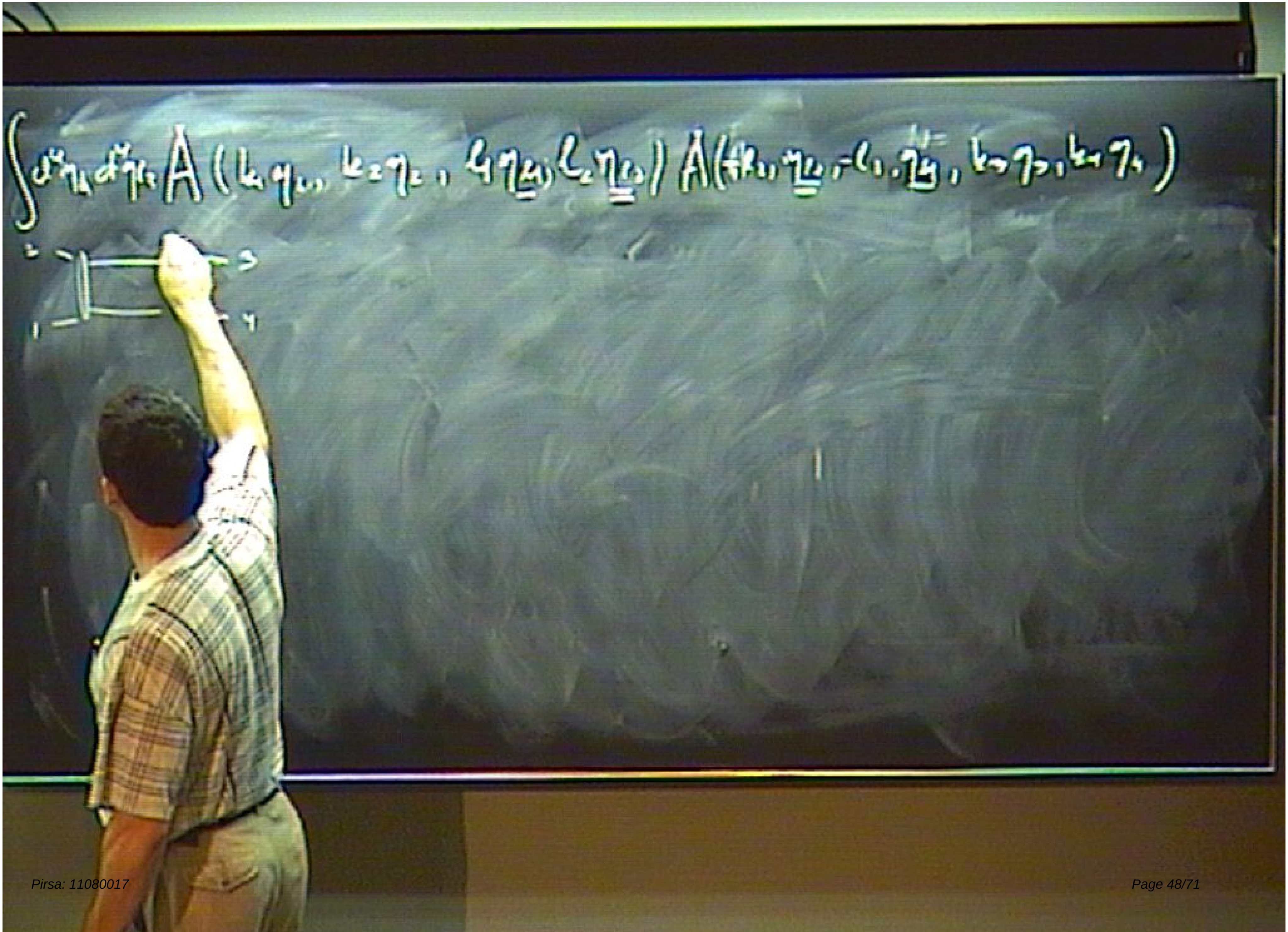

• Cuts w/ N^k MHV

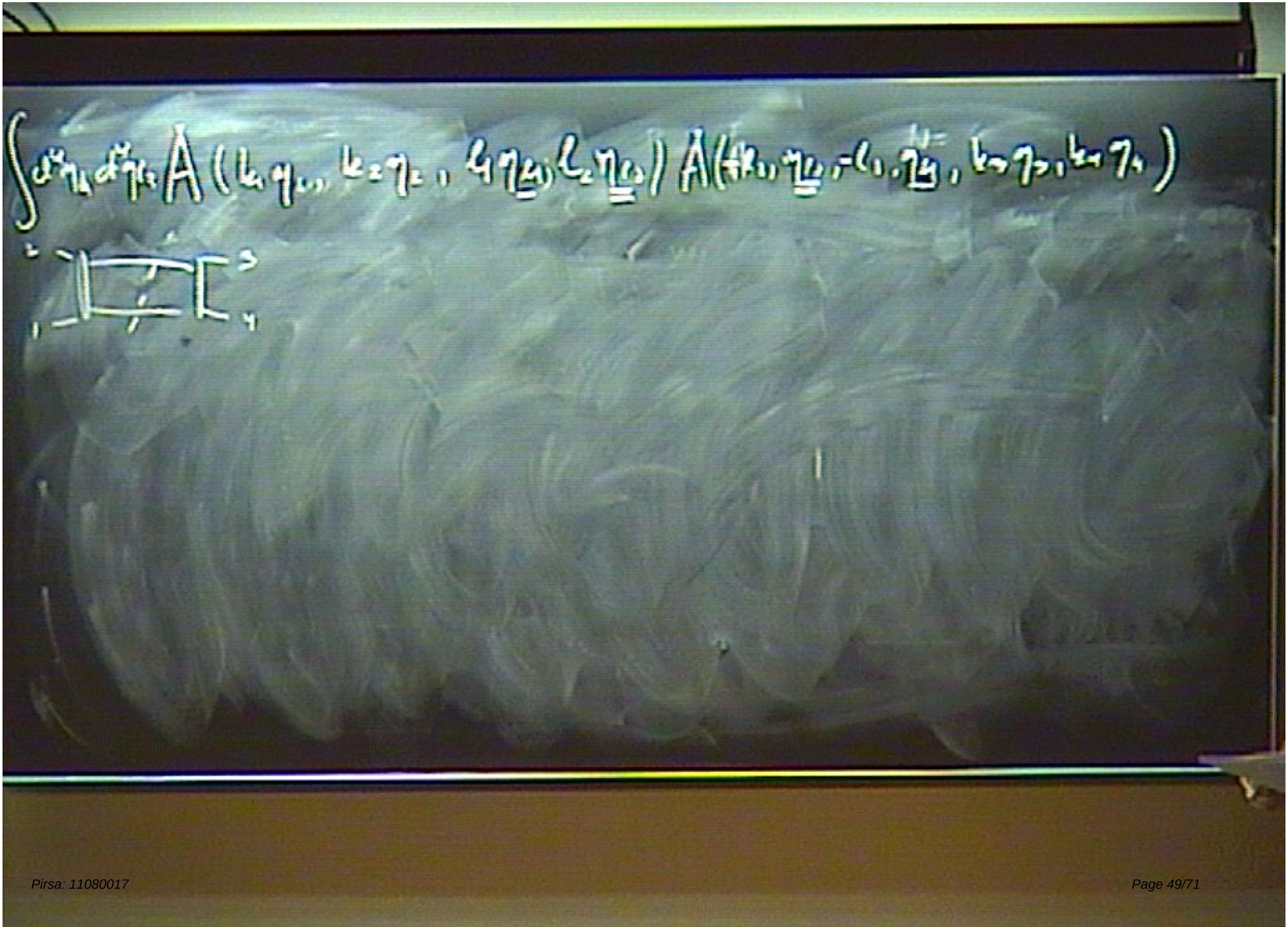

• Cuts w/ N^k MHV
• BCFW
• (S)CSW →
MHV
• MHV cuts (only sMHV or sMHV factors)

$$A^{MHV} = \delta^4(p)\,\delta^8(\textstyle\sum \lambda\eta)\,\frac{1}{\prod_{i=1}^{n}\langle i, i+1\rangle}$$

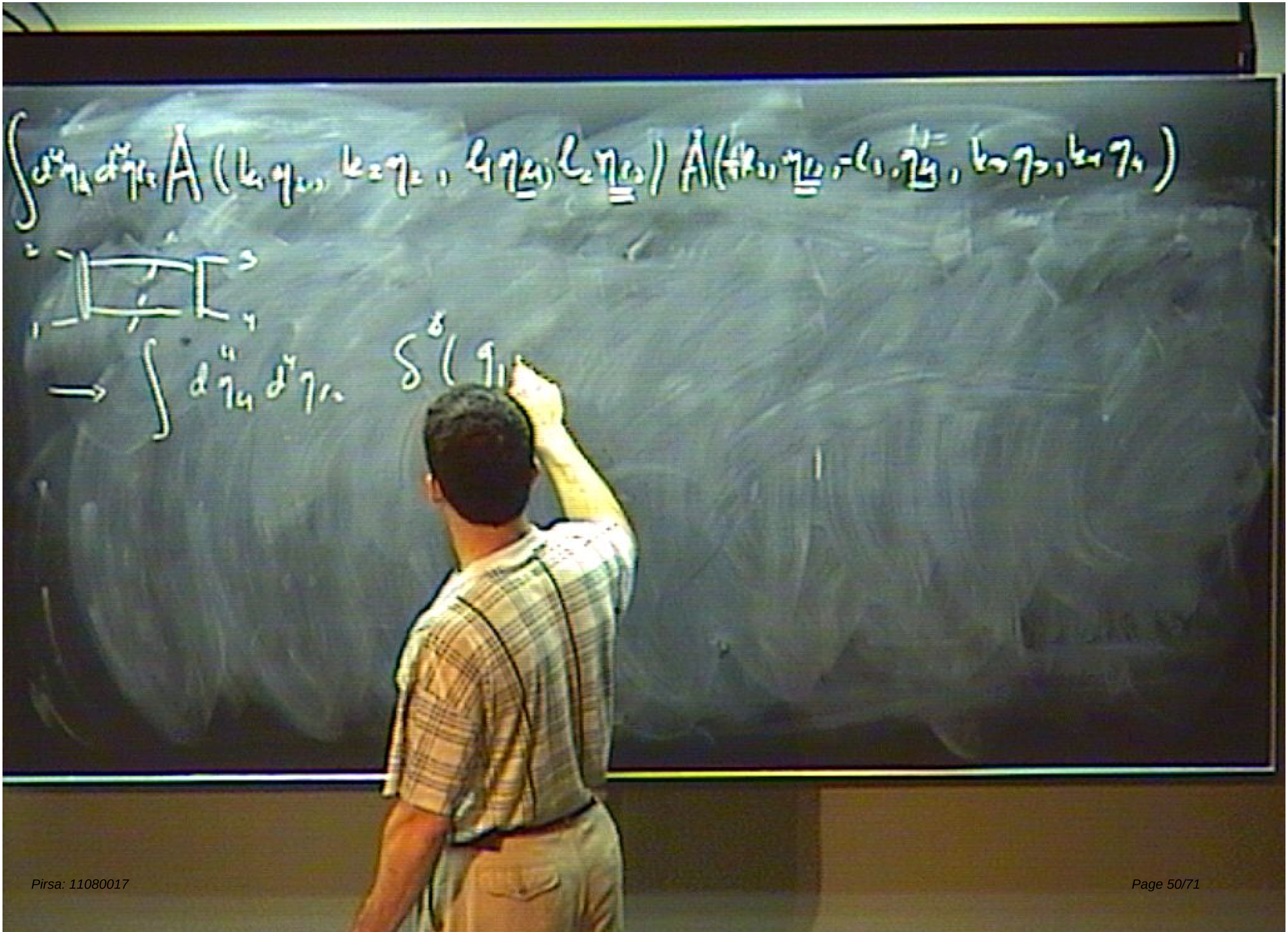
$$A^{\overline{MHV}}$$


$A^{MHV} = \delta^4(q)\,\delta^8(\xi\cdot\eta)\,\frac{1}{\prod_{i=1}^{n}\langle i,i+1\rangle}$
$A^{\overline{MHV}} = \frac{i(-)^k}{\prod_{i=1}^{n}[i,i+1]}\int d^8\omega \prod_{i=1}^{n}\delta^4(\eta_i^a - \omega_\alpha^a \xi_i^\alpha)$





∫ d⁴η_{l₁} d⁴η_{l₂} A(k₁ η₁, k₂ η₂, l₁ η_{l₁}, l₂ η_{l₂}) A(-l₂, η_{l₂}, -l₁, η_{l₁}, k₃ η₃, k₄ η₄)

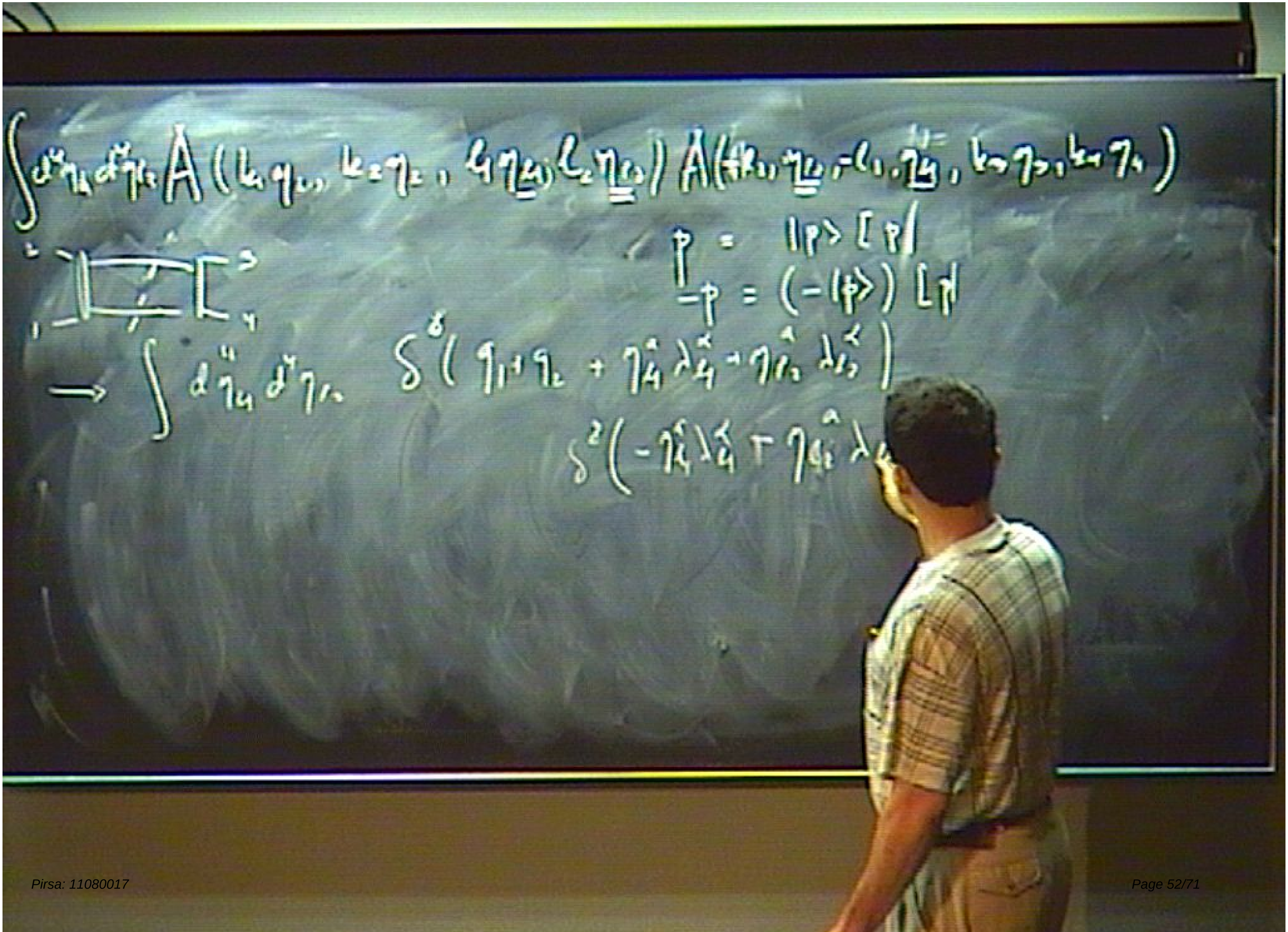


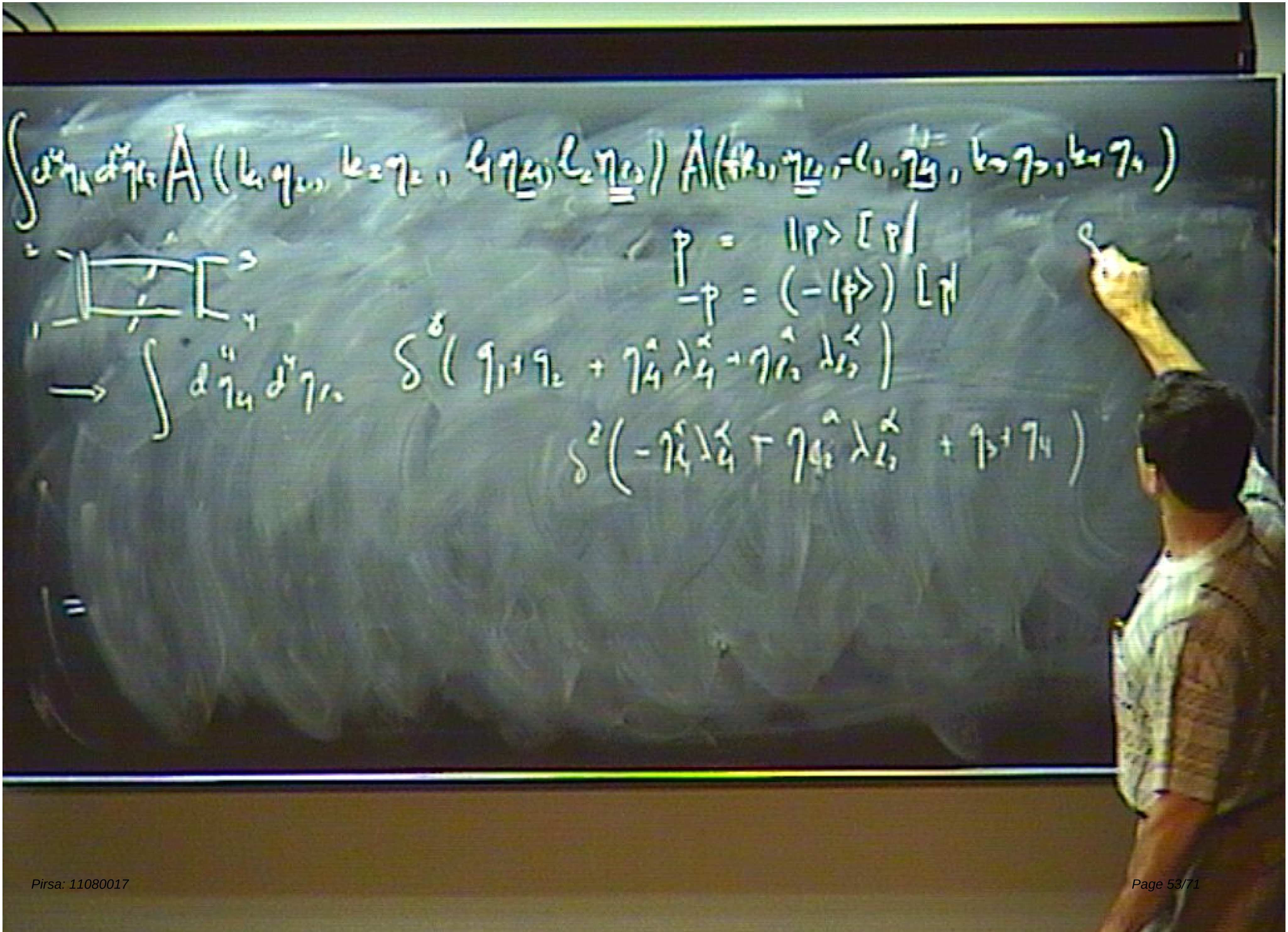
$$\int d^4\eta_{l_1} d^4\eta_{l_2}\, A\big(k_1\eta_1,\ k_2\eta_2,\ l_1\eta_{l_1},\ l_2\eta_{l_2}\big)\, A\big(-l_1,\eta_{l_1},\ -l_2,\eta_{l_2},\ k_3\eta_3,\ k_4\eta_4\big)$$

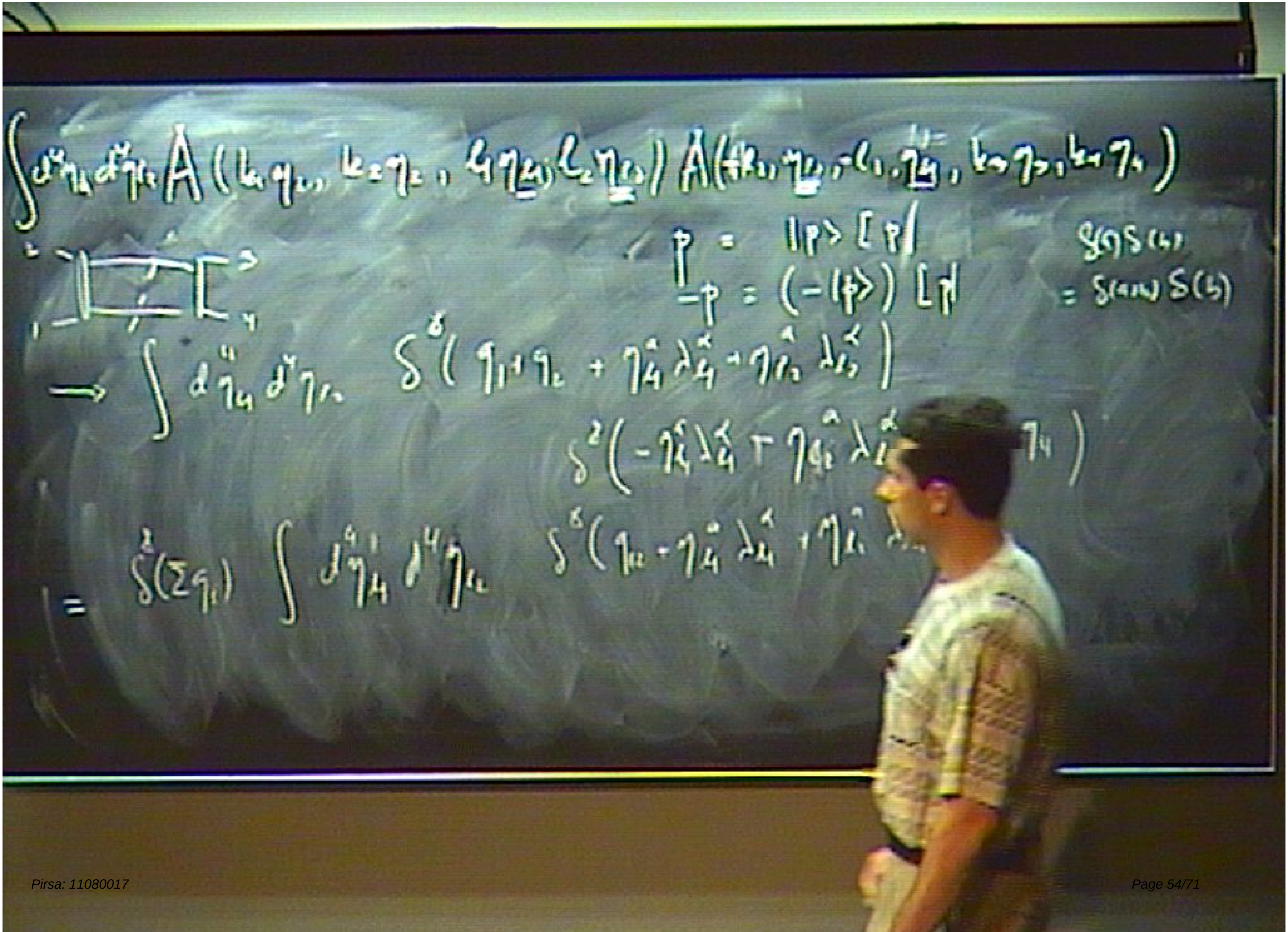
$$p = |p\rangle [p|$$

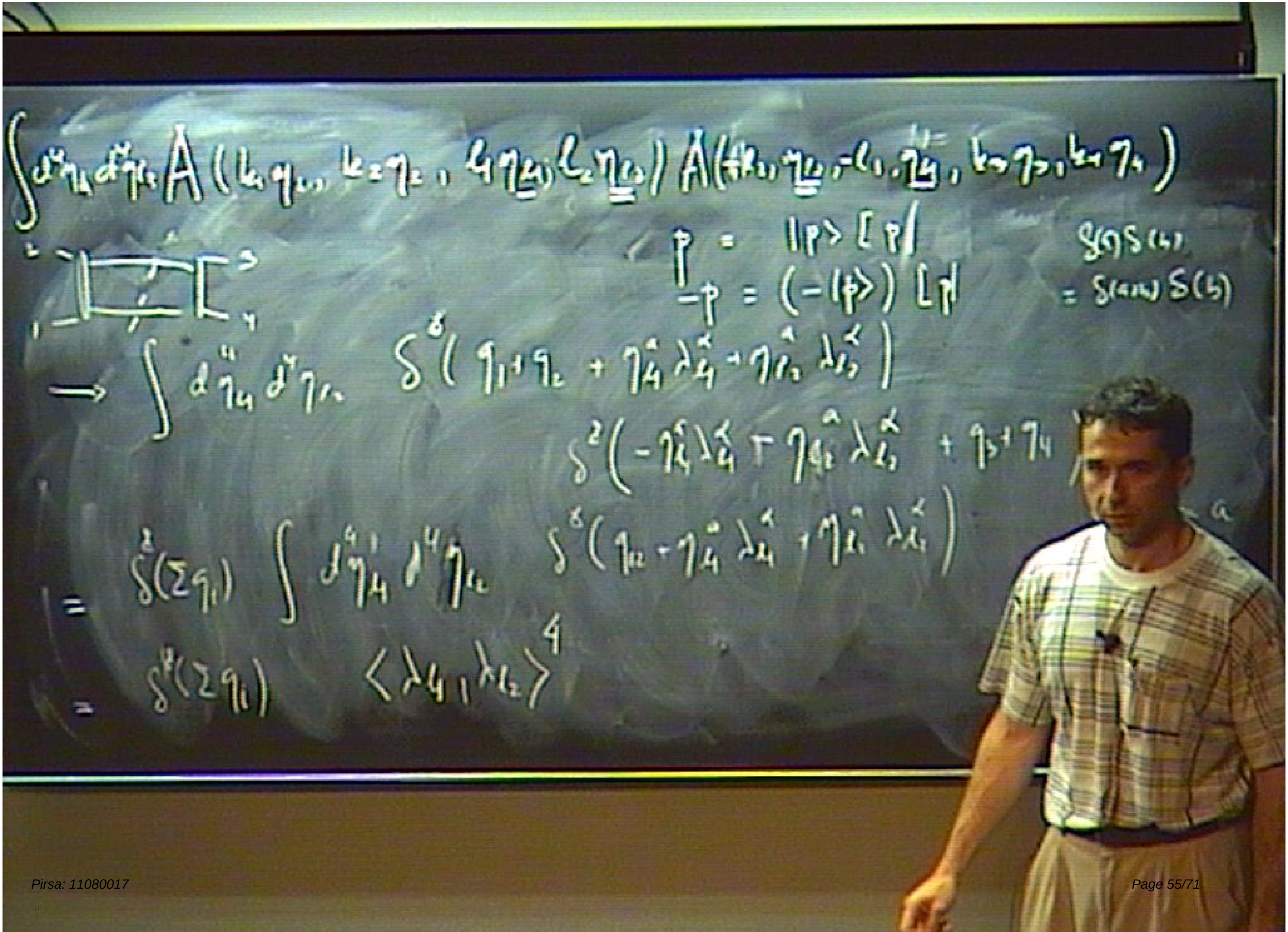
$$\longrightarrow \int d^4\eta_{l_1} d^4\eta_{l_2}\ \delta^8\big(\eta_1 + \eta_2 + \eta^a_{l_1}\lambda^\alpha_{l_1} + \eta^a_{l_2}\lambda^\alpha_{l_2}\big)$$

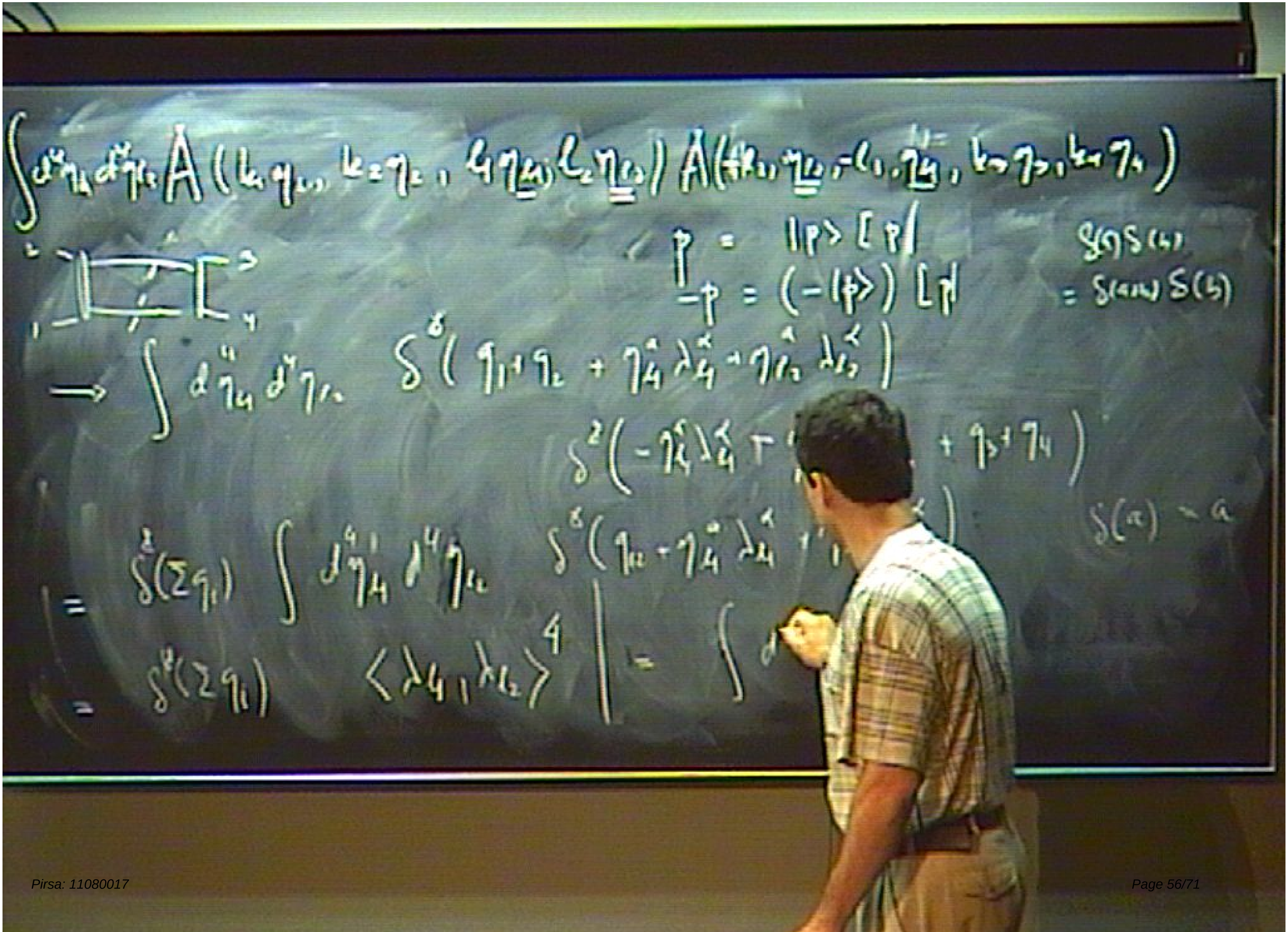
$$\delta^2\big($$


p = |p> [p|
-p = (-|p>) [p|


p = |p⟩ [p|
−p = (−|p⟩) [p|


p = |p> [p|
−p = (−|p>) [p|


p = |p> [p|
−p = (−|p>) [p|
= δ⁸(Σηi) ⟨λℓ4, λℓ2⟩⁴


p = |p> [p|
-p = (-|p>) [p|
= δ(a+b) δ(b)
δ(x) = a
=
=

$$\int d^4\eta_{l_4}\, d^4\eta_{l_2}\, A(l_1\eta_1,\ k_2\eta_2,\ l_4\eta_{l_4},\ l_2\eta_{l_2})\ A(-l_2,\eta_{l_2},\ -l_4,\eta_{l_4},\ k_3\eta_3,\ k_4\eta_4)$$

$$p = |p\rangle[p|$$

$$-p = (-|p\rangle)[p|$$

$$\delta(\eta)\,\delta(u) = \delta(u+\eta)\,\delta(5)$$

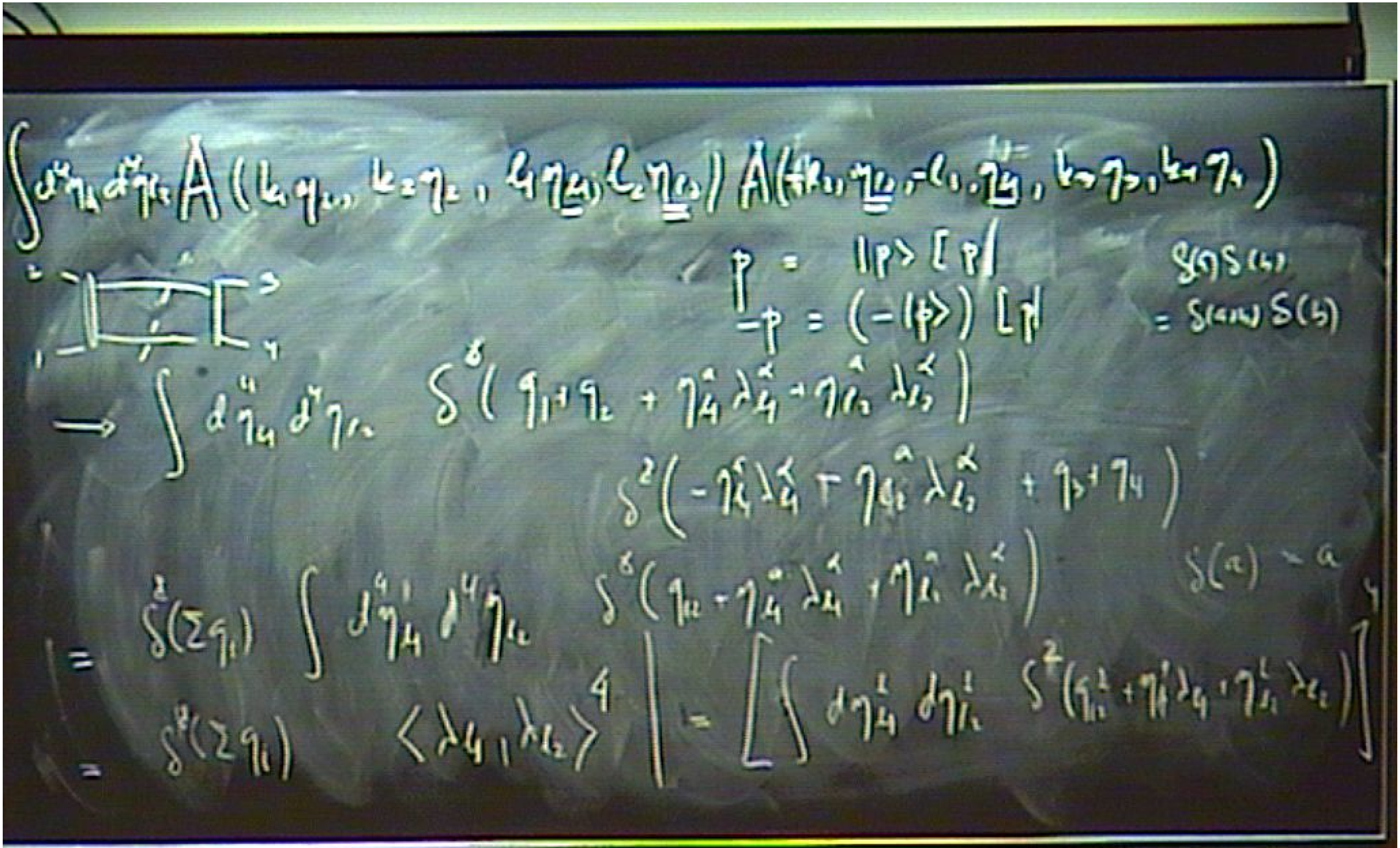
$$\longrightarrow \int d^4\eta_{l_4}\, d^4\eta_{l_2}\ \delta^8(\eta_1+\eta_2+\eta_{l_4}^a\lambda_{l_4}^\alpha+\eta_{l_2}^a\lambda_{l_2}^\alpha)$$

$$\delta^8(-\eta_{l_4}^a\lambda_{l_4}^\alpha+\eta_{l_2}^a\lambda_{l_2}^\alpha+\eta_3+\eta_4)$$

$$\delta(a) = a$$

$$= \delta^8(\Sigma\eta_i)\int d^4\eta_{l_4}\, d^4\eta_{l_2}\ \delta^8(\eta_{12}+\eta_{l_4}^a\lambda_{l_4}^\alpha+\eta_{l_2}^a\lambda_{l_2}^\alpha)$$

$$= \delta^8(\Sigma\eta_i)\ \langle\lambda_{l_4},\lambda_{l_2}\rangle^4 \quad \Big| = \Big[\int d\eta_{l_4}^1\, d\eta_{l_2}^1\ \delta^2(\eta_{12}^1+\eta_{l_4}^1\lambda_{l_4}+\eta_{l_2}^1\lambda_{l_2})\Big]^4$$


$\int d^4\eta_{\ell_1} d^4\eta_{\ell_2} A(\lambda_1\eta_1, \lambda_2\eta_2, \ell_1\eta_{\ell_1}, \ell_2\eta_{\ell_2}) A(-\ell_2, \eta_{\ell_2}, -\ell_1, \eta_{\ell_1}, \lambda_3\eta_3, \lambda_4\eta_4)$
$p = |p\rangle [p|$
$-p = (-|p\rangle) [p|$
$\delta(\eta)\delta(\zeta) = \delta(\zeta\eta)\delta(\zeta)$
$\to \int d^4\eta_{\ell_1} d^4\eta_{\ell_2}\ \delta^8(\eta_1 + \eta_2 + \eta_{\ell_1}^a \lambda_{\ell_1}^\alpha + \eta_{\ell_2}^a \lambda_{\ell_2}^\alpha)$
$\delta^8(-\eta_{\ell_1}^a \lambda_{\ell_1}^\alpha - \eta_{\ell_2}^a \lambda_{\ell_2}^\alpha + \eta_3 + \eta_4)$
$= \delta^8(\Sigma\eta_i) \int d^4\eta_{\ell_1} d^4\eta_{\ell_2}\ \delta^8(\eta_{12} + \eta_{\ell_1}^a \lambda_{\ell_1}^\alpha + \eta_{\ell_2}^a \lambda_{\ell_2}^\alpha)$
$\delta(a) = a$
$= \delta^8(\Sigma\eta_i)\ \langle \lambda_{\ell_1}, \lambda_{\ell_2}\rangle^4$
$= \left[\int d\eta_{\ell_1}^1 d\eta_{\ell_2}^1\ \delta^2(\eta_{12}^1 + \eta_{\ell_1}^1\lambda_{\ell_1} + \eta_{\ell_2}^1\lambda_{\ell_2})\right]^4$

Aspects of generalized unitarity

✓ • tree-level comments

• generalized unitarity: generalities ✓, sums ✓
  regularization ✓, type of cuts, etc

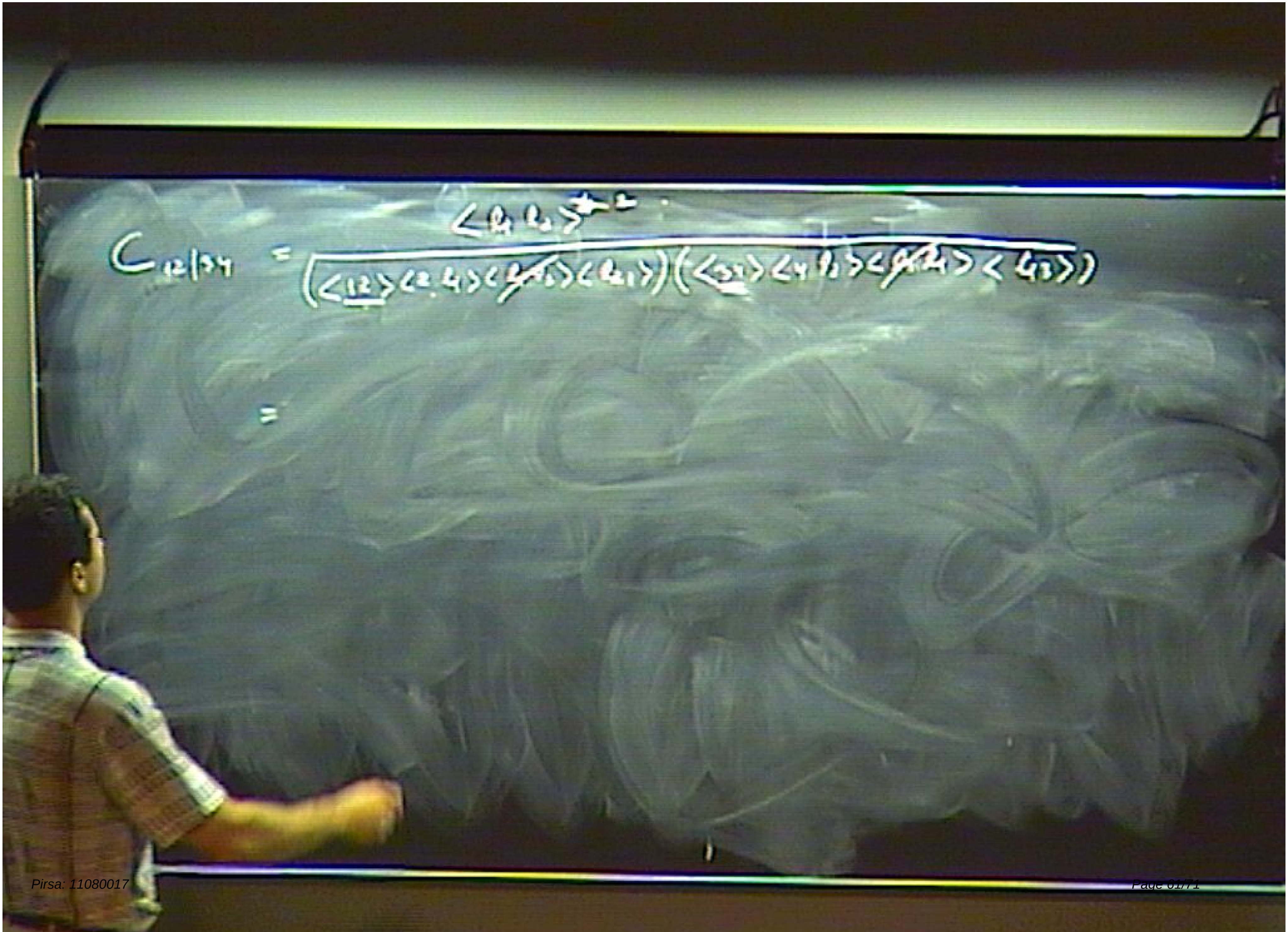
• Supersums; example, general props.

• numerics vs. analytics

• general strategy for supersums.

• ....

$$C_{12|34} = \frac{\langle \ell_1 \ell_2 \rangle^4}{\left(\langle 12\rangle \langle 2\, \ell_1\rangle \langle \ell_1 \ell_2\rangle \langle \ell_2 1\rangle\right)\left(\langle 34\rangle \langle 4\, \ell_2\rangle \langle \ell_2 \ell_1\rangle \langle \ell_1 3\rangle\right)}$$



$$C_{12|34} = \frac{\langle \ell_1 \ell_2 \rangle^{\cancel{4}\,2}}{\left(\langle \underline{12} \rangle \langle 2\, \ell_1 \rangle \langle \cancel{\ell_1 \ell_2} \rangle \langle \ell_2 1 \rangle\right)\left(\langle \underline{34} \rangle \langle 4\, \ell_1 \rangle \langle \cancel{\ell_1 \ell_2} \rangle \langle \ell_2 3 \rangle\right)}$$

$$= \frac{[34]^2}{\Delta_{12} \langle 12 \rangle^2} \; \frac{\mathrm{Tr}_+\left(\ell_1 k_2 k_1 \ell_2 k_4 k_3 \right)}{(2 k_2 \cdot \ell_1)(2 k_1 \cdot \ell_2)(2 k_3 \cdot \ell_1)(2 k_4 \cdot \ell_2)}$$

$$C_{12|34} = \frac{\langle \ell_1 \ell_2 \rangle^{2}}{\left(\langle \underline{12} \rangle \langle 2\, \ell_1 \rangle \langle \cancel{\ell_1 \ell_2} \rangle \langle \ell_2 1 \rangle\right)\left(\langle \underline{34} \rangle \langle 4\, \ell_2 \rangle \langle \cancel{\ell_2 \ell_1} \rangle \langle \ell_1 3 \rangle\right)}$$

$$= \frac{[34]^2}{\Delta_{12} \langle 12 \rangle^2} \; \frac{\mathrm{Tr}_+\left(\not{\ell}_1 \not{k}_2 \not{k}_1 \not{\ell}_2 \not{k}_4 \not{k}_3 \right)}{(2 k_2 \cdot \ell_1)(2 k_1 \cdot \ell_2)(2 k_3 \cdot \ell_1)(2 k_4 \cdot \ell_2)}$$

$$\frac{1}{\langle i\, j \rangle} = \frac{[j\, i]}{2 k_i \cdot k_j}$$

$$C_{12|34} = \frac{\langle \ell_1 \ell_2 \rangle^{2}}{\left(\langle 12\rangle\langle 2\,\ell_1\rangle\cancel{\langle \ell_1 \ell_2\rangle}\langle \ell_2 1\rangle\right)\left(\langle 34\rangle\langle 4\,\ell_1\rangle\cancel{\langle \ell_1 \ell_2\rangle}\langle \ell_2 3\rangle\right)}$$

$$= \frac{[34]^2}{\Delta_{12}\,\langle 12\rangle^2}\;\frac{\mathrm{Tr}_+\left(\not{\ell}_1 \not{k}_2 \not{k}_1 \not{\ell}_2 \not{k}_4 \not{k}_3\right)}{(2k_2\cdot\ell_1)(2k_1\cdot\ell_2)(2k_3\cdot\ell_1)(2k_4\cdot\ell_2)}$$

$$\frac{1}{\langle i\,j\rangle} = \frac{[j\,i]}{2k_i\cdot k_j}$$

$$\langle i\,j\rangle[j\,k\rangle\langle k\,\ell\rangle\cdots \longrightarrow \mathrm{Tr}_+\left(\not{k}\not{k}\not{k}\not{k}\cdots\right)$$

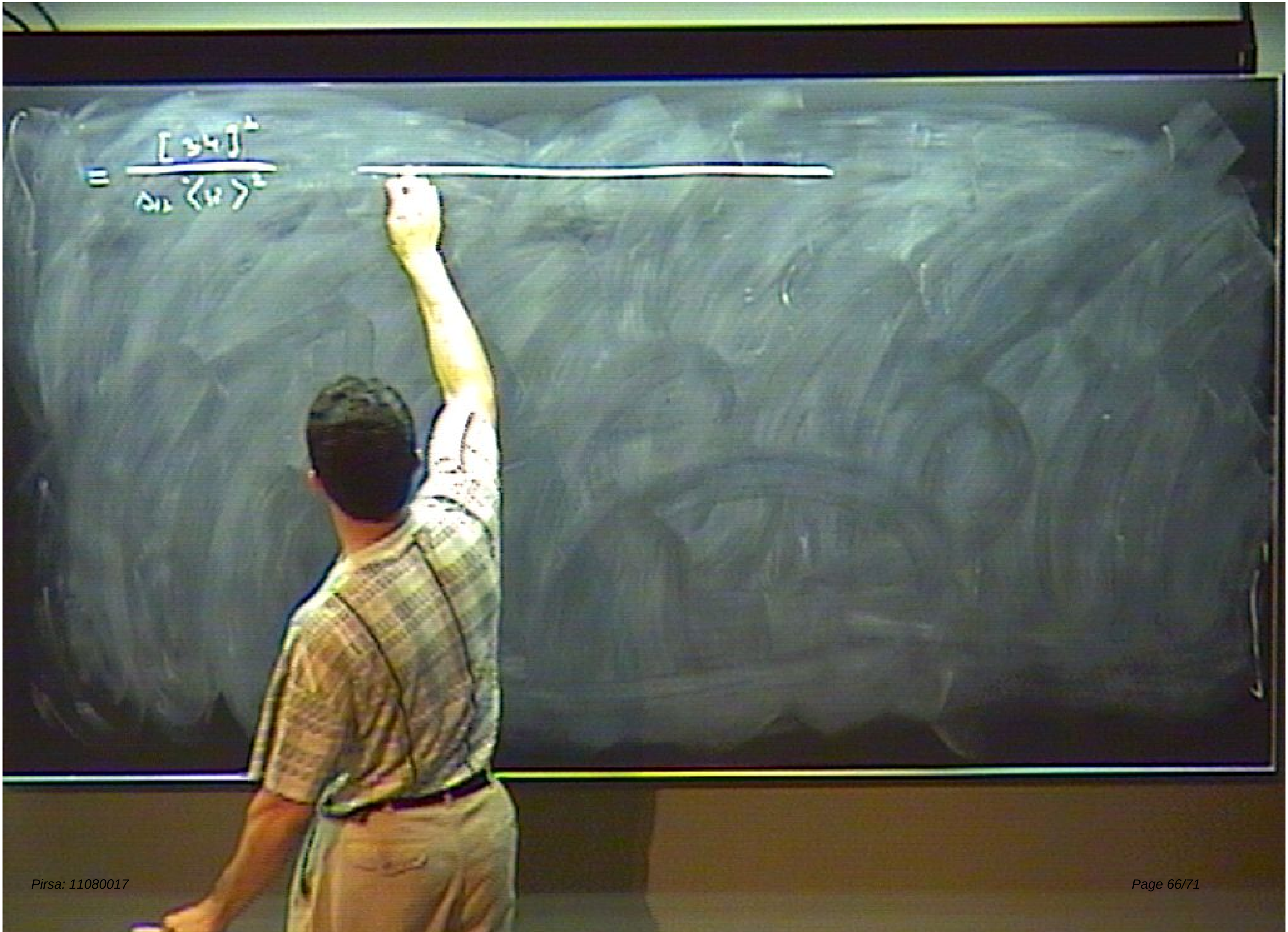
$$C_{12|34} = \frac{\langle \ell_1 \ell_2\rangle^{2}}{\left(\langle 12\rangle\langle 2\,\ell_1\rangle\langle \ell_1 \ell_2\rangle\langle \ell_2 1\rangle\right)\left(\langle 34\rangle\langle 4\,\ell_1\rangle\langle \ell_1 \ell_2\rangle\langle \ell_2 3\rangle\right)}$$

$$= \frac{[34]^2}{\Delta_{12}\,\langle 12\rangle^2}\;\frac{\mathrm{Tr}_+\left(\ell_1 k_2 k_1 \ell_2 k_4 k_3\right)}{(2k_2\cdot\ell_1)(2k_1\cdot\ell_2)(2k_3\cdot\ell_1)(2k_4\cdot\ell_2)}$$

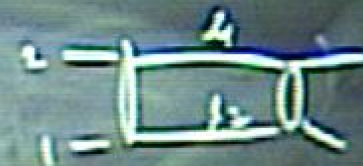
$$\frac{1}{\langle i\,j\rangle} = \frac{[j\,i]}{2k_i\cdot k_j}$$

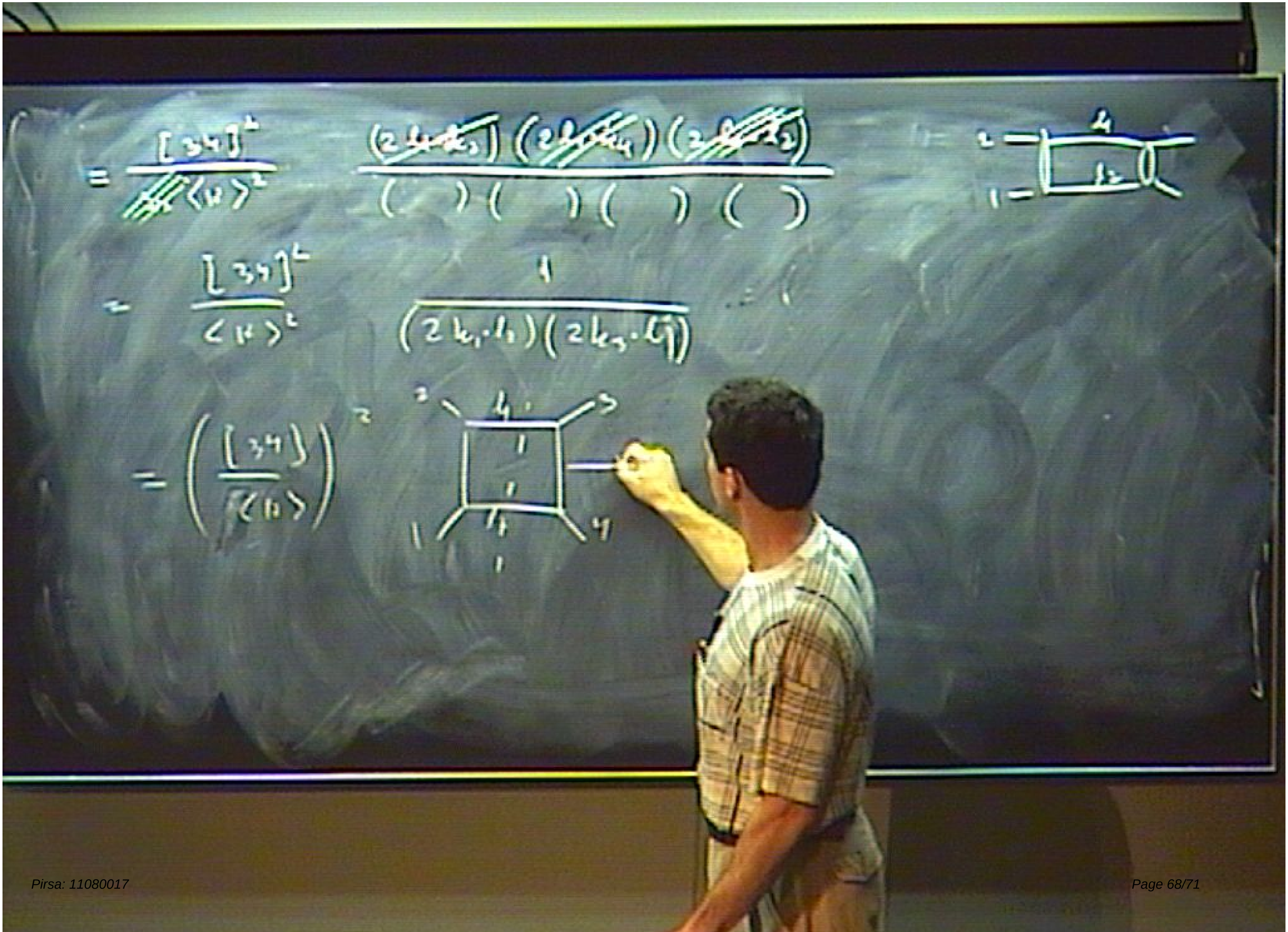
$$\langle i\,j\rangle[j\,k)\langle k\,\ell\rangle \cdots \longrightarrow \mathrm{Tr}_+\left(k\,k\,k\,k \cdots\right)$$

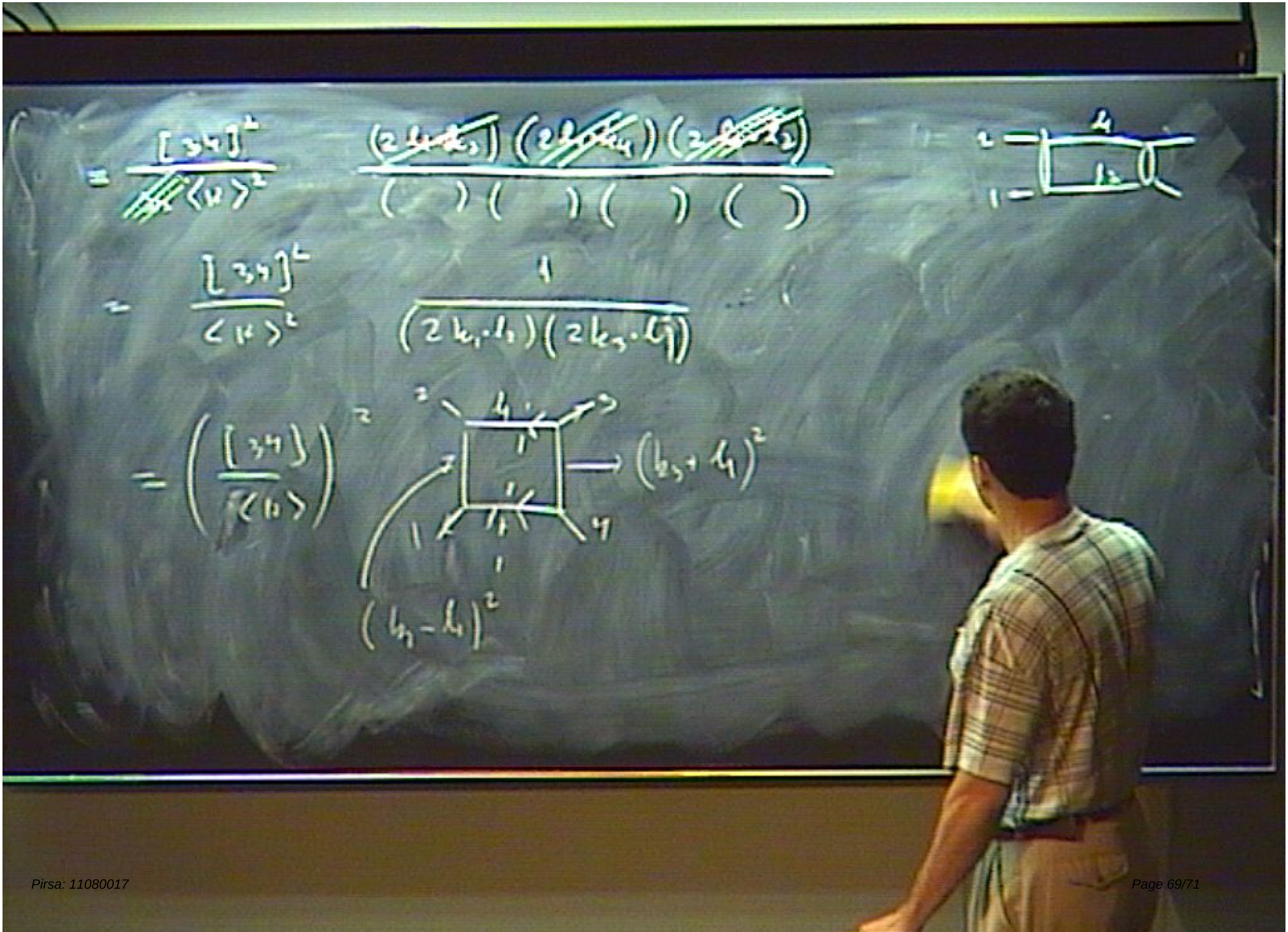
$$\langle \ell_1 k_2\rangle \cdots \langle k_2\,\ell_2\rangle[\ell_2\,k_4]\langle k_4 k_3\rangle[k_3\,\ell_1].$$

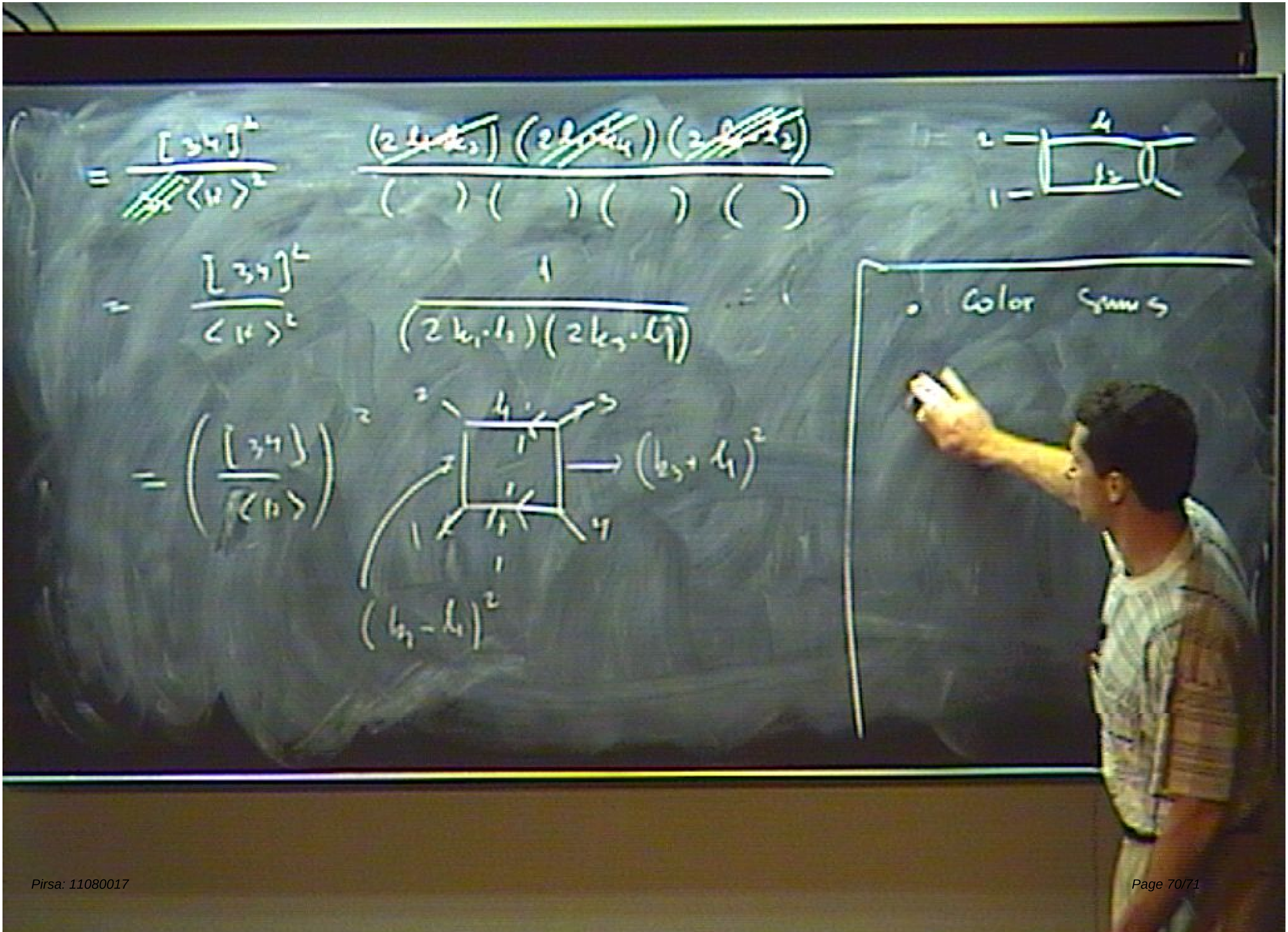


$$= \frac{[34]^2}{s_{12}\langle 12\rangle^2} \qquad \frac{\cancel{(2\,l_4\cdot k_3)}\,(2\,l_1\cdot k_4)\,(2\,l_1\cdot l_2)}{(\quad)(\quad)(\quad)(\quad)}$$




= [34]^2 / <12>^2
= [34]^2 / <12>^2 · 1 / (2k₁·l₁)(2k₃·l₃)
= ([34] / <12>)^2




$= \frac{[34]^2}{\langle 12 \rangle^2} \frac{1}{(2k_1 \cdot l_1)(2k_3 \cdot l_1)}$
$= \left(\frac{[34]}{\langle 12 \rangle}\right)^2$
$(k_3 + l_1)^2$
$(k_1 - l_1)^2$
• Color Sums