Title: Symbology and Scattering Amplitudes

Date: Aug 05, 2011 09:30 AM

URL: http://pirsa.org/11080014

Abstract:

Functional equations

- $\ln(xy) = \ln x + \ln y$
- Identities in one variable

$$\mathrm{Li}_2(z^{-1}) = -\mathrm{Li}_2(z) - \frac{1}{2}\ln^2(-z) - \frac{\pi^2}{6}$$

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \ln z \ln(1-z) + \frac{\pi^2}{6}$$

$$\mathrm{Li}_3\left(\frac{z}{1-z}\right) - \mathrm{Li}_3\left(\frac{1}{1-z}\right) = \mathrm{Li}_3(z) + \cdots$$

- inversion identity

$$\mathrm{Li}_n(z) + (-)^n \mathrm{Li}_n(z^{-1}) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-)^k \left(1-2^{1-2k}\right) B_{2k}}{(2k)!\,(n-2k)!} (2\pi)^{2k} \ln^{n-2k}(-z)$$

B_{2k}: Bernoulli numbers

$$\mathcal{S}\left(Li_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$+ (1-xy)\otimes\left(\frac{y}{1-y}\right) =$$

$$= \left(1+\frac{(1-x)y}{1-y}\right)\otimes\left(\frac{(1-x)y}{1-y}\right) + \left(1+\frac{y}{1-y}\right)\otimes\left(\frac{y}{1-y}\right) +$$

$$+ (1-x)\otimes(1-y) + (1-y)\otimes(1-x)$$

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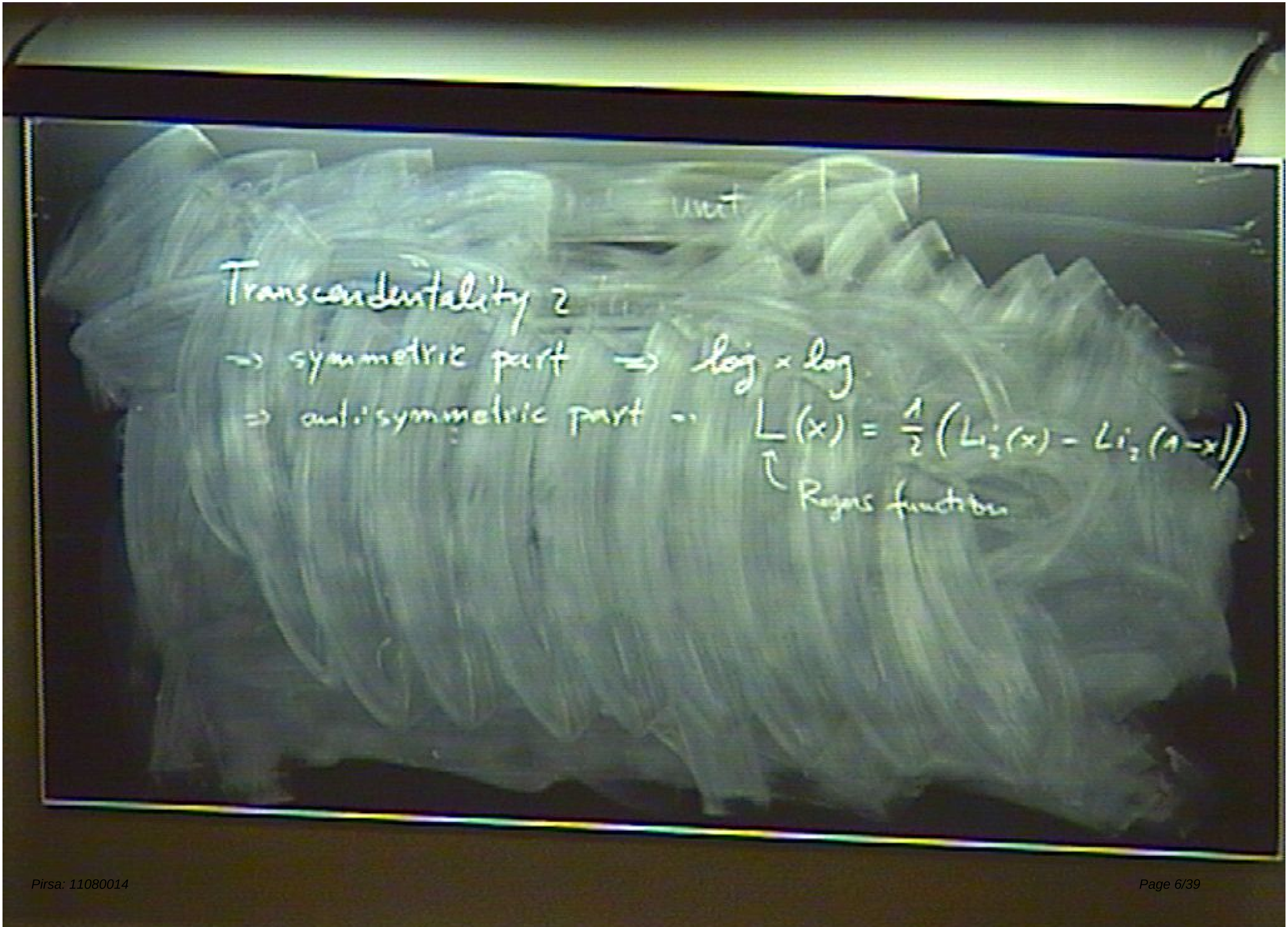
$$+ (1-x)\otimes(1-y) + (1-y)\otimes(1-x)$$

$$= \mathcal{S}\left(-Li_2\left(-\frac{(1-x)y}{1-y}\right) - Li_2\left(-\frac{y}{1-y}\right) + \ln(1-x)\ln(1-y)\right)$$

$$\mathcal{S}\left(Li_2(x)\right) = -(1-x)\otimes x$$

$$\mathcal{S}(\ln x \ln y) = x\otimes y + y\otimes x$$


Transcendentality 2
⇒ symmetric part ⇒ log × log
⇒ antisymmetric part ⇒ L(x) = 1/2 (Li2(x) − Li2(1−x))
Rogers function

$$\mathcal{S}\left(Li_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$+ (1-xy)\otimes\left(\frac{y}{1-y}\right) =$$

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$$+ (1-x)\otimes(1-y) + (1-y)\otimes(1-x)$$

$$= \mathcal{S}\left(-Li_2\left(-\frac{(1-x)y}{1-y}\right) - Li_2\left(-\frac{y}{1-y}\right) + \ln(1-x)\ln(1-y)\right)$$

Regular series expansion around $(x,y)=(0,0)$

$$\mathcal{S}(Li_2(x)) = -(1-x)\otimes x$$

$$\mathcal{S}(\ln x \ln y) = x\otimes y + y\otimes x$$

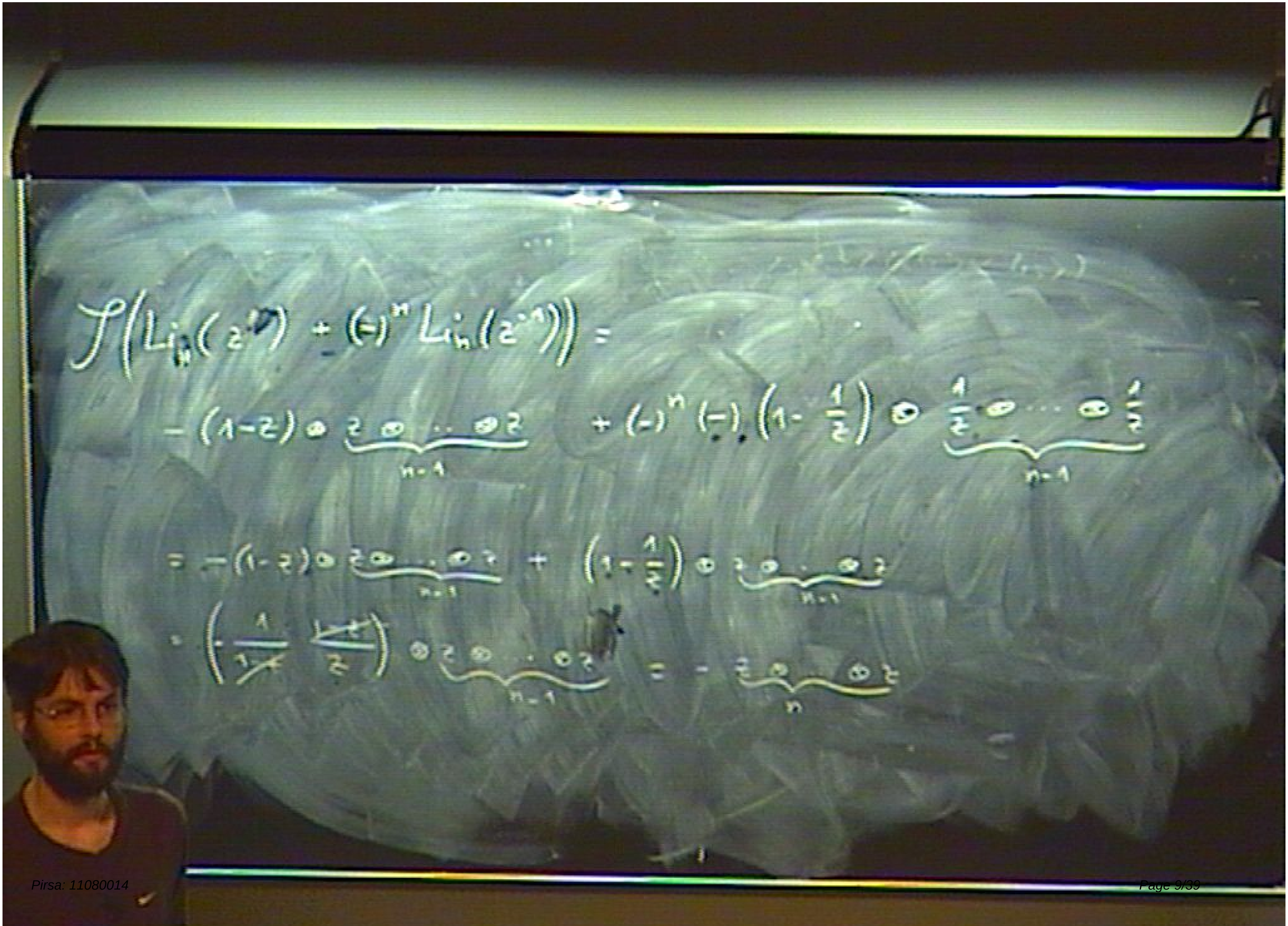
Transcendentality 2

$\Rightarrow$ symmetric part $\Rightarrow$ log × log

$\Rightarrow$ antisymmetric part $\rightarrow$ $L(x) = \frac{1}{2}\left(Li_2(x) - Li_2(1-x)\right)$

Rogers function

$$Li_{1,1}(x,y) = -Li_2\left(\frac{1-x}{1-y}\right) - Li_2\left(\frac{1}{1-y}\right) + \ln(1-x)\ln(1-y)$$



$\mathcal{S}\left(\mathrm{Li}_n(z) + (-)^n \mathrm{Li}_n(z^{-1})\right) =$
$-(1-z)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1} + (-)^n(-)\left(1-\frac{1}{z}\right)\otimes\underbrace{\frac{1}{z}\otimes\cdots\otimes\frac{1}{z}}_{n-1}$
$= -(1-z)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1} + \left(1-\frac{1}{z}\right)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1}$
$= \left(-\frac{1}{\cancel{1-z}}\,\frac{\cancel{1-z}}{z}\right)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1} = -\underbrace{z\otimes\cdots\otimes z}_{n}$

$$\mathcal{S}\left(\mathrm{Li}_n(z) + (-)^n \mathrm{Li}_n(z^{-1})\right) =$$

$$-(1-z)\otimes \underbrace{z\otimes\cdots\otimes z}_{n-1} + (-)^n(-)\left(1-\frac{1}{z}\right)\otimes \underbrace{\frac{1}{z}\otimes\cdots\otimes\frac{1}{z}}_{n-1}$$

$$= -(1-z)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1} + \left(1-\frac{1}{z}\right)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1}$$

$$= \left(-\frac{1}{\cancel{1-z}}\,\frac{\cancel{1-z}}{z}\right)\otimes\underbrace{z\otimes\cdots\otimes z}_{n-1} = -\underbrace{z\otimes\cdots\otimes z}_{n} = \mathcal{S}\left(-\frac{1}{n!}\ln^n(x)\right)$$

Five-term dilog identity

$\mathbb{Z}_5$ symmetry.

$$1 - a_n = a_{n-1}\, a_{n+1} \quad \Rightarrow \quad \text{periodic with period } 5$$

Five-term dilog identity

$\mathbb{Z}_5$ symmetry.

$1 - a_n = a_{n-1} a_{n+1} \Rightarrow$ periodic with period 5

$$\mathcal{J}\left(\sum_{n=1}^{5} \mathrm{Li}_2(a_n)\right) = \sum_{n=1}^{5} -(1-a_n)\otimes a_n = -\sum_{n=1}^{5} (a_{n-1}a_{n+1})\otimes a_n$$

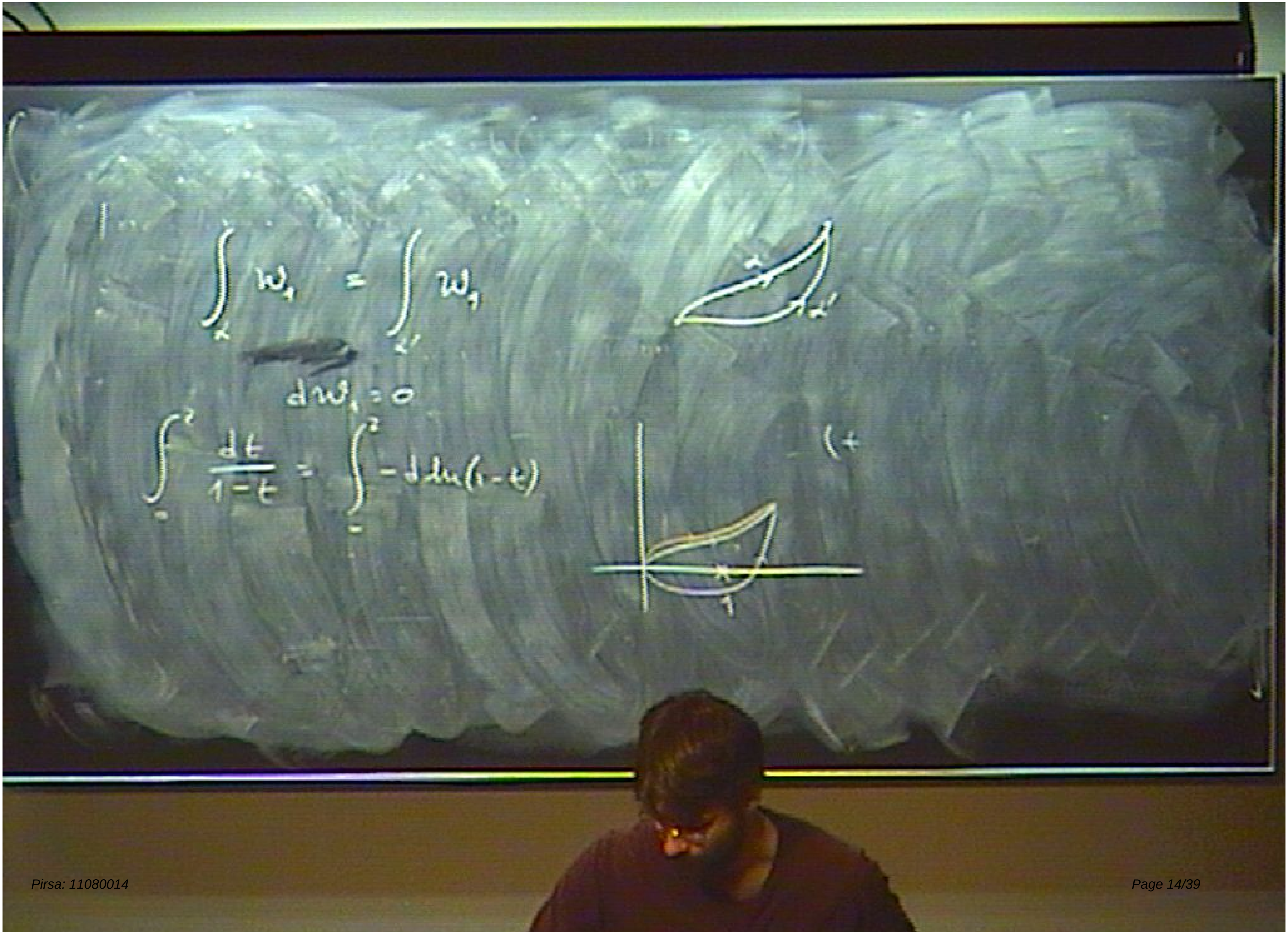
$$= -\sum_{n=1}^{5} \left(a_{n-1}\otimes a_n + a_{n+1}\otimes a_n\right) = -\sum_{n=1}^{5} \left(a_n \otimes a_{n+1} + a_{n+1}\otimes a_n\right)$$

$$= \mathcal{J}\left(-\sum_{n=1}^{5} \ln a_n \ln a_{n+1}\right)$$

$$\sum_{n=1}^{5}\left(\mathrm{Li}_2(a_n) + \ln a_n \ln a_{n+1}\right) = \frac{\pi^2}{6}$$

The usual writing of the 5-term id.

$$a_1 = x,\quad a_2 = \frac{1-x}{1-xy},\quad a_3 = \frac{1-y}{1-xy},\quad a_4 = y,\quad a_5 = 1-xy$$


$\int_{\alpha} \omega_1 = \int_{\alpha'} \omega_1$
α'
$d\omega_1 = 0$
$\int_0^z \frac{dt}{1-t} = \int_0^z -d\ln(1-t)$
1

$$\int_{\alpha} \omega_1 = \int_{\alpha'} \omega_1$$

$$\boxed{d\omega_1 = 0}$$

$$\int_0^z \frac{dt}{1-t} = \int_0^z -d\ln(1-t)$$

Multi valued function
of z.

$$I = \int_\alpha \omega_1 \circ \omega_2 \qquad F(z) = \int^z \omega_1$$

$$= \int_\alpha F \cdot \omega_2 \qquad dF = \omega_1$$

$$d(F\,\omega_2) = 0 \Rightarrow \boxed{\omega_1 \wedge \omega_2 = 0}$$ Integrability condition

unit

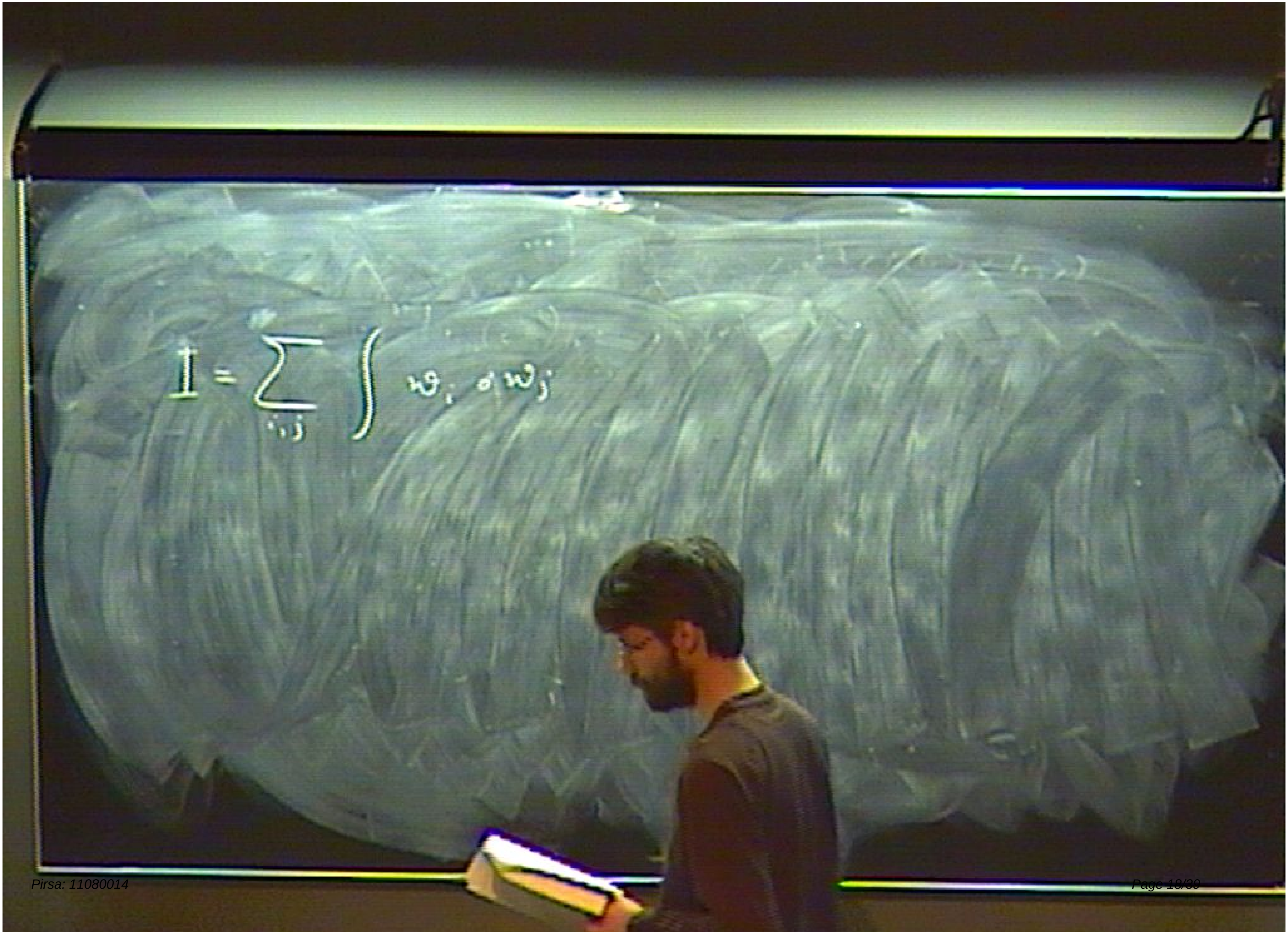
$$\boxed{d\omega_1 = d\omega_2 = 0}$$

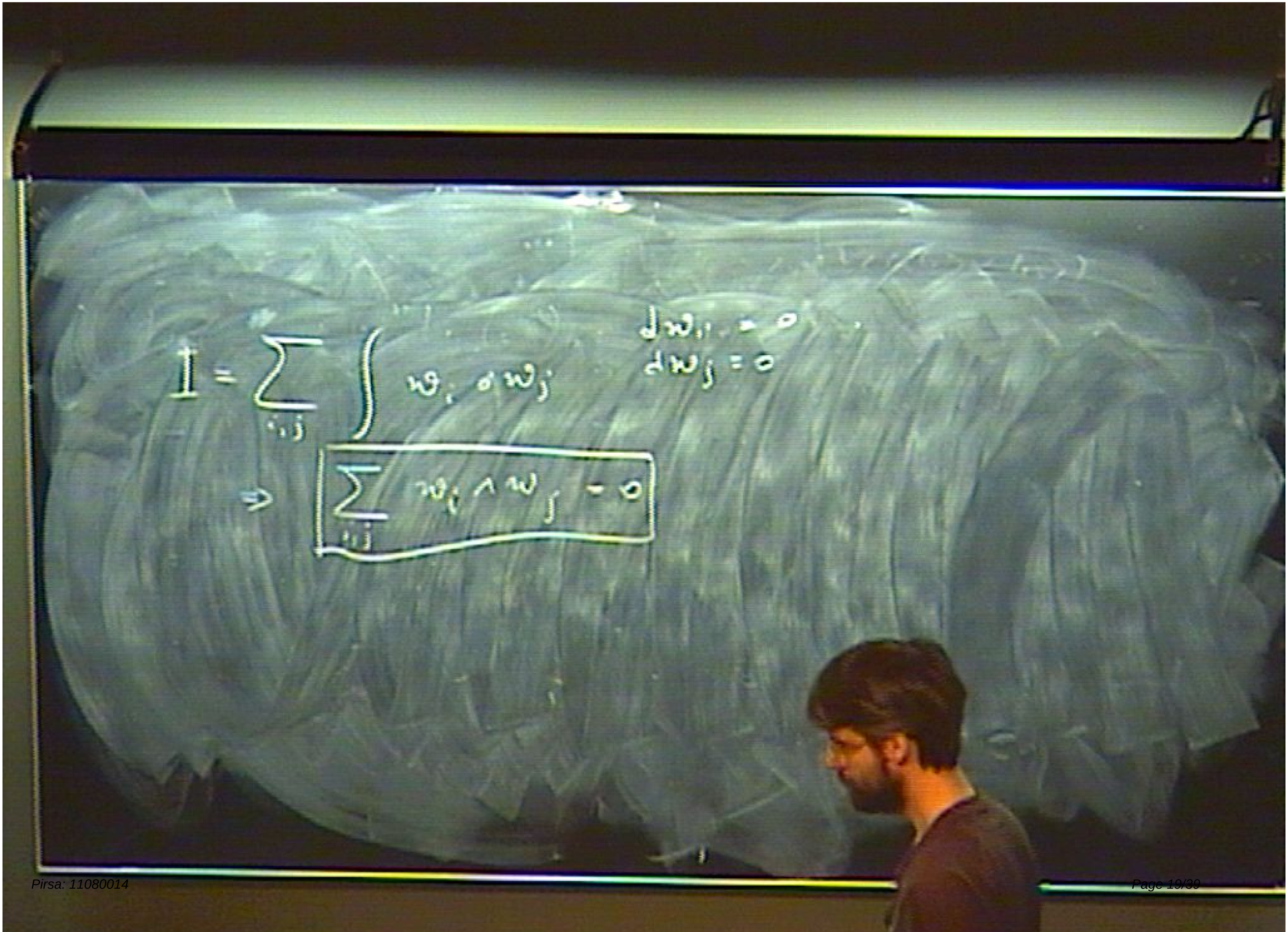
$$I = \int_\alpha \omega_1 \circ \omega_2 \qquad F(z) = \int^z \omega_1$$

$$= \int_\alpha F \cdot \omega_2 \qquad dF = \omega_1$$

$$d(F\,\omega_2) = 0 \implies \boxed{\omega_1 \wedge \omega_2 = 0}$$

Integrability condition

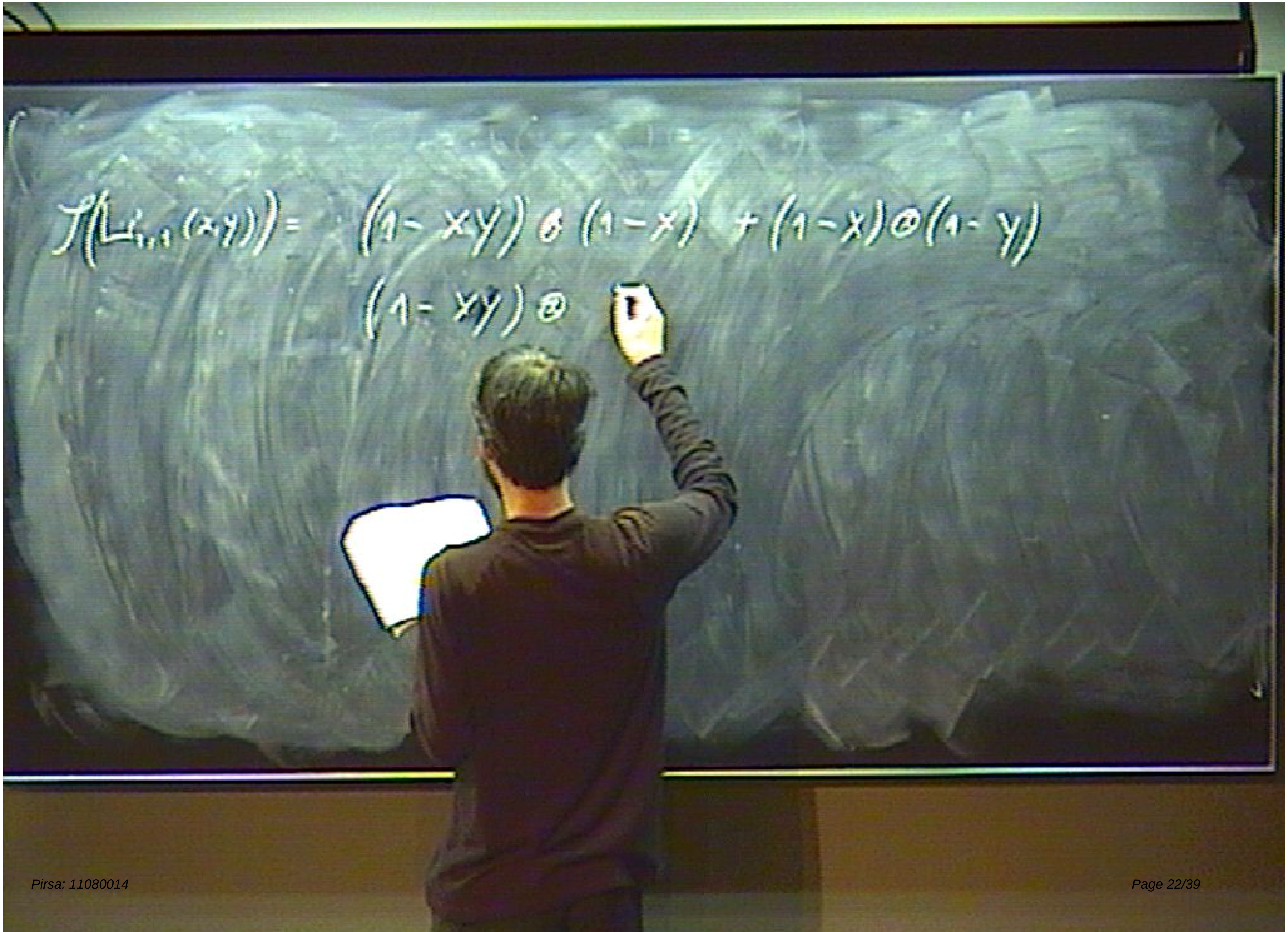

$I = \sum_{i,j} \int \omega_i \, d\omega_j$

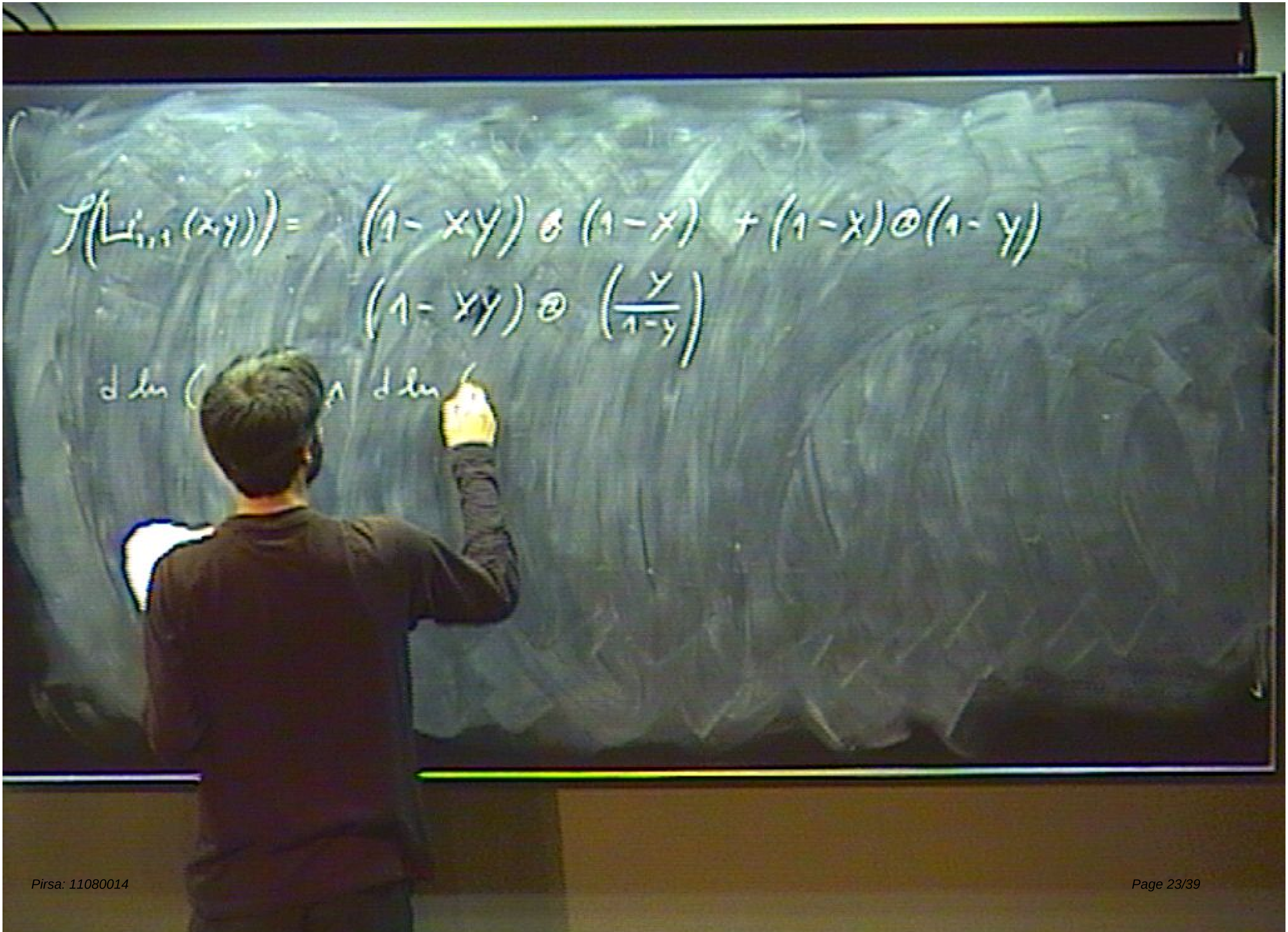

$I = \sum_{i,j} \int \omega_i \, d\omega_j$
$d\omega_i = 0$
$d\omega_j = 0$
$\Rightarrow \boxed{\sum_{i,j} \omega_i \wedge \omega_j = 0}$

$$I = \sum_{i,j} \int \omega_i \, d\omega_j \qquad \begin{array}{l} d\omega_i = 0 \\ d\omega_j = 0 \end{array}$$

$$\Rightarrow \boxed{\sum_{i,j} \omega_i \wedge \omega_j = 0}$$

$Li_{1,1}(x,y)$


J(Li_{1,1}(x,y)) = (1 - xy) ⊗ (1 - x) + (1 - x) ⊗ (1 - y)
(1 - xy) ⊗

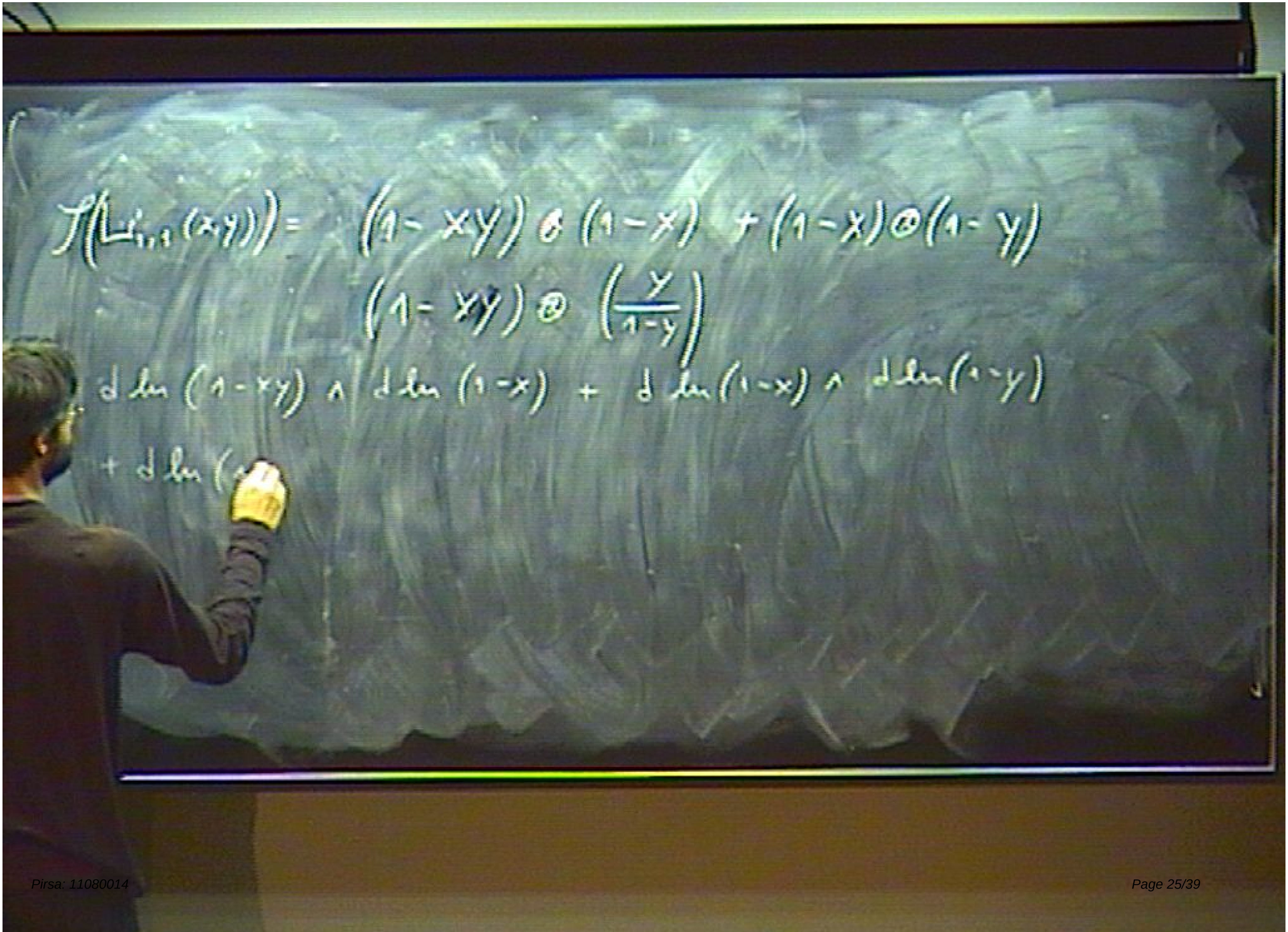

$T(\mathrm{Li}_{1,1}(xy)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$
$(1-xy)\otimes\left(\frac{y}{1-y}\right)$
$d\ln(\quad \wedge\ d\ln($

$$\mathcal{S}\left(\mathrm{Li}_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$(1-xy)\otimes\left(\frac{y}{1-y}\right)$$

$$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$$

$$+$$

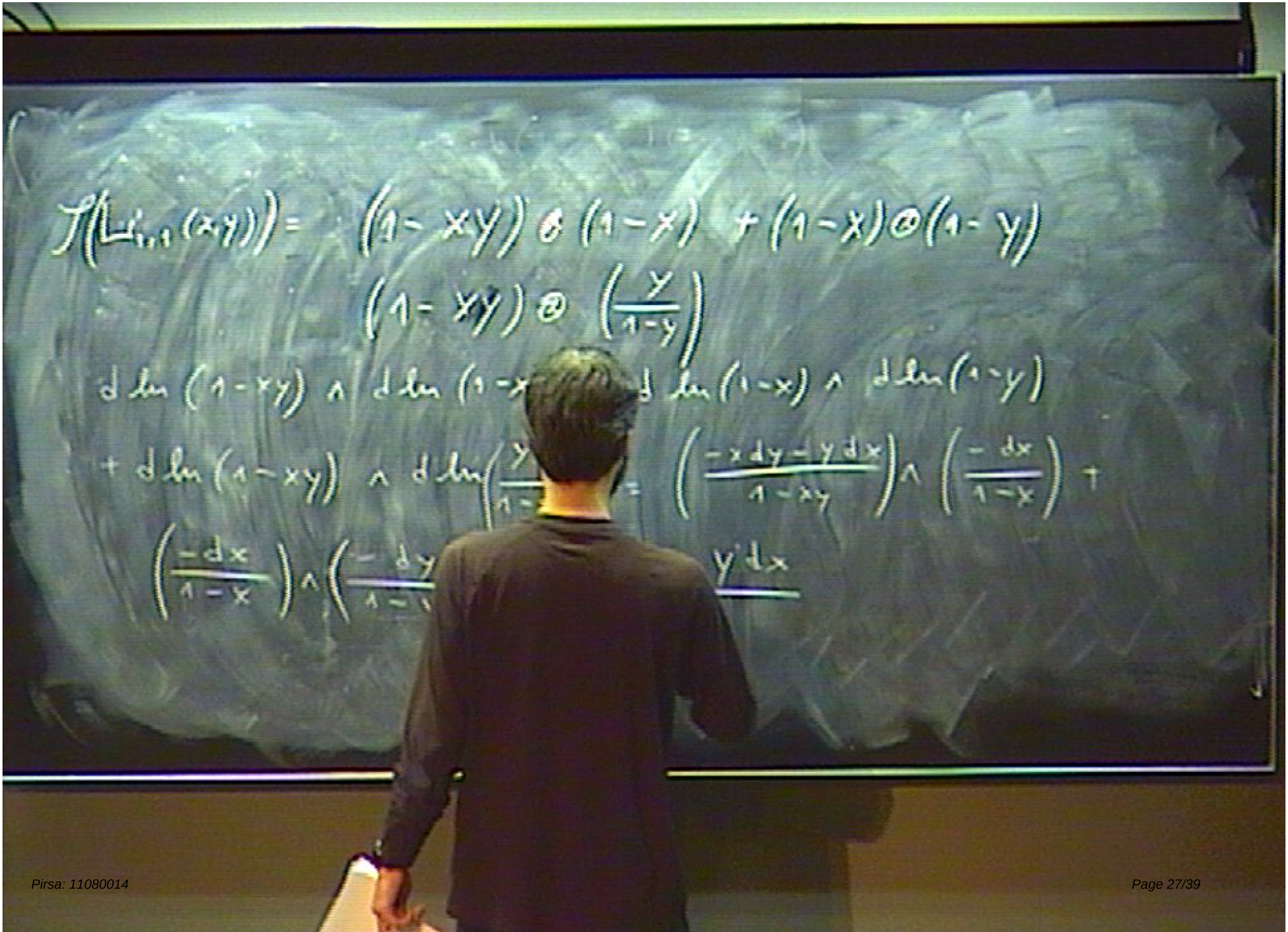

$\mathcal{T}(\mathrm{Li}_{1,1}(xy)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$
$(1-xy)\otimes\left(\frac{y}{1-y}\right)$
$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$
$+\, d\ln($

$$\mathcal{S}(\mathrm{Li}_{1,1}(x,y)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$(1-xy)\otimes\left(\frac{y}{1-y}\right)$$

$$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$$

$$+\, d\ln(1-xy)\wedge d\ln\left(\frac{y}{1-y}\right)$$


$\mathcal{S}(\mathrm{Li}_{1,1}(xy)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$
$(1-xy)\otimes\left(\frac{y}{1-y}\right)$
$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$
$+ d\ln(1-xy)\wedge d\ln\left(\frac{y}{1-y}\right) = \left(\frac{-x\,dy - y\,dx}{1-xy}\right)\wedge\left(\frac{-dx}{1-x}\right) +$
$\left(\frac{-dx}{1-x}\right)\wedge\left(\frac{-dy}{1-y}\right)$
$y\,dx$

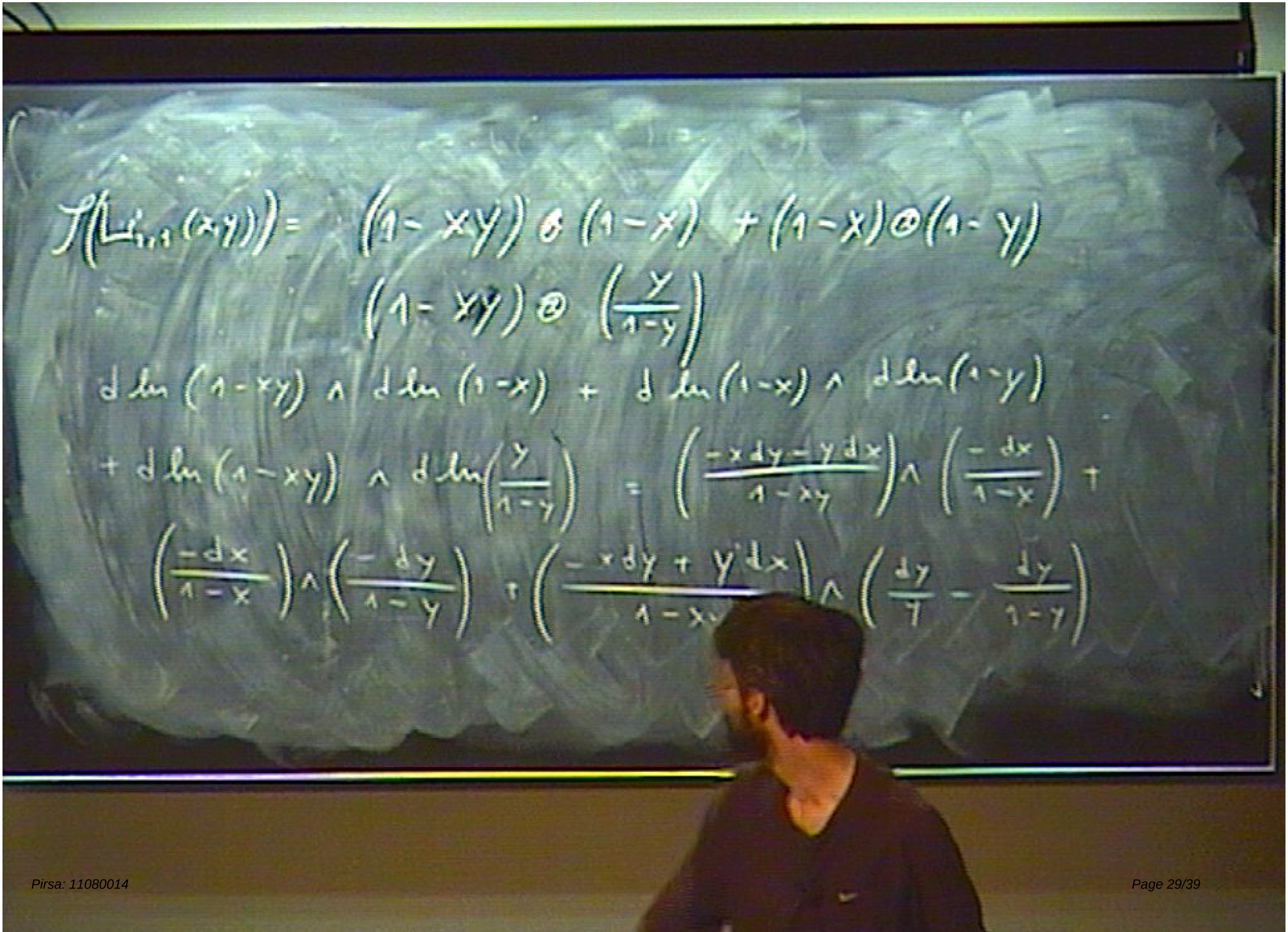
$$\mathcal{S}\left(\mathrm{Li}_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$(1-xy)\otimes\left(\frac{y}{1-y}\right)$$

$$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$$

$$+\, d\ln(1-xy)\wedge d\ln\left(\frac{y}{1-y}\right) = \left(\frac{-x\,dy-y\,dx}{1-xy}\right)\wedge\left(\ldots\right)$$

$$\left(\frac{-dx}{1-x}\right)\wedge\left(\frac{-dy}{1-y}\right) + \left(\frac{-x\,dy+y\,dx}{1-xy}\right)\wedge\left(\frac{dy}{y} - \ldots\right)$$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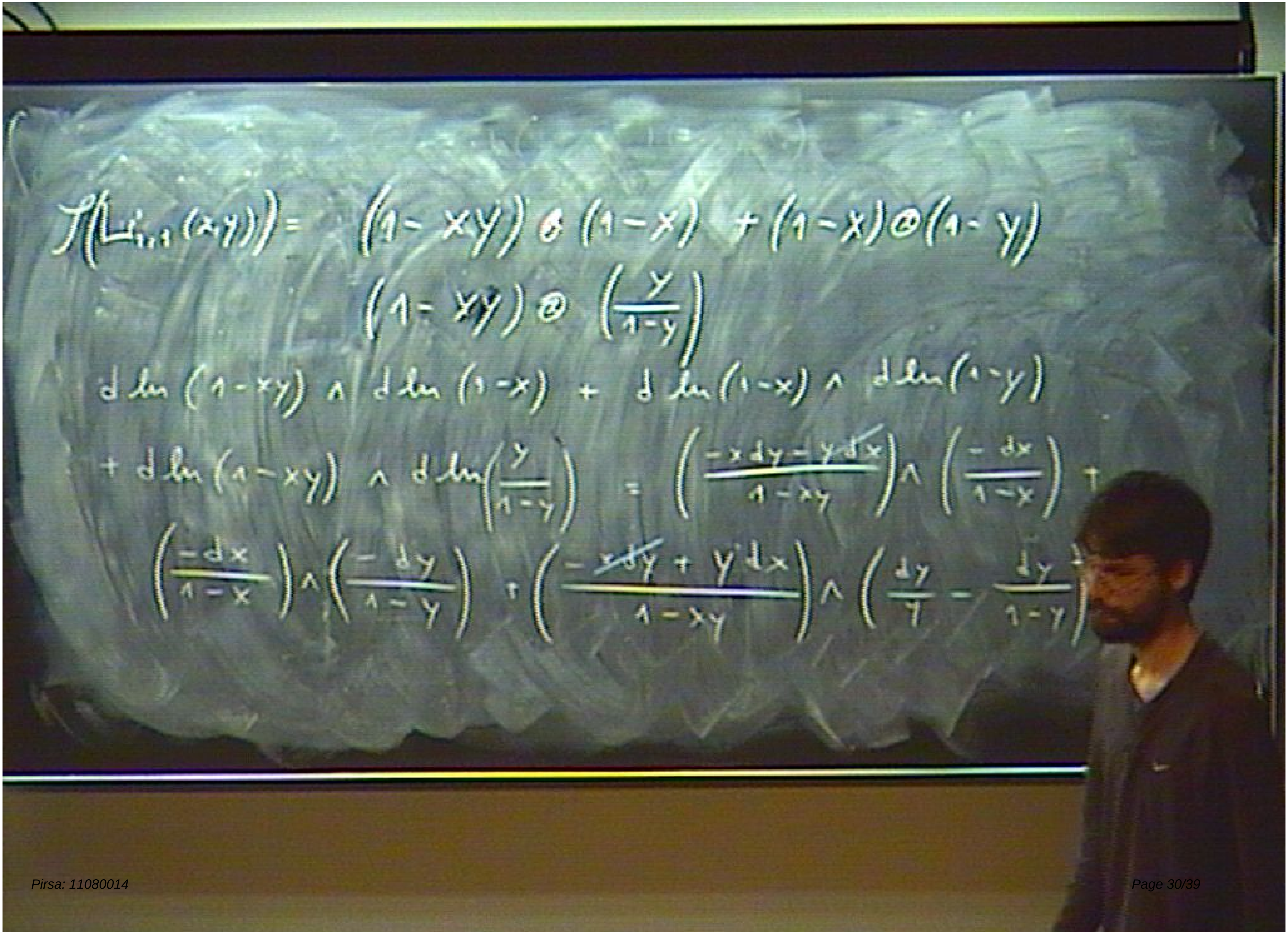
$\left(\frac{-dx}{1-x}\right)\wedge\left(\frac{-dy}{1-y}\right) + \left(\frac{-x\,dy + y\,dx}{1-xy}\right)\wedge\left(\frac{dy}{y} - \frac{dy}{1-y}\right)$


$\mathcal{S}(\mathrm{Li}_{1,1}(x,y)) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$
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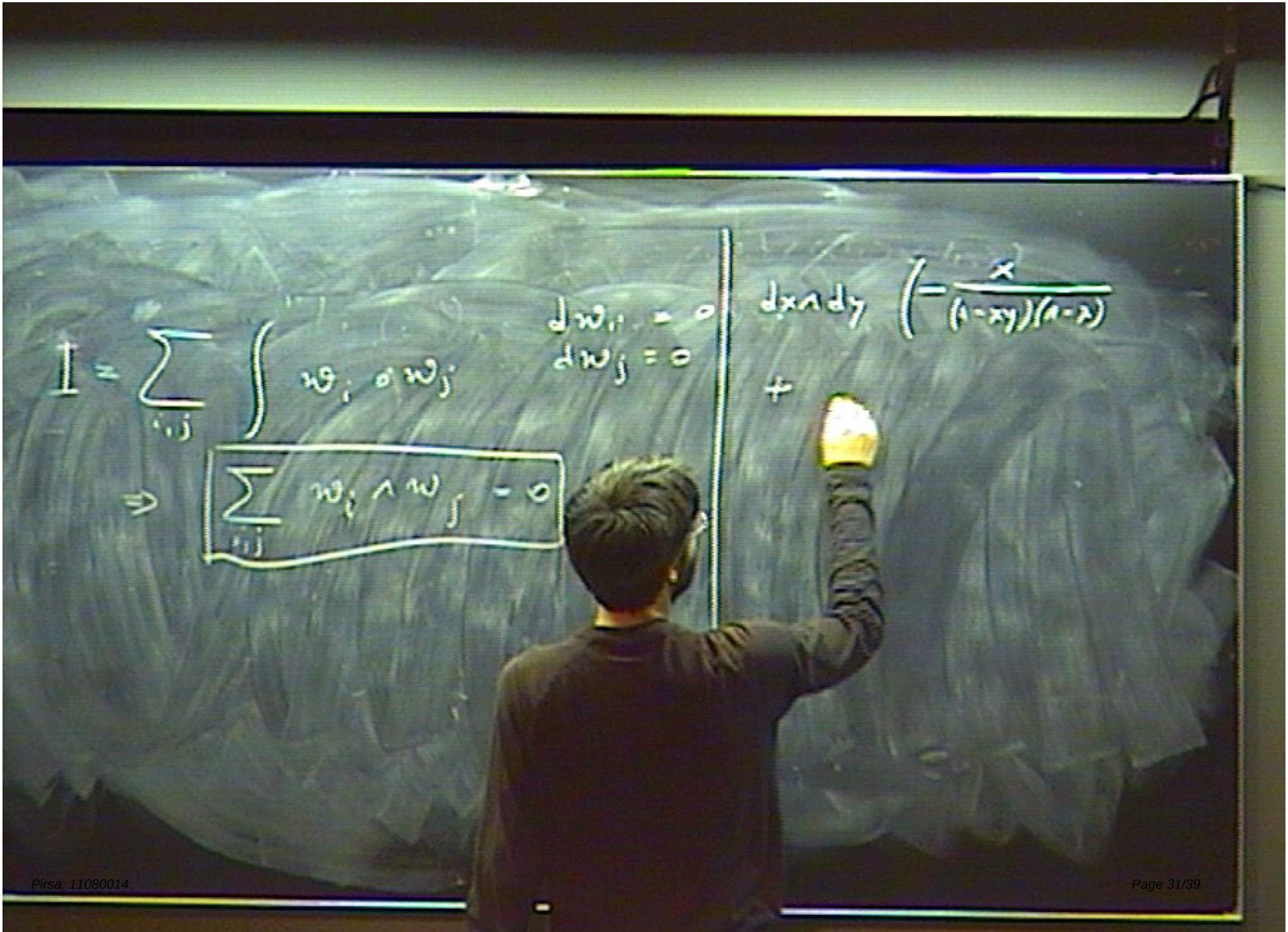
$\left(\frac{-dx}{1-x}\right)\wedge\left(\frac{-dy}{1-y}\right) + \left(\frac{-\cancel{x\,dy}+y\,dx}{1-xy}\right)\wedge\left(\frac{dy}{y}-\frac{dy}{1-y}\right)$



$I = \sum_{i,j} \int \omega_i \circ \omega_j$
$d\omega_i = 0$
$d\omega_j = 0$
$\Rightarrow \boxed{\sum_{i,j} \omega_i \wedge \omega_j = 0}$
$dx \wedge dy \left(- \frac{x}{(1-xy)(1-x)} \right.$
$+$

$$I = \sum_{i,j} \int \omega_i \circ \omega_j \qquad \begin{array}{l} d\omega_i = 0 \\ d\omega_j = 0 \end{array}$$

$$\Rightarrow \boxed{\sum_{i,j} \omega_i \wedge \omega_j = 0}$$

$$dx \wedge dy \left(- \frac{x}{(1-xy)(1-x)} + \frac{1}{(1-x)(1-y)} + \right.$$

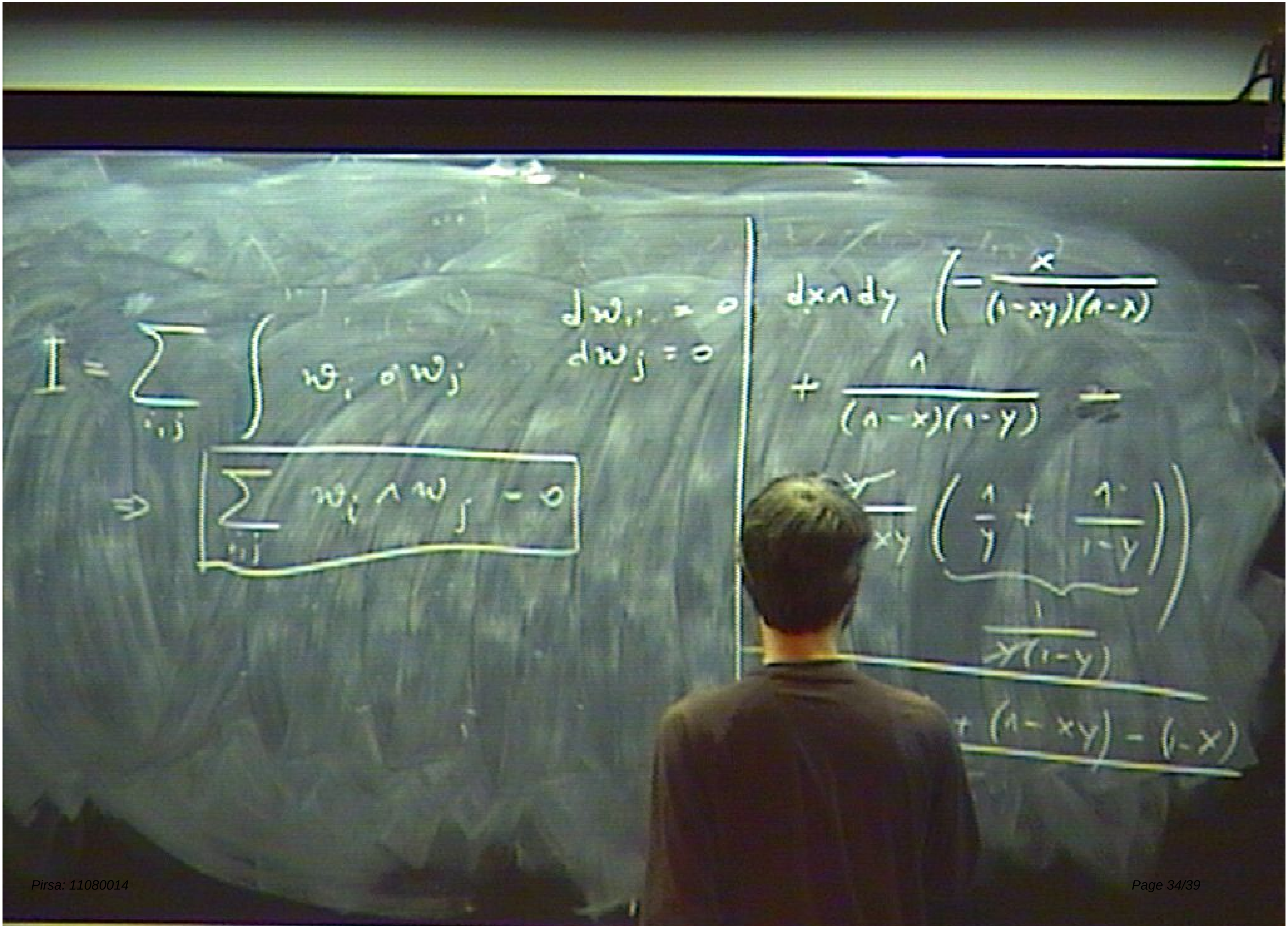
$$\mathcal{S}\left(\mathrm{Li}_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y)$$

$$(1-xy)\otimes\left(\frac{y}{1-y}\right)$$

$$d\ln(1-xy)\wedge d\ln(1-x) + d\ln(1-x)\wedge d\ln(1-y)$$

$$+\, d\ln(1-xy)\wedge d\ln\left(\frac{y}{1-y}\right) = \left(\frac{-x\,dy-\cancel{y\,dx}}{1-xy}\right)\wedge\left(\frac{-dx}{1-x}\right) +$$

$$\left(\frac{-dx}{1-x}\right)\wedge\left(\frac{-dy}{1-y}\right) + \left(\frac{-\cancel{x\,dy}-y\,dx}{1-xy}\right)\wedge\left(\frac{dy}{y}+\frac{dy}{1-y}\right)$$


I = ∑_{i,j} ∫ w_i ∧ w_j
dw_i = 0
dw_j = 0
⇒ ∑_{i,j} w_i ∧ w_j = 0
dx∧dy (− x / ((1−xy)(1−x))
+ 1 / ((1−x)(1−y))
1/y + 1/(1−y)
1 / (y(1−y))
(1 − xy) − (1 − x)

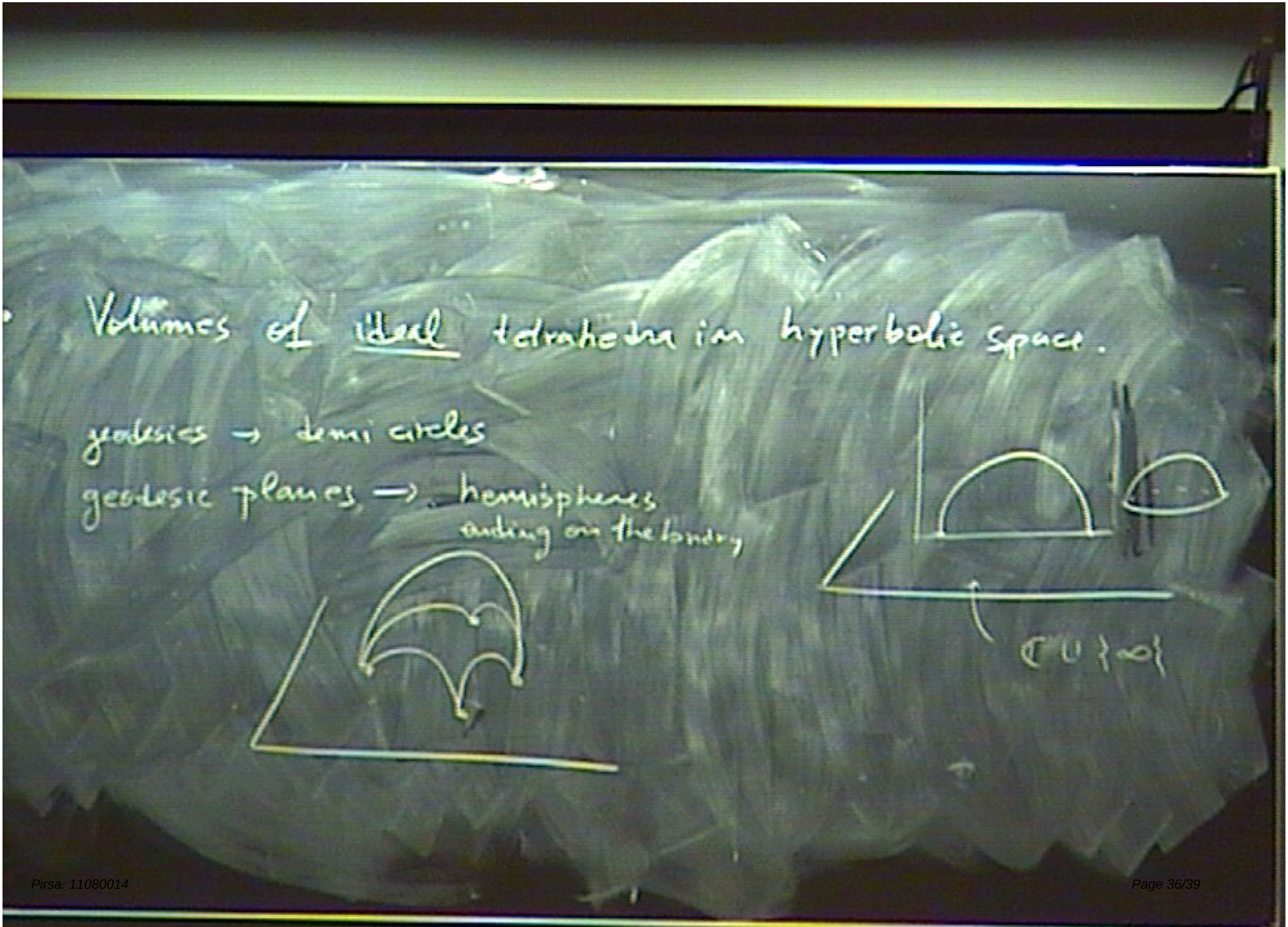
$$I = \sum_{i,j} \int \omega_i \, d\omega_j \qquad d\omega_i = 0, \quad d\omega_j = 0$$

$$\Rightarrow \boxed{\sum_{i,j} \omega_i \wedge \omega_j = 0}$$

$$dx \wedge dy \left(-\frac{x}{(1-xy)(1-x)} + \frac{1}{(1-x)(1-y)} \right)$$

$$\frac{\cancel{y}}{1-xy}\left(\underbrace{\frac{1}{y} + \frac{1}{1-y}}_{\frac{1}{\cancel{y}(1-y)}} \right)$$

$$\frac{-x(1-y) + (1-xy) - (1-x)}{(1-x)(1-y)(1-xy)} = 0$$

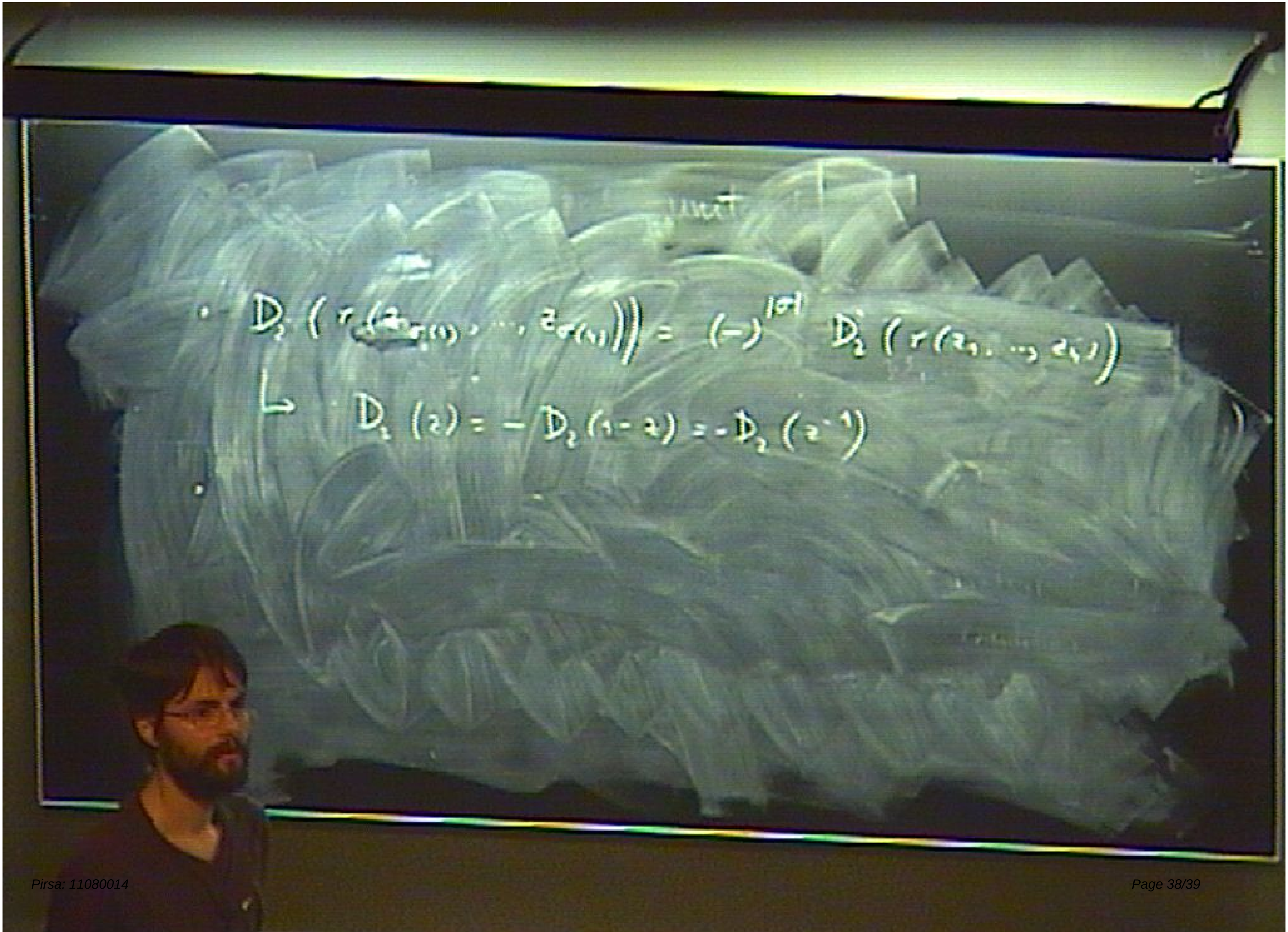

Volumes of ideal tetrahedra in hyperbolic space.
geodesics → demi circles
geodesic planes → hemispheres
ending on the boundary
$\mathbb{C} \cup \{\infty\}$

$$r(z_1, \ldots, z_4) = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)}$$

$$\mathrm{Vol} = D_2\big(r(z_1, \ldots, z_4)\big)$$

$$D_2(z) = \mathrm{Im}\big(Li_2(z)\big) + \arg(1-z)\ln|z|$$ Bloch - Wigner function

- $D_2(0) = D_2(1) = D_2(\infty) = 0$


$D_2(r(z_{\sigma(1)}, \dots, z_{\sigma(4)})) = (-)^{|\sigma|} D_2(r(z_1, \dots, z_4))$
$D_2(z) = -D_2(1-z) = -D_2(z^{-1})$

• $D_2(r(z_{\sigma(1)},\dots,z_{\sigma(4)})) = (-)^{|\sigma|} D_2(r(z_1,\dots,z_4))$

↳ $D_2(z) = -D_2(1-z) = -D_2(z^{-1})$

• $\sum_{i=1}^{5} (-)^i D_2(r(z_1,\dots,\hat{z}_i,\dots,z_5)) = 0$