

Title: Symbology and Scattering Amplitudes

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URL: <http://pirsa.org/11080012>

Abstract:

... unitarile

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$L_n(x)$

redes de computadores unitária

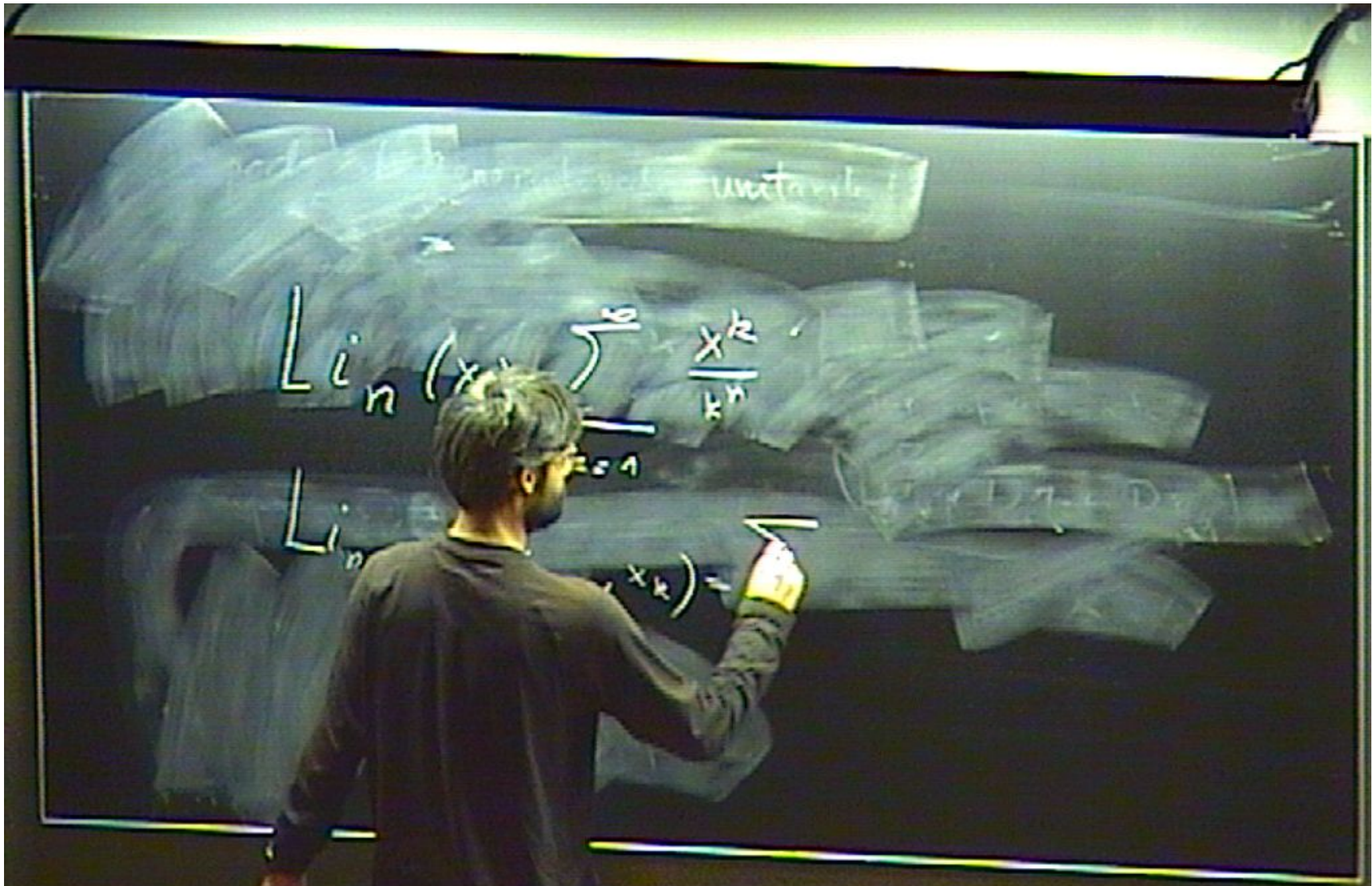
$$L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

$\ln(x)$

... unitarile

$$L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

$(x_1, \dots)$



... unitarily

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

$$L_n(x) = \sum_{k=0}^n x^k$$

podzielmy to na jednostki

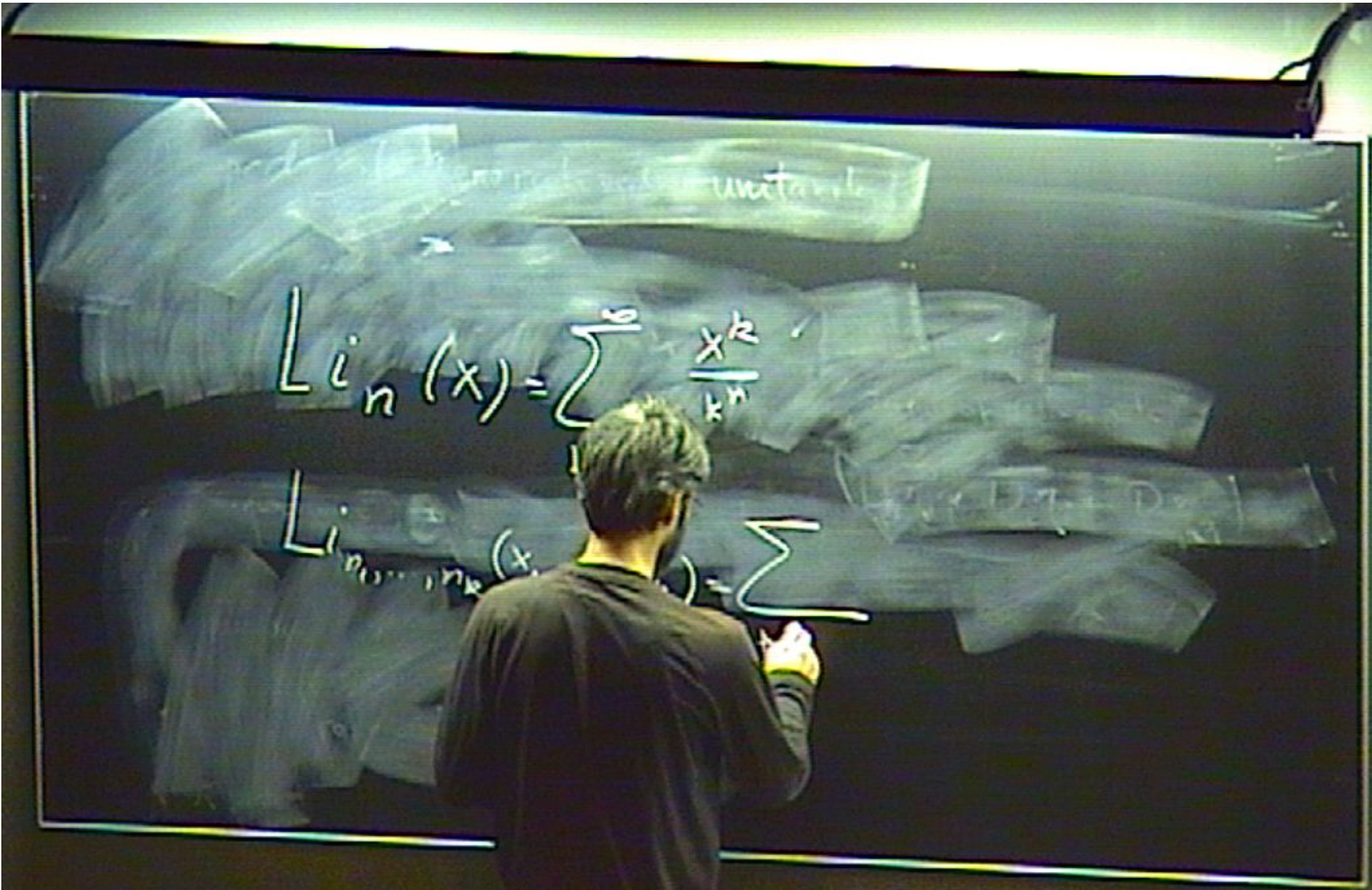
$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$$L_n(x_1, \dots, x_k) = \sum$$

... unitarily

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

$$L_n(x) = \sum_{k=0}^n x^k$$





adesso abbiamo un polinomio unitario

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

$$L_{n_1, \dots, n_k}(x) = \sum_{k=1}^n \dots$$



... unitarile

$$L_i(x) = \sum_{k=0}^n \frac{x^k}{n}$$

$$L_{i_1, \dots, i_k}(x)$$



... unitario

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$$L_{n+1}(x) = L_n(x) + \frac{x^{n+1}}{(n+1)!}$$



... unitarile

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k)$$

$$\frac{x_1^{p_1}}{p_1!}$$

... unitarile

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$$L_{n_1, \dots, n_k}(x_1, \dots)$$

$$\frac{x_1^{p_1}}{p_1!}$$

... unitaria

$$Li_n(x) = \sum_{k=1}^n \frac{x^k}{k^n}$$

$$Li_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{p_1, \dots, p_k} \frac{x_1^{p_1} \dots x_k^{p_k}}{p_1 \dots p_k}$$

... unitarile

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \dots \frac{x_k^{r_k}}{r_k!}$$

jed. složeno n-rozm. unitární

$$L i_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

$$L i_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{k=1}^{\infty} \frac{x_1^{r_1} \dots x_k^{r_k}}{k^n}$$



... und ... unitar ...

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

$$Li_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{k=1}^{\infty} \frac{x_1^{k_1} \dots x_k^{k_k}}{k^{n_1 + \dots + n_k}}$$

podzielmy to na jednostki

$$L_i(x) = \sum_{k=1}^n \frac{x^k}{k^n}$$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{k=1}^n \frac{x_k^{i_k}}{k^n}$$

podstawowe macierze unitarne

$$L_i(x) = \sum_{k=1}^n \frac{x^k}{k^n}$$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{p_1 \rightarrow \dots \rightarrow p_k} \frac{x_1^{p_1}}{p_1^{i_1}} \dots \frac{x_k^{p_k}}{p_k^{i_k}}$$

... und ...

$$L_i(x) = \sum_{k=1}^n \frac{x^k}{k^n}$$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{t_1 > \dots > t_k > 1} \frac{x_1^{t_1}}{t_1^{i_1}} \dots \frac{x_k^{t_k}}{t_k^{i_k}}$$

modul adalah matriks unitas

$$\text{Lin}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$n_1, \dots, n_k (x_1, \dots, x_k) = \sum_{i_1 \rightarrow \dots \rightarrow i_n} \frac{x_1^{i_1}}{i_1!} \dots \frac{x_k^{i_k}}{i_k!}$

polynomiali e zero real e unitario

$$L_i(x) = \sum_{k=1}^n \frac{x^k}{k^n}$$

... , n\_k (x\_1, ..., x\_k) = \sum\_{p\_1 > \dots > p\_k \geq 1} \frac{x\_1^{p\_1}}{p\_1^{n\_1}} \dots \frac{x\_k^{p\_k}}{p\_k^{n\_k}}

ipole pol

polynomial functions unitarily

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

multiple polynomials

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{p_1 \geq \dots \geq p_k \geq 1} \frac{x_1^{p_1}}{p_1!} \dots \frac{x_k^{p_k}}{p_k!}$$

polynomial approximation

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

multiple polynomials

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{t_1=0}^{n_1} \dots \frac{x_k^{t_k}}{t_k!}$$



... unitarily

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

$L_{n_1, \dots, n_k}$   
multiple

$$\frac{x_k^{r_k}}{r_k!}$$

redes de potencias unitarias

$$L_i(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad |x| < 1$$

$L_{n_1, \dots, n_k}(x_1, \dots)$   
multiple polynoms

$$\frac{x_1^{p_1}}{p_1! x}$$
$$\frac{x_k^{p_k}}{p_k! x}$$

pod... unitary

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{|x|^{k+1}}$$

$L_{n_1, \dots, n_k}(x)$   
multiple poles

$$\frac{x_k^{r_k}}{p_k^{n_k}}$$

~~... unitarily~~

$$L_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Convergent,  
 $|x| < 1$

$L_{n_1, \dots, n_k}$   
multiple p

$$\frac{x_1^{p_1}}{p_1!} \dots \frac{x_k^{p_k}}{p_k!}$$

polynomial is unitary

$$L_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) =$$

multiple poles

$$\frac{x_1^{n_1}}{1^{n_1}} \dots \frac{x_k^{n_k}}{k^{n_k}}$$

polynomial unitary

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k)$$

multiple polylogs

$$\frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

redes de computadores unitária

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum \frac{x_1^{p_1}}{p_1^{i_1}} \dots \frac{x_k^{p_k}}{p_k^{i_k}}$$

multiple polylogs  
Convergent

redaktionelle Notwendigkeit: univariabel

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \quad \text{Convergent, } |x| < 1$$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{k=1}^{\infty} \frac{x_1^{k_1} \dots x_k^{k_k}}{k^{n_1} \dots n_k}$$

multiple polylog

$$\frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

convergent in a



... unitarile

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent,  
 $|x| < 1$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum$$

multiple polylogs

$$\frac{x_1^{i_1}}{1^{i_1}} \dots \frac{x_k^{i_k}}{k^{i_k}}$$

convergent in a poly

... unitarile

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum \frac{x_k^{i_k}}{k^{i_k}}$$

multiple polylogs

... gent in a

power series with unit circle

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^i}$$

Convergent.  
 $|x| < 1$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{n=1}^{\infty} \frac{x_1^{n_1} \dots x_k^{n_k}}{n^{i_1} \dots n^{i_k}}$$

multiple polylogs

convergent in a  
 $|x_i| < 1$

redaktionell bearbeitet und in unitalen

$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$Li_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{p=1}^{\infty} \frac{x_1^{p^{n_1}} \dots x_k^{p^{n_k}}}{p^{n_1 + \dots + n_k}}$$

multiple polylogs

represent in a  
 $|x_i| < 1$   
 $i=1, \dots, k$

... unitarily

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{k=1}^{\infty} \frac{x_k^{n_k}}{k^{n_k}}$$

multiple polylogs

$$\frac{x_k^{n_k}}{k^{n_k}}$$

... in a  
 $|x_k| < 1$   
interval

~~... unitarily~~

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \quad \text{Convergent. } |x| < 1$$

multiple polylogs

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

convergent in a polydisc  $|x_i| < 1$  is limit

redaktionell überarbeitet und in der Einheit

$$L_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

multiple polylogs

convergent in a polydisc  $|x_i| < 1$

podobno je normalno unitarno

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent,  $|x| < 1$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{p_1 > \dots > p_k > 1} \frac{x_1^{p_1}}{p_1^{i_1}}$$

multiple polylogs

Convergent  
Polylog



... unitarile

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1,$

$$L_{i_1, \dots, i_k}(x_1, \dots, x_k) = \sum_{k=1}^{\infty} \dots$$

multiple polylogs

gent in a  
se  $|x_i| < 1,$   
is, it

polynomial approximation of unit circle

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

Convergent.  
 $|x| < 1, n$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum_{t=0}^{\infty} \dots$$

multiple polynomials

$\frac{1}{1-x}$   
 $x^k$

gent in a  
se  $|x| < 1$   
is limit

power series with unit circle

$$L_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{k^n}$$

Converges

$$|x| < 1, n \in \mathbb{N}$$

$$L_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum$$

multiple polylogs

$$\frac{x_1^{n_1}}{1} \dots \frac{x_k^{n_k}}{1}$$

converges in a disc  $|x_i| < 1$  is finite

unitarile

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k^n}$$

Convergent.

$$|x| < 1, n \in \mathbb{N}^+$$

$$L_{n_1, \dots, n_k}(x) = \sum_{p_1 \geq \dots \geq p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

convergent in a polydisc  $|x_i| < 1$  is limit

power series expansion - unit circle

$$L_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergent.

$$|x| < 1, n \in \mathbb{N}^+$$

$$L(x_1, \dots, x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

Multiple polylogs

convergent in a polydisc  $|x_i| < 1$  is limit

... und ... unitarisch

$$L_{i_n}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

Convergenz.  
 $|x| < 1, n \in \mathbb{N}^+$

$$L_{i_{n_1, \dots, n_k}}(x_1, \dots, x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \dots \frac{x_k^{p_k}}{p_k^{n_k}}$$

polylogs

convergent in a polydisc  $|x_i| < 1$

redaktionelle Mitarbeit: unitarisch

$$L_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

Convergenz  
 $|x| < 1, n \in \mathbb{N}^+$

polynoms

$$L_n(x_1, \dots, x_k) = \sum_{p_1 \geq \dots \geq p_k \geq 0} \frac{x_1^{p_1}}{p_1!} \dots \frac{x_k^{p_k}}{p_k!}$$

Convergenz in einem Polydisc  $|x_i| < 1$

A

•  $L_n(\dots)$



•  $L_n(1) = \zeta(n)$

$L_{n_1, \dots, n_k}(1, \dots, 1) = \zeta(n_1, \dots, n_k)$

Number theory.

•  $L_{n,1}(1) = \zeta(n)$

$$L_{n_1, \dots, n_k}(1, \dots, 1) = \zeta(n_1, \dots, n_k)$$

Number theory.

$$\zeta(2, 1) = \zeta(3) \quad \text{Euler.}$$

•  $L_{n_1}(1) = \zeta(n_1)$

$L_{n_1, \dots, n_k}(1, \dots, 1) = \zeta(n_1, \dots, n_k)$

Number theory.

$\zeta(2, 1) = \zeta(3)$  Euler.

• Iterated integrals:

Topology  $\rightarrow$  Multiple linking numbers.

•  $L_{n,1}(1) = \zeta(n)$

$L_{n_1, \dots, n_k}(1, \dots, 1) = \zeta(n_1, \dots, n_k)$

Number theory.

$\zeta(2,1) = \zeta(3)$  Euler.

• Iterated integrals.

Topology  $\rightarrow$  Multiple linking numbers.

• Motives, K-theory.

- In physics  $\rightarrow$  loop integrals.
- Integrable models, etc.

- In physics  $\rightarrow$  loop integrals.
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- 

Iterated integrals

• In physics  $\rightarrow$  loop integrals.

• Integrable models, etc.

Iterated integrals (K.T. Chen) Algebras of iterated integrals  
and functions

• In physics  $\rightarrow$  loop integrals.

• Integrable models, etc.

Iterated integrals (K.T. Chen.

"Algebras of iterated integrals  
and fundamental groups"



- In physics  $\rightarrow$  loop integrals.
- Integrable models, etc.

Iterated integrals (K.T. Chen "Algebras of iterated integrals and fundamental groups")

$$\int_a^b f_1(t) dt \circ \dots \circ f_n(t) dt = \int_a^b \left( \int_a^t f_1(u) du \circ \dots \circ f_{n-1}(u) du \right) f_n(t) dt$$

- In physics  $\rightarrow$  loop integrals.
- Integrable models, etc.

Iterated integrals (K.T. Chen "Algebras of iterated integrals and fundamental groups")

$$\int_a^b f_1(t) dt \circ \dots \circ f_n(t) dt = \int_a^b \left( \int_a^t f_1(u) du \circ \dots \circ f_{n-1}(u) du \right) f_n(t) dt$$

- In physics  $\rightarrow$  loop integrals.
- Integrable models, etc.

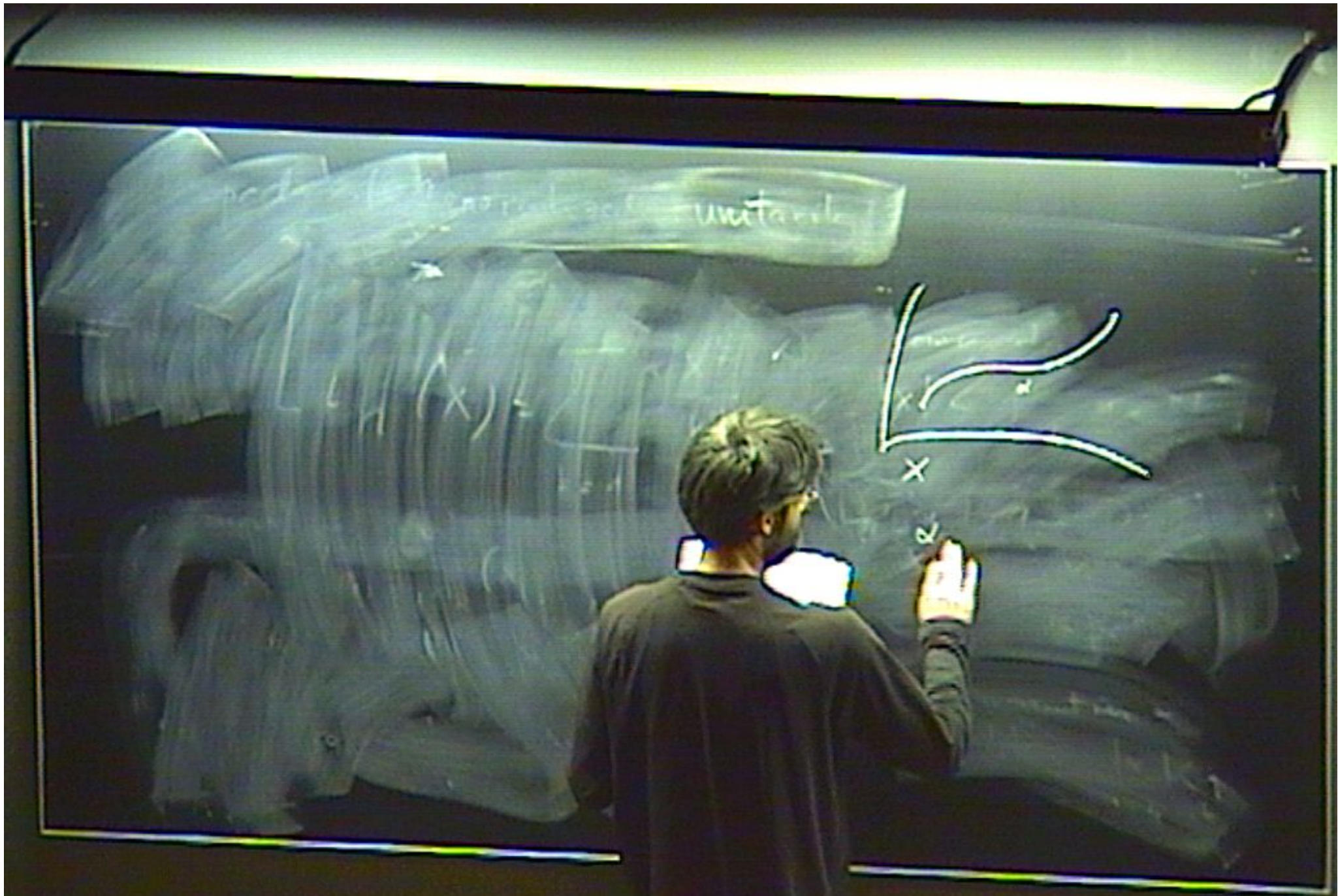
Iterated integrals (K.T. Chen "Algebras of iterated integrals and fundamental groups")

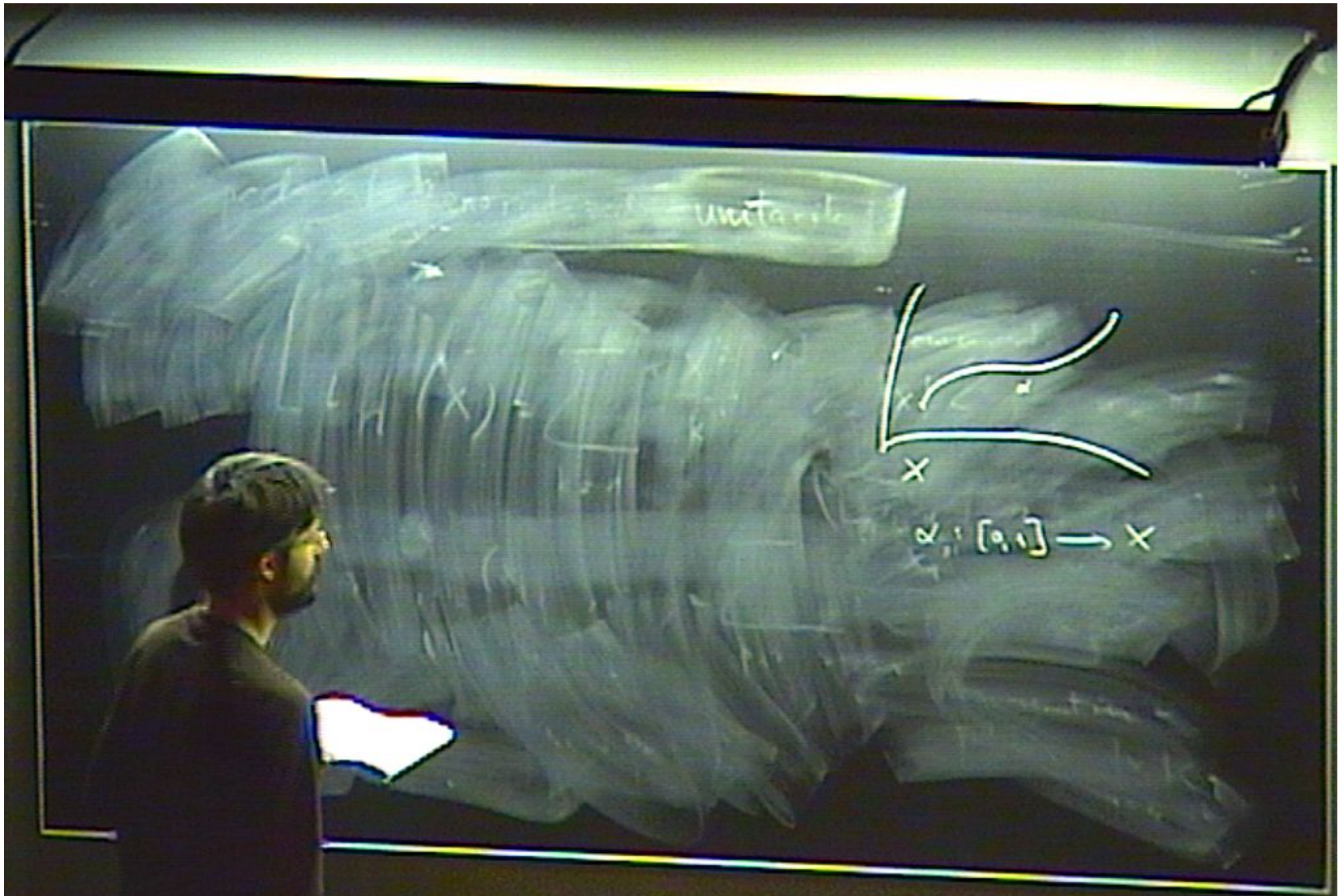
$$\int_a^b f_1(t) dt \circ \dots \circ f_n(t) dt = \int_a^b \left( \int_a^t f_1(u) du \circ \dots \circ f_{n-1}(u) du \right) f_n(t) dt$$

- In physics  $\rightarrow$  loop integrals.
- Integrable models, etc.

Iterated integrals (K.T. Chen "Algebras of iterated integrals and fundamental groups")

$$\int_a^b f_1(t) dt \circ \dots \circ f_n(t) dt = \int_a^b \left( \int_a^t f_1(u) du \circ \dots \circ f_{n-1}(u) du \right) f_n(t) dt$$





paths homotopic to unit circle



$$\alpha: [0,1] \rightarrow X$$

$$\alpha, \beta: [0,1] \rightarrow X$$

$\alpha \cdot \beta$  composition

$\alpha^{-1}$  inverse path

unitary



$X$ -manifold

$$\alpha: [0,1] \rightarrow X$$

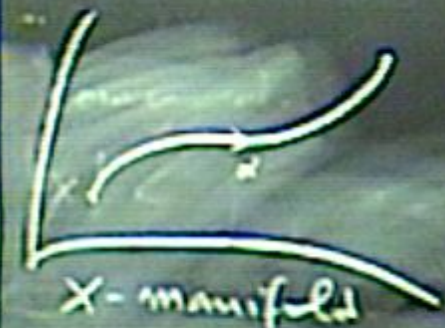
$$\alpha, \beta: [0,1] \rightarrow X$$

composition

inverse path.



path in  $X$  is called a unitary



$X$ -manifold

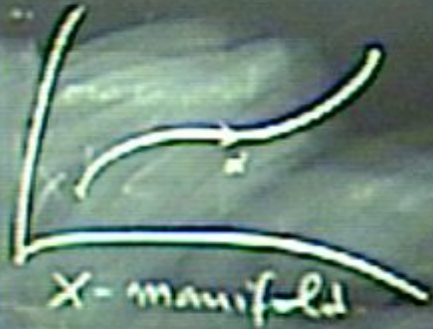
$$\alpha: [0,1] \rightarrow X$$

$$a, \beta: [0,1] \rightarrow X$$

composition

$\alpha^{-1}$  inverse path.

$\omega_1, \dots, \omega_r$  differential forms on  $X$



$$\alpha: [0, 1] \rightarrow X$$

$$\alpha, \beta: [0, 1] \rightarrow X$$

$\alpha \cdot \beta$  composition

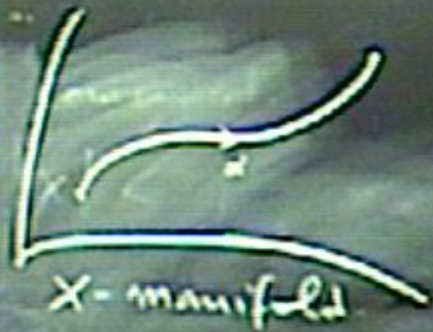
$\alpha^{-1}$  inverse path.

$\omega_1, \dots, \omega_r$  differential forms on  $X$

$x^i$  are local coordinates on  $X$

$$f = \sum f_i dx^i$$

$$\alpha^{-1} f = \sum f_i \frac{dx^i}{dt} dt$$



$X$ -manifold

$$\alpha: [0,1] \rightarrow X$$

$$\alpha, \beta: [0,1] \rightarrow X$$

$\alpha \circ \beta$  composition

$\alpha^{-1}$  inverse path.

$$\int_{\alpha} w_{\alpha} \circ \circ w_{\alpha} = \int_0^1 (\alpha^x w_{\alpha}) \circ$$

$$\int_{\alpha} w_A \cdot \dots \cdot w_r = \int_0^1 (\alpha^x w_A) \cdot \dots \cdot (\alpha^x w_r)$$

$$\int_{\alpha} w_A \cdot w_r = \int_0^1 (\alpha^x w_A) \cdot (\alpha^x w_r)$$

$$\int_{\alpha} w_1 \circ \dots \circ w_r = \int_0^1 (\alpha^x w_1) \circ \dots \circ (\alpha^x w_r)$$

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$$\int_{\alpha} w_1 \circ \dots \circ w_r = \int_0^1 \alpha^x w_1 \circ \dots \circ w_r$$

$$\int_{\alpha} w_1 \circ \dots \circ w_r = \int_0^1 (\alpha^x w_1) \circ \dots \circ (\alpha^x w_r)$$

$$\int_{\alpha} w_1 \circ \dots \circ w_r \int_{\alpha} w_{r+1} \circ \dots \circ w_{r+s} =$$

$\sum_{\sigma \in S_{r+s}}$   
 (r,s) shuffles



$$\int_{\alpha} w_1 \circ \dots \circ w_r = \int_0^1 (\alpha^x w_1) \circ \dots \circ (\alpha^x w_r)$$

$$\int_{\alpha} w_1 \circ \dots \circ w_r \int_{\alpha} w_{r+1} \circ \dots \circ w_{r+s} =$$

$$\sum_{\sigma \in \text{shuffles}(r,s)} \int_{\alpha} w_{\sigma(1)} \circ \dots \circ w_{\sigma(r+s)}$$

$(r, s)$  shuffles

$$\sigma(1) < \dots < \sigma(r)$$

$$\sigma(r+1) < \dots < \sigma(r+s)$$

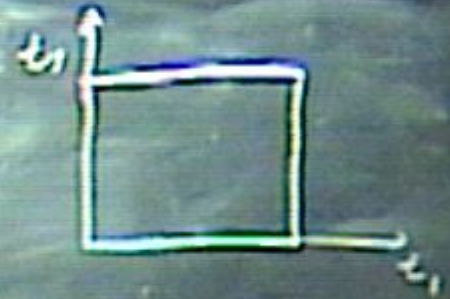
$(r, s)$  shuffles

$$\sigma(1) < \dots < \sigma(r)$$

$$\sigma(r+1) < \dots < \sigma(r+s)$$

---

$$\int_0^1 f_1(t_1) dt_1 \int_0^1 f_2(t_2) dt_2 = \int_{0 \leq t_1 < t_2 \leq 1}$$



$(r, s)$  shuffles

$$\sigma(1) < \dots < \sigma(r)$$

$$\sigma(r+1) < \dots < \sigma(r+s)$$



$$\int_0^1 f_1(t_1) dt_1 \int_0^1 f_2(t_2) dt_2 = \int_{0 \leq t_1 < t_2 \leq 1} f_1(t_1) f_2(t_2) dt_1 dt_2$$

$$\int_0^1 f_1(t) dt \cdot \int_0^1 f_2(t) dt + \int_{0 \leq t_2 < t_1 \leq 1} f_1(t_1) f_2(t_2) dt_1 dt_2$$
$$\int_0^1 f_2(t) dt \cdot \int_0^1 f_1(t) dt$$

$(r, s)$  shuffles

$$\sigma(1) < \dots < \sigma(r)$$

$$\sigma(r+1) < \dots < \sigma(r+s)$$



$$\int_0^1 f_1(t_1) dt_1 \int_0^1 f_2(t_2) dt_2 =$$

$$\int_{0 < t_1 < t_2 \leq 1} f_1(t_1) f_2(t_2) dt_1 dt_2$$

$$\int_0^1 f_1(t) dt \cdot \int_0^1 f_2(t) dt + \int_0^1 f_2(t) dt \cdot \int_0^1 f_1(t) dt$$

$$+ \int_{0 \leq t_2 < t_1 \leq 1} f_1(t_1) f_2(t_2) dt_1 dt_2$$

# Shuffle product

$$(w_1 \circ \dots \circ w_r) \sqcup (w_{r+1} \circ \dots \circ w_{r+s})$$

↑  
shuffle product

$$= \sum_{\sigma \in (r,s)\text{-shuffles}} w_{\sigma(1)} \circ \dots \circ w_{\sigma(r+s)}$$

$$= w_1 \circ \left( (w_2 \circ \dots \circ w_r) \sqcup (w_{r+1} \circ \dots \circ w_{r+s}) \right) + w_{r+1} \circ \left( (w_1 \circ \dots \circ w_r) \sqcup (w_{r+2} \circ \dots \circ w_{r+s}) \right) + \dots$$

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$$(w_1 \circ \dots \circ w_r) \sqcup (w_{r+1} \circ \dots \circ w_{r+s})$$

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$$\begin{aligned}
 \int_{\mathbb{R}^p} w_1 \circ \dots \circ w_r &= \int_{\mathbb{R}^p} w_1 \circ \dots \circ w_r + \dots \\
 &+ \int_{\mathbb{R}^p} w_1 \circ \dots \circ w_i \int_{\mathbb{R}^p} w_{i+1} \circ \dots \circ w_r + \\
 &\dots \int_{\mathbb{R}^p} w_1 \circ \dots \circ w_r
 \end{aligned}$$

$$\int_{\alpha^{-1}} w_1 \circ \dots \circ w_r = (\alpha)^r \int_{\alpha} w_r \circ \dots \circ w_1$$

$$\int_{x^{-1}} w_1 \circ \dots \circ w_r = (-1)^r \int_x w_r \circ \dots \circ w_1$$

---

$$F[x] = \int_x w_1 \circ \dots \circ w_r$$

$$\int_{\alpha^{-1}} w_1 \circ \dots \circ w_r = (\tau)^r \int_{\alpha} w_r \circ \dots \circ w_1$$

---

$$[\alpha] = \int_{\alpha} w_1 \circ \dots \circ w_r$$

unit



unit





$$F[\alpha] = \int_{\alpha} w_1 \dots w_T$$



unit



$$F[\alpha] = \int_{\alpha} w_1 \circ \dots \circ w_T$$

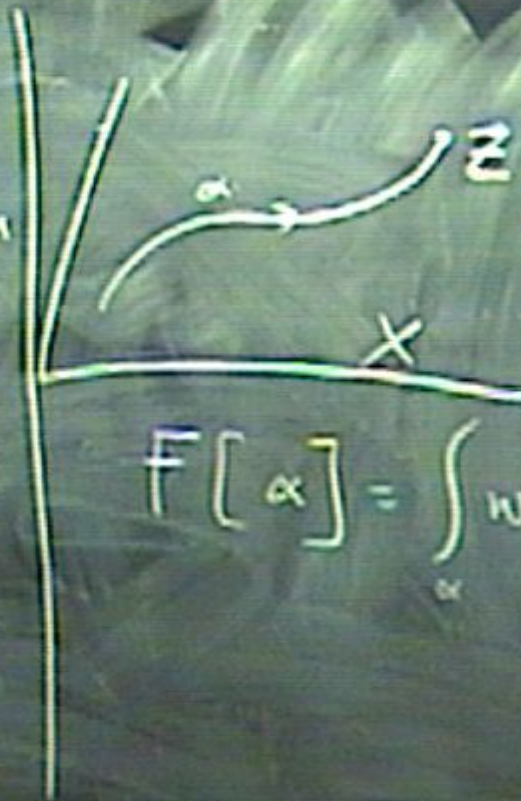
$\sigma$



$$F[\alpha] = \int_{\alpha} w_1 \sigma \dots \sigma w_T$$

unit

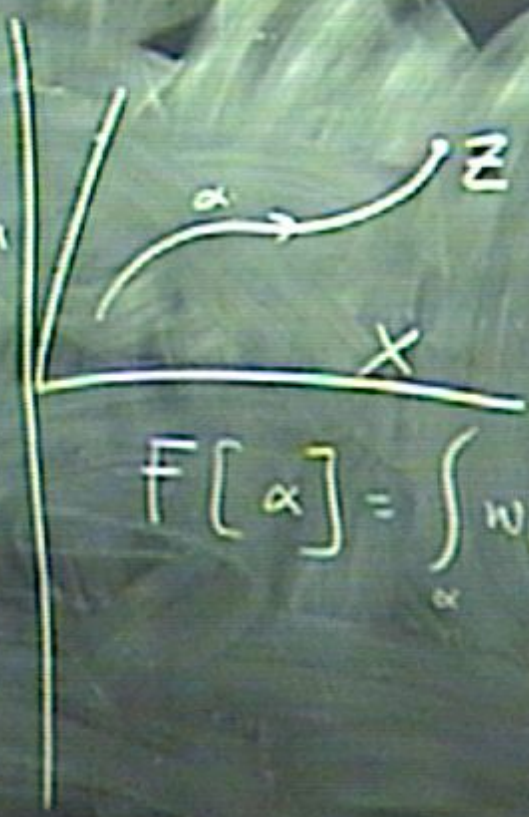
$$dF = w_r(z) \int_x w_1 \circ \dots \circ w_{r-1}$$



$$F[\alpha] = \int_x w_1 \circ \dots \circ w_r$$

$$dF = w_r(z) \int w_1 \circ \dots \circ w_{r-1}$$

$$d \int^z F(t) dt = \int F(z) dz$$



$$F[\alpha] = \int_{\alpha} w_1 \circ \dots \circ w_r$$

$$Li_2(z) = \int_0^z \left( \int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = - \int_0^z \frac{\ln(1-t) dt}{t}$$

diagonaliter.

$$-\ln(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}$$

$$Li_n(z) = \int_0^z \left( \frac{dt}{t} \right) Li_{n-1}(t)$$

$$z \frac{\partial}{\partial z} L_n(z) = L_{n-1}(z)$$

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$$Li_2(z) = \int_0^z \left( \int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = - \int_0^z \frac{\ln(1-t) dt}{t}$$

dilogarithmus

$$Li_2(z) = \int_0^z \frac{dt}{1-t} = \int_0^z \frac{dt}{t} = - \ln(1-t)$$

$$Li_2(z) = - \int_0^z d \ln(1-t) = d \ln t = \int_0^z \frac{dt}{1-t} = \frac{dt}{t}$$



$$z \frac{\partial}{\partial z} L_n(z) = L_{n-1}(z)$$

$$z \frac{\partial}{\partial z} L_n(z) = L_{n-1}(z)$$

$$= \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n-1 \text{ times}}$$

$$= - \int_0^z d \ln(1-t) \circ \underbrace{d \ln t \circ \dots \circ d \ln t}_{n-1}$$

$$z \frac{\partial}{\partial z} \text{Li}_n(z) = \text{Li}_{n-1}(z)$$

$$\begin{aligned} \text{Li}_n(z) &= \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n-1 \text{ times}} \\ &= - \int_0^z d \ln(1-t) \circ \underbrace{d \ln t \circ \dots \circ d \ln t}_{n-1} \end{aligned}$$

Symbol

Unit

$$z \frac{\partial}{\partial z} L_n(z) = L_{n-1}(z)$$

$$L_n(z) = \int_0^z \frac{dt}{1-t} \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n-1 \text{ times}}$$

$$= - \int_0^z d \ln(1-t) \circ \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n-1}$$

Symbol

$$f_r(s) = \int_{R_1}^{\dots} \dots \text{den } R_{r+1}$$

$$J(f_r) = R_1 \otimes \dots \otimes R_r$$

If the  $\Omega$  has le

Symbol

$$f_r = \int d \ln R_r \cdot \ln R_{r,d}$$

$$J(f_r) = R_r \circ$$

If the integral b  $r \rightarrow$

Symbol

$$f_r(x) = \int \dots d \ln R_1 \cdot \dots \cdot d \ln R_r$$

$$\mathcal{J}(f_r) = R_1 \circ \dots \circ R_r$$

If the integral has length  $r \rightarrow$  Transcendental function



Symbol

$$f_r(x) = \int d \ln R_1 \cdot \dots \cdot d \ln R_r$$

$$\mathcal{J}(f_r) = R_1 \circ \dots \circ R_r$$

If the integral has length  $r \rightarrow$  Transcendental function  
with degree  $r$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

1)  $\odot(R_1 R_2) \odot \dots = \dots \odot R_1 \odot \dots +$   
 $\dots \odot R_2 \odot \dots$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \dots \odot (R_1 R_2) \odot \dots = \dots \odot R_1 \odot \dots + \dots \odot R_2 \odot \dots$$

$$2) \odot \odot \dots = \odot$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \dots \circ (R_1 R_2) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$2) \dots \circ = 0 \quad \forall c \in \mathbb{Q}$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \quad \dots \circ (R_1 R_2) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$2) \quad \dots \circ \subset \circ \dots = \circ \quad \text{if } \subset \in \mathbb{Q}$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \dots \circ (R_1, D) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$2) \dots \circ \circ = \circ \quad \text{if } c \in \mathbb{Q}$$

$$\frac{x^n}{n^2}$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \dots \circ (R_1 R_2) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$2) \dots \circ c \circ \dots = c \dots \quad \text{if } c \in \mathbb{Q}$$

---

$$\prod_{n=1}^{\infty} (x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^n} \frac{y^n}{n^n}$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \dots \circ (R_1 R_2) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$\circ \subset \circ \dots = \circ \quad f \subset \mathbb{Q}$$

$$(x, y) = \sum_{n > m \geq 1} \frac{x^n}{n^a} \frac{y^m}{m^b}$$



$$z \frac{\partial}{\partial z} \text{Li}_n(z) = \text{Li}_{n-1}(z)$$

$$\text{Li}_n(z) = \int_0^z \frac{t^{n-1}}{1-t} dt = \int_0^z \underbrace{\frac{dt}{t}}_{n-1} \dots$$
$$= - \int_0^z d \ln(1-t) = \underline{\underline{\int_0^z \frac{dt}{1-t}}}$$

Symbol

$$f_r(x) = \int d \ln R_1 \cdot \dots \cdot \underline{d \ln R_r}$$

$$\mathcal{J}(f_r) = R_1 \circ \dots \circ R_r$$

If the integral has length  $r \rightarrow$  Transcendental function  
with degree  $r$

$a, b > 1$

$$dL_{a,b}(x,y) = d \ln x L_{a-1,b}(x,y) + d \ln y L_{a,b-1}(x,y)$$

$L_{a,b}(x,y)$

$$a, b > 1$$

$$d \operatorname{Li}_{a,b}(x,y) = d \ln x \operatorname{Li}_{a-1,b}(x,y) + d \ln y \operatorname{Li}_{a,b-1}(x,y)$$

$$\operatorname{Li}_{a,0}(x,y) = \frac{1}{1-y} (y \operatorname{Li}_a(x) - \operatorname{Li}_a(xy))$$

$$\operatorname{Li}_{a,b}(x,y) = \frac{x}{1-x} \operatorname{Li}_b(xy)$$

$$a, b > 1$$

$$d \operatorname{Li}_{a,b}(x,y) = d \ln x \operatorname{Li}_{a-1,b}(x,y) + d \ln y \operatorname{Li}_{a,b-1}(x,y)$$

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$$\operatorname{Li}_{a,b}(x,y) = \frac{x}{1-x} \operatorname{Li}_b(xy)$$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

$$1) \quad \circ(R_1 R_2) \circ \dots = \dots \circ R_1 \circ \dots + \dots \circ R_2 \circ \dots$$

$$\circ \circ \dots = \circ \quad \text{if } \circ \in \mathbb{Q}$$

$$(x, y) = \sum_{n > m \geq 1} \frac{x^n}{n^a} \frac{y^m}{m^b}$$

$$a, b > 1$$

$$d \text{Li}_{a,b}(x,y) = d \ln x \text{Li}_{a-1,b}(x,y) + d \ln y \text{Li}_{a,b-1}(x,y)$$

$$\text{Li}_{a,0}(x,y) = \frac{1}{1-y} \left( y \text{Li}_a(x) = \text{Li}_a(xy) \right)$$

$$\underline{\text{Li}_{0,b}(x,y)} = \frac{x}{1-x} \text{Li}_b(xy)$$

$$d \text{Li}_{0,b}(xy) = -d \ln(1-x) \text{Li}_b(xy) - d \ln(1-y) \text{Li}_b(xy) - d \ln\left(\frac{y}{1-y}\right) \text{Li}_b(xy)$$

$$a, b > 1$$

$$d \operatorname{Li}_{a,b}(x,y) = d \ln x \operatorname{Li}_{a-1,b}(x,y) + d \ln y \operatorname{Li}_{a,b-1}(x,y)$$

$$\operatorname{Li}_{a,0}(x,y) = \frac{1}{1-y} \left( y \operatorname{Li}_a(x) = \operatorname{Li}_a(xy) \right)$$

$$\underline{\operatorname{Li}_{0,b}(x,y)} = \frac{x}{1-x} \operatorname{Li}_b(xy)$$

$$d \operatorname{Li}_{0,b}(xy) = -d \ln(1-x) \operatorname{Li}_b(xy) - d \ln(1-y) \operatorname{Li}_b(xy) - d \ln\left(\frac{y}{1-y}\right) \operatorname{Li}_b(xy)$$

$$\operatorname{Li}_1(x) = -\ln(1-x)$$



$$a, b > 1$$

$$d \text{Li}_{a,b}(x,y) = d \ln x \text{Li}_{a-1,b}(x,y) + d \ln y \text{Li}_{a,b-1}(x,y)$$

$$\text{Li}_{a,0}(x,y) = \frac{1}{1-y} (y \text{Li}_a(x) = \text{Li}_a(xy))$$

$$\underline{\underline{\text{Li}_{0,b}(x,y) = \frac{x}{1-x} \text{Li}_b'(xy)}}$$

$$d \text{Li}_{0,b}(xy) = -d \ln(1-x) \text{Li}_b'(xy) - d \ln(1-y) \text{Li}_b'(xy) - d \ln\left(\frac{y}{1-y}\right) \text{Li}_b'(xy)$$

$$\text{Li}_1'(x) = -\ln(1-x)$$

unit

$$\mathcal{J}(L_{i,i}(x,y)) = (1-xy) \otimes (1-x) + (1-x) \otimes (1-y) + (1-xy) \otimes \left(\frac{y}{1-y}\right)$$

$$\begin{aligned}
 \mathcal{J}(L_{i,i}(x,y)) &= (1-xy) \otimes (1-x) + \\
 &\quad (1-x) \otimes (1-y) + \\
 &\quad \underbrace{(1-xy) \otimes \left(\frac{y}{1-y}\right)}_{(1-xy) \otimes}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{I}(L_{i,i}(x,y)) &= (1-xy) \otimes (1-x) + \\
 &\quad (1-x) \otimes (1-y) + \\
 &\quad \underbrace{(1-xy) \otimes \left(\frac{y}{1-y}\right)} \\
 &= (1-xy) \otimes y + (1-xy) \otimes \frac{1}{1-y} \\
 &= (1-xy) \otimes y - (1-xy) \otimes (1-y)
 \end{aligned}$$