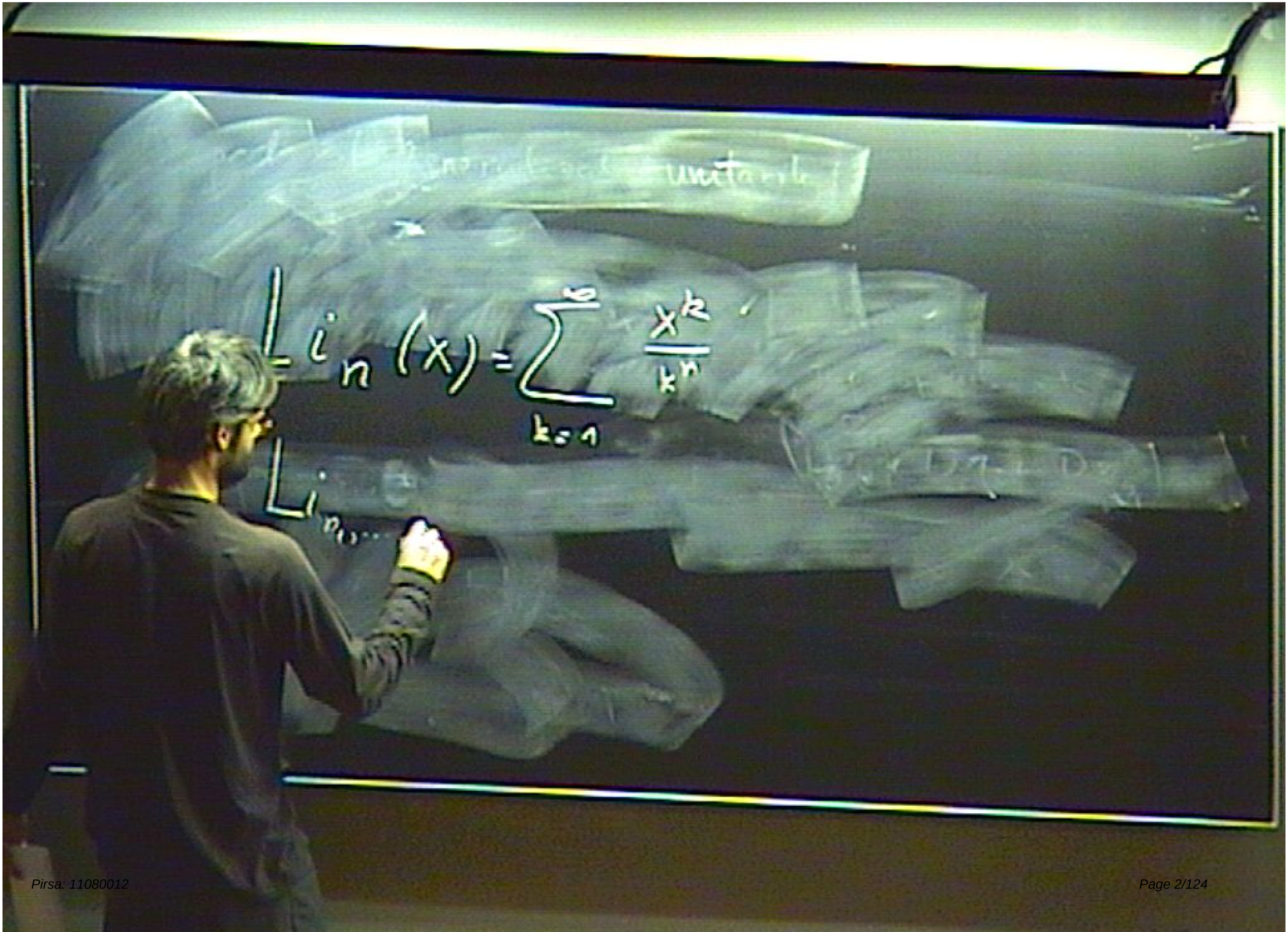
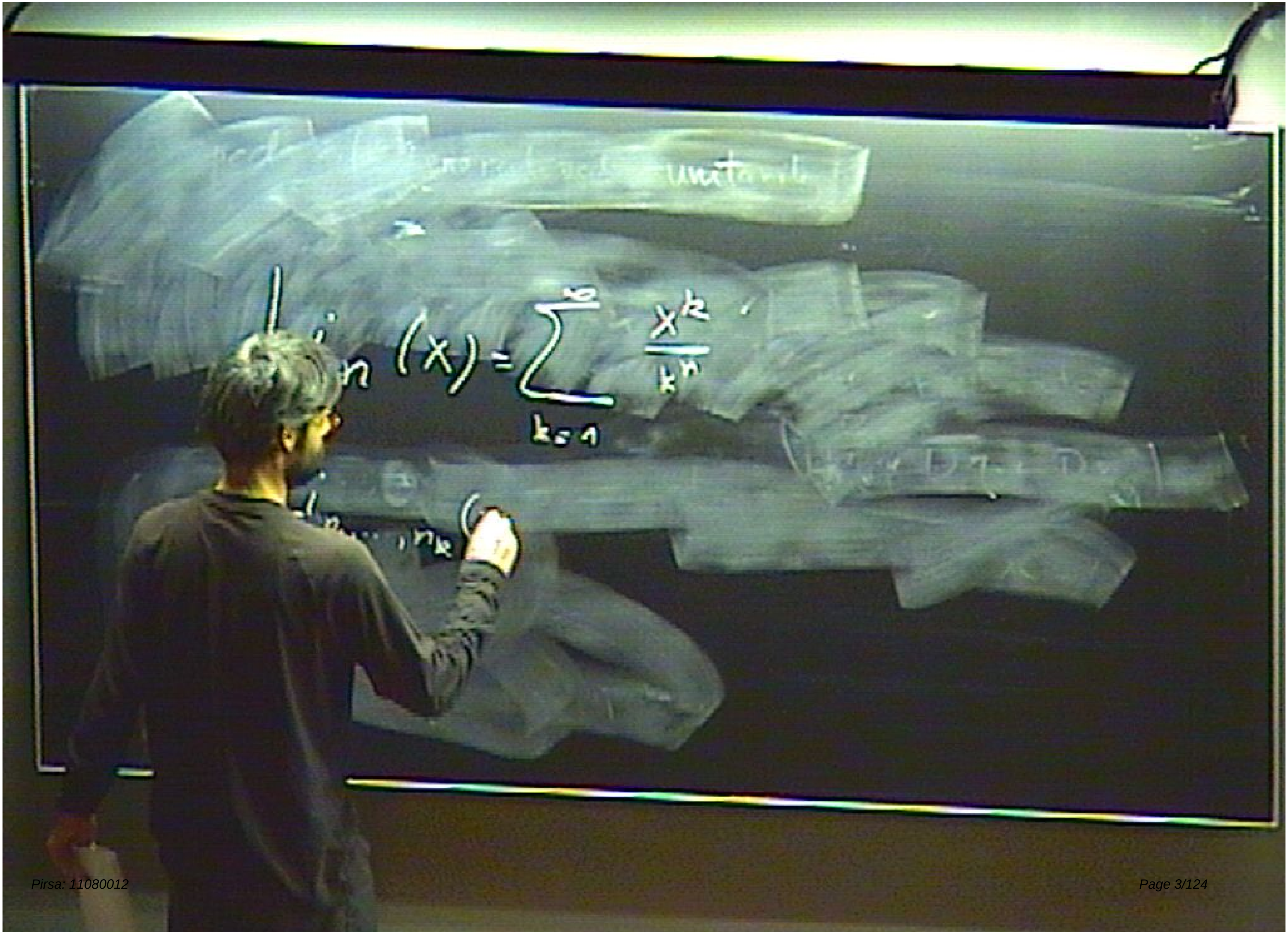
Title: Symbology and Scattering Amplitudes

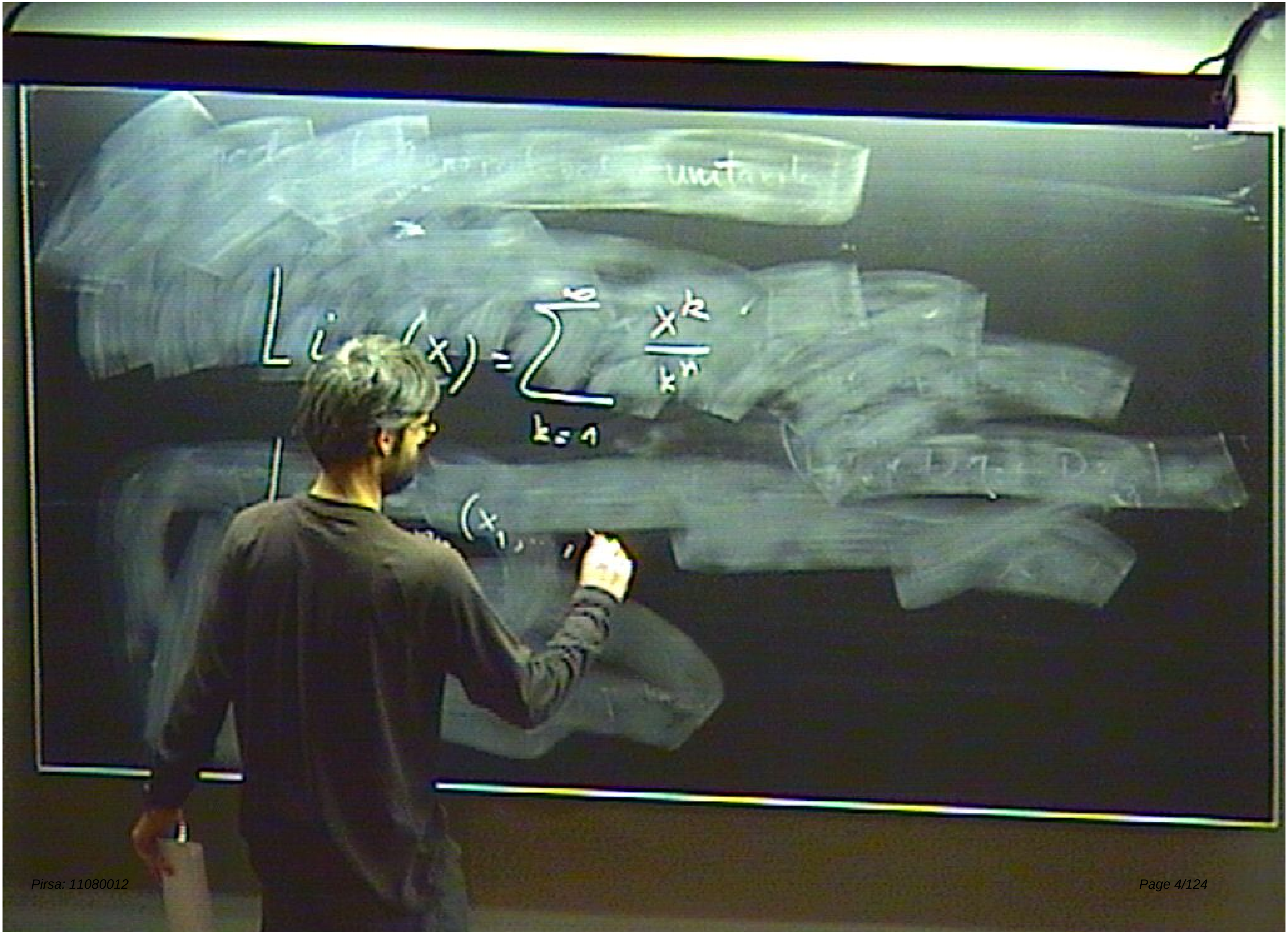
Date: Aug 04, 2011 12:15 PM

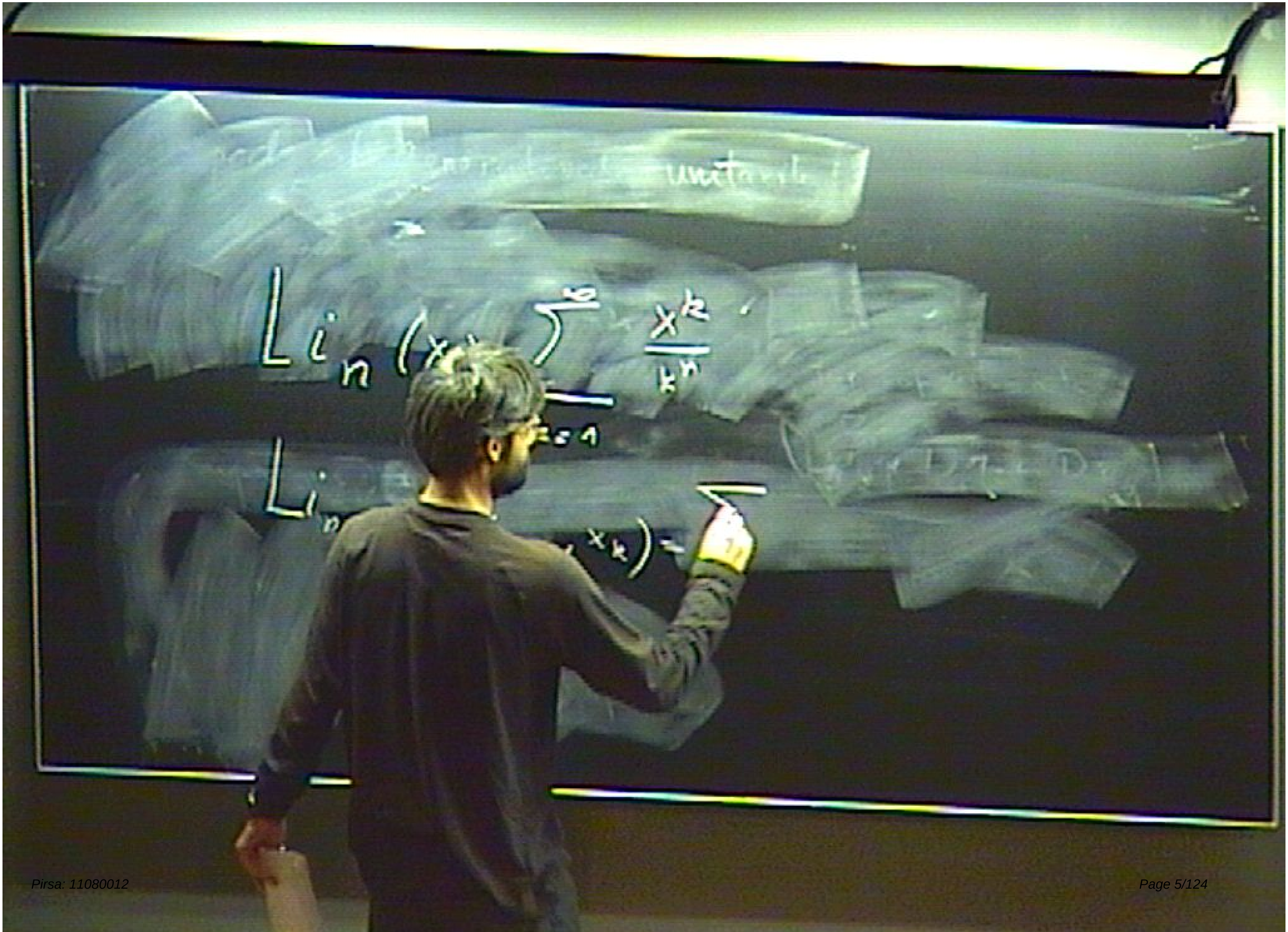
URL: http://pirsa.org/11080012

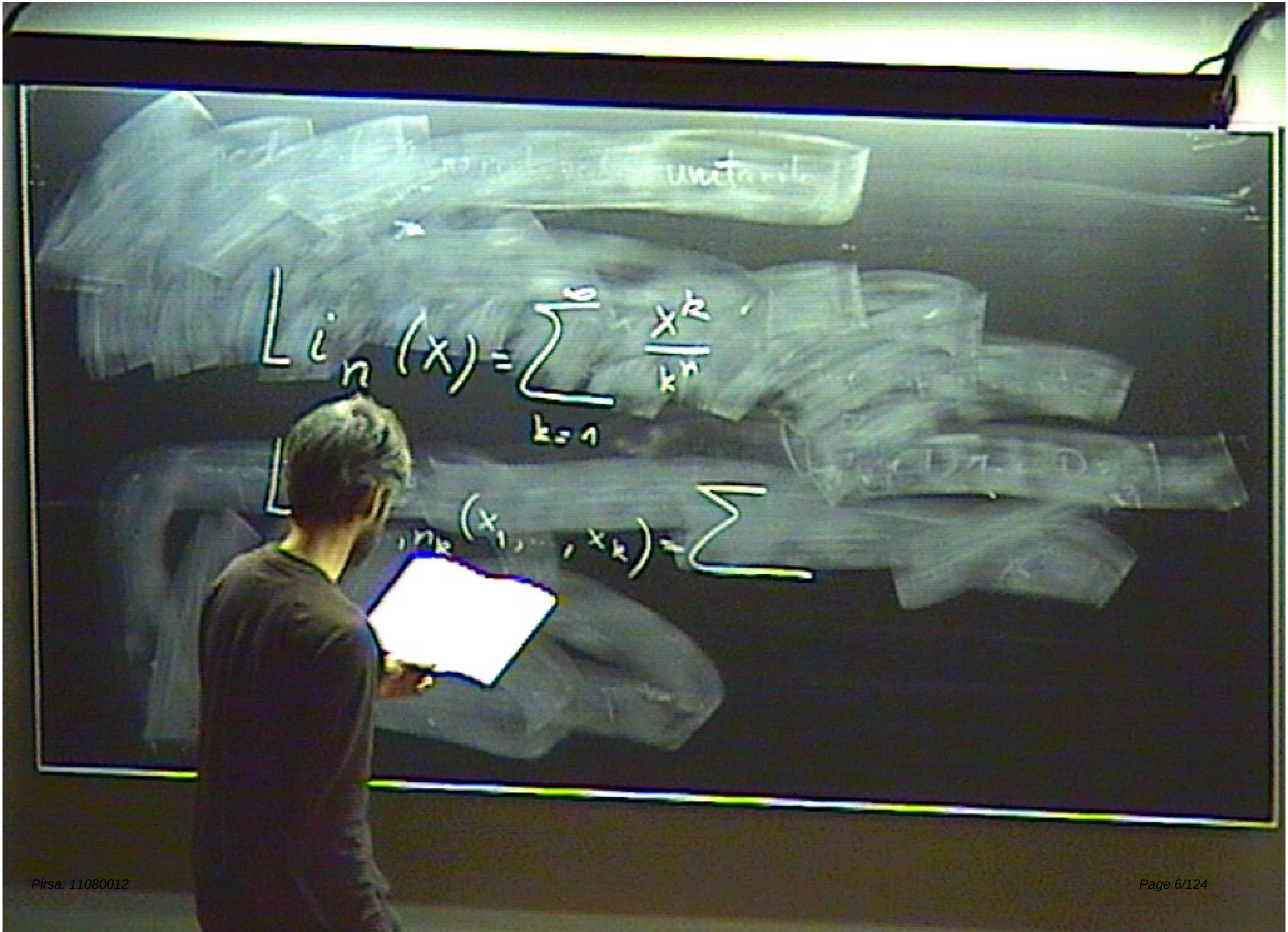
Abstract:

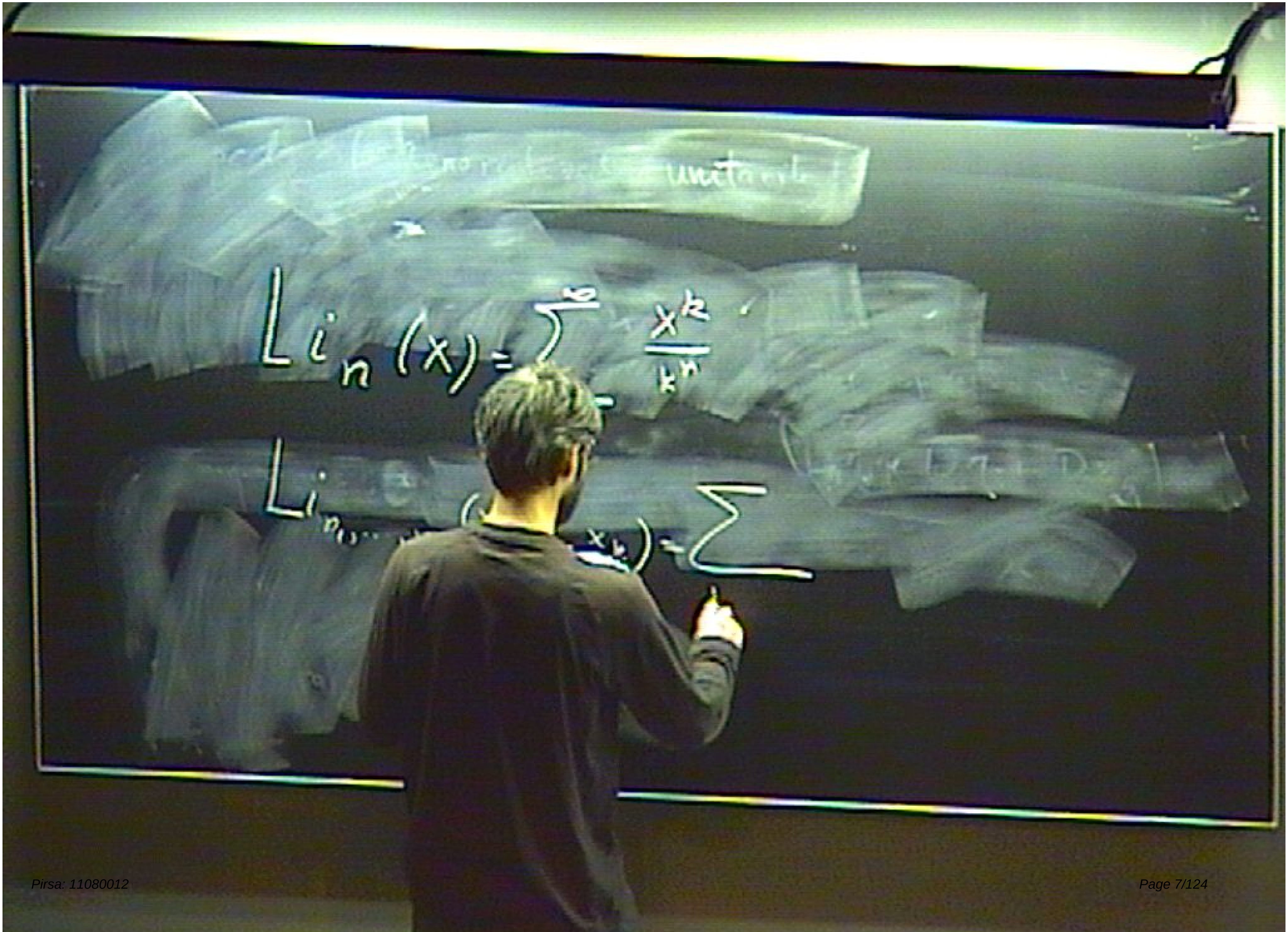

$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1 \ldots}$

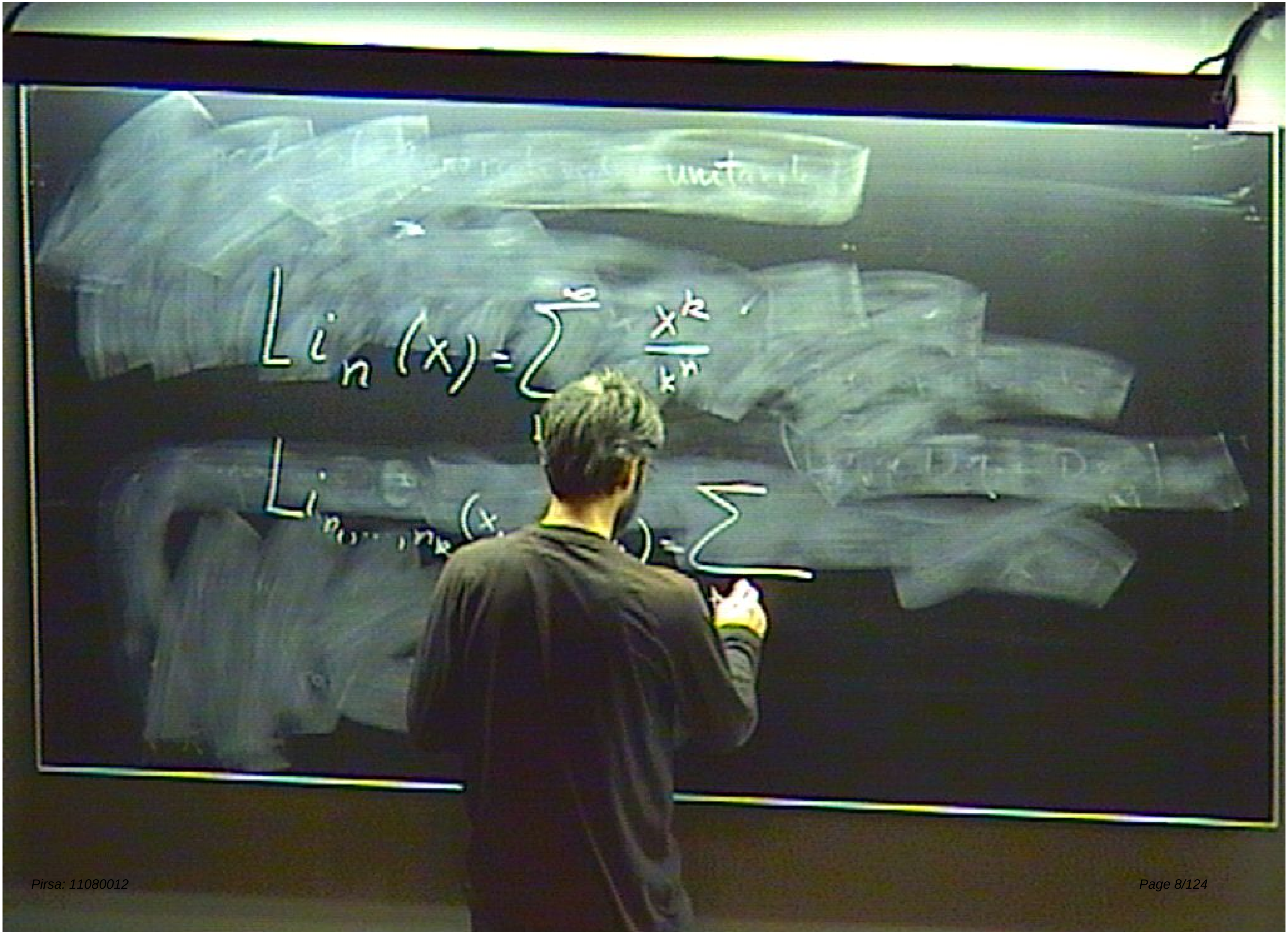

n (x) = $\sum_{k=1}^{\infty} \frac{x^k}{k^n}$

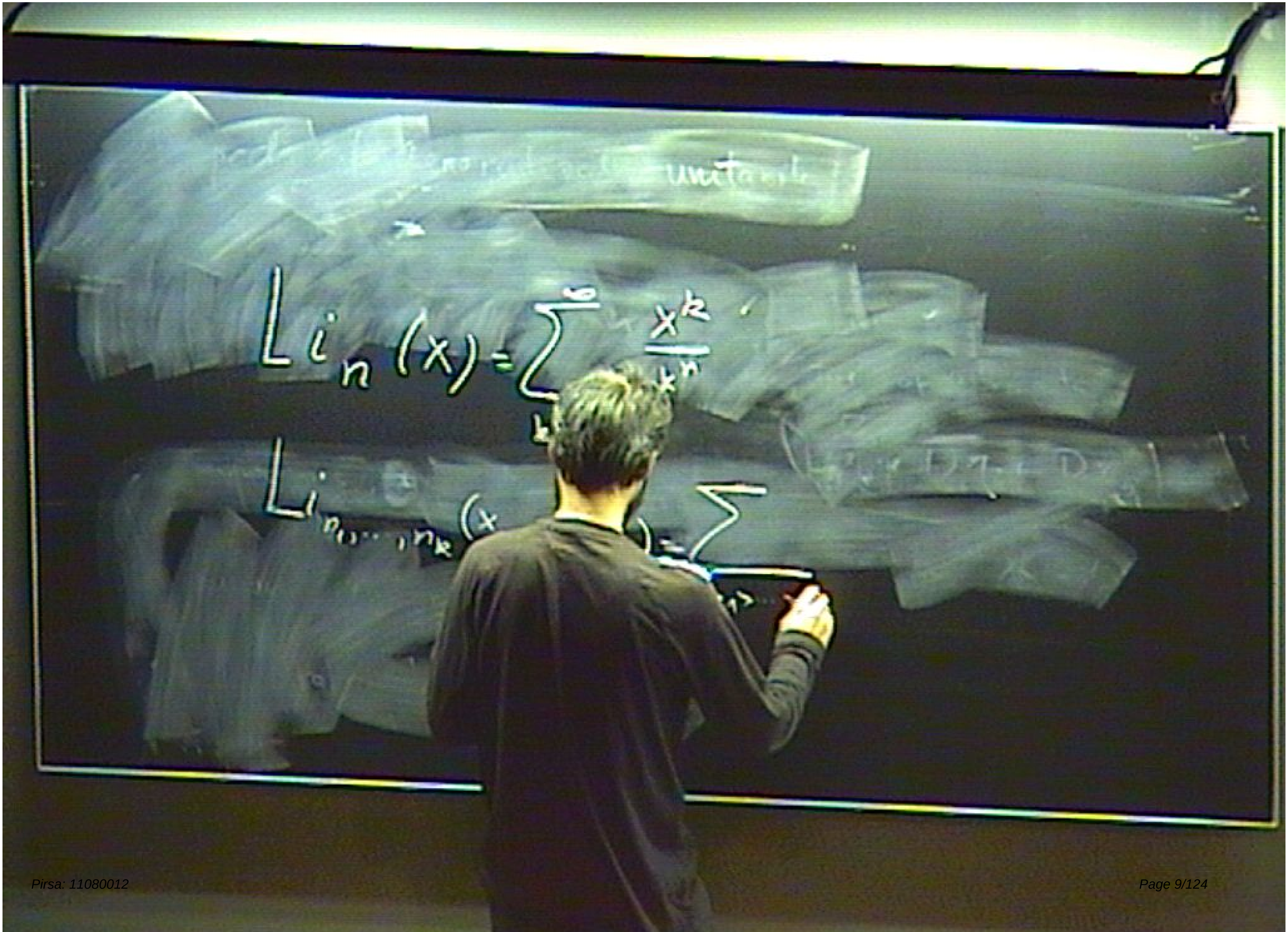

$L i \; (x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$

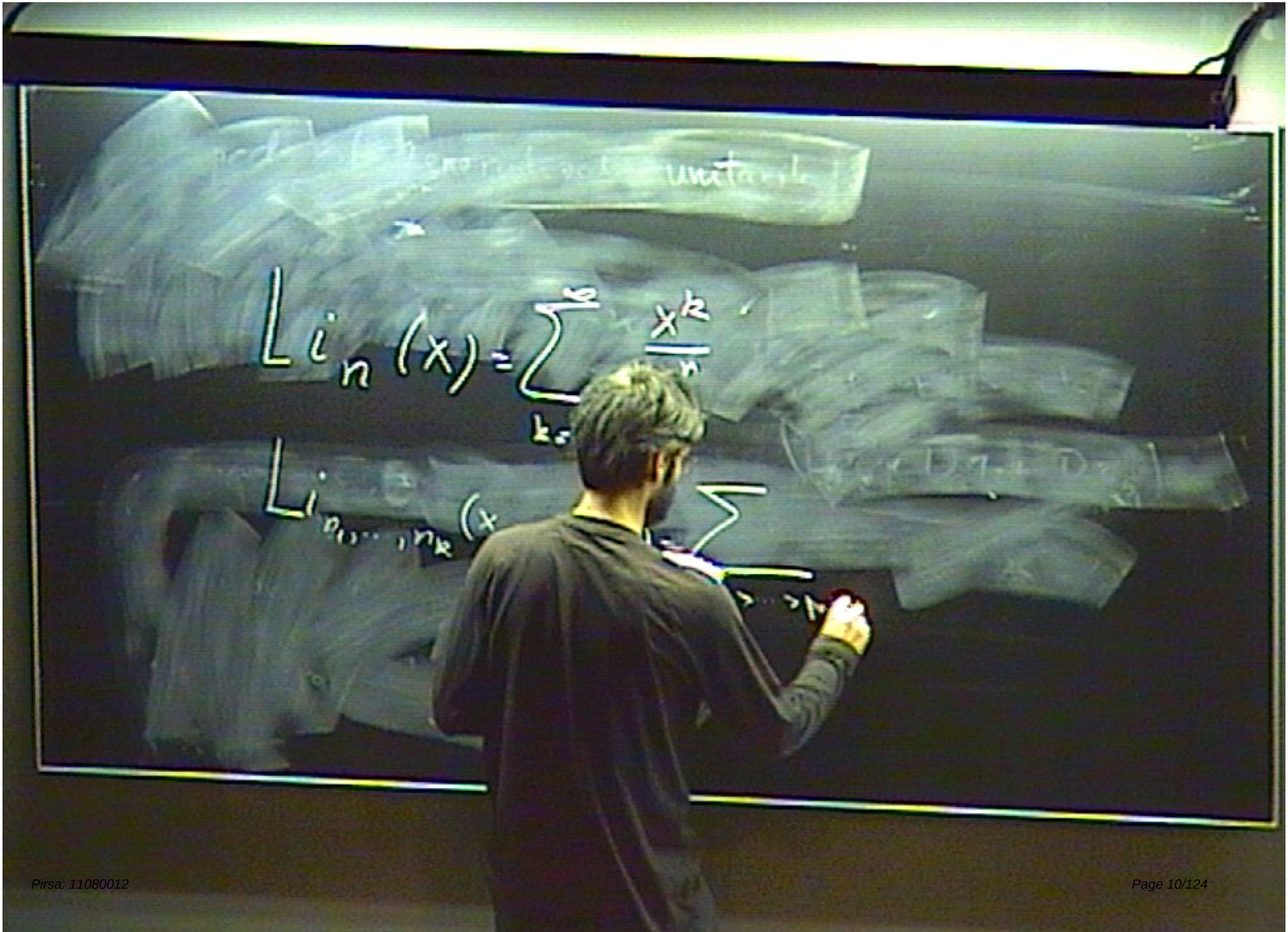

Li n (x
x k
k n
Li n
x k)

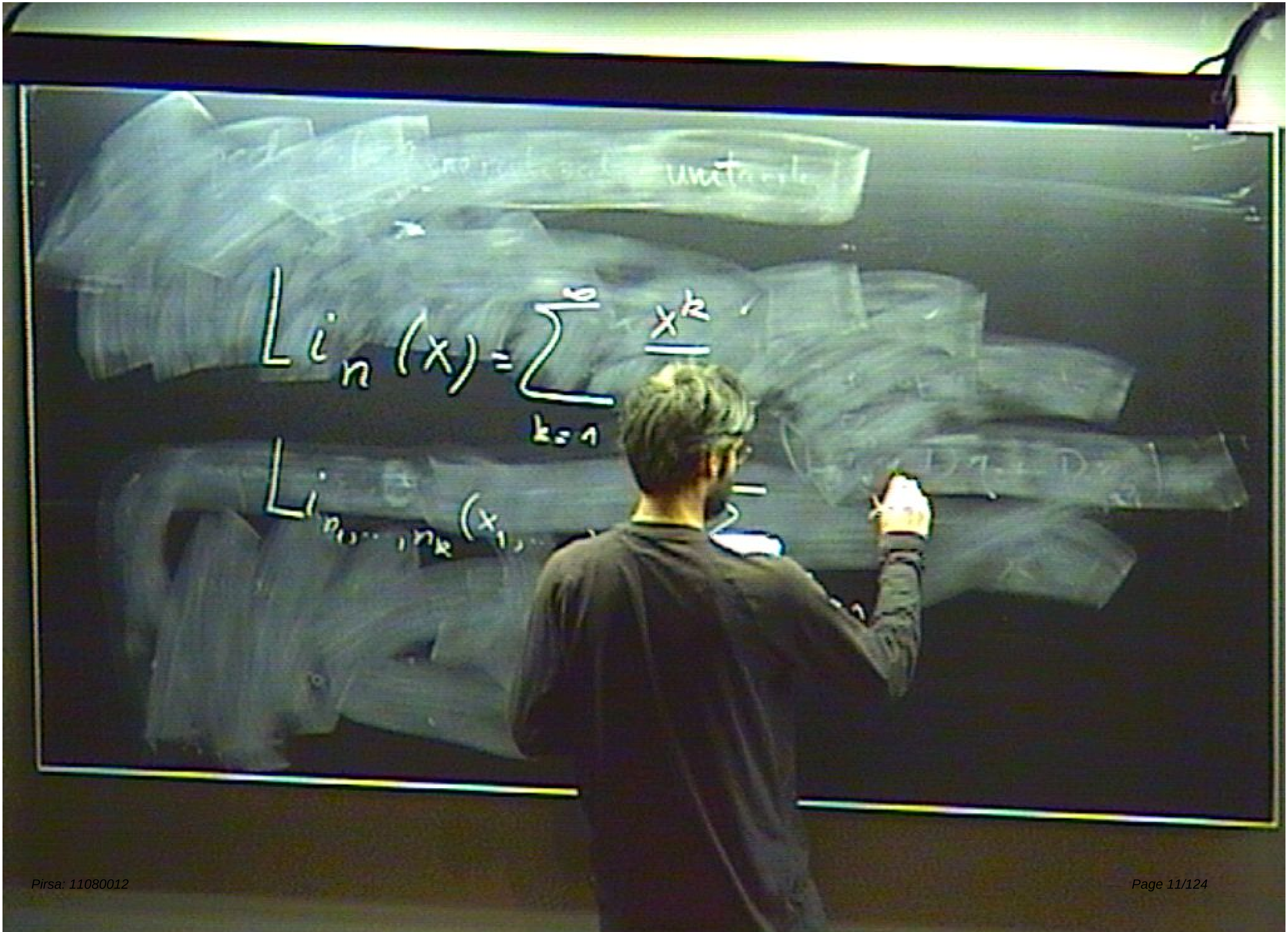

Lin(x) = ∑ k=1 ∞ x^k / k^n
,nk (x1, … , xk) = ∑

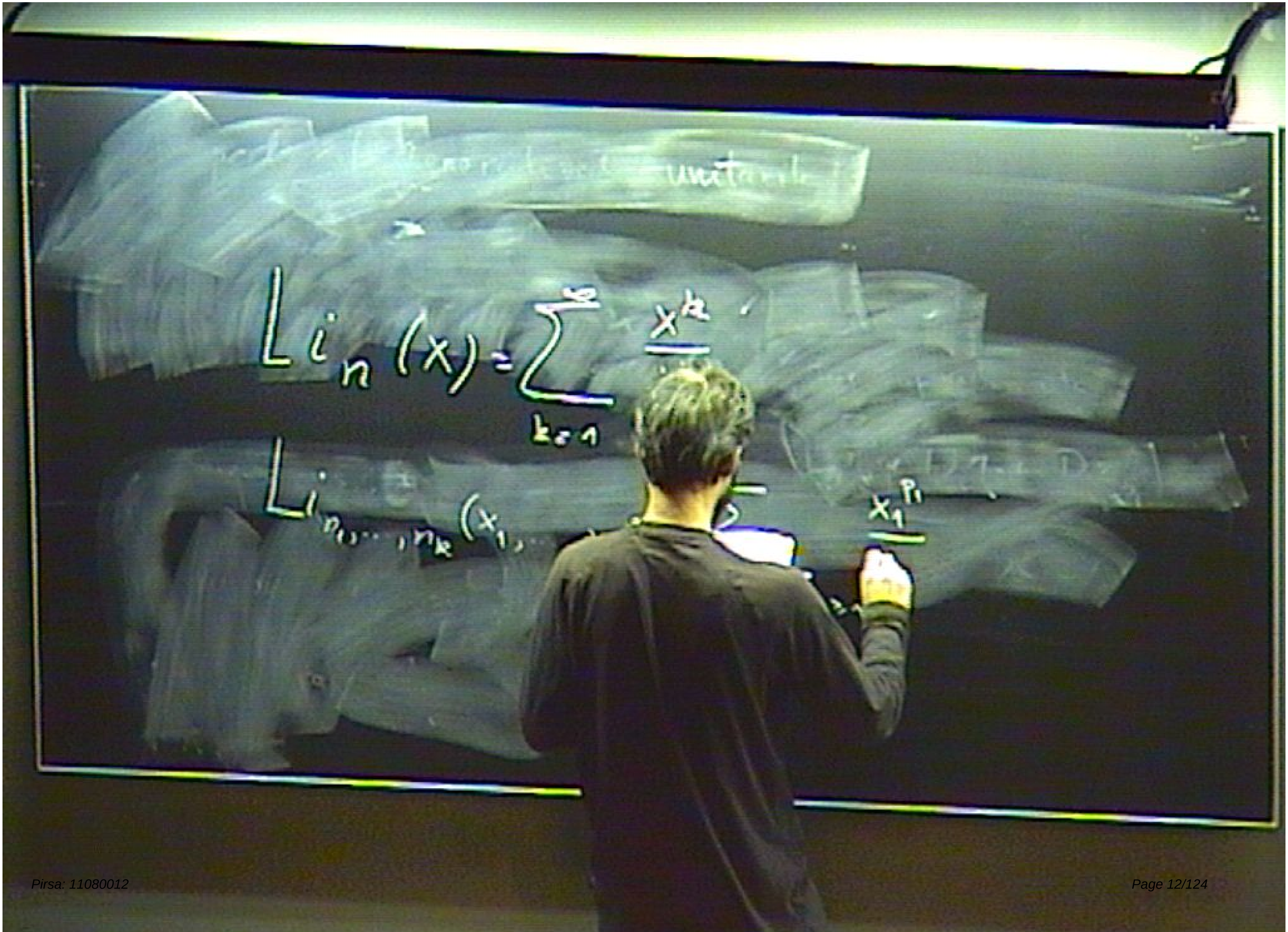

Li_n(x) = ∑ x^k / k^n

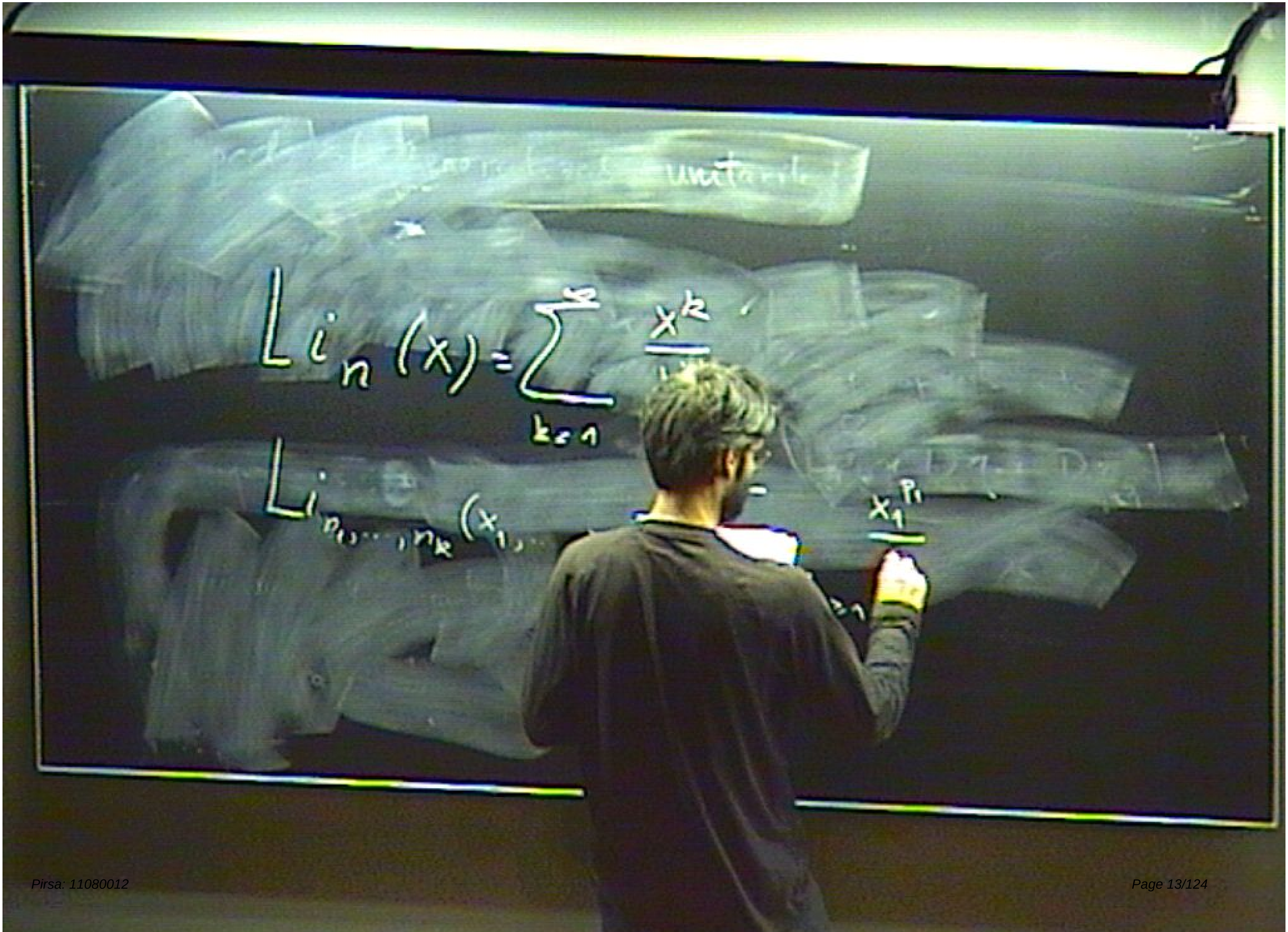

$Li_n(x) = \sum^{\infty} \frac{x^k}{k^n}$

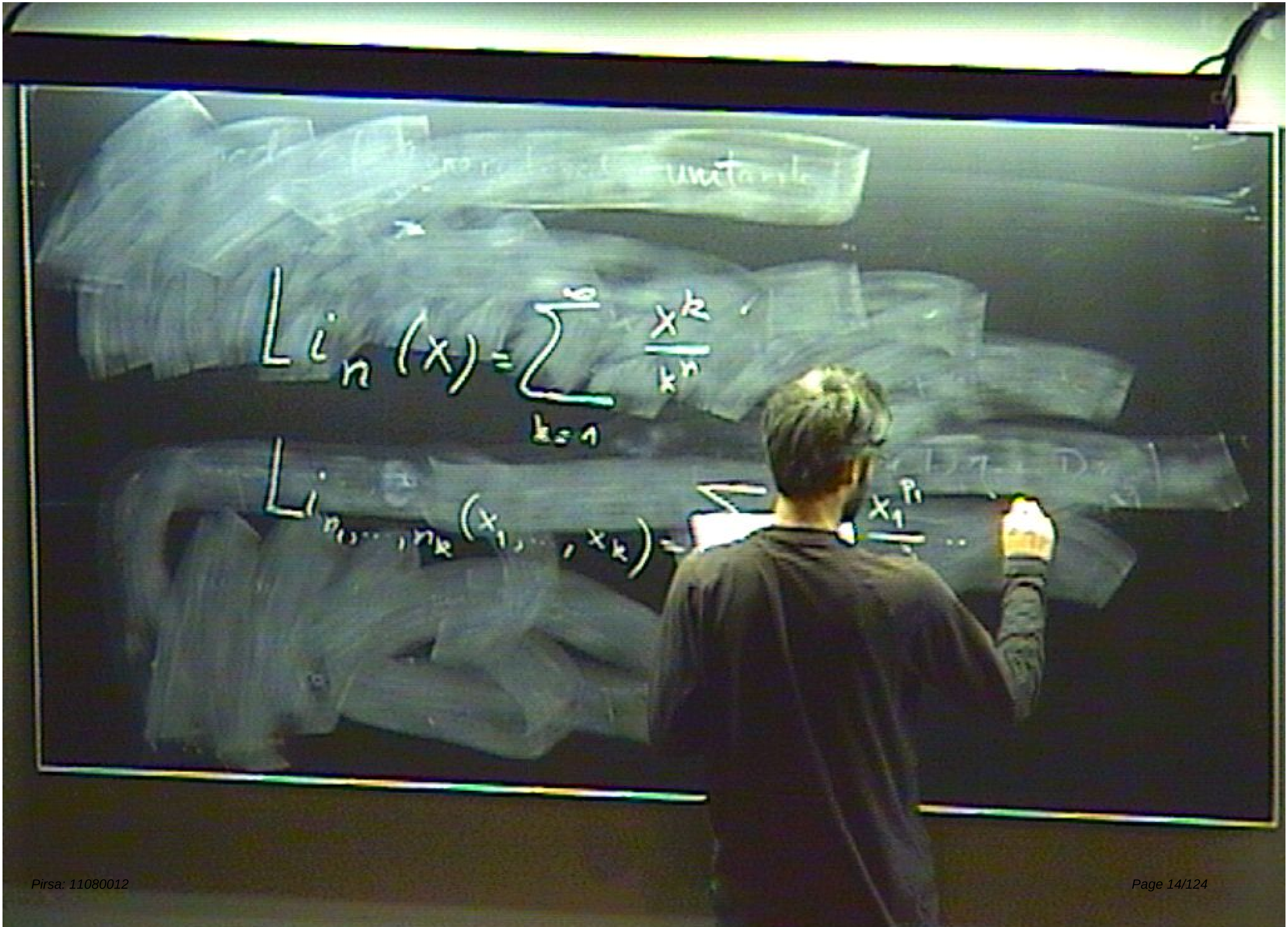

$Li_n(x) = \sum \frac{x^k}{k^n}$
$Li_{n_1,\ldots,n_k}(x$

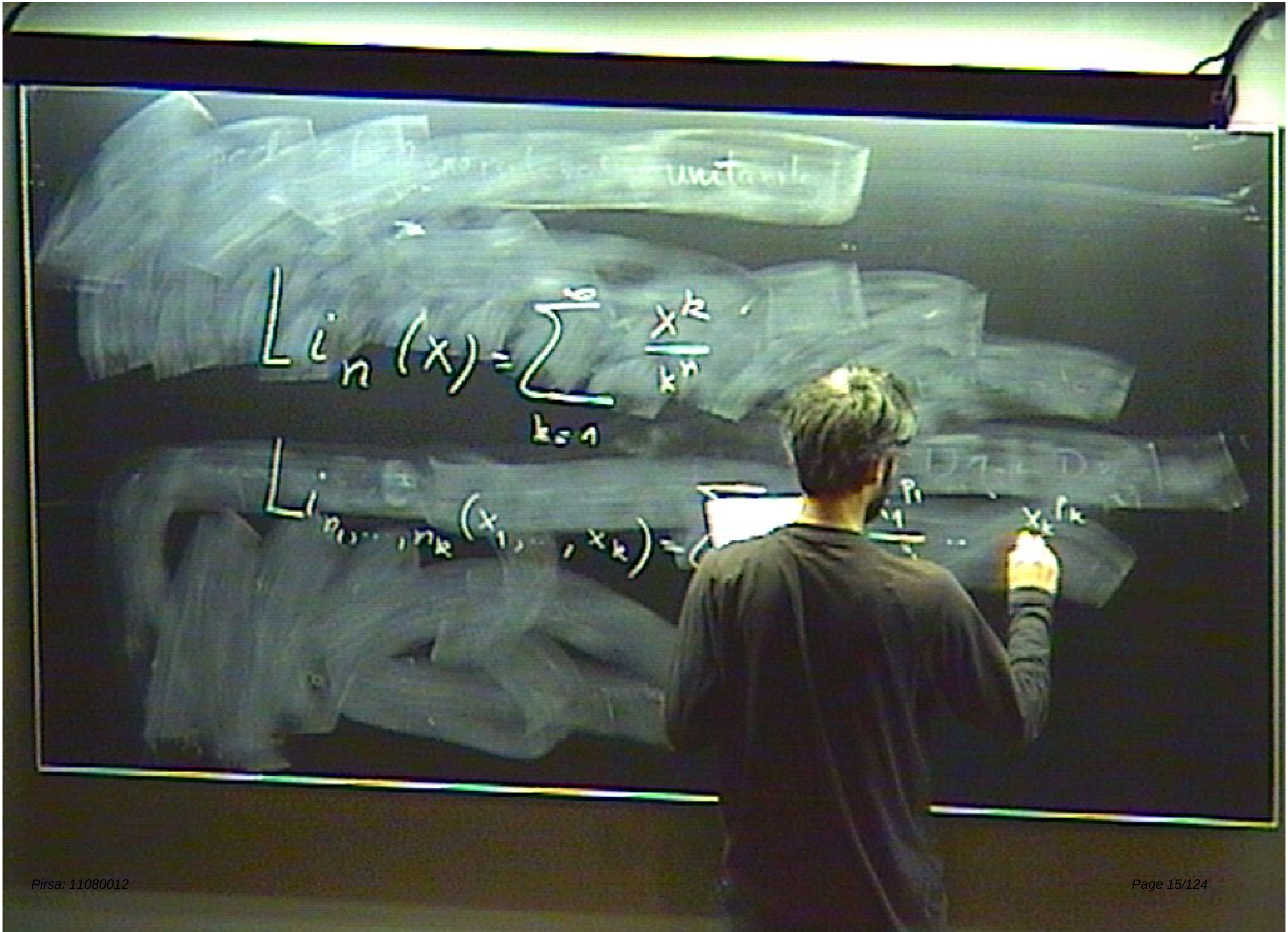

unitary
$Li_n(x) = \sum \frac{x^k}{n}$

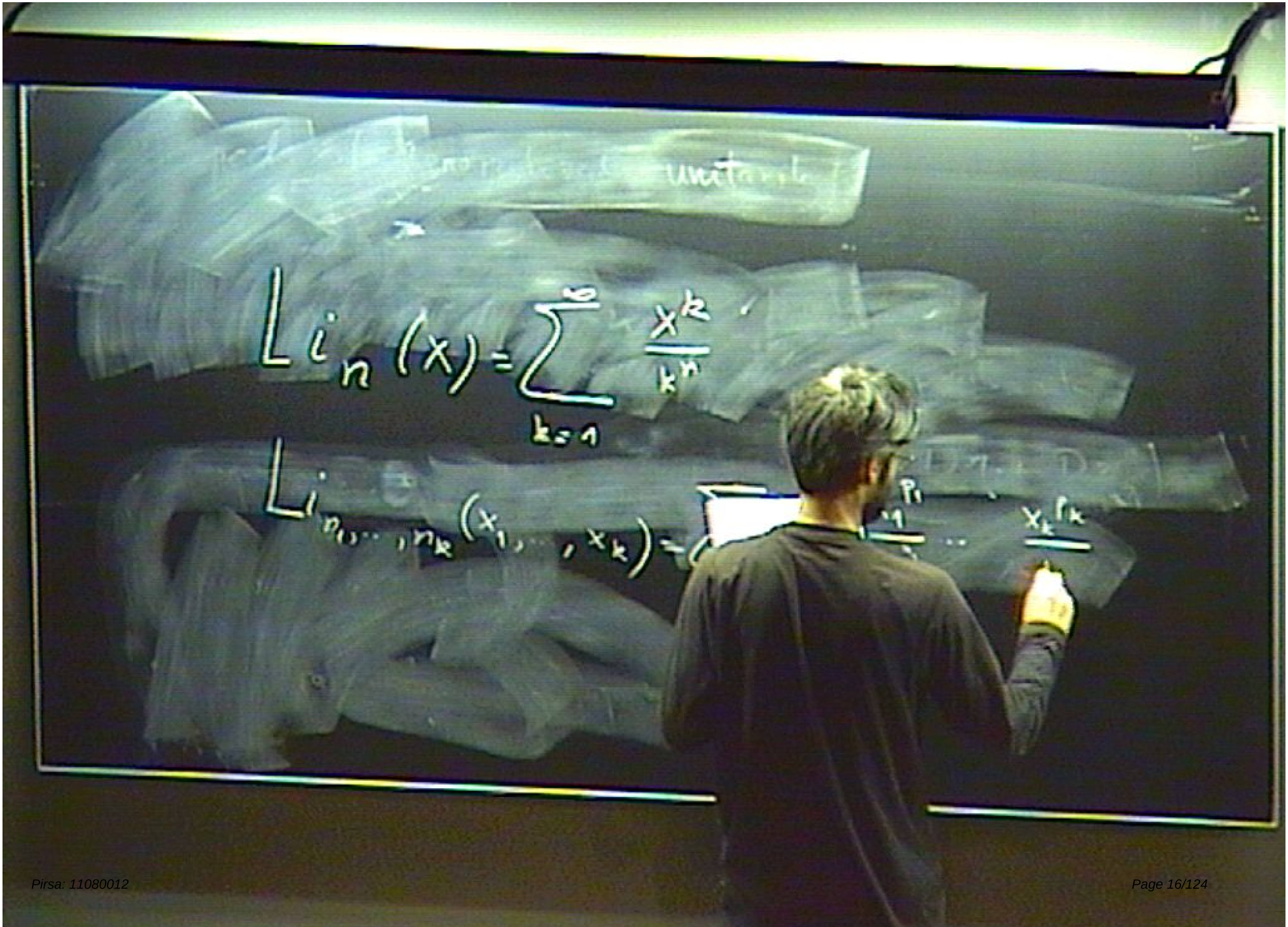

Li n (x) = Σ k=1 x^k
Li n1,...,nk (x1,...


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{\ }$
$Li_{n_1,\dots,n_k}(x_1,\dots$
$\frac{x_1^{p_1}}{\ }$

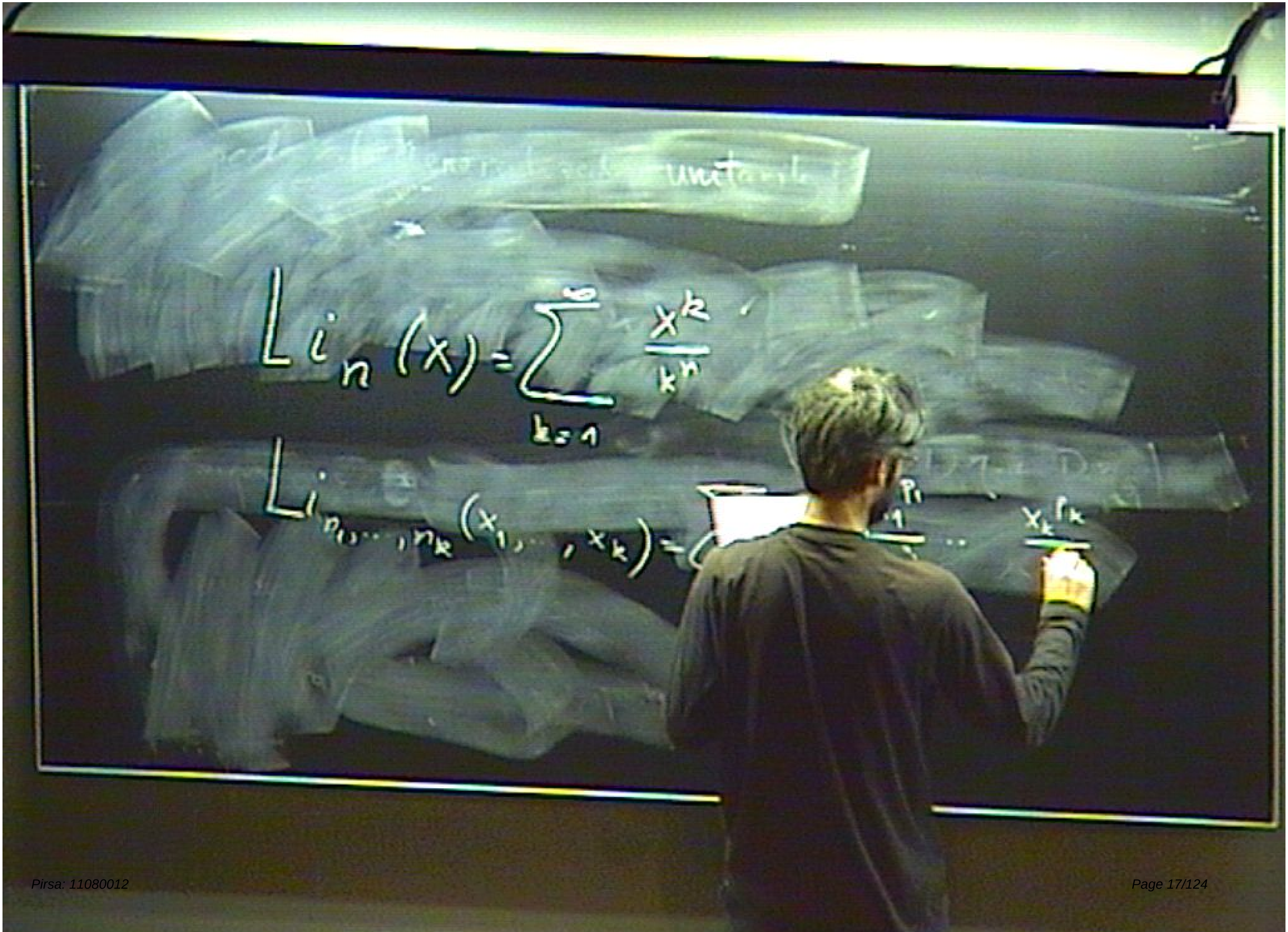

Li n (x) =
k=1
x k


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) =$

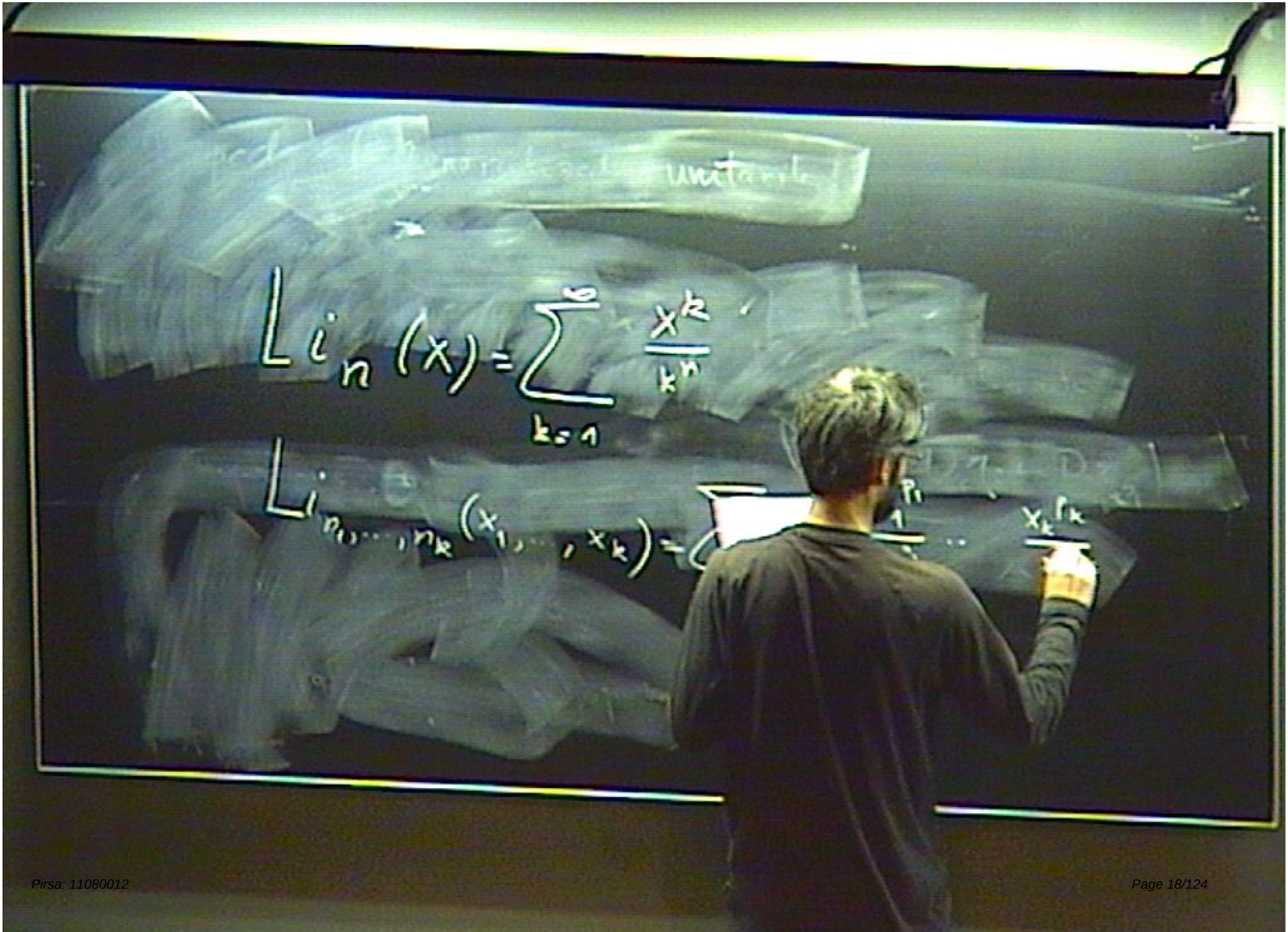

Li_n(x) = Σ_{k=1}^∞ x^k / k^n
Li_{n_1,...,n_k}(x_1, ..., x_k) =

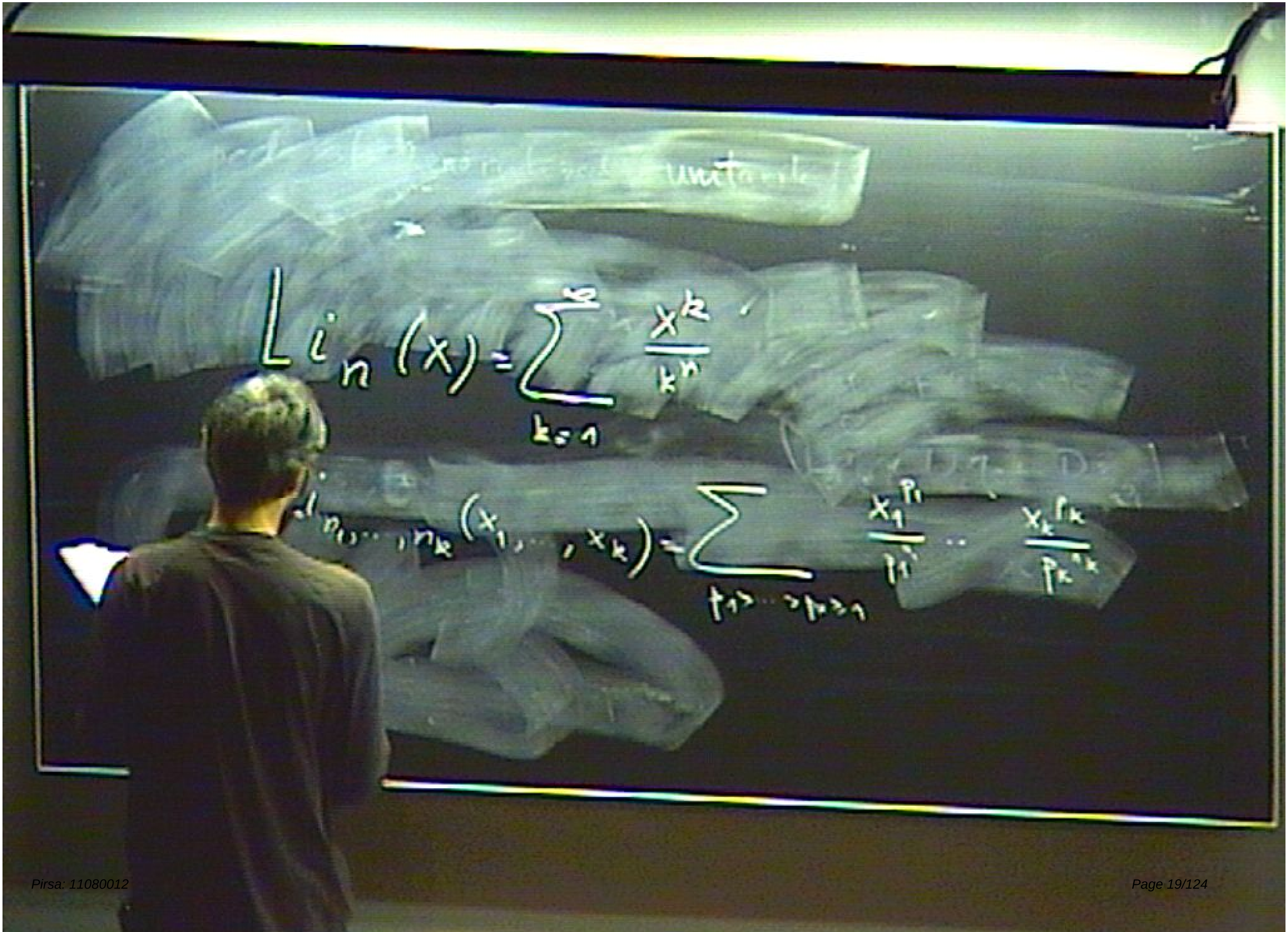


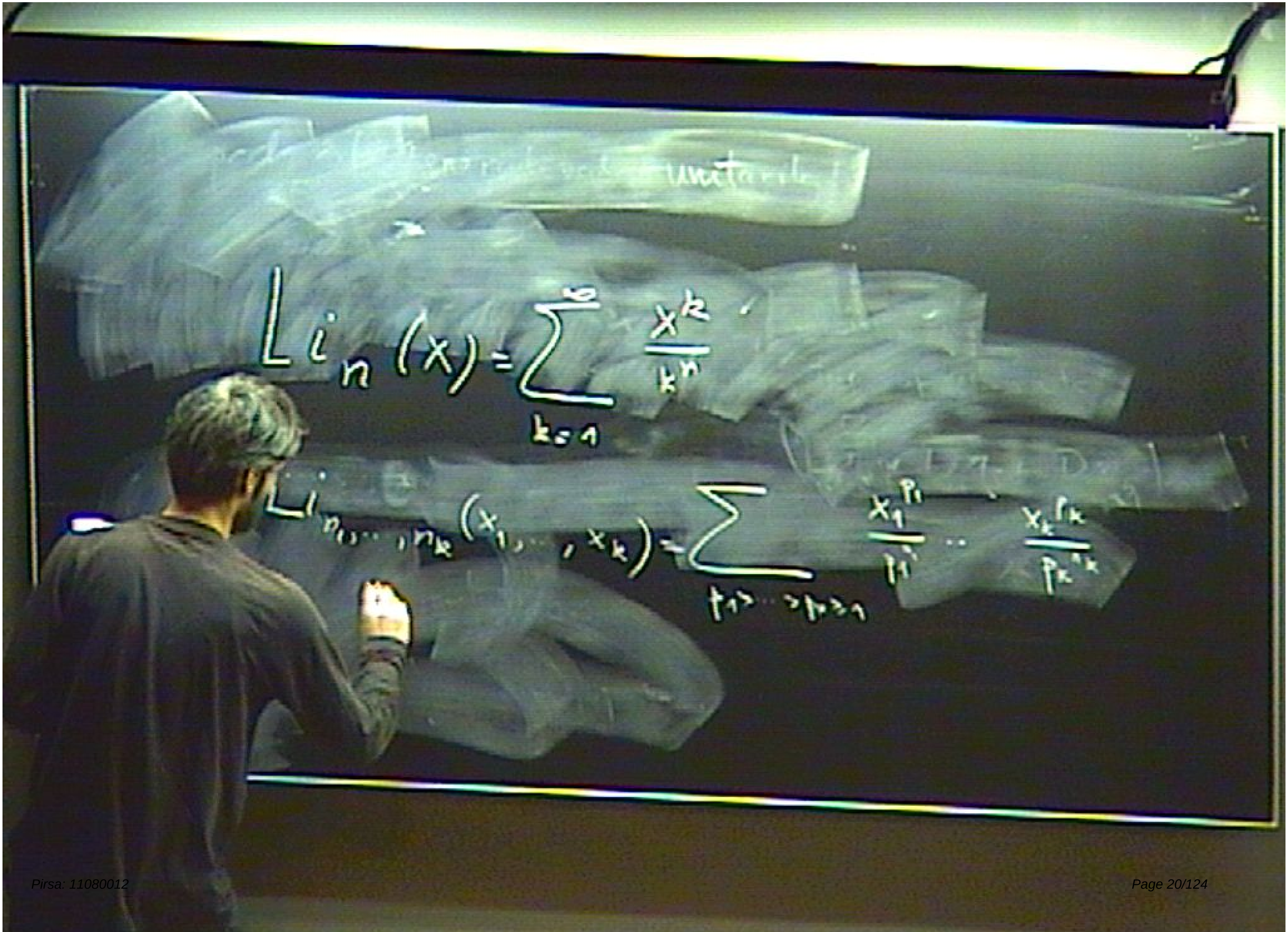
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) =$

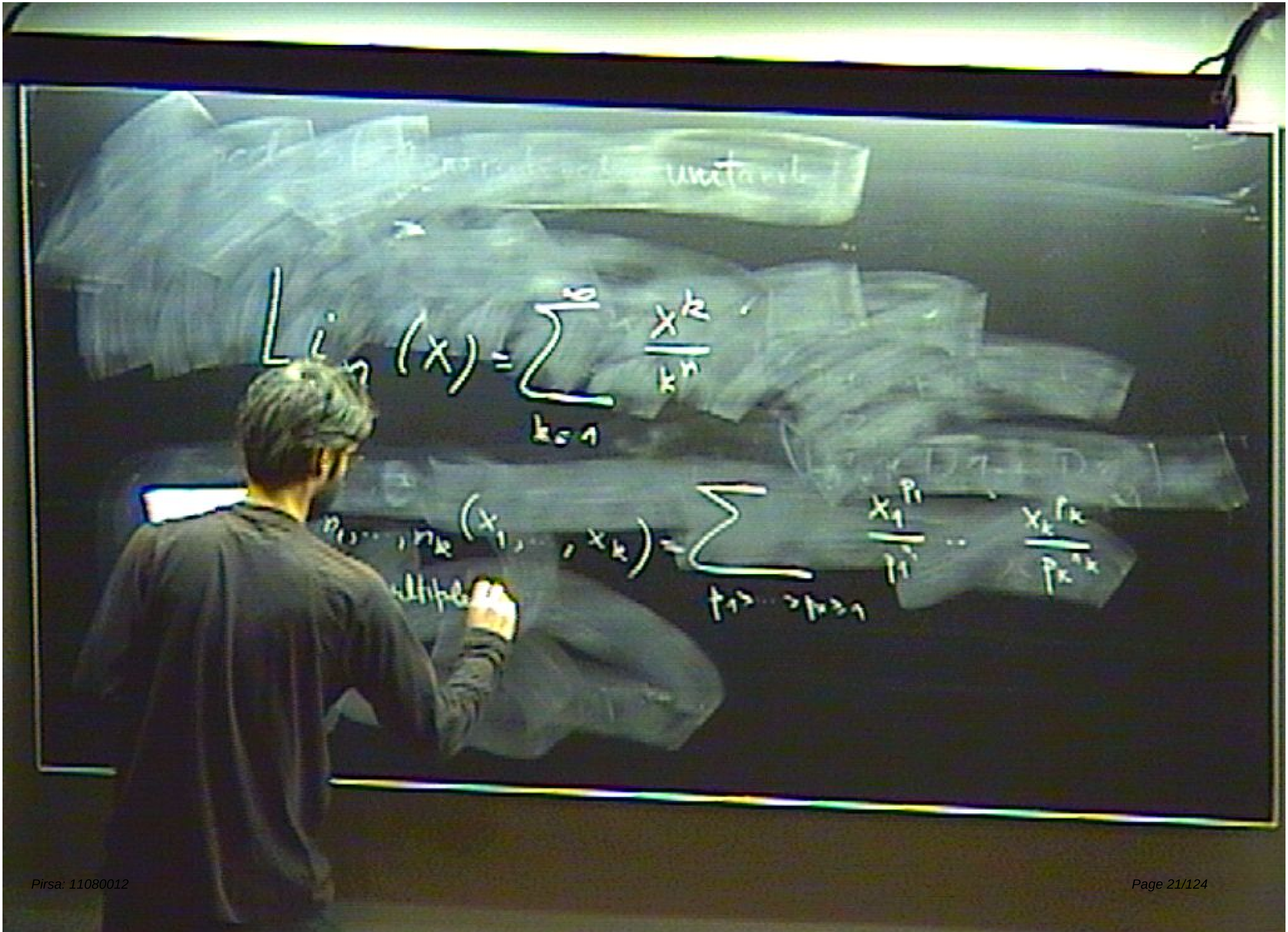


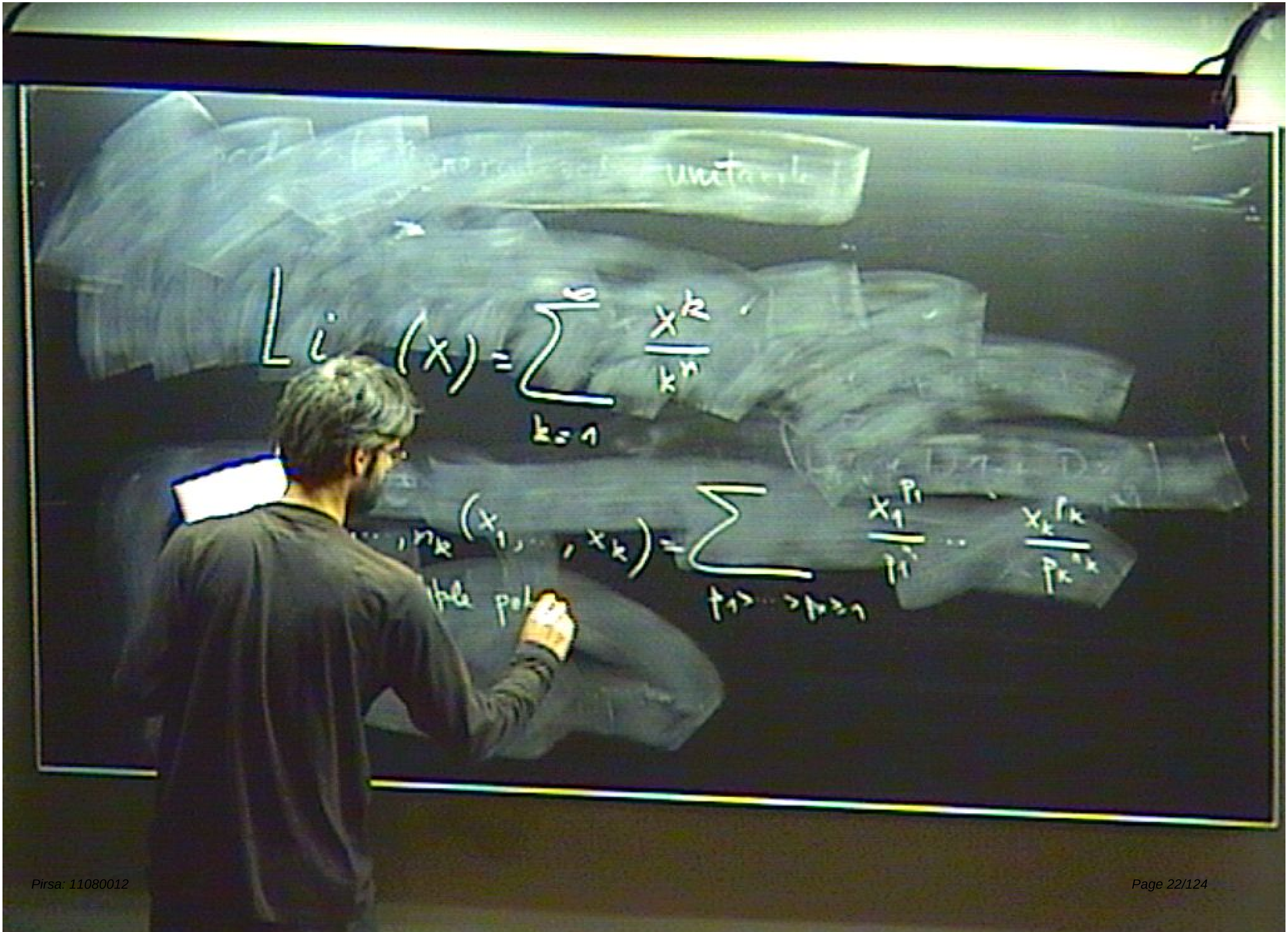
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) =$

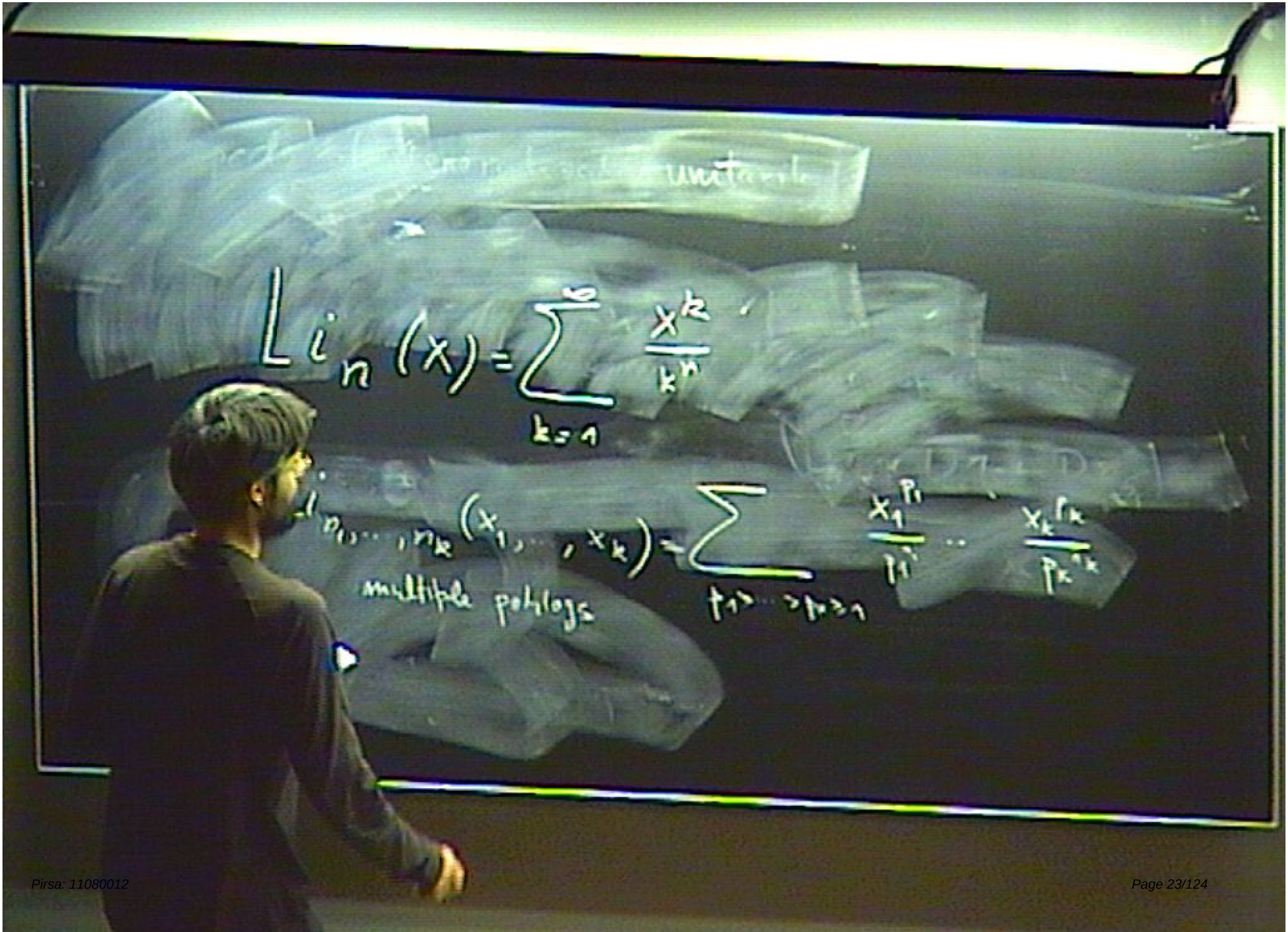

Li_n(x) = Σ_{k=1}^∞ x^k / k^n


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1>\dots>p_k\geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$

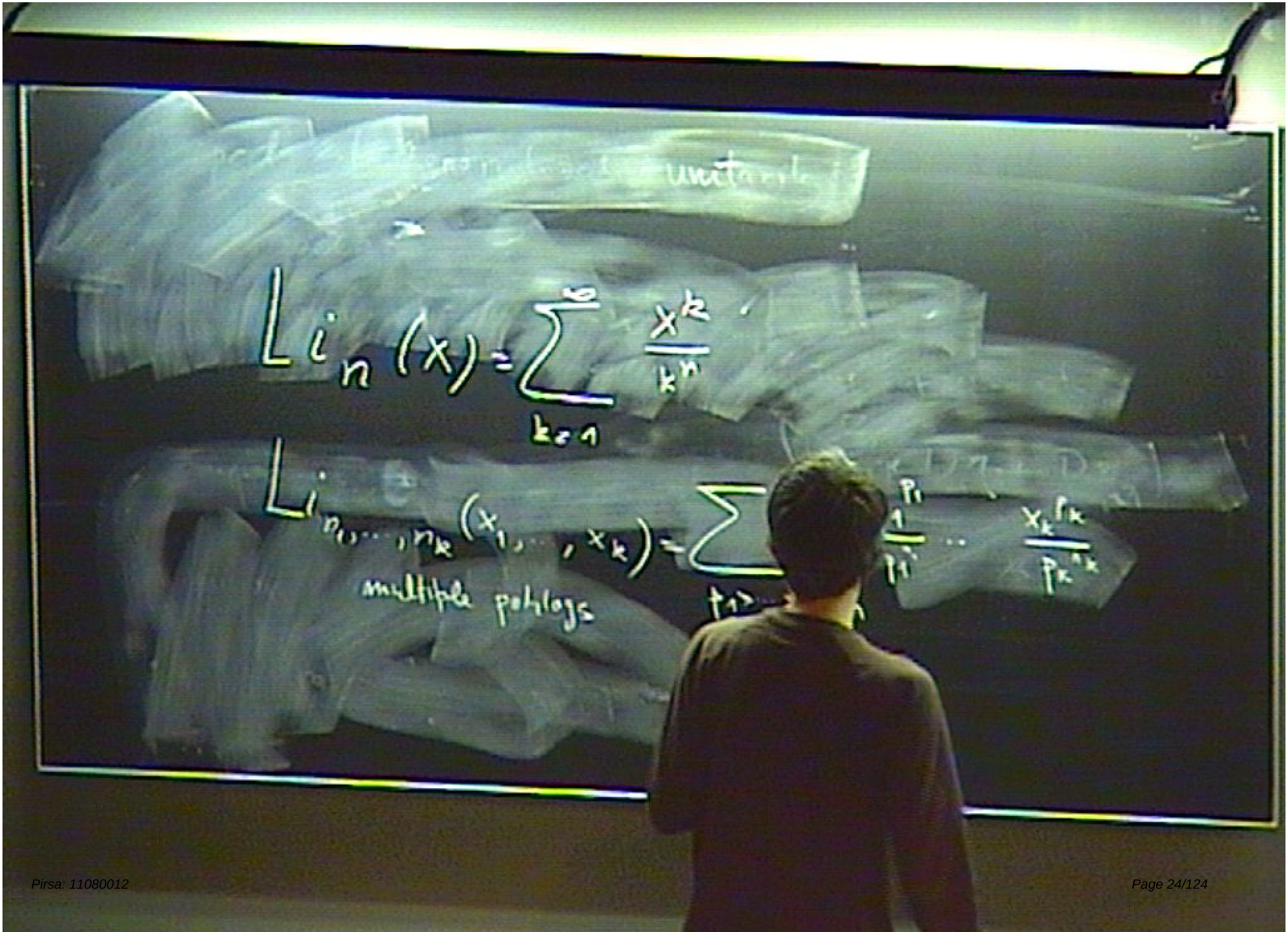

$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$

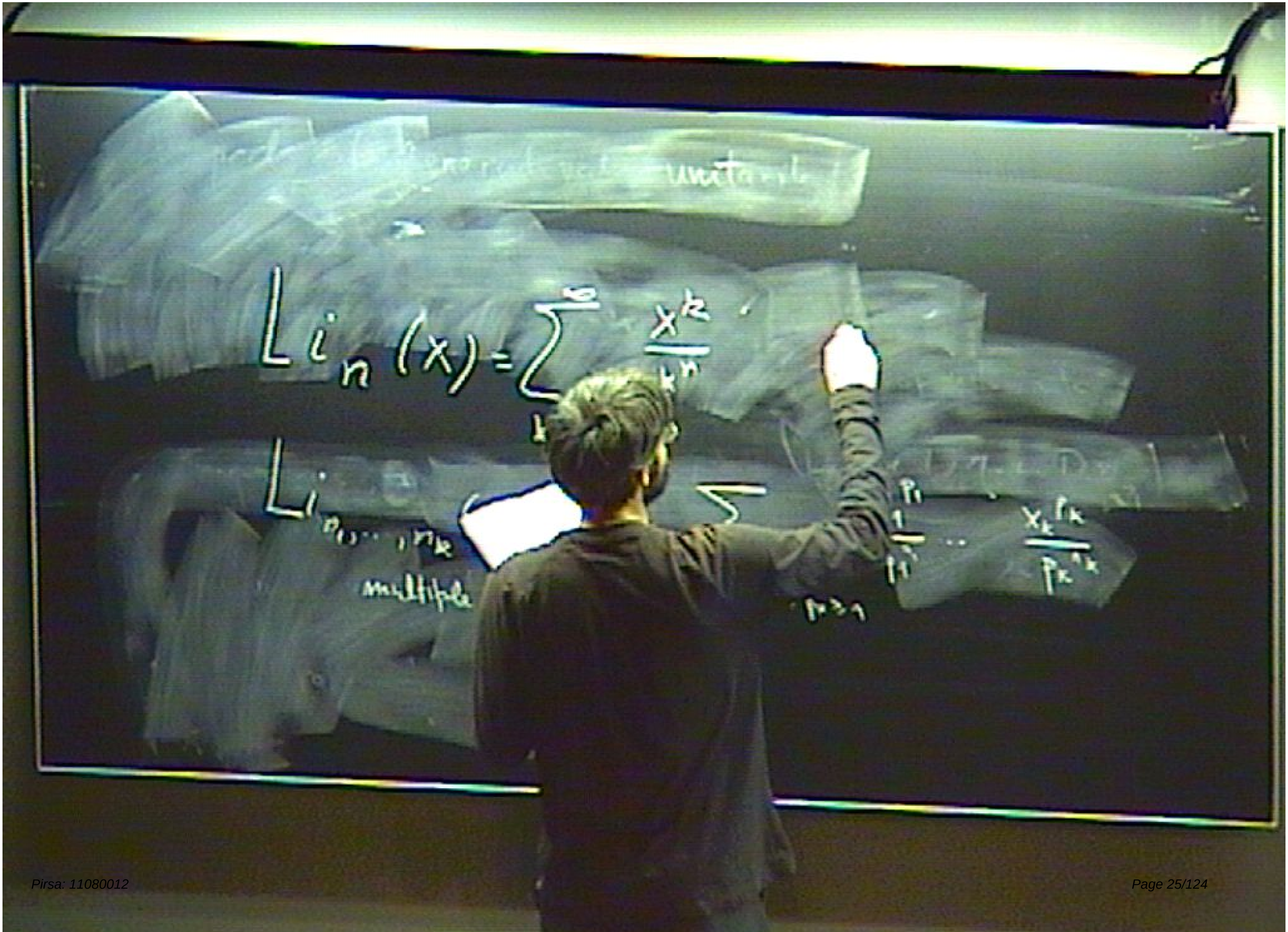

Li_n (x) = Σ_{k=1}^∞ x^k / k^n

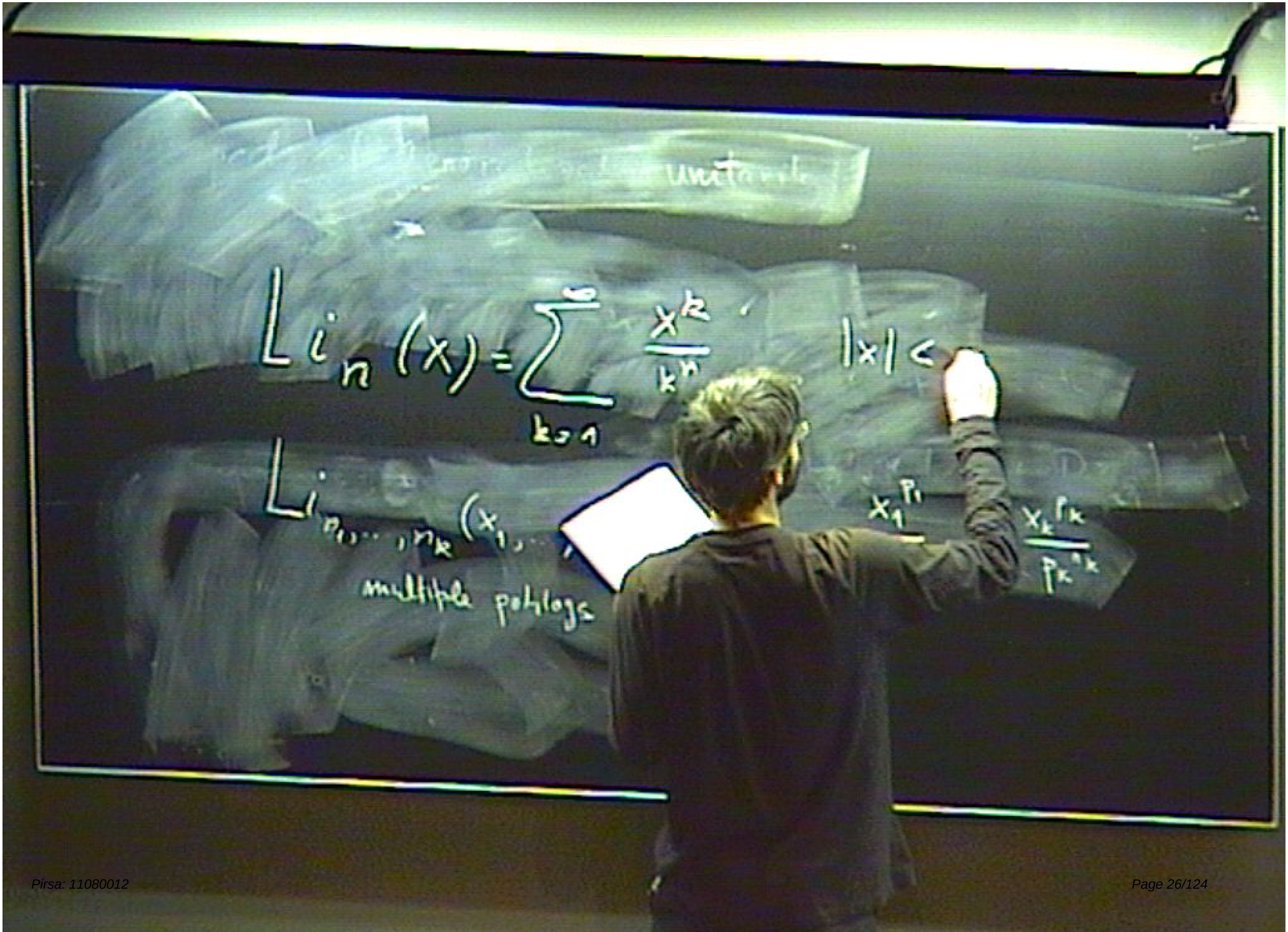

k=1
p1>…>pk≥1

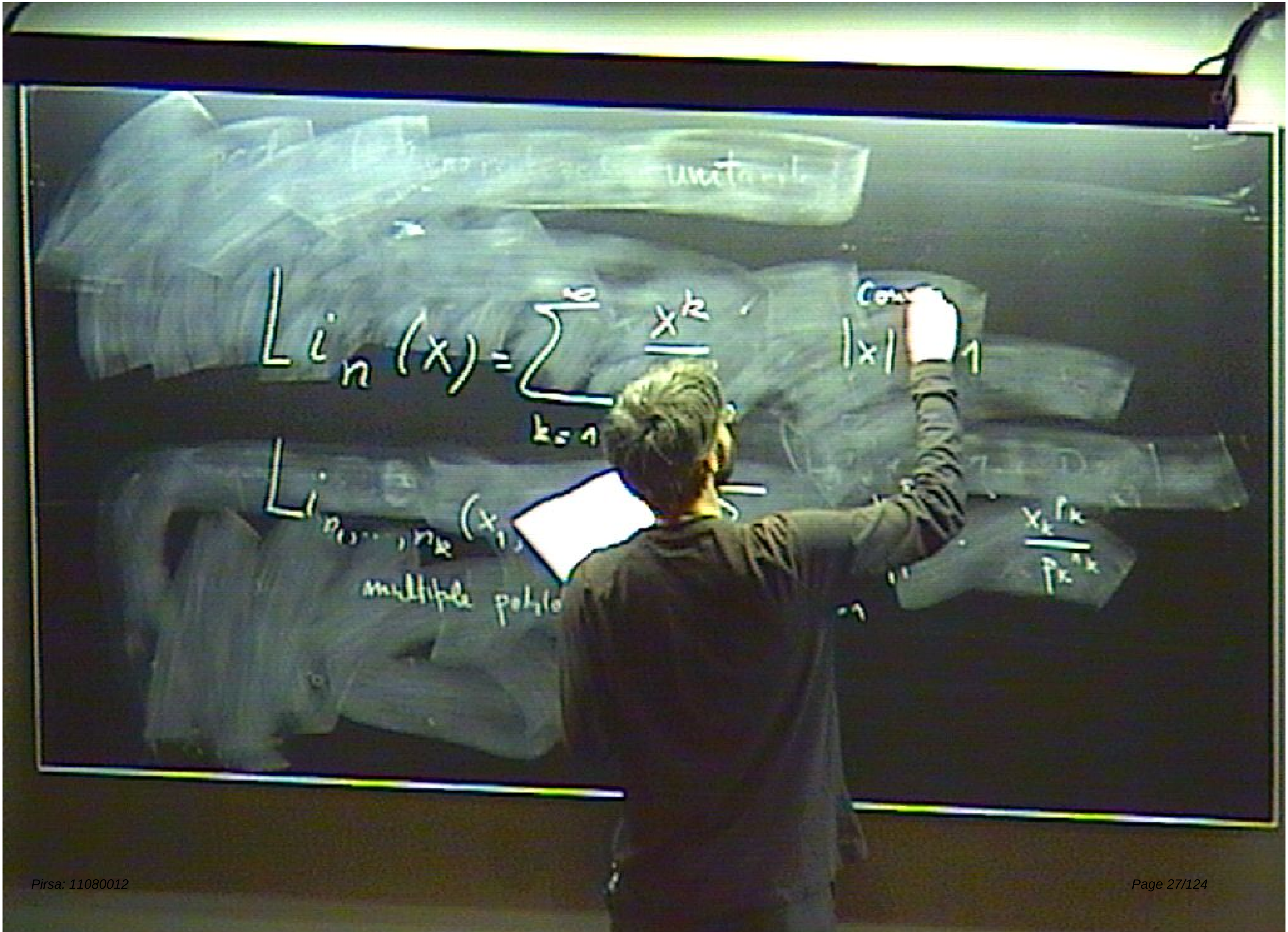


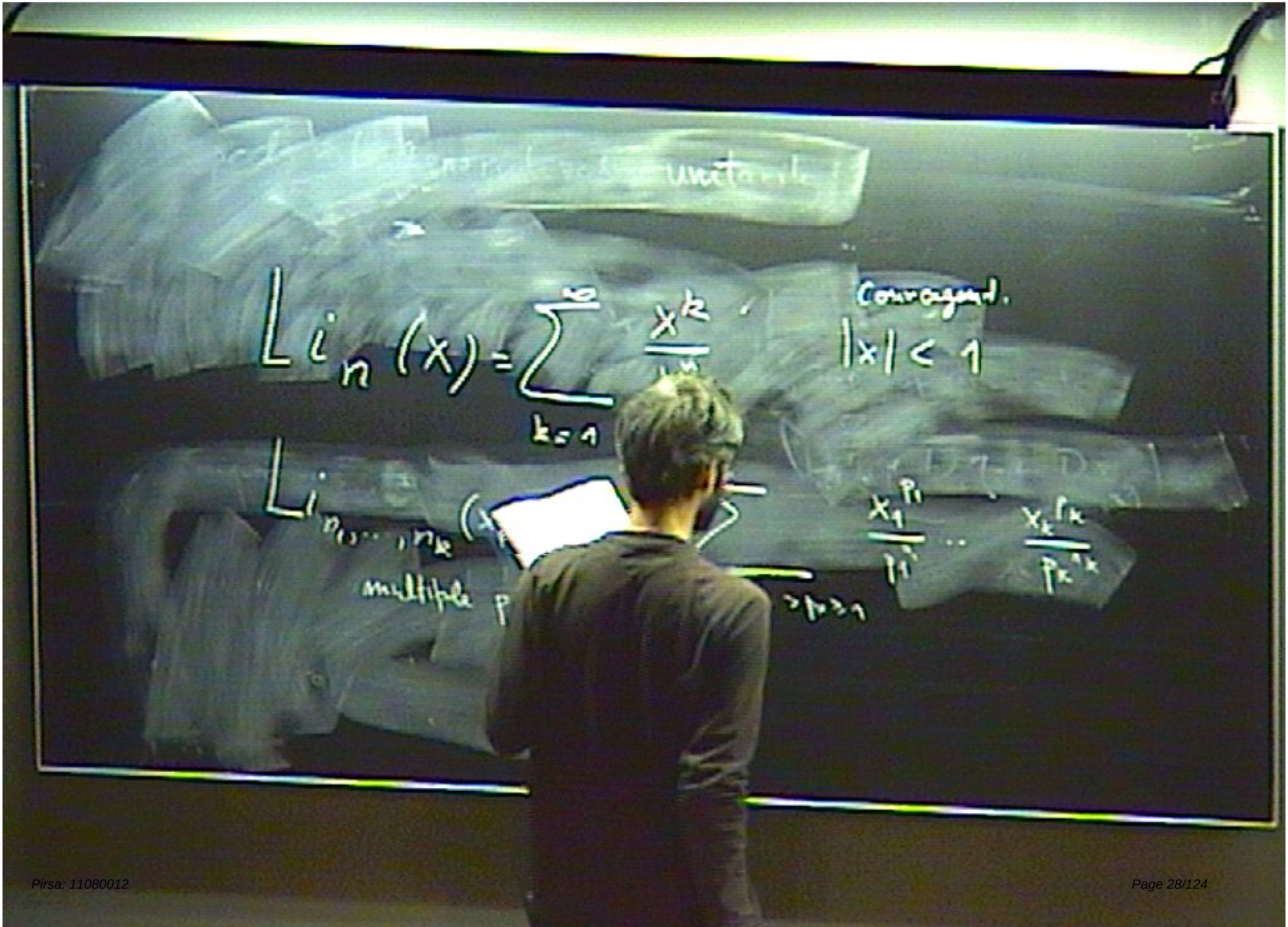
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1>\dots>p_k\geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
multiple polylogs

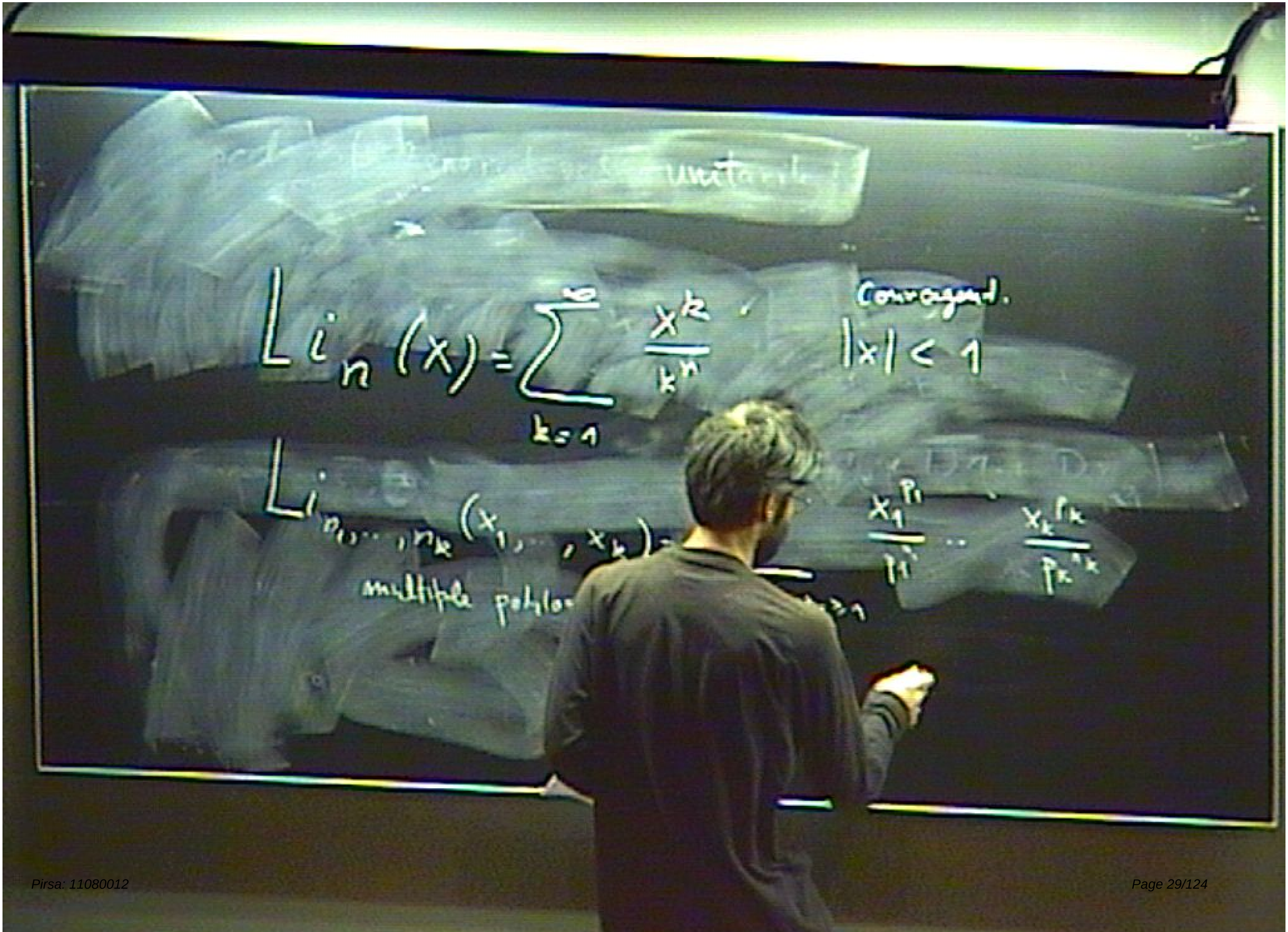

$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
multiple polylogs

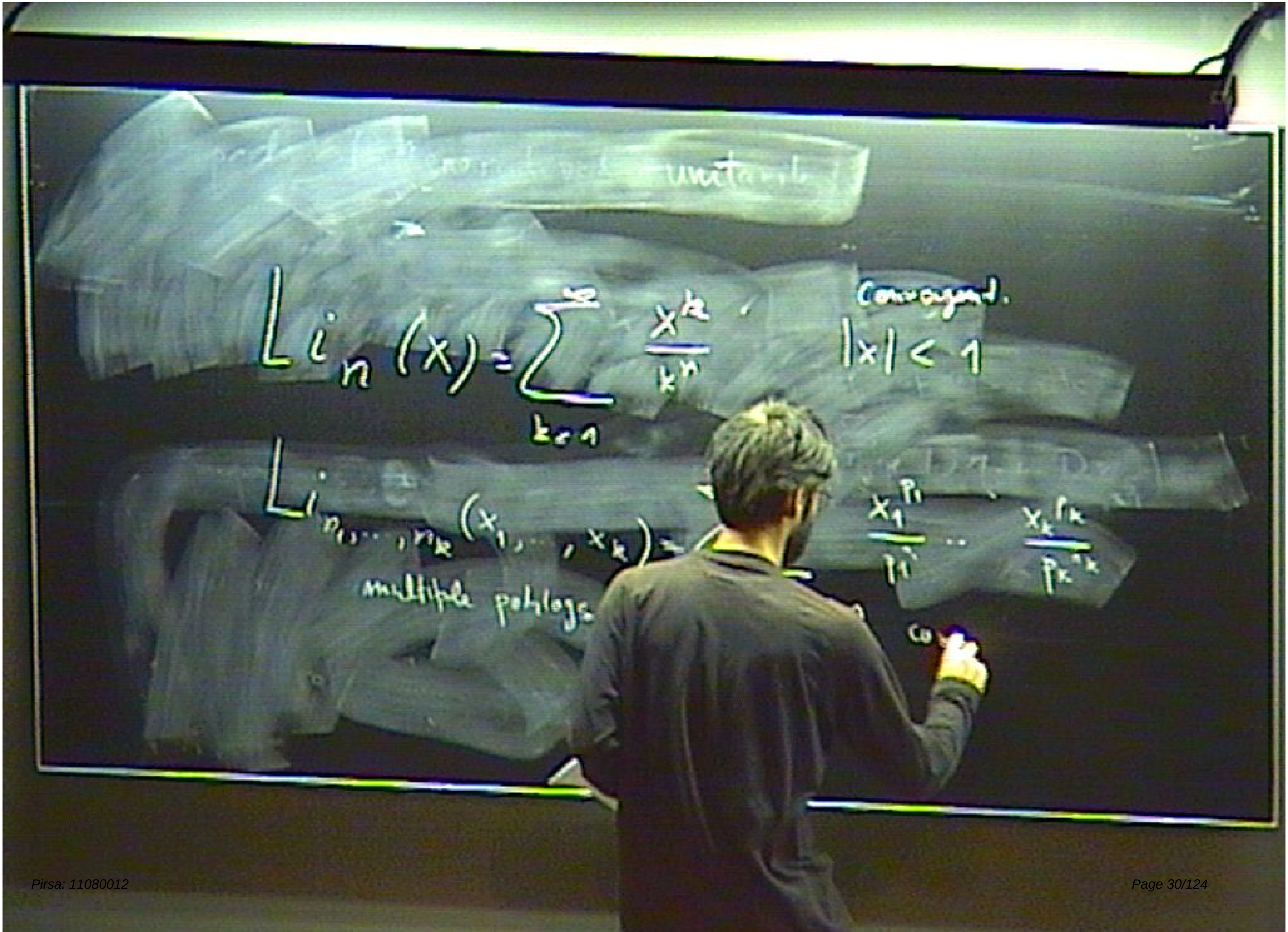

Li_n(x) = Σ x^k / k^n
multiple

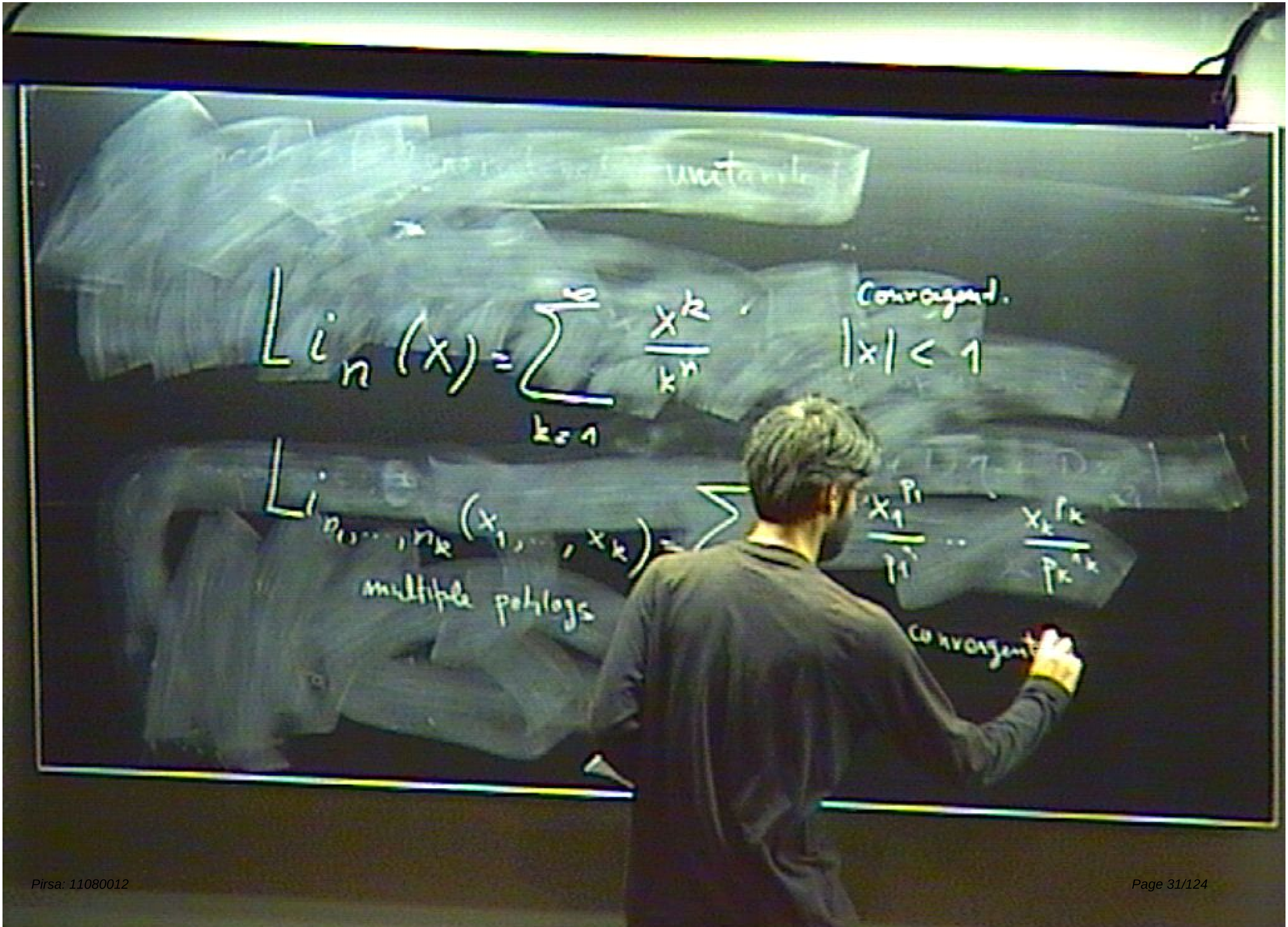

Li_n(x) = Σ_{k≥1}^∞ x^k / k^n , |x| <
Li_{n_1,...,n_k}(x_1, ...
multiple polylogs
x_1^{p_1}
x_k^{p_k} / p_k^{n_k}

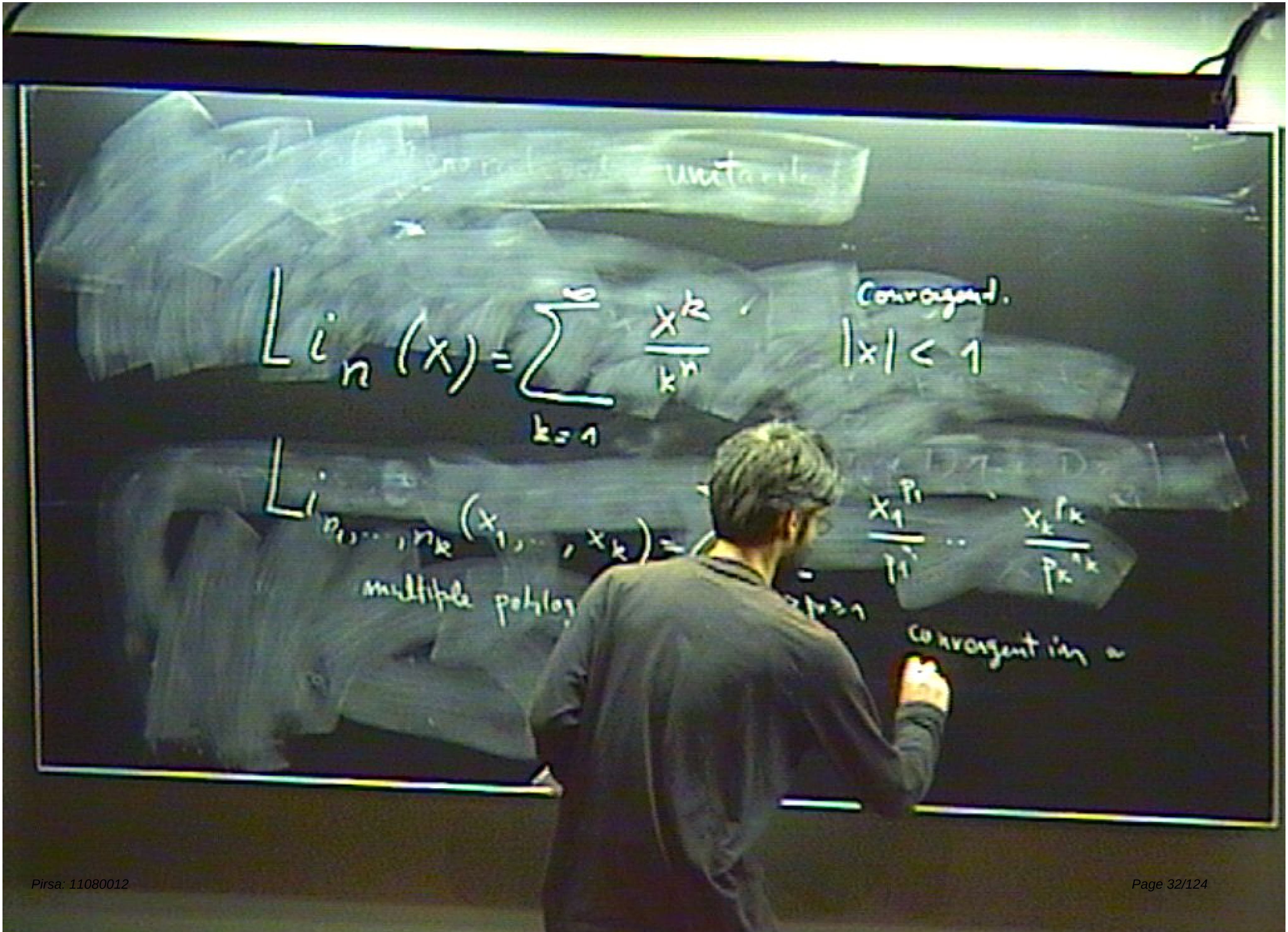

Li n (x) = ∑ x^k
k=1
|x|
multiple

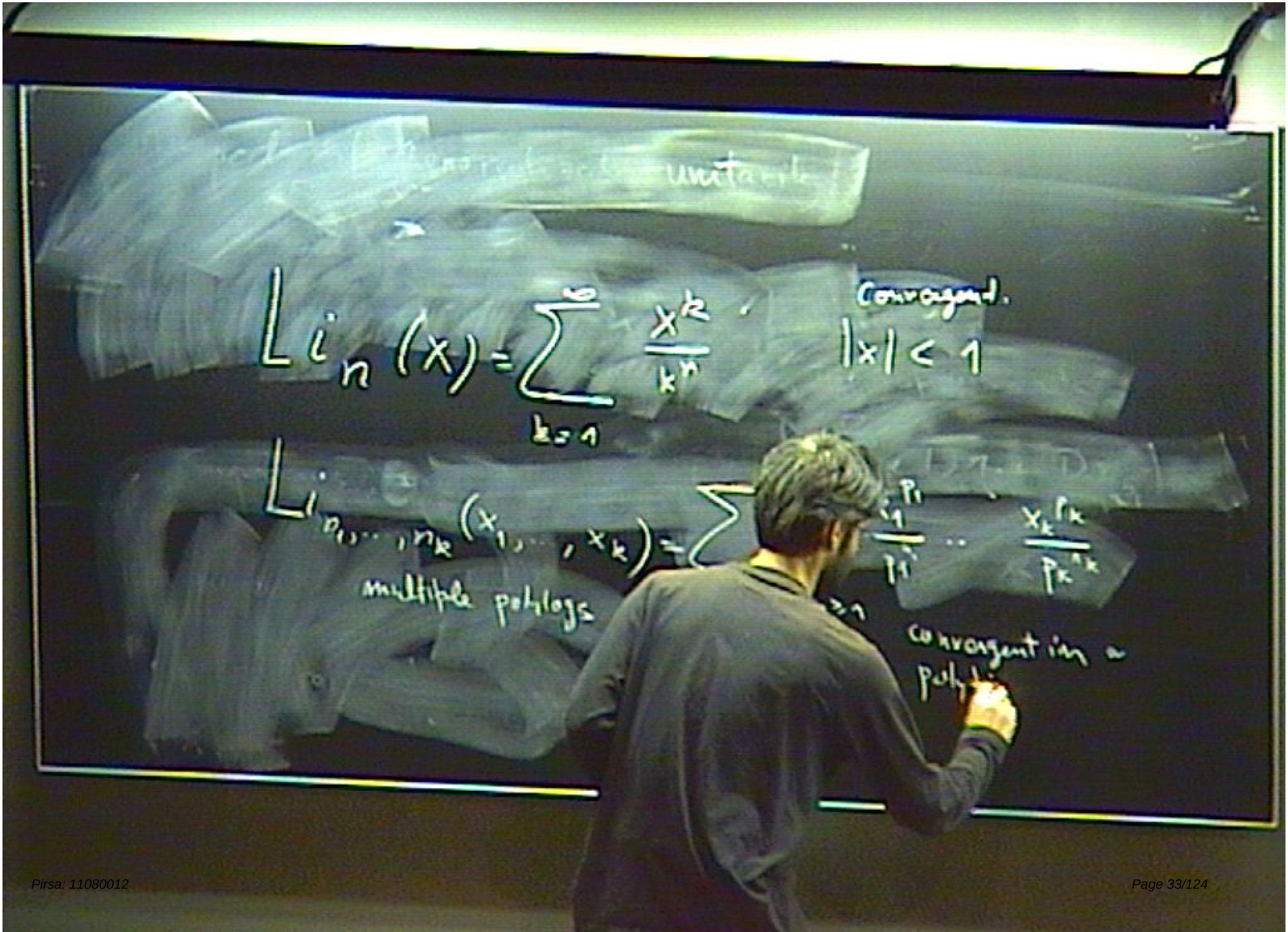

$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{\ldots}$
Convergent.
$|x| < 1$
multiple


Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$|x| < 1$
multiple

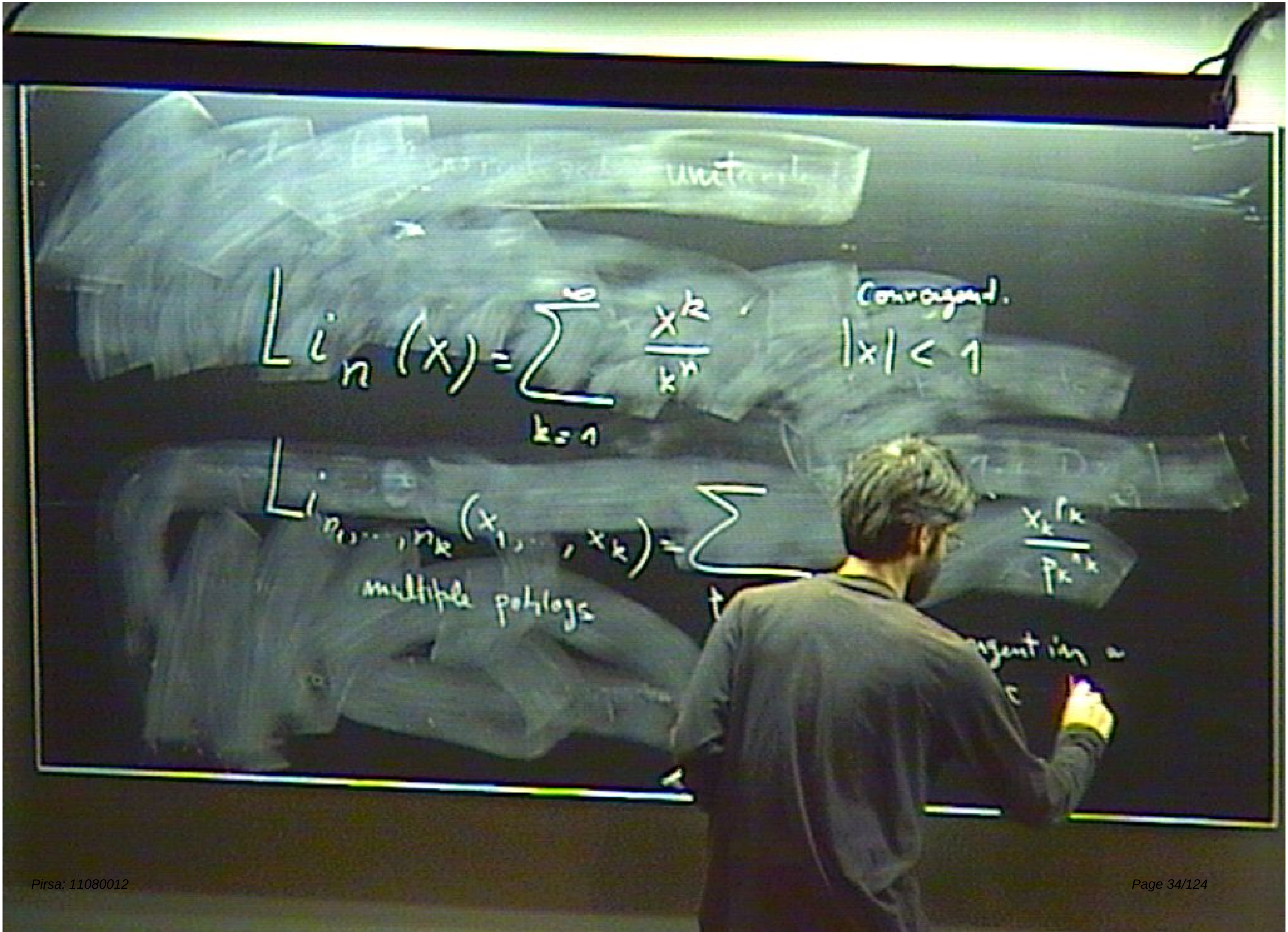

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$|x| < 1$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) =$
$\frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
multiple polylogs

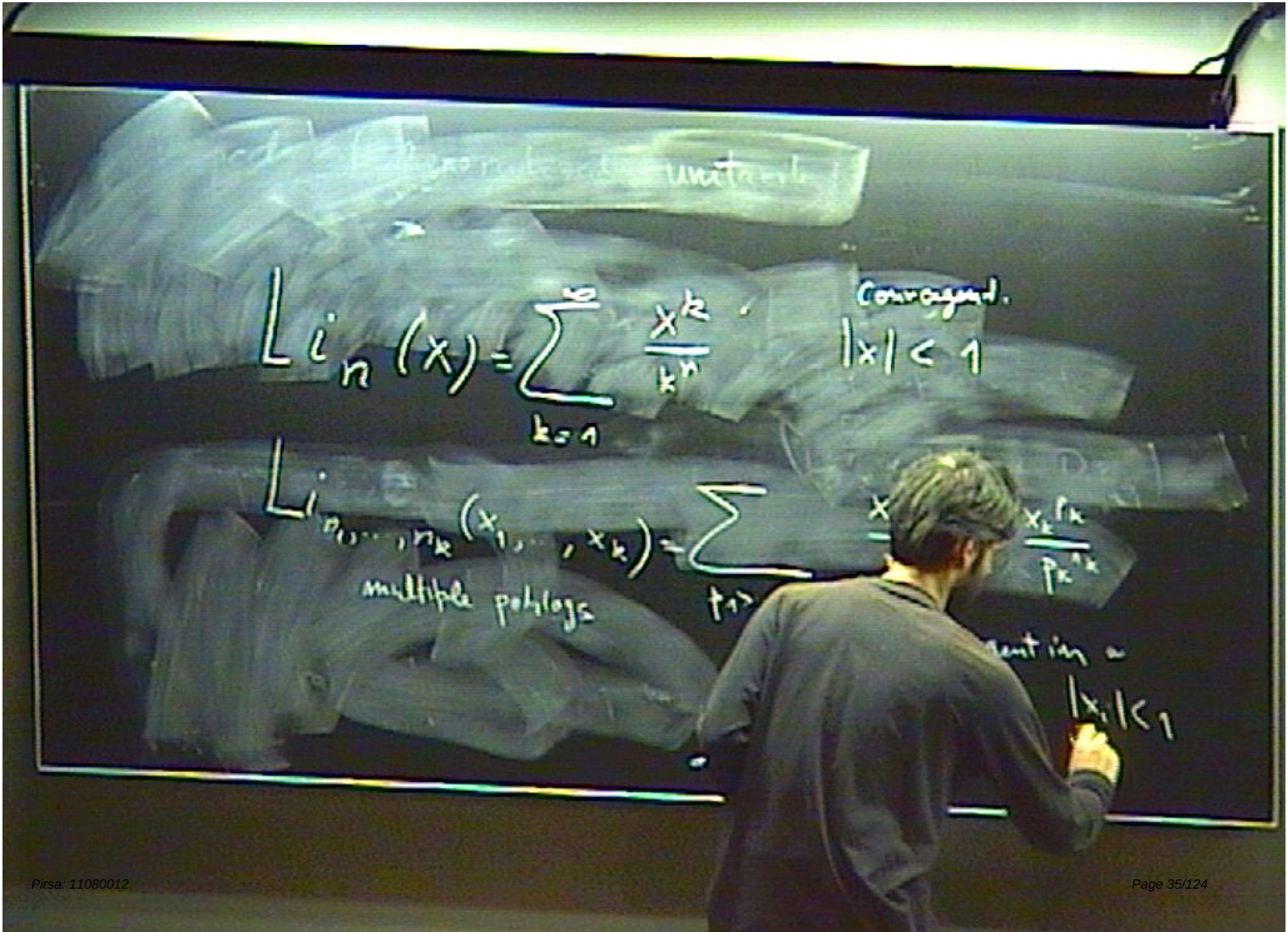

Li_n(x) = Σ_{k=1}^∞ x^k / k^n
convergent.
|x| < 1
Li_{n_1,…,n_k}(x_1,…,x_k) = Σ x_1^{p_1}/p_1^{n_1} … x_k^{p_k}/p_k^{n_k}
multiple polylogs
convergent

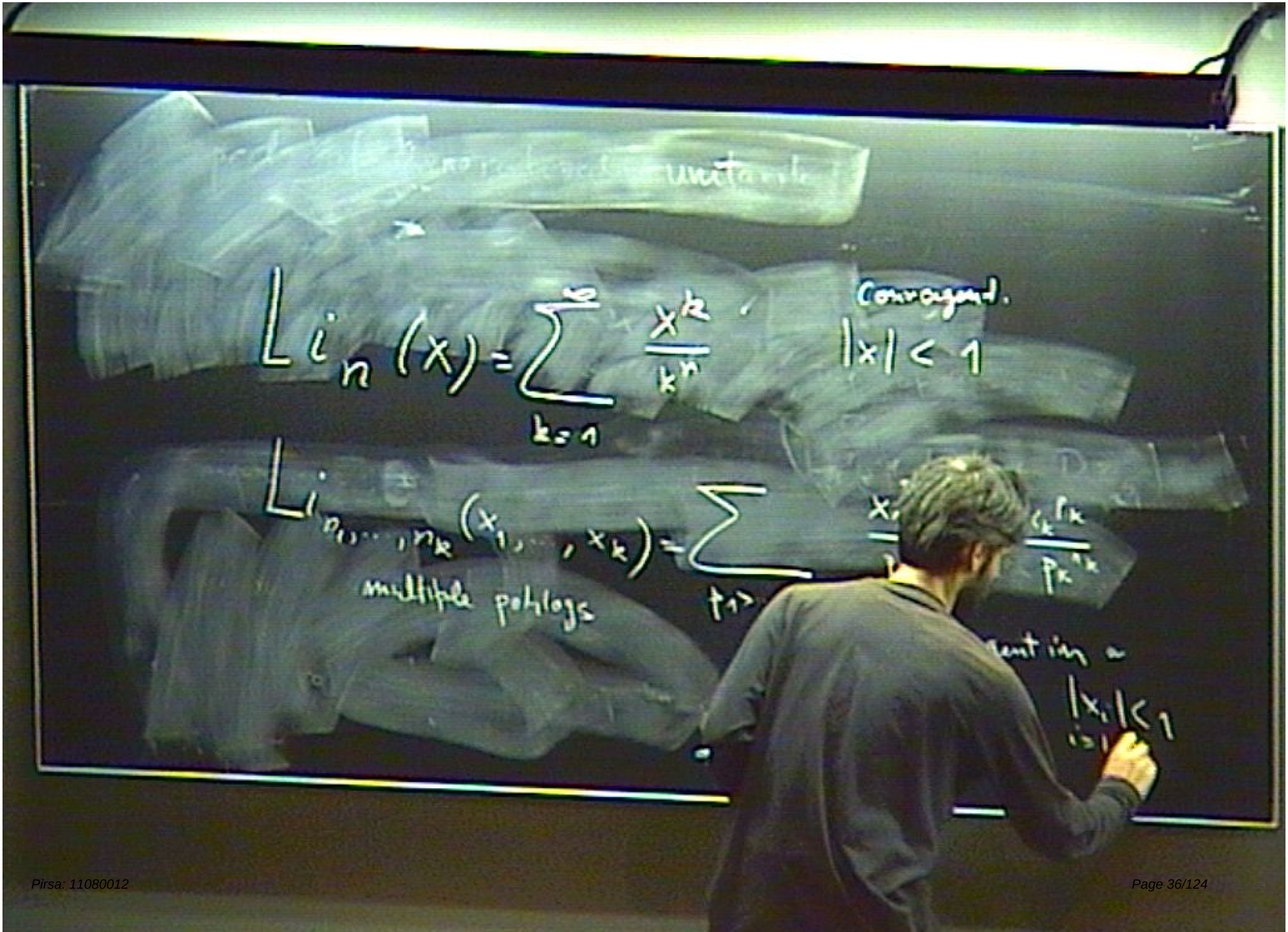

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
$|x| < 1$
multiple polylog
convergent in a

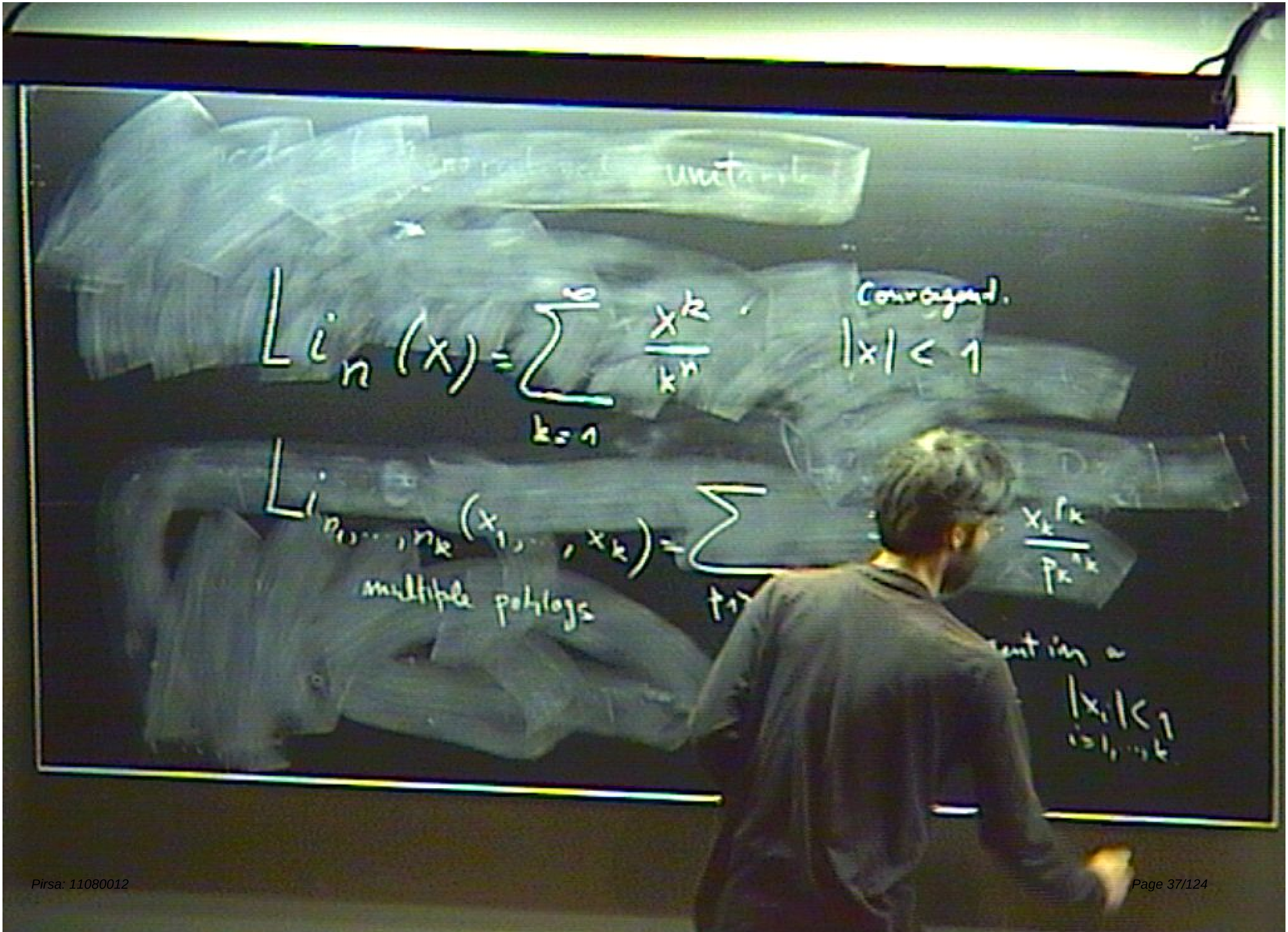


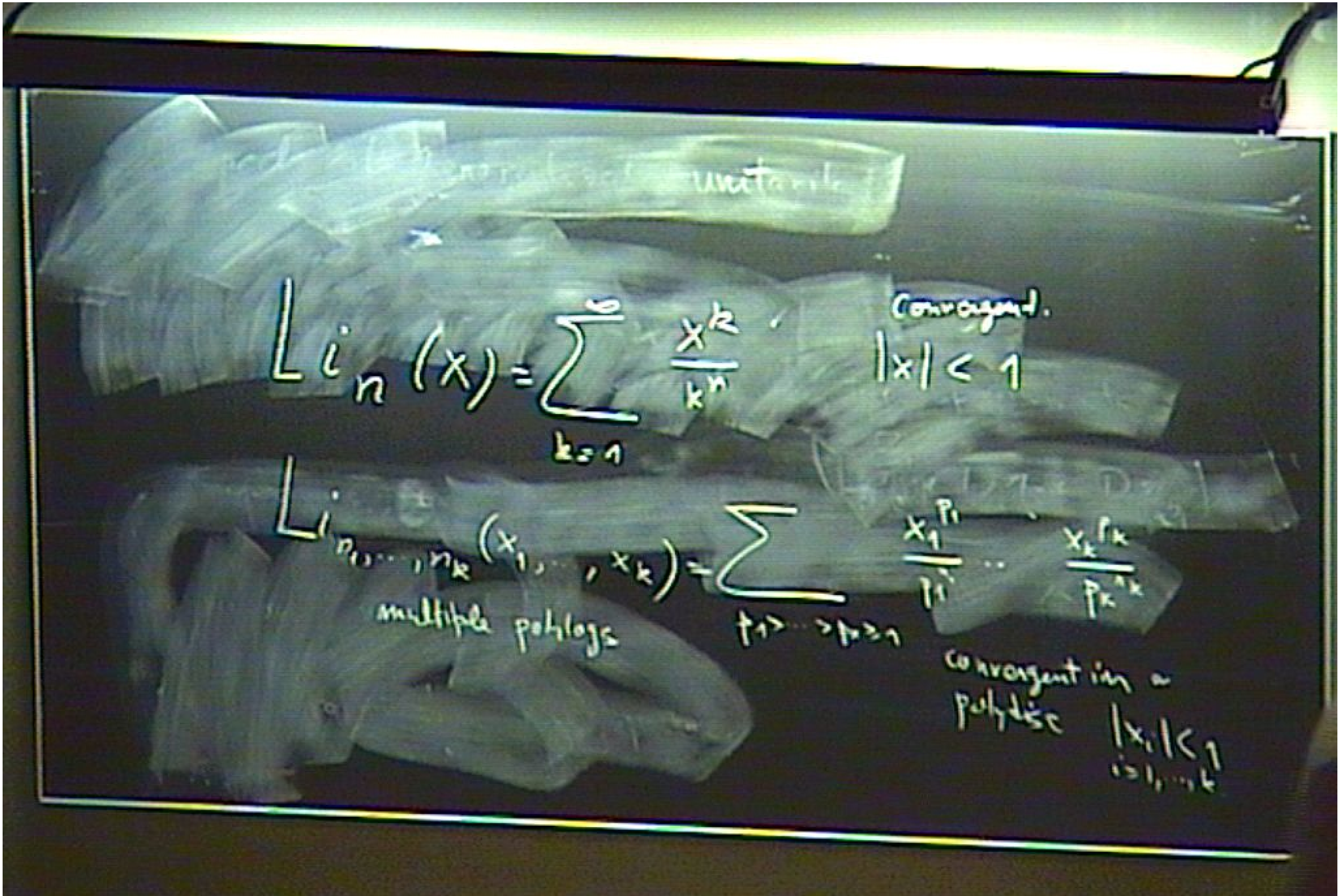
Li_n (x) = ∑_{k=1}^∞ x^k / k^n
Convergent.
|x| < 1
Li_{n_1,…,n_k}(x_1,…,x_k) = ∑ x_1^{p_1}/p_1^{n_1} … x_k^{p_k}/p_k^{n_k}
multiple polylogs
convergent in a


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
Convergent.
$|x| < 1$
multiple polylogs


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
Convergent.
$|x| < 1$
multiple polylogs
ent in a
$|x_i| < 1$


Convergent.
Li_n(x) = Σ_{k=1}^∞ x^k / k^n
|x| < 1
multiple polylogs
ent in a
|x_i| < 1


Li_n(x) = Σ_{k=1}^∞ x^k / k^n
convergent.
|x| < 1
Li_{n_1,...,n_k}(x_1,...,x_k) = Σ
x_k^{p_k} / p_k^{n_k}
multiple polylogs
|x_i| < 1
i=1,...,k


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
convergent.
$|x| < 1$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
multiple polylogs
convergent in a
polydisc $|x_i| < 1$
$i = 1, \dots, k$

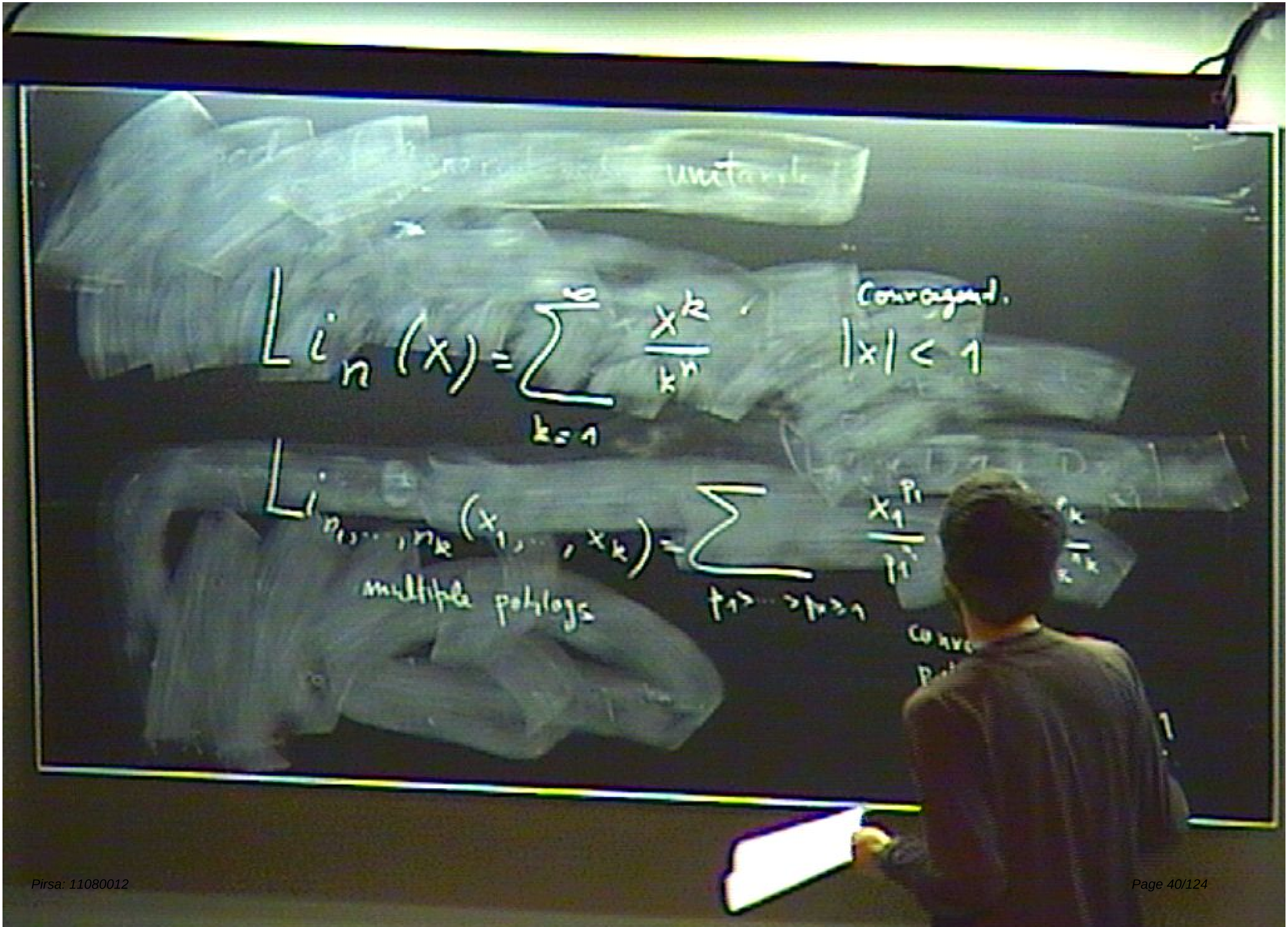
$$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

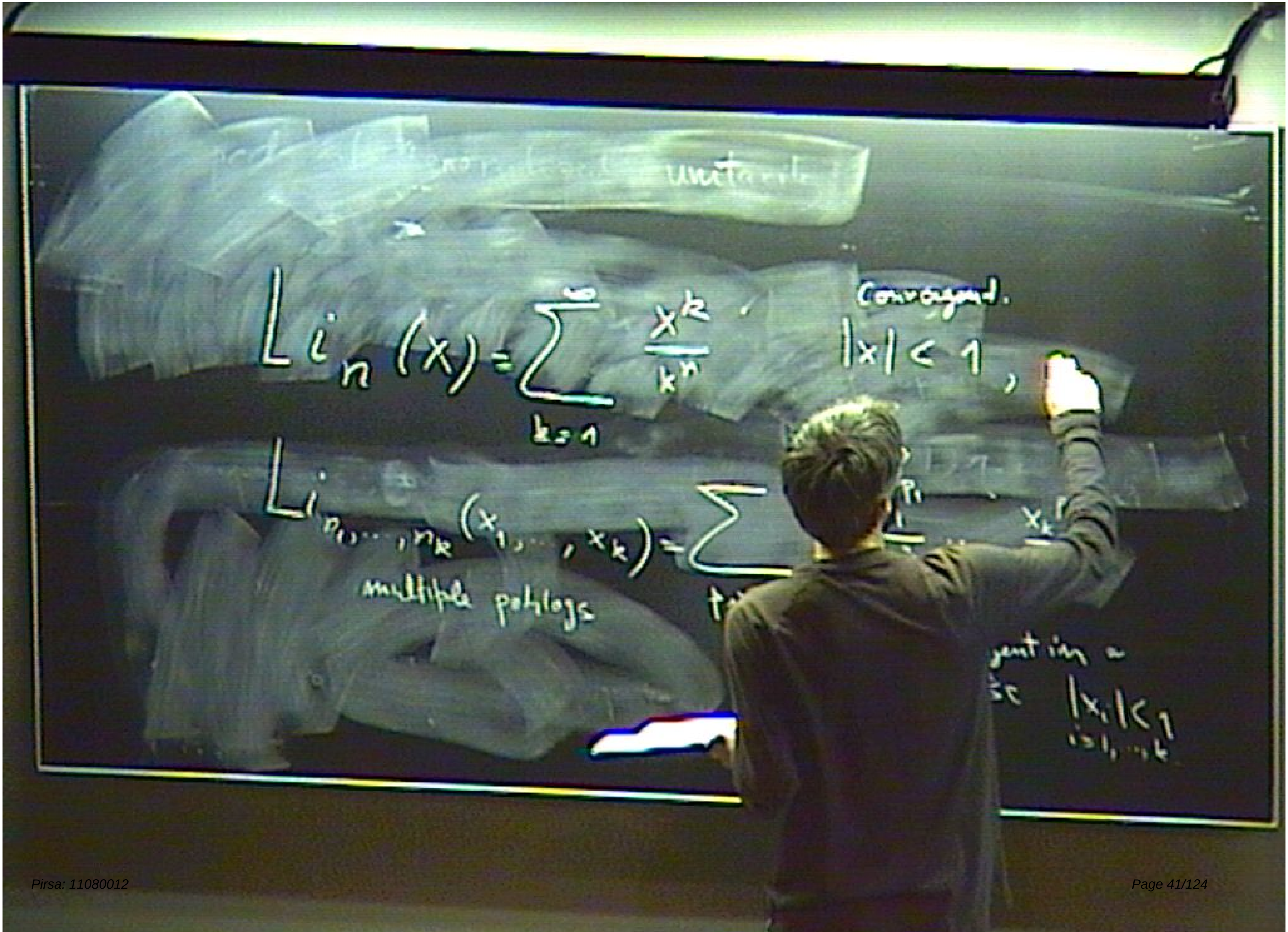
convergent. $|x| < 1$

$$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$$

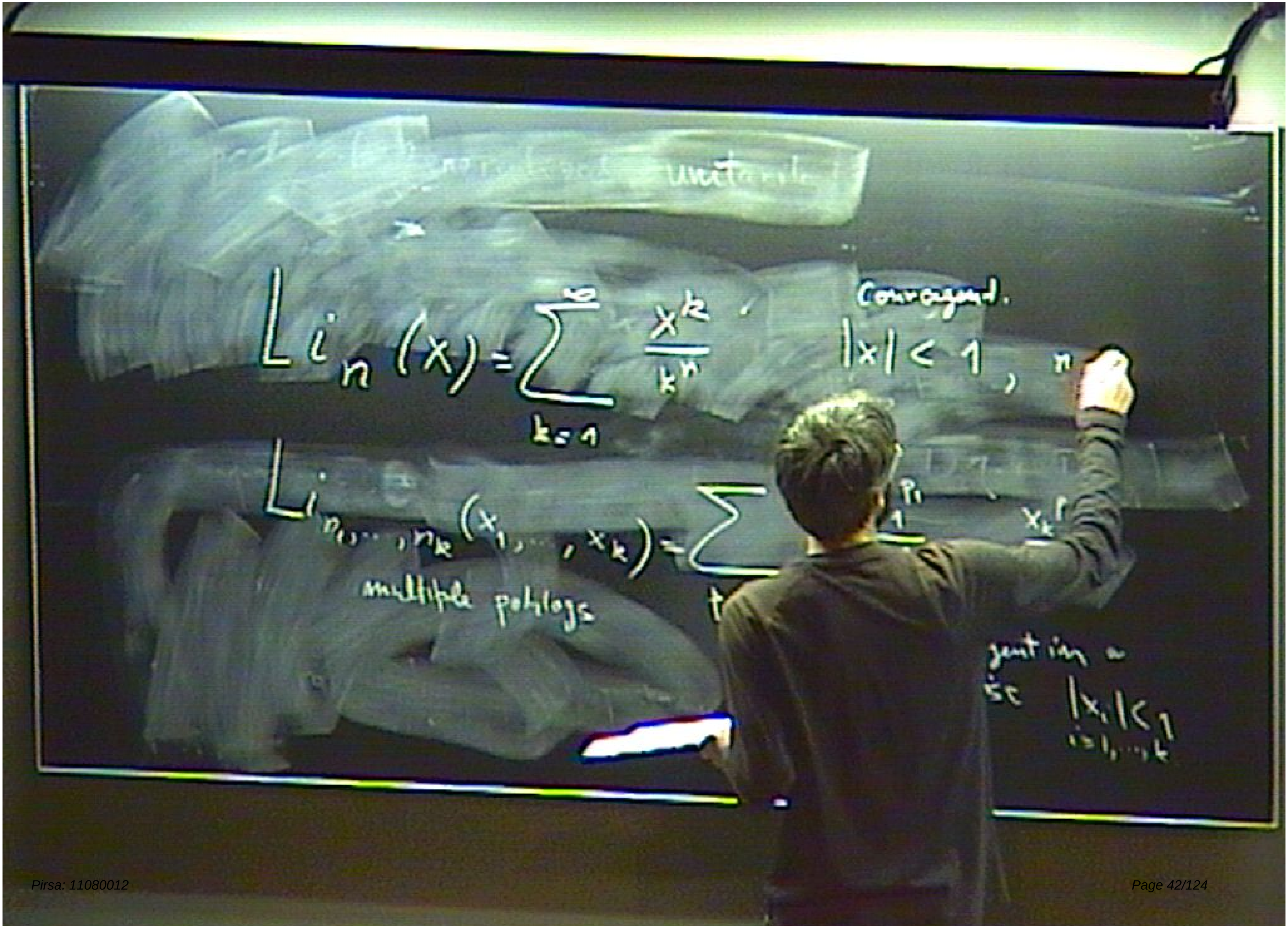
multiple polylogs

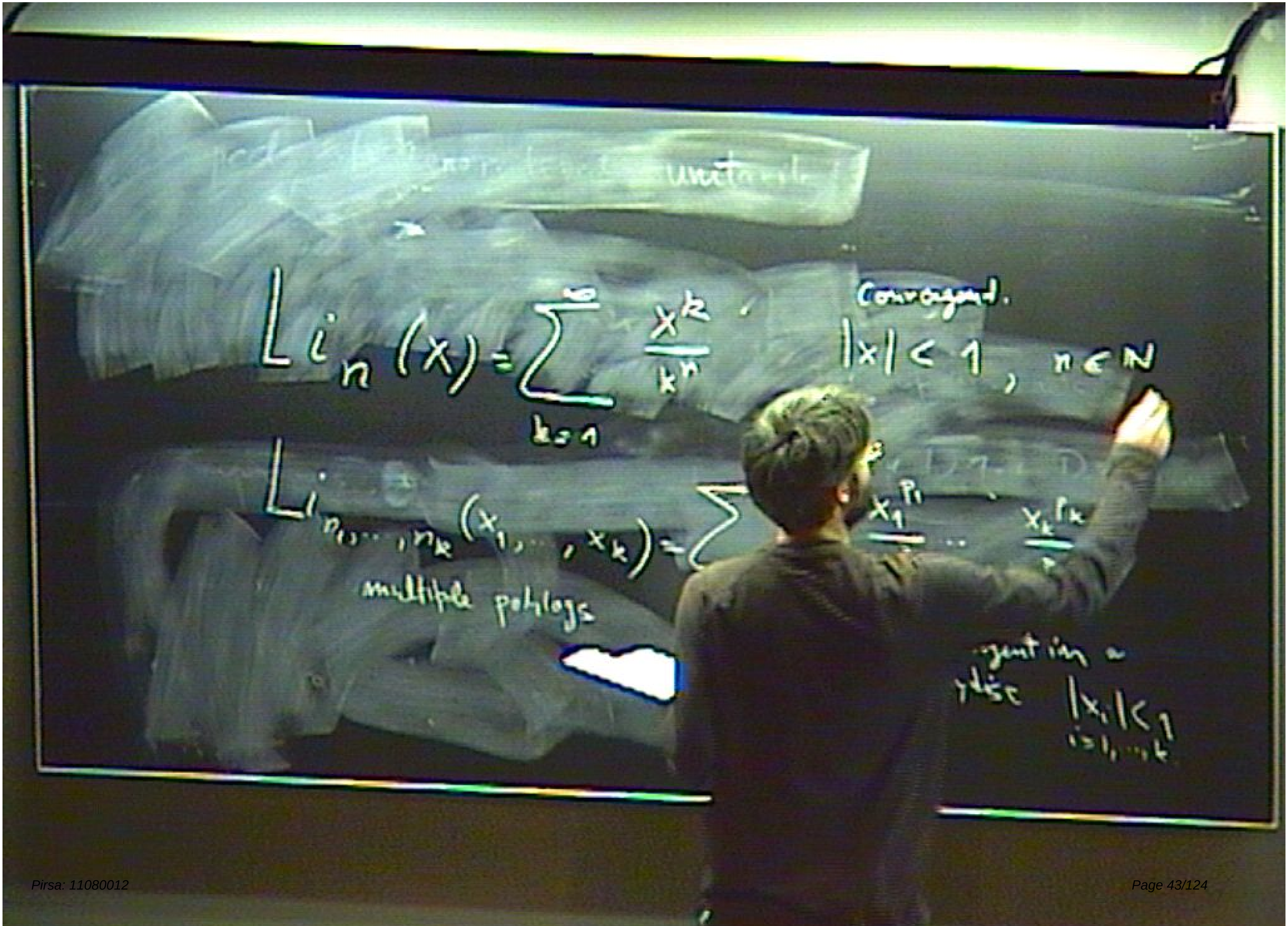
convergent in a polydisc $|x_i| < 1$, $i = 1, \dots, k$

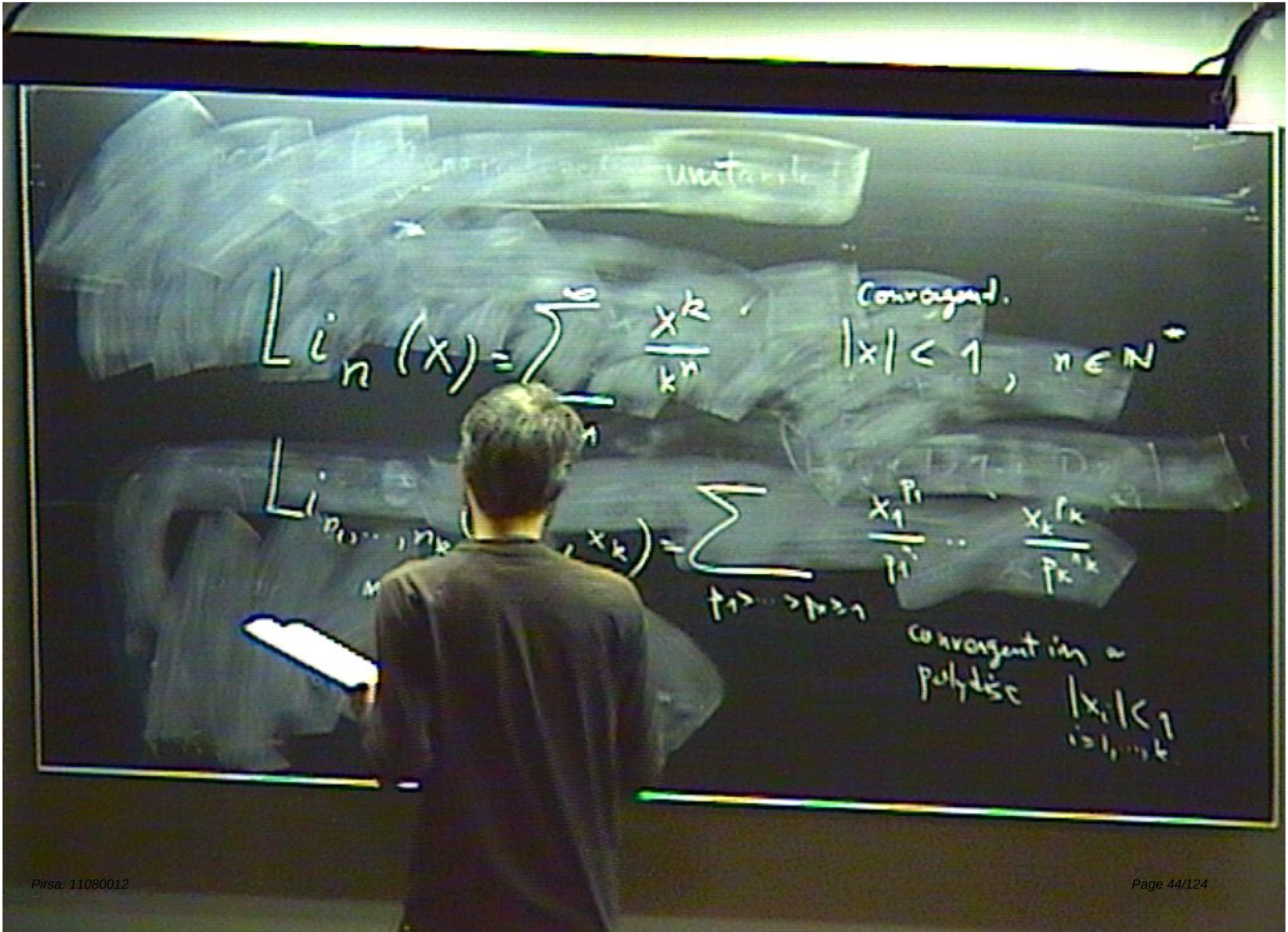

$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
convergent.
$|x| < 1$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots$
multiple polylogs

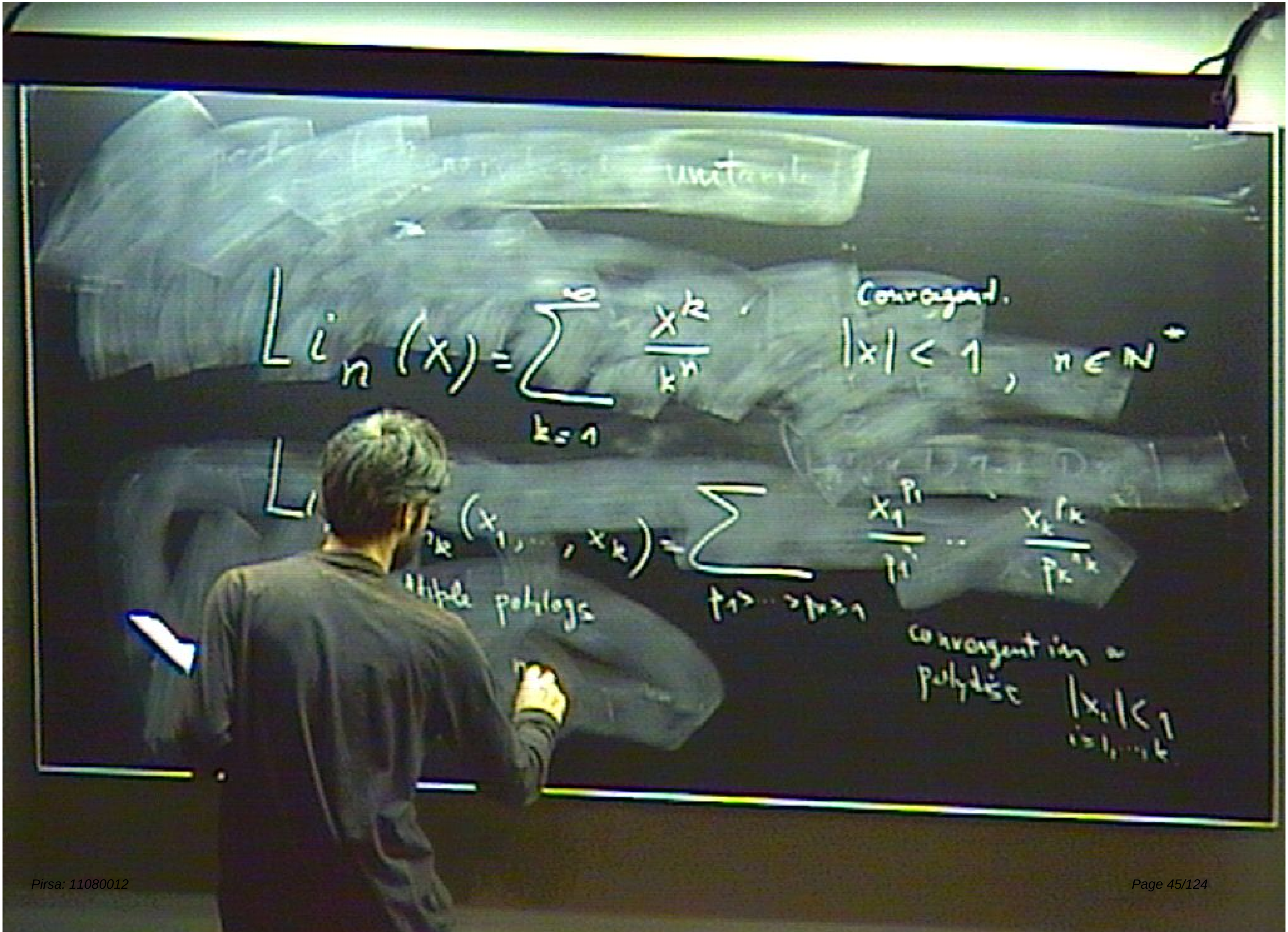


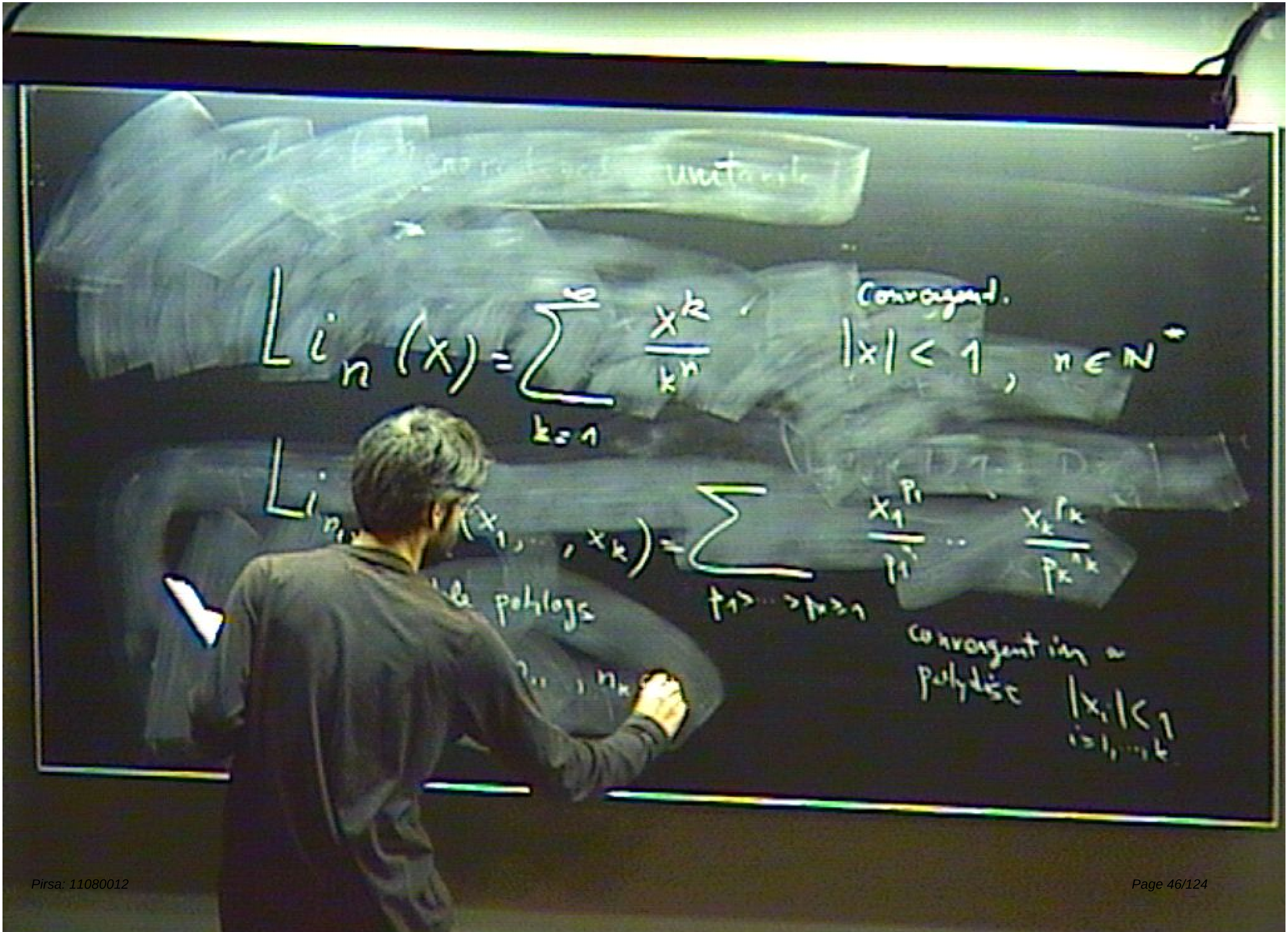
Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, $|x| < 1$,
$Li_{n_1, \dots, n_k}(x_1, \dots, x_k) = \sum$
multiple polylogs
$|x_i| < 1$
$i = 1, \dots, k$

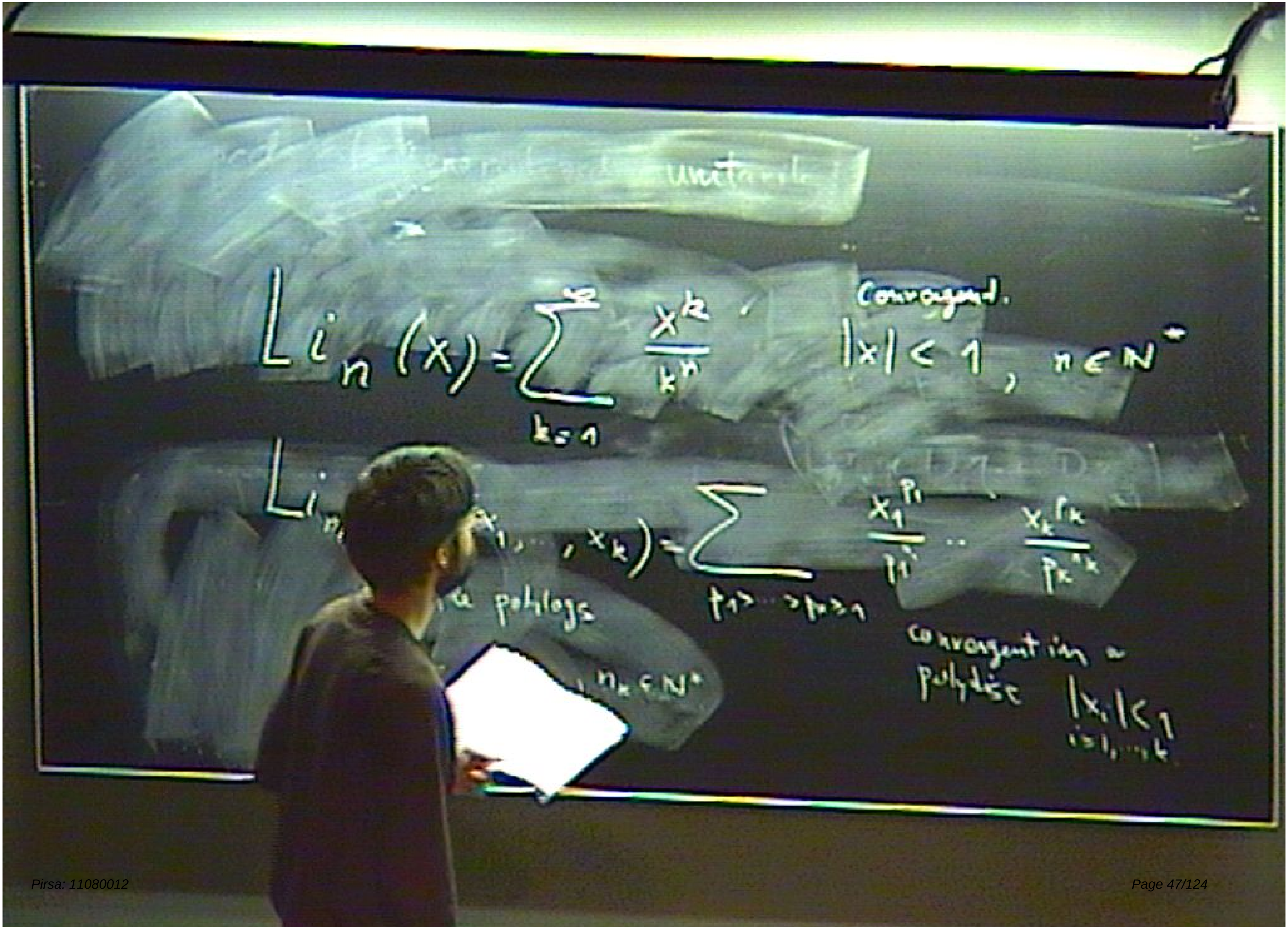

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$ $|x| < 1$,
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum$
multiple polylogs
$|x_i| < 1$
$i = 1,\dots,k$

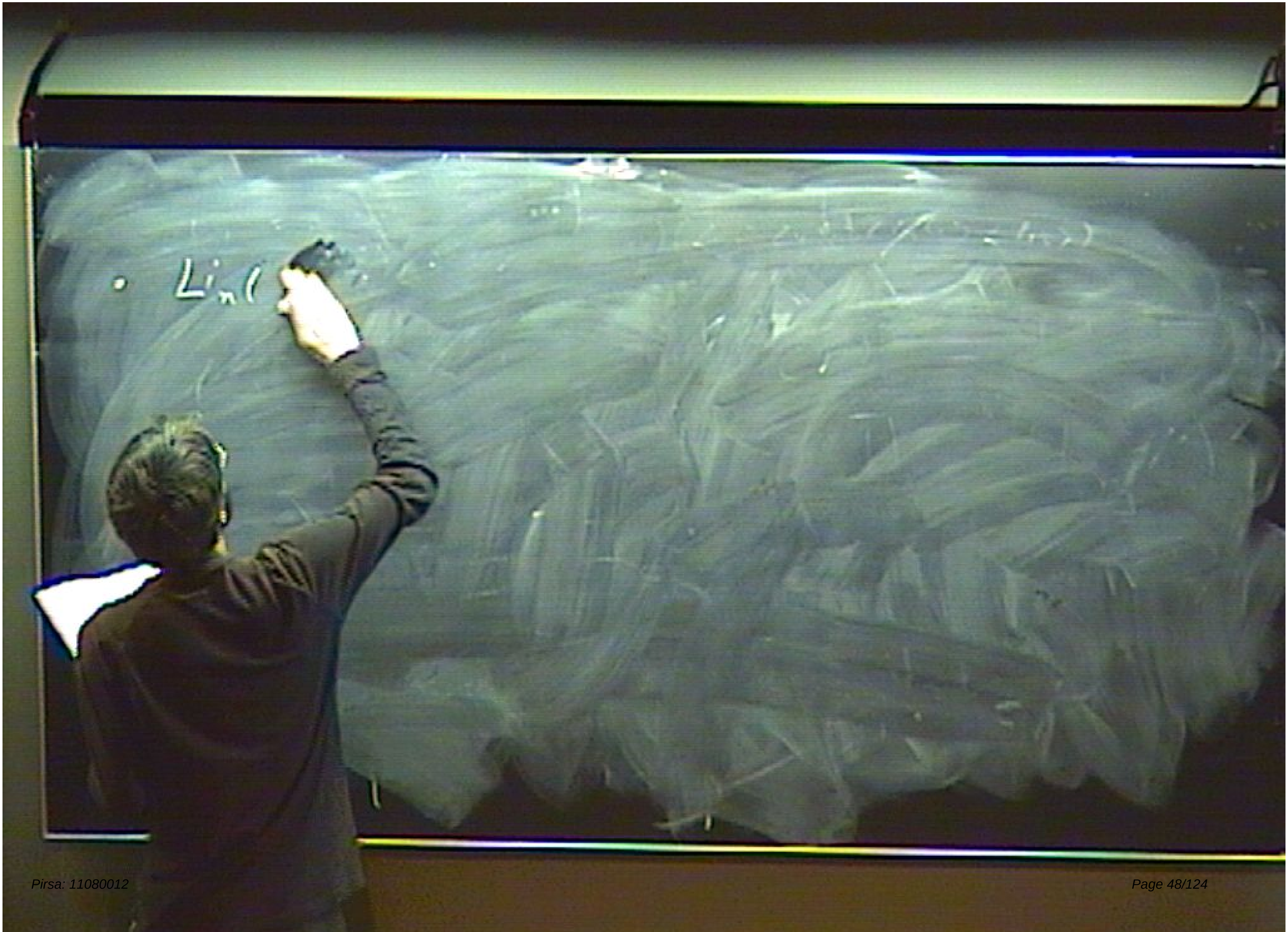

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, $|x| < 1$, $n \in N$
$Li_{n_1,\ldots,n_k}(x_1,\ldots,x_k) = \sum$
multiple polylogs
$|x_i| < 1$
$i = 1,\ldots,k$

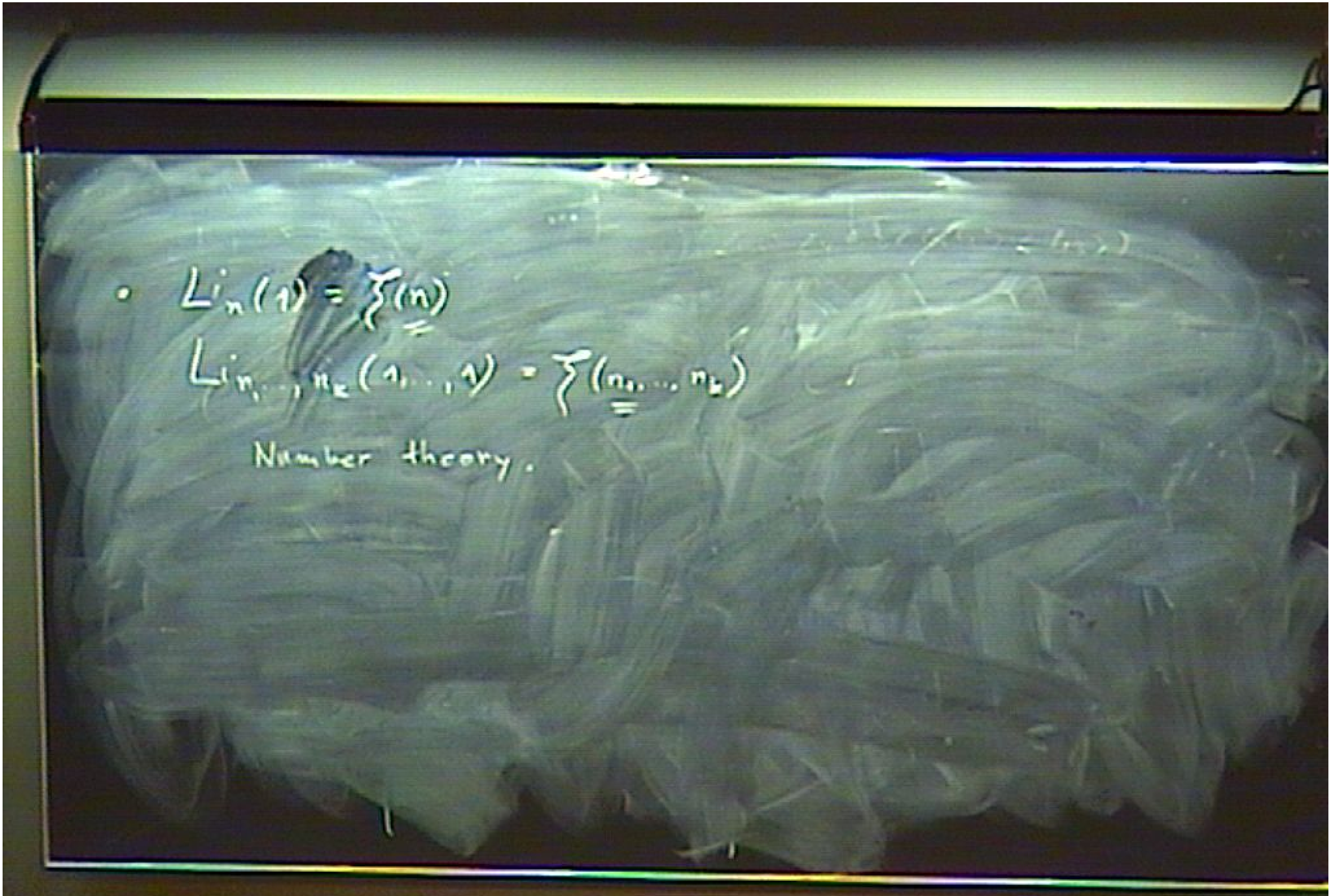

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, $|x| < 1$, $n \in \mathbb{N}^*$
$Li_{n_1,\dots,n_k}(\dots, x_k) = \sum_{p_1 > \dots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
convergent in a polydisc $|x_i| < 1$ $i=1,\dots,k$

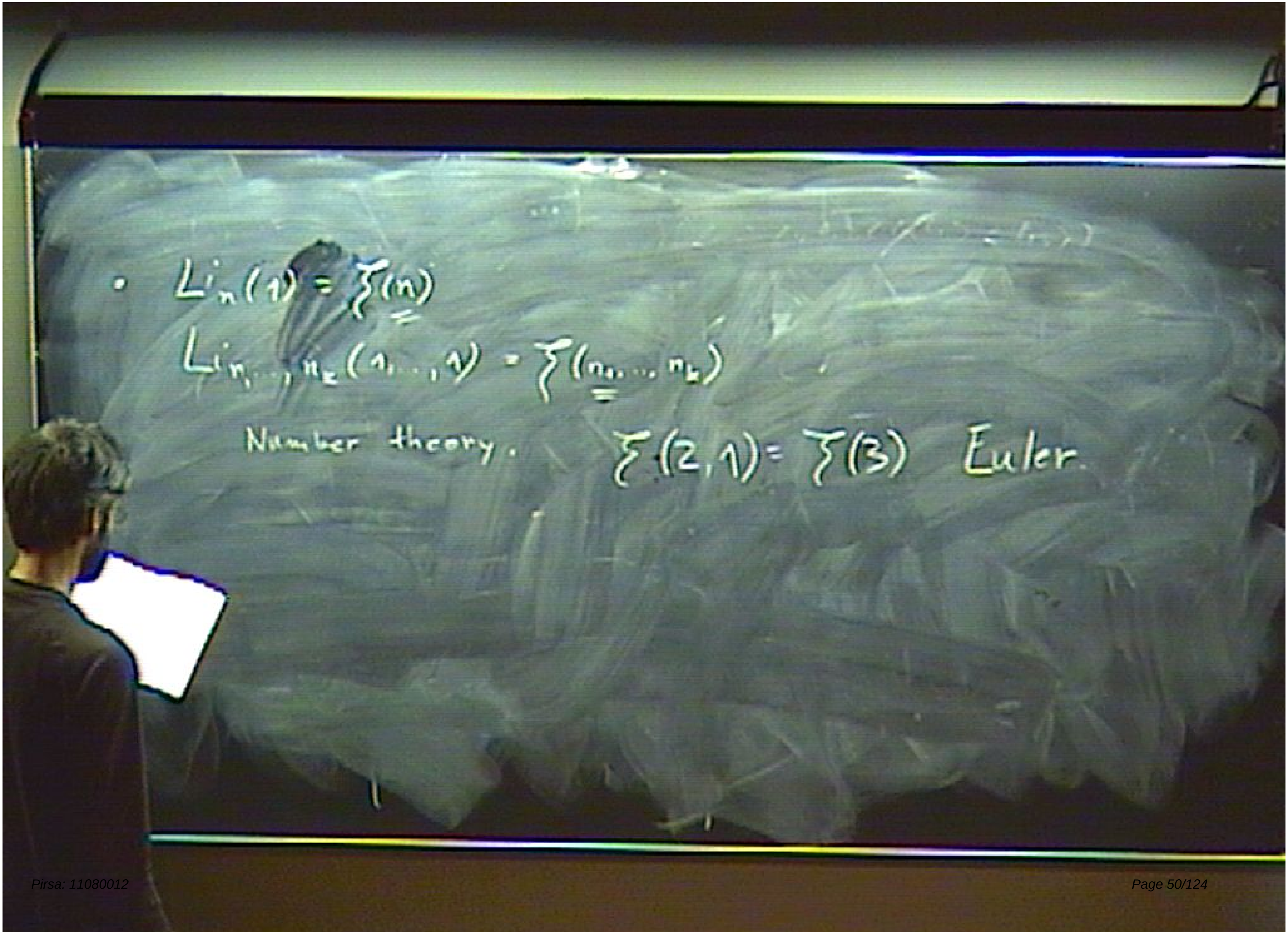

Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, $|x| < 1$, $n \in N^*$
$(x_1, \ldots, x_k) = \sum_{p_1 > \cdots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
polylogs
convergent in a
polydisc $|x_i| < 1$
$i = 1, \ldots, k$


$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$
convergent.
$|x| < 1$, $n \in N^*$
$Li_{n_1,\dots,n_k}(x_1,\dots,x_k) = \sum_{p_1>\dots>p_k\geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
convergent in a polydisc $|x_i| < 1$
$i=1,\dots,k$


Convergent.
$Li_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$, $|x| < 1$, $n \in \mathbb{N}^*$
$x_k) = \sum_{p_1 > \cdots > p_k \geq 1} \frac{x_1^{p_1}}{p_1^{n_1}} \cdots \frac{x_k^{p_k}}{p_k^{n_k}}$
convergent in a
polydisc $|x_i| < 1$
$i = 1, \ldots, k$


• Li_n(


• $Li_n(1) = \zeta(n)$
$Li_{n_1,\dots,n_k}(1,\dots,1) = \zeta(n_1,\dots,n_k)$
Number theory.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$\zeta(2,1) = \zeta(3)$ Euler.

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- Iterated integrals.

  Topology → Multiple linking numbers.

- $Li_n(1) = \zeta(n)$

  $Li_{n_1,\dots,n_k}(1,\dots,1) = \zeta(n_1,\dots,n_k)$

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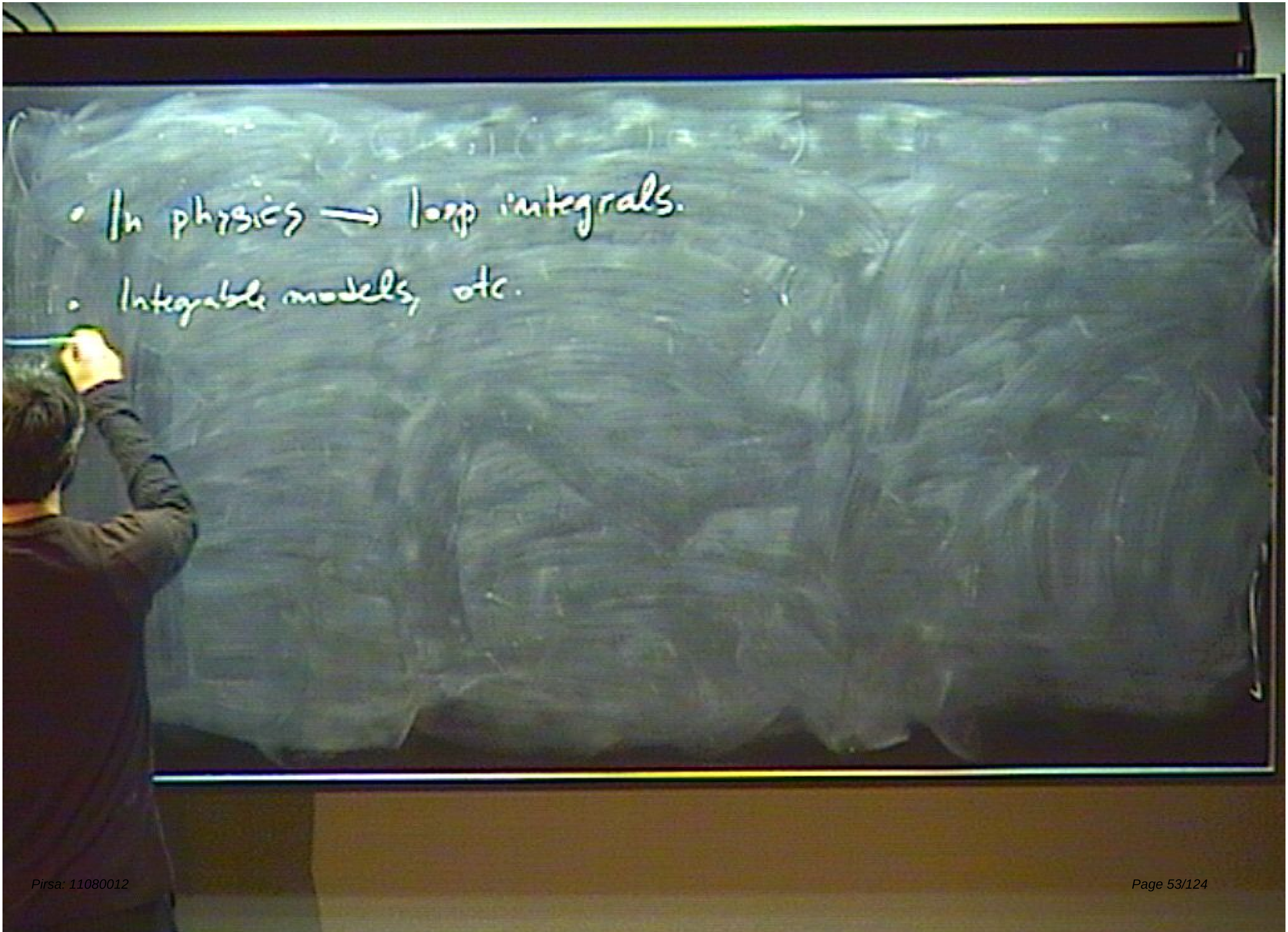
- Iterated integrals.

  Topology $\longrightarrow$ Multiple linking numbers.

- Motives, K-theory.


• In physics ⟶ loop integrals.
• Integrable models, etc.

- In physics ⟶ loop integrals.
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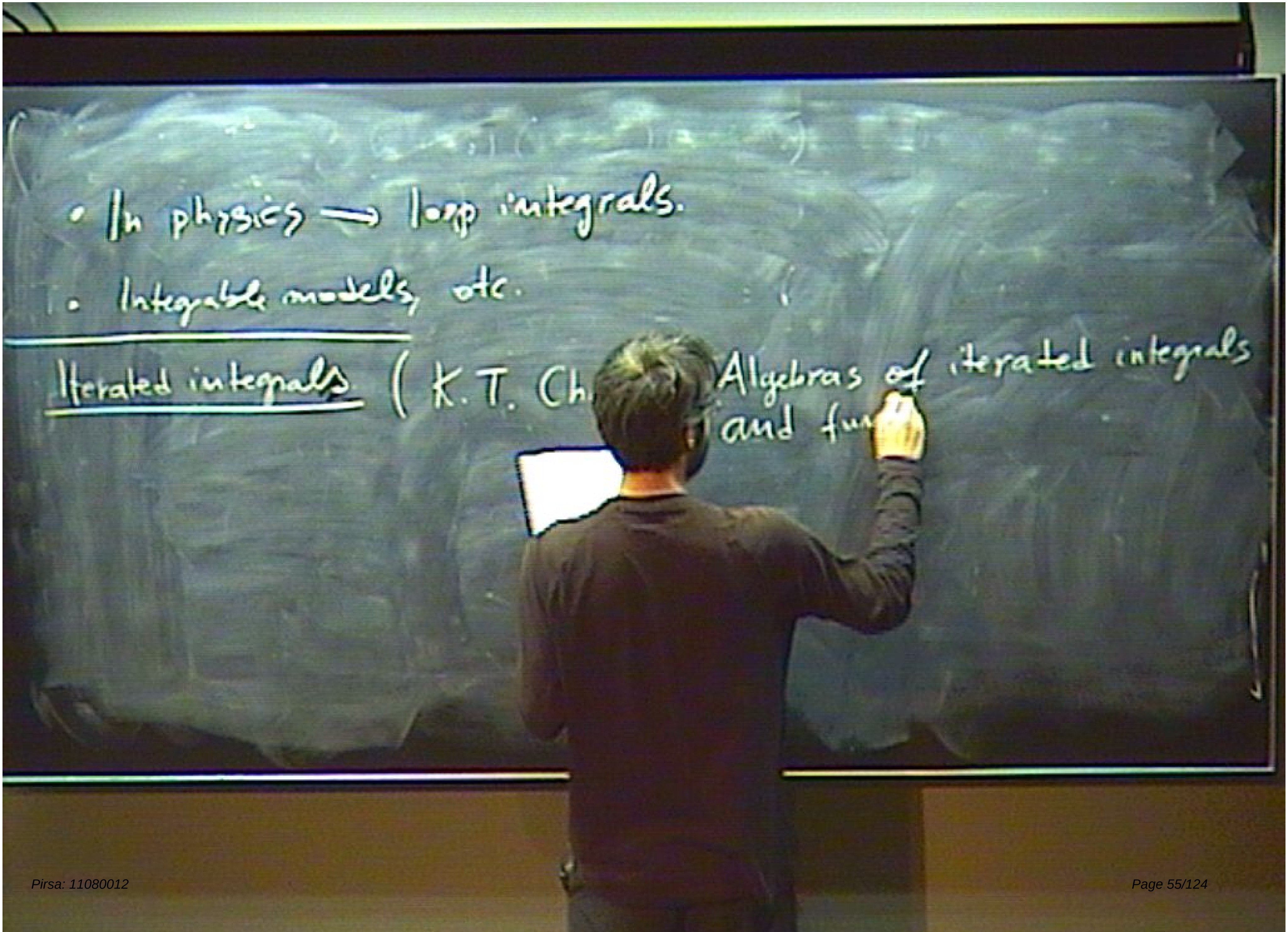
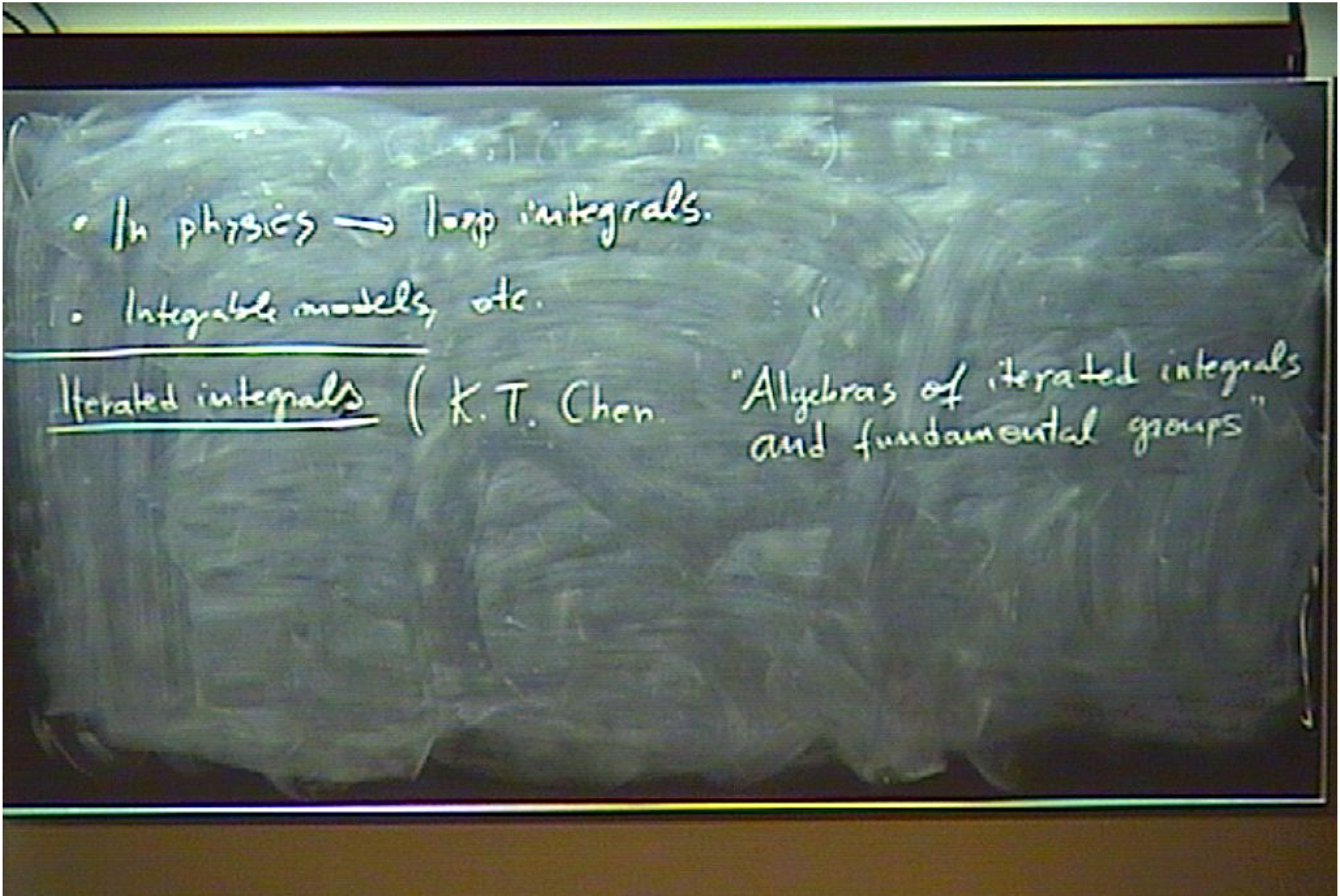
Iterated integrals


• In physics → loop integrals.
• Integrable models, etc.
Iterated integrals (K.T. Ch
Algebras of iterated integrals
and fun


• In physics → loop integrals.
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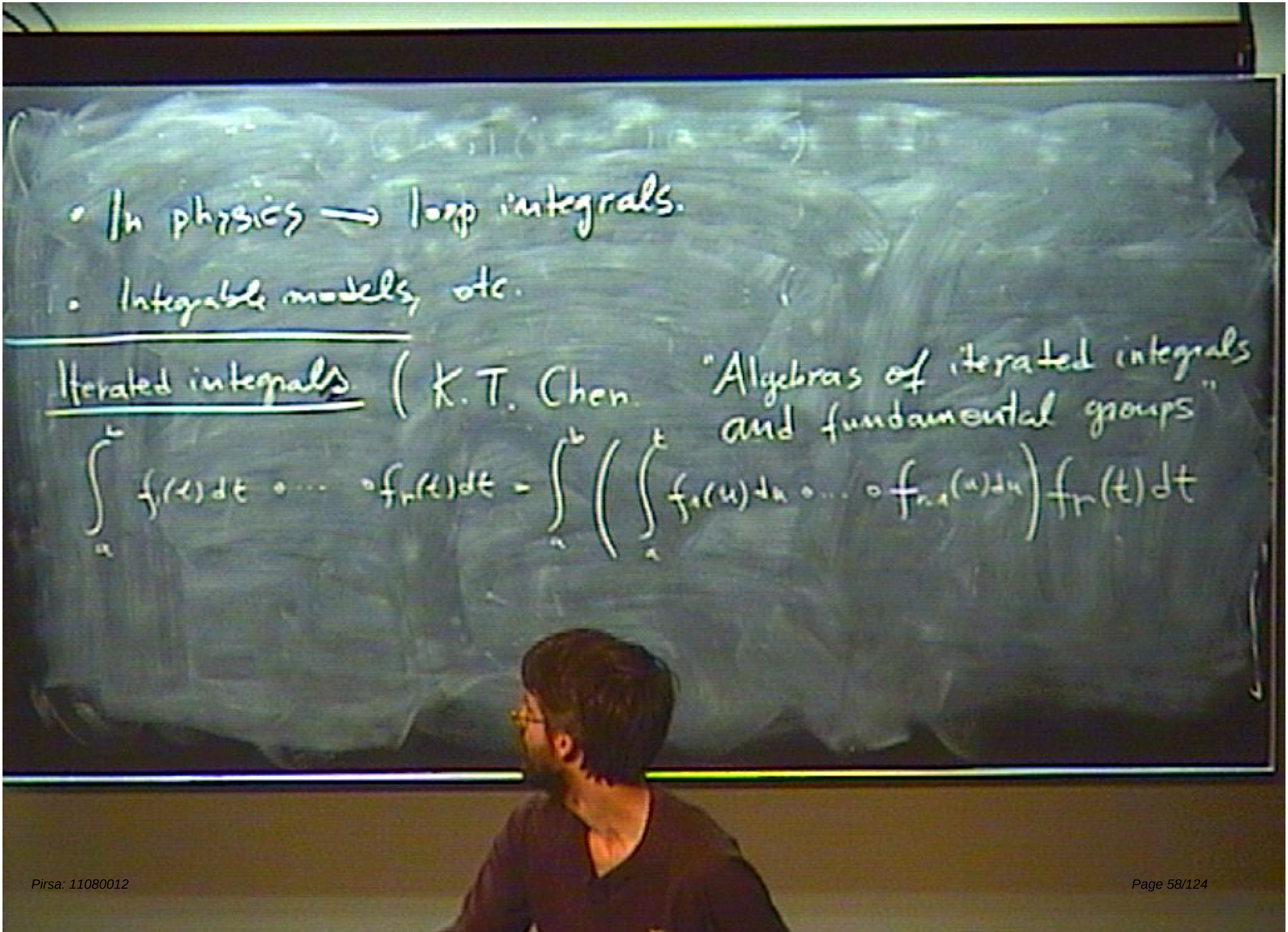
"Algebras of iterated integrals
and fundamental groups"

- In physics ⟶ loop integrals.
- Integrable models, etc.

Iterated integrals (K. T. Chen "Algebras of iterated integrals and fundamental groups"

$$\int_a^b f_1(t)\,dt \circ \cdots \circ f_r(t)\,dt = \int_a^b \left(\int_a^t f_1(u)\,du \circ \cdots \circ f_{r-1}(u)\,du \right) f_r(t)\,dt$$


• In physics → loop integrals.
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Iterated integrals (K.T. Chen
"Algebras of iterated integrals and fundamental groups"
$\int_a^b f_1(t)dt \circ \cdots \circ f_r(t)dt = \int_a^b \left(\int_a^t f_1(u)du \circ \ldots \circ f_{r-1}(u)du \right) f_r(t)dt$

- In physics $\longrightarrow$ loop integrals.
- Integrable models, etc.

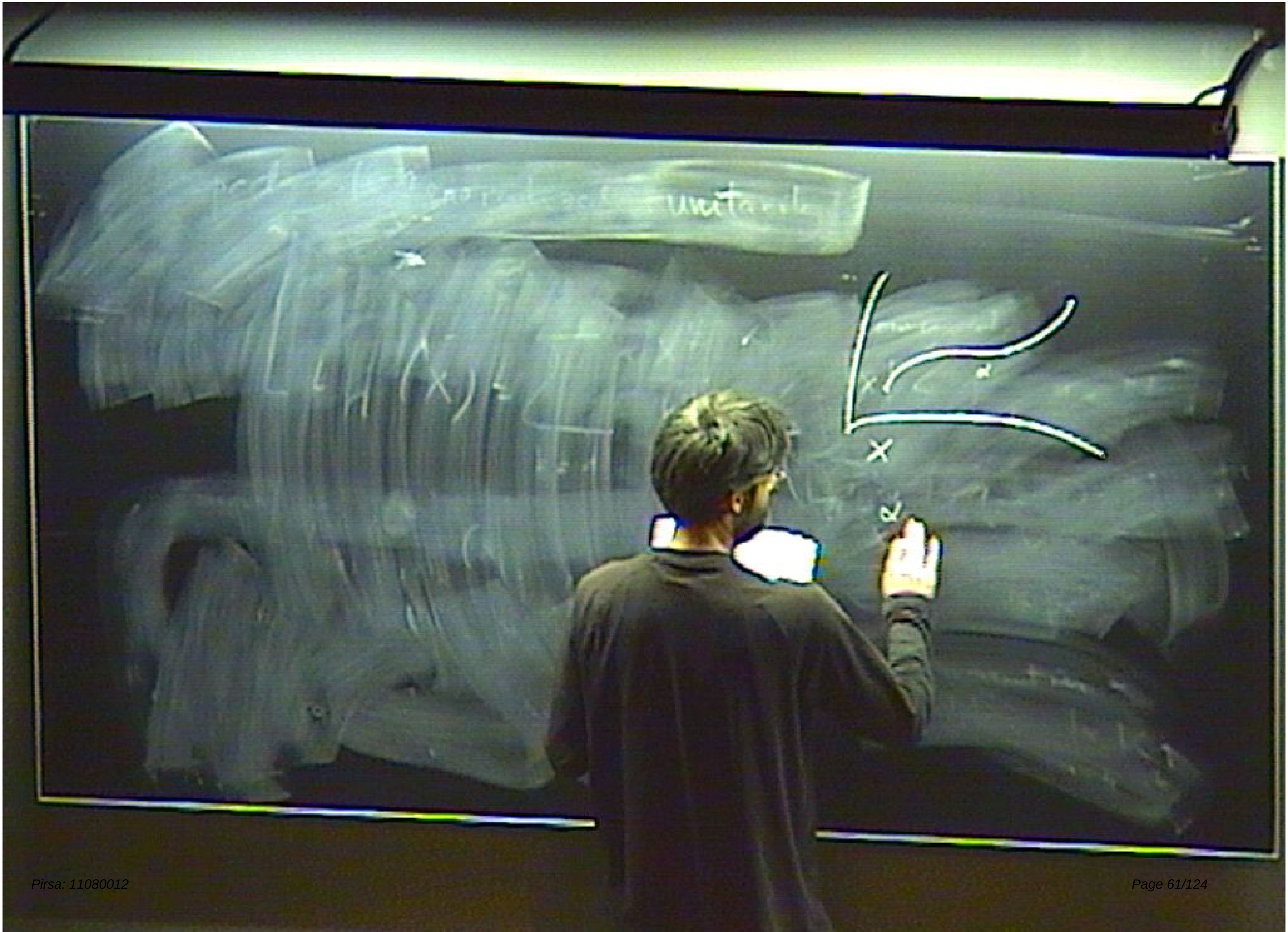
<u>Iterated integrals</u> (K. T. Chen "Algebras of iterated integrals and fundamental groups"

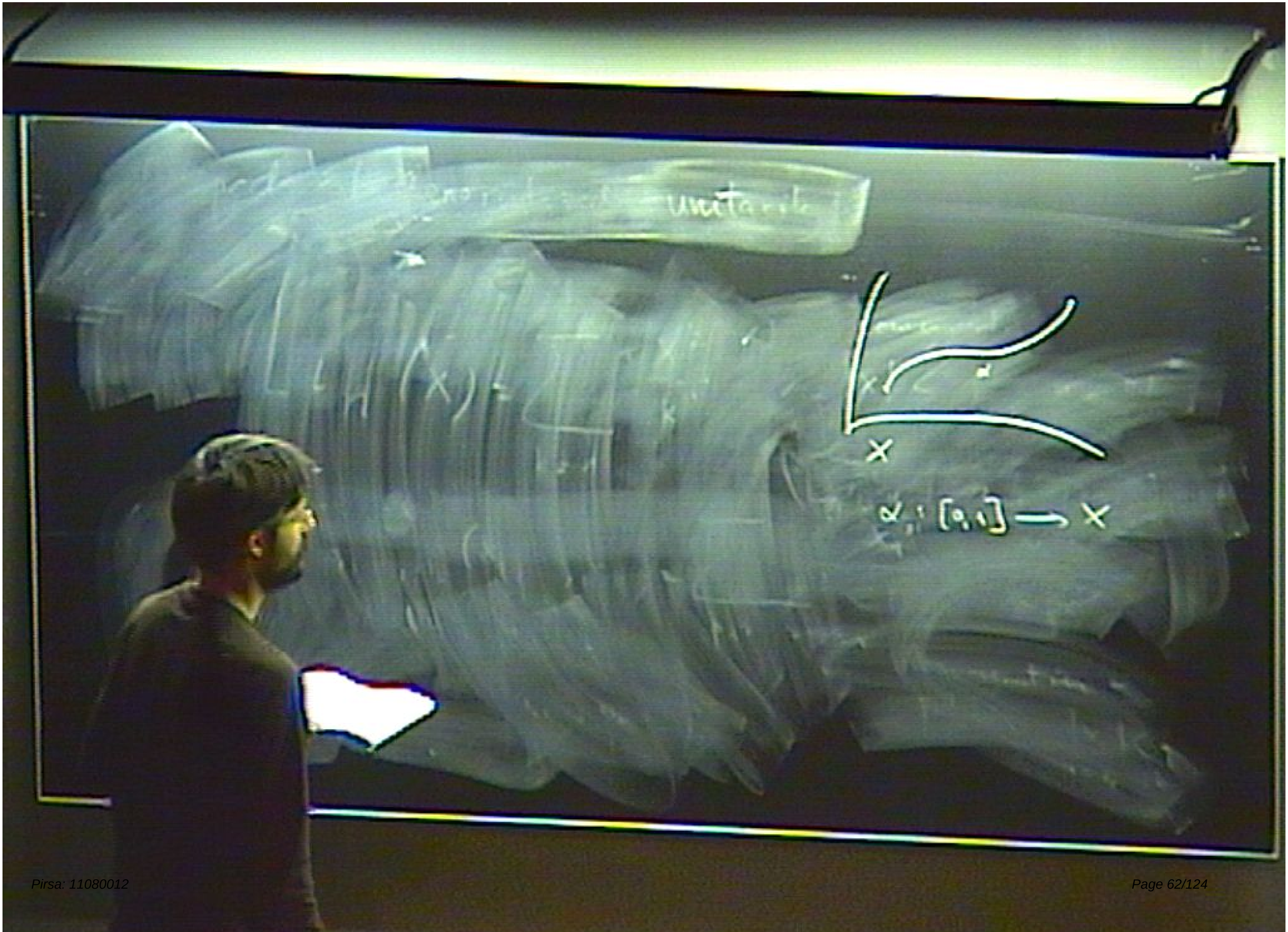
$$\int_a^b f_1(t)\,dt \circ \cdots \circ f_r(t)\,dt = \int_a^b \left(\int_a^t f_1(u)\,du \circ \cdots \circ f_{r-1}(u)\,du \right) f_r(t)\,dt$$

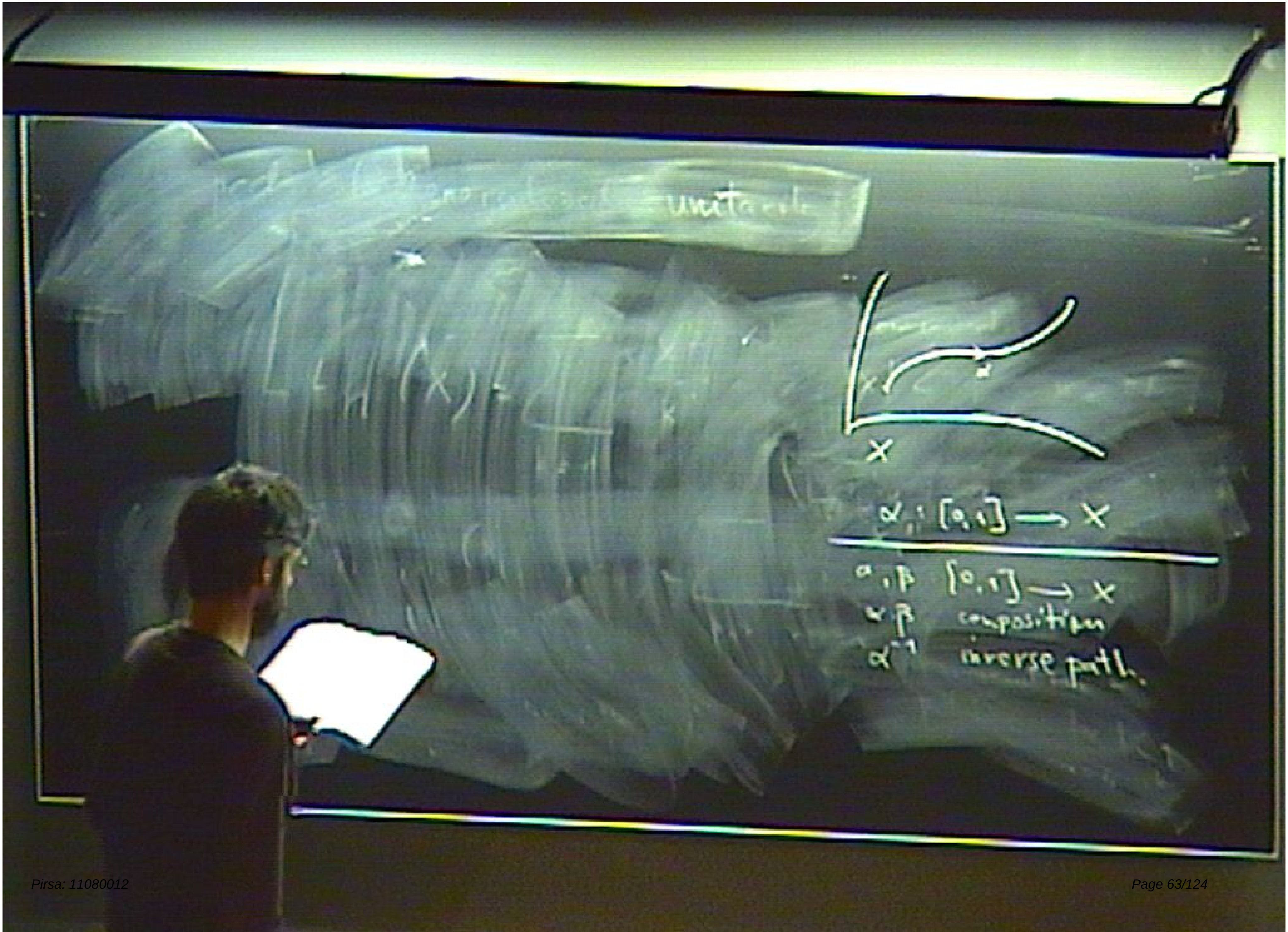
- In physics → loop integrals.
- Integrable models, etc.

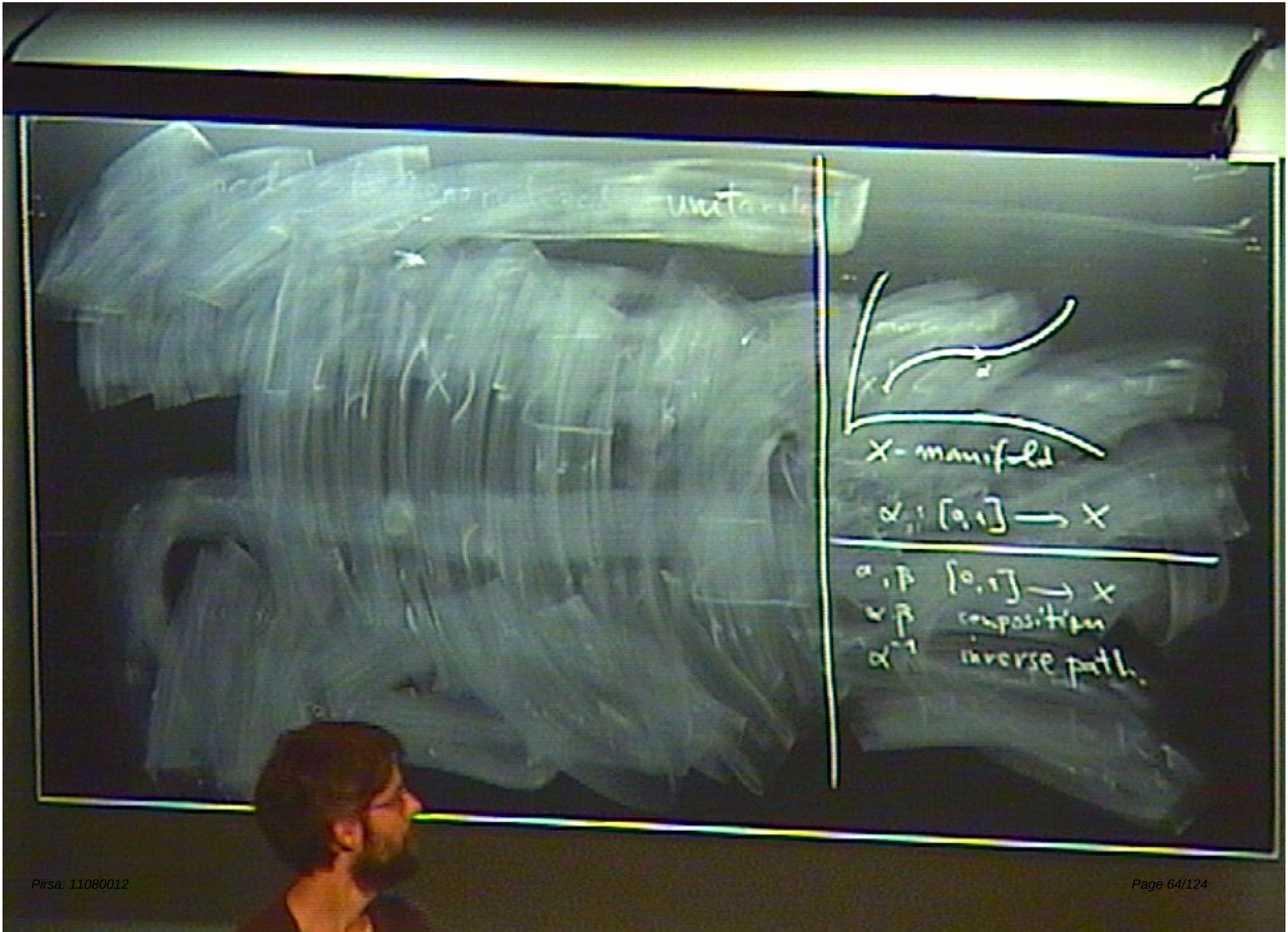
Iterated integrals (K.T. Chen "Algebras of iterated integrals and fundamental groups"

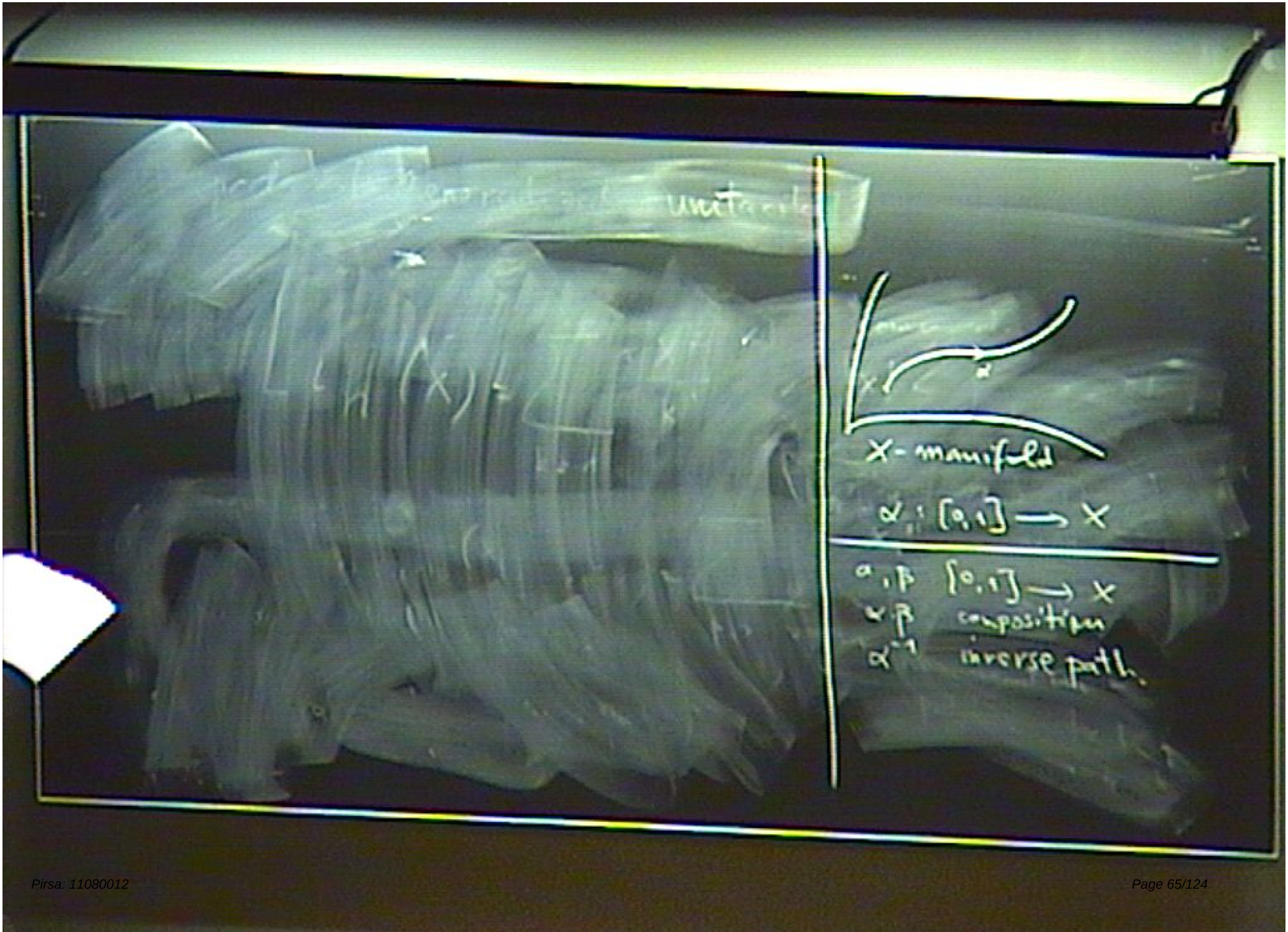
$$\int_a^t f_1(t)\,dt \circ \cdots \circ f_n(t)\,dt = \int_a^t \left(\int_a^t f_1(u)\,du \circ \cdots \circ f_{n-1}(u)\,du \right) f_n(t)\,dt$$




X
α : [0,1] ⟶ X


X
α, : [0,1] ⟶ X
α, β [0,1] ⟶ X
α·β composition
α⁻¹ inverse path.


X - manifold
α : [0,1] ⟶ X
α, β [0,1] ⟶ X
α·β composition
α⁻¹ inverse path.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α : [0,1] ⟶ X
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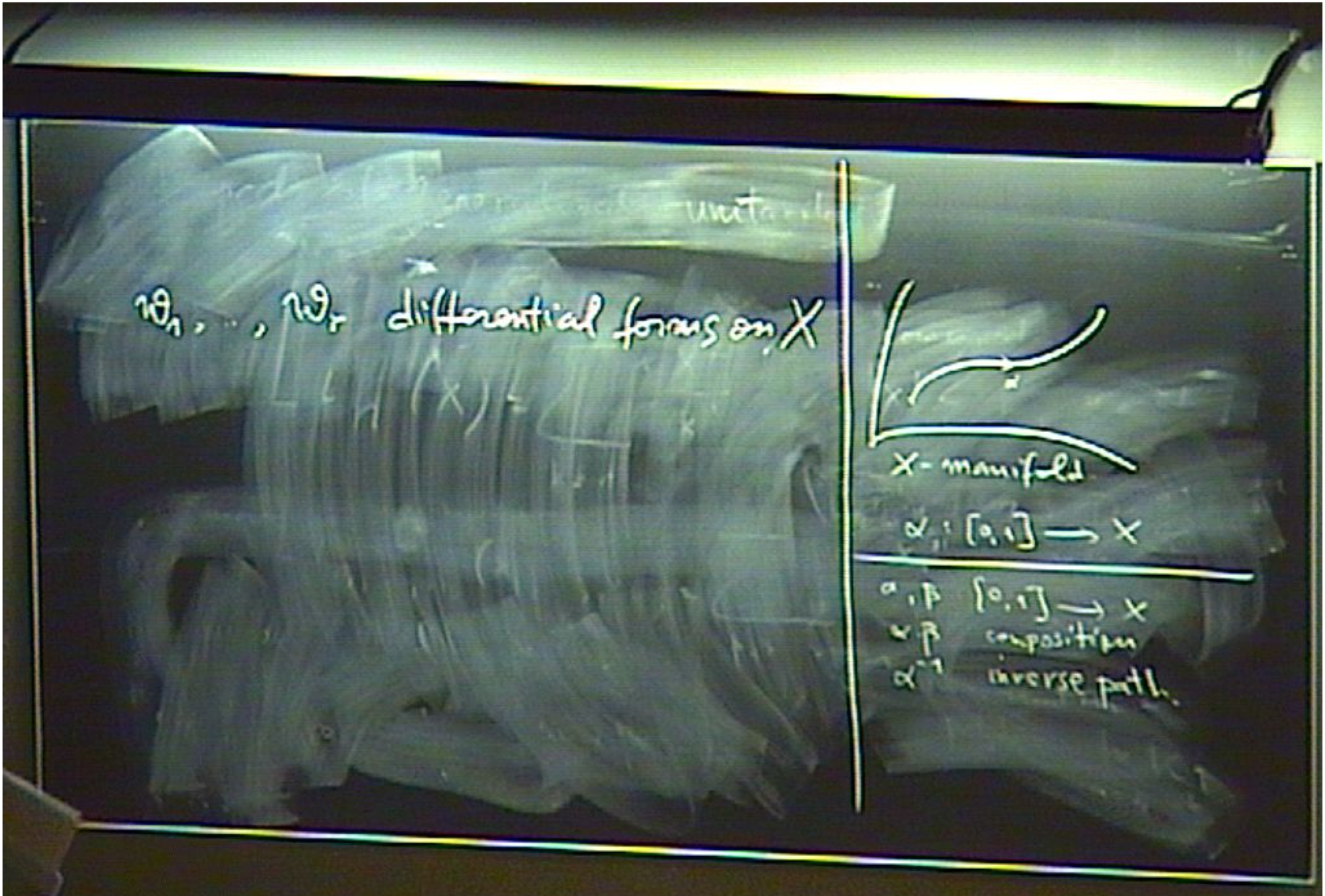

ϑ1, ..., ϑr differential forms on X
X - manifold
α : [0,1] ⟶ X
α, β [0,1] ⟶ X
α·β composition
α⁻¹ inverse path.

$\vartheta_1, \ldots, \vartheta_r$ differential forms on X

x^i are local coordinates on X

$f = \sum f_i \, dx^i$

$\alpha^* f = \sum f_i \frac{dx^i}{dt} dt$

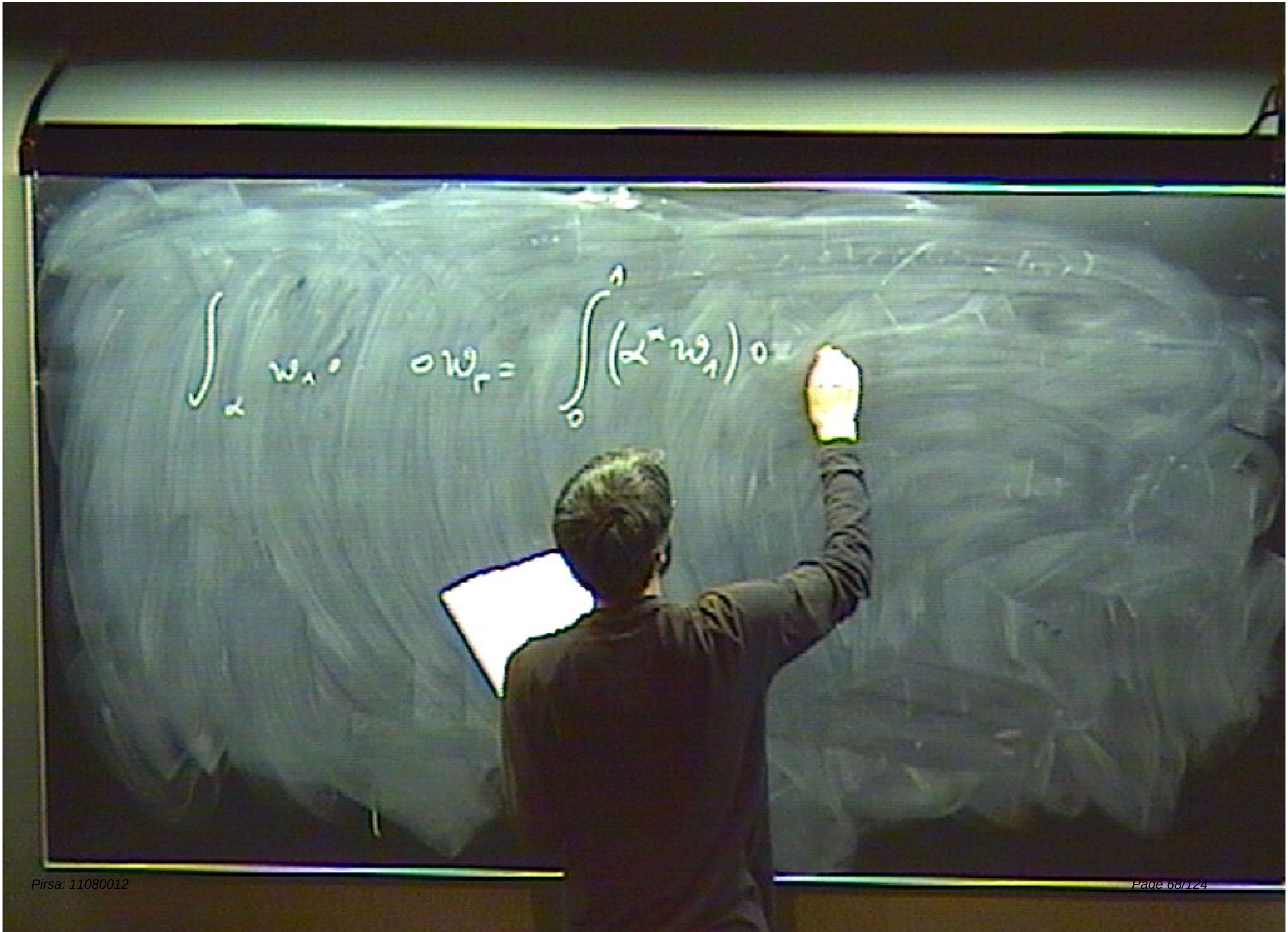
X - manifold

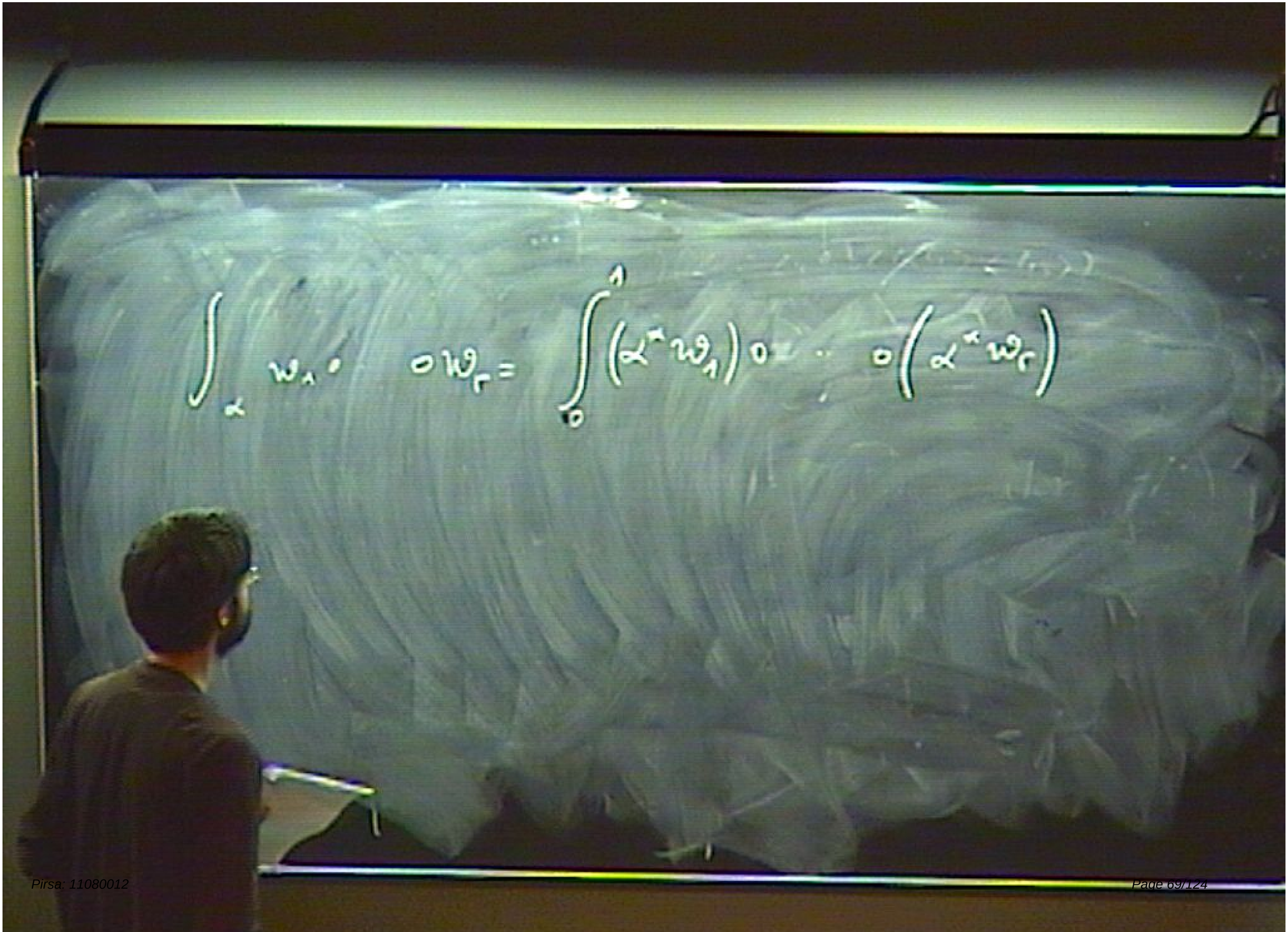
$\alpha : [0,1] \longrightarrow X$

$\alpha, \beta \ [0,1] \longrightarrow X$

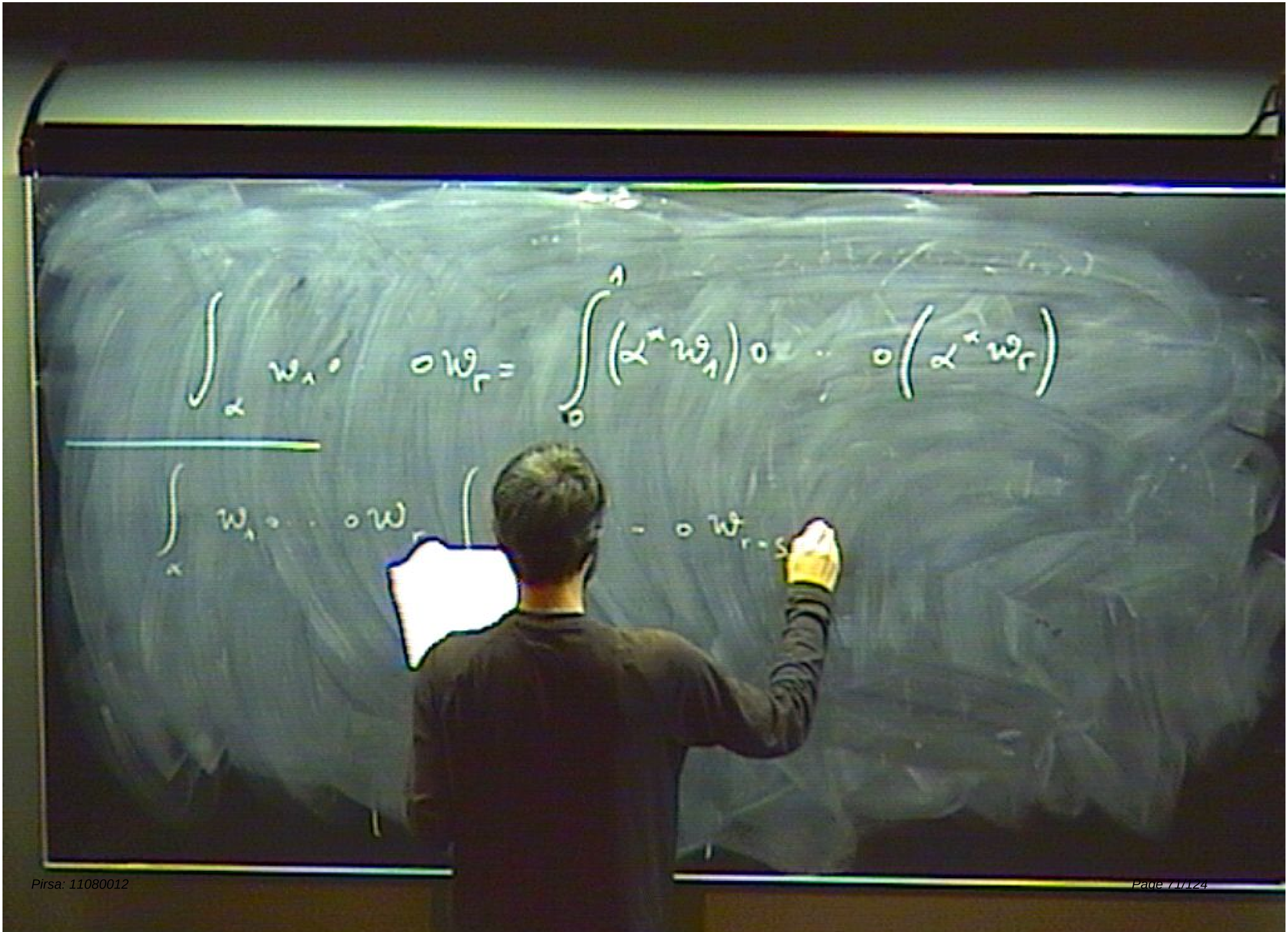
$\alpha \cdot \beta$ composition

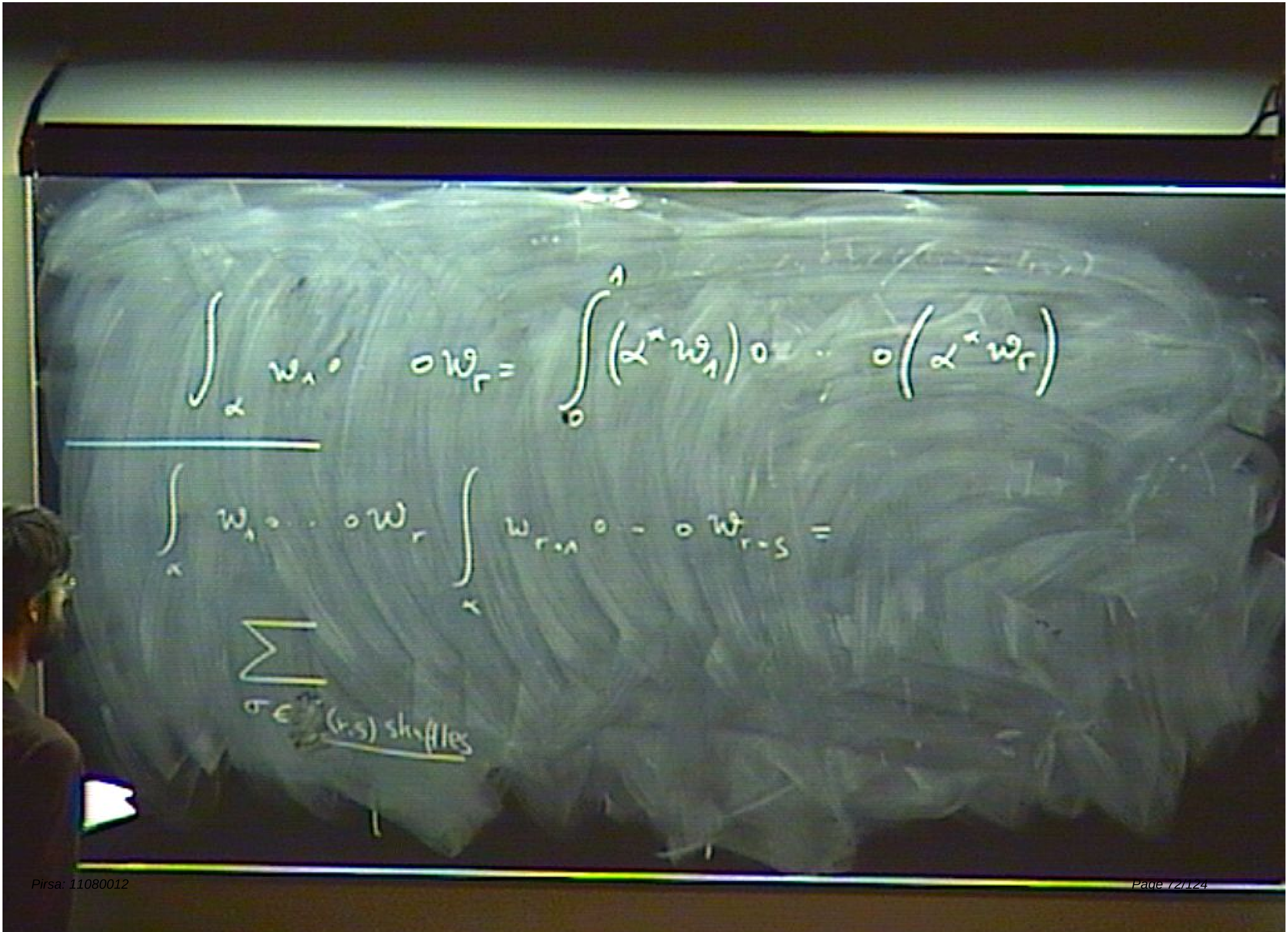
α^{-1} inverse path.


∫_α w_1 ∘ ∘ w_r = ∫_0^1 (α* w_1) ∘


$\int_\alpha w_1 \circ \cdots \circ w_r = \int_0^1 (\alpha^* w_1) \circ \cdots \circ (\alpha^* w_r)$

$$\int_\alpha w_1 \circ \cdots \circ w_r = \int_0^1 (\alpha^* w_1) \circ \cdots \circ (\alpha^* w_r)$$


$\int_\alpha \omega_1 \circ \cdots \circ \omega_r = \int_0^1 (\alpha^* \omega_1) \circ \cdots \circ (\alpha^* \omega_r)$
$\int_\alpha \omega_1 \circ \cdots \circ \omega_r$
$\cdots \circ \omega_{r-s}$


∫_α ω_1 ∘ ⋯ ∘ ω_r = ∫_0^1 (α^* ω_1) ∘ ⋯ ∘ (α^* ω_r)
∫_α ω_1 ∘ ⋯ ∘ ω_r ∫_α ω_{r+1} ∘ ⋯ ∘ ω_{r+s} =
Σ_{σ ∈ (r,s) shuffles}

$$\int_\alpha \omega_1 \circ \cdots \circ \omega_r = \int_0^1 (\alpha^* \omega_1) \circ \cdots \circ (\alpha^* \omega_r)$$

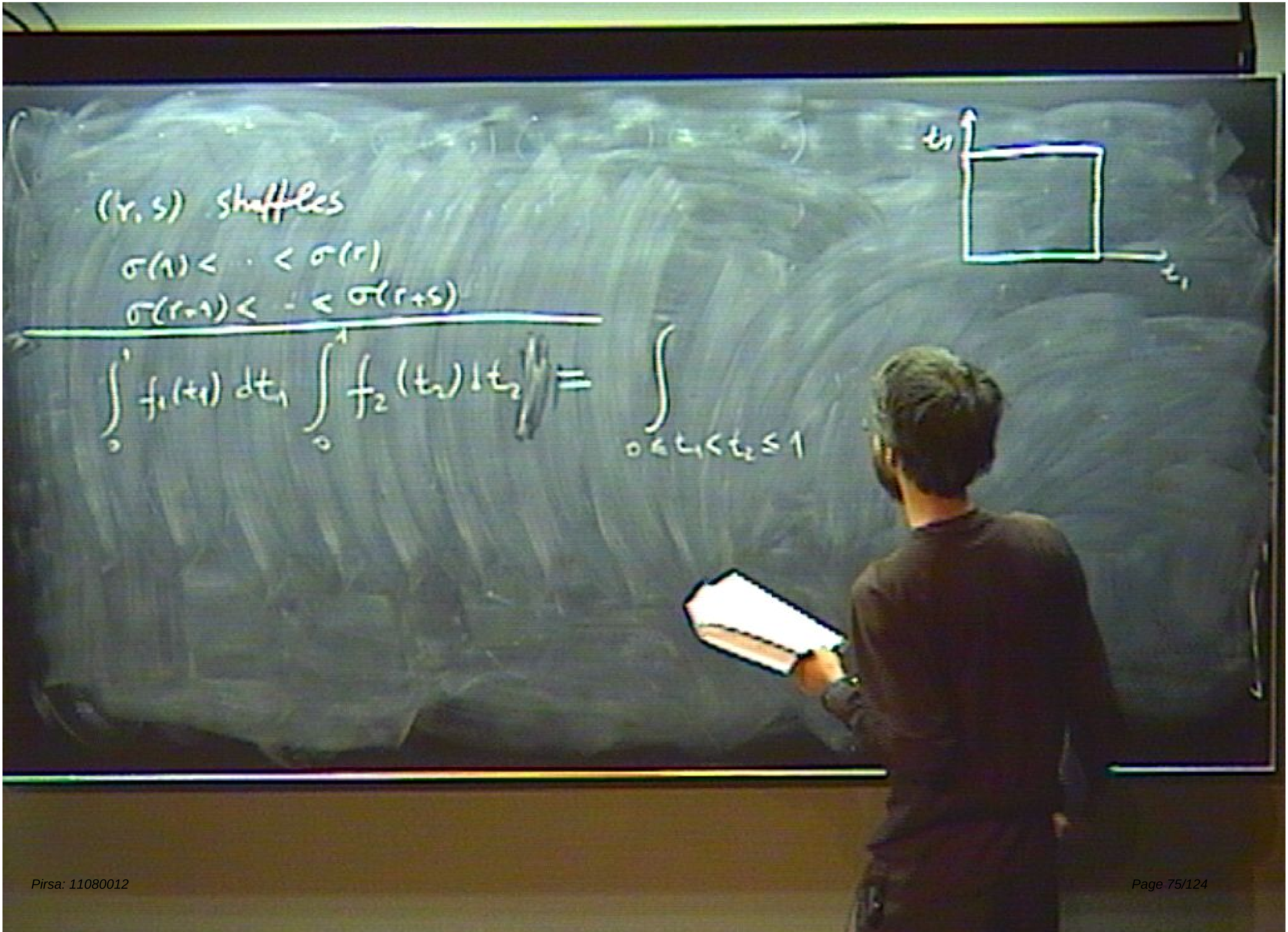
$$\int_\alpha \omega_1 \circ \cdots \circ \omega_r \int_\alpha \omega_{r+1} \circ \cdots \circ \omega_{r+s} =$$

$$\sum_{\sigma \in (r,s)\text{ shuffles}} \int_\alpha \omega_{\sigma(1)} \circ \cdots \circ \omega_{\sigma(r+s)}$$

(r, s) shuffles

$\sigma(1) < \cdots < \sigma(r)$

$\sigma(r+1) < \cdots < \sigma(r+s)$


(r, s) shuffles
σ(1) < · < σ(r)
σ(r+1) < · < σ(r+s)
∫₀¹ f₁(t₁) dt₁ ∫₀¹ f₂(t₂) dt₂ = ∫
0 ≤ t₁ < t₂ ≤ 1
t₁
t₁

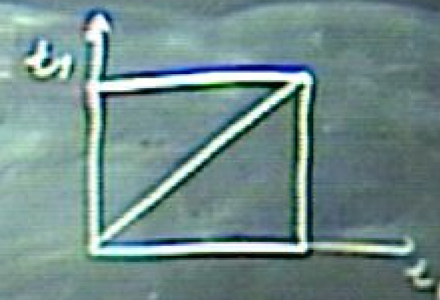
(r, s) shuffles

$\sigma(1) < \cdots < \sigma(r)$

$\sigma(r+1) < \cdots < \sigma(r+s)$

$$\int_0^1 f_1(t_1)\, dt_1 \int_0^1 f_2(t_2)\, dt_2 = \int_{0 \le t_1 < t_2 \le 1} f_1(t_1) f_2(t_2)\, dt_1\, dt_2 + \int_{0 \le t_2 < t_1 \le 1} f_1(t_1) f_2(t_2)\, dt_1\, dt_2$$

$$\int_0^1 f_1(t)\, dt \circ f_2(t)\, dt + \int_0^1 f_2(t)\, dt \circ f_1(t)\, dt$$



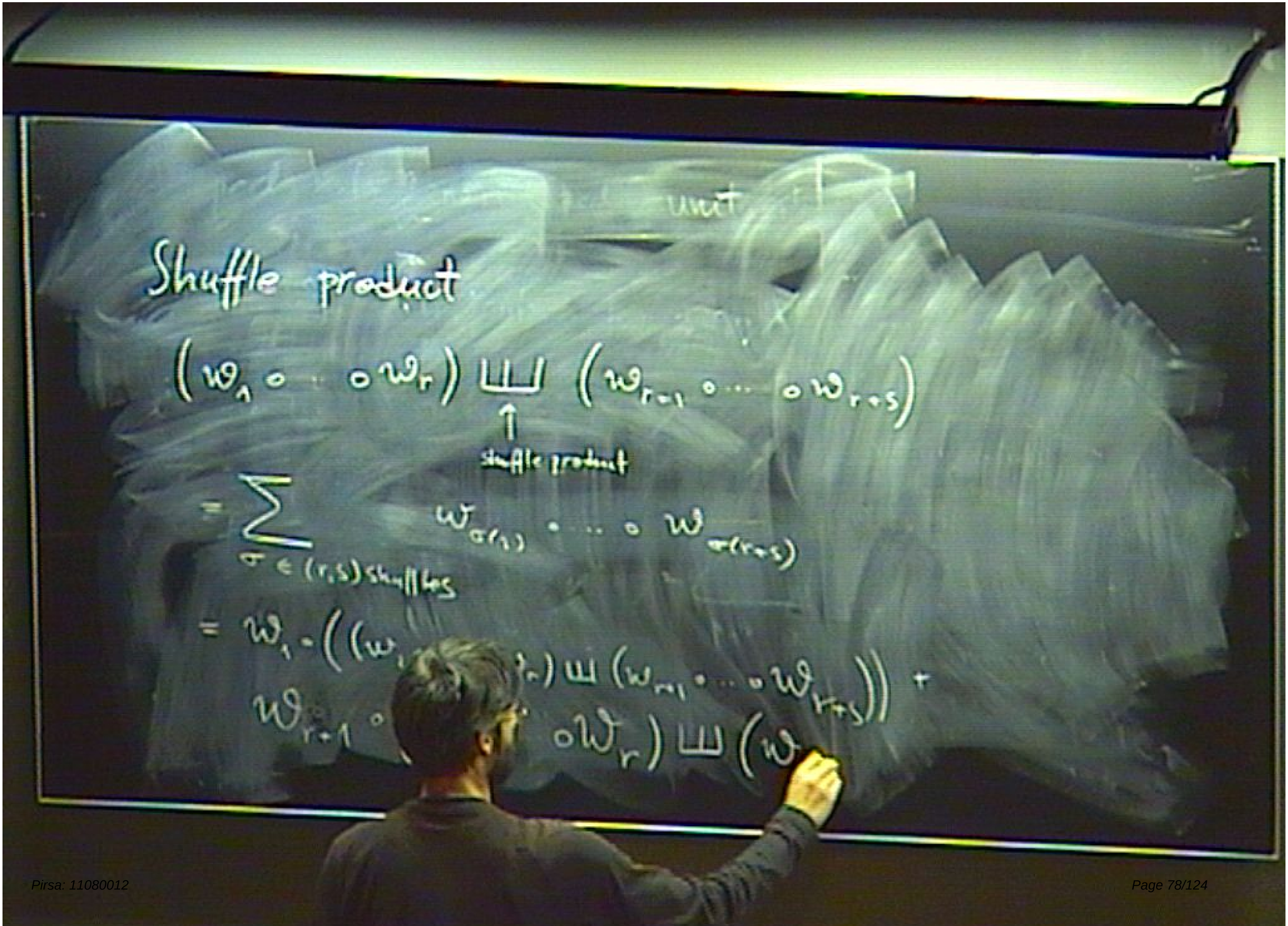
(r, s) shuffles

$\sigma(1) < \cdots < \sigma(r)$

$\sigma(r+1) < \cdots < \sigma(r+s)$

$$\int_0^1 f_1(t_1)\, dt_1 \int_0^1 f_2(t_2)\, dt_2 = \int_{0 \le t_1 < t_2 \le 1} f_1(t_1)\, f_2(t_2)\, dt_1\, dt_2 + \int_{0 \le t_2 < t_1 \le 1} f_1(t_1)\, f_2(t_2)\, dt_1\, dt_2$$

$$\int_0^1 f_1(t)\, dt \circ f_2(t)\, dt + \int_0^1 f_2(t)\, dt \circ f_1(t)\, dt$$


Shuffle product
(w_1 ∘ ⋯ ∘ w_r) ⧢ (w_{r+1} ∘ ⋯ ∘ w_{r+s})
shuffle product
= Σ_{σ ∈ (r,s) shuffles} w_{σ(1)} ∘ ⋯ ∘ w_{σ(r+s)}
= w_1 ∘ ((w_2 ∘ ⋯ ∘ w_r) ⧢ (w_{r+1} ∘ ⋯ ∘ w_{r+s})) +
w_{r+1} ∘ ((w_1 ∘ ⋯ ∘ w_r) ⧢ (w

Shuffle product

$$(w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s})$$

↑ shuffle product

$$= \sum_{\sigma \in (r,s)\text{ shuffles}} w_{\sigma(1)} \circ \cdots \circ w_{\sigma(r+s)}$$

$$= w_1 \circ \left((w_2 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s}) \right) + w_{r+1} \circ \left((w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+2} \circ \cdots \circ w_{r+s}) \right)$$

Shuffle product

$$(w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s})$$

↑ shuffle product

$$= \sum_{\sigma \in (r,s)\text{ shuffles}} w_{\sigma(1)} \circ \cdots \circ w_{\sigma(r+s)}$$

$$= w_1 \circ \left((w_2 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s}) \right) +$$

$$w_{r+1} \circ \left((w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+2} \circ \cdots \circ w_{r+s}) \right)$$

Shuffle product

$$(w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s})$$

↑ shuffle product

$$= \sum_{\sigma \in (r,s)\text{ shuffles}} w_{\sigma(1)} \circ \cdots \circ w_{\sigma(r+s)}$$

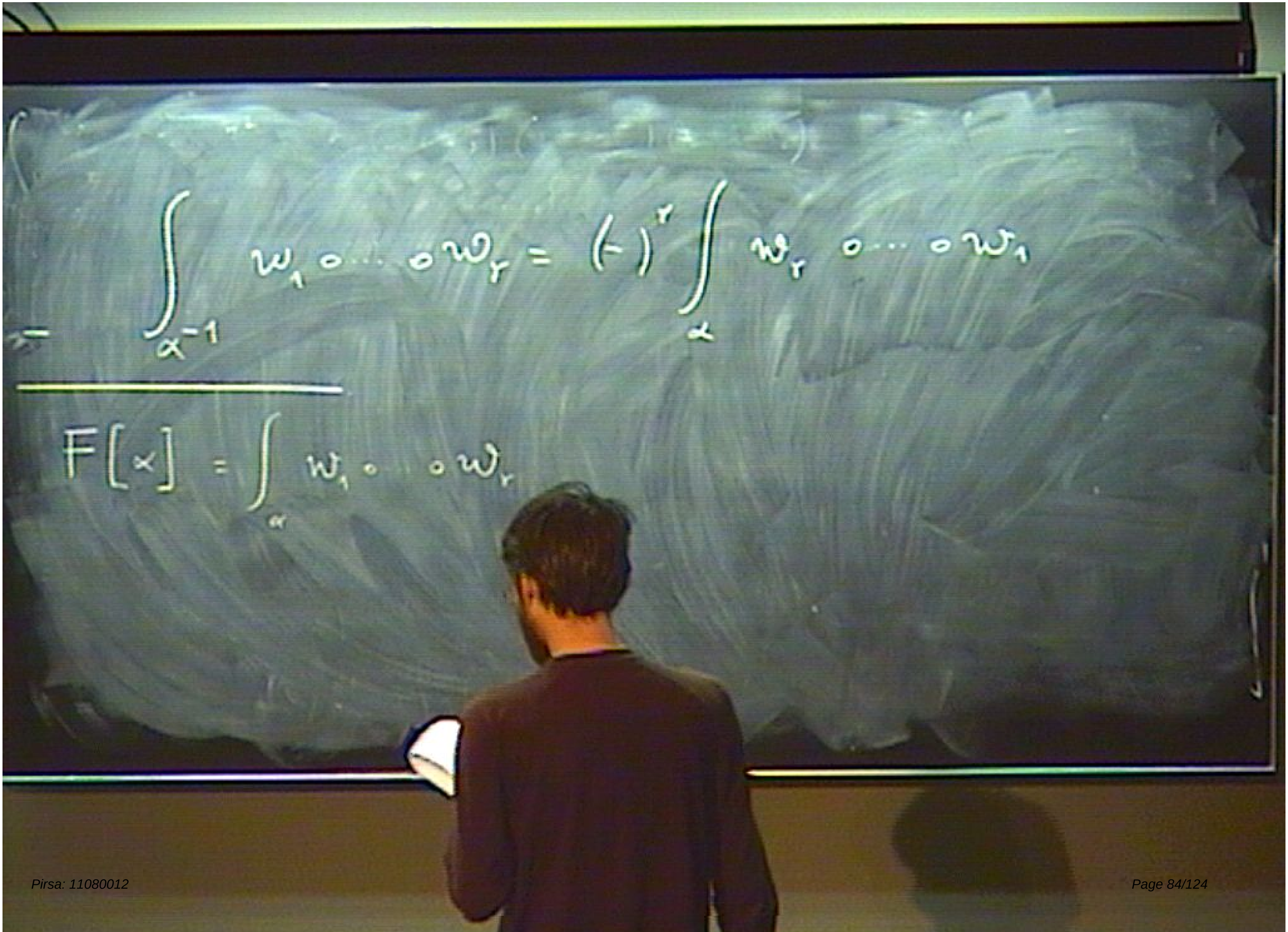
$$= w_1 \circ \big((w_2 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+1} \circ \cdots \circ w_{r+s}) \big) + w_{r+1} \circ \big((w_1 \circ \cdots \circ w_r) \sqcup\!\sqcup (w_{r+2} \circ \cdots \circ w_{r+s}) \big)$$

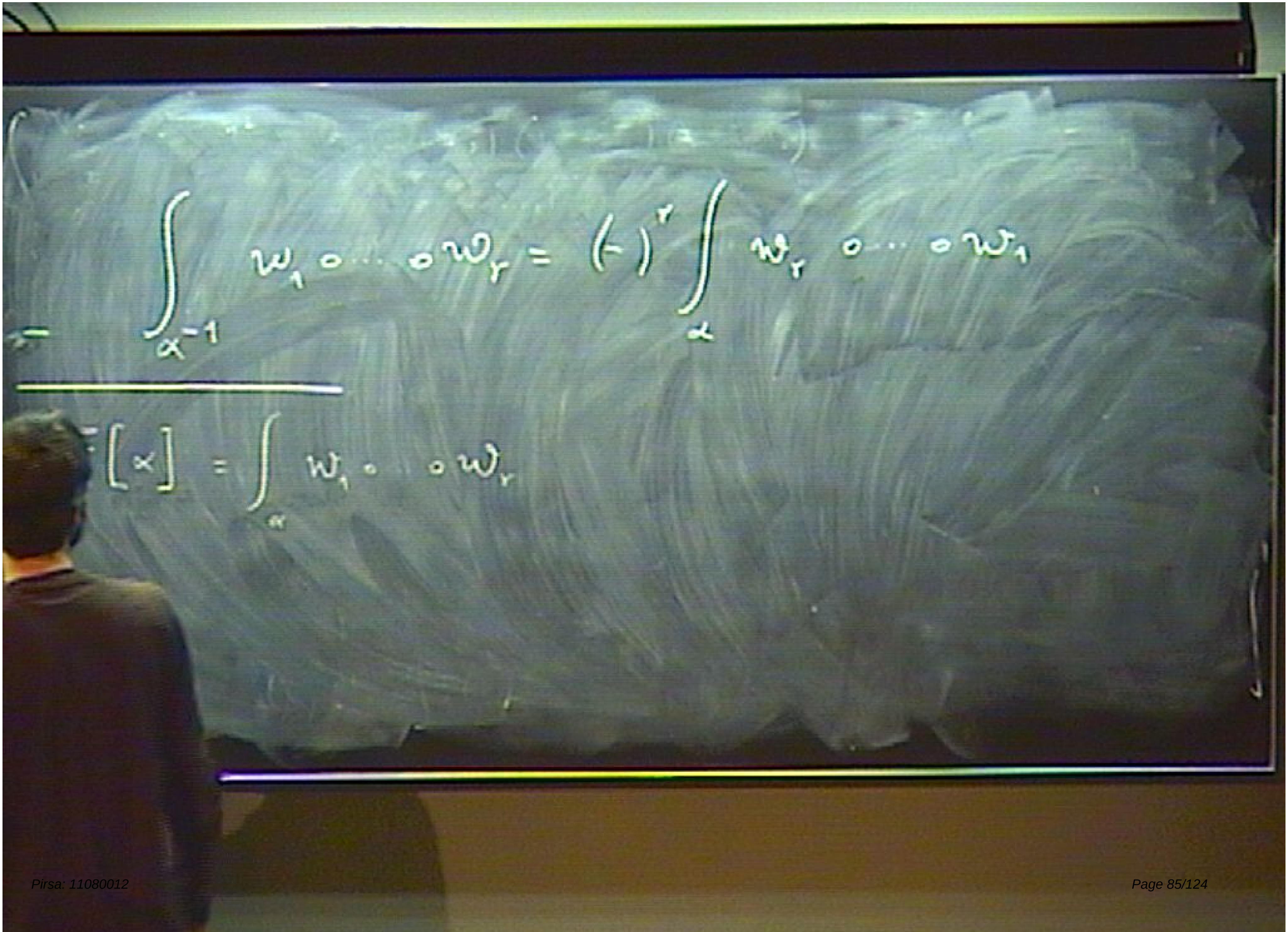
$$\int_{\alpha\beta} \omega_1 \circ \cdots \circ \omega_r = \int_{\alpha} \omega_1 \circ \cdots \circ \omega_r + \cdots$$

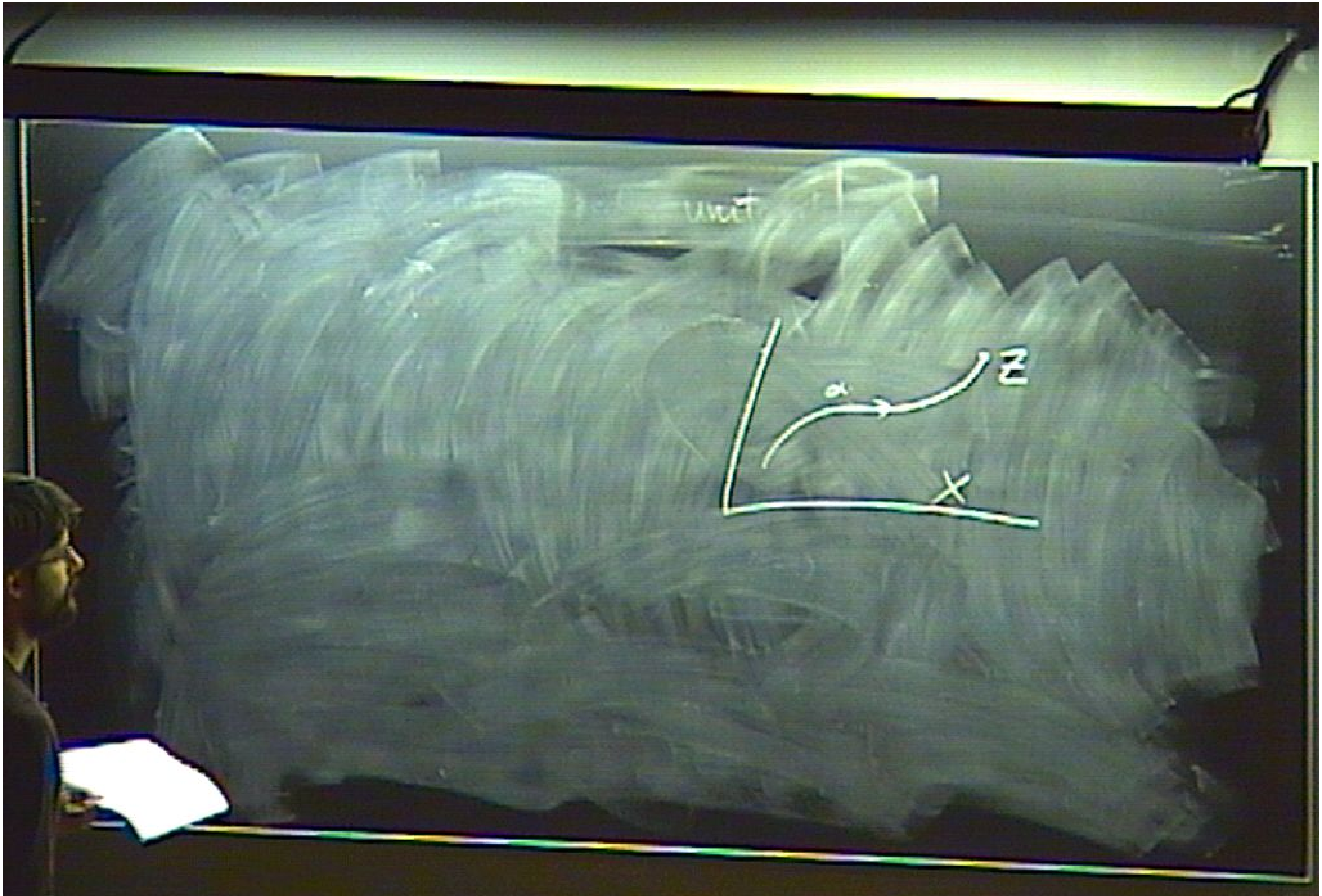
$$+ \int_{\alpha} \omega_1 \circ \cdots \circ \omega_i \int_{\beta} \omega_{i+1} \circ \cdots \circ \omega_r + \cdots$$

$$\cdots \int_{\beta} \omega_1 \circ \cdots \circ \omega_r$$

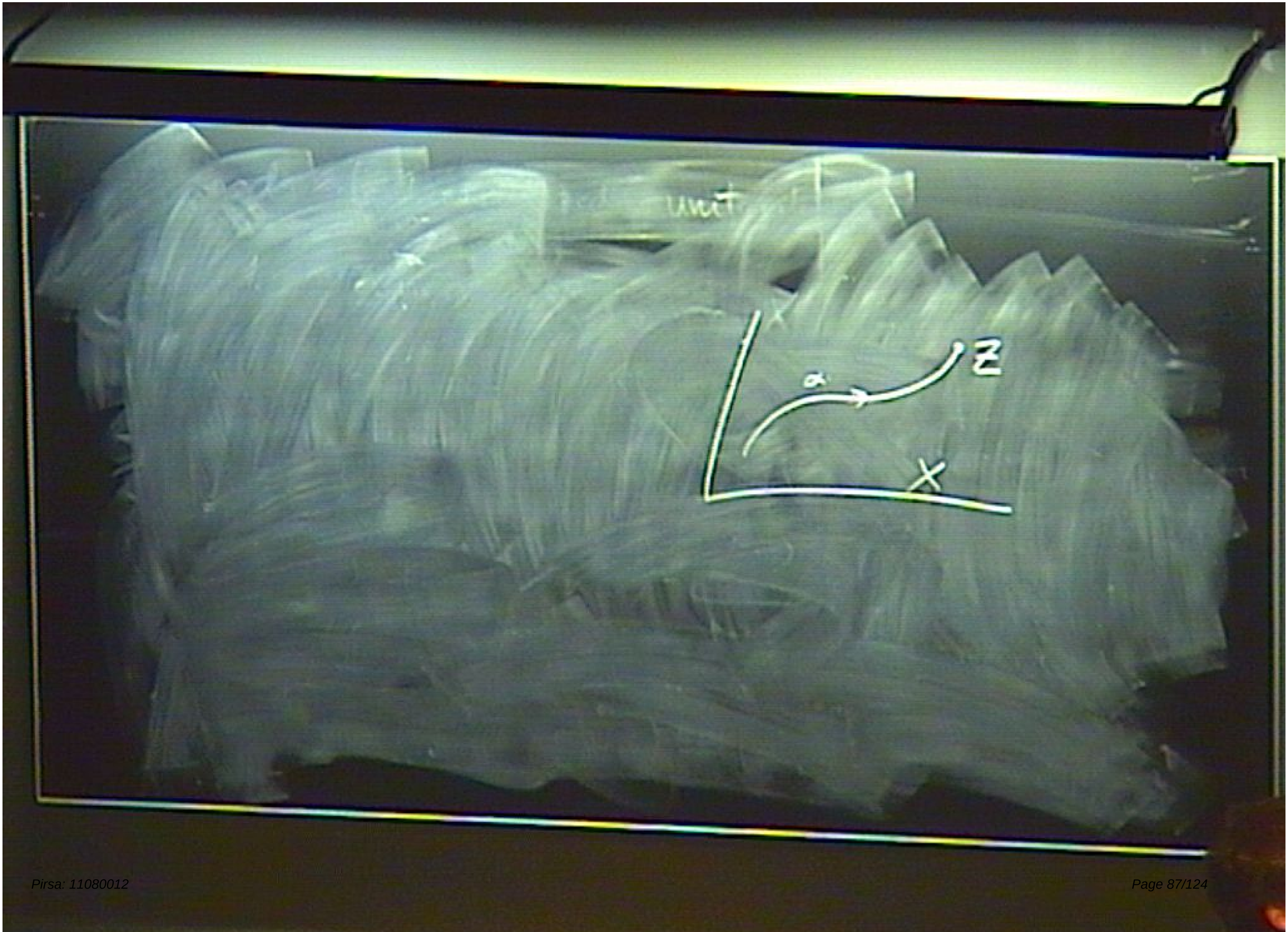
$$\int_{\alpha^{-1}} \omega_1 \circ \cdots \circ \omega_r = (-)^r \int_{\alpha} \omega_r \circ \cdots \circ \omega_1$$

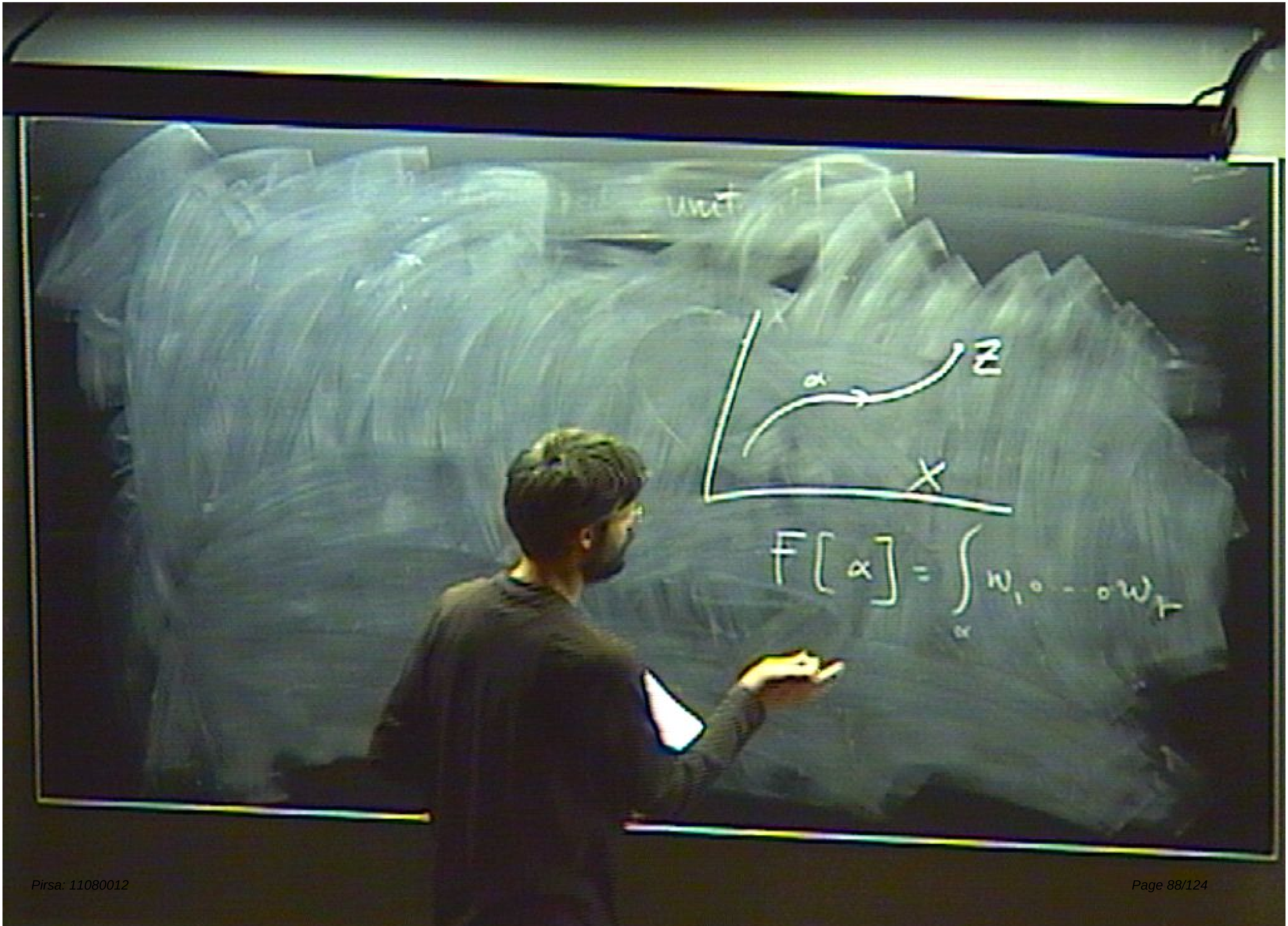

$\int_{\alpha^{-1}} w_1 \circ \cdots \circ w_r = (-)^r \int_{\alpha} w_r \circ \cdots \circ w_1$
$F[\alpha] = \int_{\alpha} w_1 \circ \cdots \circ w_r$


$\int_{\alpha^{-1}} \omega_1 \circ \cdots \circ \omega_r = (-)^r \int_{\alpha} \omega_r \circ \cdots \circ \omega_1$
$[\alpha] = \int_{\alpha} \omega_1 \circ \cdots \circ \omega_r$

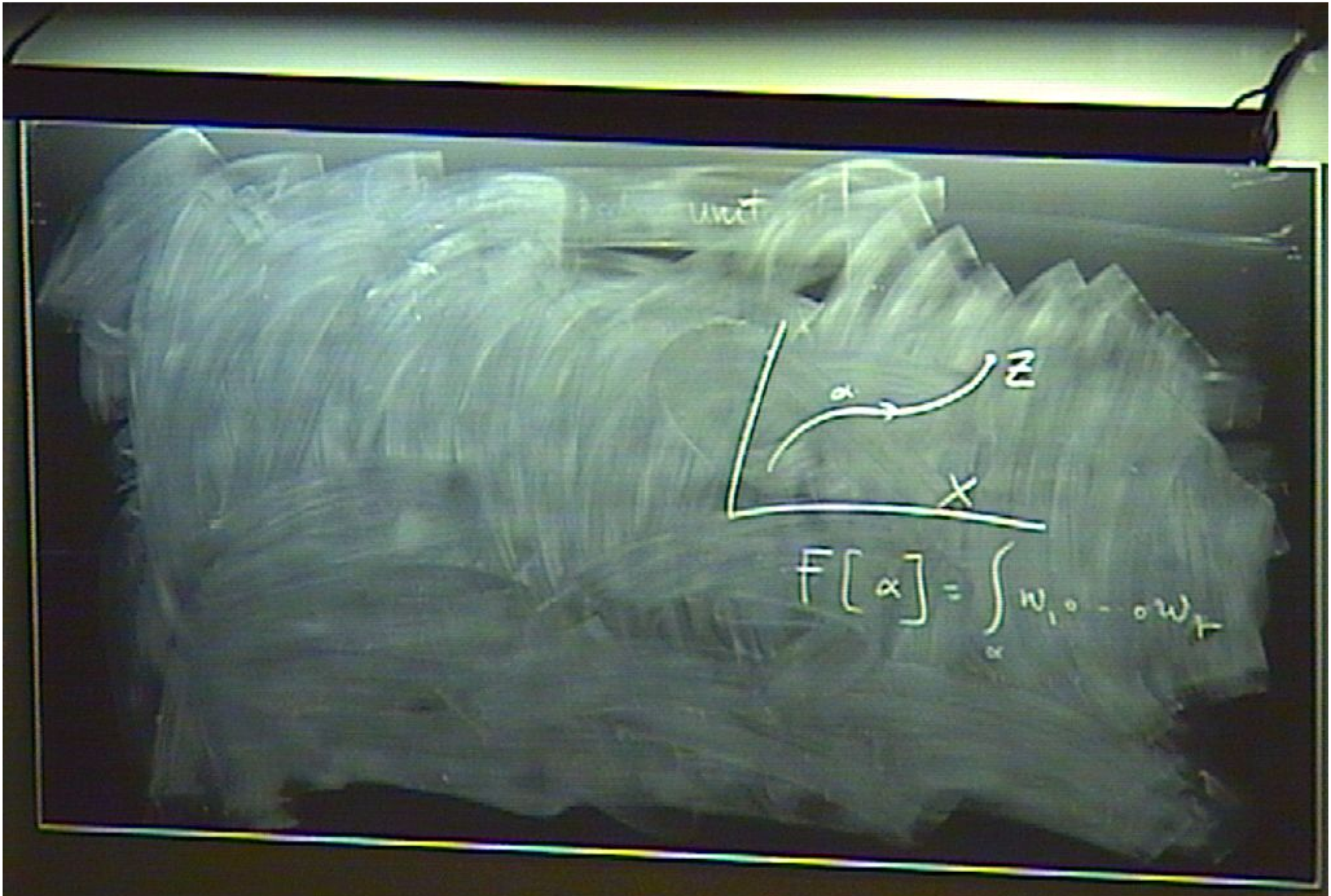


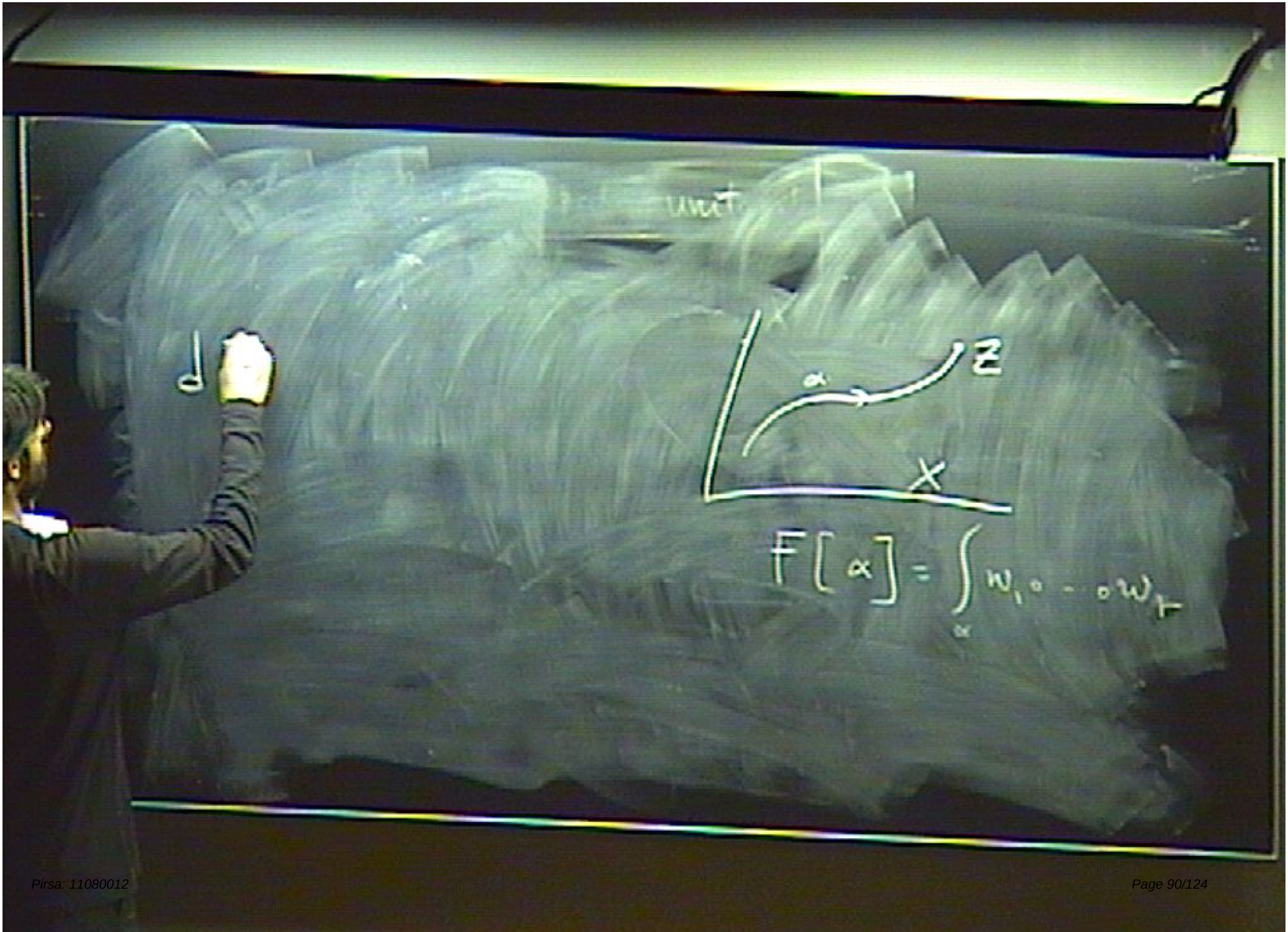
Z
X

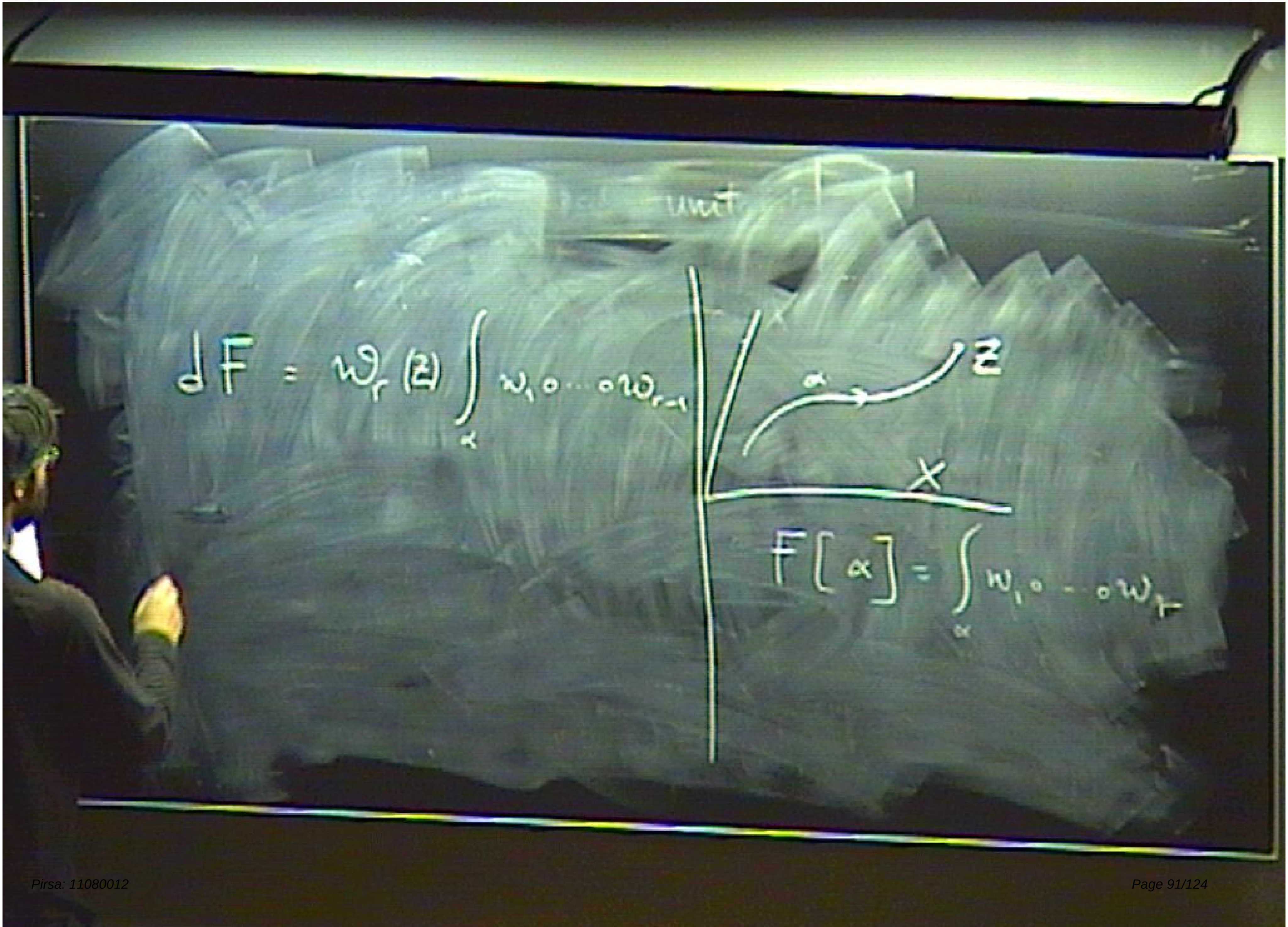

Z
α
X



Z
α
X
$F[\alpha] = \int_\alpha \omega_1 \circ \cdots \circ \omega_r$


Z
α
X
$F[\alpha] = \int_\alpha w_1 \circ \cdots \circ w_r$

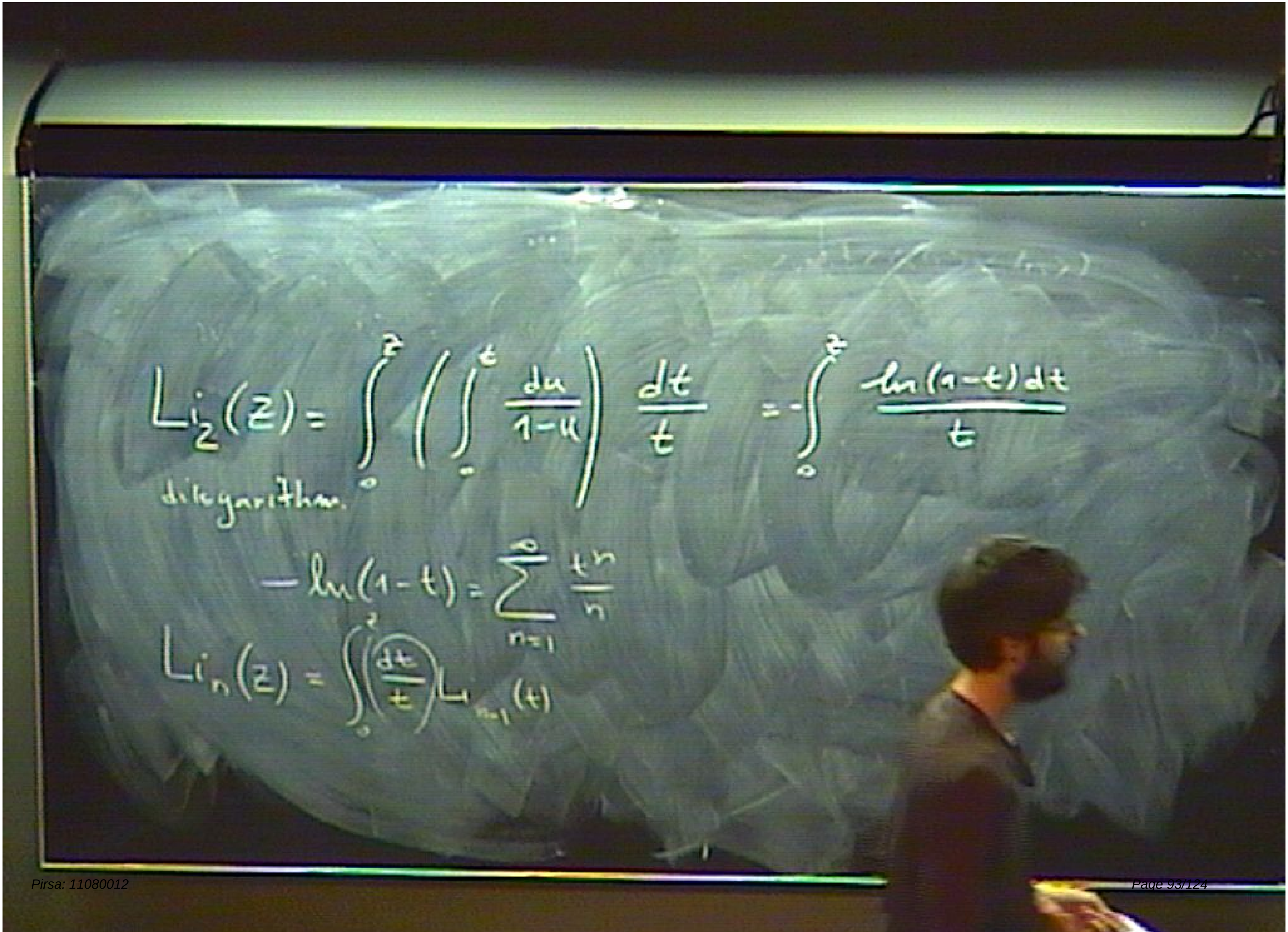

d
z
α
x
$F[\alpha] = \int_\alpha w_1 \circ \cdots \circ w_r$


unit
$dF = \omega_r(z) \int_\alpha \omega_1 \circ \cdots \circ \omega_{r-1}$
z
α
X
$F[\alpha] = \int_\alpha \omega_1 \circ \cdots \circ \omega_r$

$$dF = \omega_r(z) \int_\alpha \omega_1 \circ \cdots \circ \omega_{r-1}$$

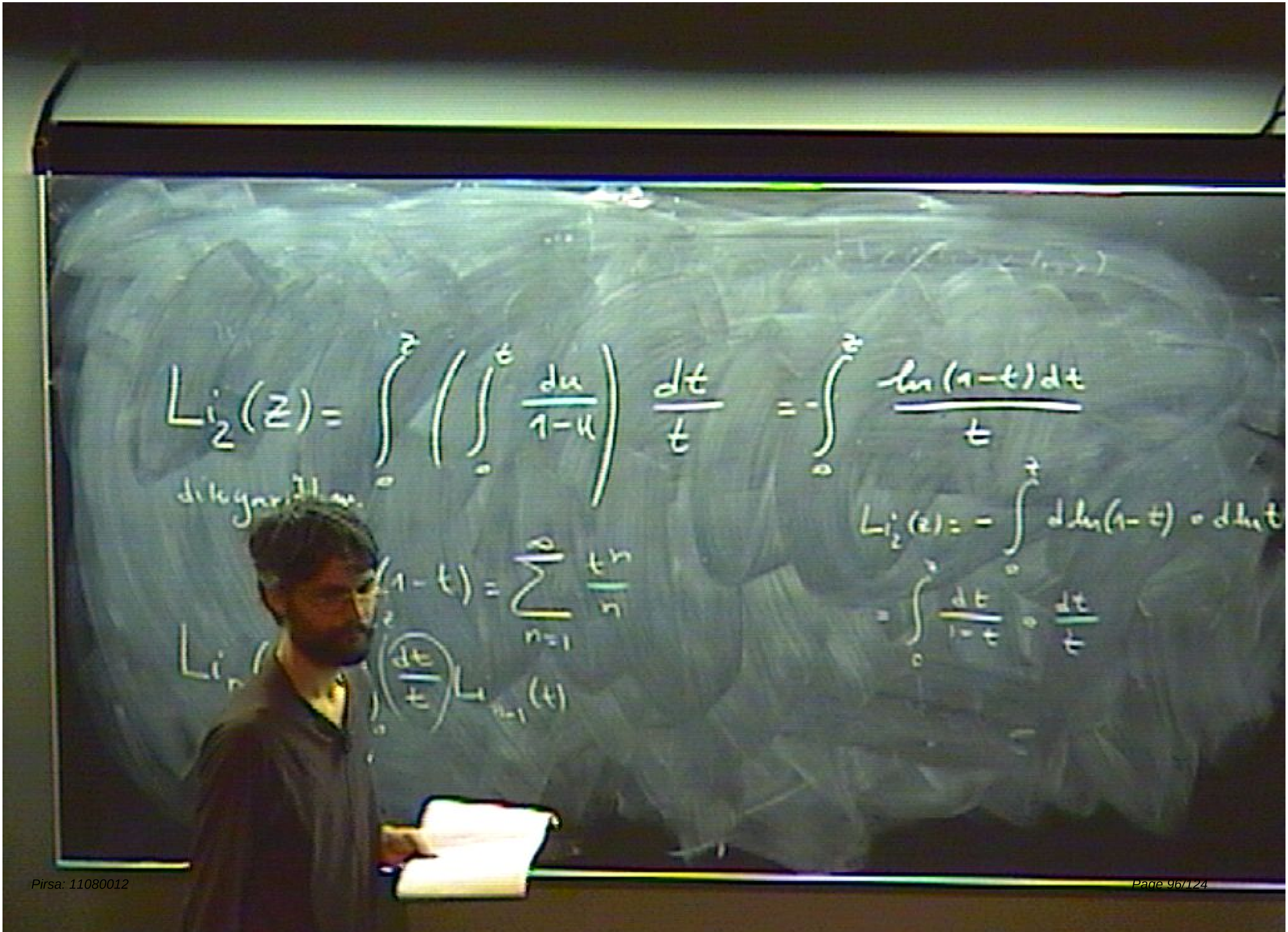
$$d \int^z F(t)\,dt = F(z)\,dz$$

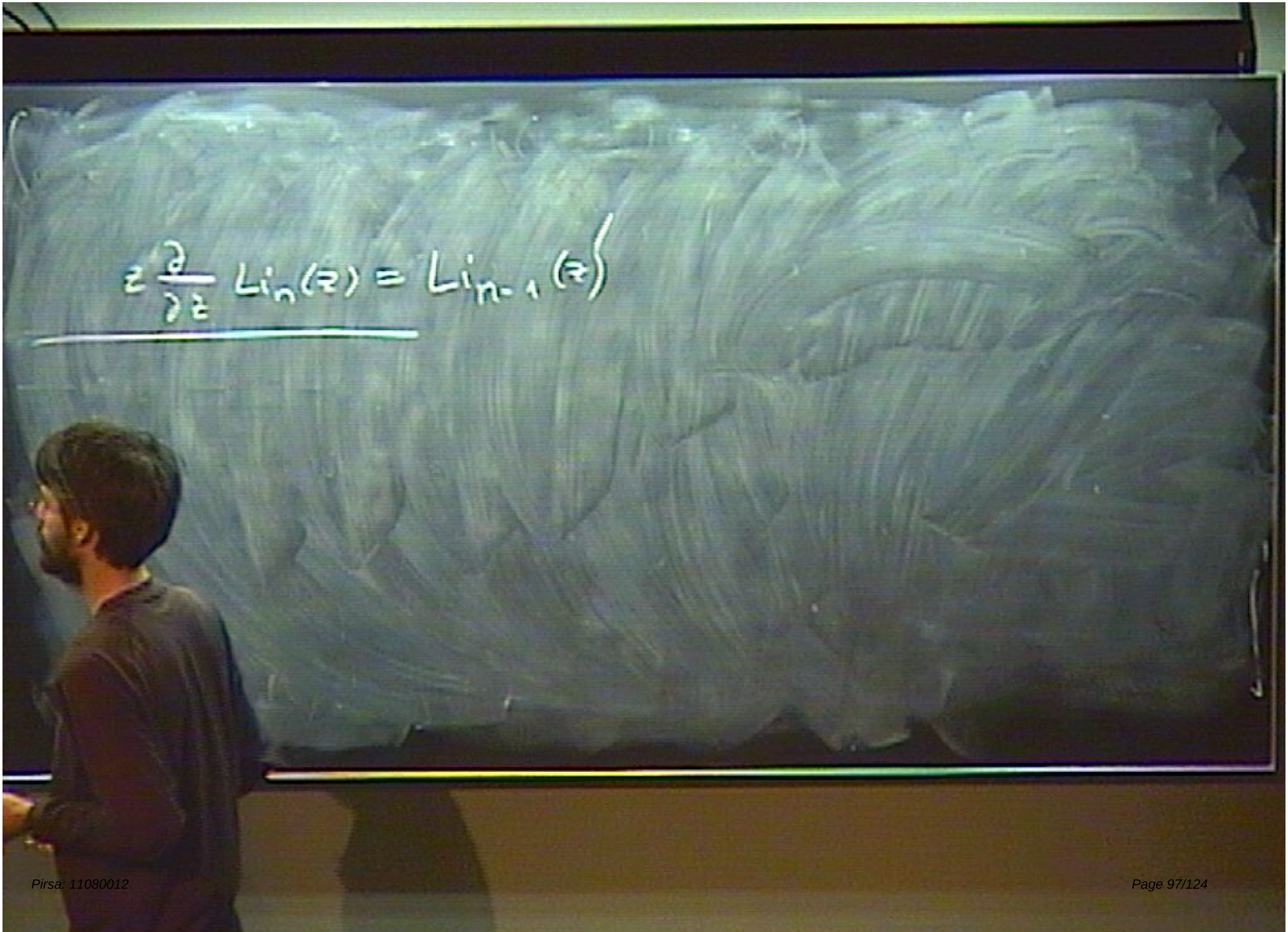
$$F[\alpha] = \int_\alpha \omega_1 \circ \cdots \circ \omega_r$$

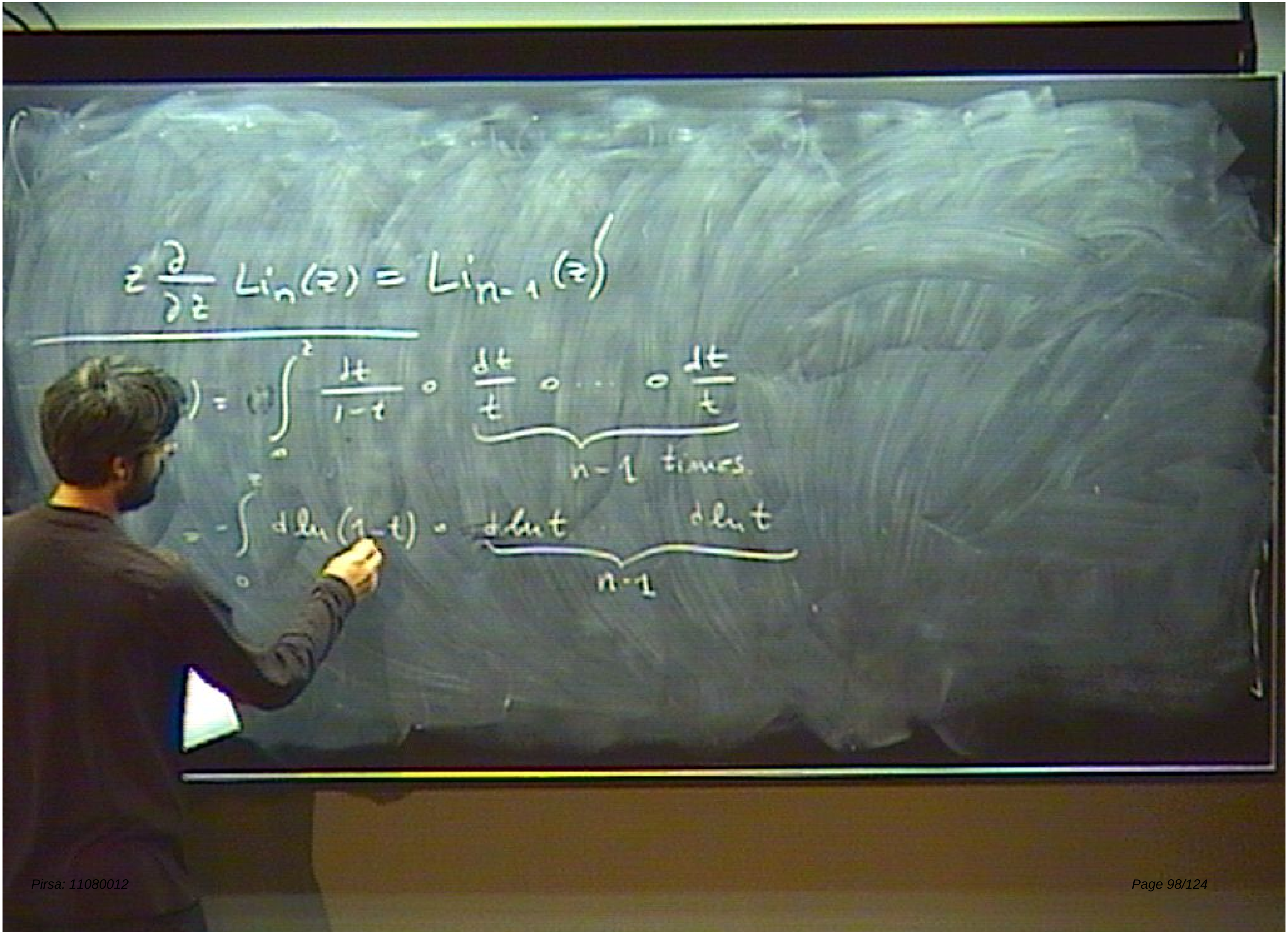

$Li_2(z) = \int_0^z \left(\int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = -\int_0^z \frac{\ln(1-t)\,dt}{t}$
dilogarithm
$-\ln(1-t) = \sum_{n=1}^{\infty} \frac{t^n}{n}$
$Li_n(z) = \int_0^z \frac{dt}{t} Li_{n-1}(t)$

$$z \frac{\partial}{\partial z} \mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z)$$

$$z \frac{\partial}{\partial z} Li_n(z) = Li_{n-1}(z)$$


$Li_2(z) = \int_0^z \left(\int_0^t \frac{du}{1-u} \right) \frac{dt}{t} = -\int_0^z \frac{\ln(1-t)\,dt}{t}$
dilogarithm
$Li_2(z) = -\int_0^z d\ln(1-t) \circ d\ln t$
$= \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t}$

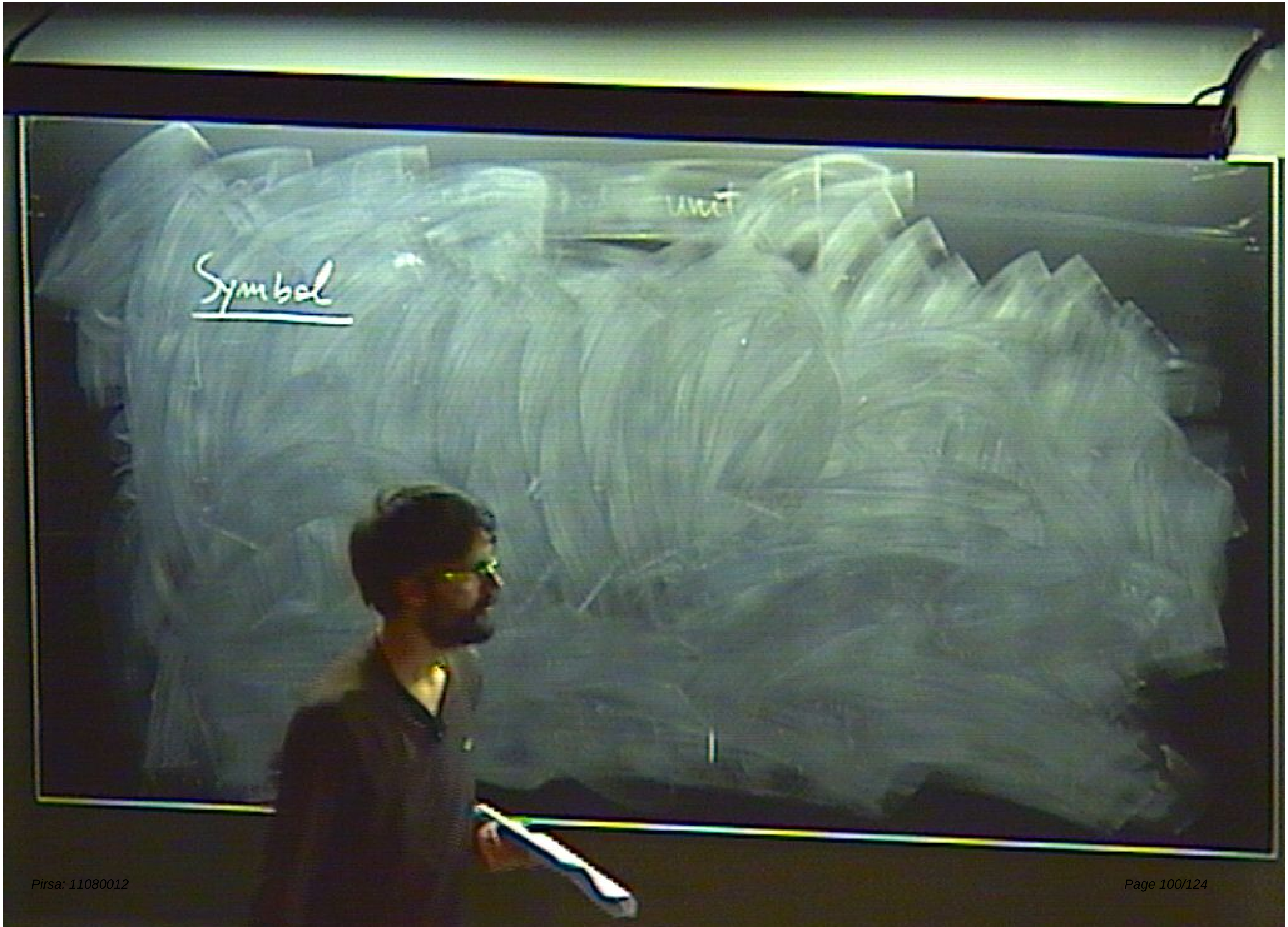

$z \frac{\partial}{\partial z} Li_n(z) = Li_{n-1}(z)$

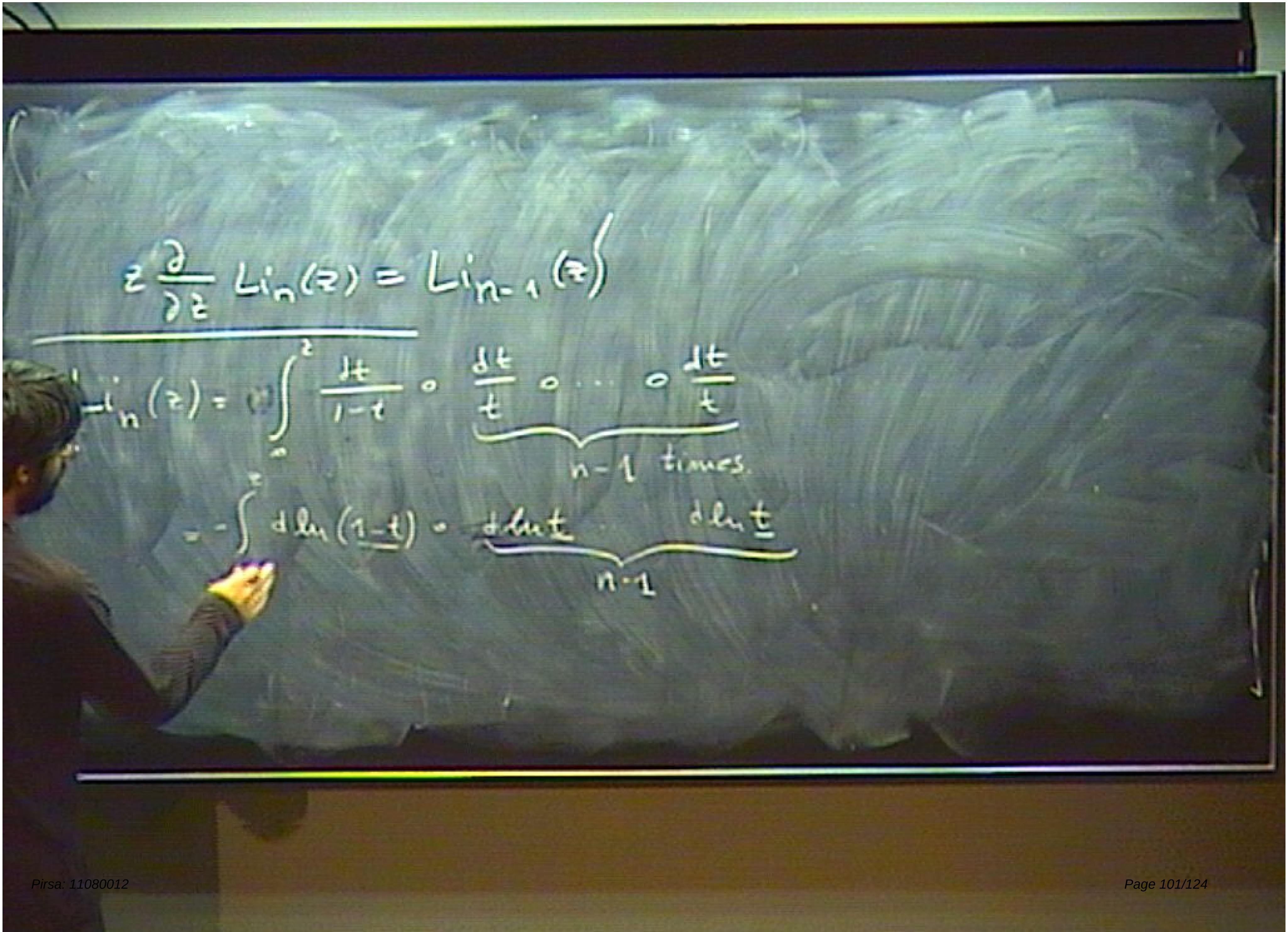

$z \frac{\partial}{\partial z} Li_n(z) = Li_{n-1}(z)$
$= \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}$
n-1 times
$= -\int_0^z d\ln(1-t) \circ d\ln t \quad d\ln t$
n-1

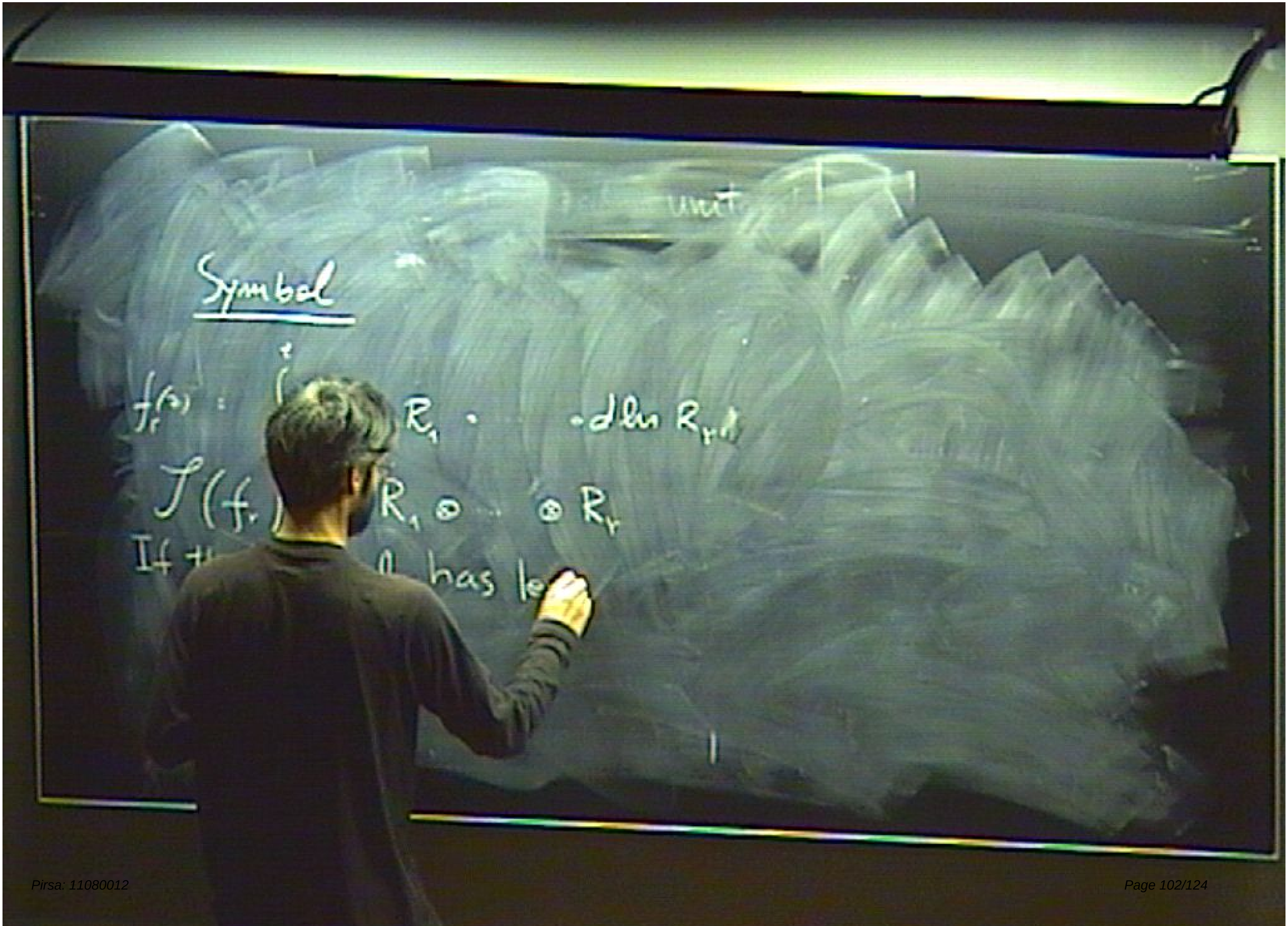
$$z\frac{\partial}{\partial z}\mathrm{Li}_n(z) = \mathrm{Li}_{n-1}(z)$$

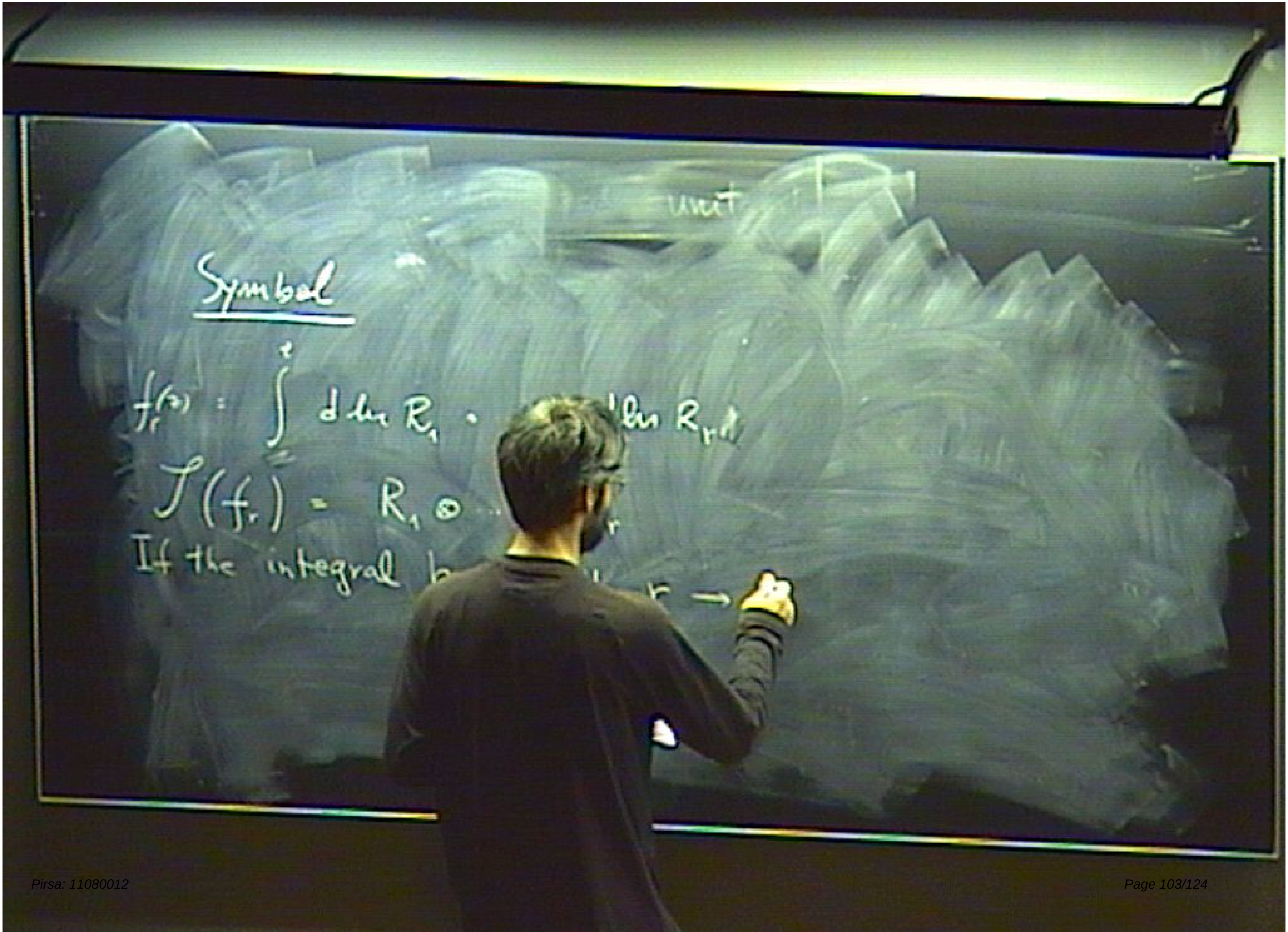
$$\mathrm{Li}_n(z) = \int_0^z \frac{dt}{1-t}\circ\underbrace{\frac{dt}{t}\circ\cdots\circ\frac{dt}{t}}_{n-1\ \text{times}}$$

$$= -\int_0^z d\ln(1-t)\circ\underbrace{d\ln t \quad \cdots \quad d\ln t}_{n-1}$$


unit
Symbol


$z \frac{\partial}{\partial z} Li_n(z) = Li_{n-1}(z)$
$Li_n(z) = \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t} \circ \cdots \circ \frac{dt}{t}$
n-1 times
$= -\int^z d\ln(1-t) \circ d\ln t \quad d\ln t$
n-1


Symbol
If
has

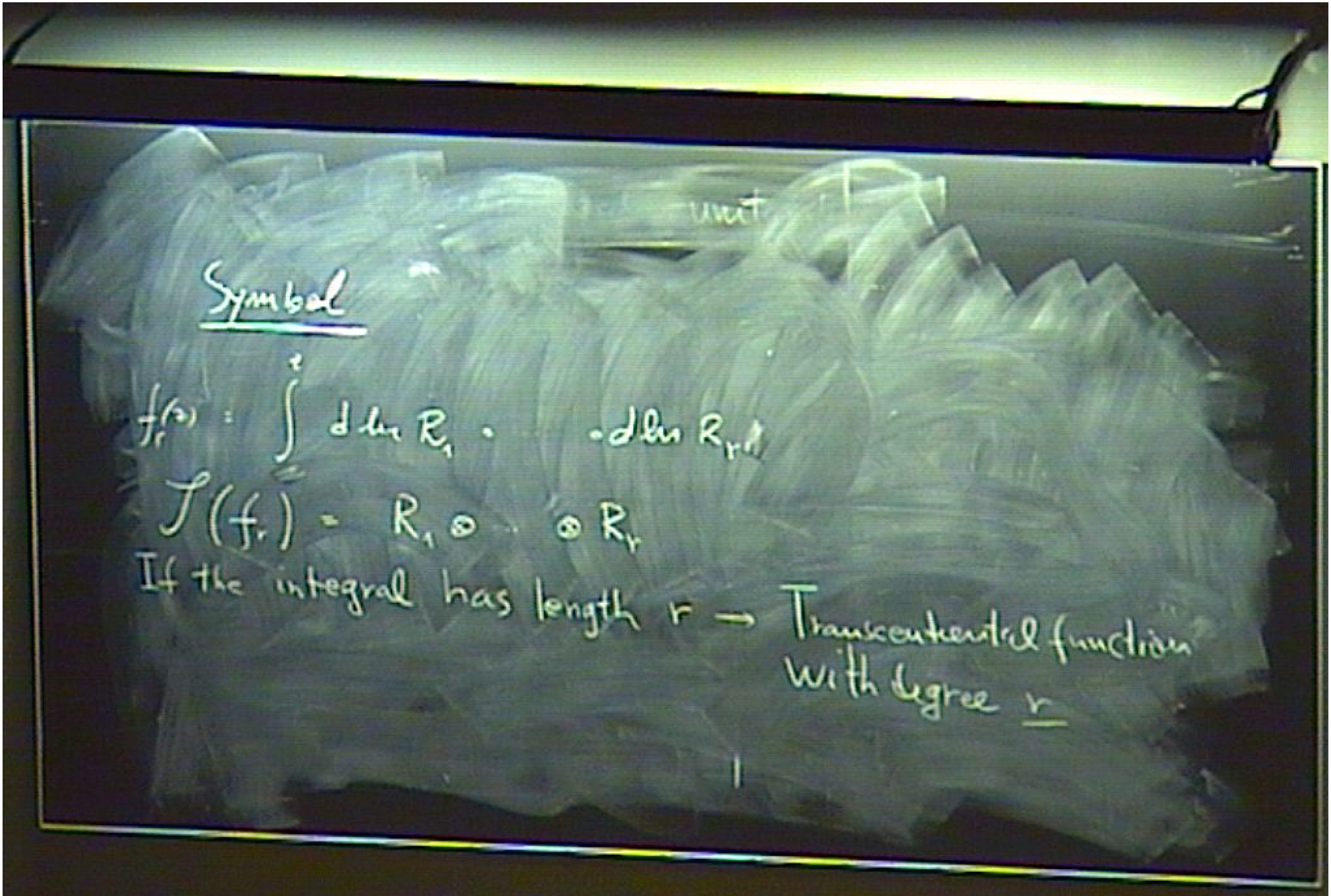

Symbol
If the integral

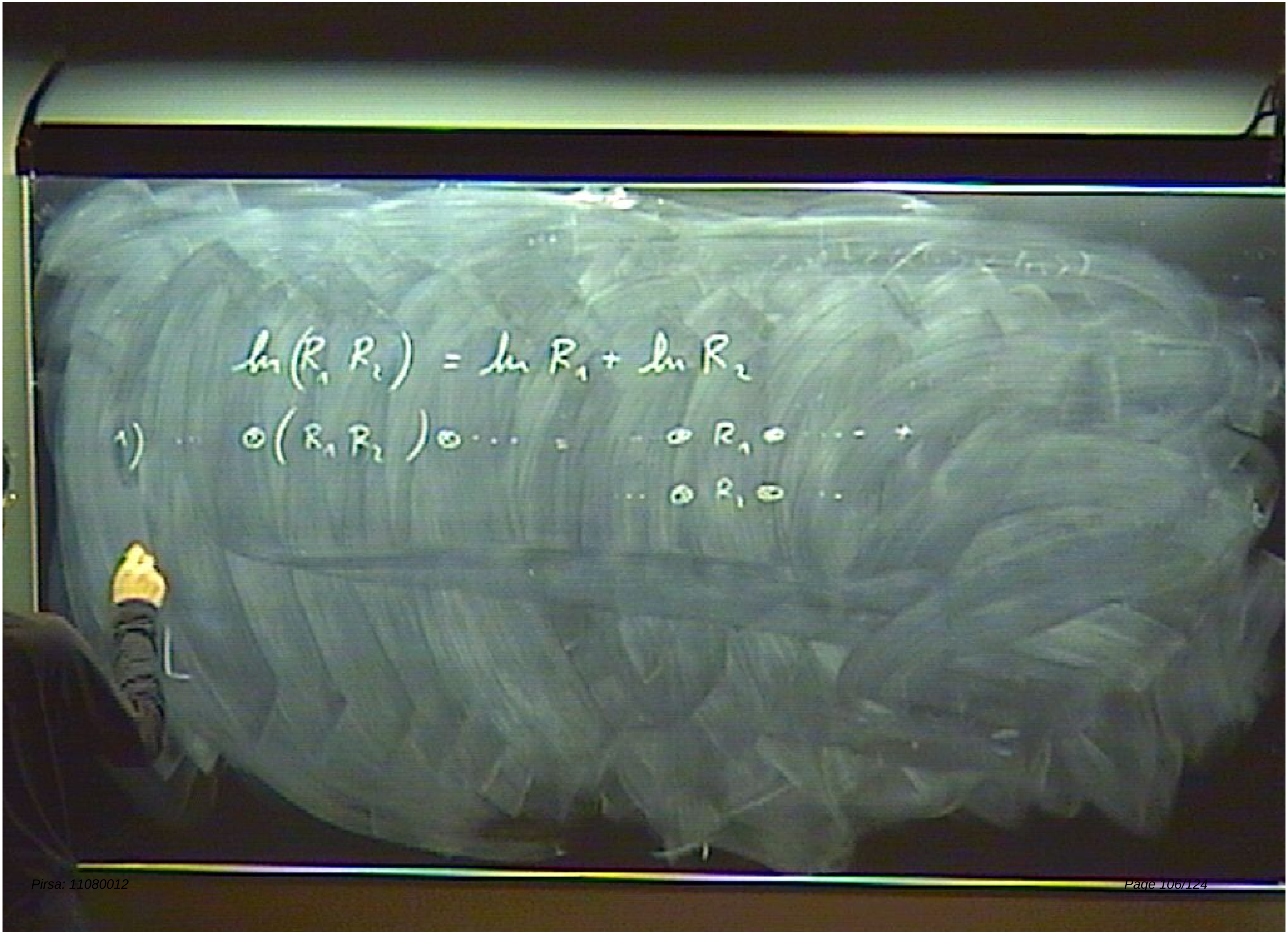
Symbol

$$f_r^{(x)} = \int^{x} d\ln R_1 \circ \cdots \circ d\ln R_r$$

$$\mathcal{S}(f_r) = R_1 \otimes \cdots \otimes R_r$$

If the integral has length r → Transcendental function


Symbol
$f_r^{(x)} = \int_0^x d\ln R_1 \circ \cdots \circ d\ln R_r$
$\mathcal{S}(f_r) = R_1 \otimes \cdots \otimes R_r$
If the integral has length r → Transcendental function
with degree r

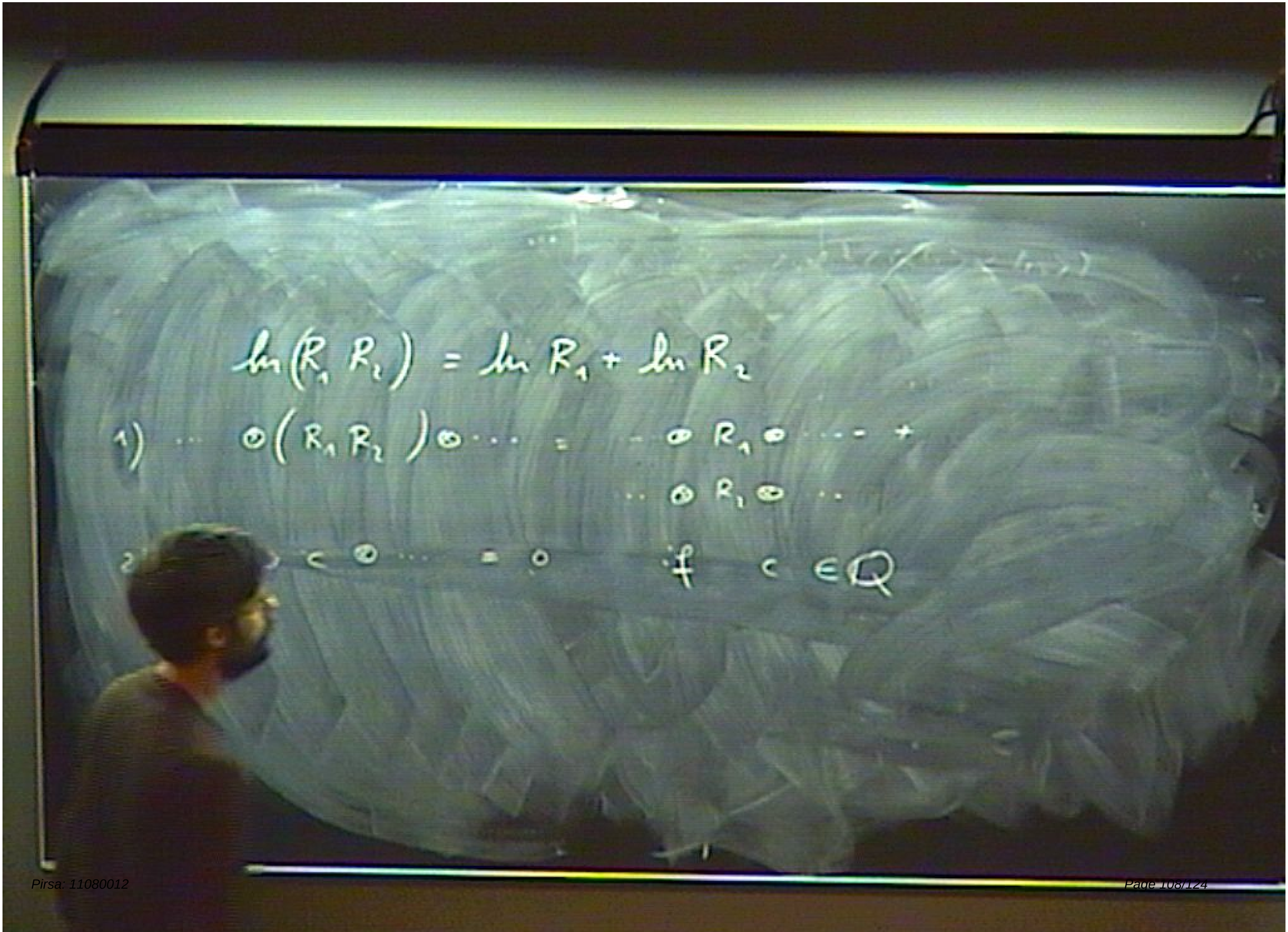

ln(R1 R2) = ln R1 + ln R2
1) ... ⊗(R1 R2)⊗ ... = ... ⊗ R1 ⊗ ... +
... ⊗ R2 ⊗ ..

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

1) $\cdots \otimes (R_1 R_2) \otimes \cdots = \cdots \otimes R_1 \otimes \cdots +$

$\qquad \cdots \otimes R_2 \otimes \cdots$

2) $\cdots \otimes \subset \otimes \cdots = 0$


$\ln(R_1 R_2) = \ln R_1 + \ln R_2$
1) $\cdots \otimes (R_1 R_2) \otimes \cdots = \cdots \otimes R_1 \otimes \cdots +$
$\cdots \otimes R_2 \otimes \cdots$
2) $\cdots c \otimes \cdots = 0$ if $c \in \mathbb{Q}$

$$\ln(R_1 R_2) = \ln R_1 + \ln R_2$$

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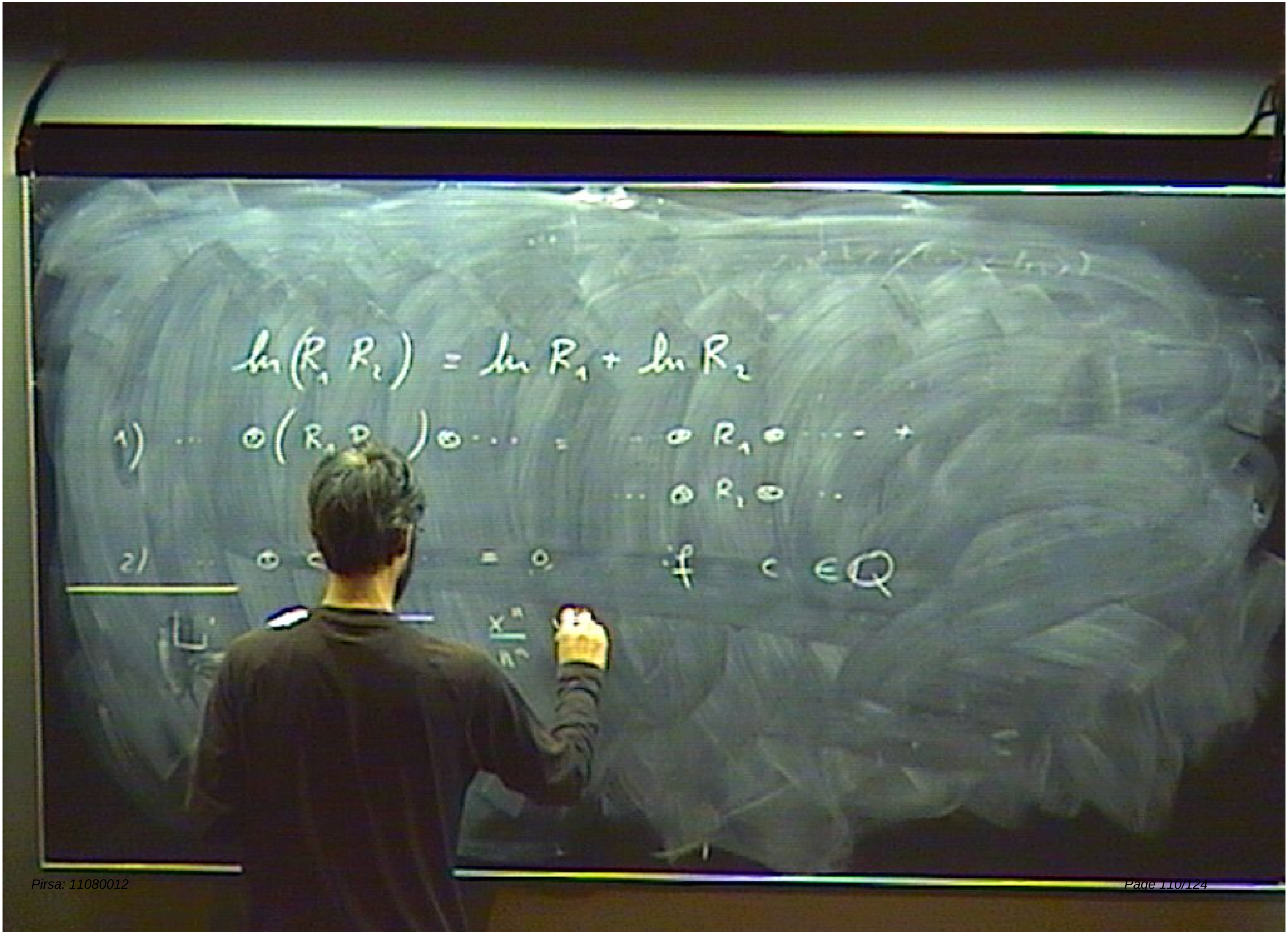
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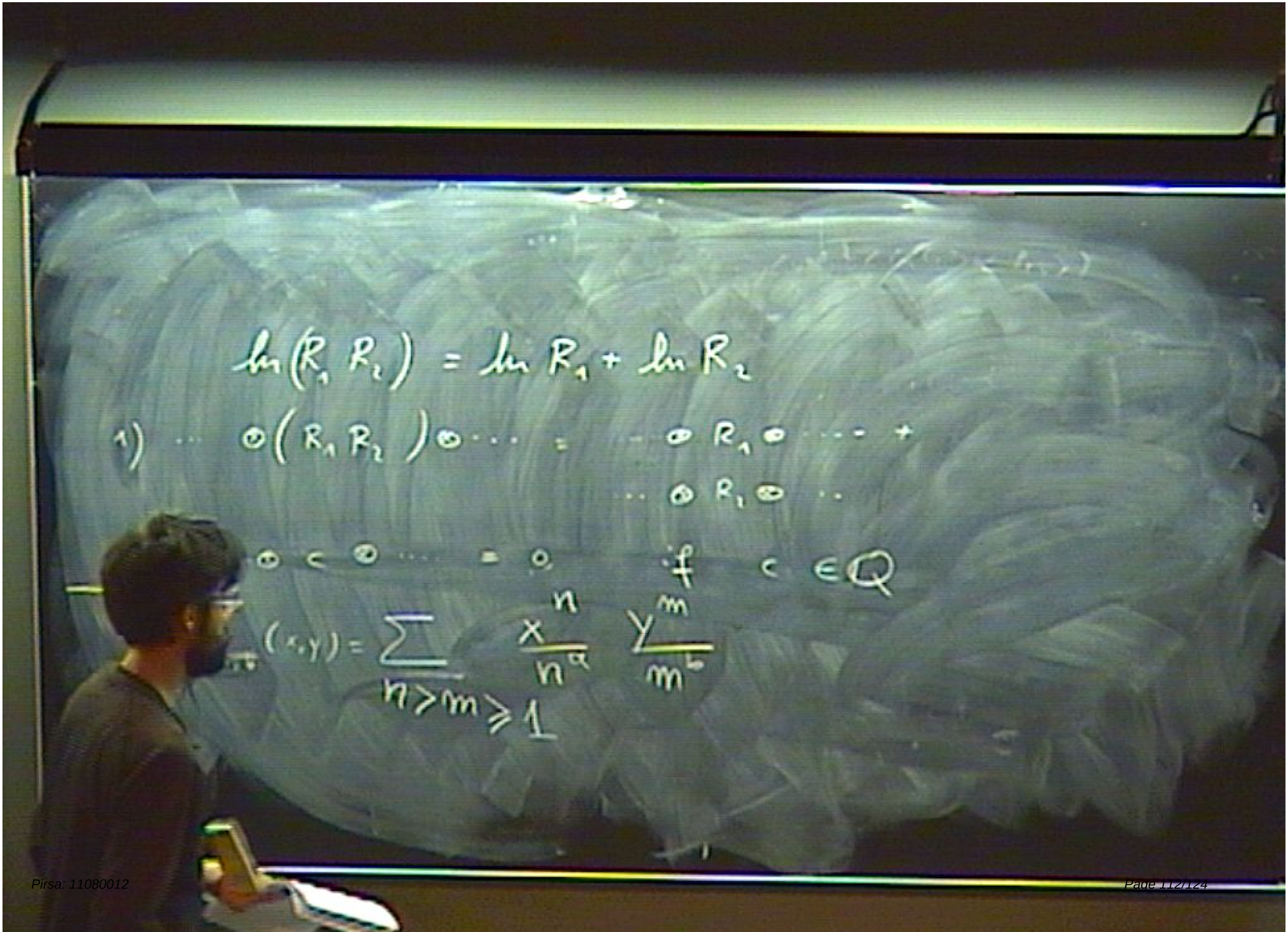
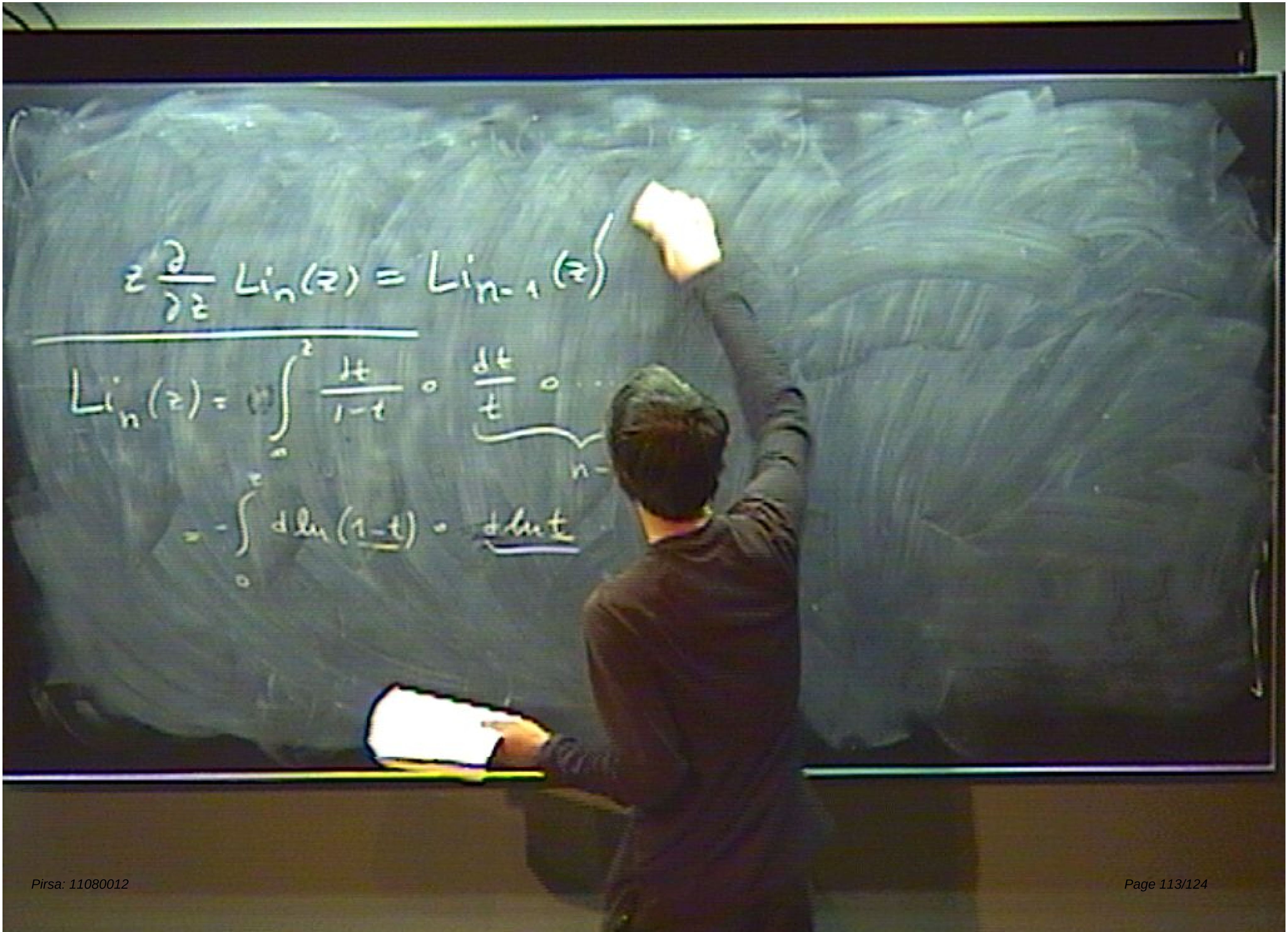
$$\mathrm{Li}_{a,b}(x,y) = \sum_{n>m>0} \frac{x^n}{n^a} \frac{y^m}{m^b}$$


ln(R1 R2) = ln R1 + ln R2
1) ... ⊗(R1 R2)⊗ ... = ... ⊗ R1 ⊗ ... +
... ⊗ R2 ⊗ ...
⊗ c ⊗ ... = 0 if c ∈ Q
(x, y) = Σ x^n/n^a y^m/m^b
n > m ≥ 1


z ∂/∂z Li_n(z) = Li_{n-1}(z)
Li_n(z) = (-) ∫_0^z dt/(1-t) ∘ dt/t ∘ ...
n-
= -∫_0^z d ln(1-t) ∘ d ln t

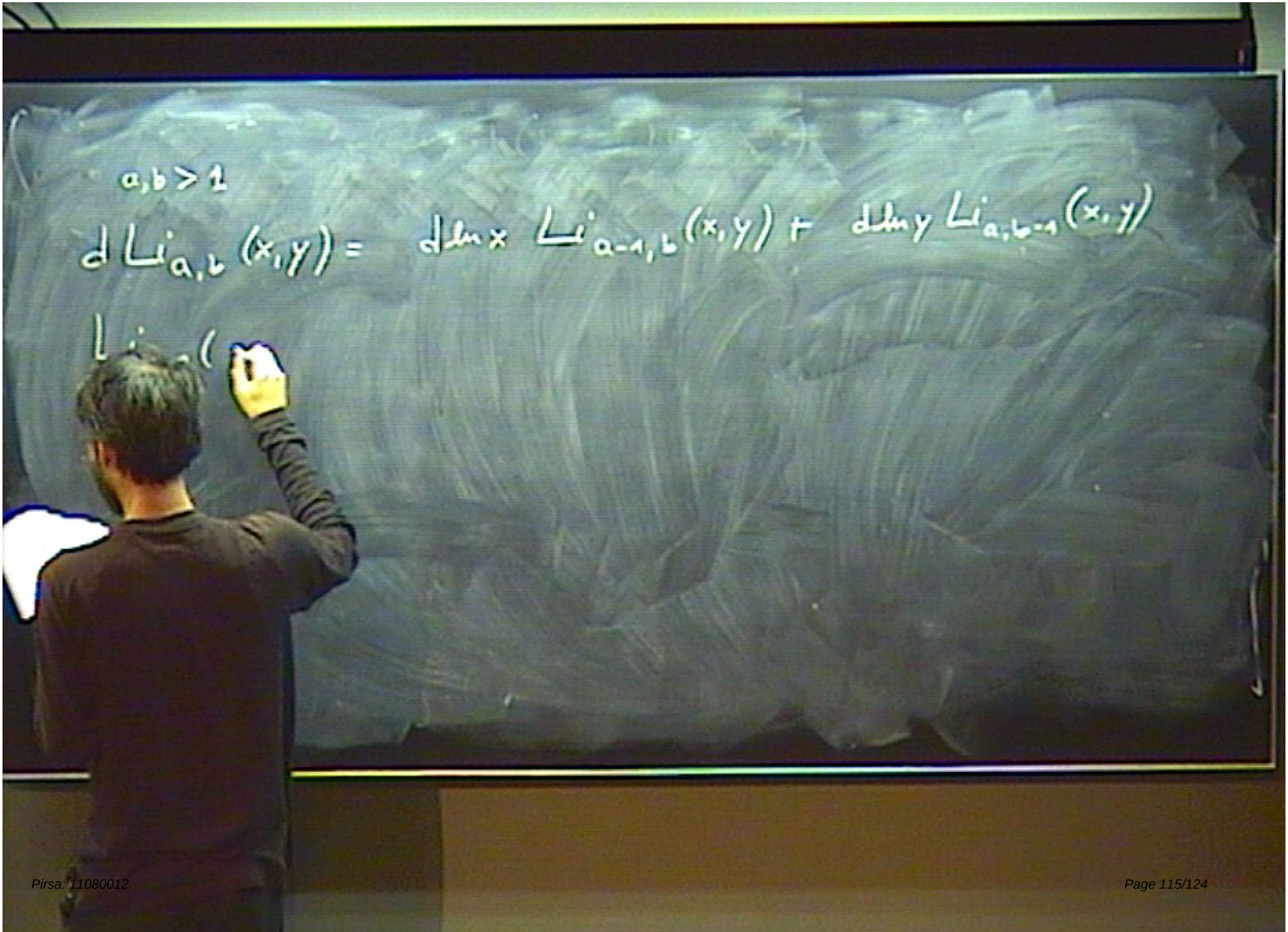
unit

Symbol

$$f_r^{(z)} = \int^{z} d\ln R_1 \circ \cdots \circ d\ln R_r$$

$$\mathcal{S}(f_r) = R_1 \otimes \cdots \otimes R_r$$

If the integral has length r → Transcendental function with degree r


$a, b > 1$
$d\, Li_{a,b}(x,y) = d\ln x\, Li_{a-1,b}(x,y) + d\ln y\, Li_{a,b-1}(x,y)$

$a, b > 1$

$$d\,Li_{a,b}(x,y) = d\ln x \; Li_{a-1,b}(x,y) + d\ln y \; Li_{a,b-1}(x,y)$$

$$Li_{a,0}(x,y) = \frac{1}{1-y}\left(y\, Li_a(x) - Li_a(xy) \right)$$

$$Li_{0,b}(x,y) = \frac{x}{1-x}\, Li_b(xy)$$

$a,b > 1$

$$dLi_{a,b}(x,y) = d\ln x \; Li_{a-1,b}(x,y) + d\ln y \; Li_{a,b-1}(x,y)$$

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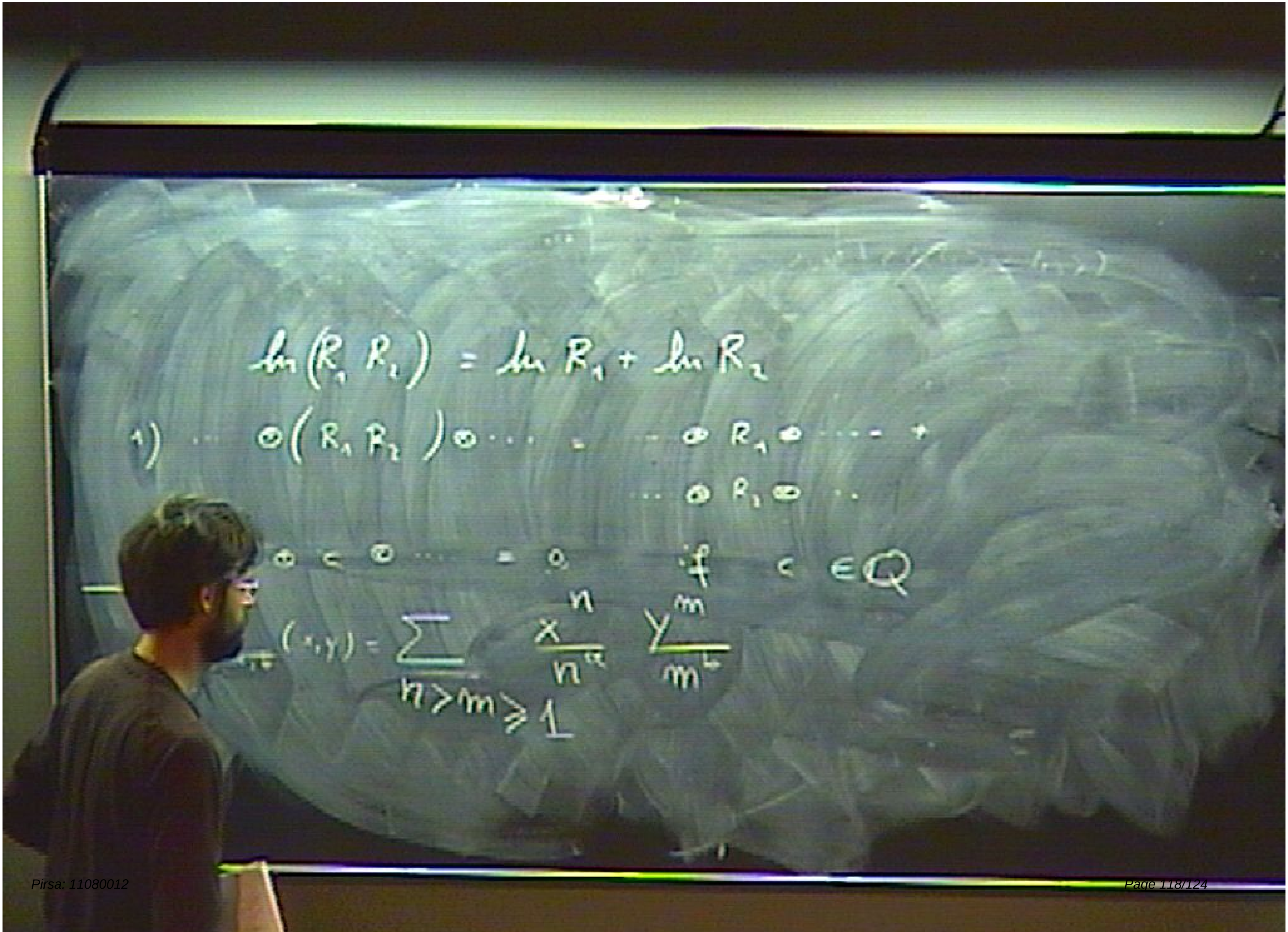
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⊗ c ⊗ ... = 0, if c ∈ Q
(x, y) = Σ x^n y^m
n>m≥1

$a, b > 1$

$$d\,Li_{a,b}(x,y) = d\ln x \; Li_{a-1,b}(x,y) + d\ln y \; Li_{a,b-1}(x,y)$$

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$$d\,Li_{1,1}(x,y) = -d\ln(1-x)\,Li_1(xy) - d\ln(1-y)\,Li_1(x) - d\ln\left(\frac{y}{1-y}\right)Li_1(xy)$$

$a, b > 1$

$$d\,\mathrm{Li}_{a,b}(x,y) = d\ln x\; \mathrm{Li}_{a-1,b}(x,y) + d\ln y\; \mathrm{Li}_{a,b-1}(x,y)$$

$$\mathrm{Li}_{a,0}(x,y) = \frac{1}{1-y}\Big(y\,\mathrm{Li}_a(x) - \mathrm{Li}_a(xy)\Big)$$

$$\mathrm{Li}_{0,b}(x,y) = \frac{x}{1-x}\,\mathrm{Li}_b(xy)$$

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$$\mathrm{Li}_1(x) = -\ln(1-x)$$

$a,b > 1$

$$d\,Li_{a,b}(x,y) = d\ln x \; Li_{a-1,b}(x,y) + d\ln y \; Li_{a,b-1}(x,y)$$

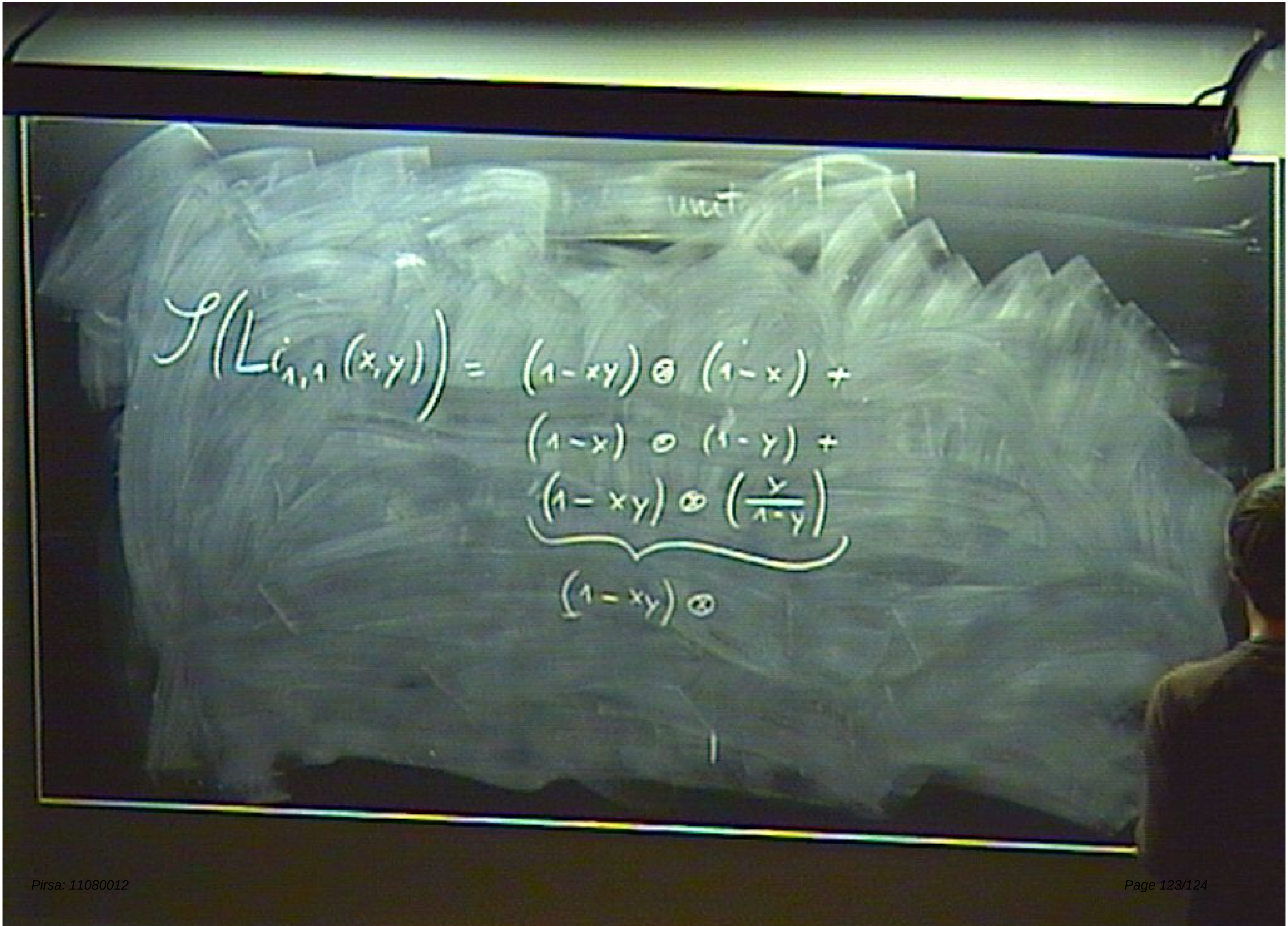
$$Li_{a,0}(x,y) = \frac{1}{1-y}\left(y\, Li_a(x) - Li_a(xy) \right)$$

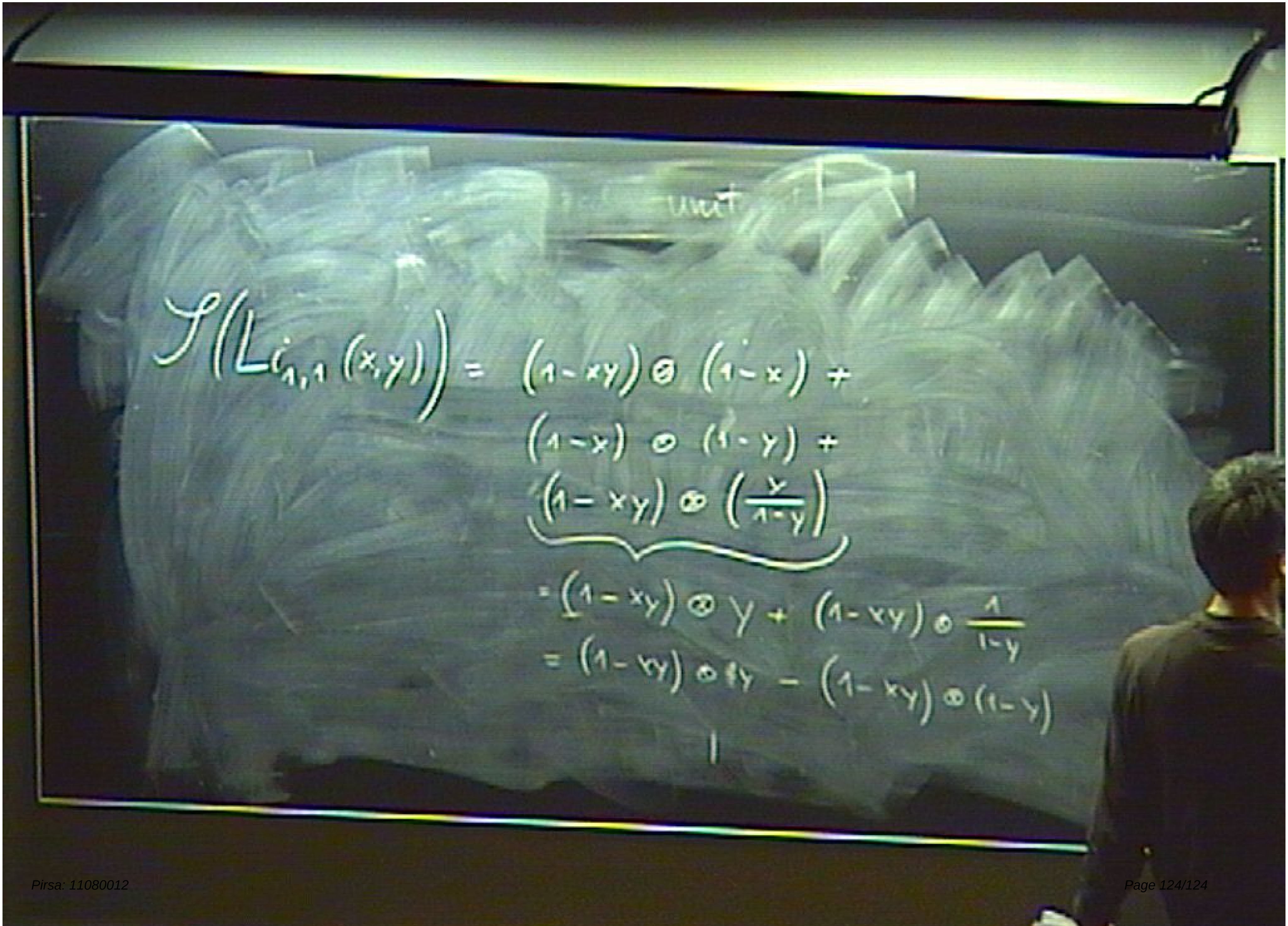
$$Li_{0,b}(x,y) = \frac{x}{1-x}\, Li_b(xy)$$

$$d\,Li_{1,1}(x,y) = -\,d\ln(1-x)\, Li_1(xy) - d\ln(1-y)\, Li_1(x) - d\ln\left(\frac{y}{1-y}\right) Li_1(xy)$$

$$Li_1(x) = -\ln(1-x)$$

$$\mathcal{S}\left(Li_{1,1}(x,y)\right) = (1-xy)\otimes(1-x) + (1-x)\otimes(1-y) + (1-xy)\otimes\left(\frac{y}{1-y}\right)$$


$\mathcal{S}(Li_{1,1}(x,y)) = (1-xy) \otimes (1-x) +$
$(1-x) \otimes (1-y) +$
$(1-xy) \otimes \left(\frac{y}{1-y}\right)$
$(1-xy) \otimes$


$\mathcal{S}(Li_{1,1}(x,y)) = (1-xy) \otimes (1-x) +$
$(1-x) \otimes (1-y) +$
$(1-xy) \otimes \left(\frac{y}{1-y}\right)$
$= (1-xy) \otimes y + (1-xy) \otimes \frac{1}{1-y}$
$= (1-xy) \otimes y - (1-xy) \otimes (1-y)$