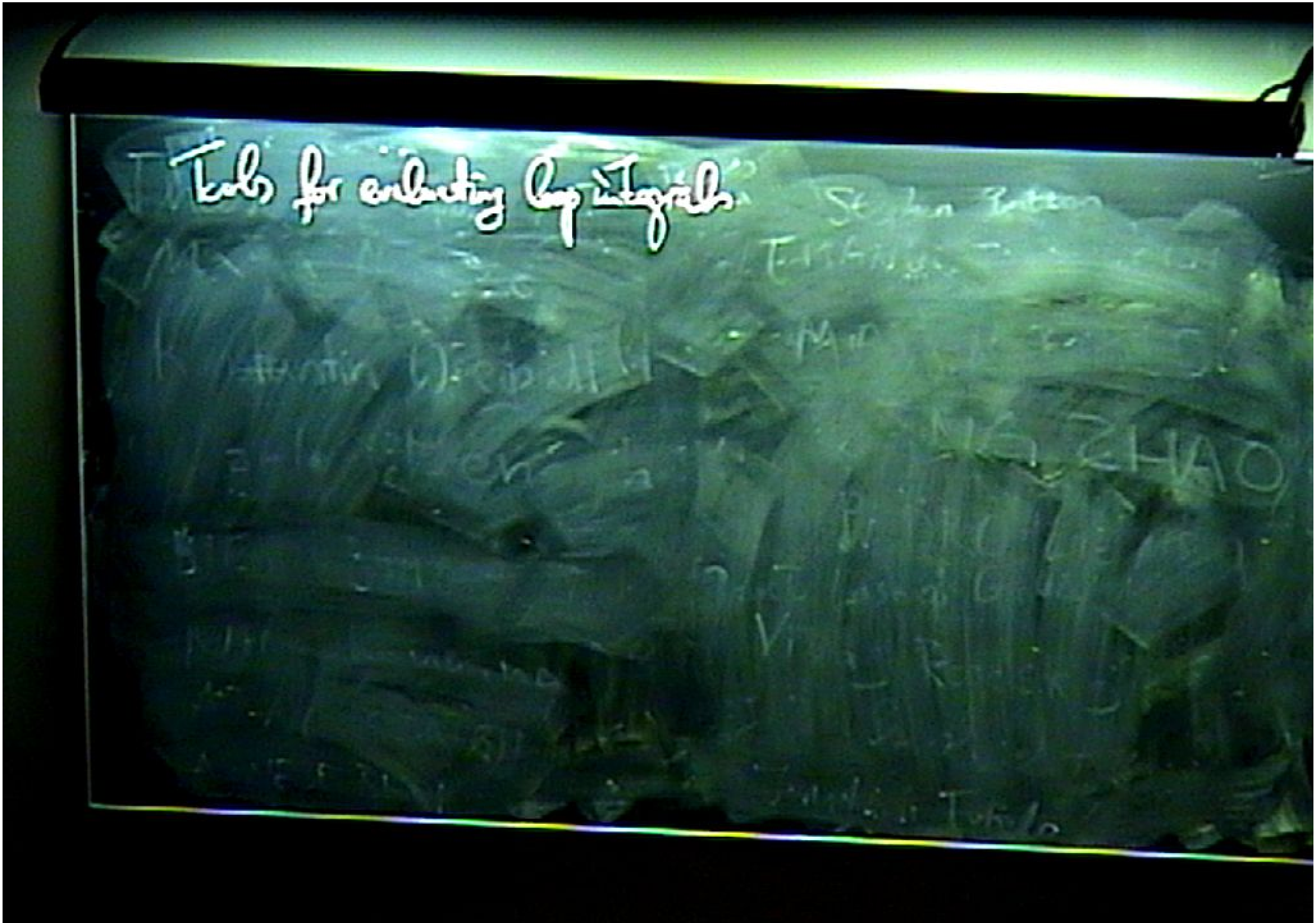
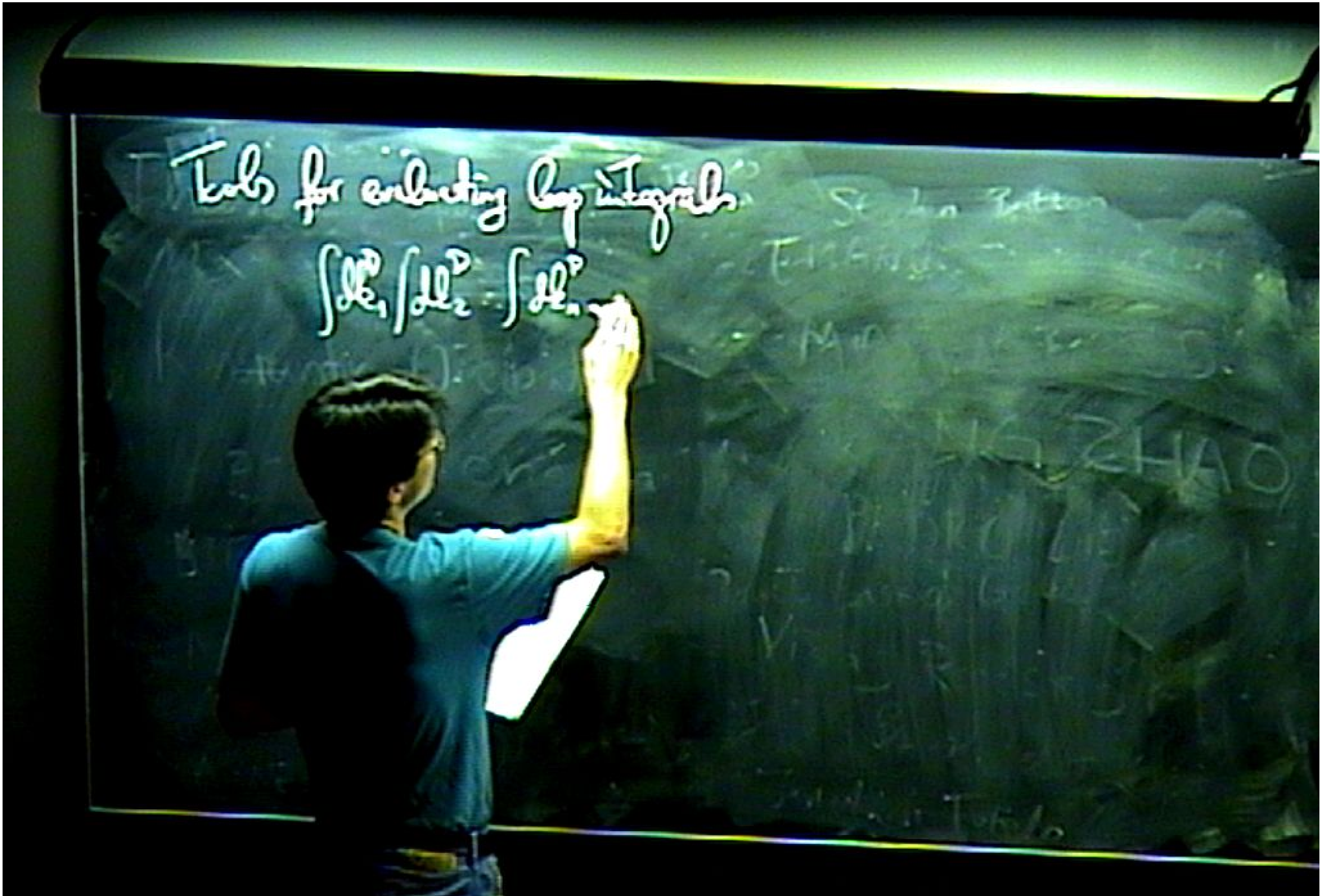
Title: Tools for Evaluating Loop Integrals

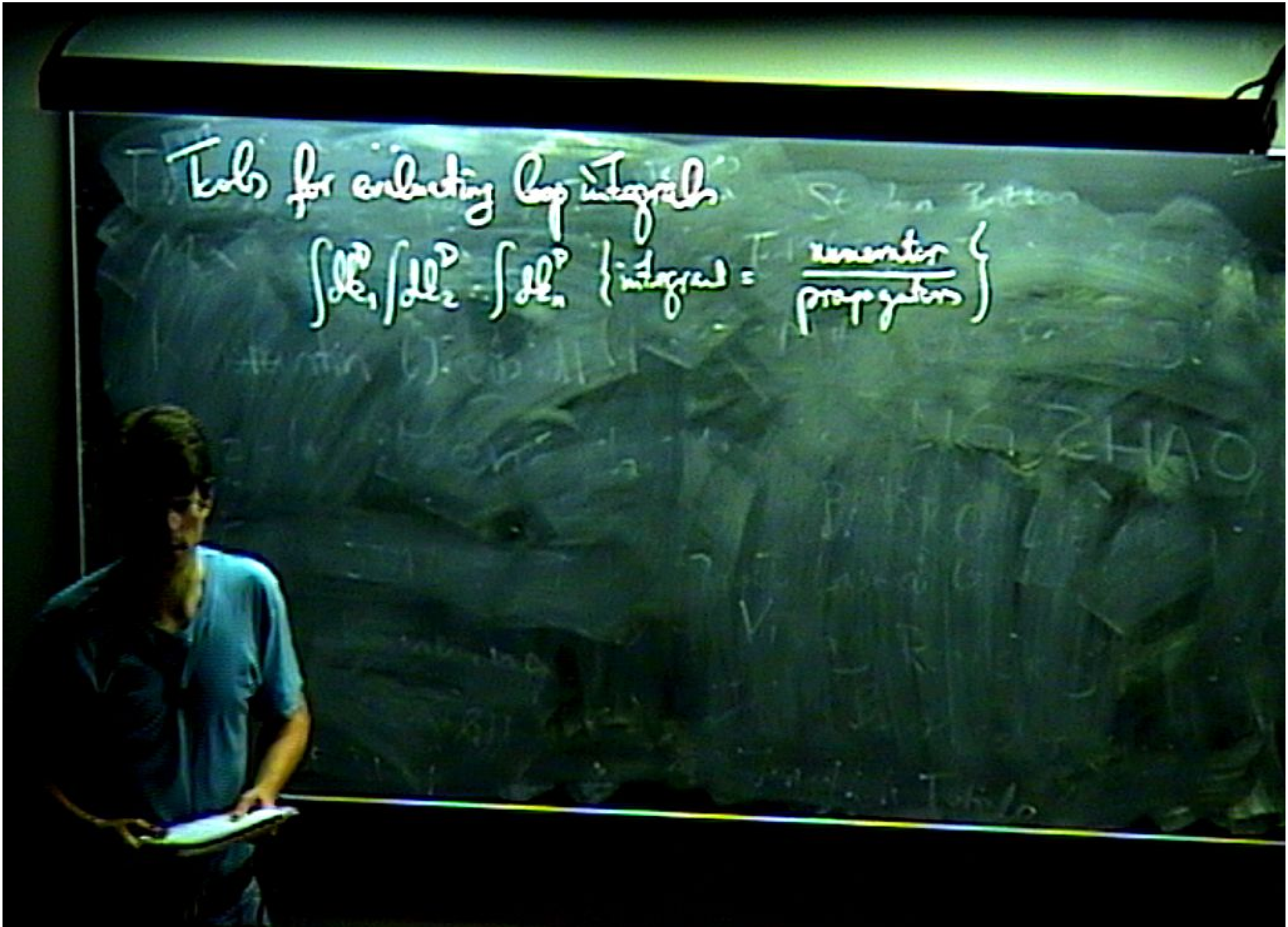
Date: Aug 01, 2011 09:30 AM

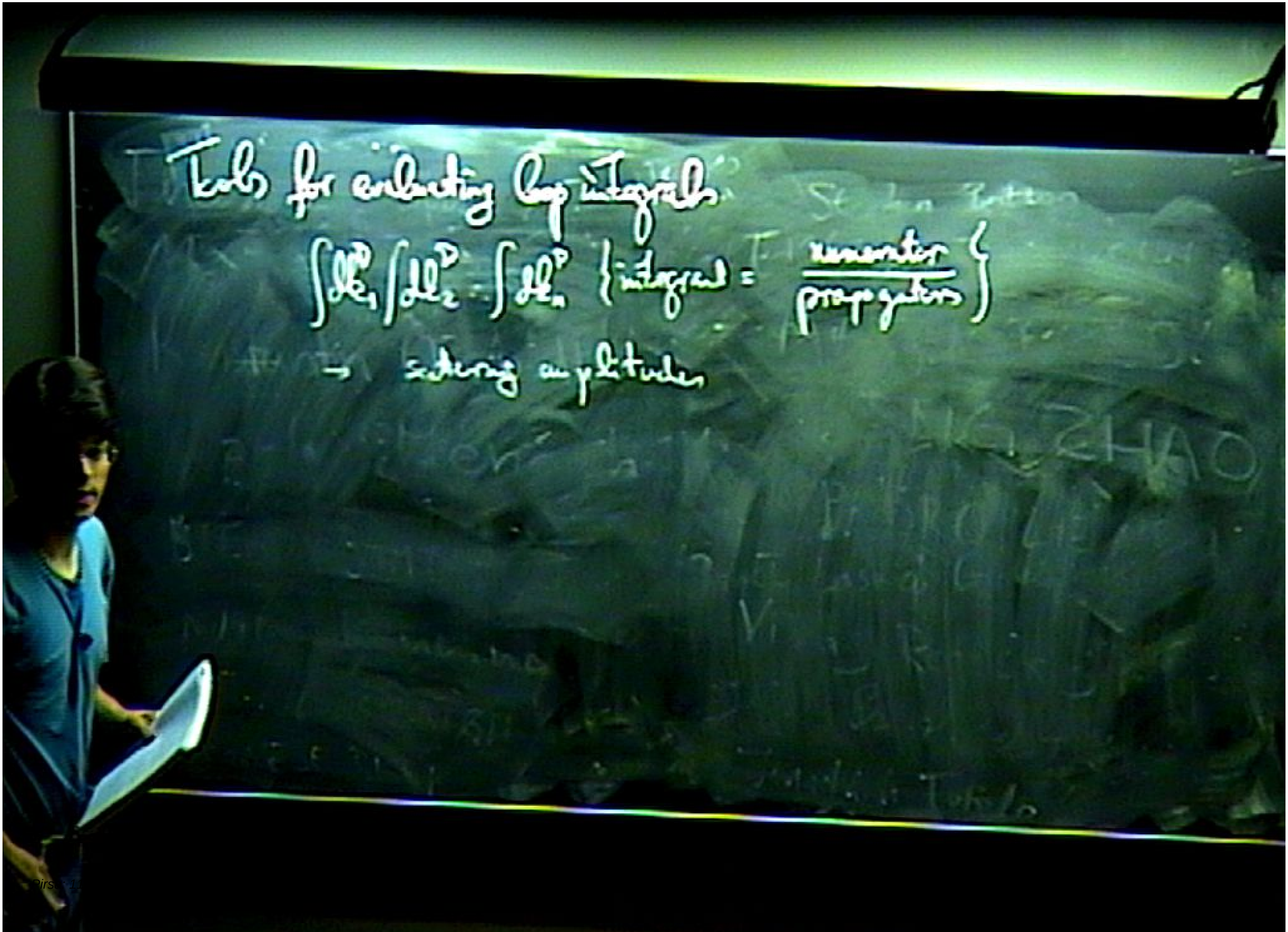
URL: http://pirsa.org/11080003

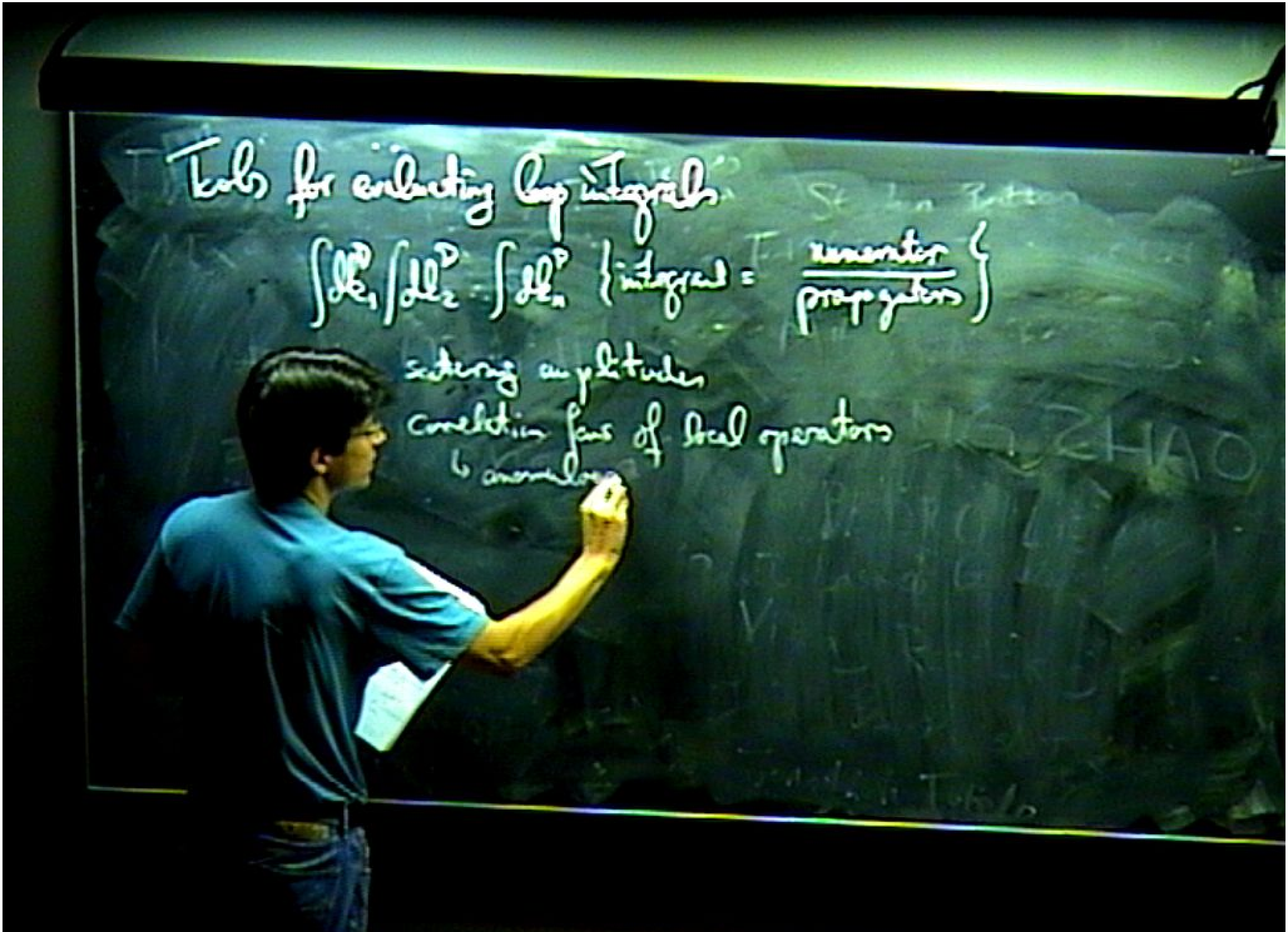
Abstract:

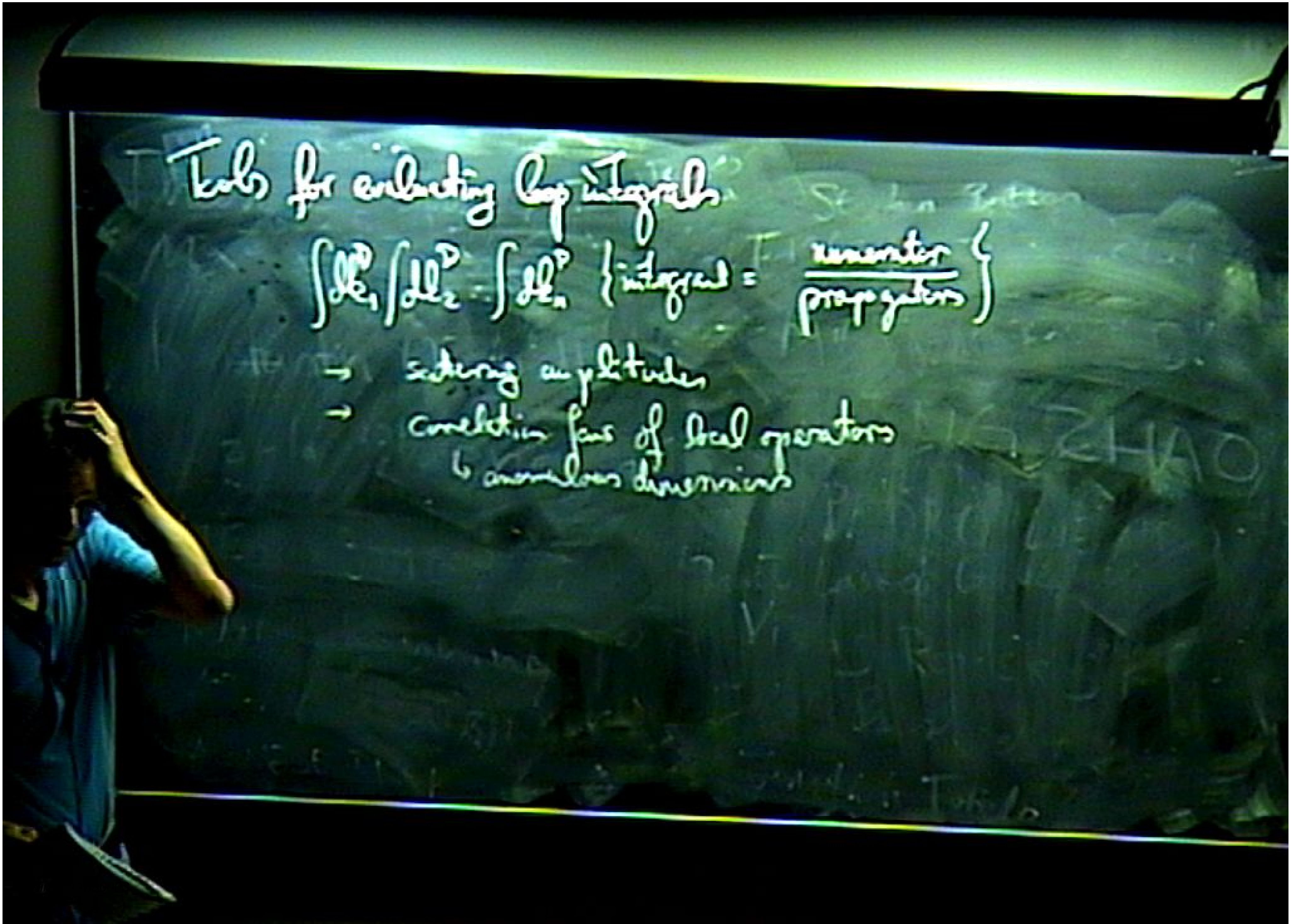

Tools for evaluating loop integrals


Tools for evaluating loop integrals
∫dl_1^D ∫dl_2^D ... ∫dl_n^D


Tools for evaluating loop integrals
$\int dl_1^D \int dl_2^D \cdots \int dl_n^D \left\{ \text{integrand} = \frac{\text{numerator}}{\text{propagators}} \right\}$


Tools for evaluating loop integrals
∫d^D l_1 ∫d^D l_2 … ∫d^D l_n {integrand = numerator / propagators}
→ scattering amplitudes


Tools for evaluating loop integrals
∫dℓ₁^D ∫dℓ₂^D … ∫dℓₙ^D {integrand = numerator / propagators}
scattering amplitudes
correlation fns of local operators
↳ anomalous


Tools for evaluating loop integrals
∫dl₁ᴰ ∫dl₂ᴰ … ∫dlₙᴰ {integrand = numerator / propagators}
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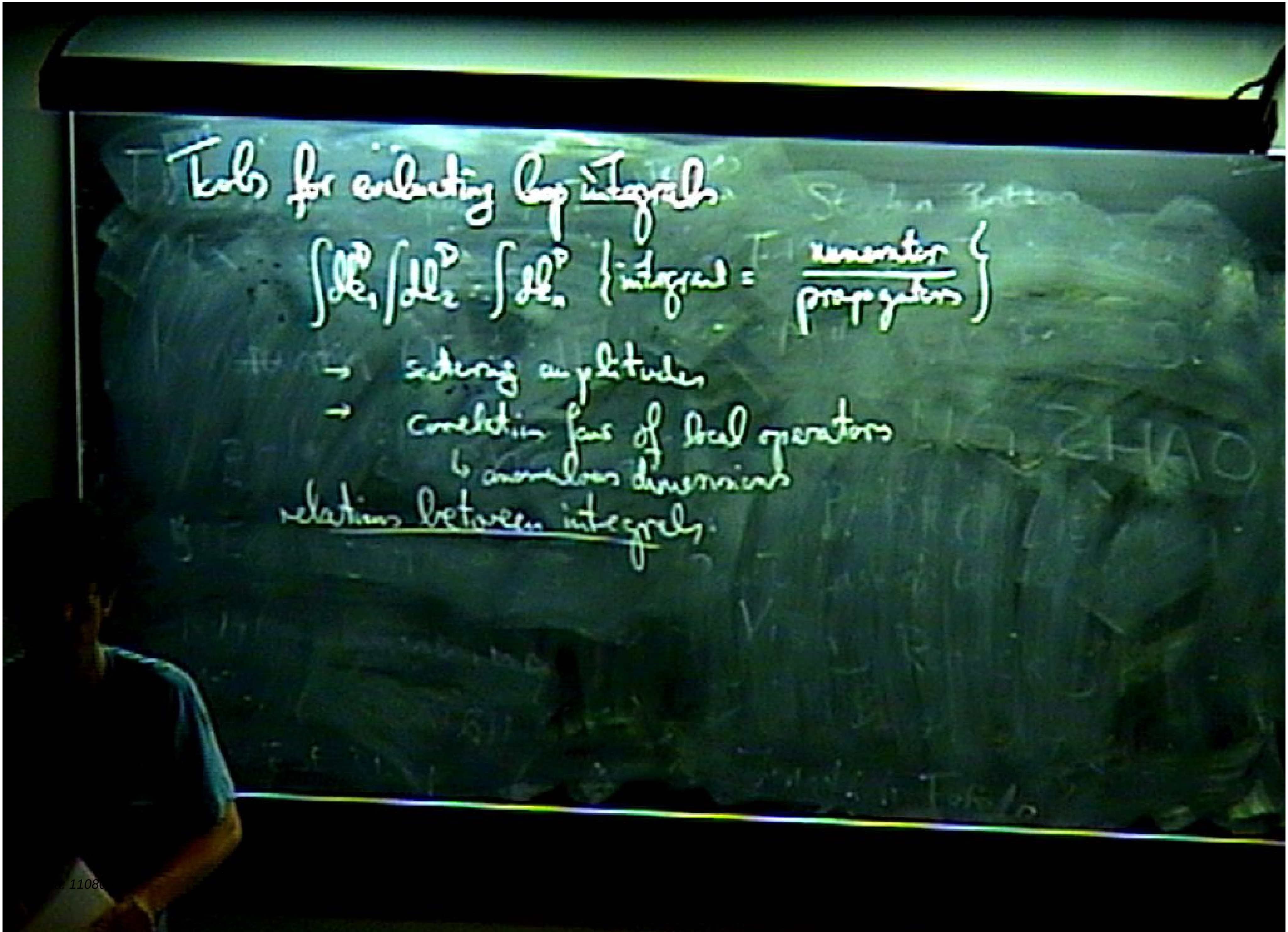
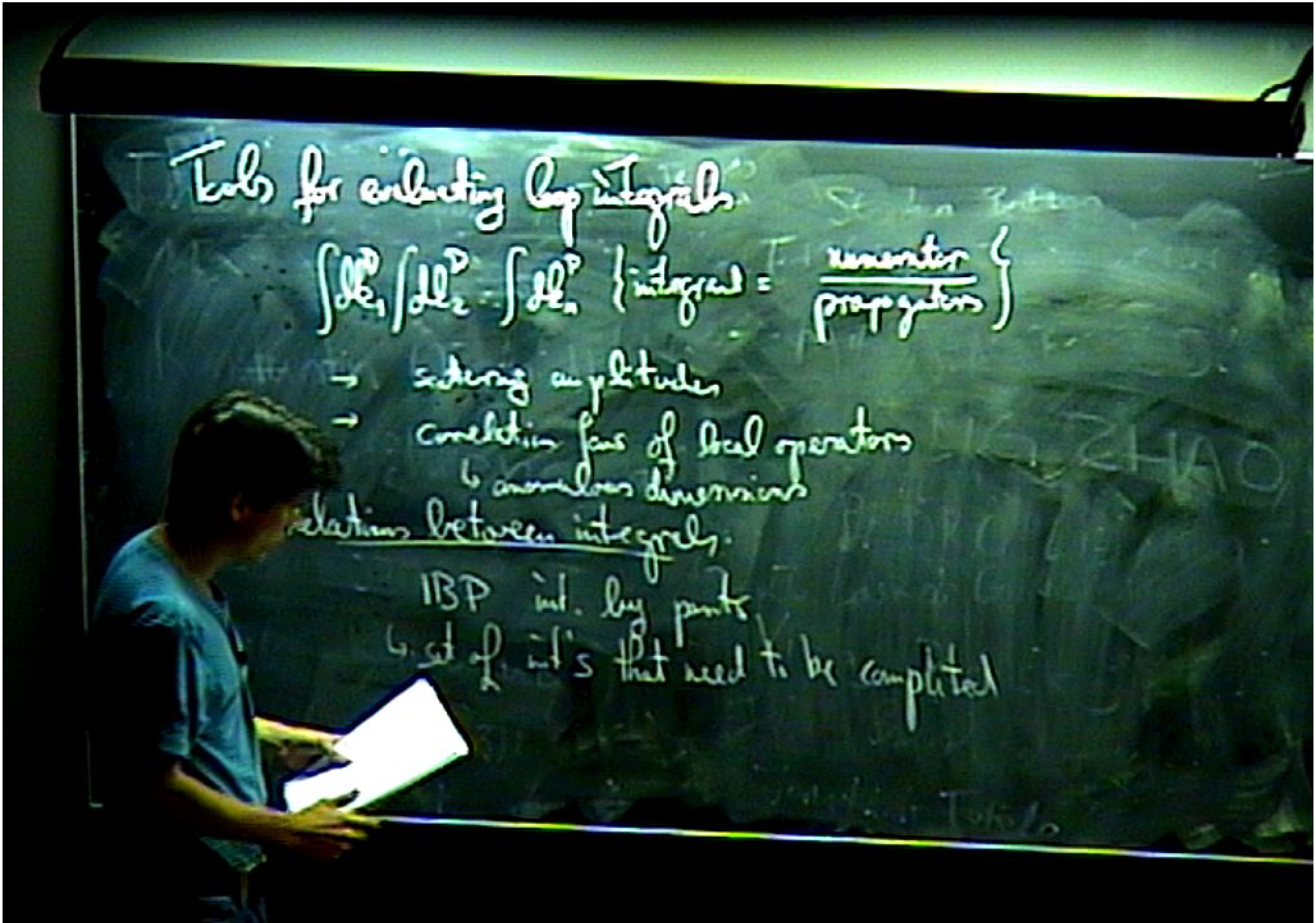
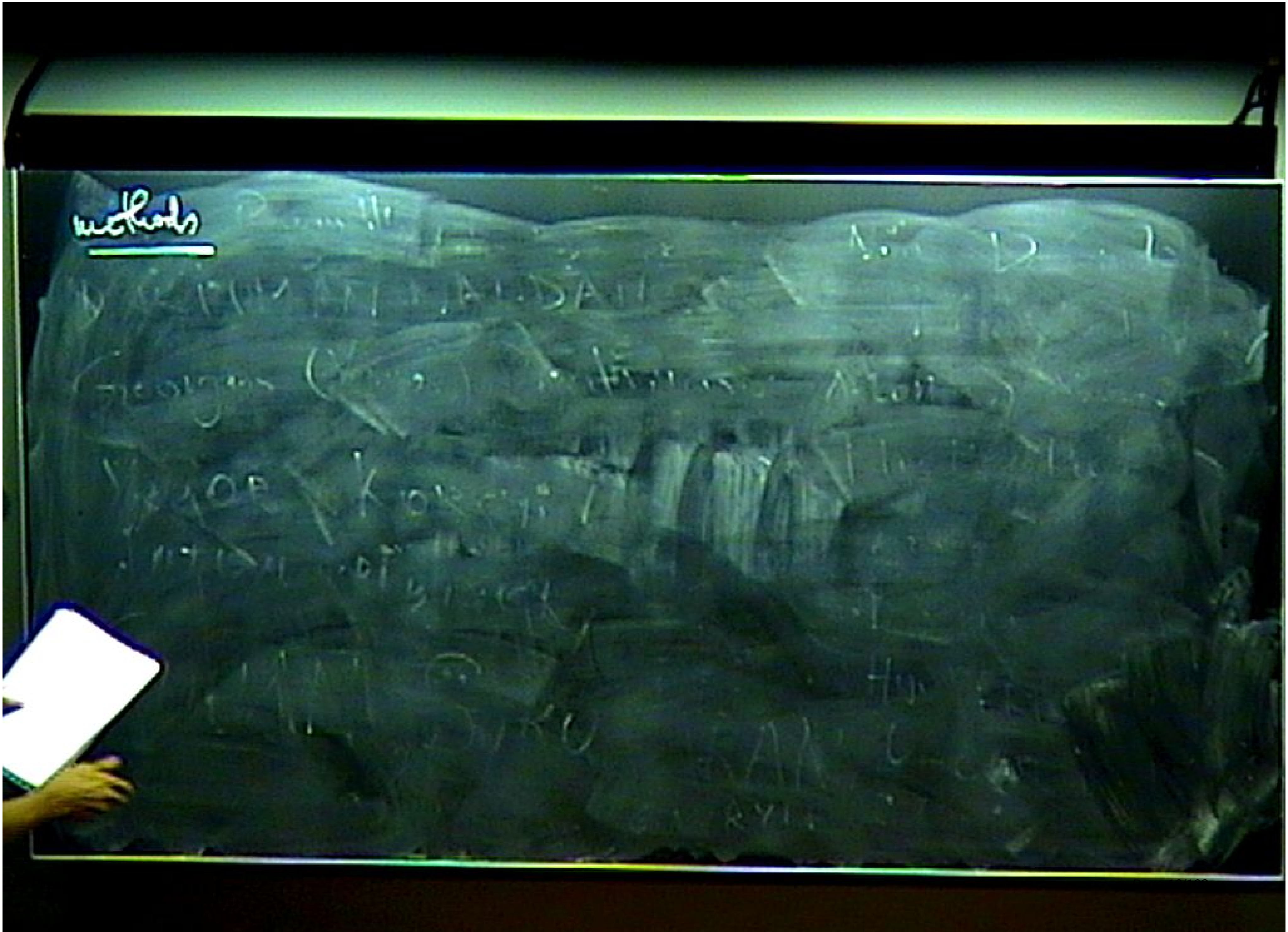
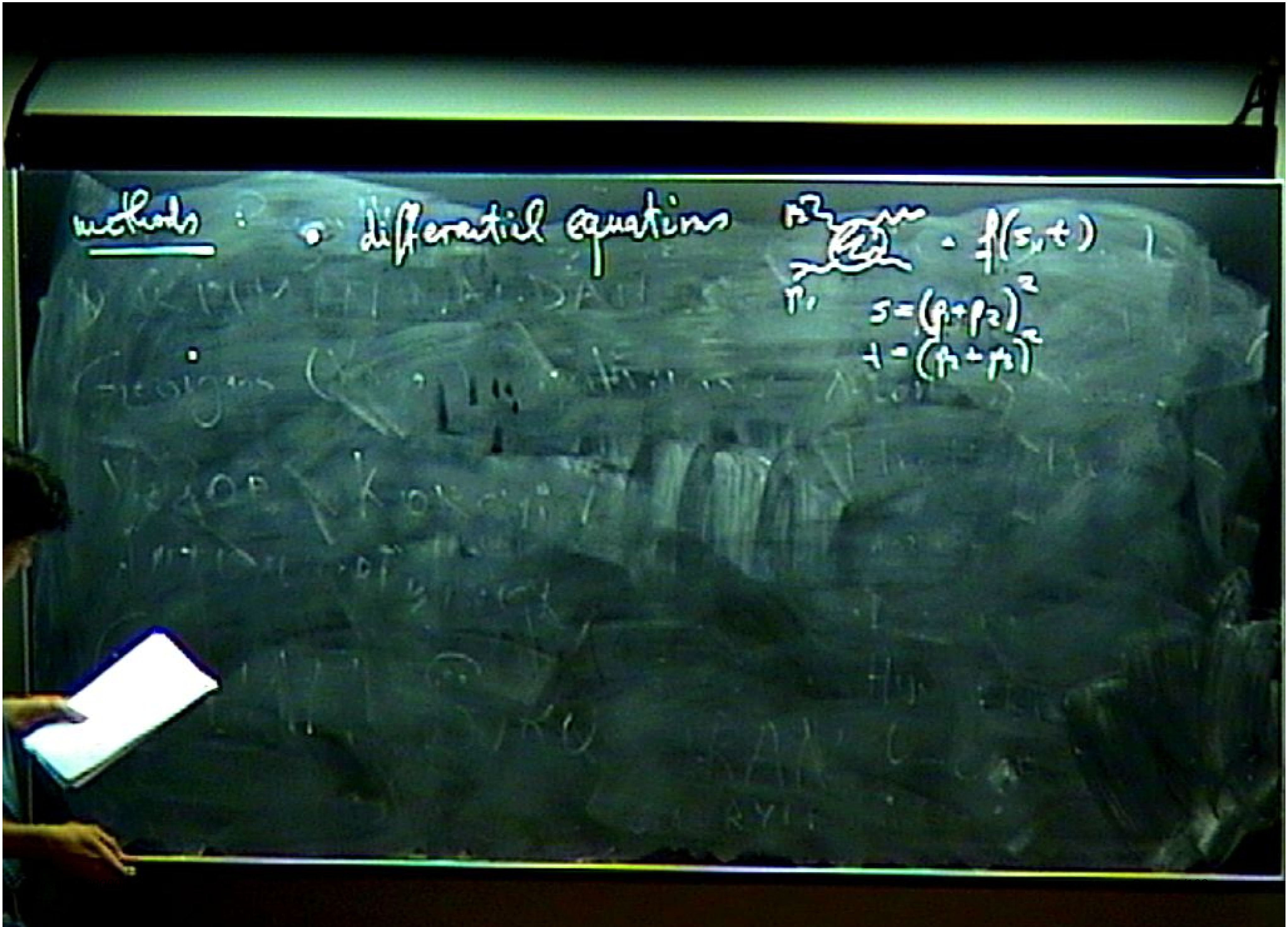
↳ anomalous dimensions


Tools for evaluating loop integrals
∫dl1^D ∫dl2^D ... ∫dln^D {integrand = numerator / propagators}
→ scattering amplitudes
→ correlation fns of local operators
↳ anomalous dimensions
relations between integrals:



Tools for evaluating loop integrals
$\int dl_1^D \int dl_2^D \cdots \int dl_n^D \left\{ \text{integrand} = \frac{\text{numerator}}{\text{propagators}} \right\}$
→ scattering amplitudes
→ correlation fns of local operators
↳ anomalous dimensions
elations between integrals.
IBP int. by parts
↳ set of int's that need to be computed


methods


methods :
• differential equations
f(s,t)
s = (p1+p2)^2
t = (p1-p3)^2

methods:

- differential equations  $= f(s,t)$

$$s=(p_1+p_2)^2$$
$$t=(p_2+p_3)^2$$

- Gegenbauer, conformal symmetry

methods:

- differential equations 

$= f(s,t)$

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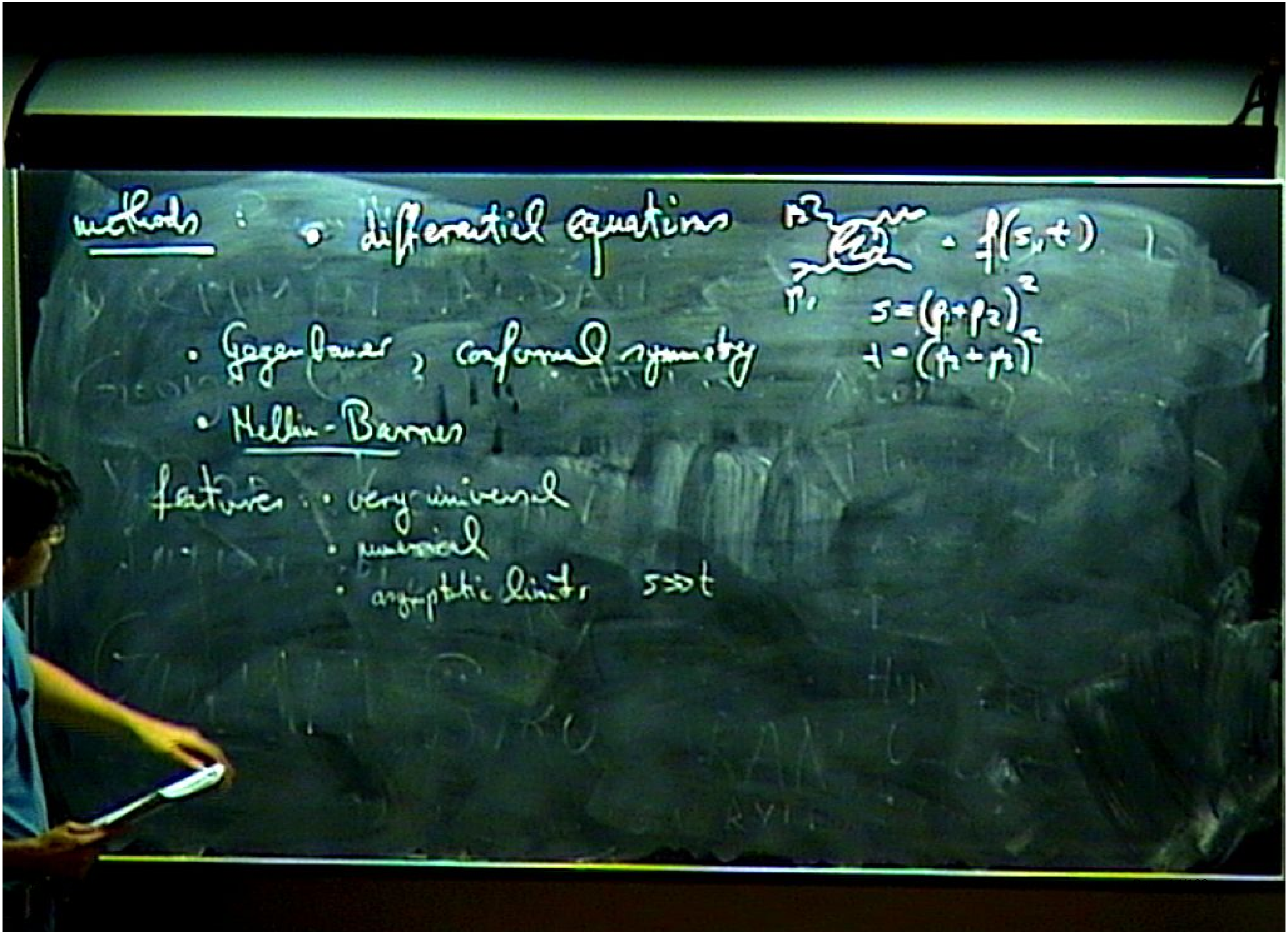
- Gegenbauer, conformal symmetry
- Mellin-Barnes

features:

- very universal
- numerical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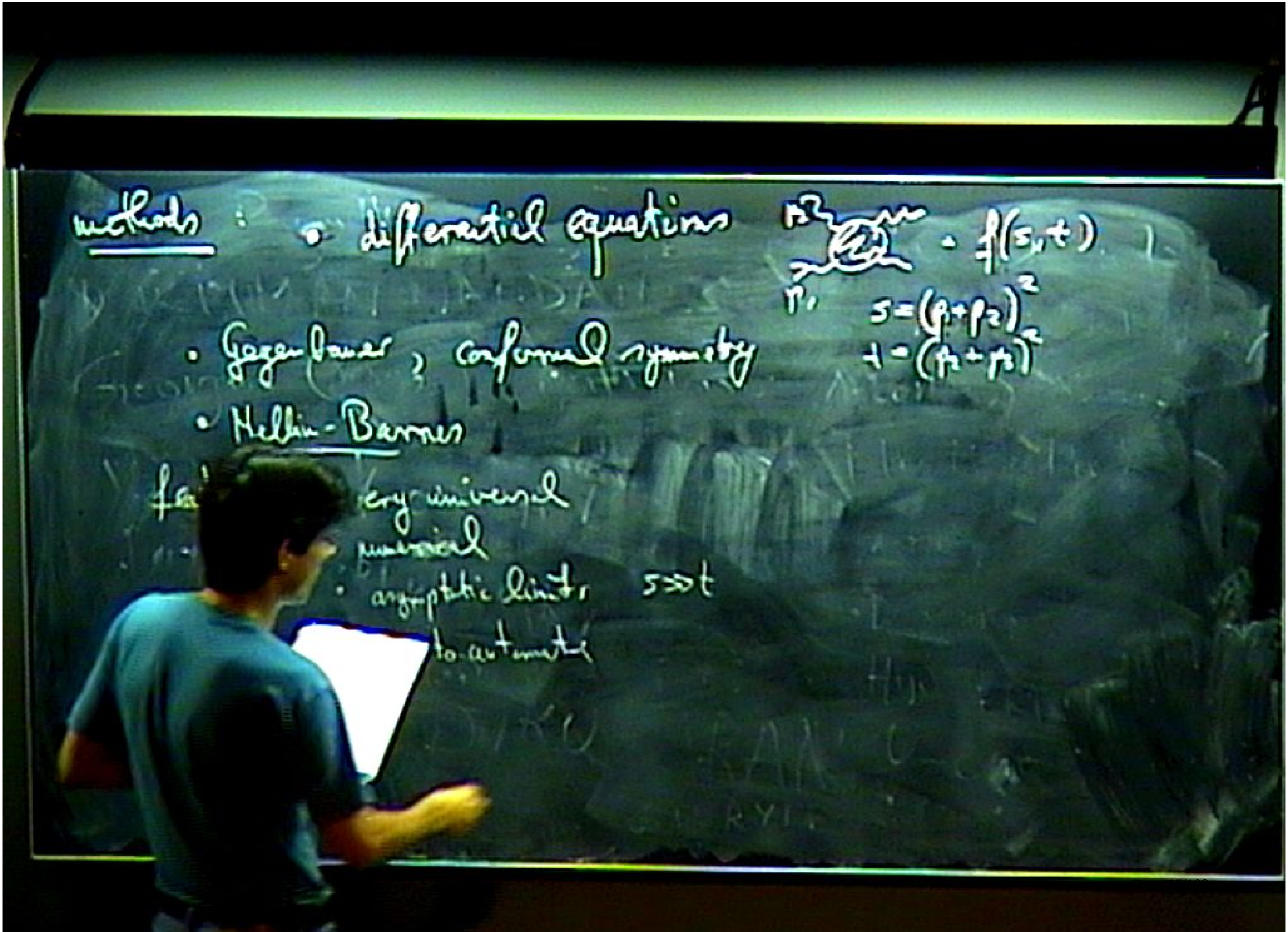
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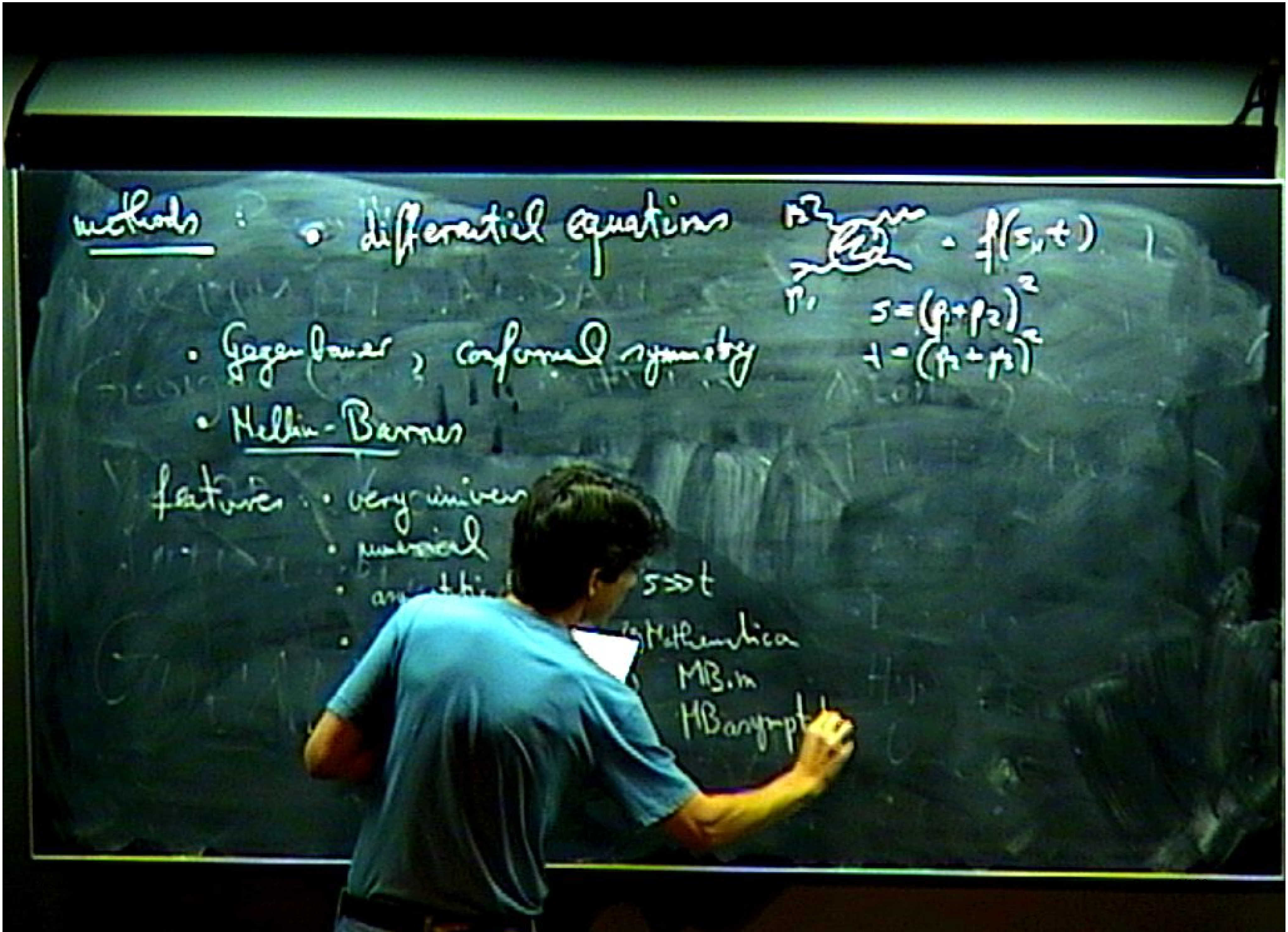
to automate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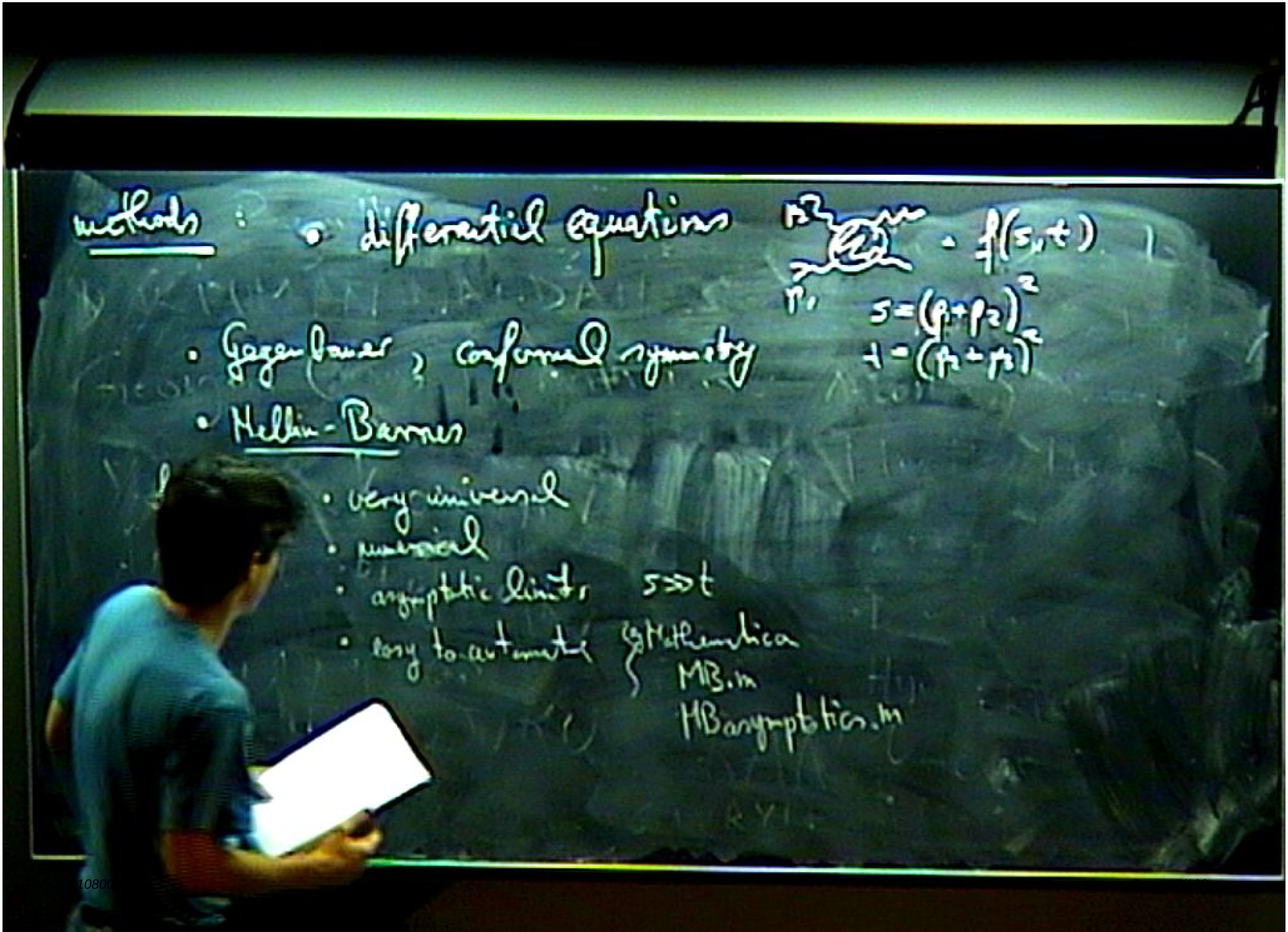
Mathematica
MB.m
MBasympt


methods :
• differential equations = f(s,t)
$s = (p_1+p_2)^2$
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methods :

- differential equations  $= f(s,t)$

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- **Mellin-Barnes**

features :

- very universal
- numerical
- asymptotic limits, $s \gg t$
- easy to automate: Mathematica
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$s = (p_1+p_2)^2$

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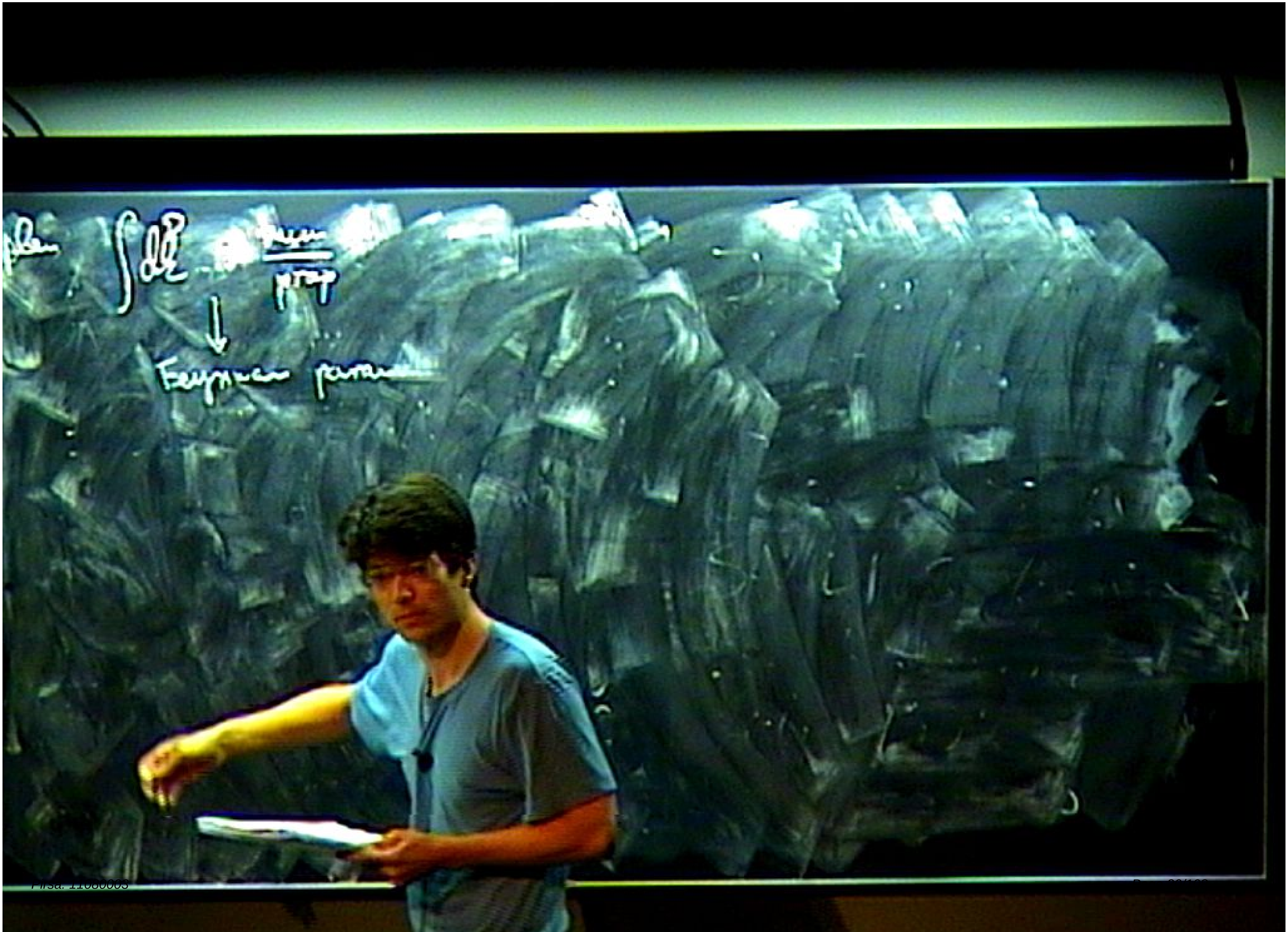
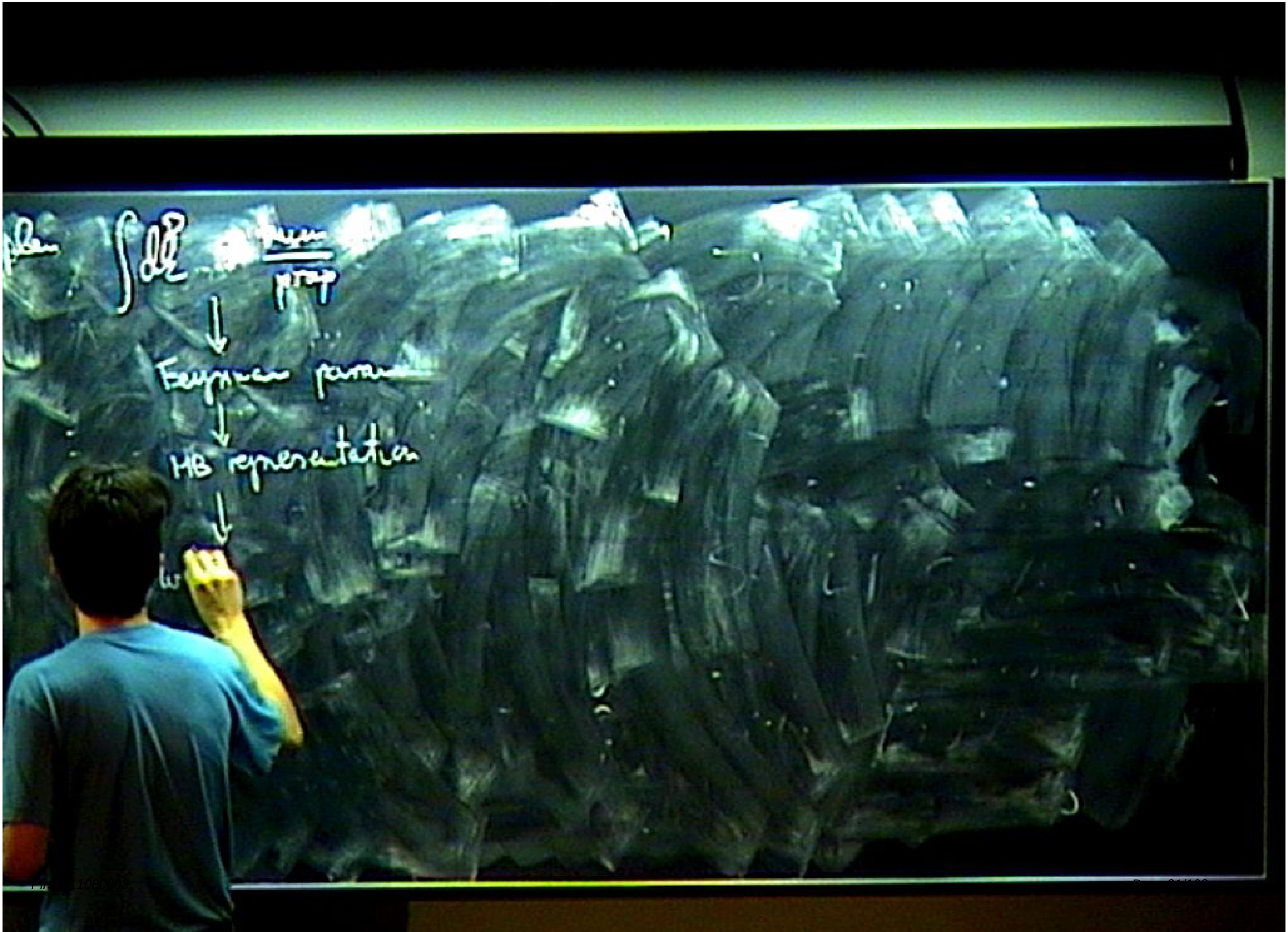
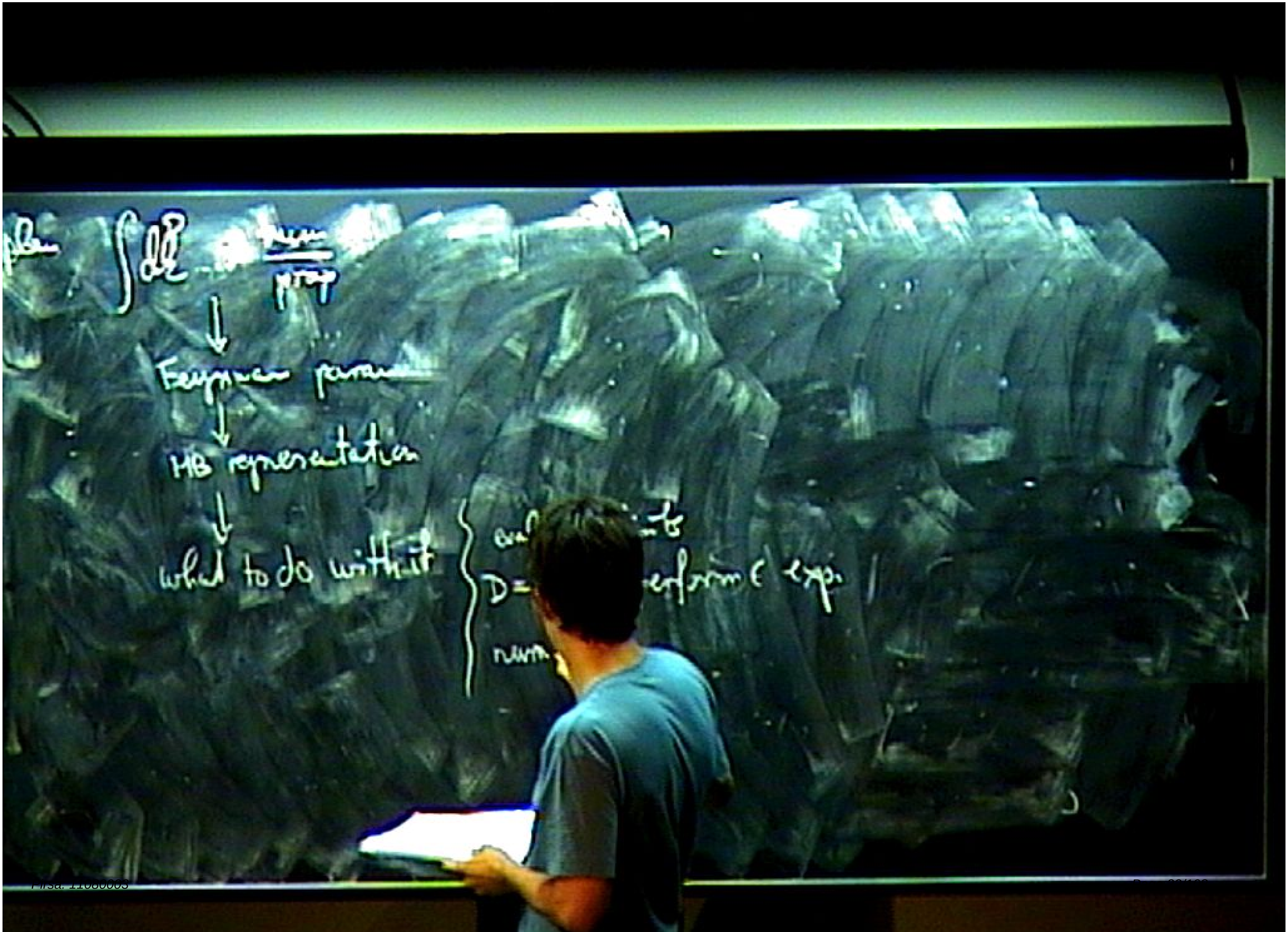
⊖ simplicity of specific N.
can be hidden




Feynman param
HB representation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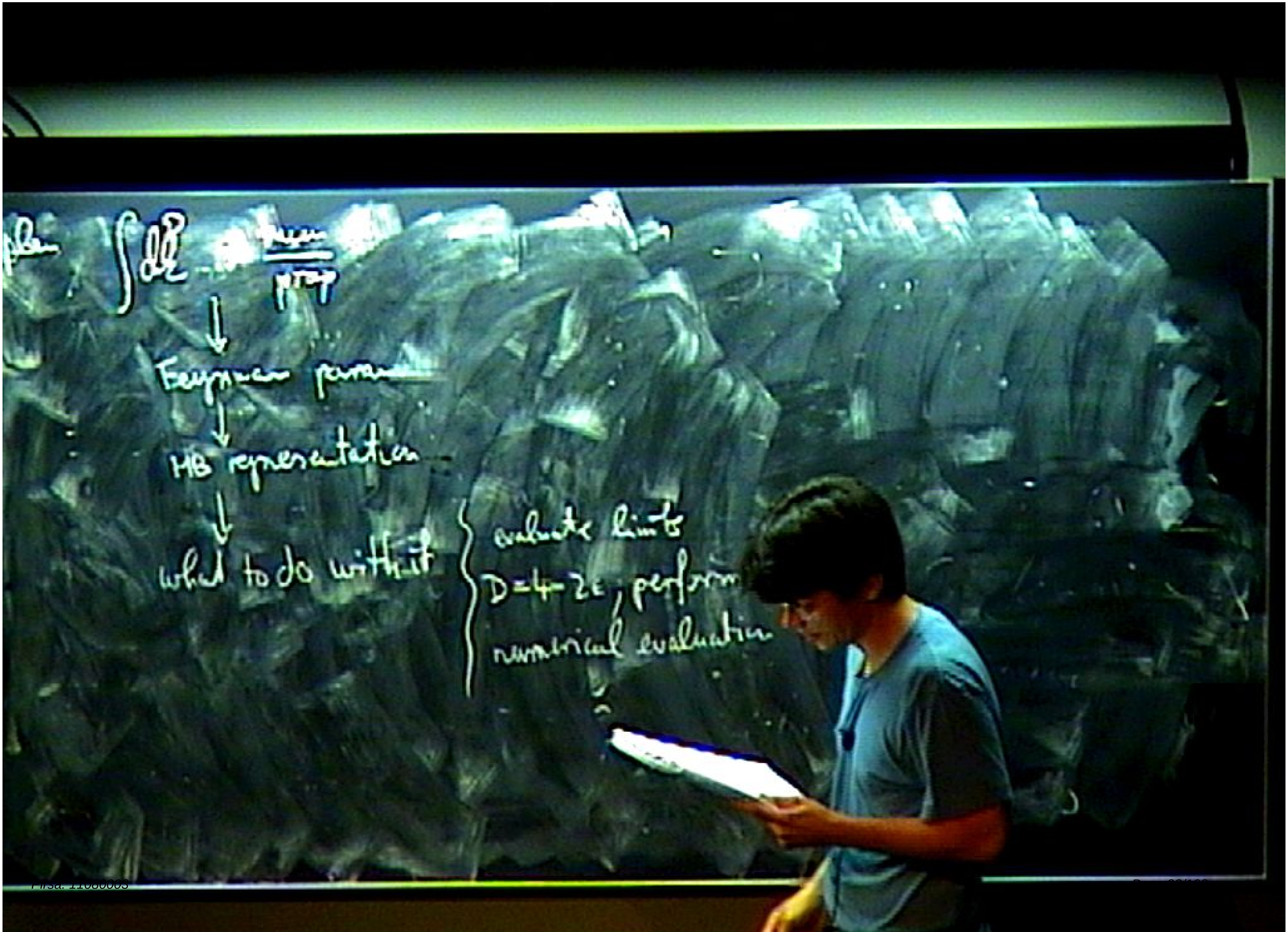
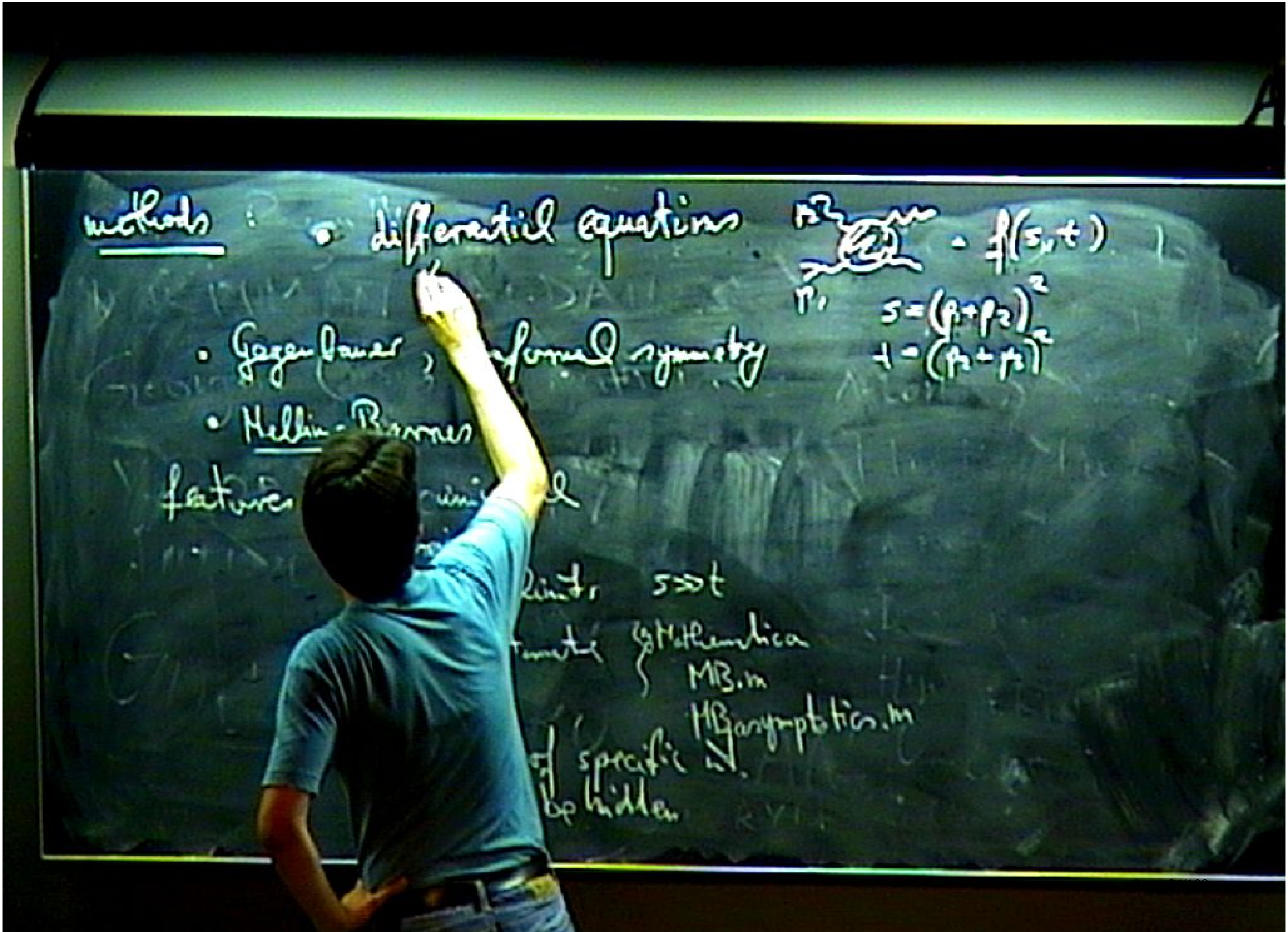
what to do with it


Feynman param
MB representation
what to do with it
evaluate limits
D=4-2ε, perform
numerical evaluation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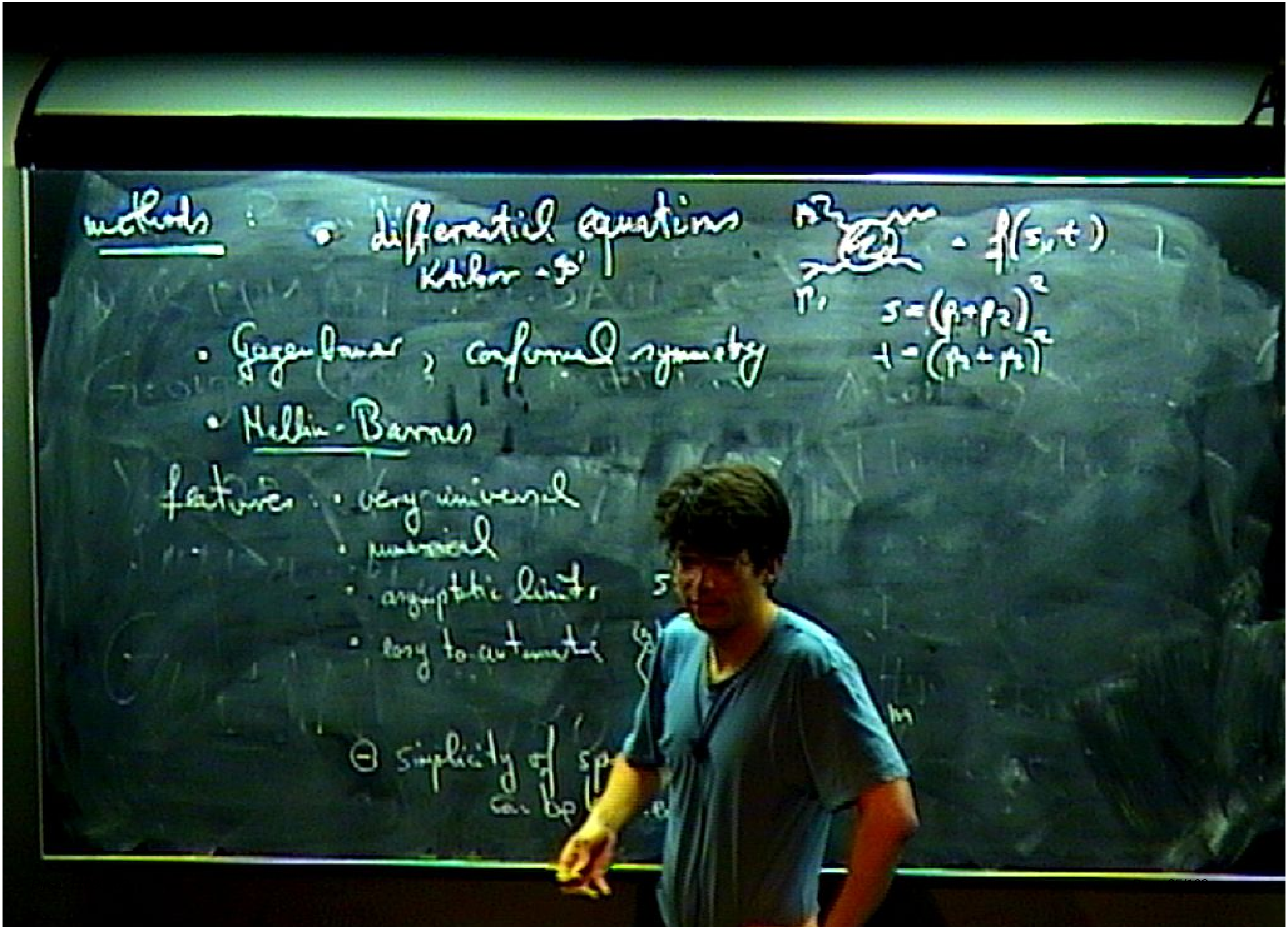
methods :
• differential equations
Kotikov -'91
f(s,t)
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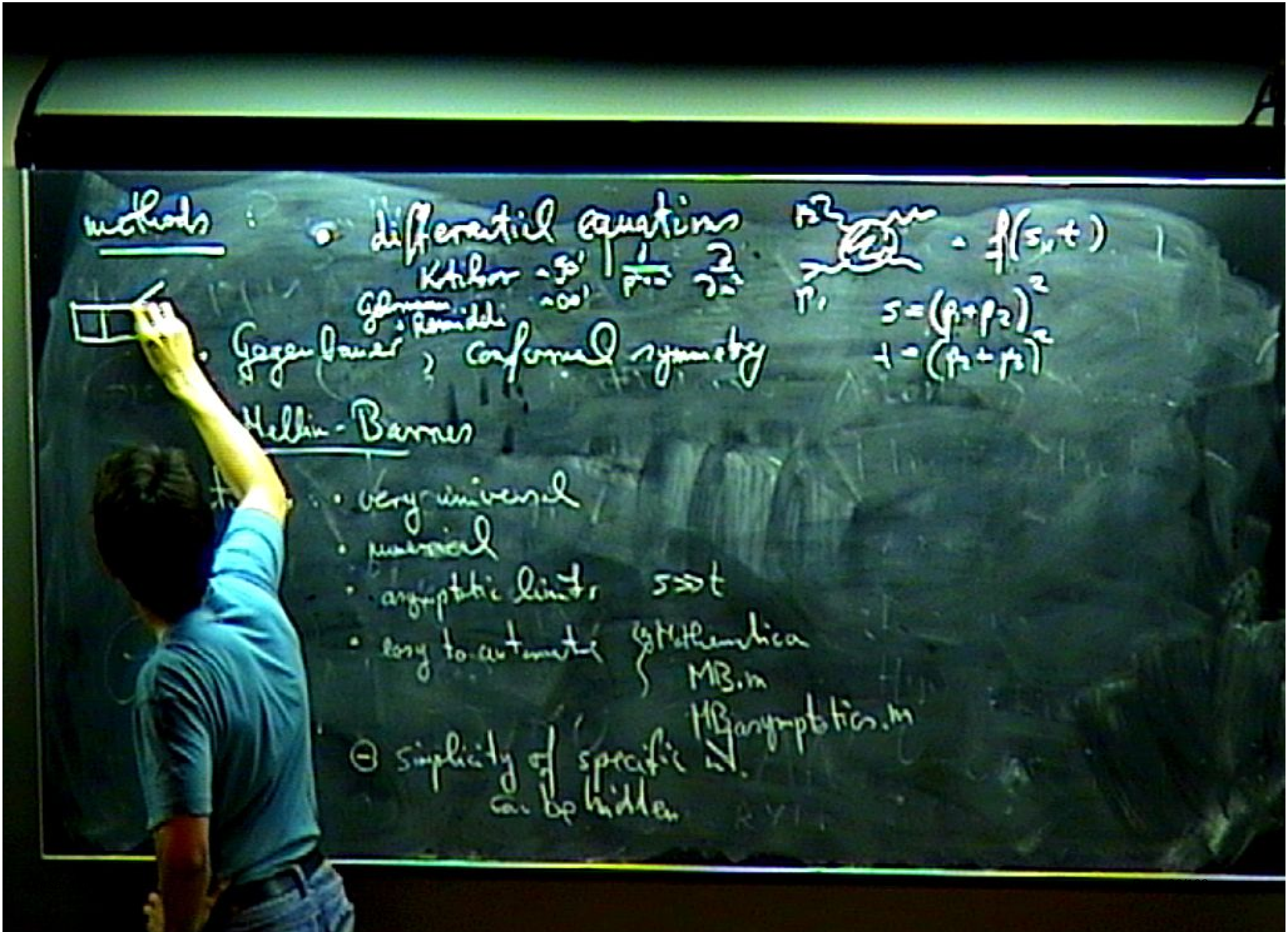
methods :
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Kotikov ~90'
Gehrmann Remiddi ~00'
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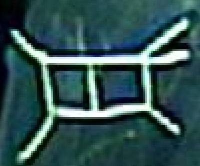
methods:

- differential equations $= f(s,t)$
  - Kotikov ~'90' $\frac{1}{p^2-m^2}$ $\frac{\partial}{\partial p^2}$
  - Gehrmann-Remiddi ~'00'



- Gegenbauer, conformal symmetry

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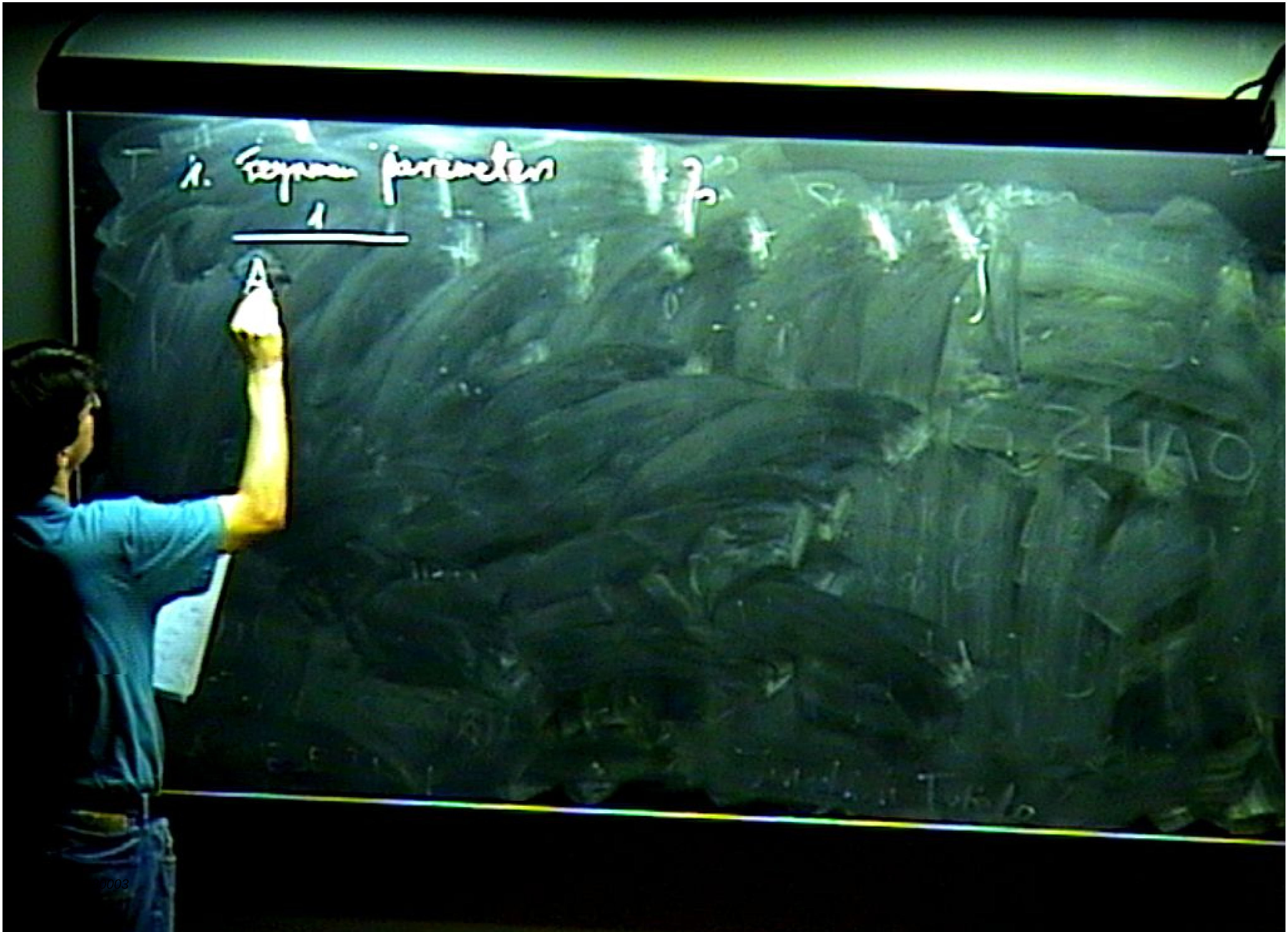
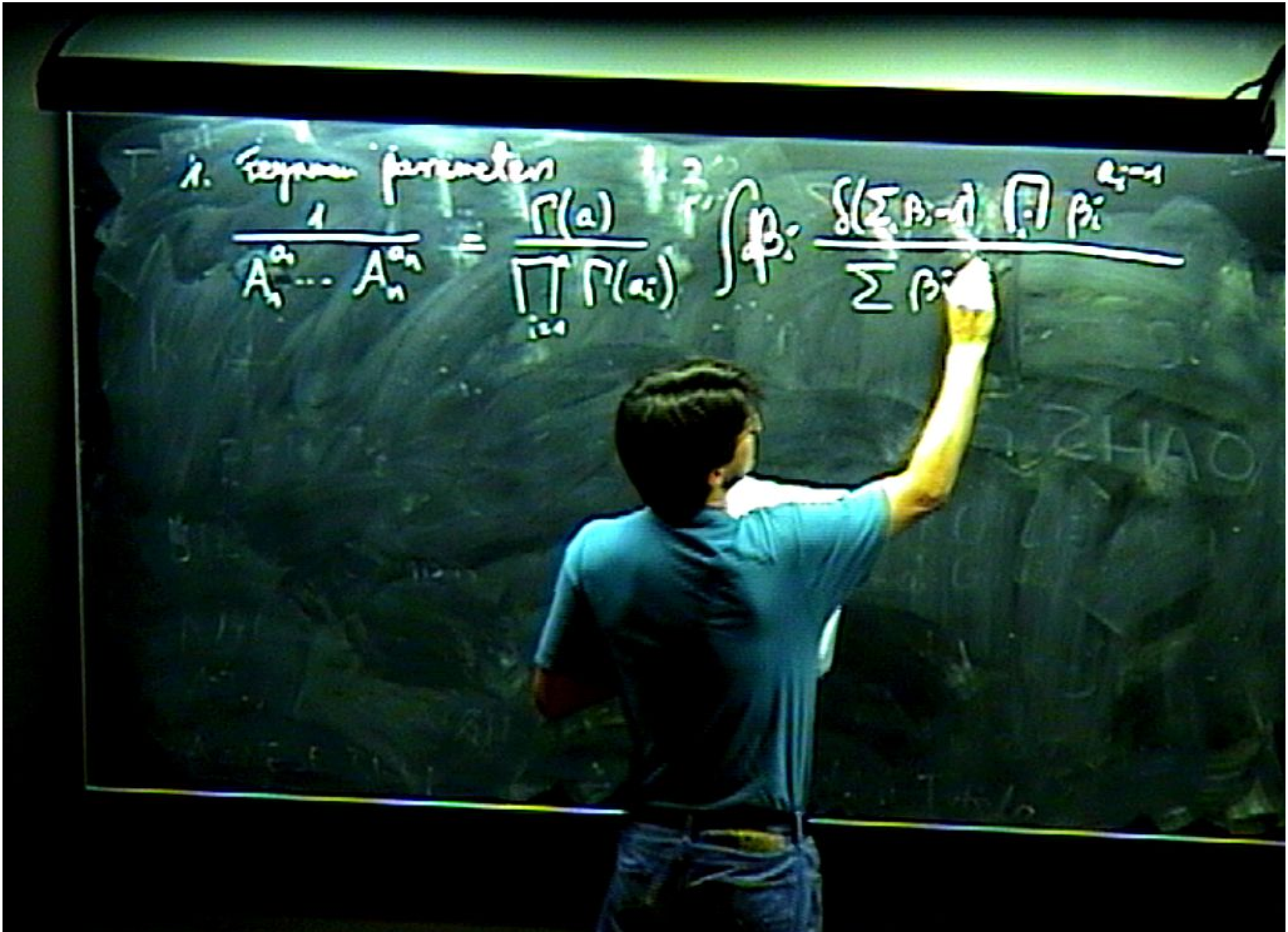
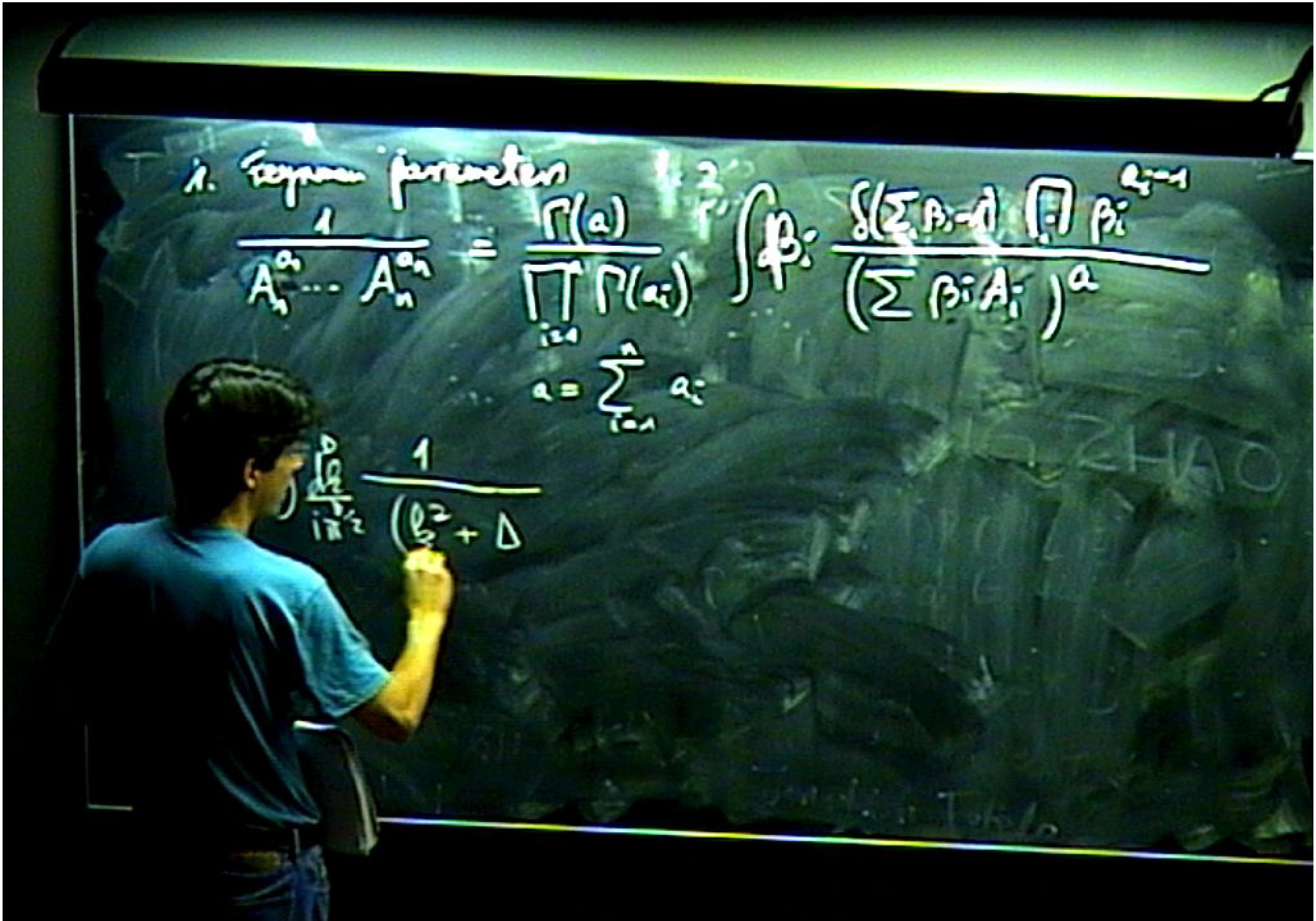

1. Feynman parameters
1
A
ZHAO


1. Feynman parameters
1/(A_1^{a_1} … A_n^{a_n}) = Γ(a)/∏_{i=1} Γ(a_i) ∫ dβ_i δ(Σ β_i − 1) ∏ β_i^{a_i−1} / Σ β_i
ZHAO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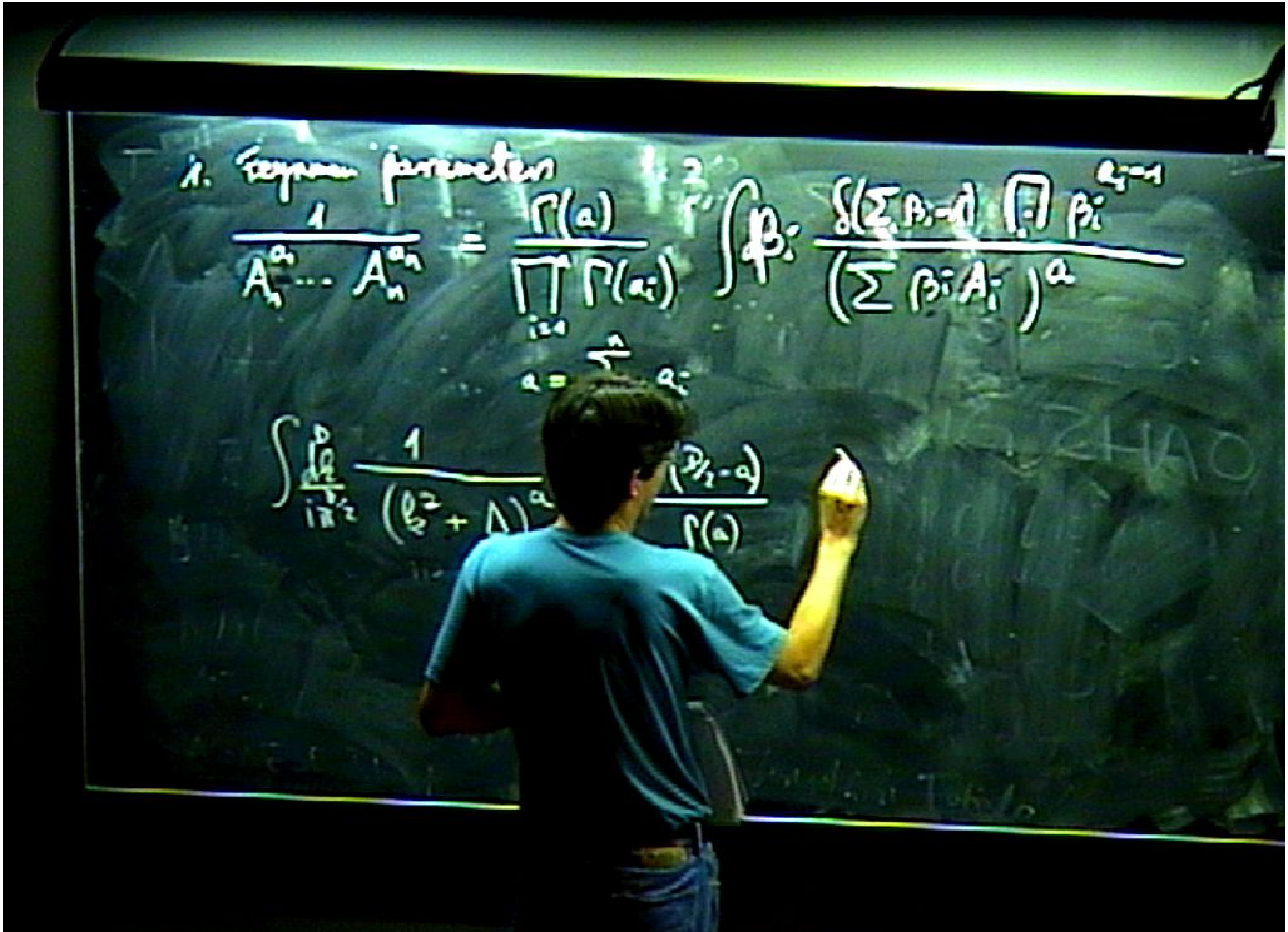
$\frac{1}{A_1^{a_1}\cdots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^{n}\Gamma(a_i)} \int d\beta_i \frac{\delta(\sum \beta_i - 1)\prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$
$a = \sum_{i=1}^{n} a_i$
2.
$\int \frac{d^D \ell}{i\pi^{D/2}} \frac{1}{(\ell^2 + \Delta}$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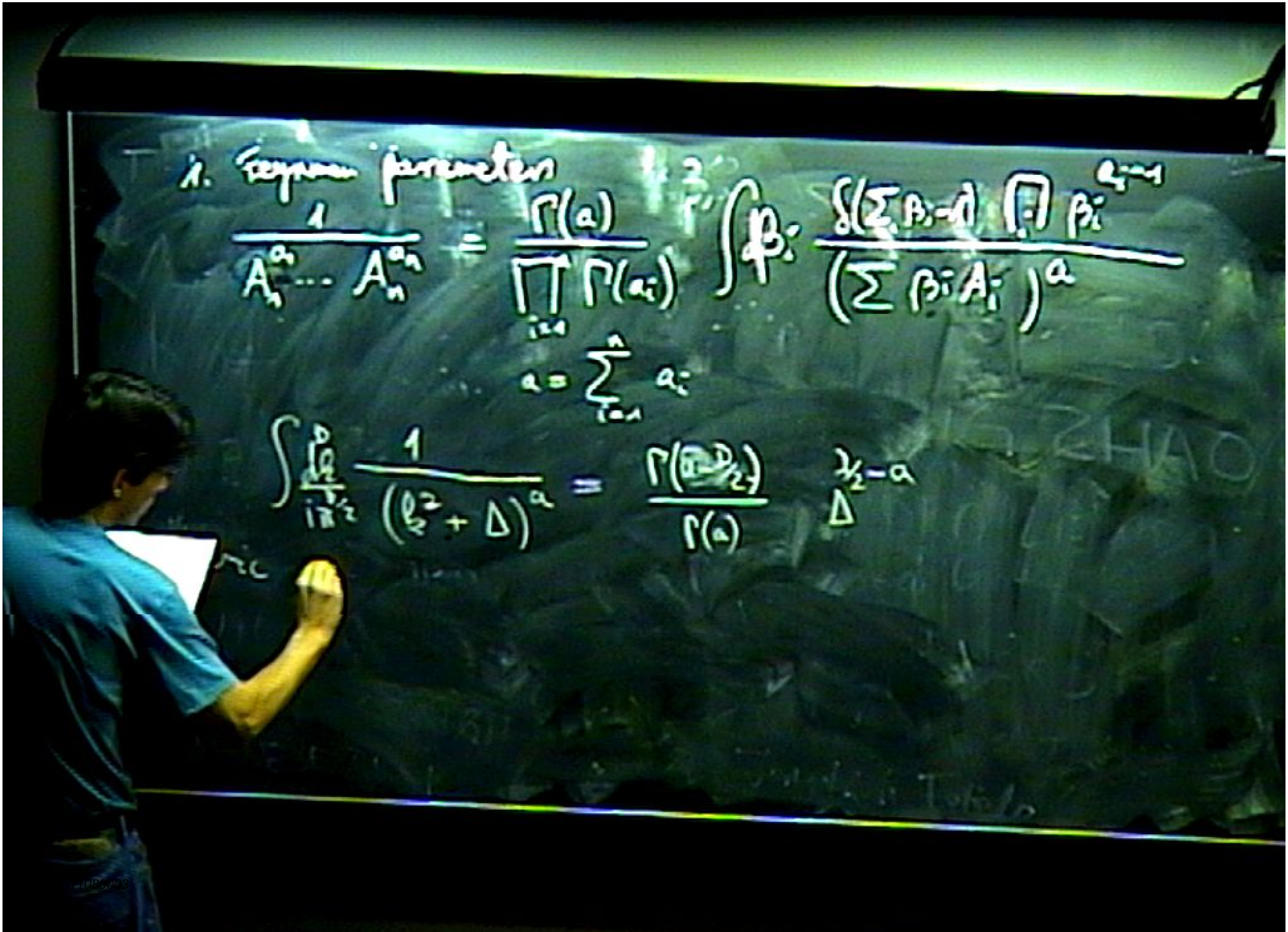
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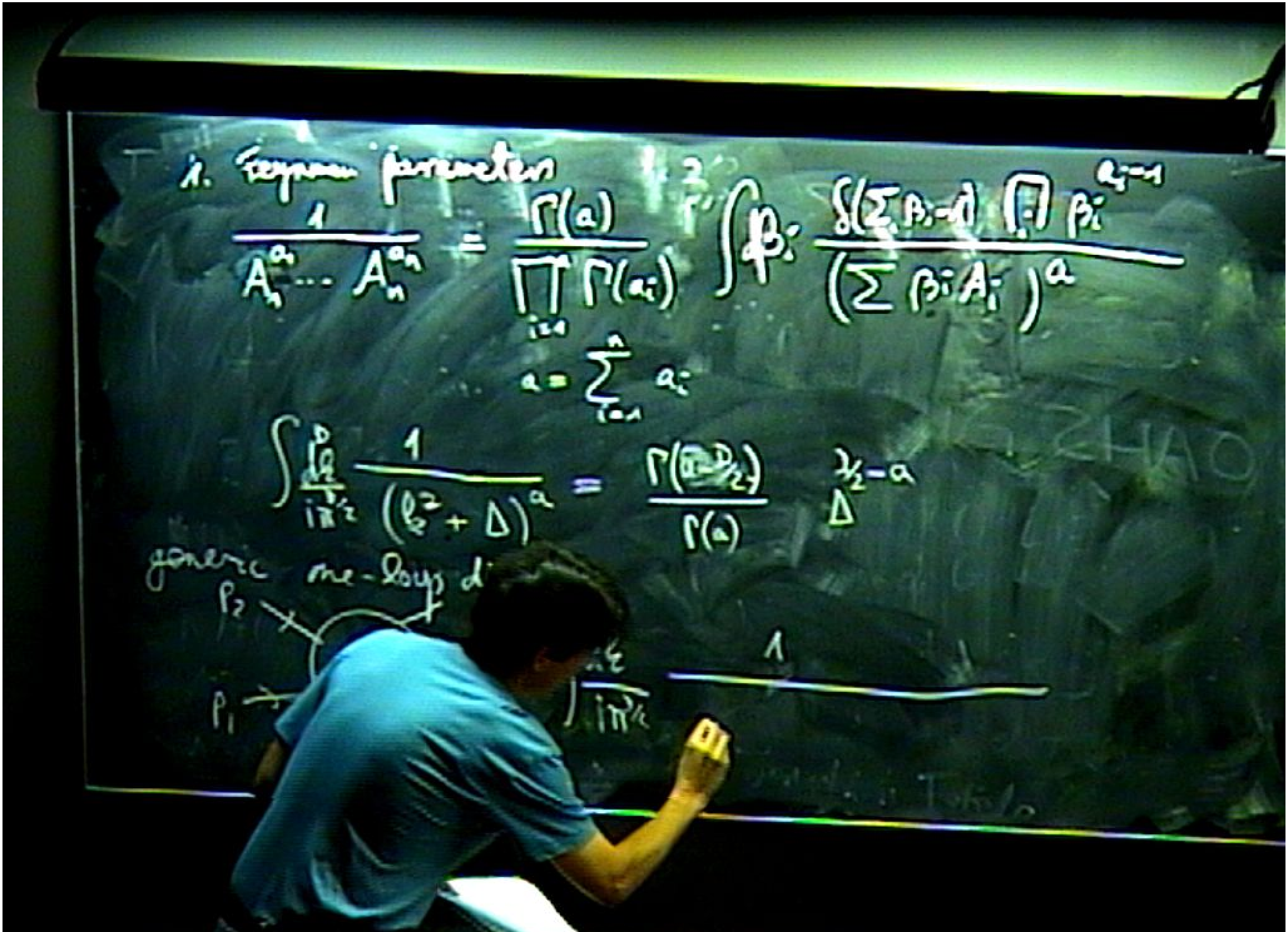
$a = \sum_{i=1}^{n} a_i$
$\int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{(\ell^2+\Delta)^a} = \frac{\Gamma(a-D/2)}{\Gamma(a)} \Delta^{D/2-a}$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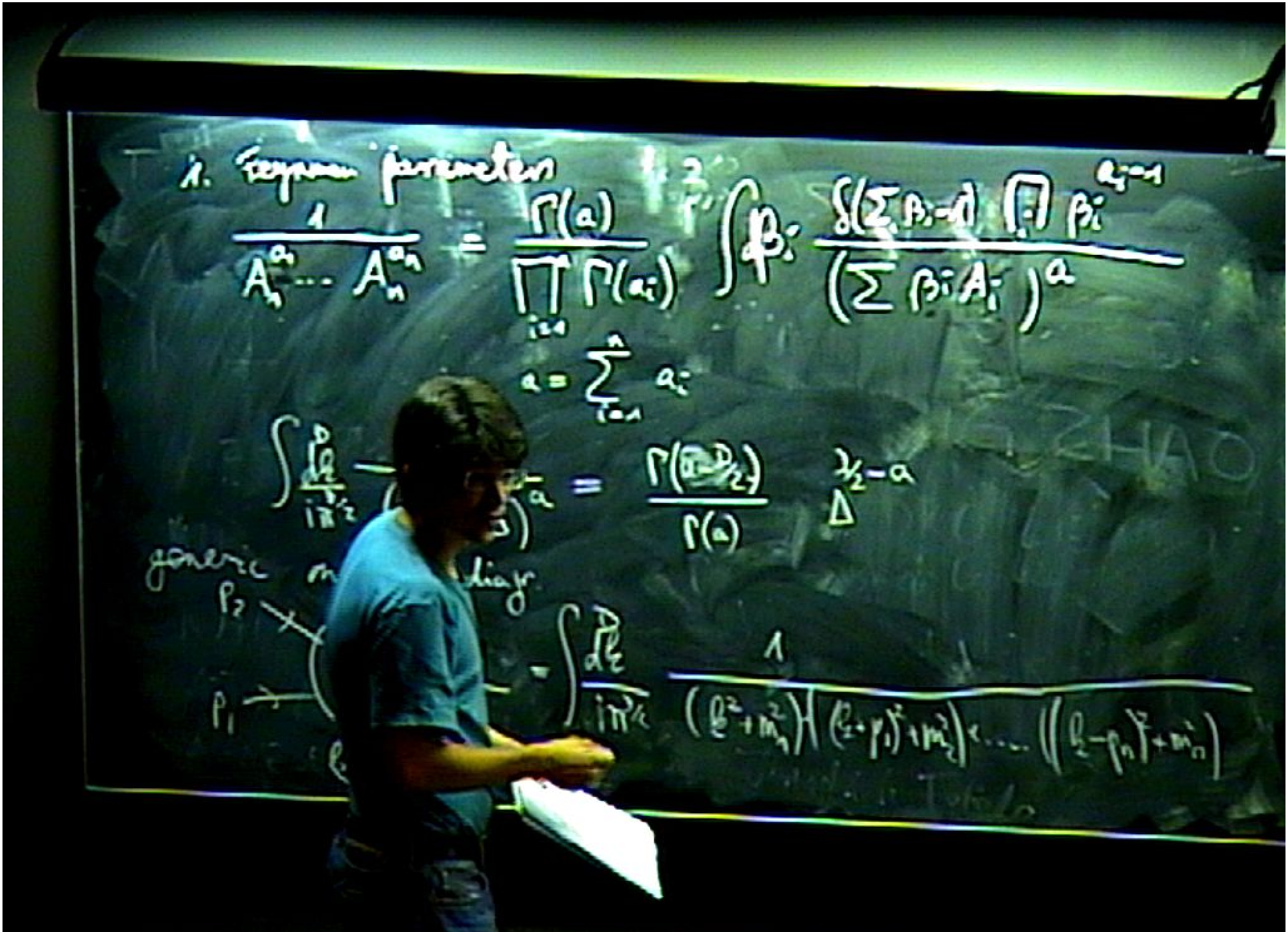
generic one-loop di
p_2
p_1
$\int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\quad}$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$= \frac{\Gamma(a-D/2)}{\Gamma(a)} \Delta^{D/2-a}$
generic
diagr.
p_2
p_1
$= \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2+m_1^2)((l+p_1)^2+m_2^2)\cdots((l-p_n)^2+m_n^2)}$

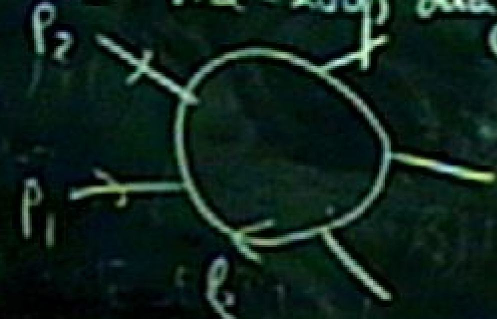
1. Feynman parameters

$$\frac{1}{A_1^{a_1}\cdots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^{n}\Gamma(a_i)} \int d\beta_i \, \frac{\delta\left(\sum \beta_i - 1\right)\prod \beta_i^{a_i-1}}{\left(\sum \beta_i A_i\right)^{a}}$$

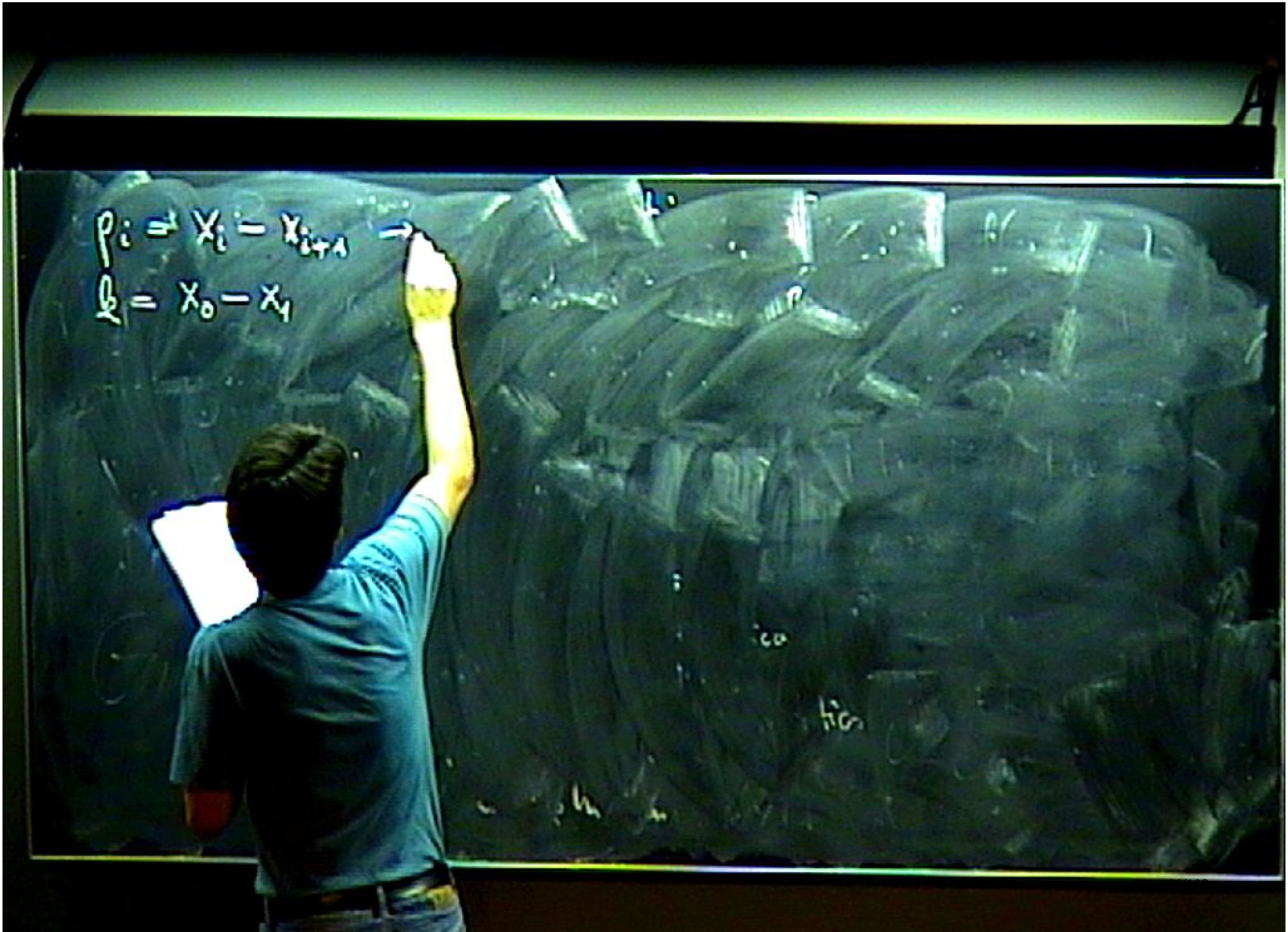
$$a = \sum_{i=1}^{n} a_i$$

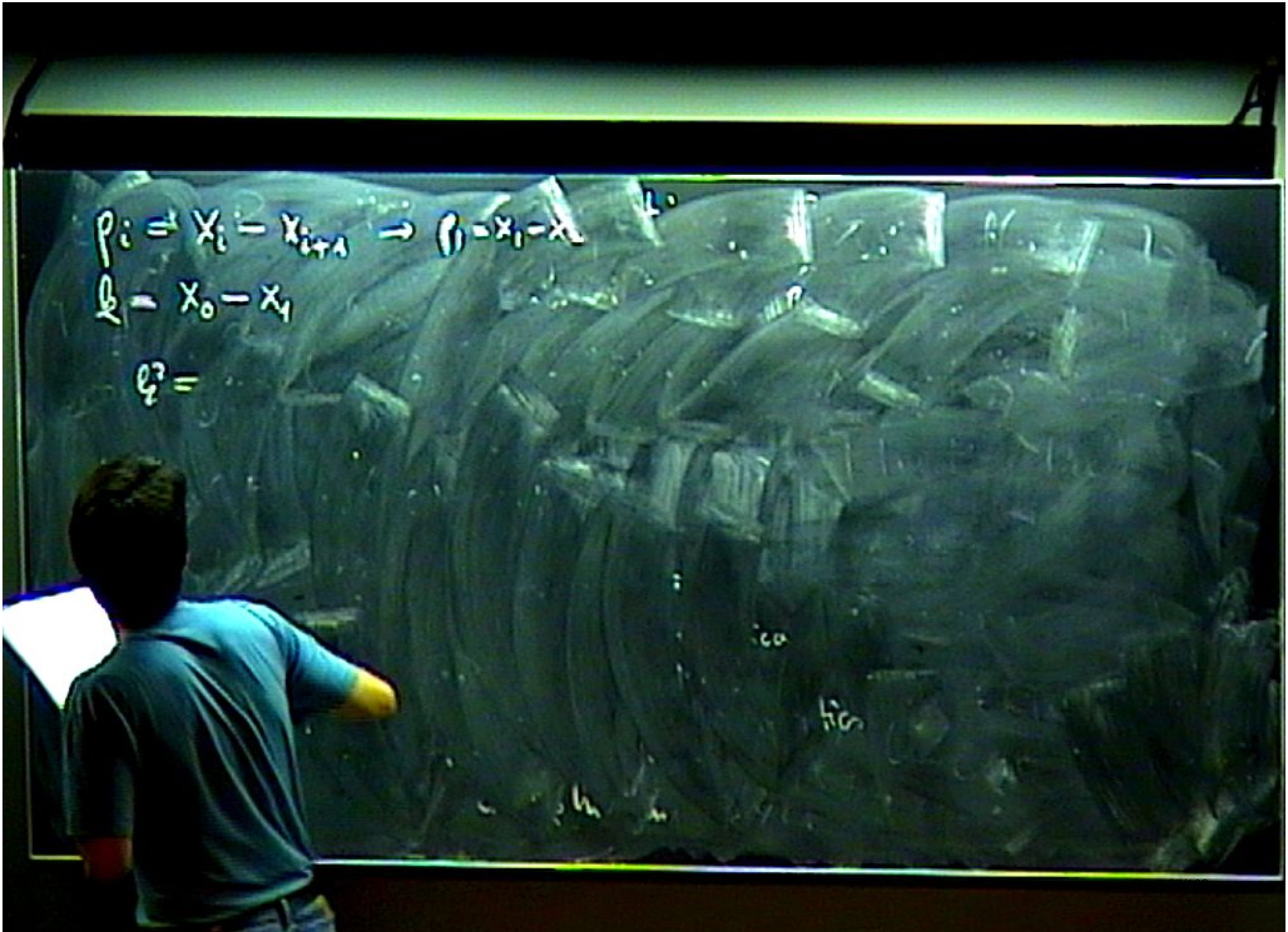
$$\int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\left(\ell^2+\Delta\right)^{a}} = \frac{\Gamma\left(a-D/2\right)}{\Gamma(a)} \, \Delta^{D/2-a}$$

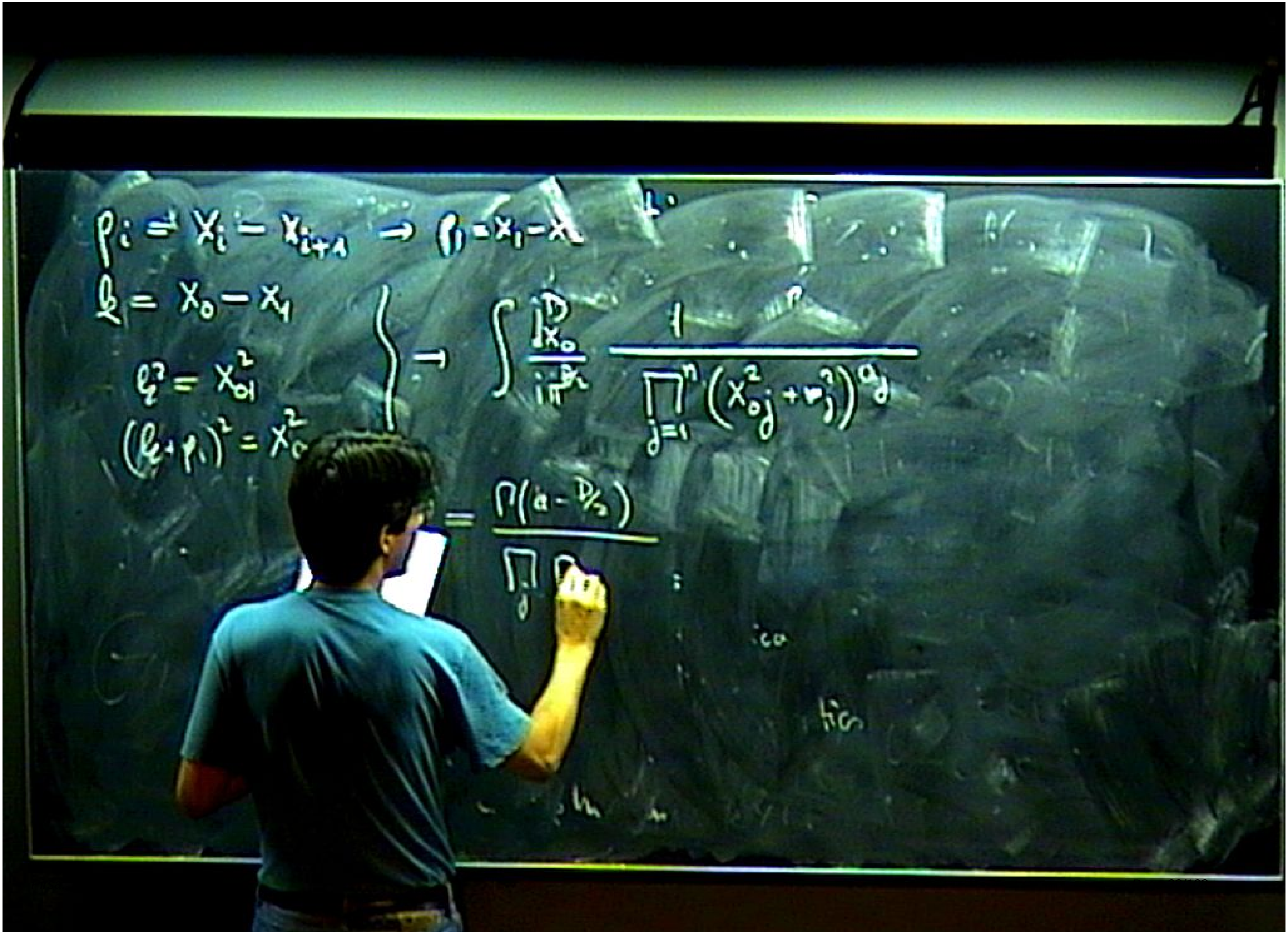
generic one-loop diagr.

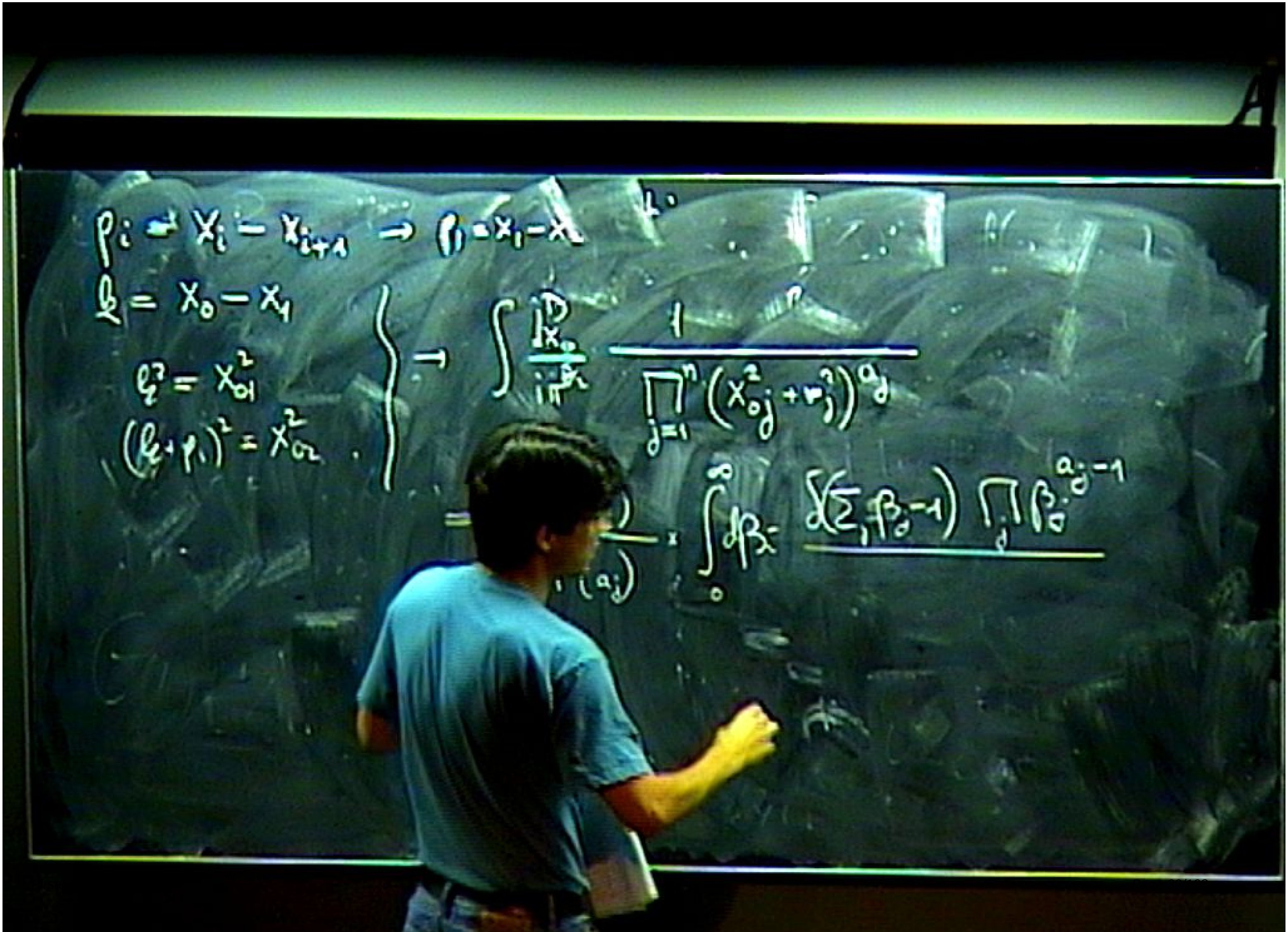


$$= \int \frac{d^D\ell}{i\pi^{D/2}} \frac{1}{\left(\ell^2+m_1^2\right)^{a_1}\left((\ell+p_1)^2+m_2^2\right)^{a_2}\cdots\left((\ell-p_n)^2+m_n^2\right)^{a_n}}$$


$\rho_i = X_i - X_{i+1} \rightarrow$
$\ell = X_0 - X_1$


ρi := Xi − Xi+1 → ρ1 = x1 − x
ρ = X0 − X1
ρ² =


$p_i = x_i - x_{i+1} \to p_1 = x_1 - x_2$
$k = x_0 - x_1$
$k^2 = x_{01}^2$
$(k+p_1)^2 = x_{02}^2$
$\to \int \frac{d^D x_0}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^{n} (x_{0j}^2 + m_j^2)^{a_j}}$
$= \frac{\Gamma(a - D/2)}{\prod_j \Gamma}$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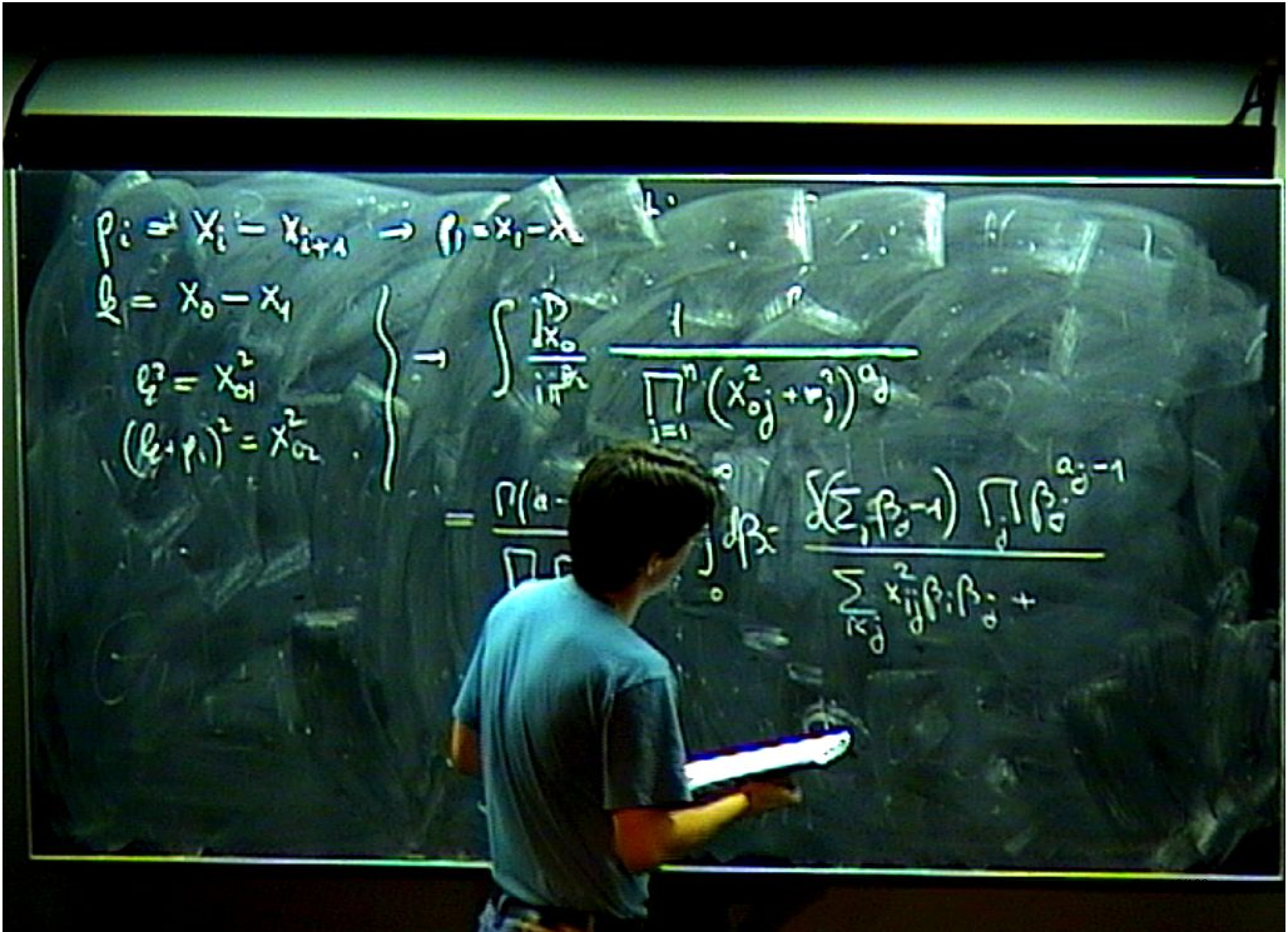

$\rho_i = x_i - x_{i+1} \to \rho_1 = x_1 - x$
$k = x_0 - x_1$
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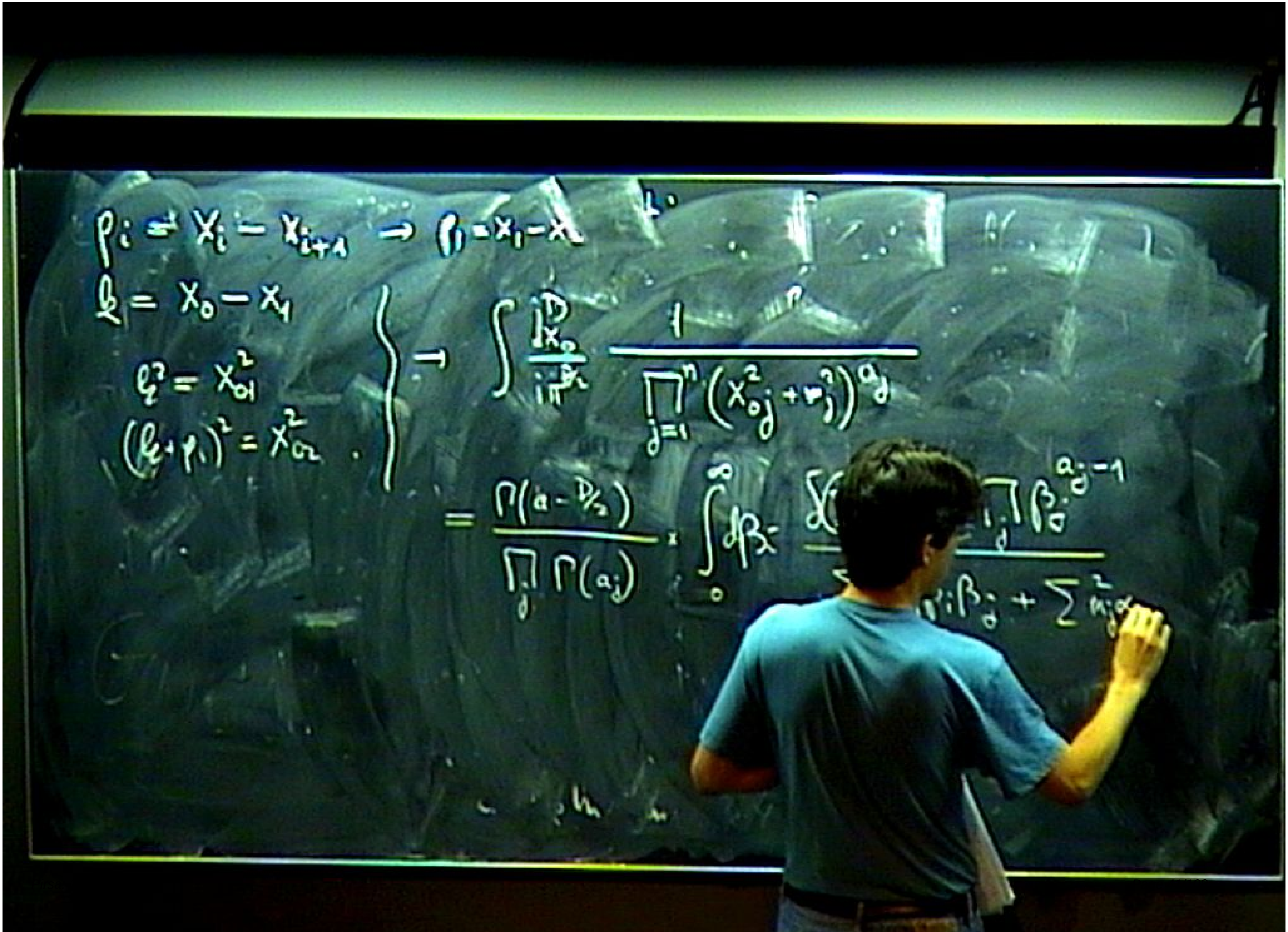
$(k + \rho_1)^2 = x_{02}^2$
$\prod_{j=1}^{n} (x_{0j}^2 + m_j^2)^{a_j}$
$\delta(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}$
$\sum_{i<j} x_{ij}^2 \beta_i \beta_j +$


$p_i = x_i - x_{i+1}$
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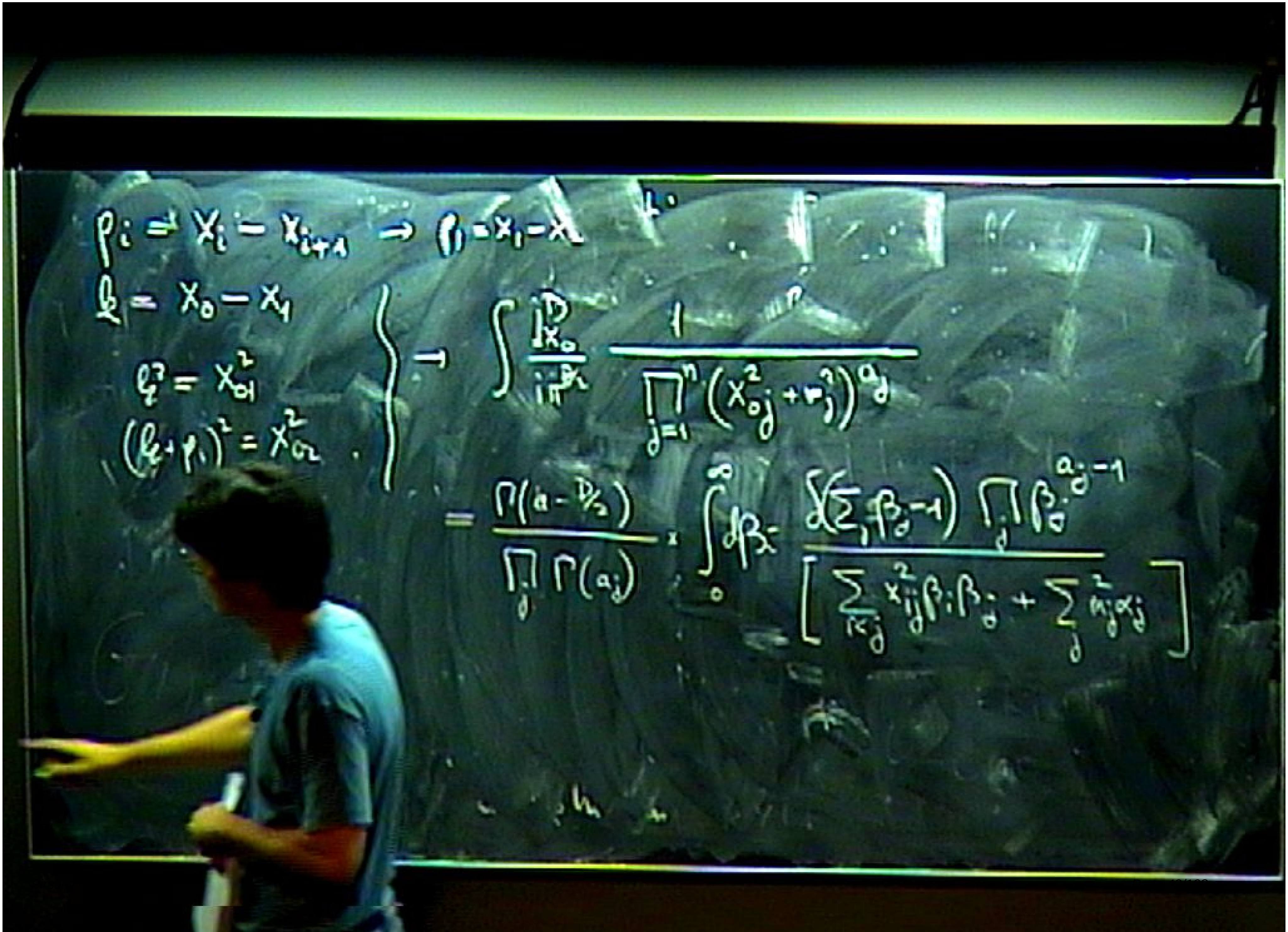
$\int \frac{d^D x_0}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^n (x_{0j}^2 + m_j^2)^{a_j}}$
$= \frac{\Gamma(a - D/2)}{\prod_j \Gamma(a_j)}$


$\rho_i = x_i - x_{i+1} \to \rho_1 = x_1 - x_2$
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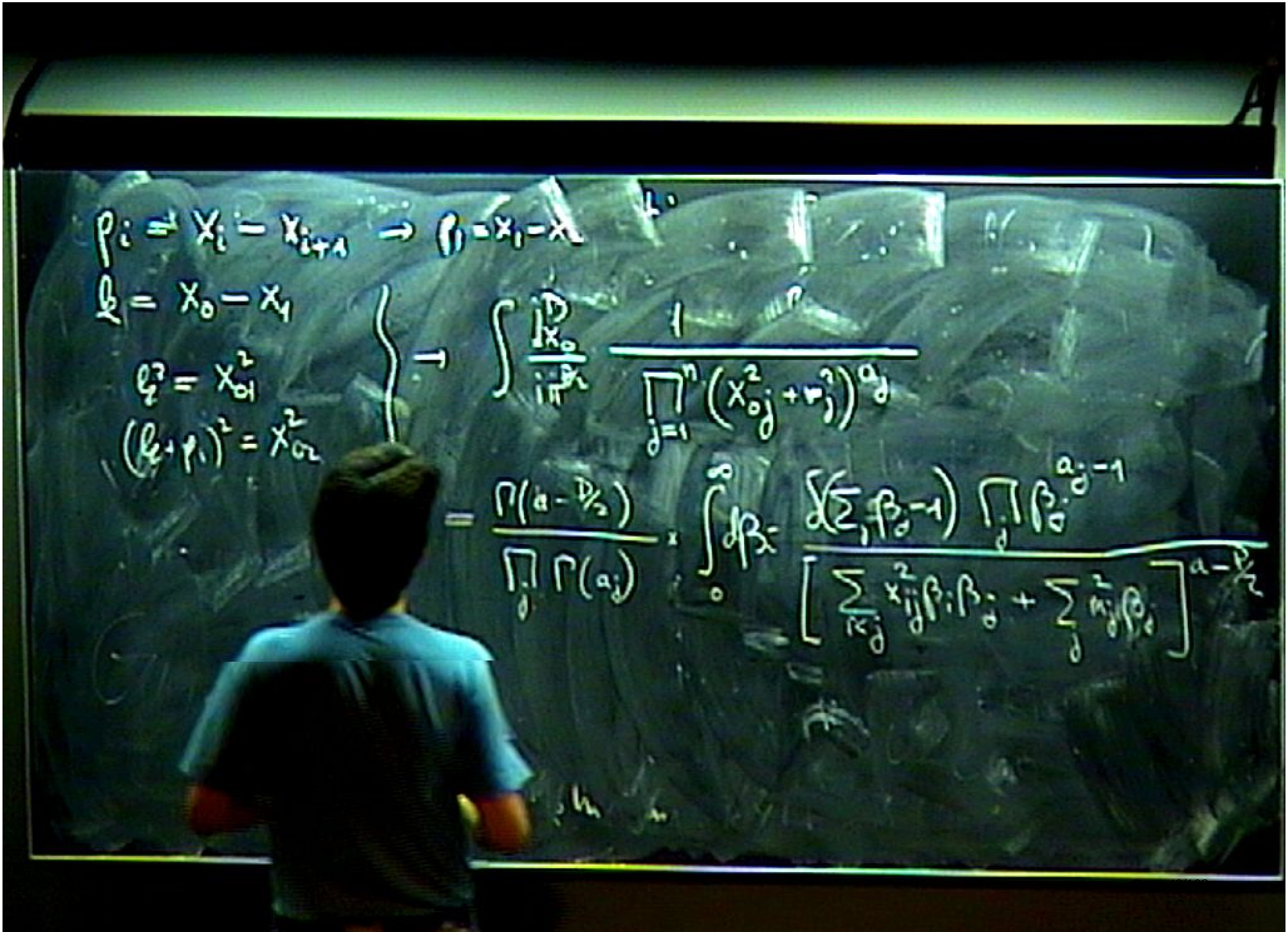
$= \frac{\Gamma(a - D/2)}{\prod_j \Gamma(a_j)} \times \int_0^\infty d\beta_j \frac{\delta(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{i<j} x_{ij}^2 \beta_i \beta_j + \sum_j m_j^2 \alpha_j \right]}$


$\beta_i = x_i - x_{i+1} \to \beta_1 = x_1 - x$
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$k^2 = x_{01}^2$
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$\to \int \frac{d^D k_0}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^{n} (x_{0j}^2 + m_j^2)^{a_j}}$
$= \frac{\Gamma(a - D/2)}{\prod_j \Gamma(a_j)} \times \int_0^\infty d\beta_j \frac{\delta(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{i<j} x_{ij}^2 \beta_i \beta_j + \sum_j m_j^2 \beta_j\right]^{a - D/2}}$

$\rho_i = X_i - X_{i+1} \to \rho_1 = x_1 - x$

$k_1 = X_0 - X_1$

$k^2 = X_{01}^2$

$(k + \rho_1)^2 = X_{02}^2$

$$\to \int \frac{d^D X_0}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^{n} \left(X_{0j}^2 + m_j^2\right)^{a_j}} =$$

$$= \boxed{\frac{\Gamma(a - D/2)}{\prod_j \Gamma(a_j)} \times \int_0^\infty d\beta_j \, \frac{\delta\left(\sum_j \beta_j - 1\right) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{i<j} x_{ij}^2 \beta_i \beta_j + \sum_j m_j^2 \beta_j\right]^{a - D/2}}}$$

1. Feynman parameters

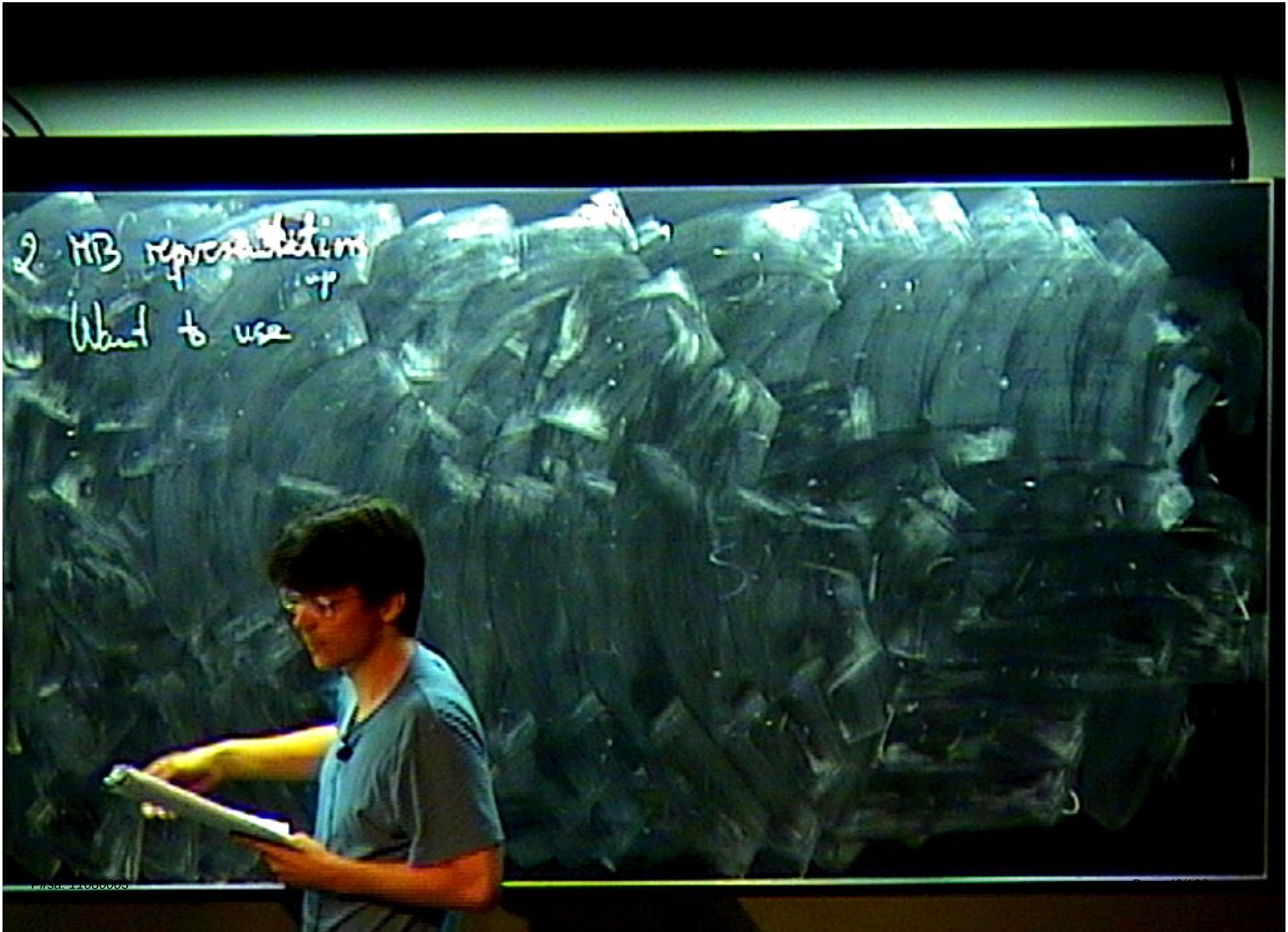
$$\frac{1}{A_1^{a_1}\cdots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^{n}\Gamma(a_i)} \int d\beta_i \, \frac{\delta\left(\sum \beta_i - 1\right)\prod \beta_i^{a_i-1}}{\left(\sum \beta_i A_i\right)^{a}}$$

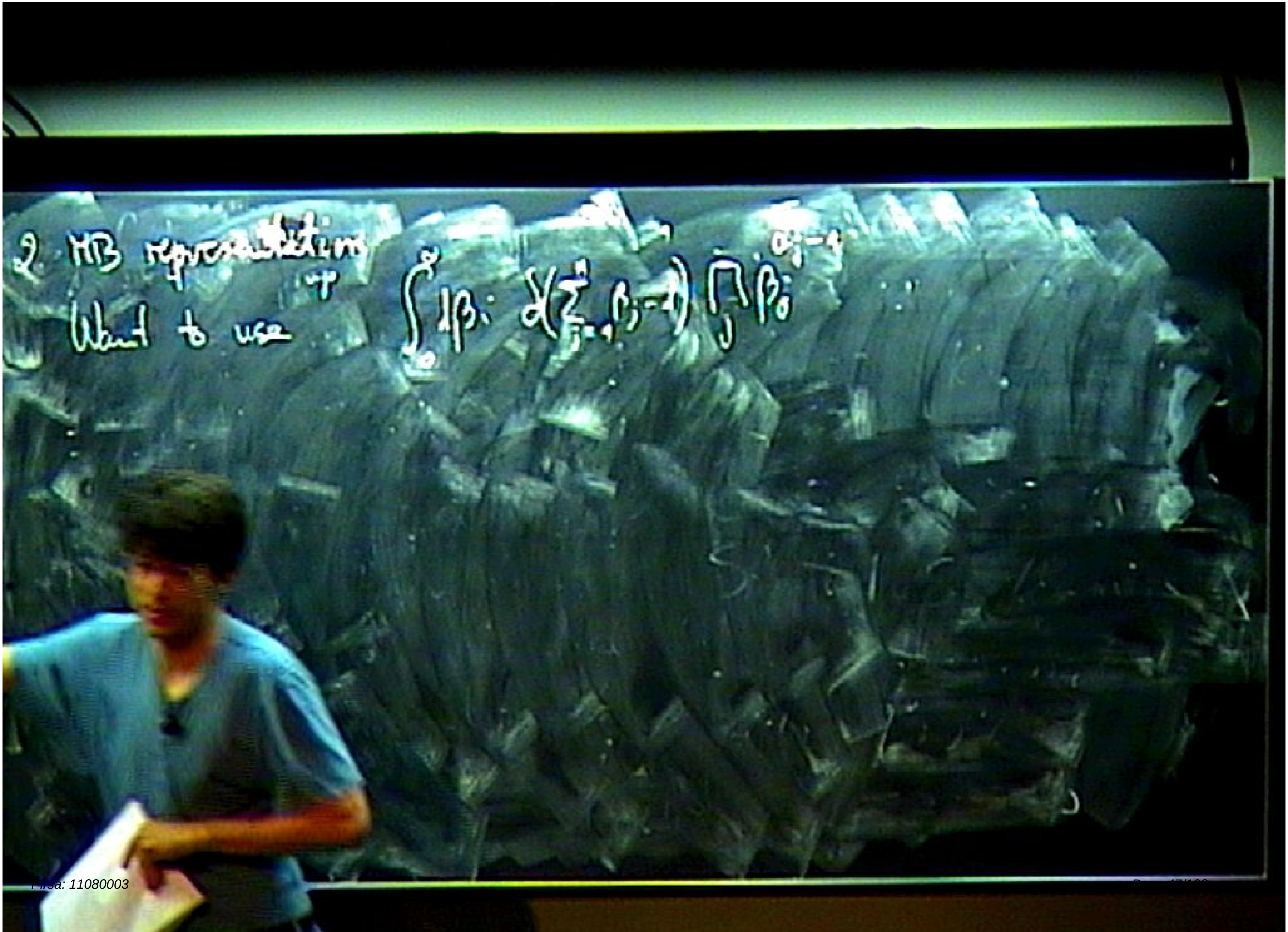
$$a = \sum_{i=1}^{n} a_i$$

$$\int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left(l^2+\Delta\right)^{a}} = \frac{\Gamma\left(a-\frac{D}{2}\right)}{\Gamma(a)} \, \Delta^{\frac{D}{2}-a}$$

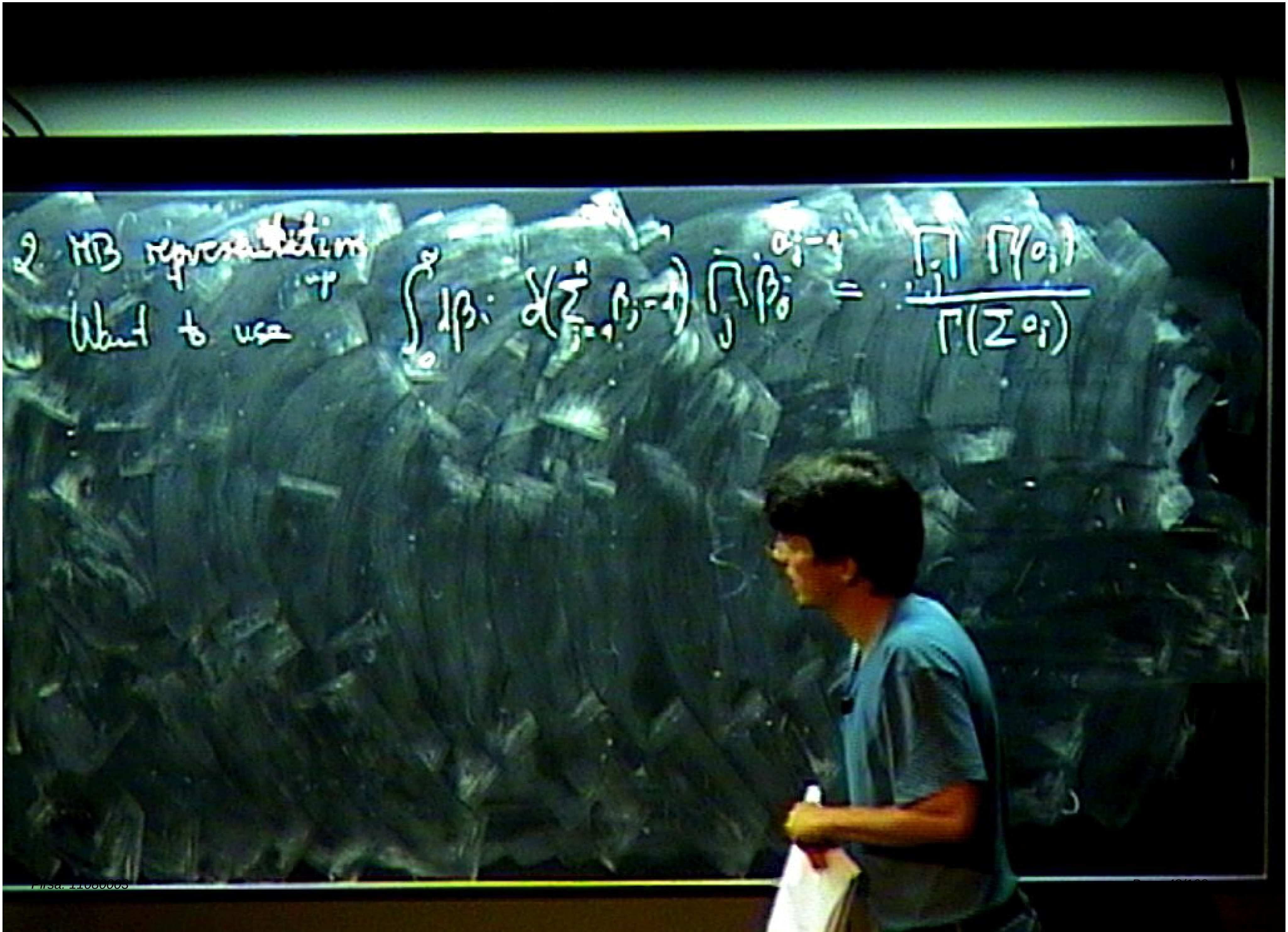
generic one-loop diagr.

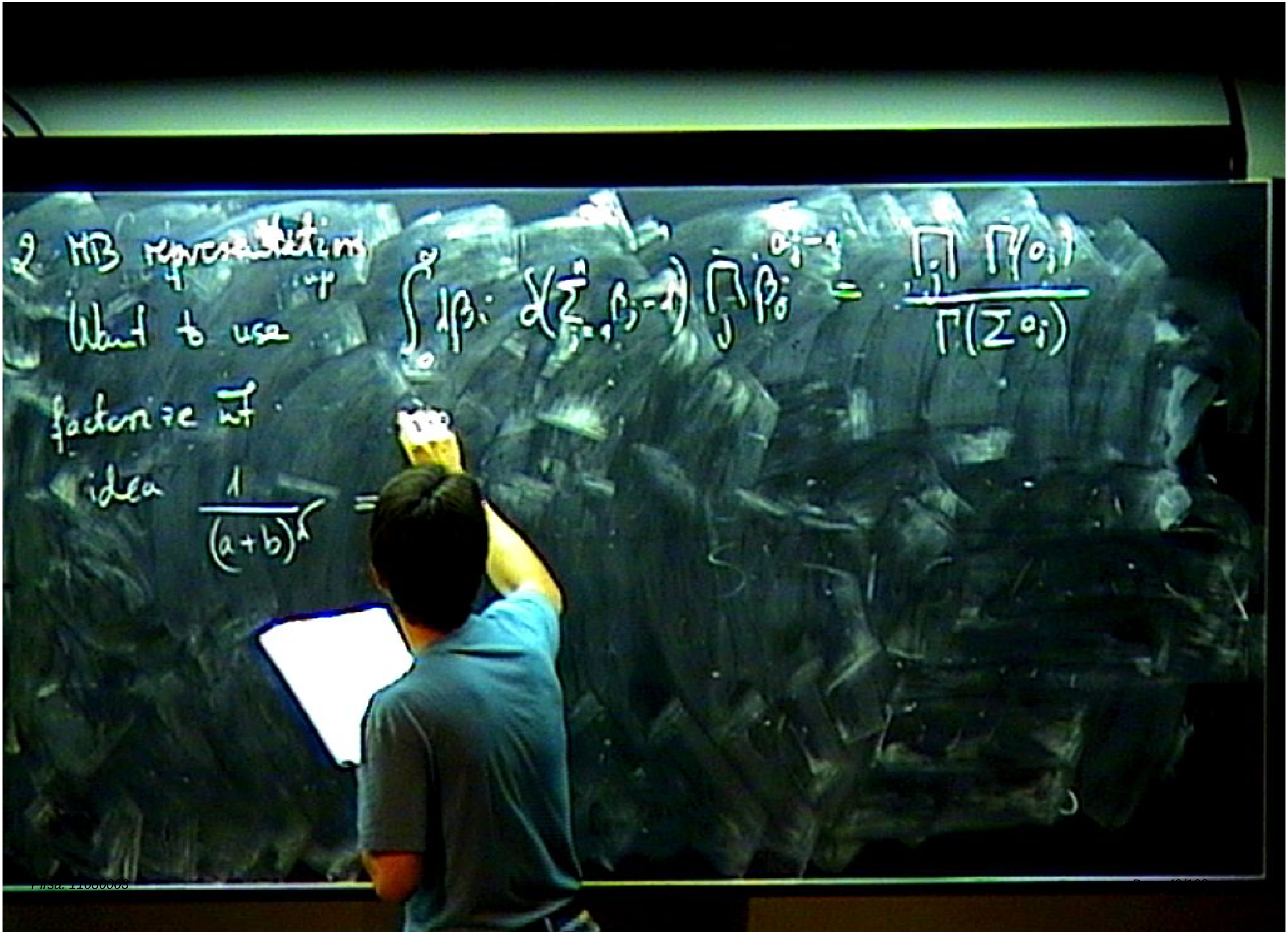
$$= \int \frac{d^D l}{i\pi^{D/2}} \frac{1}{\left(l^2+m_1^2\right)^{a_1}\left((l+p_1)^2+m_2^2\right)^{a_2}\cdots\left((l-p_n)^2+m_n^2\right)^{a_n}}$$

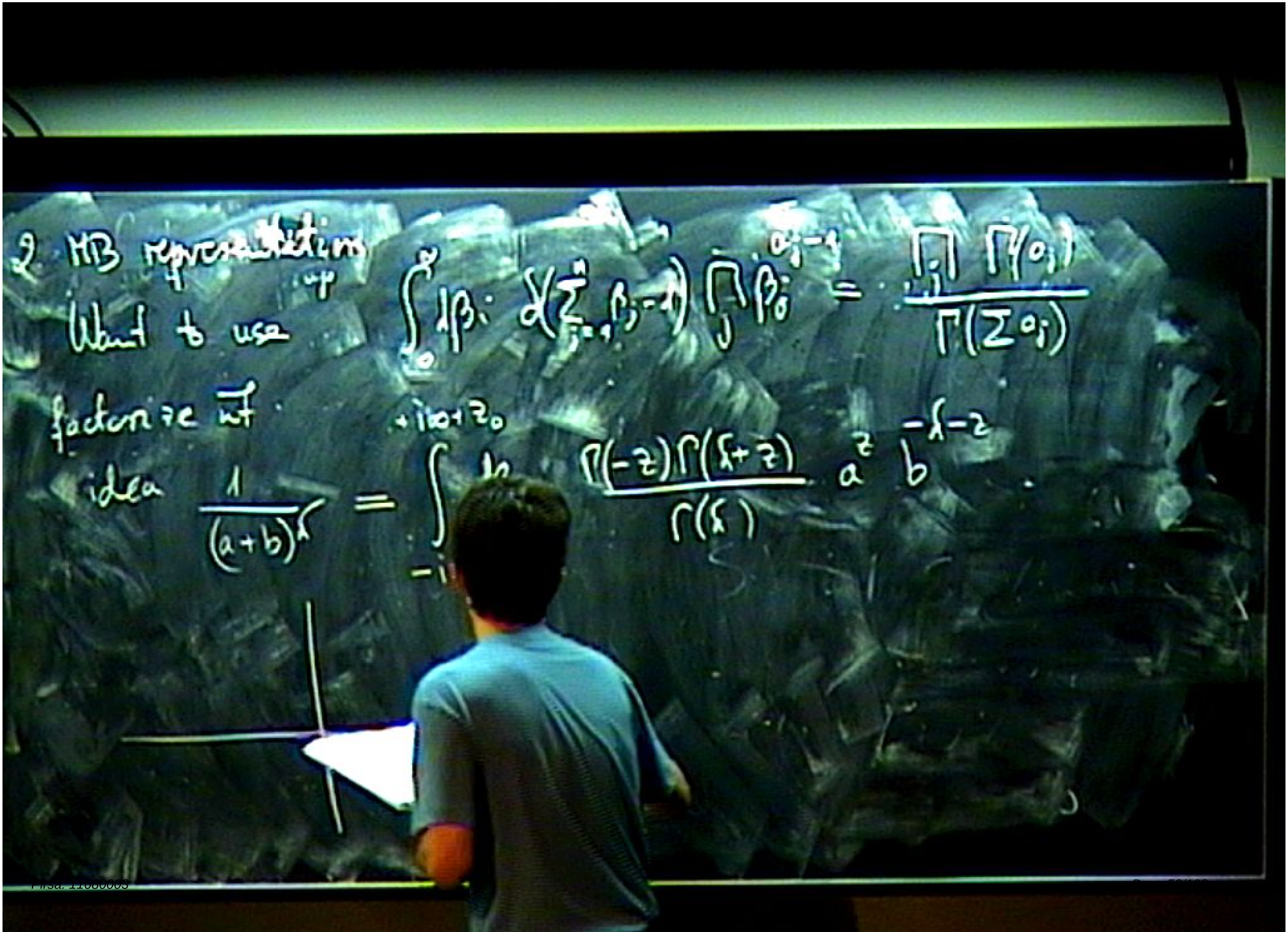

2. MB representation
up
Want to use

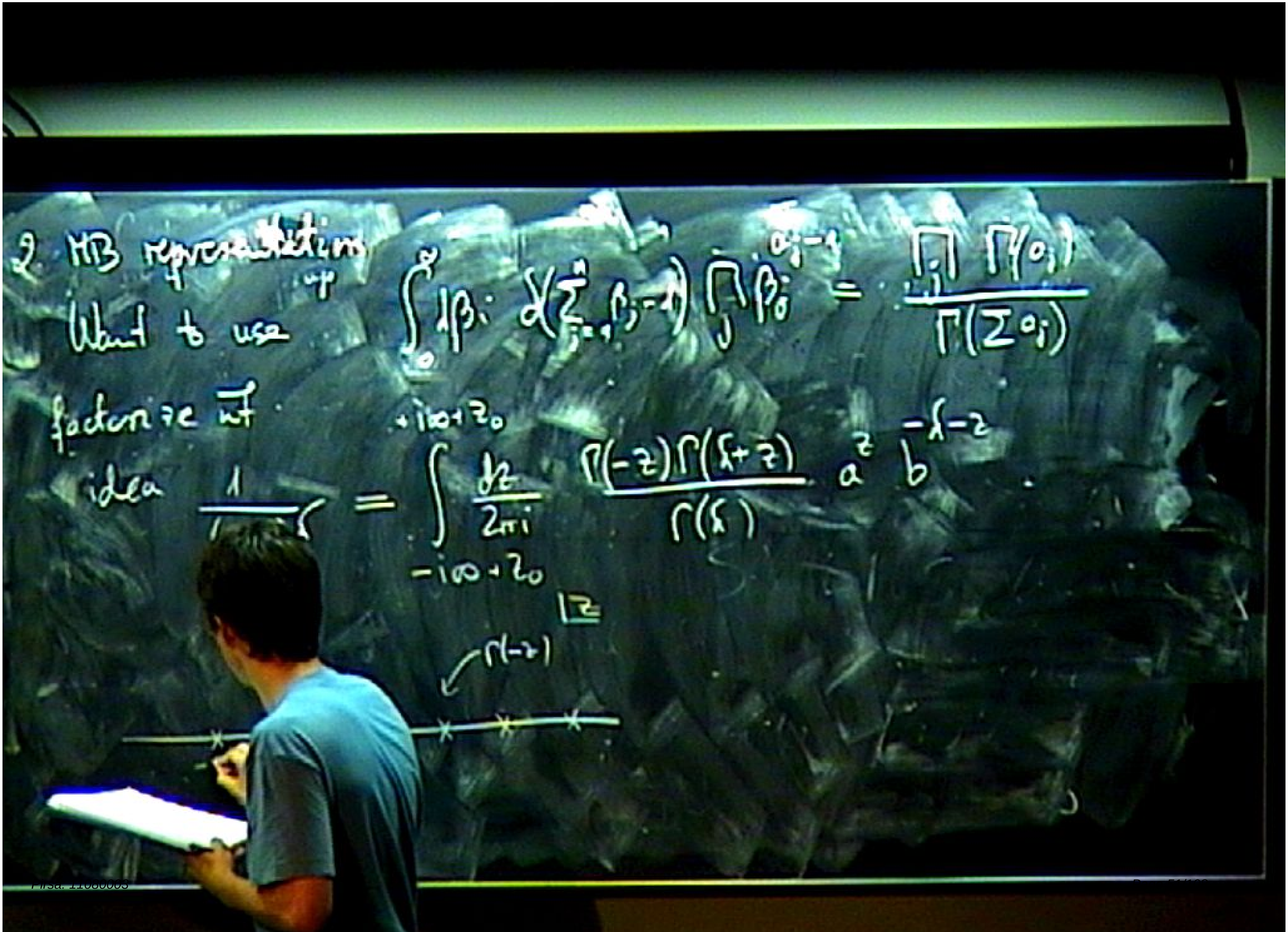


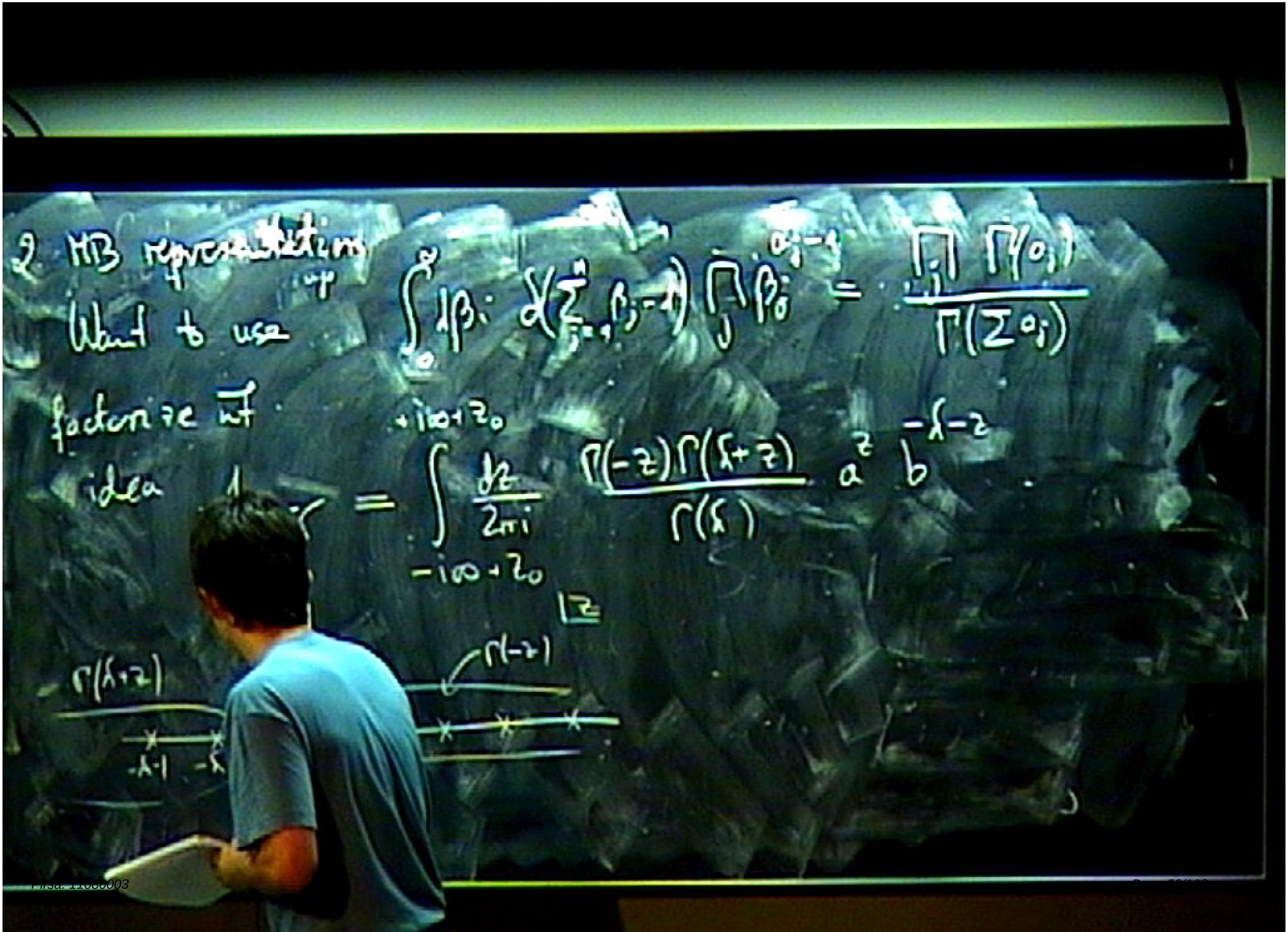
2. MB representation
Want to use $\int_0^\infty d\beta_i\, \delta(\sum_{j=1}^{n}\beta_j - 1) \prod_j \beta_j^{a_j - 1}$

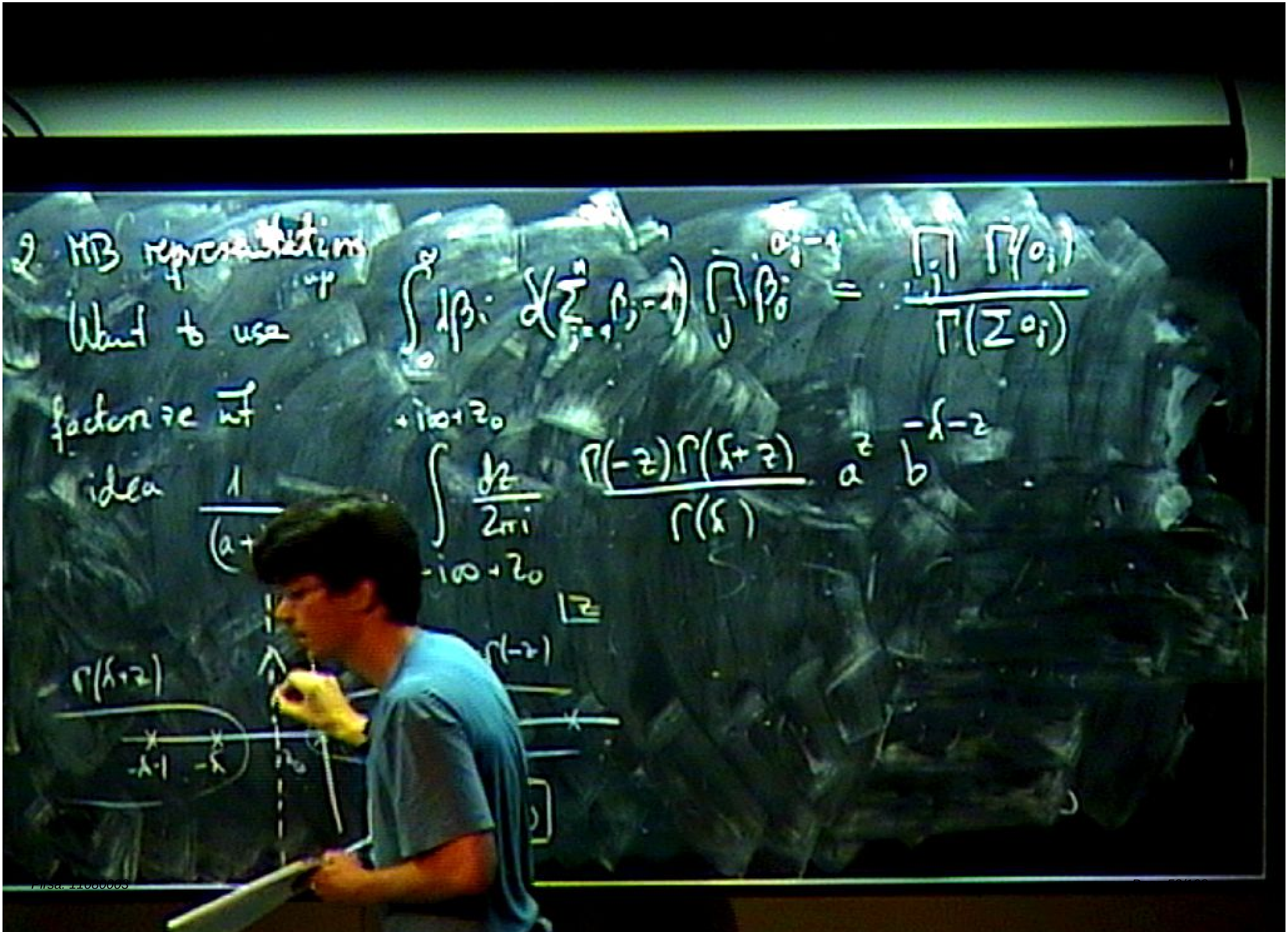

2. MB representation
Want to use $\int_0^\infty d\beta_i \; \delta\left(\sum_{j=1}^{n}\beta_j - 1\right) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma\left(\sum a_j\right)}$


2. MB representation
Want to use $\int_0^\infty d\beta_i\, \delta(\sum_{i=1}^{n} \beta_i - 1) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma(\sum a_j)}$
factorize int
idea $\frac{1}{(a+b)^\lambda} =$

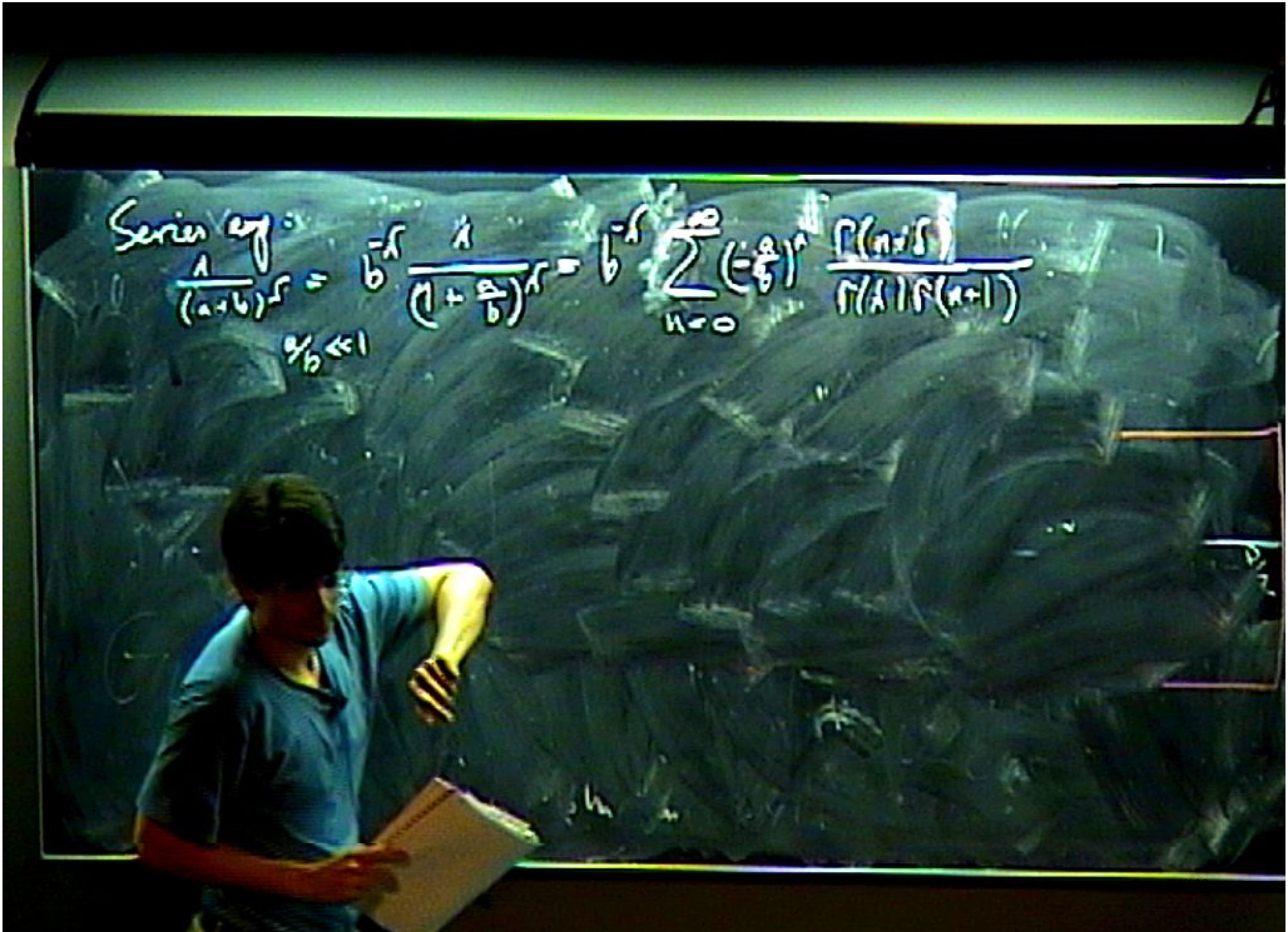

2. MB representation
Want to use $\int_0^\infty d\beta_i\, \delta(\sum_{j=1}^{n} \beta_j - 1) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma(\sum a_j)}$
factorize wrt
idea $\frac{1}{(a+b)^\lambda} = \int_{-i}^{+i\infty+z_0} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)}\, a^z\, b^{-\lambda-z}$

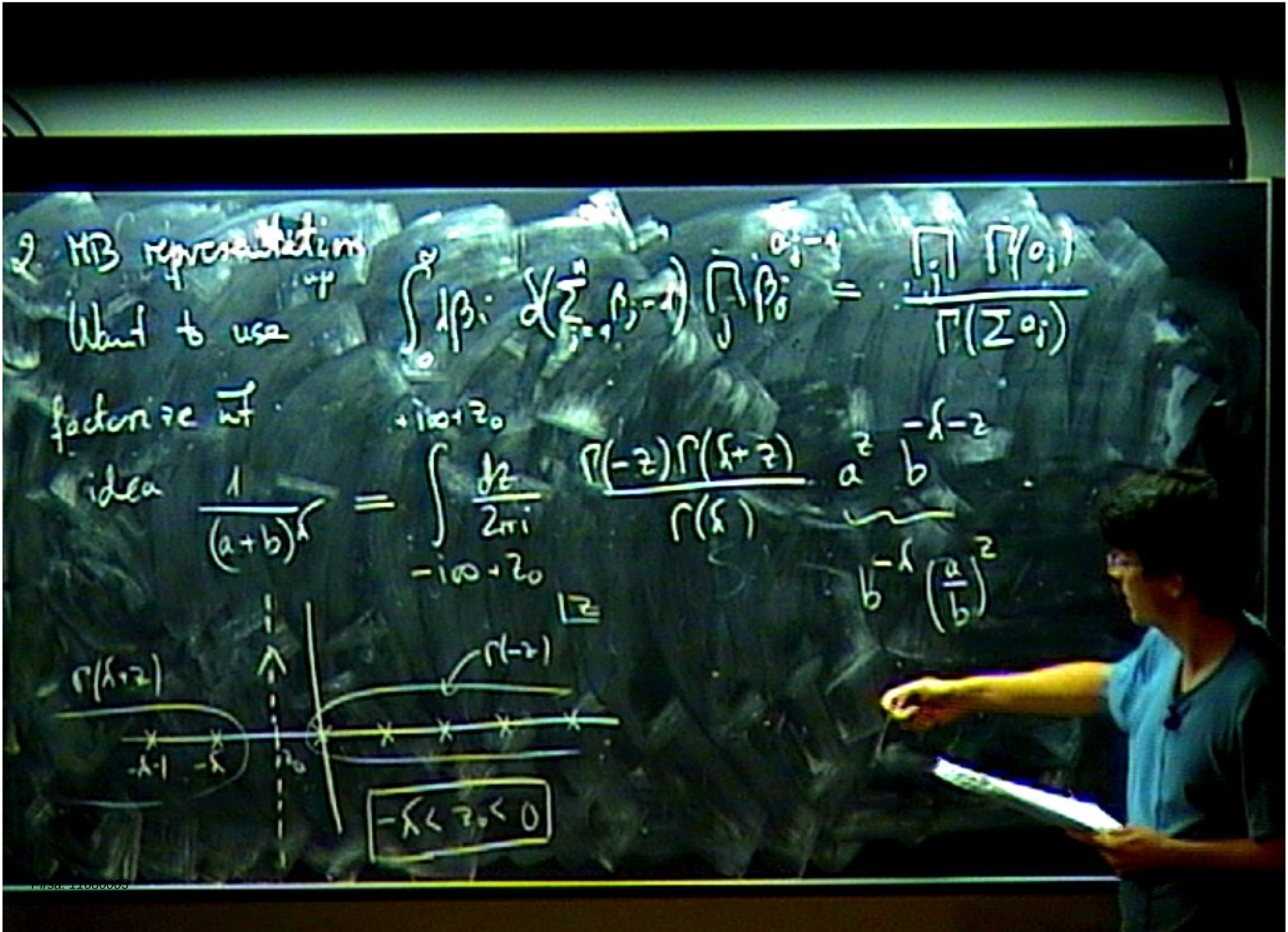

2. MB representation
Want to use $\int_0^\infty d\beta_i\, \delta\left(\sum_{j=1}^n \beta_j - 1\right) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma(\sum a_j)}$
idea $= \int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z}$
$\Gamma(-z)$


2. MB representation
Want to use
factorize int
idea
Γ(−z)Γ(λ+z)
Γ(λ)
Γ(λ+z)
Γ(−z)
−λ−1
−λ

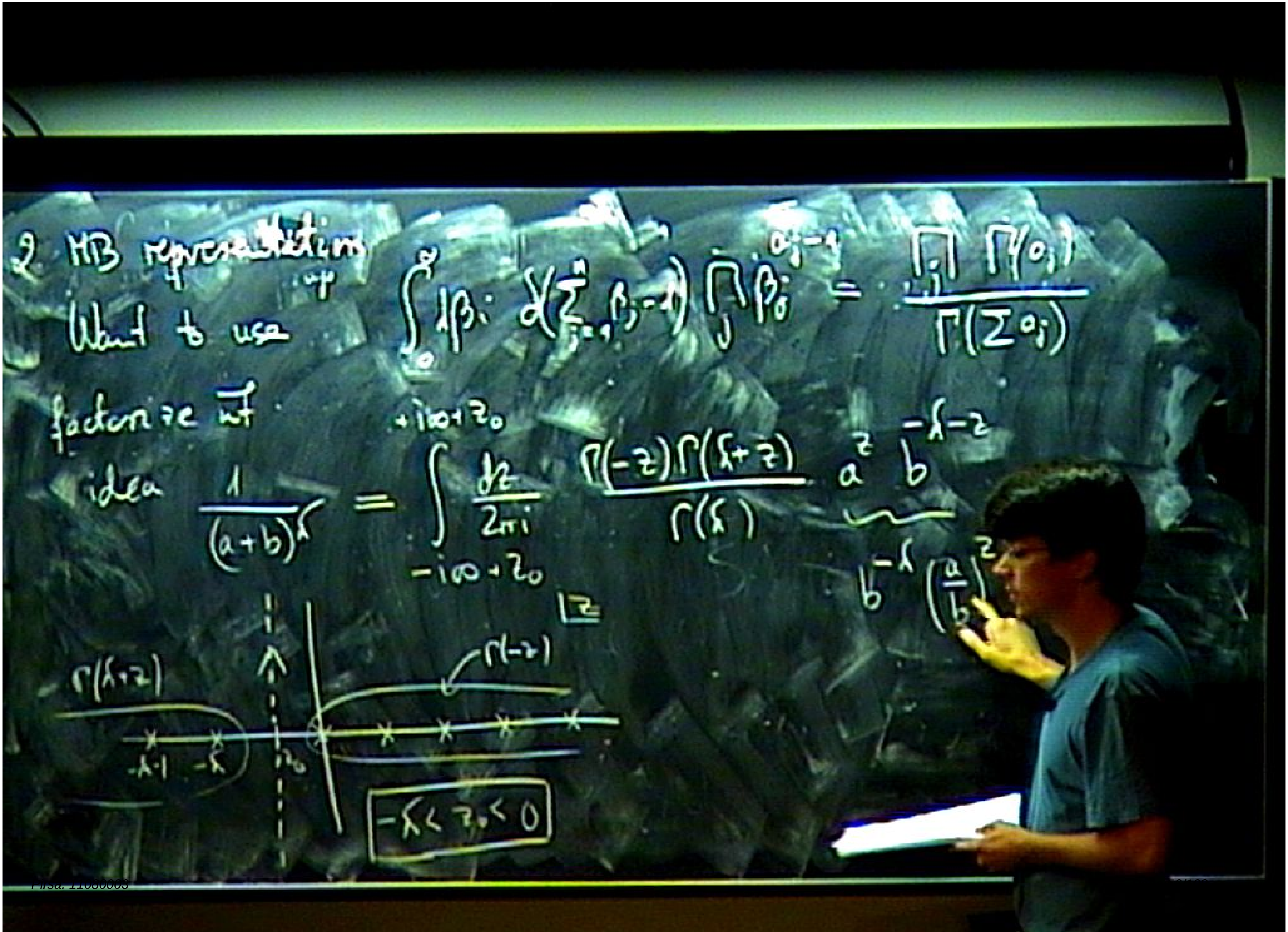


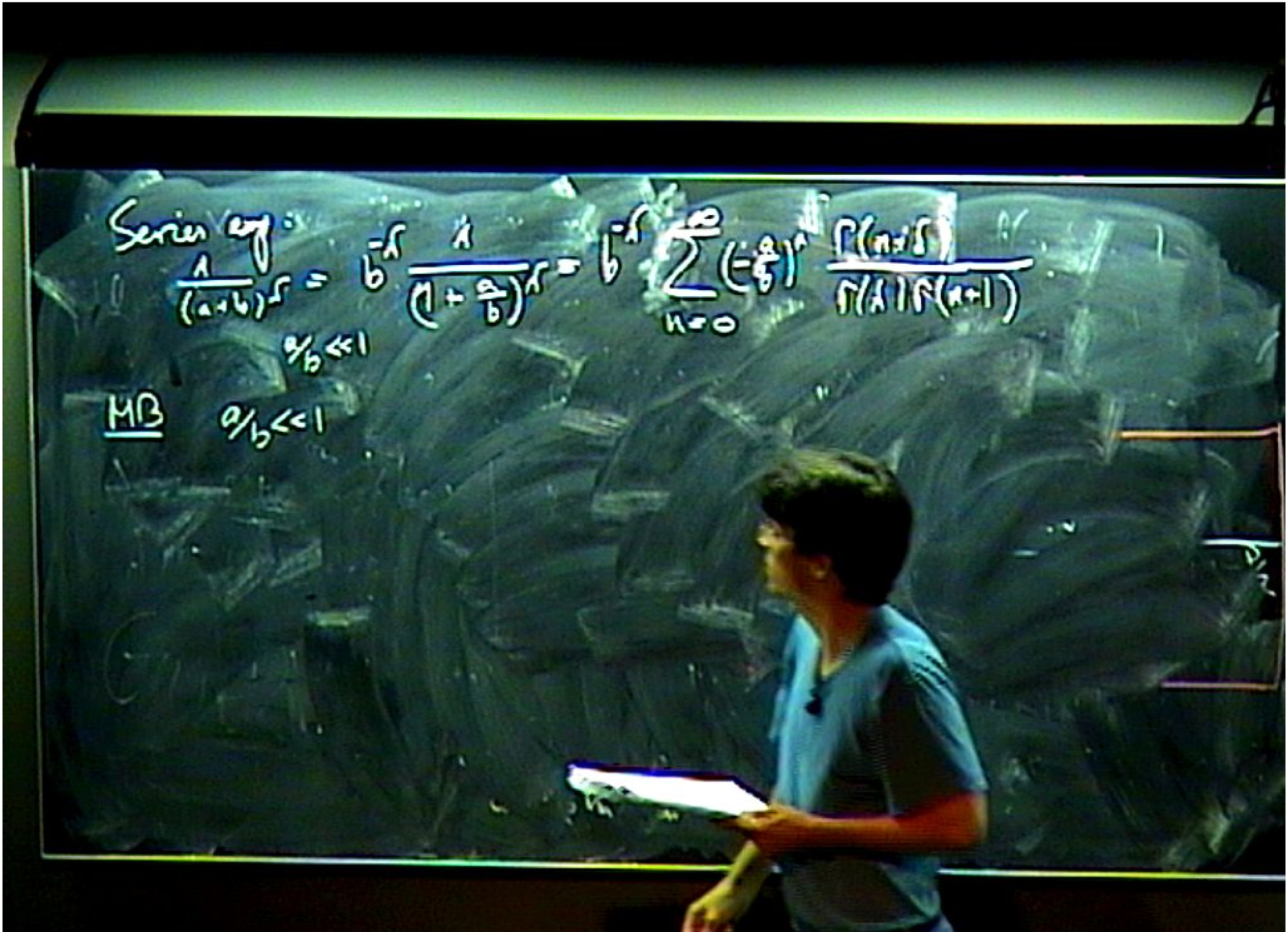
2. MB representation
Want to use $\int_0^\infty d\beta_i\, \delta\left(\sum_{j=1}^n \beta_j - 1\right) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma(\sum a_j)}$
factorize int
idea
$\frac{1}{(a+}$
$\int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z}$
$\Gamma(\lambda+z)$
$-\lambda-1$
$-\lambda$
z_0
$\Gamma(-z)$

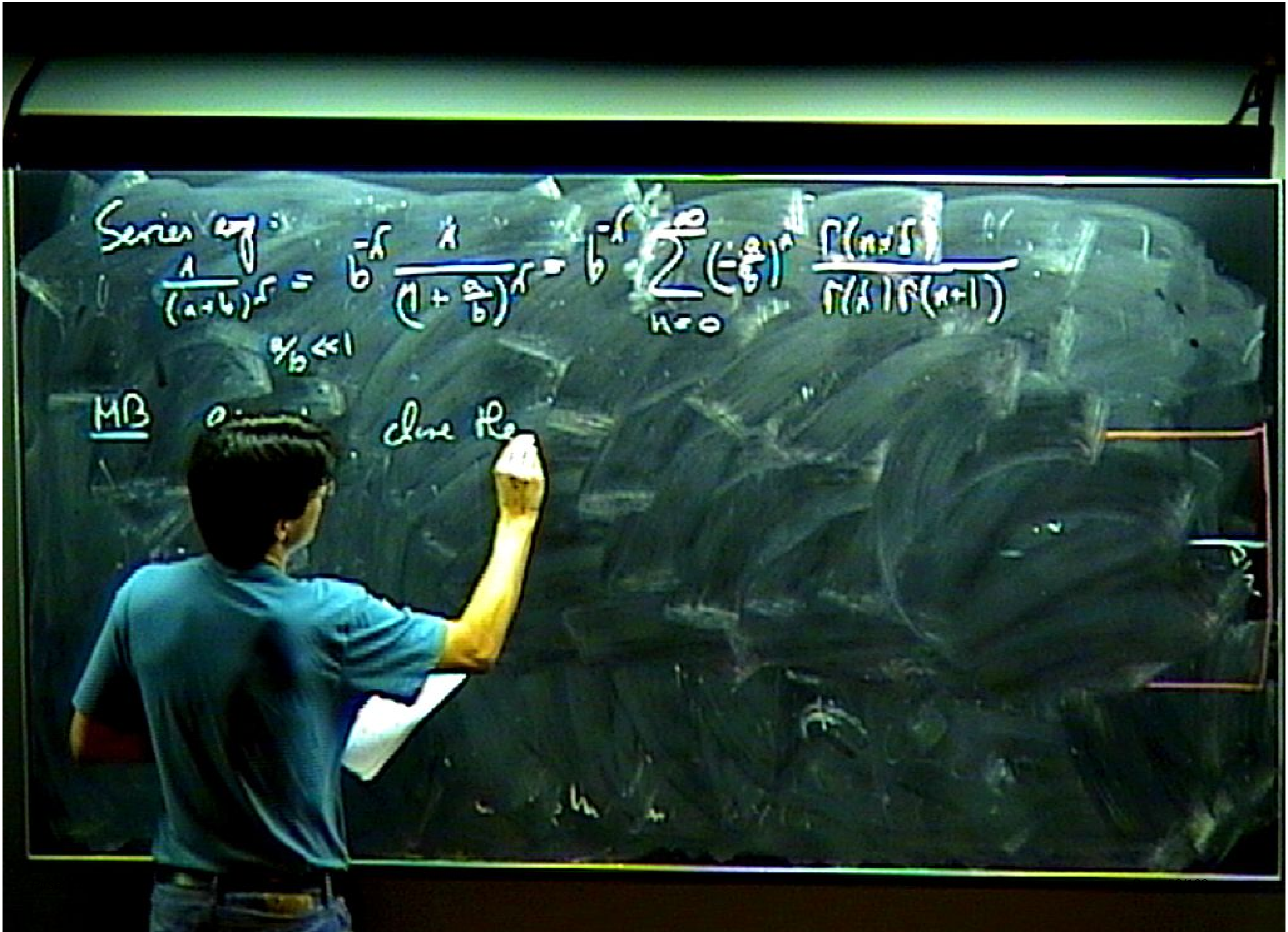

Series eg:
$\frac{1}{(a+b)^\lambda} = b^{-\lambda}\frac{1}{(1+\frac{a}{b})^\lambda} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$
$a/b \ll 1$



Pirsa: 11080003


2. MB representation
Want to use $\int_0^\infty d\beta_i \, \delta\left(\sum_{j=1}^n \beta_j - 1\right) \prod_j \beta_j^{a_j-1} = \frac{\prod_j \Gamma(a_j)}{\Gamma(\sum a_j)}$
factorize wrt
idea $\frac{1}{(a+b)^\lambda} = \int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} \underbrace{a^z b^{-\lambda-z}}_{b^{-\lambda}\left(\frac{a}{b}\right)^z}$
z
$\Gamma(\lambda+z)$
$\Gamma(-z)$
$-\lambda-1$
$-\lambda$
z_0
$\boxed{-\lambda < z_0 < 0}$


Series exp:
$\frac{1}{(a+b)^\lambda} = b^{-\lambda} \frac{1}{(1+\frac{a}{b})^\lambda} = b^{-\lambda} \sum_{n=0}^{\infty} (-\frac{a}{b})^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$
$a/b \ll 1$
MB $a/b \ll 1$


Series exp:
$\frac{1}{(a+b)^{\lambda}} = b^{-\lambda}\frac{1}{(1+\frac{a}{b})^{\lambda}} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^{n}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$
$a/b \ll 1$
MB
close the

Series exp.:

$$\frac{1}{(a+b)^{\lambda}} = b^{-\lambda}\frac{1}{\left(1+\frac{a}{b}\right)^{\lambda}} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^{n}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\,\Gamma(n+1)}$$

$a/b \ll 1$

<u>MB</u> <u>$a/b \ll 1$</u> close the int. contour on the right

Series exp:

$$\frac{1}{(a+b)^{\lambda}} = b^{-\lambda}\frac{1}{\left(1+\frac{a}{b}\right)^{\lambda}} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^{n}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$$

$a/b \ll 1$

MB $a/b \ll 1$: close the in. contour on the right

$=$

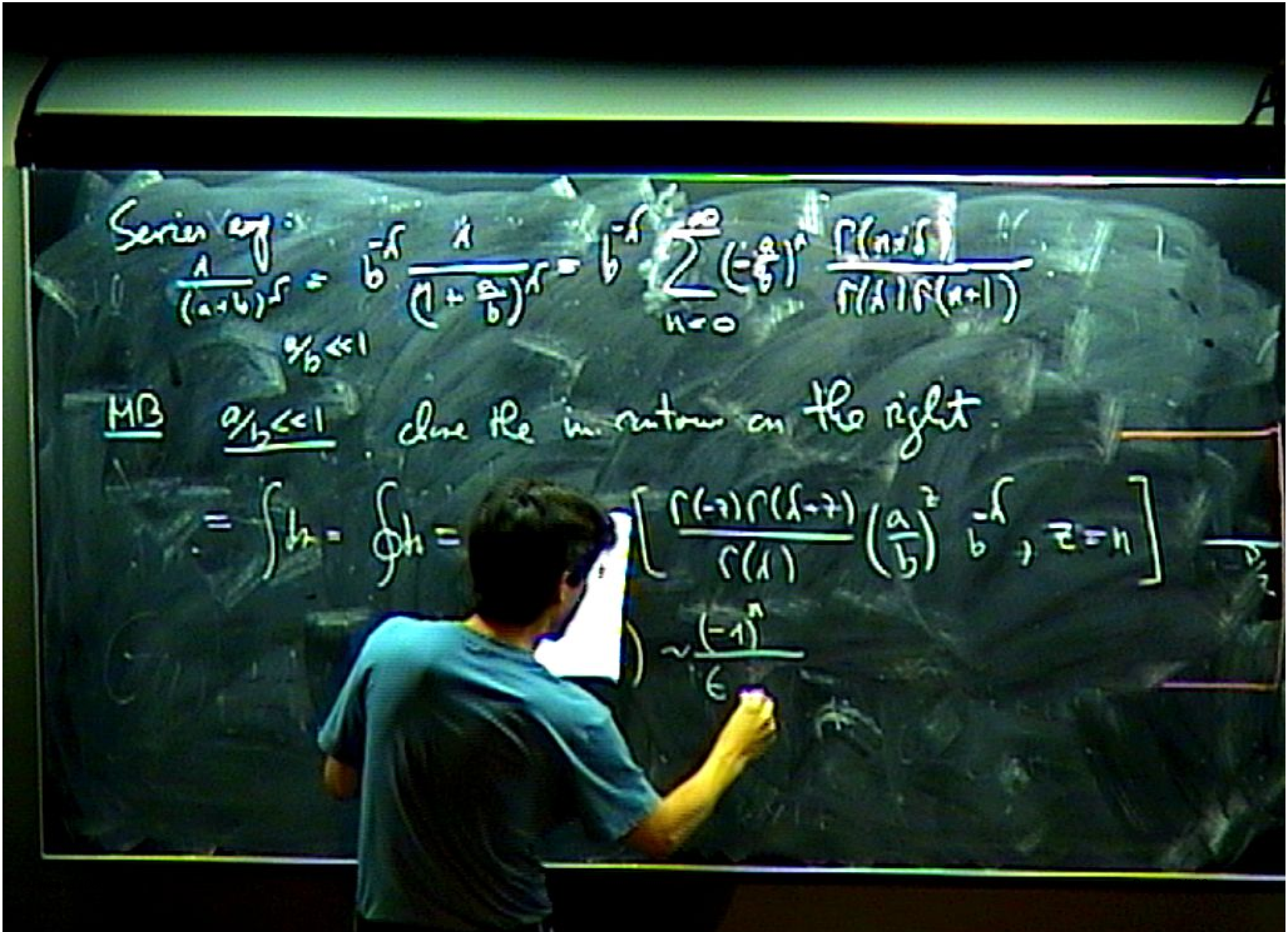
Series exp:

$$\frac{1}{(a+b)^\lambda} = b^{-\lambda}\frac{1}{(1+\frac{a}{b})^\lambda} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$$

$a/b \ll 1$

<u>MB</u> <u>$a/b \ll 1$</u> close the int. contour on the right

$$\ldots = \oint \ldots = \sum_{n=0}^{\infty} \mathrm{Res}\left[\frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)}\left(\frac{a}{b}\right)^z b^{-\lambda}, z=n\right]$$


Series exp:
$\frac{1}{(a+b)^\lambda} = b^{-\lambda}\frac{1}{(1+\frac{a}{b})^\lambda} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$
$a/b \ll 1$
MB $a/b \ll 1$ close the in. contour on the right
$= \int h = \oint h = \left[\frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)}\left(\frac{a}{b}\right)^z b^{-\lambda},\ z=n\right]$
$\sim \frac{(-1)^n}{\epsilon}$

Series eq:

$$\frac{1}{(a+b)^{\lambda}} = b^{-\lambda}\frac{1}{\left(1+\frac{a}{b}\right)^{\lambda}} = b^{-\lambda}\sum_{n=0}^{\infty}\left(-\frac{a}{b}\right)^{n}\frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$$

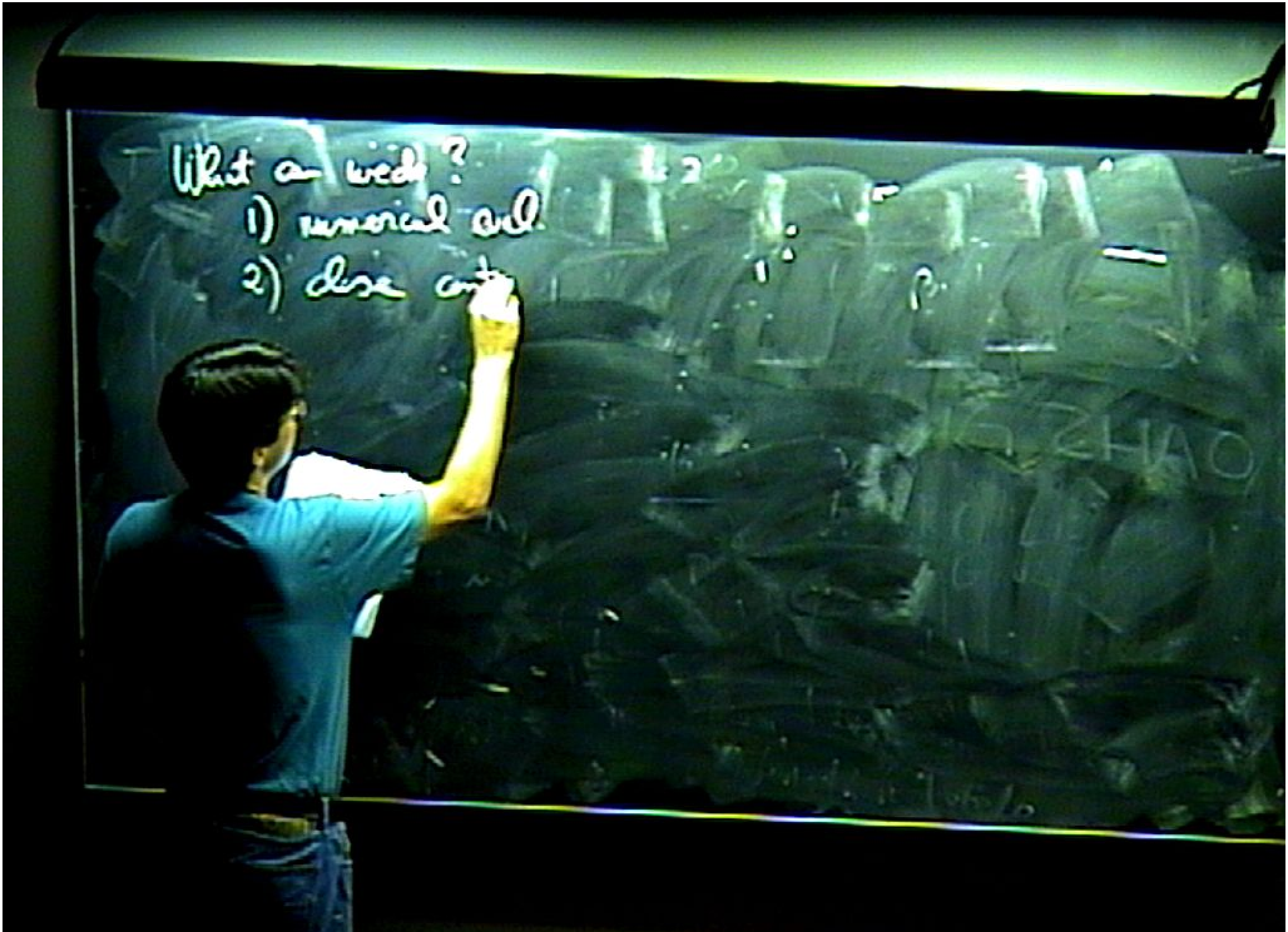
$a/b \ll 1$

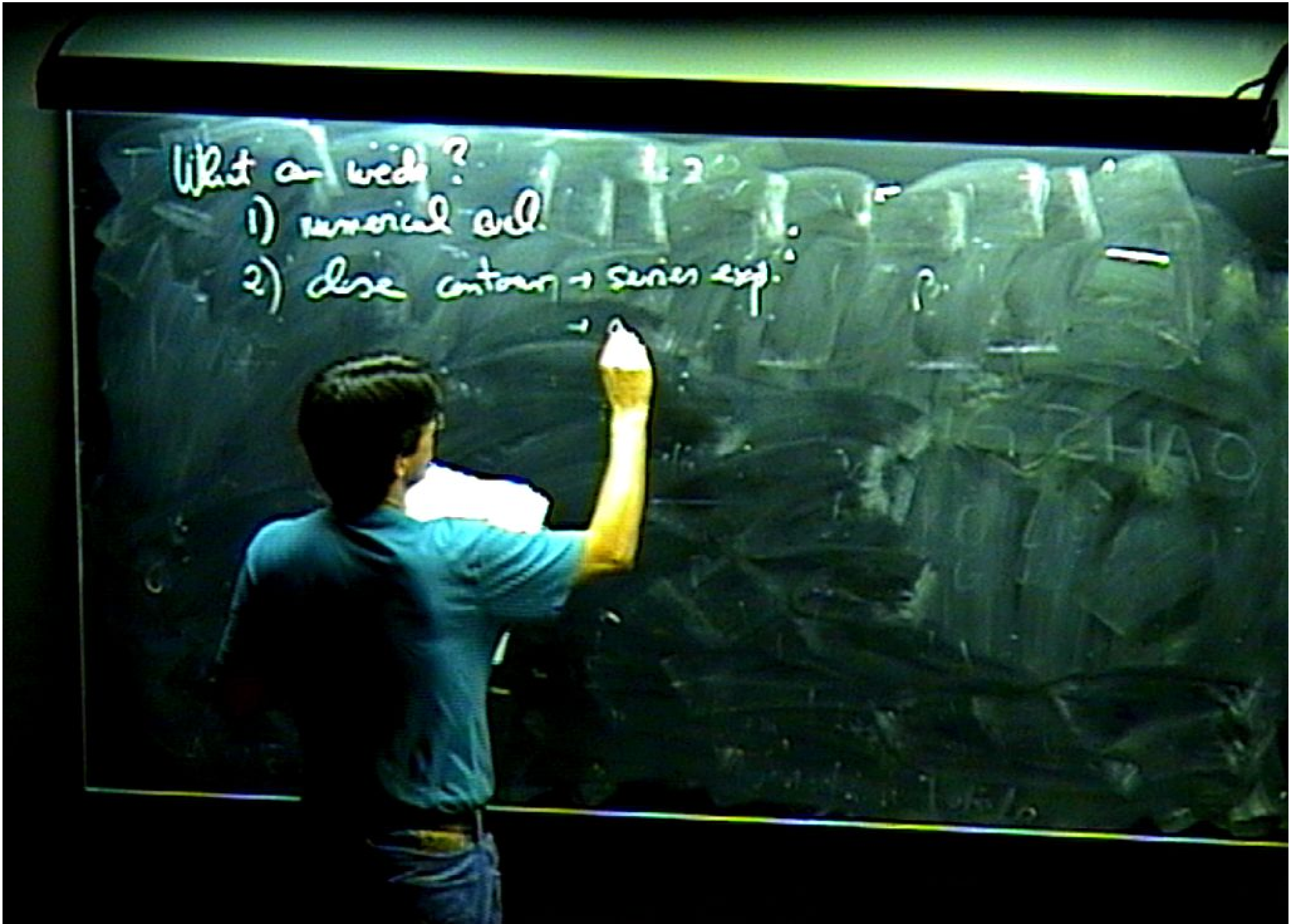
MB $\quad$ $a/b \ll 1$ $\quad$ close the in. contour on the right

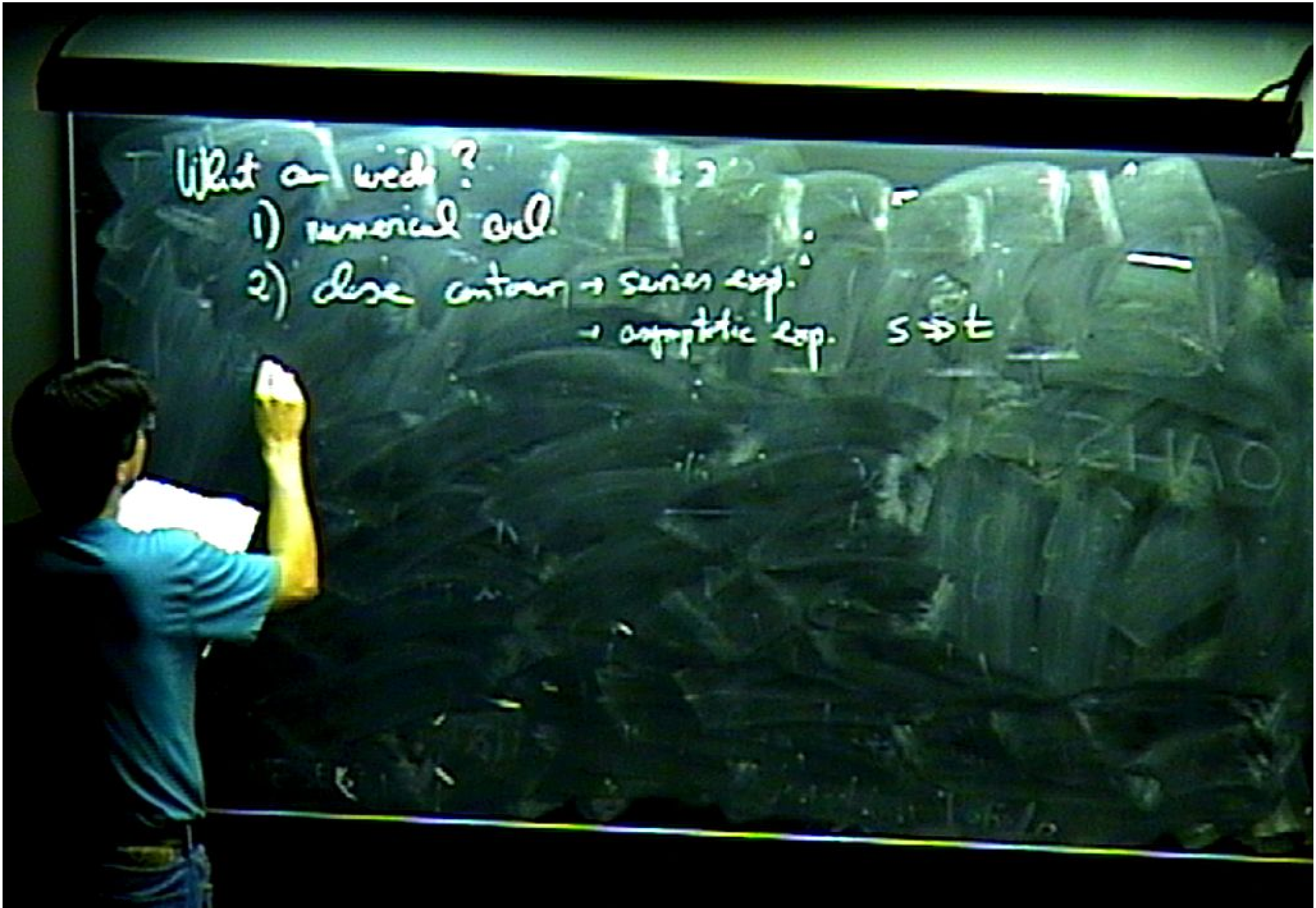
$$= \int h - \oint h = \sum_{n=0}^{\infty} \mathrm{Res}\left[\frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)}\left(\frac{a}{b}\right)^{z} b^{-\lambda},\ z=n\right]$$

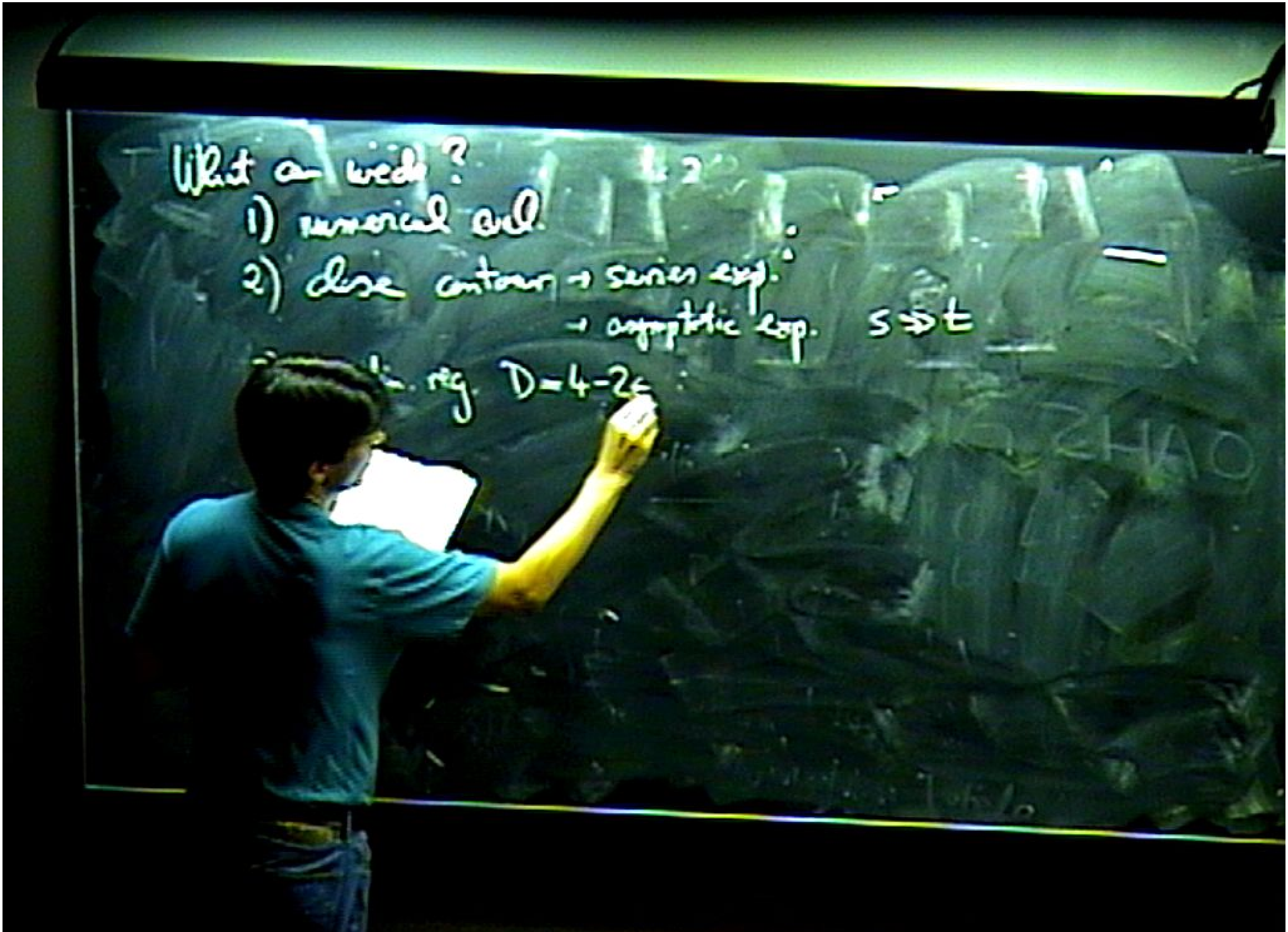
$$\Gamma(-n+\epsilon) \sim \frac{(-1)^{n}}{\epsilon\,\Gamma(n+1)}$$

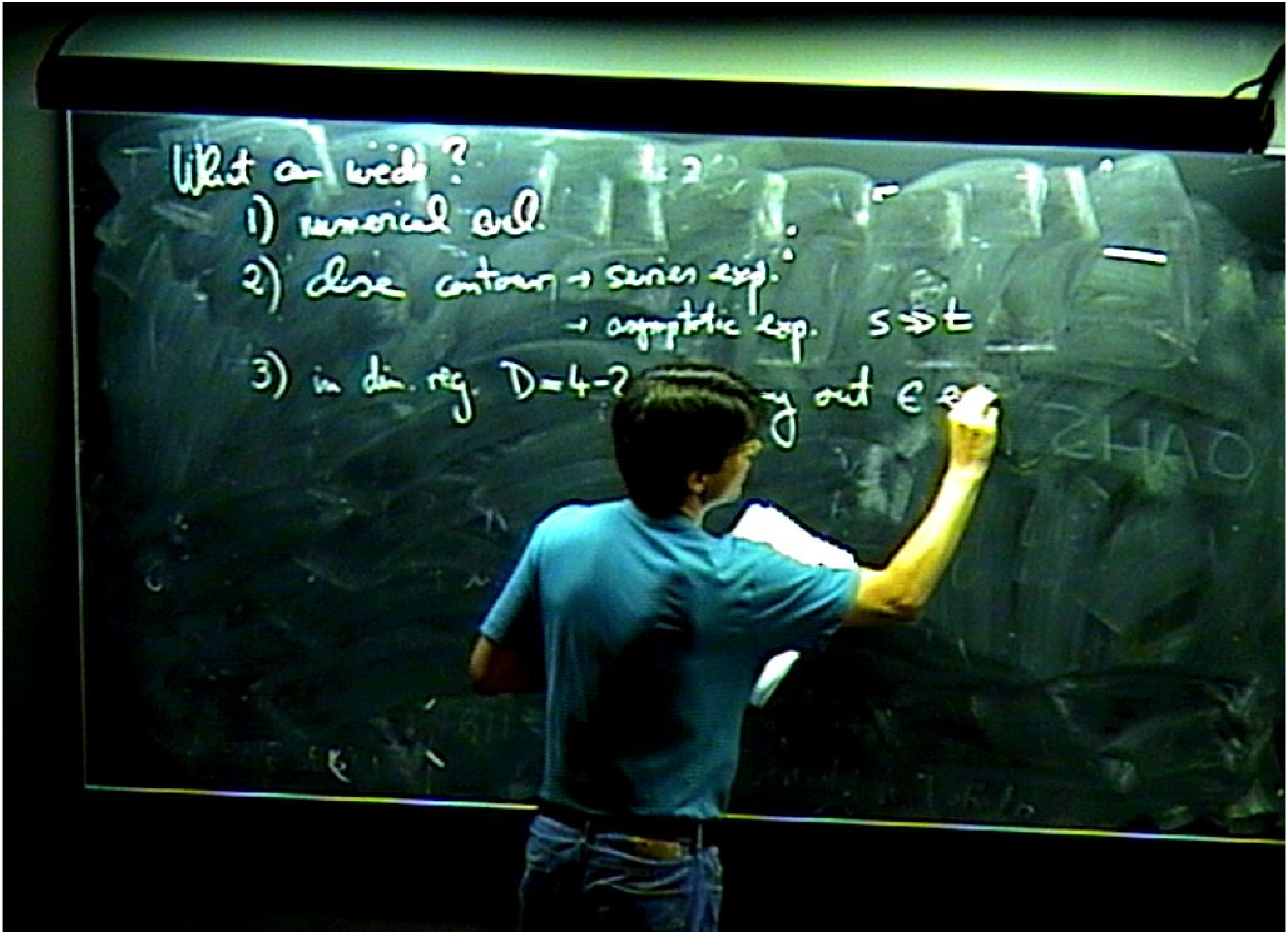
Pirsa: 11080003


ZHAO


What are weds ?
1) numerical eval.
2) close contour → series exp.


What can we do?
1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. $s \gg t$


1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. s >> t
reg. D=4-2ε

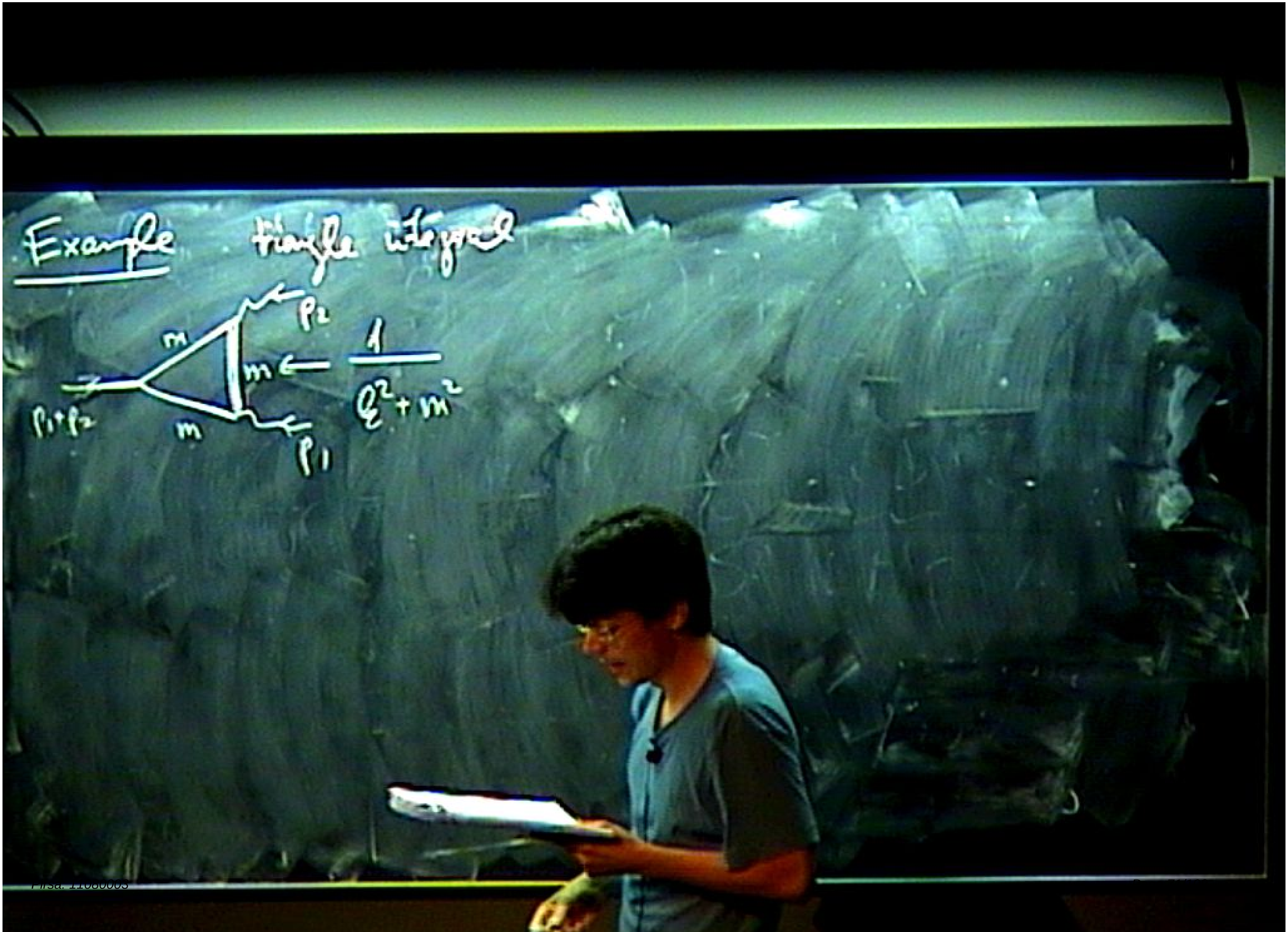

1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. s >> t
3) in dim. reg. D=4-2
out ε

What can we do?

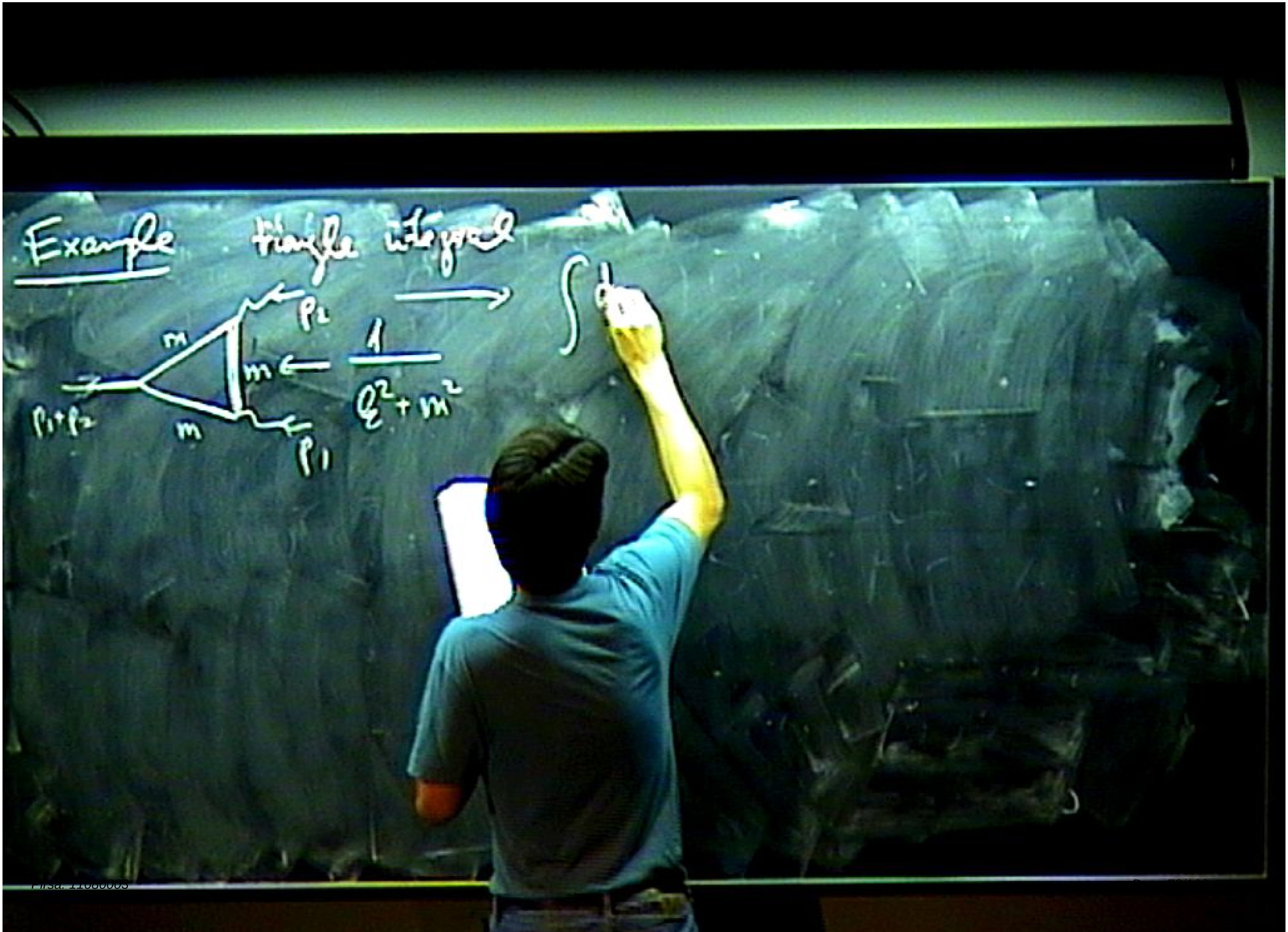
1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. $s \gg t$
3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion

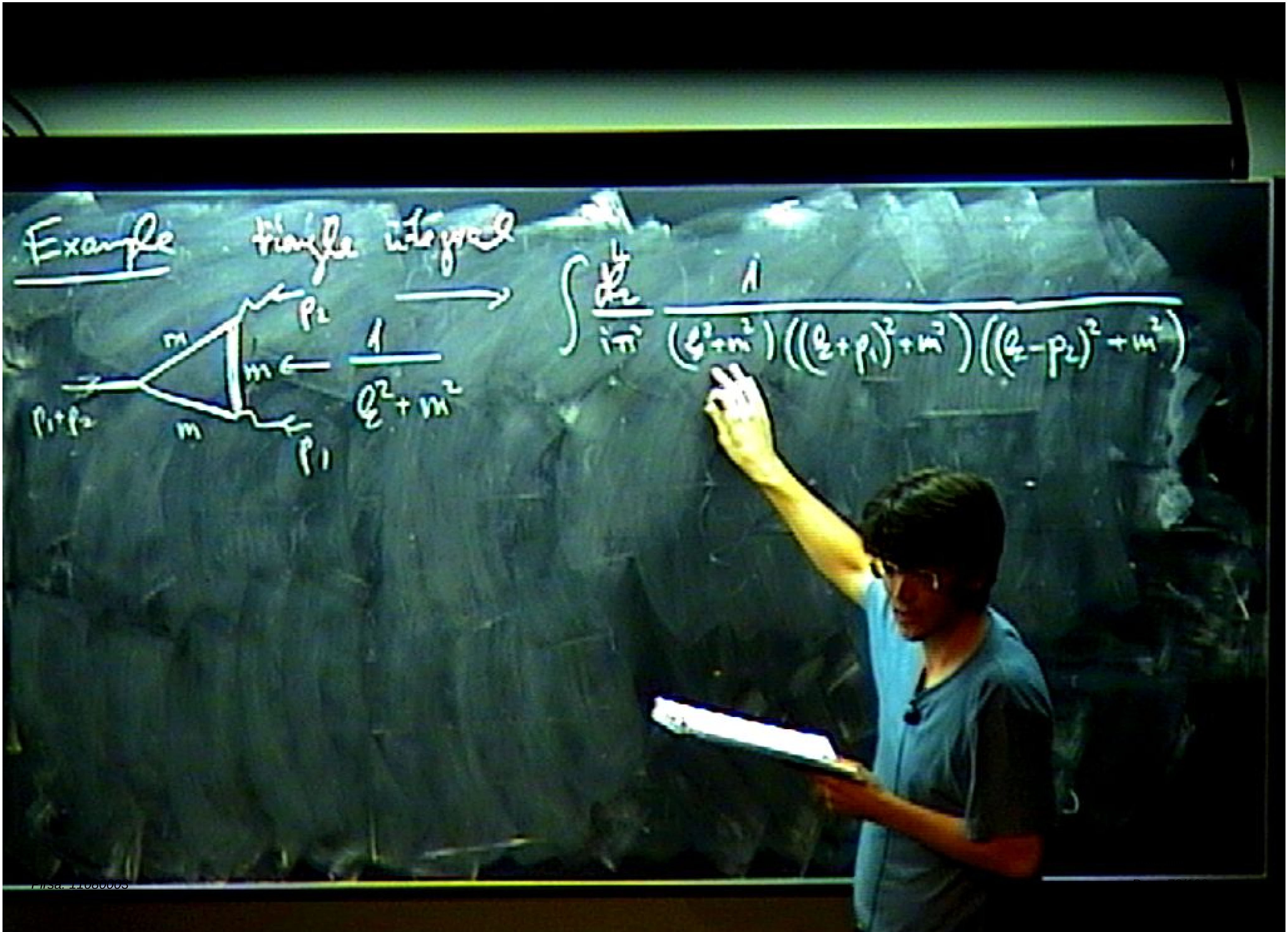
1)


Example

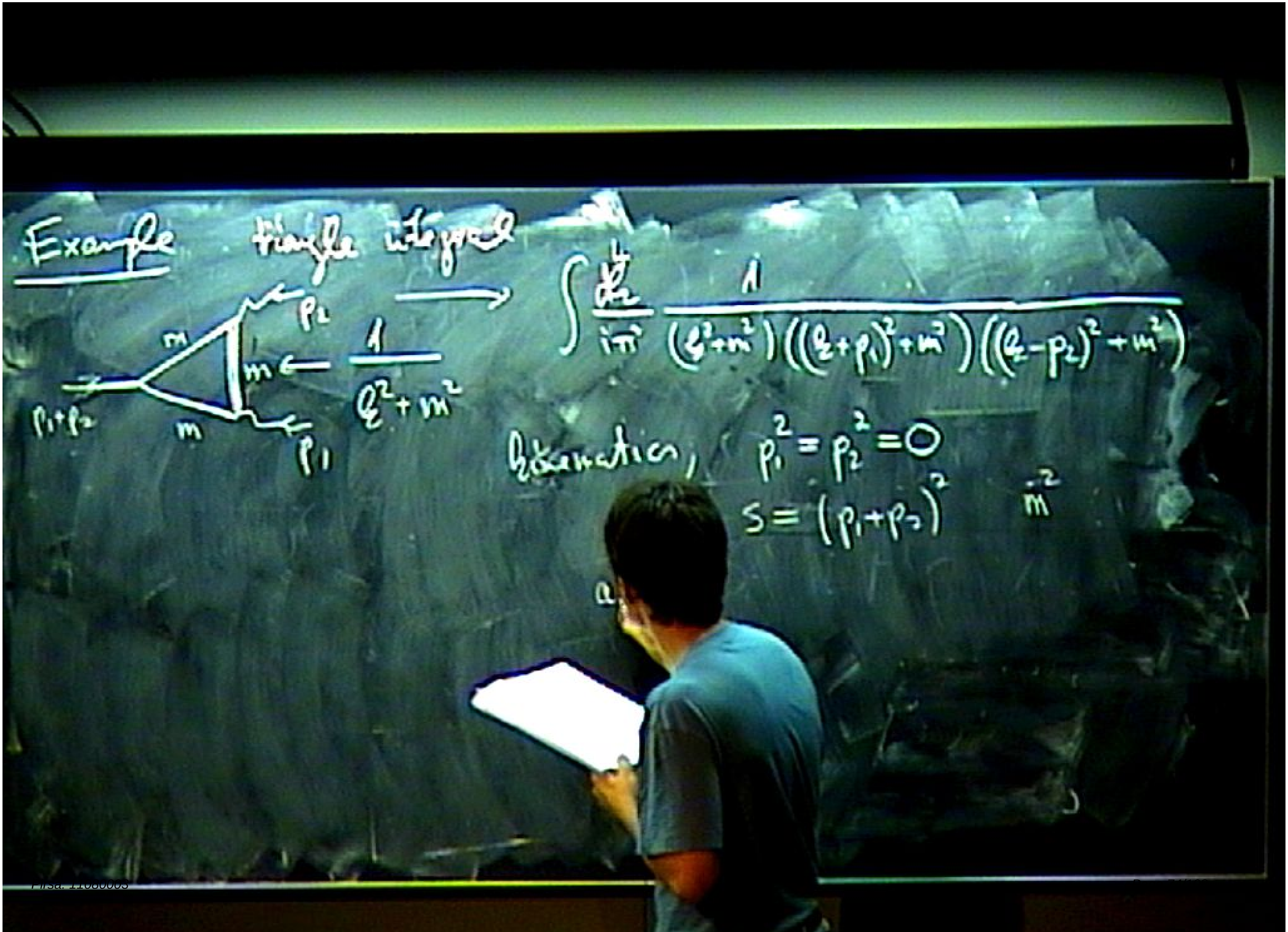


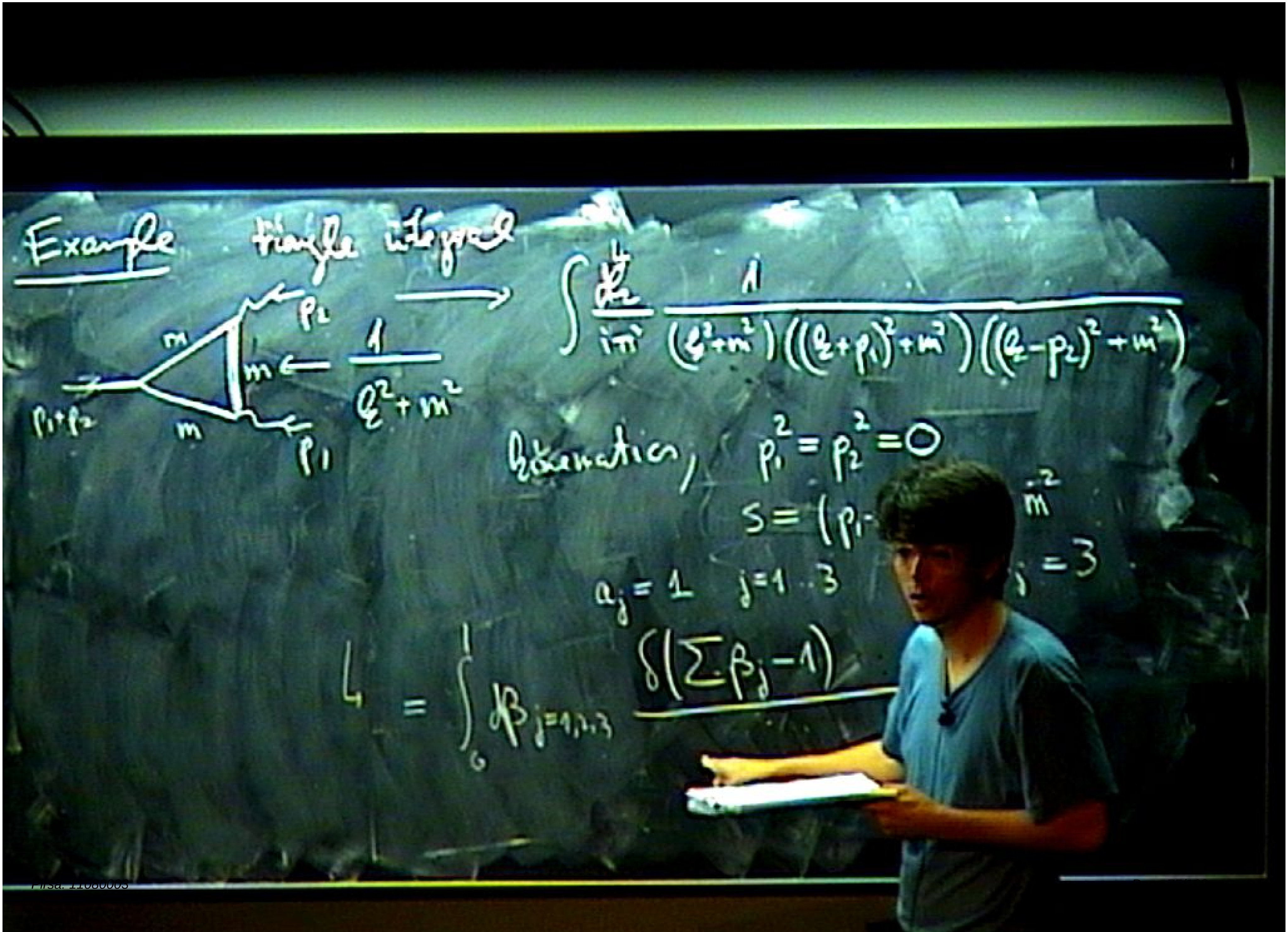
Example
triangle integral
p2
m
m
1
ℓ² + m²
p1+p2
m
p1


Example triangle integral
p2
p1+p2
m
m
m
p1
1
l2 + m2

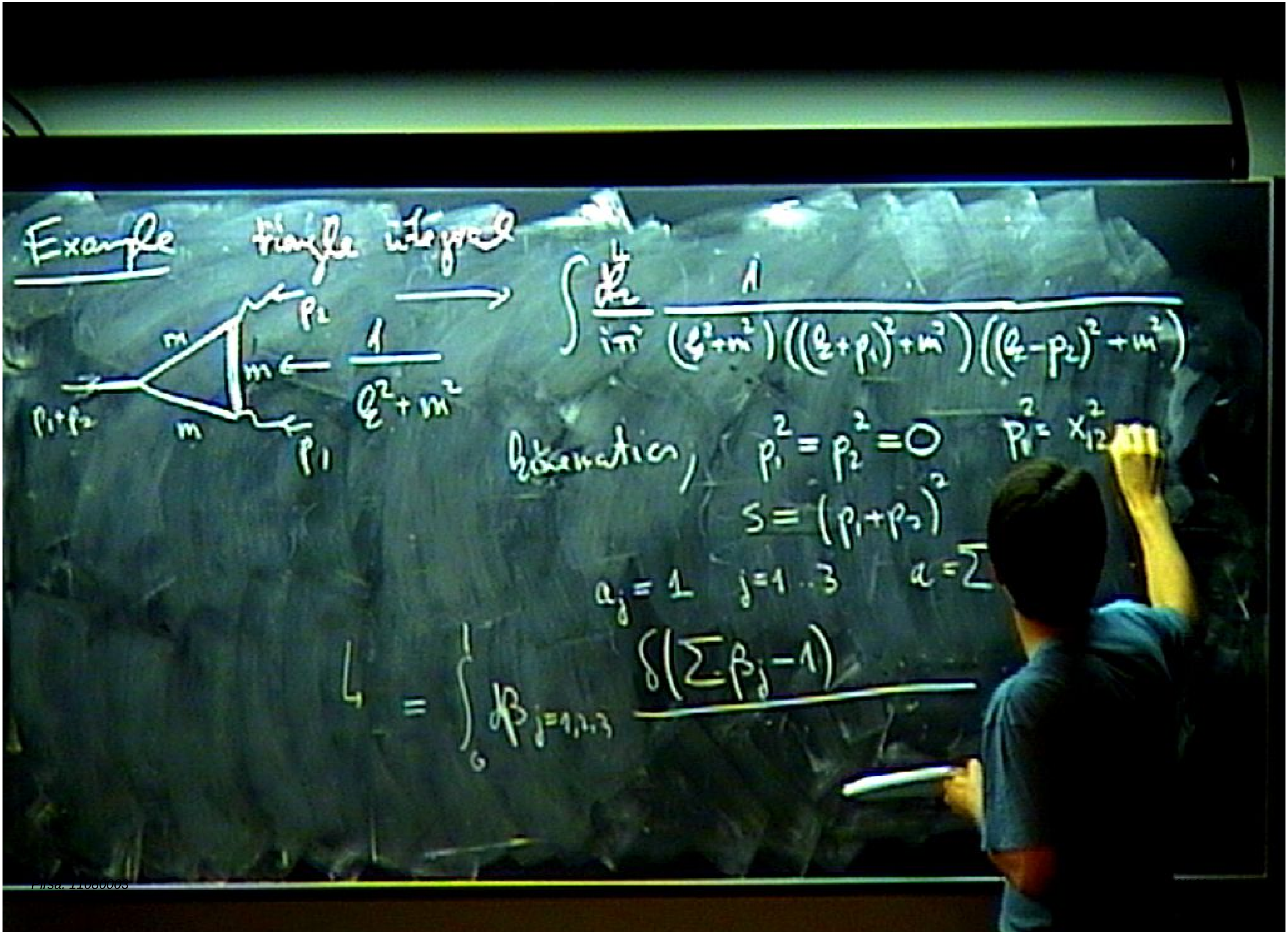


Example triangle integral
p_2
m
m
$m \leftarrow \frac{1}{\ell^2+m^2}$
p_1+p_2
p_1
$\int \frac{d^4\ell}{i\pi^2} \frac{1}{(\ell^2+m^2)((\ell+p_1)^2+m^2)((\ell-p_2)^2+m^2)}$

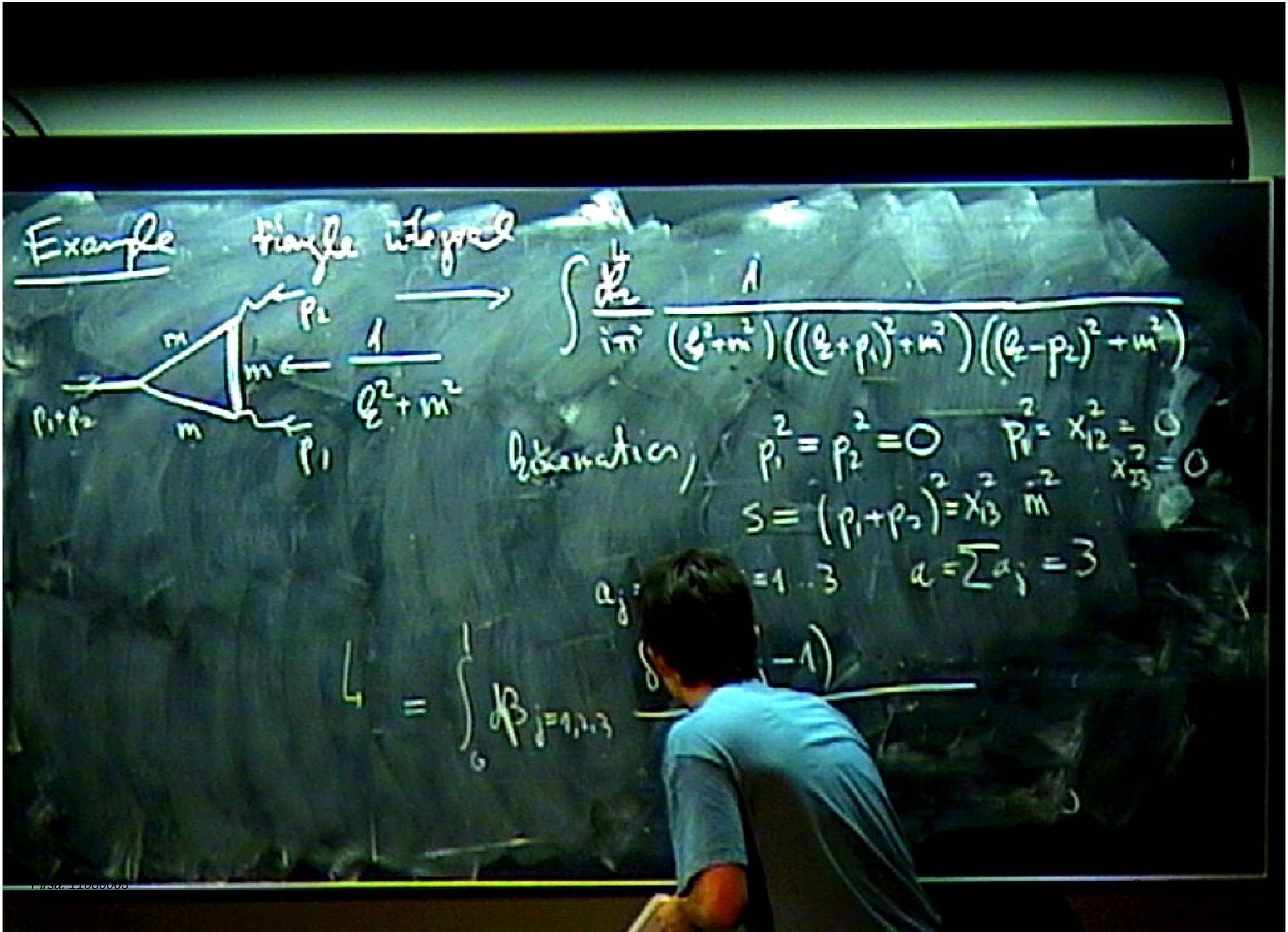

Example triangle integral
p_2
m
$m \leftarrow \frac{1}{\ell^2+m^2}$
m
p_1+p_2
p_1
$\int \frac{d^4\ell}{i\pi^2} \frac{1}{(\ell^2+m^2)((\ell+p_1)^2+m^2)((\ell-p_2)^2+m^2)}$
Kinematics, $p_1^2 = p_2^2 = 0$
$s = (p_1+p_2)^2$ m^2

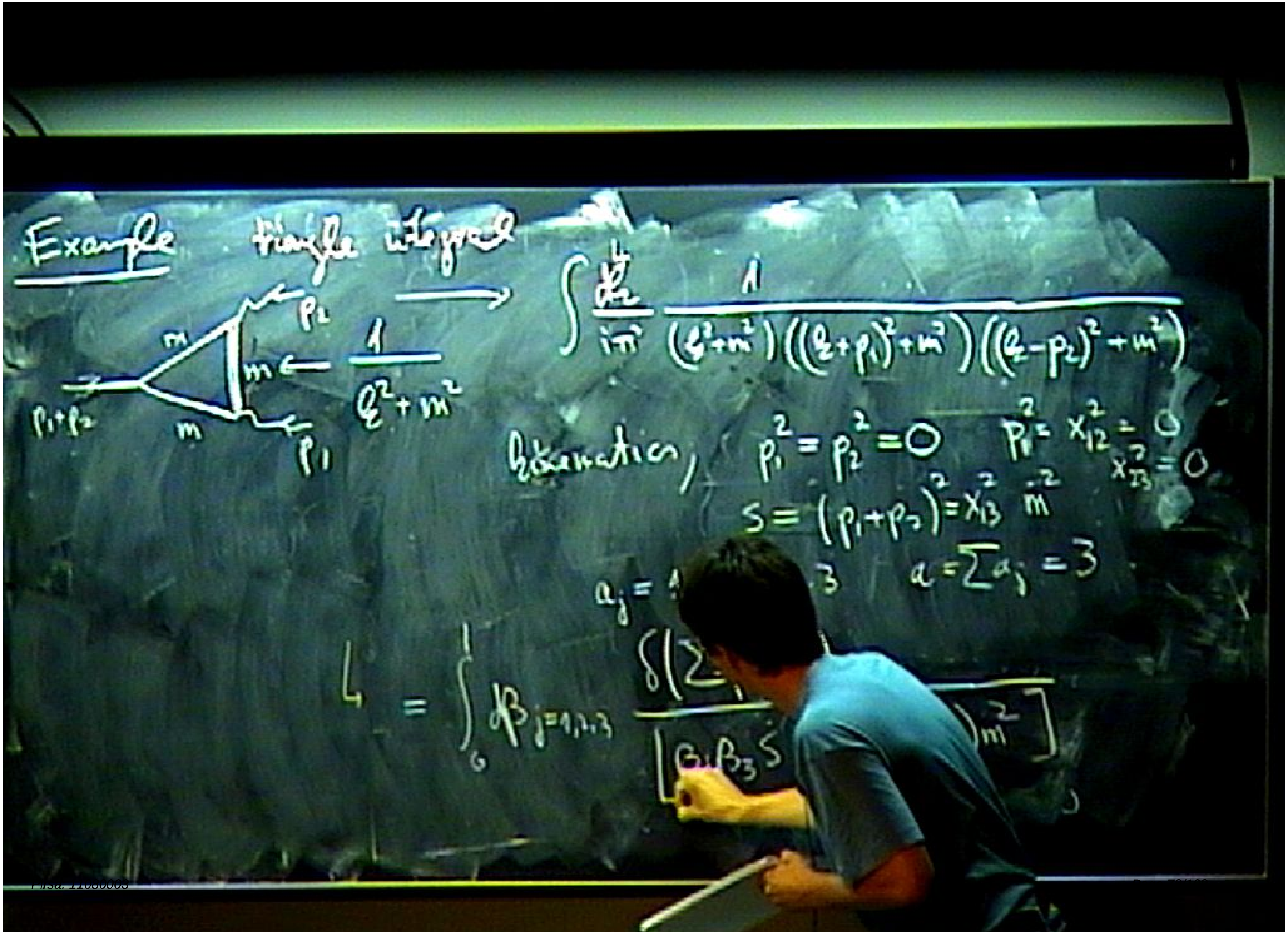


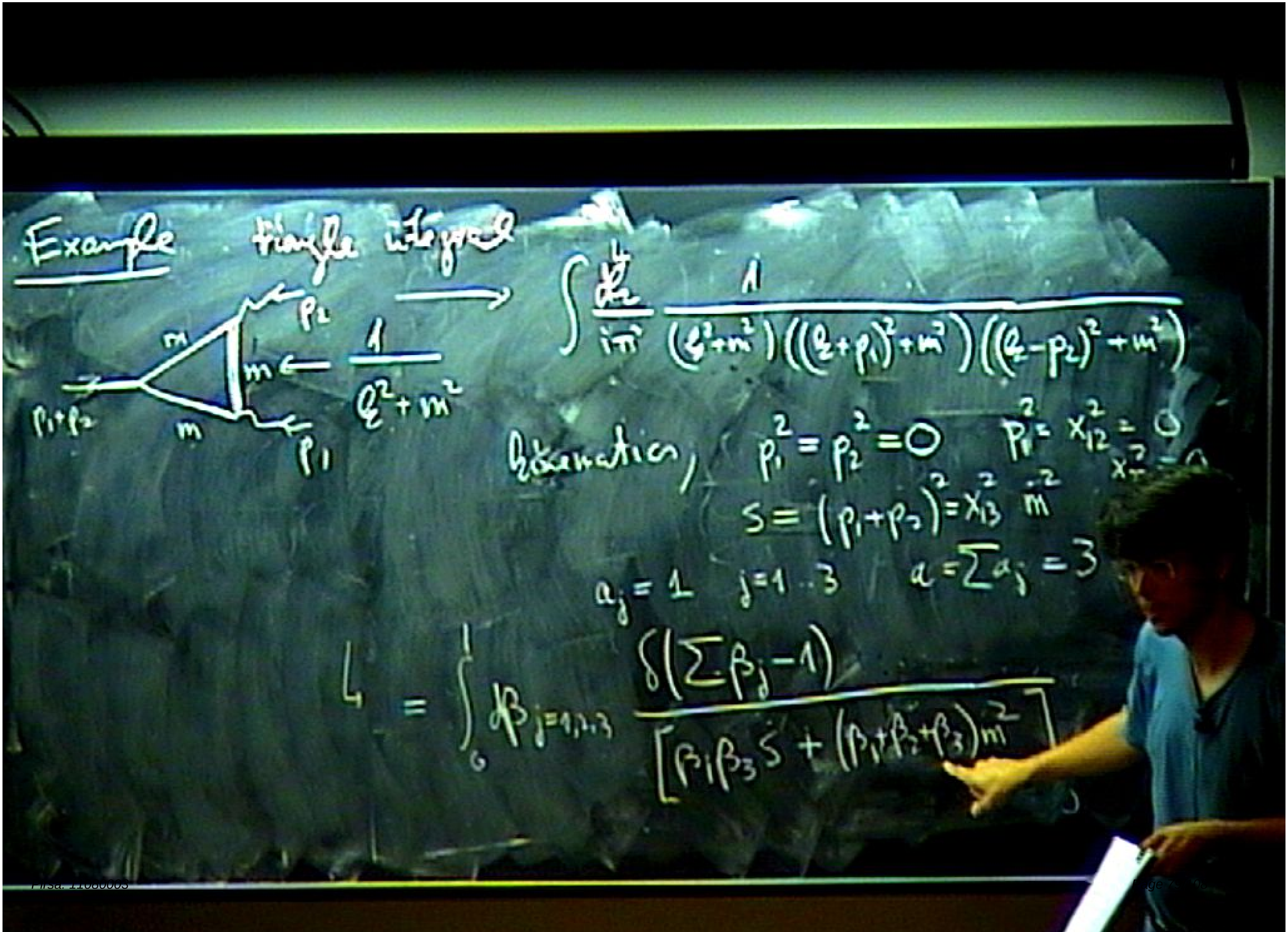
Pirsa: 11080063

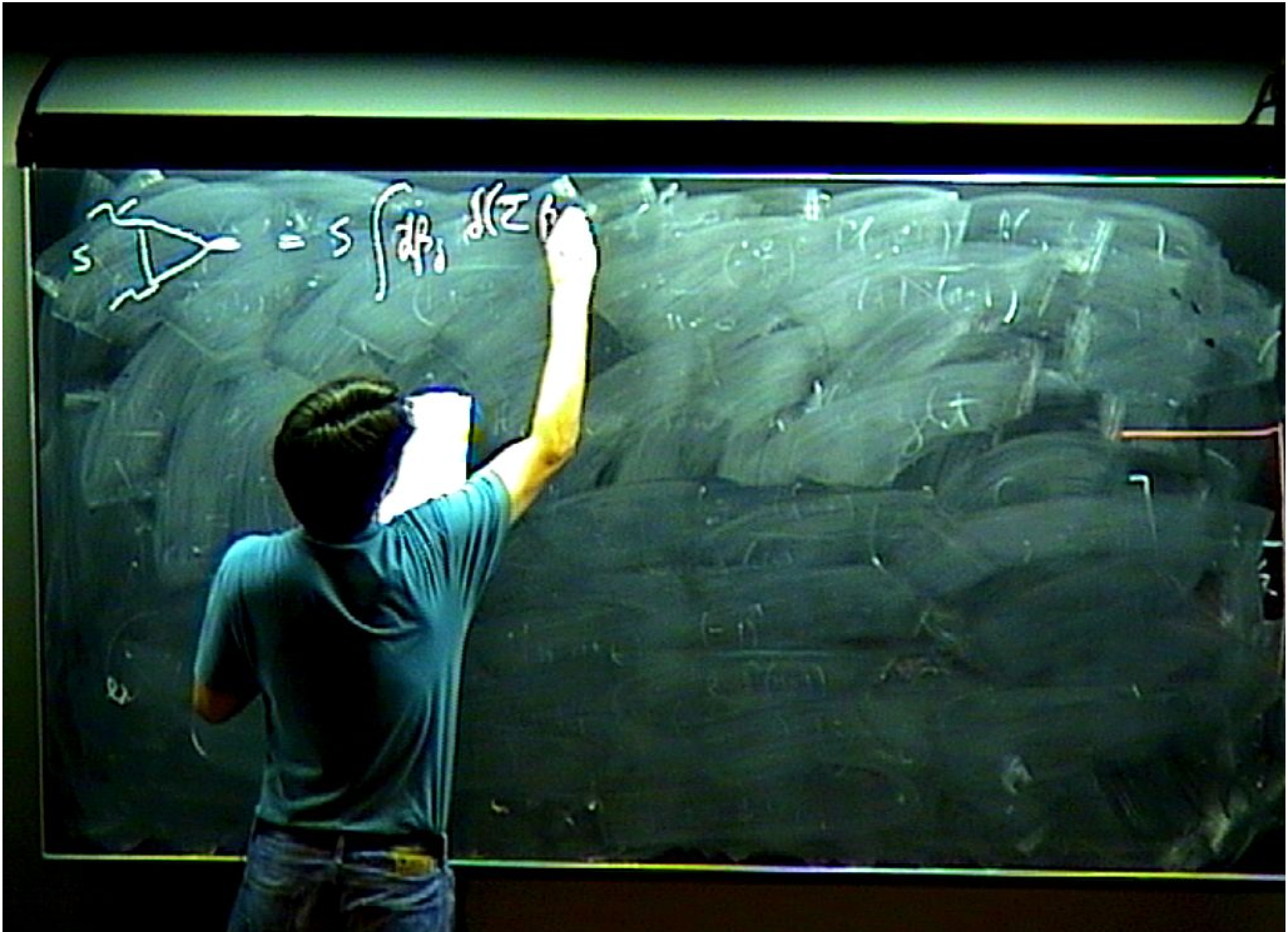


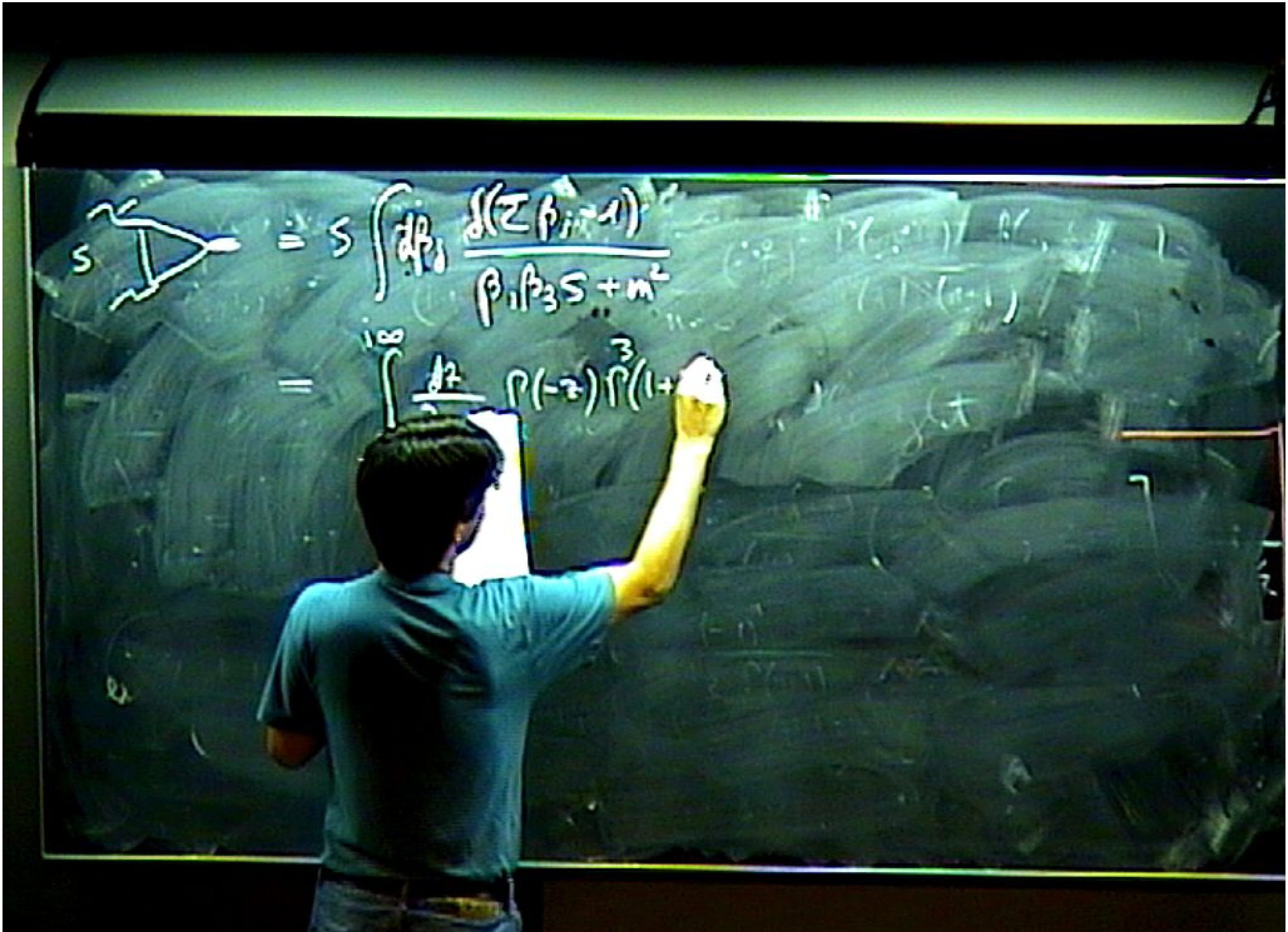
Pirsa: 11080005


Example triangle integral
p_2
m
m
p_1+p_2
m
p_1
$\frac{1}{\ell^2+m^2}$
$\int \frac{d^4\ell}{i\pi^2} \frac{1}{(\ell^2+m^2)((\ell+p_1)^2+m^2)((\ell-p_2)^2+m^2)}$
Kinematics, $p_1^2 = p_2^2 = 0$ $p_1^2 = x_{12}^2 = 0$ $x_{23}^2 = 0$
$s = (p_1+p_2)^2 = x_{13}^2$ m^2
$a = \sum a_j = 3$
$I = \int_0^1 d\beta_{j=1,2,3}$


Example triangle integral
p2
m
m
m
p1+p2
p1
1/(ℓ^2+m^2)
∫ d^4ℓ/(iπ^2) 1/((ℓ^2+m^2)((ℓ+p1)^2+m^2)((ℓ−p2)^2+m^2))
Kinematics
p1^2 = p2^2 = 0
p1^2 = x12^2 = 0
x23^2 = 0
s = (p1+p2)^2 = x13^2 m^2
a = Σ aj = 3
aj = 1
I = ∫_0^1 dβ j=1,2,3
m^2


Example triangle integral
p_2
p_1+p_2
m
m
m
p_1
$\frac{1}{\ell^2+m^2}$
$\int \frac{d^4\ell}{i\pi^2} \frac{1}{(\ell^2+m^2)((\ell+p_1)^2+m^2)((\ell-p_2)^2+m^2)}$
Kinematics
$p_1^2 = p_2^2 = 0$
$s = (p_1+p_2)^2 = x_{13}^2$
$a_j = 1 \quad j=1..3 \quad a = \sum a_j = 3$
$I = \int_0^1 d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{[\beta_1\beta_3 s + (\beta_1+\beta_2+\beta_3)m^2]}$





$$= s \int d\beta_i \, \frac{\delta(\sum \beta_i - 1)}{\beta_1 \beta_3 s + m^2}$$

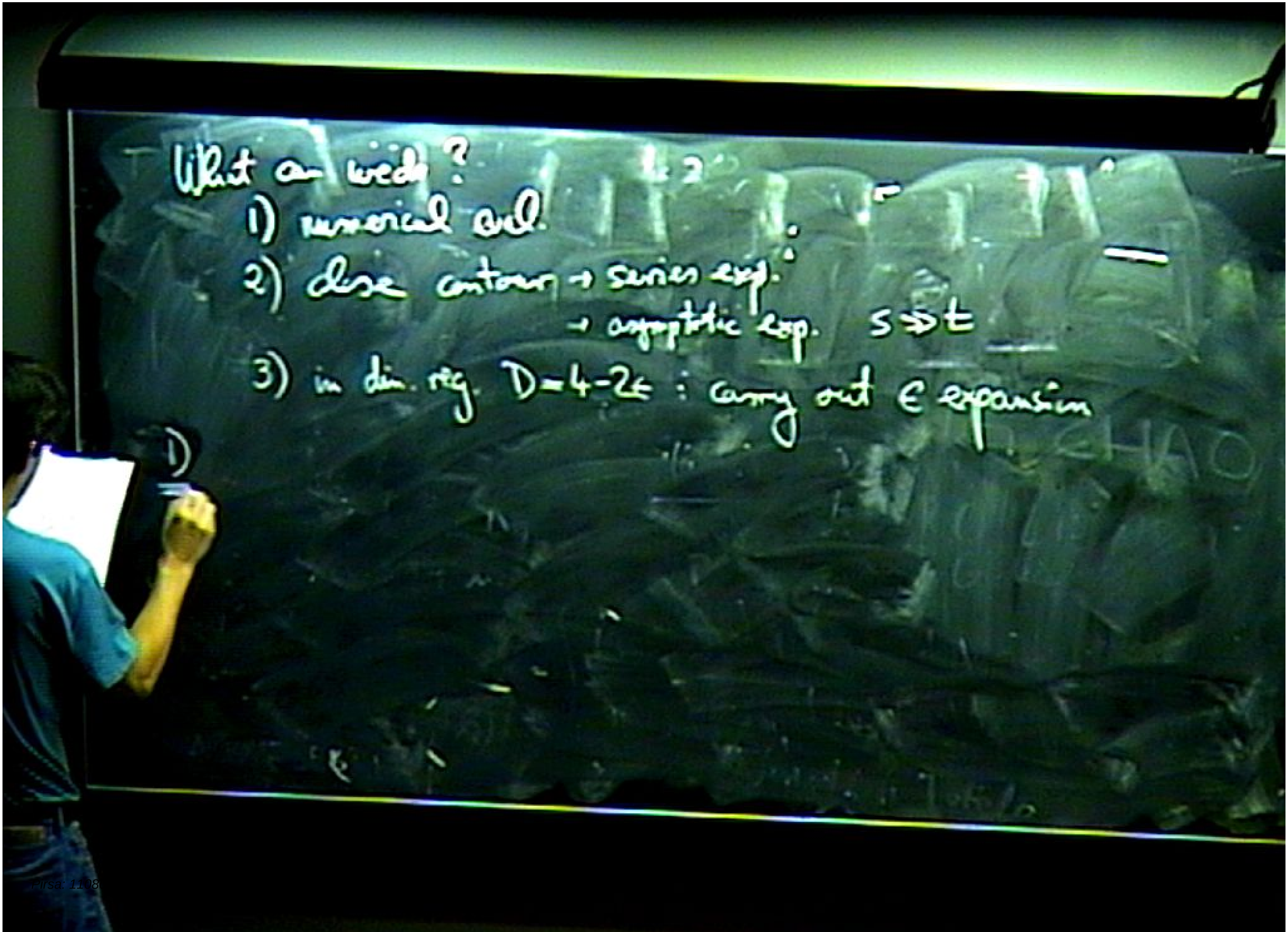
$$= \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \, \frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}$$

$$-1 < \mathrm{Re}\, z_0 < 0$$

$$= s \int d\beta_i \, \frac{\delta\left(\sum \beta_i - 1\right)}{\beta_1 \beta_3 s + m^2}$$

$$= \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \, \frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}$$

$-1 < \mathrm{Re}\, z < 0$


What can we do?
1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. s >> t
3) in dim. reg. D=4-2ε : carry out ε expansion
1)

What can we do?

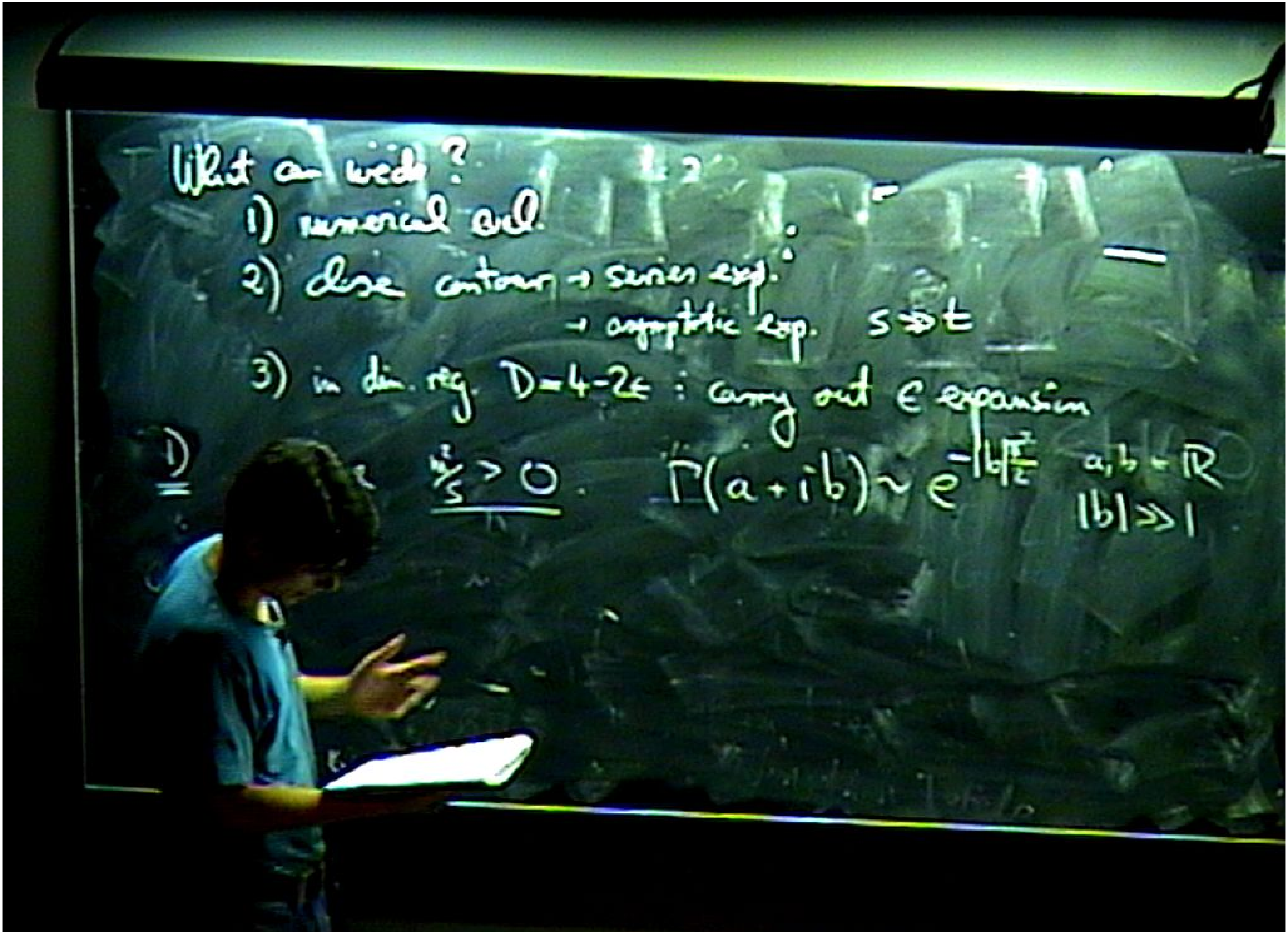
1) numerical eval.

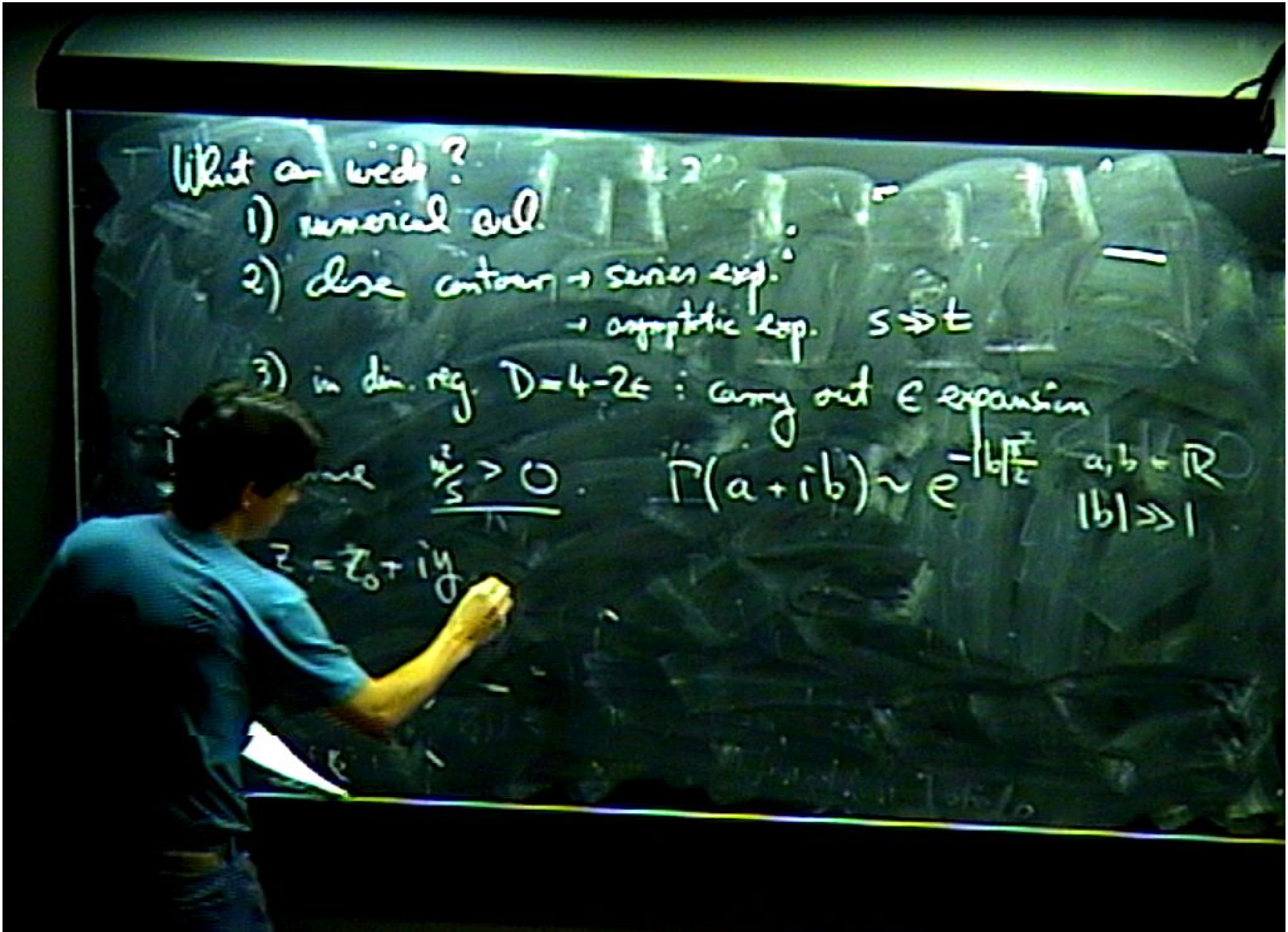
2) close contour → series exp.

→ asymptotic exp. $s \gg t$

3) in dim. reg. $D = 4 - 2\epsilon$: carry out ϵ expansion

1) assume $\frac{m^2}{s} > 0$. $\Gamma(a + ib) \sim e^{-|b|\frac{\pi}{2}}$ $a, b \in \mathbb{R}$, $|b| \gg 1$


What can we do?
1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. s >> t
3) in dim. reg. D = 4 − 2ε : carry out ε expansion
1)
m²/s > 0
Γ(a + ib) ~ e^{−|b|π/2}
a, b ∈ R
|b| >> 1


What can we do ?
1) numerical eval.
2) close contour → series exp.
→ asymptotic exp. $s \gg t$
3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion
$\frac{u}{s} > 0$
$\Gamma(a+ib) \sim e^{-|b|\frac{\pi}{2}}$
$a,b \in \mathbb{R}$
$|b| \gg 1$
$z = z_0 + iy$

What can we do?

1) numerical eval.

2) close contour $\to$ series exp.

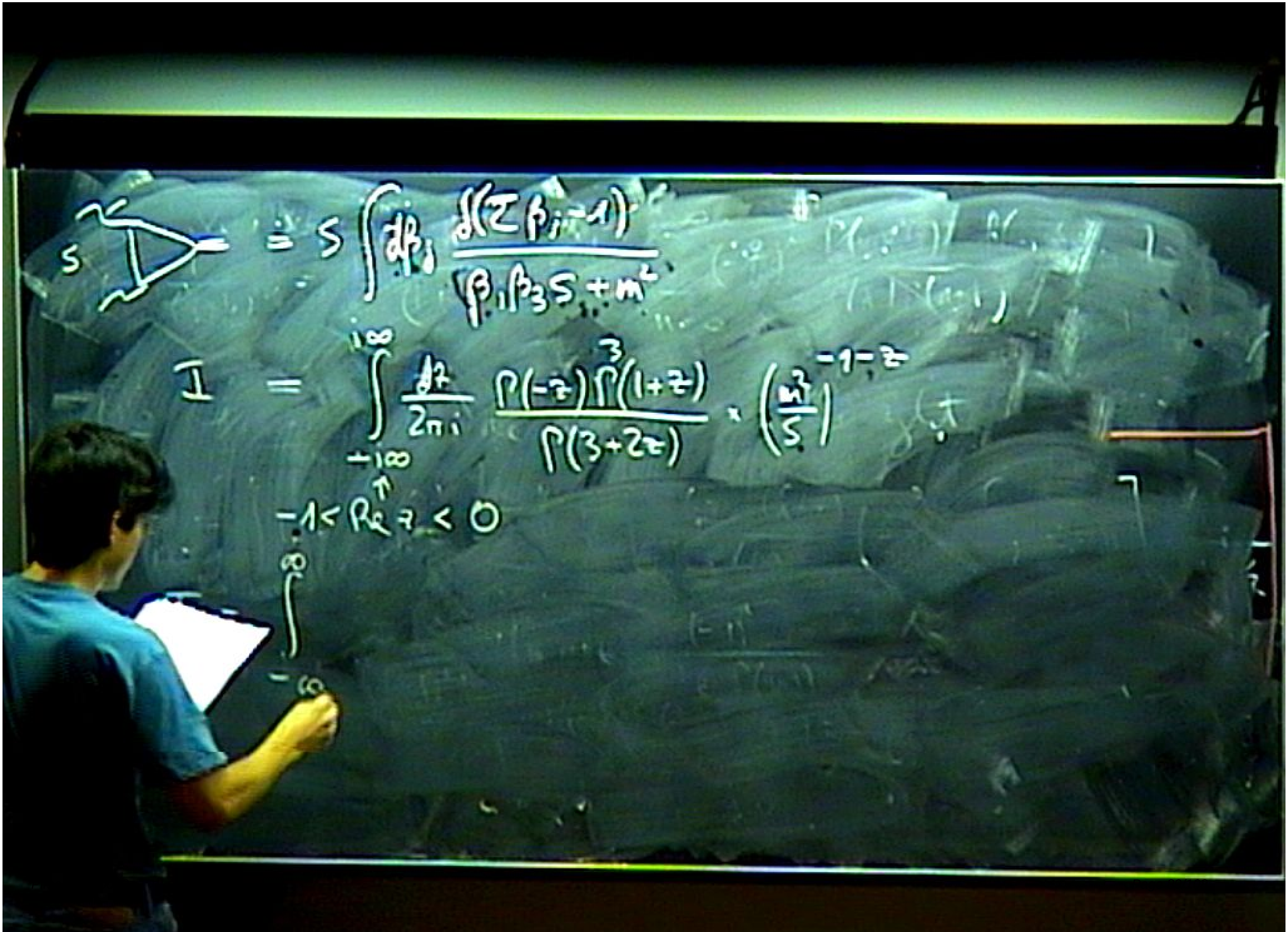
$\to$ asymptotic exp. $\quad s \gg t$

3) in dim. reg. $D = 4 - 2\epsilon$: carry out ϵ expansion

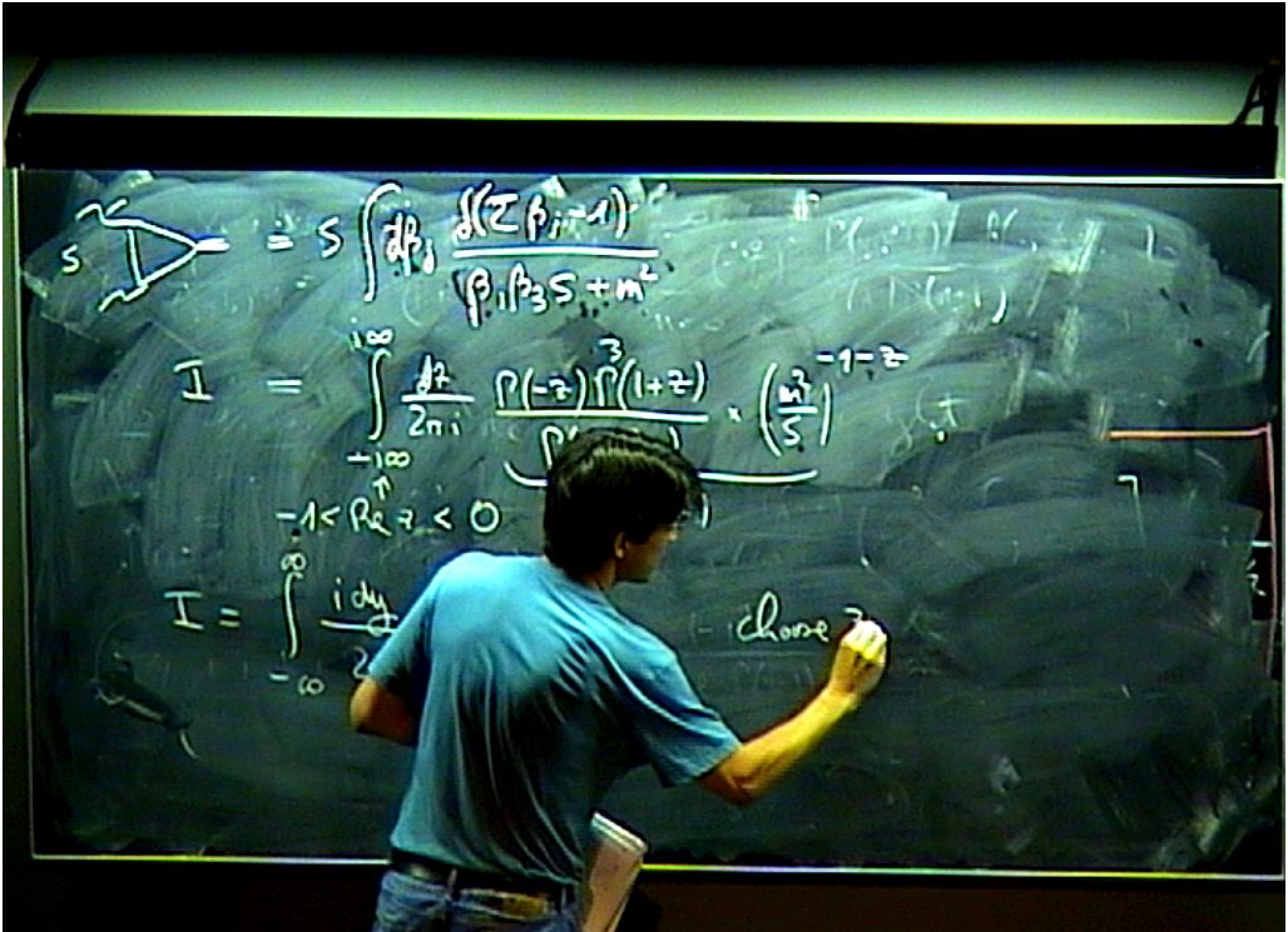
1) assume $\underline{\frac{m^2}{s} > 0}$. $\qquad \Gamma(a + ib) \sim e^{-|b|\frac{\pi}{2}} \quad a, b \in \mathbb{R}$, $|b| \gg 1$

$z = z_0 + iy$

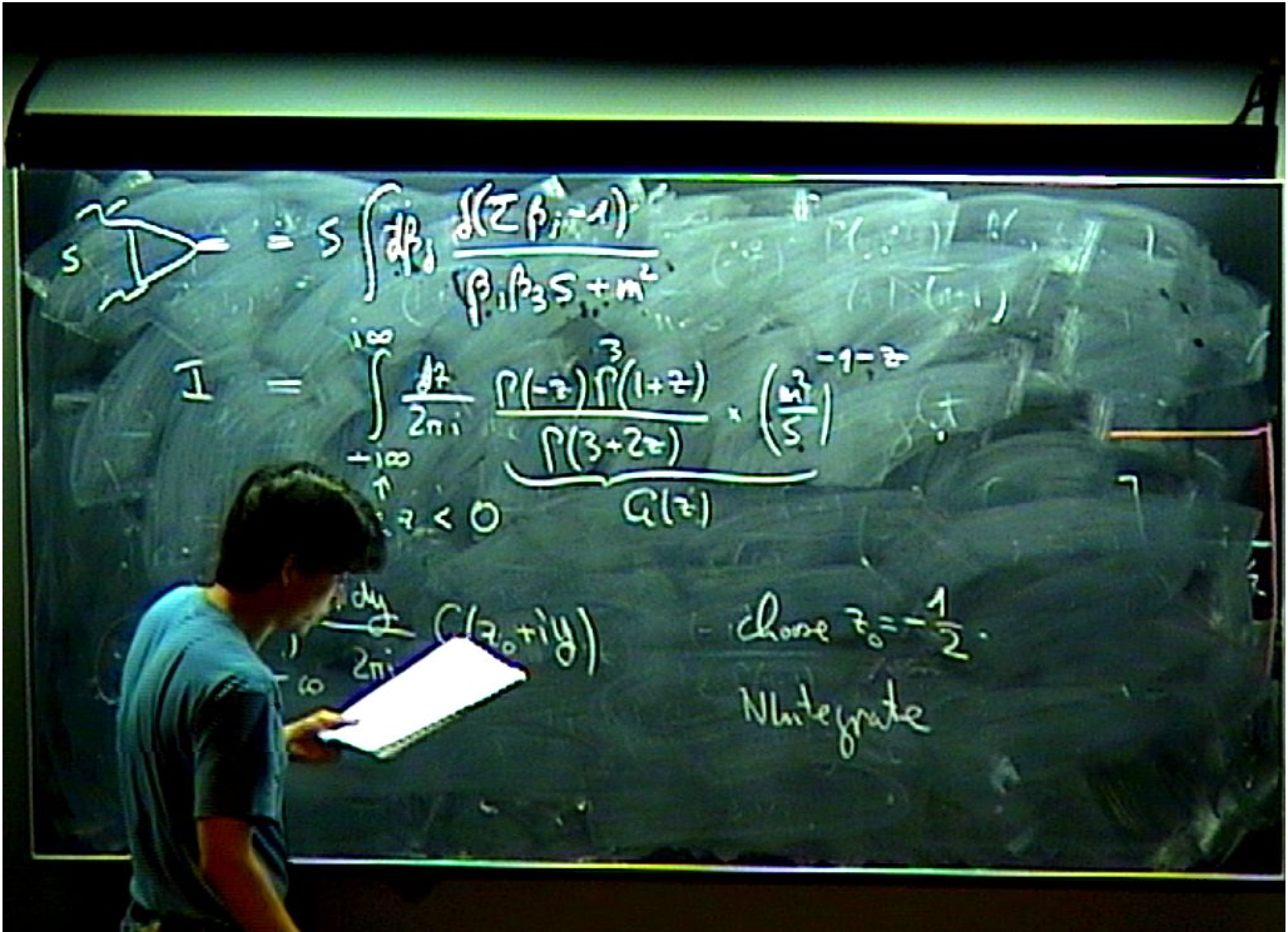
$dz = i\,dy$



s
$= s \int d\beta_j \, \frac{\delta(\sum \beta_j - 1)}{\beta_1 \beta_3 s + m^2}$
$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \, \frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}$
$-1 < \mathrm{Re}\, z < 0$
$\int_{-\infty}^{\infty}$



= s ∫ dβj δ(Σ βj − 1) / (β1β3 s + m²)
I = ∫ dz/2πi Γ(−z) Γ³(1+z) · (m²/s)^(−1−z)
−1 < Re z < 0
I = ∫ i dy
close



s
$= s \int d\beta_j \frac{\delta(\sum \beta_i - 1)}{\beta_1 \beta_3 s + m^2}$
$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}}_{G(z)}$
$z < 0$
$\frac{dy}{2\pi i} G(z_0+iy)$
choose $z_0 = -\frac{1}{2}$.
NIntegrate

$$= s\int d\beta_i \, \frac{\delta(\sum \beta_i - 1)}{\beta_1\beta_3 s + m^2}$$

$$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}}_{G(z)}$$

$-1 < \mathrm{Re}\, z < 0$

$$I = \int_{-\infty}^{\infty} \frac{i\,dy}{2\pi i}\, G(z_0 + iy)$$

choose $z_0 = -\frac{1}{2}$.

NIntegrate

MB.m $\rightarrow$ MBintegrate

$$= s\int d\beta_j \, \frac{\delta\left(\sum \beta_i - 1\right)}{\beta_1\beta_3 s + m^2}$$

$$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}}_{G(z)}$$

$-1 < \mathrm{Re}\, z < 0$

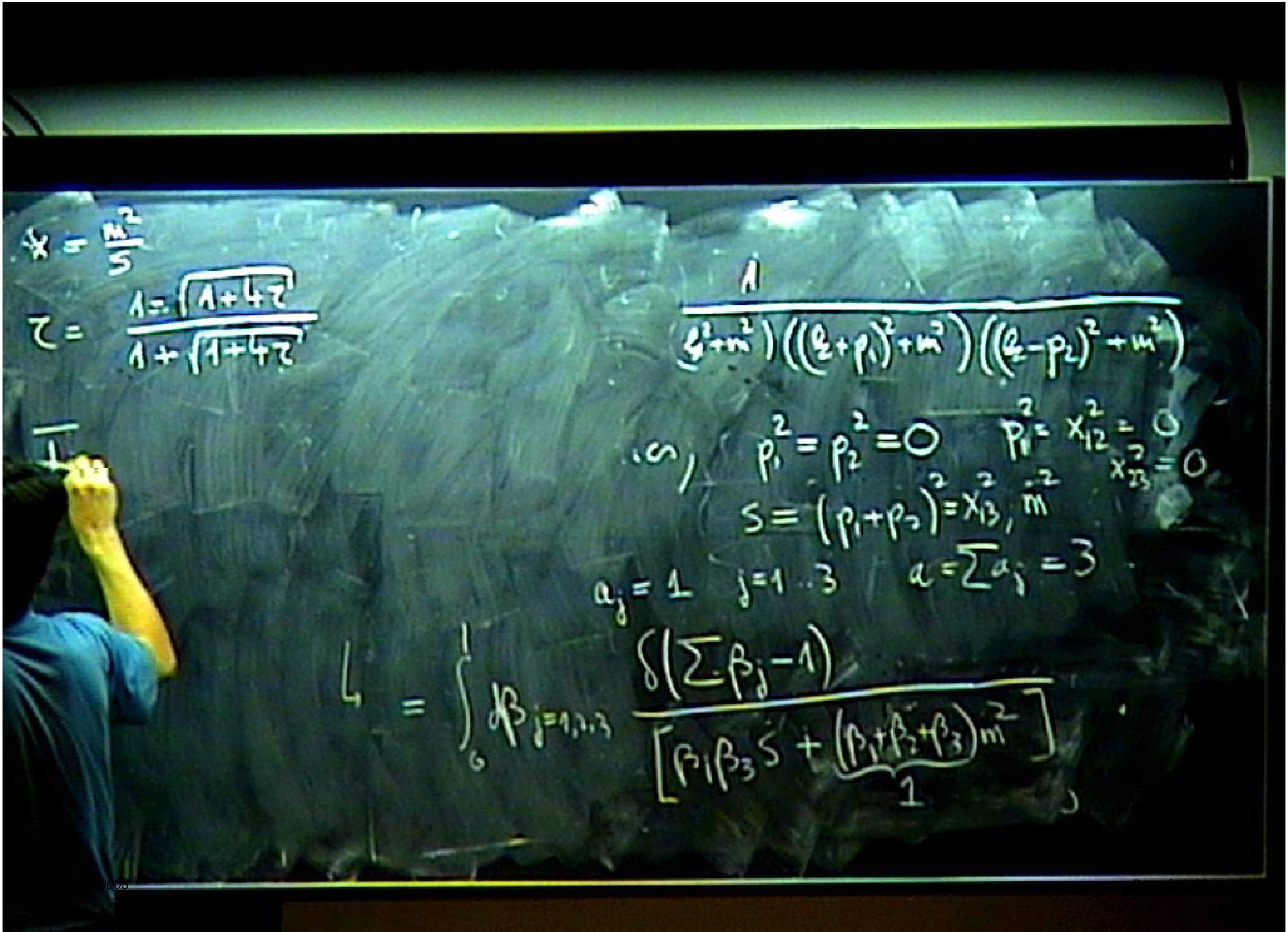
$$I = \int_{-\infty}^{\infty} \frac{i\,dy}{2\pi i}\, G(z_0 + iy)$$

choose $z_0 = -\frac{1}{2}$.

$$\approx \int_{-5}^{5} \frac{i\,dy}{2\pi i}\, G\left(-\tfrac{1}{2} + iy\right)$$

NIntegrate

MB.m $\to$ MBintegrate


$x = \frac{m^2}{s}$
$\tau = \frac{1 - \sqrt{1+4z}}{1 + \sqrt{1+4z}}$
$\frac{1}{(k^2+m^2)((k+p_1)^2+m^2)((k-p_2)^2+m^2)}$
$p_1^2 = p_2^2 = 0 \quad p_1^2 = x_{12}^2 = 0 \quad x_{23}^2 = 0$
$s = (p_1+p_2)^2 = x_{13}^2, \; m^2$
$a_j = 1 \quad j = 1..3 \quad a = \sum a_j = 3$
$I = \int_0^1 d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{[\beta_1\beta_3 s + \underbrace{(\beta_1+\beta_2+\beta_3)}_{1} m^2]}$

$$= s \int d\beta_j \, \frac{\delta\left(\sum \beta_j - 1\right)}{\beta_1 \beta_3 s + m^2} = I(x)$$

$$I = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\,\Gamma^3(1+z)}{\Gamma(3+2z)} \cdot \left(\frac{m^2}{s}\right)^{-1-z}}_{G(z)}$$

$-1 < \mathrm{Re}\, z < 0$

$$I = \int_{-\infty}^{\infty} \frac{i\,dy}{2\pi i}\, G(z_0 + iy)$$

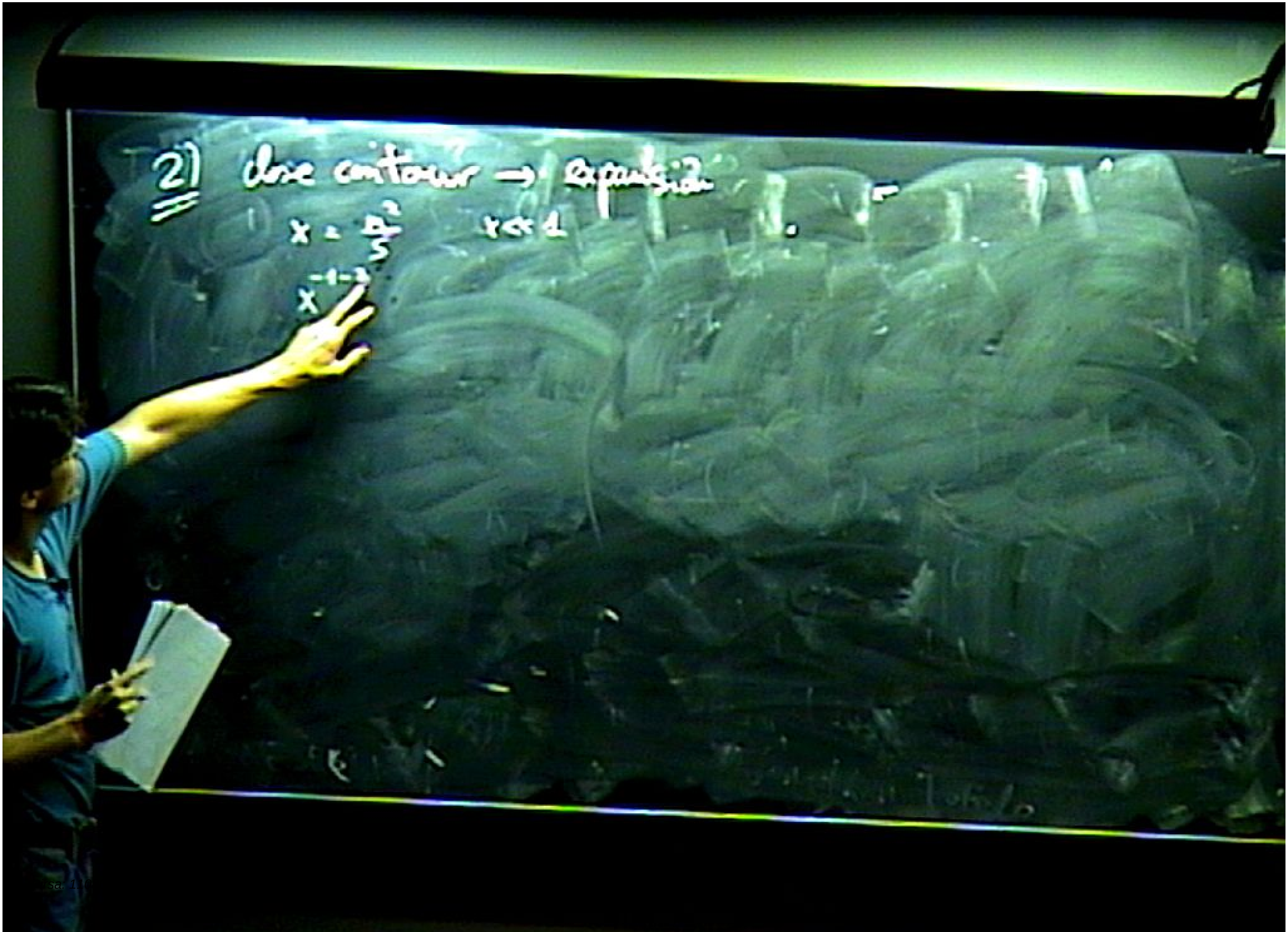
choose $z_0 = -\frac{1}{2}$.

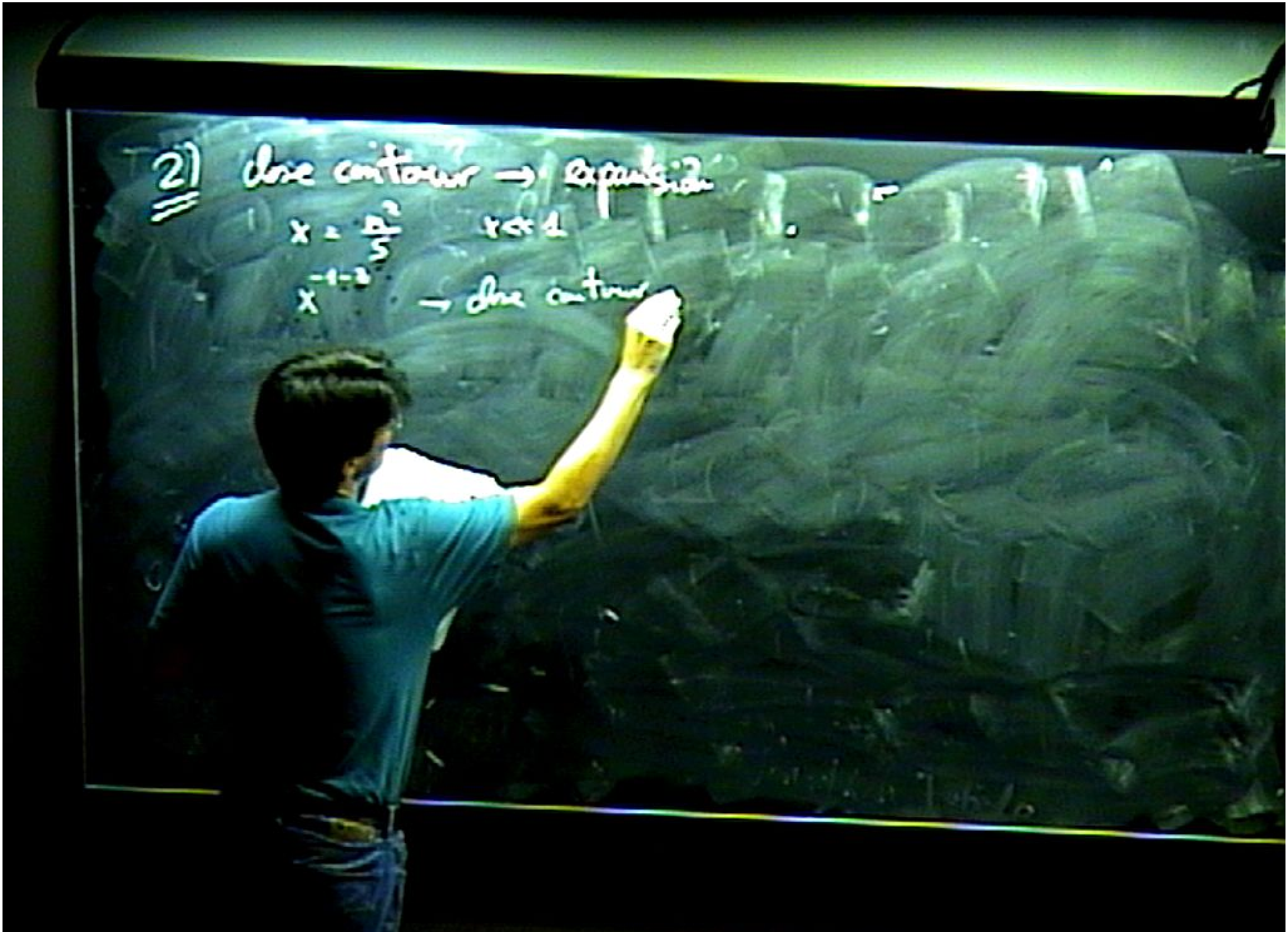
$$\approx \int_{-5}^{5} \frac{i\,dy}{2\pi i}\, G\left(-\frac{1}{2} + iy\right)$$

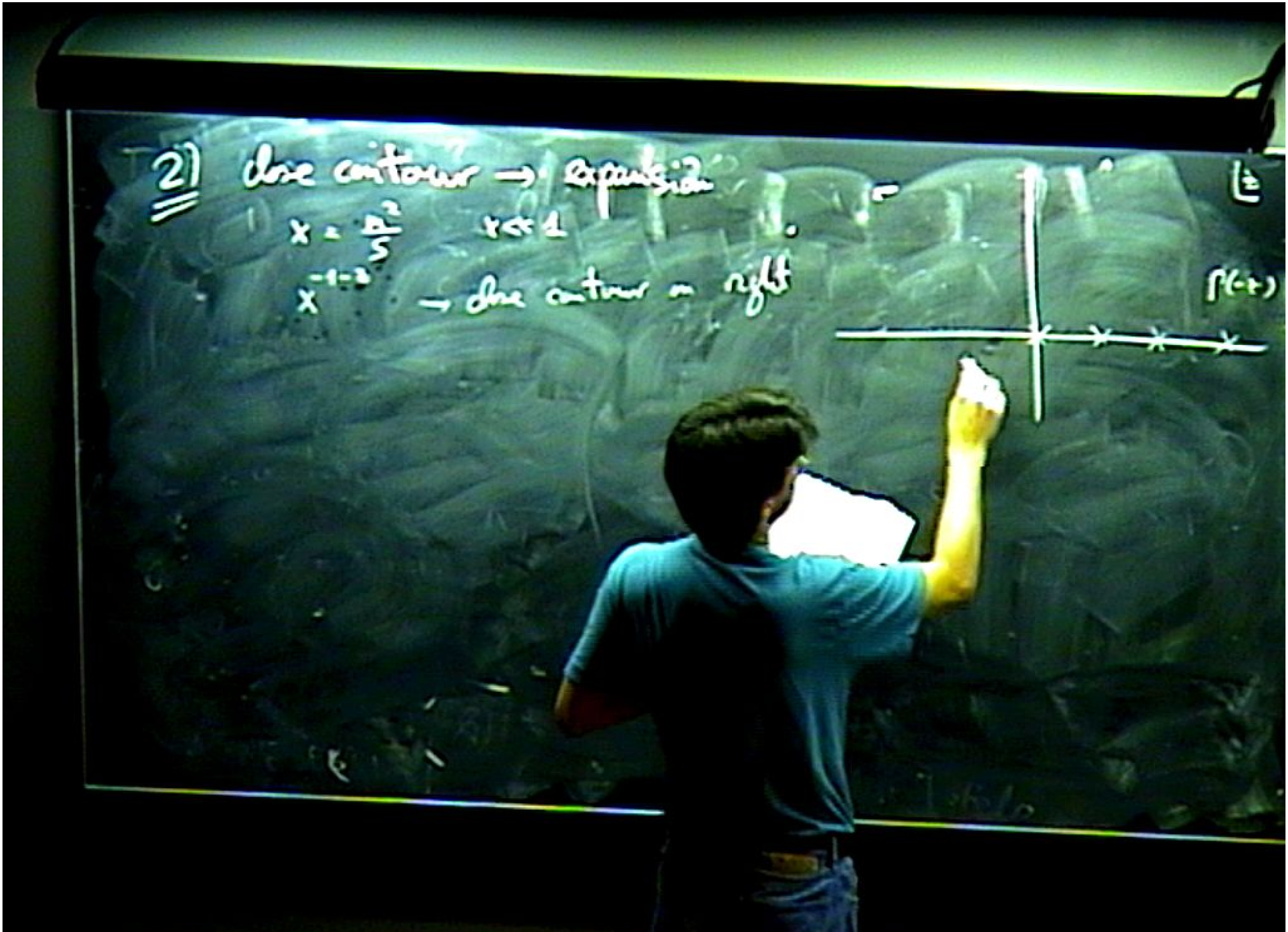
NIntegrate

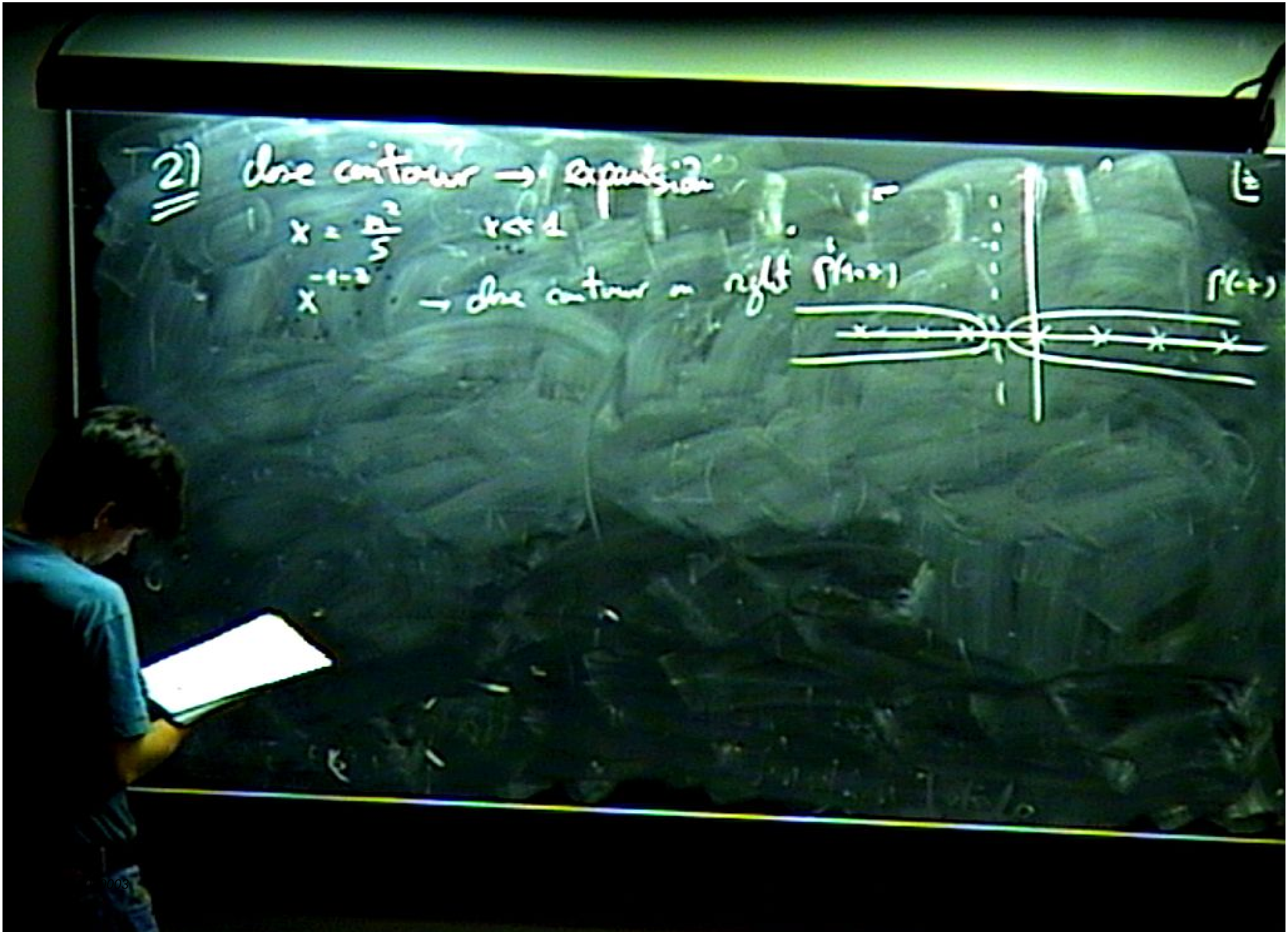
MB.m → MBintegrate

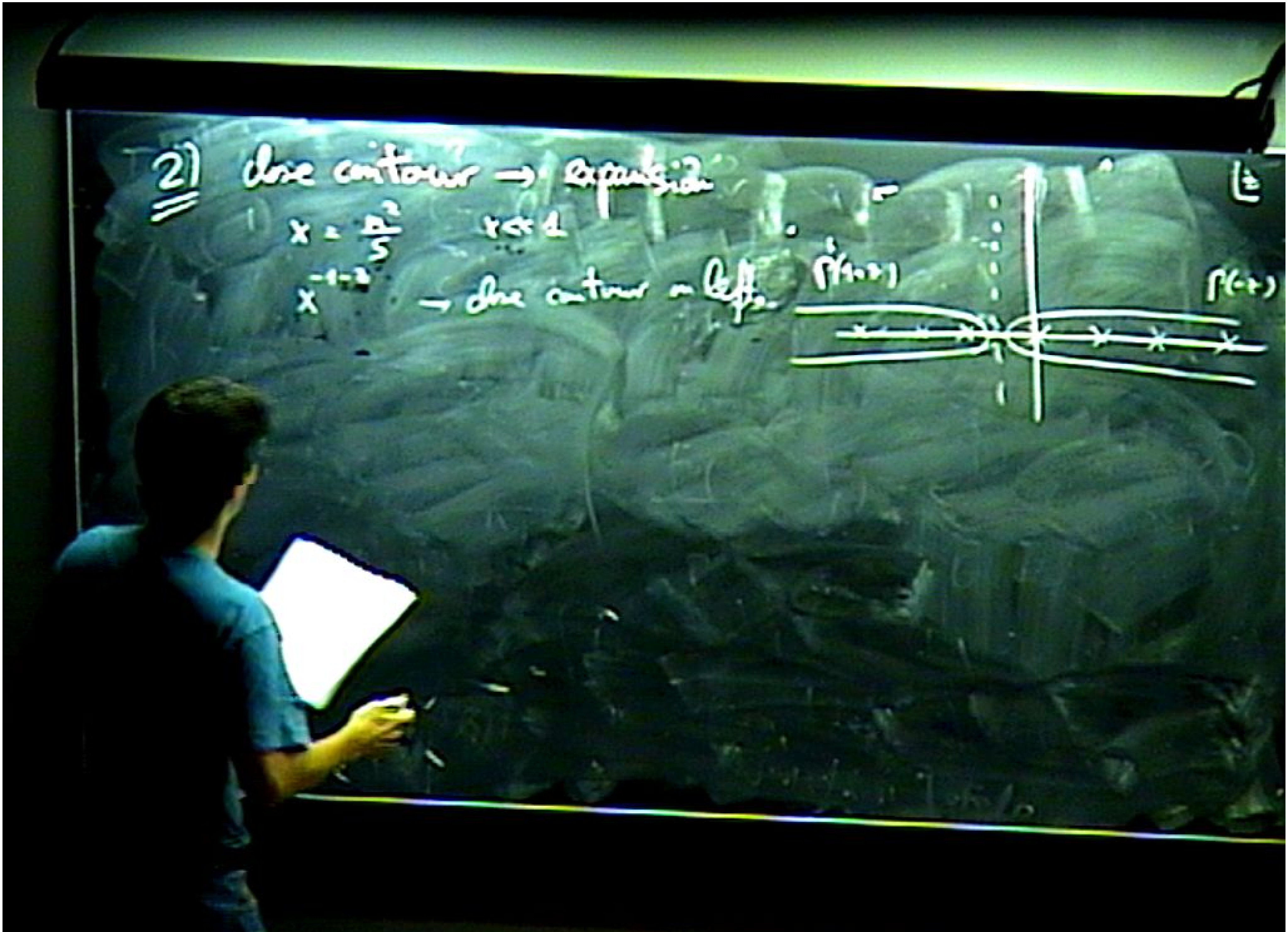
2) close contour → expulsion

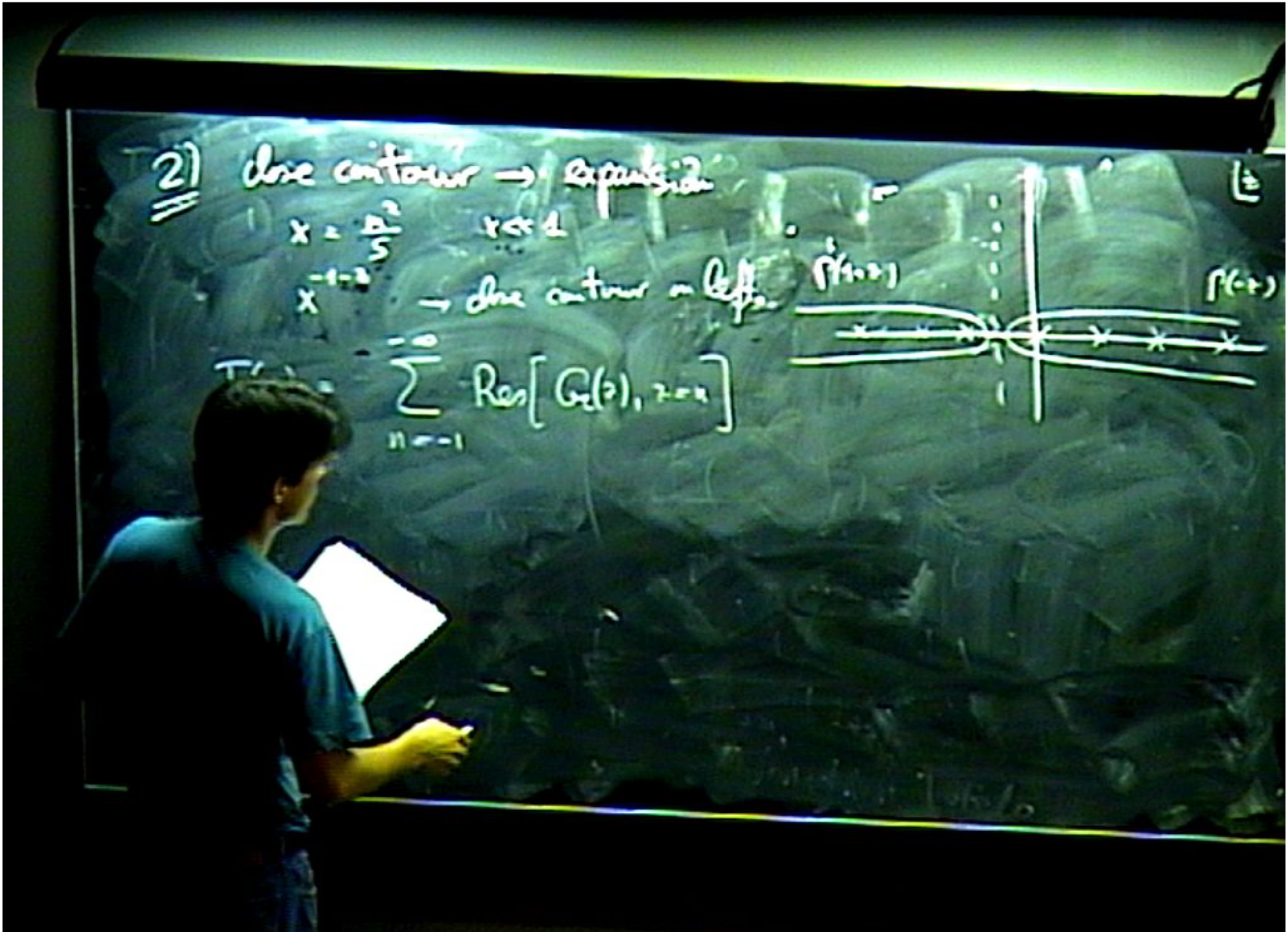

2) close contour → expansion

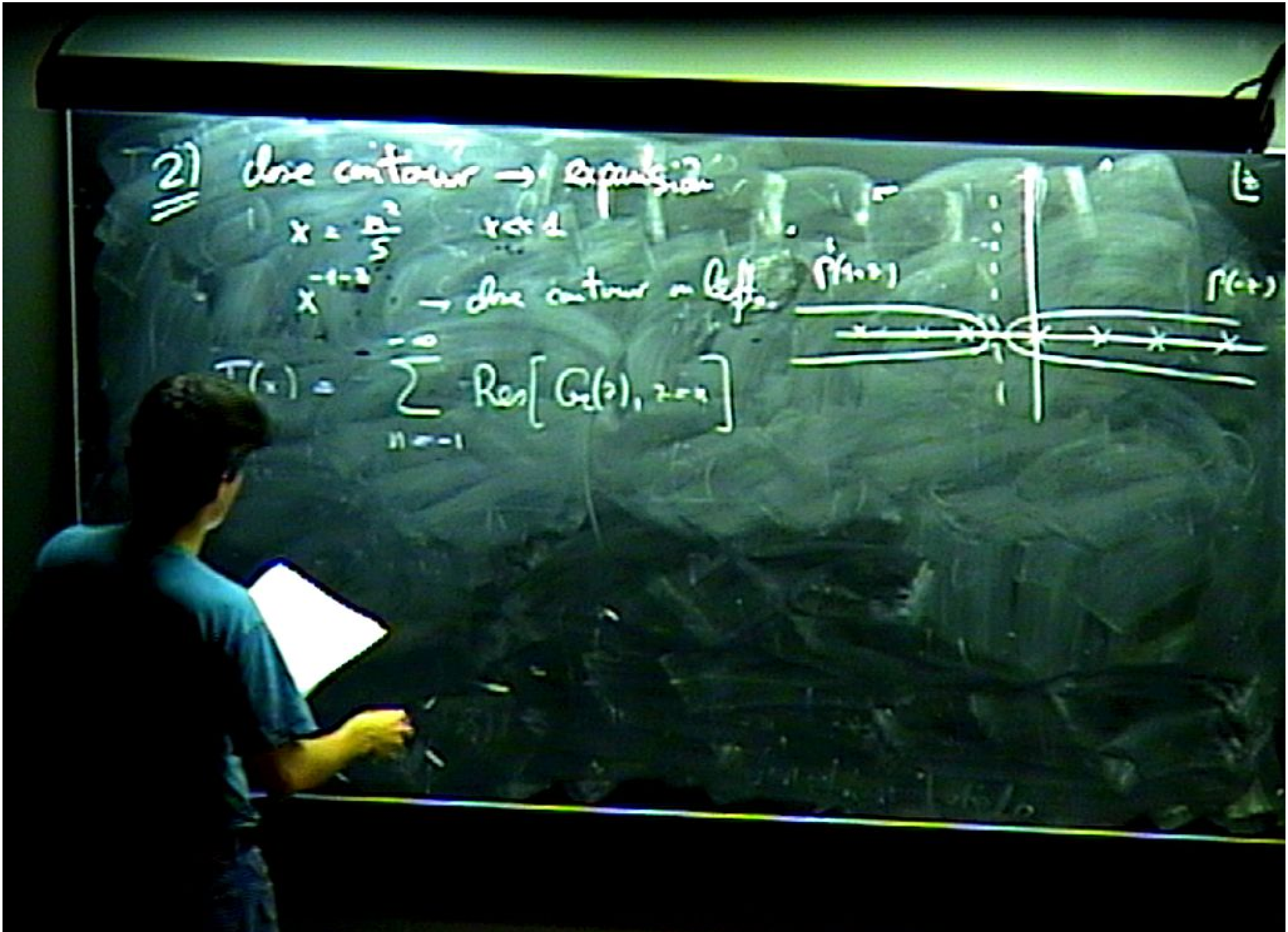

2) close contour → expansion
→ close contour


2) close contour → expansion
x << 1
→ close contour on right
Γ(-z)

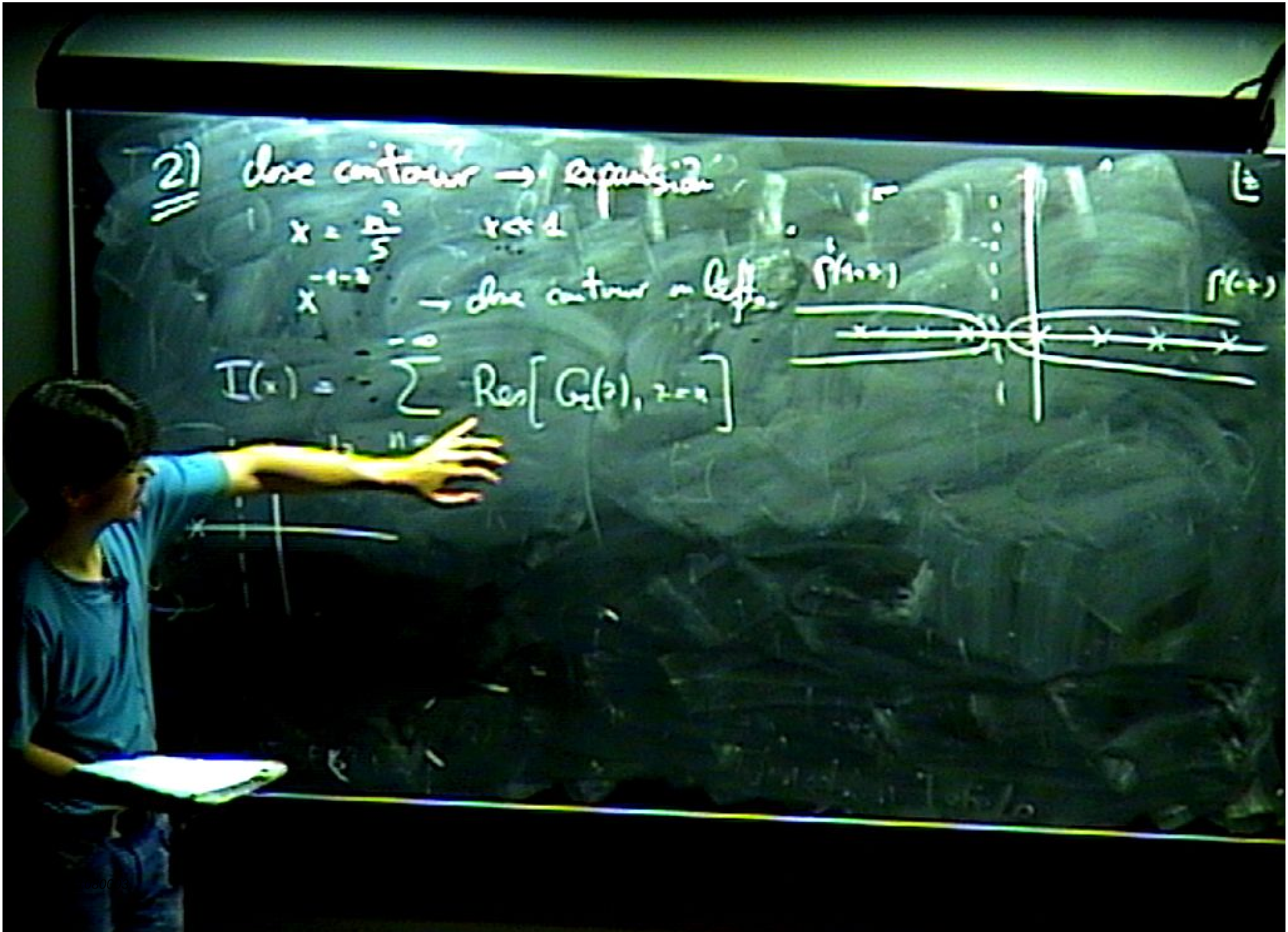

2) close contour → expansion
close contour on right

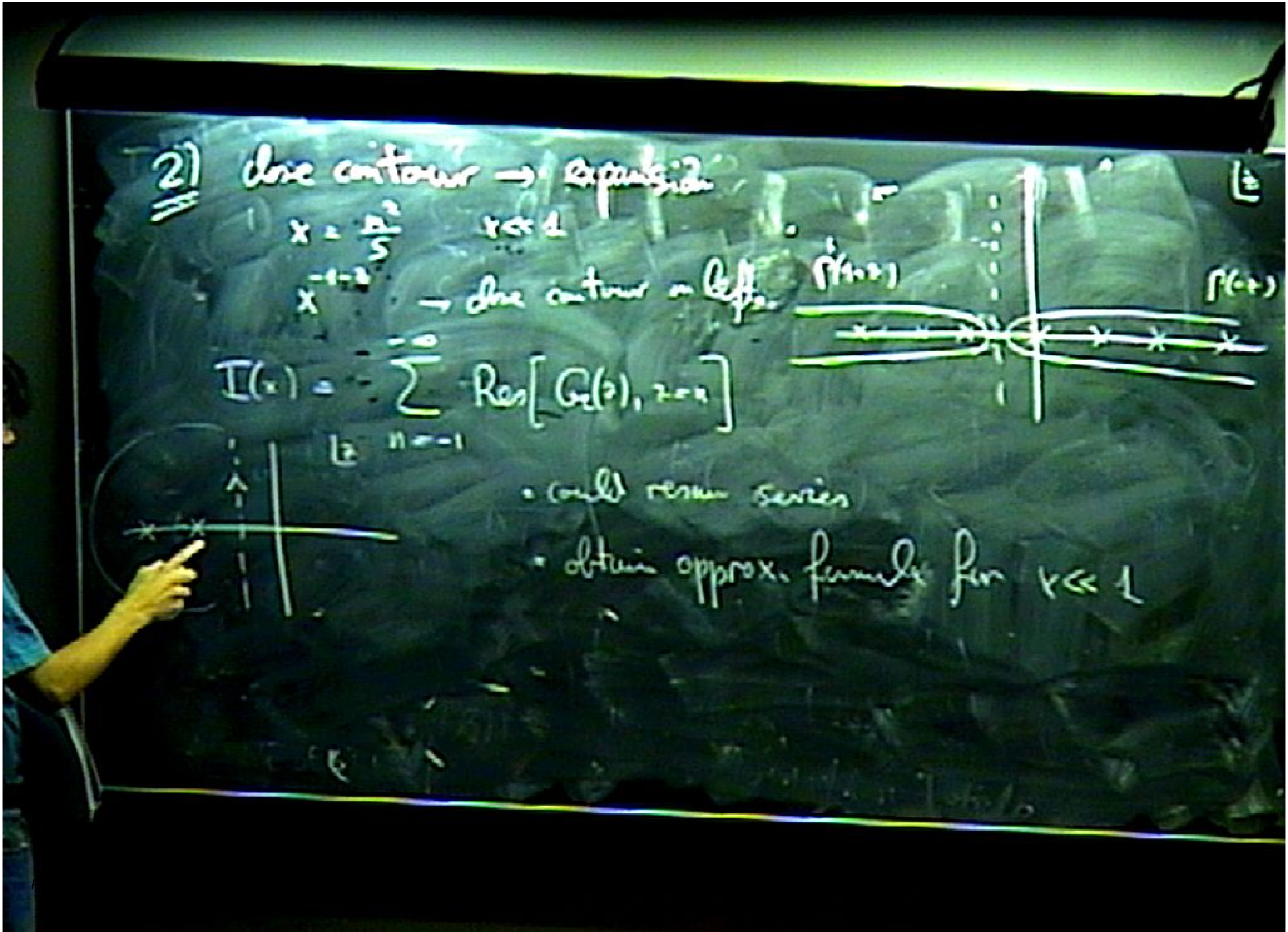

2) close contour → expansion
close contour on left


2) close contour → expansion
→ close contour on left
Γ(1+z)
Γ(-z)
Res[G(z), z=n]



2) close contour → expansion
close contour on left
Res
Γ(1+z)
Γ(-z)

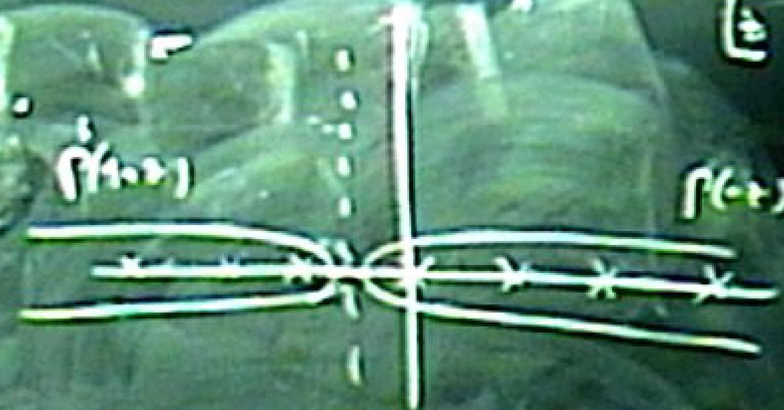

2) close contour → expansion
x^{-1-z} → close contour on left
I(x) = ∑ Res[G(z), z=n]
Γ(1+z)
Γ(-z)


2) close contour → expansion
→ close contour on left
(could) resum series
obtain approx. formula for r << 1

2) close contour → expansion

$x = \frac{\alpha^2}{5}$ $\quad x \ll 1$

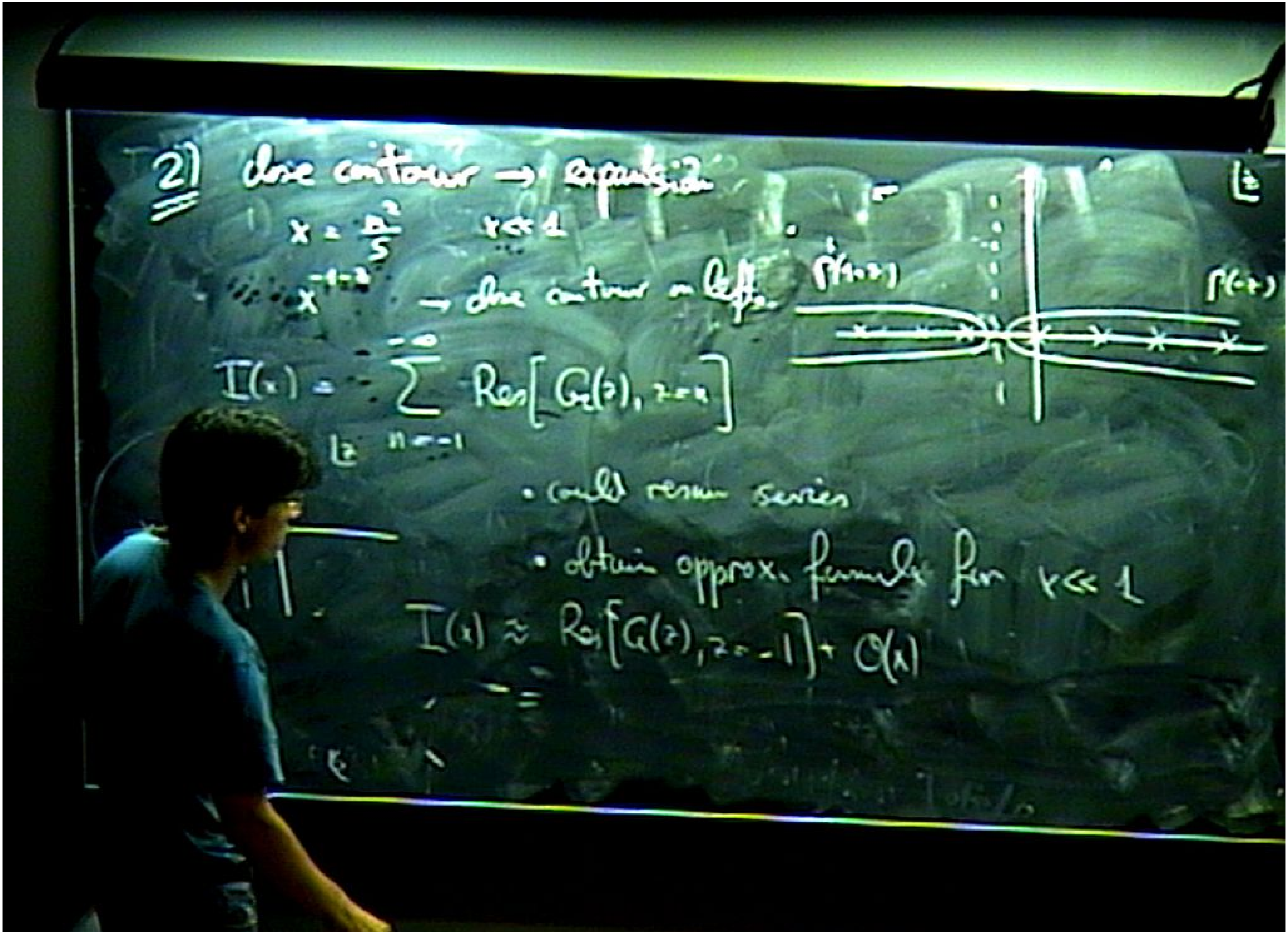
x^{-1-z} → close contour on left.



$$I(x) = \sum_{n=-1}^{-\infty} \mathrm{Res}\left[G(z), z=n\right]$$

• (could) resum series

• obtain approx. formula for $x \ll 1$


2) close contour → expansion
x << 1
→ close contour on left.
Γ(1+z)
Γ(-z)
I(x) = Σ Res[G(z), z=n]
• (could) resum series
• obtain approx. formula for x << 1
I(x) ≈ Res[G(z), z=-1] + O(x)

$$x = \frac{m^2}{s}$$

$$\tau = \frac{1 - \sqrt{1+4x}}{1 + \sqrt{1+4x}}$$

$$\Phi(x) = \frac{1}{2} \log^2\left(\tau(x)\right)$$

$$\frac{1}{(k^2+m^2)\left((k+p_1)^2+m^2\right)\left((k-p_2)^2+m^2\right)}$$

$p_1^2 = p_2^2 = 0 \qquad p_1^2 = x_{12}^2 = 0 \quad x_{23}^2 = 0$

$s = (p_1+p_2)^2 = x_{13}^2 , \ m^2$

$a_j = 1 \quad j = 1 \dots 3 \qquad a = \sum a_j = 3$

$$I = \int_0^1 d\beta_{j=1,2,3} \frac{\delta\left(\sum \beta_j - 1\right)}{\left[\beta_1 \beta_3 s + \underbrace{(\beta_1+\beta_2+\beta_3)}_{1} m^2\right]}$$