

Title: Tools for Evaluating Loop Integrals

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Abstract:

Tools for evaluating loop integrals

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Tools for evaluating loop integrals

$$\int d\ell_1, \int d\ell_2, \dots, \int d\ell_n$$

Stephen Britton

Falk Richter

Marcelo D. Schwartz

IG ZHAO



Tools for evaluating loop integrals

$$\int d\ell_1 \int d\ell_2 \dots \int d\ell_n \left\{ \text{integrand} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

Student Zitation

DAH2 BRIG ZHAO



Tools for evaluating loop integrals:

$$\int d\ell_1, \int d\ell_2, \dots, \int d\ell_n \quad \left\{ \text{integral} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

→ scattering amplitudes

Student Zhen

NG ZHAO

Tools for evaluating loop integrals

$$\int d\ell_1, \int d\ell_2, \dots, \int d\ell_n \quad \left\{ \text{integral} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

scattering amplitudes

correlation functions of local operators

↳ anomalies

Stefan Weinzierl

ZHAO

Tools for evaluating loop integrals

$$\int d\ell_1, \int d\ell_2, \dots, \int d\ell_n \quad \left\{ \text{integral} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

- scattering amplitudes
- correlation functions of local operators
 - ↳ anomalous dimensions

Tools for evaluating loop integrals

$$\int d\ell_1, \int d\ell_2, \int d\ell_n \quad \left\{ \text{integral} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

- scattering amplitudes
 - correlation functions of local operators
 - ↳ anomalous dimensions
- relations between integrals.

Steven Britton

ZHAO

Tools for evaluating loop integrals

$$\int d\ell_1, \int d\ell_2, \dots, \int d\ell_n \quad \left\{ \text{integral} = \frac{\text{numerator}}{\text{propagators}} \right\}$$

→ scattering amplitudes

→ correlation functions of local operators
↳ anomalous dimensions

relations between integrals.

IBP int. by parts

↳ set of int's that need to be computed

Methods

1. Primary
 2. Secondary
 3. Tertiary
 4. Quaternary
 5. Quinary
 6. Sextary
 7. Septary
 8. Octary
 9. Nonary
 10. Decary
 11. Undecary
 12. Dodecary
 13. Tridecary
 14. Quattuordecary
 15. Quindecary
 16. Sextodecary
 17. Septodecary
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methods : • differential equations

$$P_2 = f(s, t)$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 + p_2)^2$$

methods : • differential equations

• Gegenbauer, conformal symmetry

$$\mathbb{R}^2 \rightarrow f(s, t)$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_2)^2$$

methods :

• differential equations

• Gegenbauer, conformal symmetry

• Helic-Barnes

features : • very universal
• universal

$$\mathbb{R}^2 \times \mathbb{R} = f(s, t)$$

$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_2)^2$$

methods :

- differential equations

- Gegenbauer, conformal symmetry

- Mellin-Barnes

features :

- very universal
- numerical
- asymptotic limits $s \gg t$

$$\frac{d^2}{ds^2} = f(s, t)$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_2)^2$$

methods : • differential equations

• Goursat, conformal symmetry

• Helmholtz-Barnes

for very universal
numerical

• asymptotic limits $s \gg t$
to automate

$$\mathbb{R}^2 \rightarrow f(s, t)$$
$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_2)^2$$

methods : • differential equations

• Gegenbauer, conformal symmetry

• Mellin-Barnes

features : • very universal

• numerical

• asymptotic

• ...

$$\mathbb{R}^2 \rightarrow \mathbb{C} = f(s, t)$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_2)^2$$

$s \gg t$

Mathematica

MB.m

HBasymp

methods :

• differential equations

~~$\frac{\partial^2 \psi}{\partial s^2} = f(s, t)$~~

• Gegenbauer, conformal symmetry

$s = (p_1 + p_2)^2$
 $t = (p_1 - p_2)^2$

• Hellin-Barnes

• very universal

• numerical

• asymptotic limits

$s \gg t$

• easy to automate

{ Mathematica
MB.m

HBasymptotics.m

methods

- differential equations
- Gegenbauer, conformal symmetry
- Helic-Barnes

$$\begin{aligned}
 & \mathbb{R}^2 \times \mathbb{R} = f(s, t) \\
 & s = (p_1 + p_2)^2 \\
 & t = (p_1 - p_2)^2
 \end{aligned}$$

features

- very universal
- numerical
- asymptotic limits $s \gg t$
- easy to automate
 - ↳ Mathematica
 - ↳ MB.m
 - ↳ HB asymptotics.m

methods :

- differential equations

- Gegenbauer, conformal symmetry

- Mellin-Barnes

features :

- very universal

- numerical

- asymptotic limits $s \gg t$

- easy to automate (Mathematica
MB.m
HBasymptotics.m)

⊖ simplicity of specific int.
can be hidden

$$f(z) = f(s, t)$$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_2)^2$$





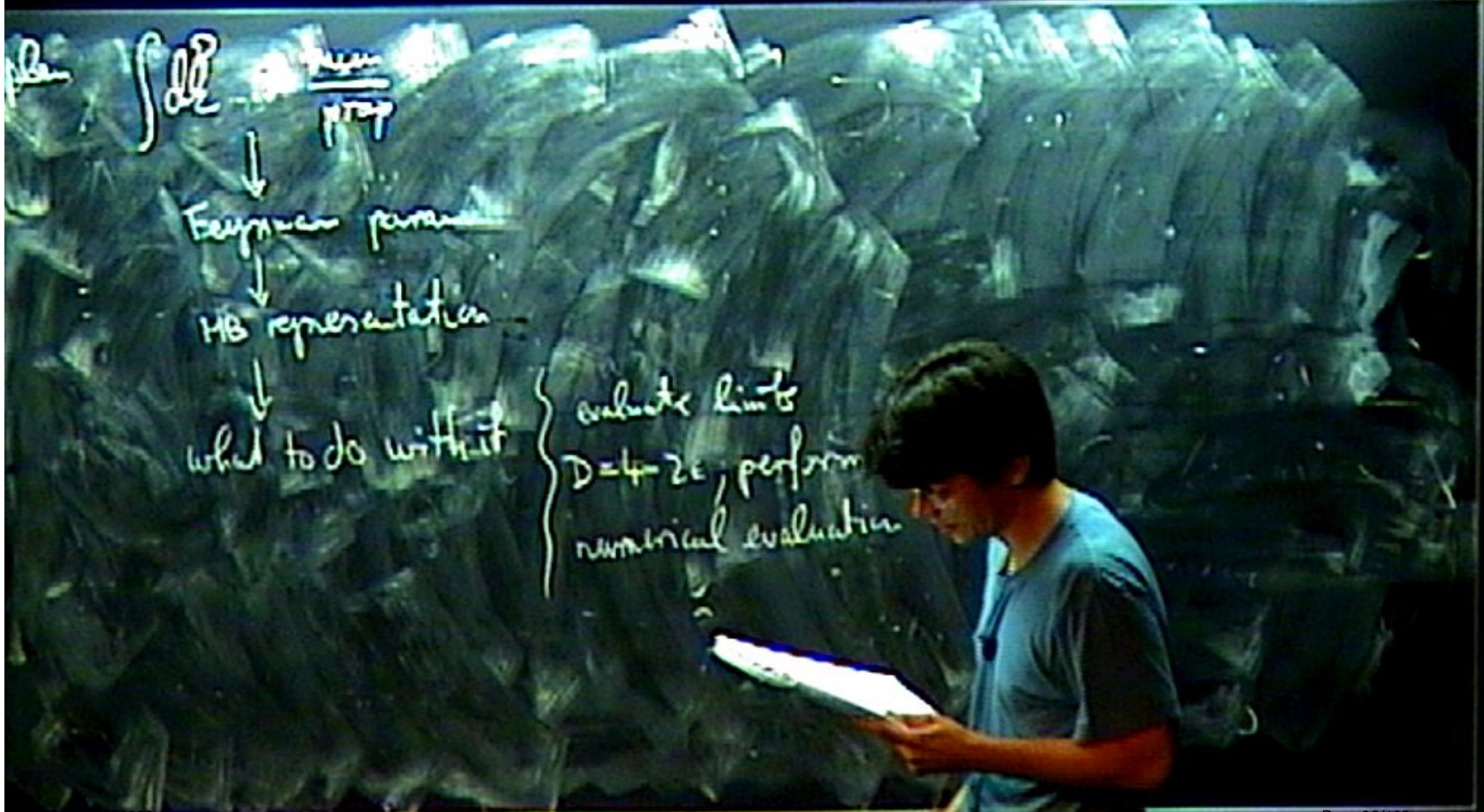
plan $\int d\mathbf{l}$ $\frac{d\mathbf{l}}{|\mathbf{l}|^3}$

↓
Feynman parameter

↓
MB representation

↓
what to do with it

} eval. integrals
D = perform exp.
num.



methods

• differential equations

~~$\frac{d^2 s}{dt^2}$~~ = $f(s, t)$

• Gegenbauer, conformal symmetry

$s = (p_1 + p_2)^2$
 $t = (p_1 - p_2)^2$

• Mellin-Barnes

features

minimal

limits $s \gg t$

tools (Mathematica, MB.m)

of specific int. be hidden MB asymptotics.m

methods

• differential equations
Klein - G

~~Δ~~ = $f(z, t)$
 $s = (p_1 + p_2)^2$
 $t = (p_1 - p_2)^2$

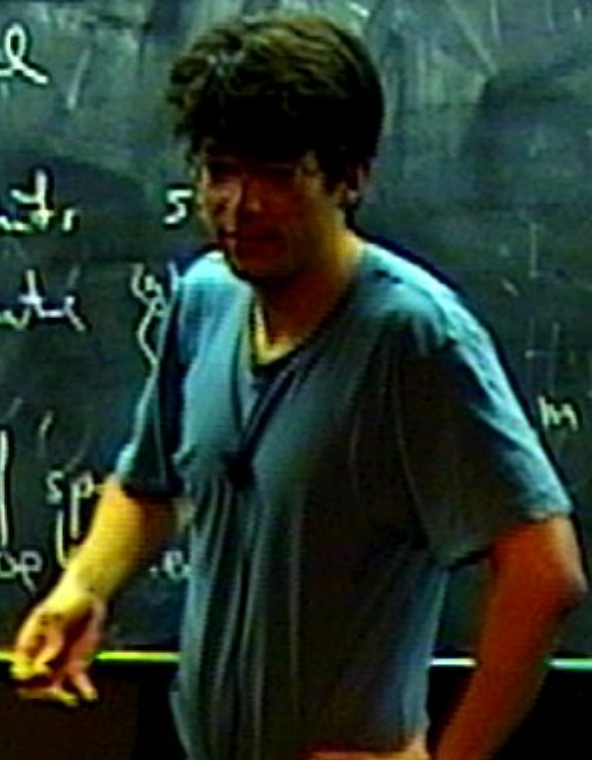
• Gegenbauer, conformal symmetry

• Mellin-Barnes

features

- very universal
- numerical
- asymptotic limits
- easy to automate

⊖ simplicity of sp
can be l



methods

• differential equations

Katkov - 90' | $\frac{1}{p+2}$ $\frac{2}{p+2}$
Gelman - 100' | Remiddi

~~123~~ $= f(s, t)$



• Gegenbauer, conformal symmetry

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_2)^2$$

Helix-Barnes

- very universal
- numerical
- asymptotic limits $s \gg t$
- easy to automate (Mathematica, MB.m)

⊖ simplicity of specific int. can be hidden
MBasymptotics.m

methods

• differential equations

Kähler $\sim \frac{1}{p+q}$ $\frac{1}{p+q}$ $\frac{2}{2}$

Gömbösi

~~math~~ $= f(s, t)$

$s = (p_1 + p_2)^2$
 $t = (p_1 - p_2)^2$



• Gegenbauer, conformal symmetry

• Mellin-Barnes

features

- very universal
- numerical
- asymptotic limits $s \gg t$
- easy to automate (Mathematica, MB.m, HBasymptics.m)

⊖ simplicity of specific int. can be hidden

1. System parameter

1

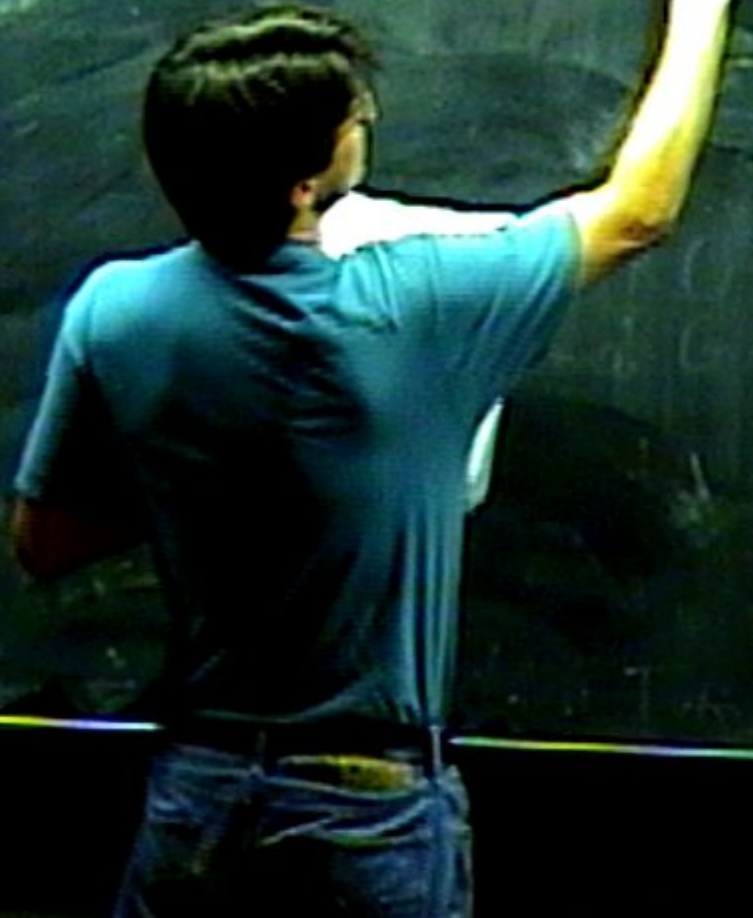
A

ZHAO

Tokyo

1. Gamma parameter

$$\frac{1}{A_1^{a_1} \cdots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i - 1}}{\sum \beta_i}$$



1. Gamma parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \prod \beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i - 1}}{(\sum \beta_i A_i)^a}$$
$$a = \sum_{i=1}^n a_i$$

$$\frac{1}{i\pi^{1/2} (\beta^2 + \Delta)}$$

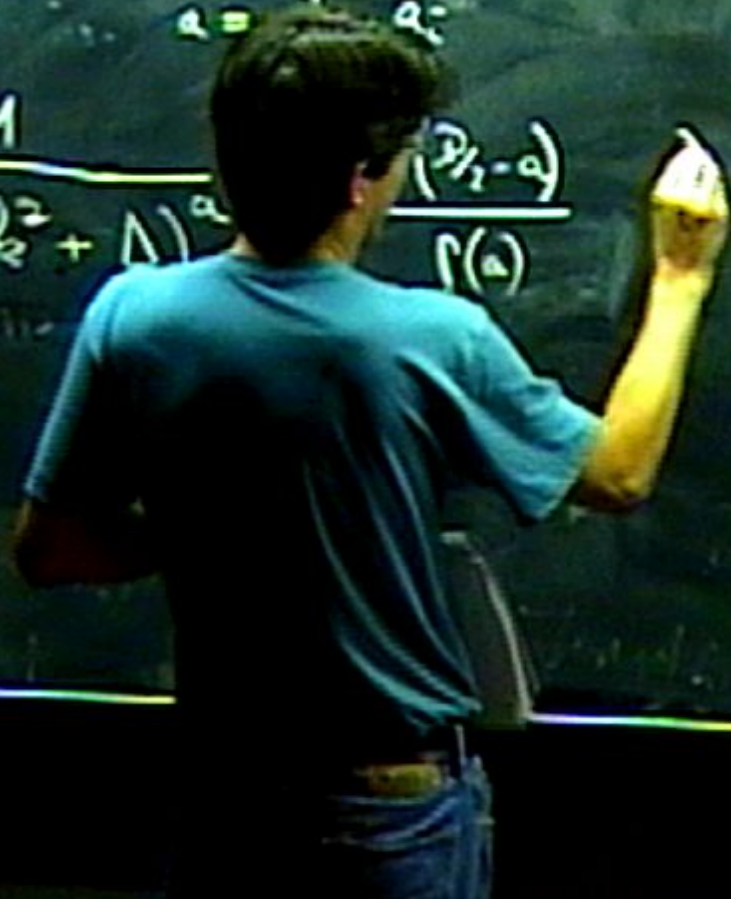
ZHAO

1. Gamma parameter

$$\frac{1}{A_1^{\alpha_1} \dots A_n^{\alpha_n}} = \frac{\Gamma(\alpha)}{\prod_{i=1}^n \Gamma(\alpha_i)} \int \beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{\alpha_i - 1}}{(\sum \beta_i A_i)^{\alpha}}$$

$\alpha = \sum \alpha_i$

$$\int_{-\infty}^{\infty} \frac{1}{(b^2 + \Lambda)^{\alpha}} \frac{\Gamma(\frac{D}{2} - \alpha)}{\Gamma(\alpha)}$$

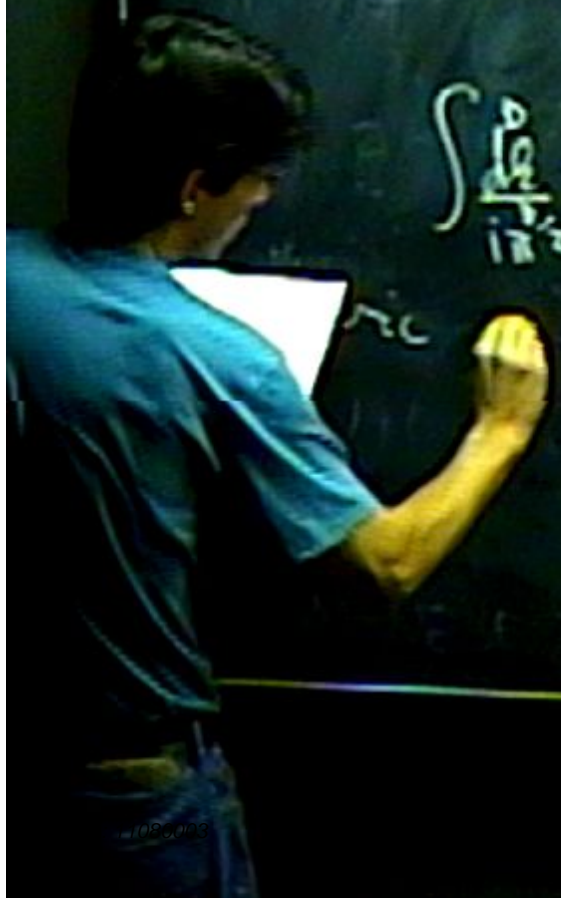


1. Gamma parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \prod \beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$$

$$a = \sum_{i=1}^n a_i$$

$$\int_{-\infty}^{\infty} \frac{1}{(b^2 + \Delta)^a} = \frac{\Gamma(\frac{1}{2})}{\Gamma(a)} \Delta^{\frac{1}{2}-a}$$



ZHAO

1. Gamma parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \prod \beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$$

$$a = \sum_{i=1}^n a_i$$

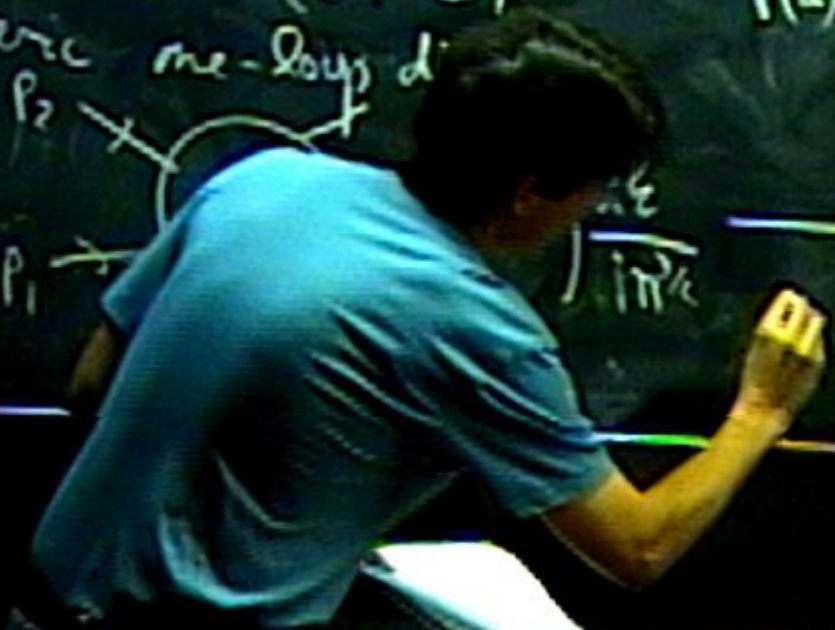
$$\int \frac{d^D l}{i\pi^{D/2}} \frac{1}{(l^2 + \Delta)^a} = \frac{\Gamma(\frac{D}{2} - a)}{\Gamma(a)} \Delta^{a - \frac{D}{2}}$$

generic me-loop d



$$\frac{1}{i\pi^{D/2}}$$

1



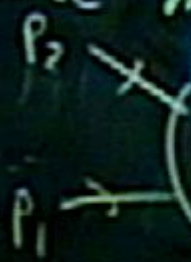
1. Feynman parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \mathcal{D}\beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$$

$$a = \sum_{i=1}^n a_i$$

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(\Delta)^a} = \frac{\Gamma(\frac{D}{2}-a)}{\Gamma(a)} \Delta^{a-\frac{D}{2}}$$

generic m diag.



$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + m_1^2) ((k+p_1)^2 + m_2^2) \dots ((k-p_n)^2 + m_n^2)}$$

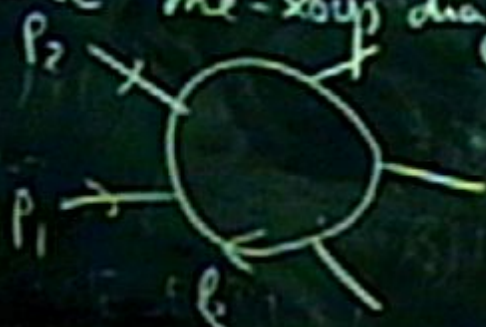
1. Feynman parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \mathcal{D}\beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$$

$$a = \sum_{i=1}^n a_i$$

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + \Delta)^a} = \frac{\Gamma(D/2 - a)}{\Gamma(a)} \Delta^{a - D/2}$$

generic one-loop diag.



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + m_1^2) \left((k+p_1)^2 + m_2^2 \right)^{a_2} \dots \left((k-p_n)^2 + m_n^2 \right)^{a_n}}$$

$$p_i \Rightarrow x_i = x_{i+1} \rightarrow$$

$$q_i = x_0 - x_1$$



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$$p_i = x_i - x_{i+1} \rightarrow p = x_1 - x_2$$

$$q = x_0 - x_1$$

$$q^2 =$$

ica

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$$p_i \Rightarrow x_i - x_{i+1} \rightarrow p_1 = x_1 - x_2$$

$$k_1 = x_0 - x_1$$

$$k_2^2 = x_{01}^2$$

$$(k_2 + p_1)^2 = x_{02}^2$$

$$\int \frac{dx_0}{i \neq k_1} \frac{1}{\prod_{j=1}^n (x_{0j}^2 + v_j^2)^{a_j}}$$

$$= \frac{\Gamma(a - D/2)}{\prod_{j=1}^n \Gamma(a_j)}$$



$$p_i \Rightarrow x_i - x_{i+1} \rightarrow \beta = x_1 - x_0$$

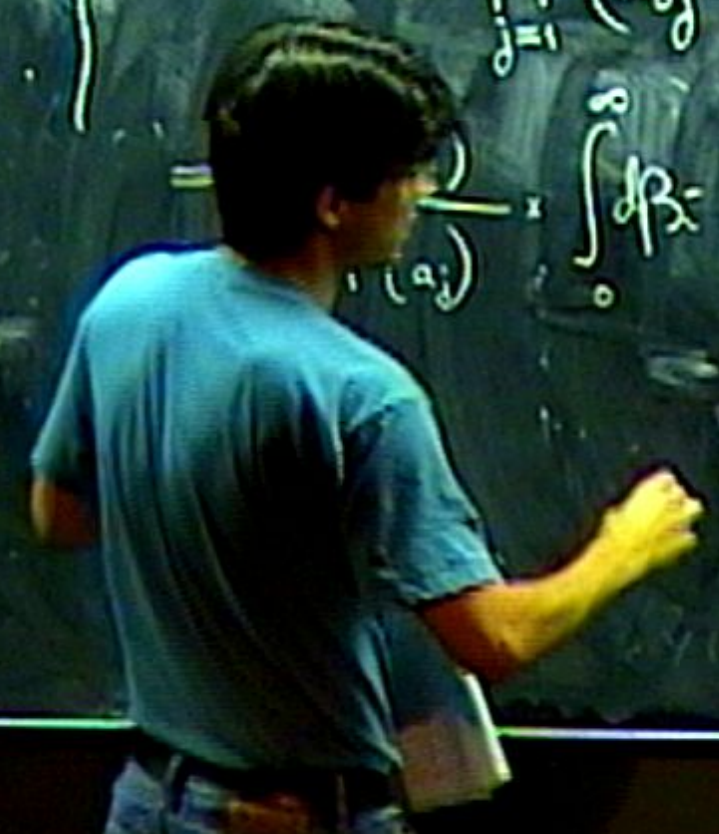
$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\left. \begin{array}{l} k = x_0 - x_1 \\ k^2 = x_{01}^2 \\ (k+p_1)^2 = x_{02}^2 \end{array} \right\} \rightarrow \int \frac{dx}{\prod_{j=1}^n (x^2 + a_j^2)^{a_j}}$$

$$= \frac{1}{\prod_{j=1}^n a_j} \int_0^{\infty} d\beta = \frac{\mathcal{L}(\sum, \beta_0 - 1) \prod_{j=1}^n \beta_0^{a_j - 1}}{\dots}$$



$$p_i \Rightarrow x_i - x_{i+1} \rightarrow \beta = x_1 - x_0$$

$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\int \frac{1}{\prod_{j=1}^n (x_{0j}^2 + v_j^2)^{a_j}}$$

$$= \frac{\Gamma(a_1 - 1) \dots \Gamma(a_n - 1)}{\Gamma(a_1 + \dots + a_n - 1)} \frac{\prod_{j=1}^n \beta_j^{a_j - 1}}{\sum_{k,j} x_{ij}^2 \beta_i \beta_j + \dots}$$

$$p_i = x_i - x_{i+1} \rightarrow p_1 = x_1 - x_2$$

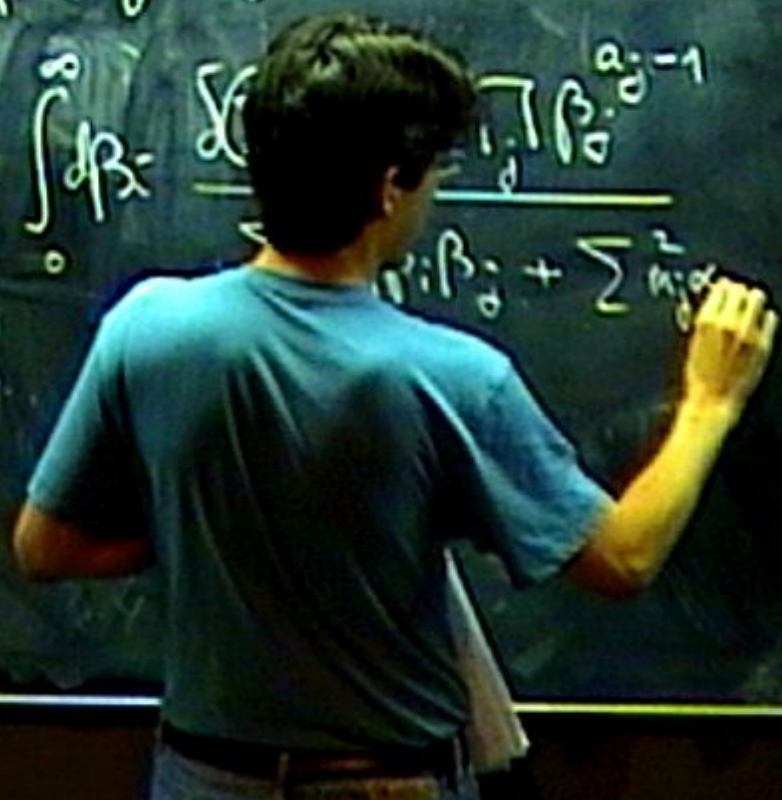
$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\int \frac{1}{\prod_{j=1}^n (x_{0j}^2 + v_j^2)^{a_j}}$$

$$= \frac{\Gamma(a - \frac{1}{2})}{\prod_j \Gamma(a_j)} \int_0^\infty d\beta \frac{\prod_j \beta_j^{a_j - 1}}{\beta + \sum_j k_{ij}^2}$$



$$p_i \Rightarrow x_i - x_{i+1} \rightarrow \beta = x_1 - x_2$$

$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\int \frac{1}{\prod_{j=1}^n (x_{0j}^2 + \alpha_j^2)^{a_j}}$$

$$= \frac{\Gamma(a - \frac{1}{2})}{\prod_j \Gamma(a_j)} \int_0^\infty d\beta \frac{\delta(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{i,j} x_{ij}^2 \beta_i \beta_j + \sum_j \alpha_j^2 \right]}$$

$$p_i = x_i - x_{i+1} \rightarrow \beta = x_1 - x_2$$

$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\int \frac{1}{\prod_{j=1}^n (x_{0j}^2 + v_j^2)^{a_j}}$$

$$= \frac{\Gamma(a - \frac{p}{2})}{\prod_j \Gamma(a_j)} \int_0^\infty d\beta = \frac{\Gamma(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{i,j} x_{ij}^2 \beta_i \beta_j + \sum_j v_j^2 \beta_j \right]^{a - \frac{p}{2}}}$$

$$p_i = x_i - x_{i+1} \rightarrow \beta = x_1 - x_0$$

$$k = x_0 - x_1$$

$$k^2 = x_{01}^2$$

$$(k+p_1)^2 = x_{02}^2$$

$$\int \frac{dx_0}{\prod_{j=1}^n (x_{0j}^2 + v_j^2)^{a_j}}$$

$$= \frac{\Gamma(a - \frac{1}{2})}{\prod_j \Gamma(a_j)} \int_0^\infty d\beta = \frac{\Gamma(\sum_j \beta_j - 1) \prod_j \beta_j^{a_j - 1}}{\left[\sum_{ij} x_{ij}^2 \beta_i \beta_j + \sum_j v_j^2 \beta_j \right]^{a - \frac{1}{2}}}$$

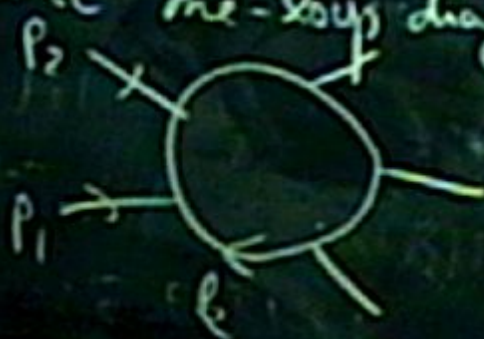
1. Feynman parameter

$$\frac{1}{A_1^{a_1} \dots A_n^{a_n}} = \frac{\Gamma(a)}{\prod_{i=1}^n \Gamma(a_i)} \int \mathcal{D}\beta_i \frac{\delta(\sum \beta_i - 1) \prod \beta_i^{a_i-1}}{(\sum \beta_i A_i)^a}$$

$$a = \sum_{i=1}^n a_i$$

$$\int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + \Delta)^a} = \frac{\Gamma(D/2 - a)}{\Gamma(a)} \Delta^{a - D/2}$$

generic one-loop diag.



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{(k^2 + m_1^2) \left((k+p_1)^2 + m_2^2 \right)^{a_2} \dots \left((k-p_n)^2 + m_n^2 \right)^{a_n}}$$

2. HB representation
up
Want to use

2. HB representation
Want to use ^{up}

$$\int_0^{\infty} \beta_i \lambda \left(\sum_{j=1}^n \beta_j - 1 \right) \prod_{j=1}^n \beta_j^{\alpha_j - 1}$$

2. HB representation
Want to use \uparrow

$$\int_0^\infty \prod \beta_i \delta\left(\sum_{j=1}^n \beta_j - 1\right) \prod \beta_i^{\alpha_i - 1} = \frac{\prod \Gamma(\alpha_i)}{\Gamma(\sum \alpha_i)}$$



2. MB representations

Want to use

factorize w.r.t.

idea

$$\frac{1}{(a+b)^k} =$$

$$\int_0^{\infty} \prod \beta_i \delta\left(\sum_{i=1}^n \beta_i - 1\right) \prod \beta_i^{\alpha_i - 1} = \frac{\prod \Gamma(\alpha_i)}{\Gamma(\sum \alpha_i)}$$

2. MB representations

Want to use

factorize \overline{w}

idea

$$\frac{1}{(a+b)^{\lambda}}$$

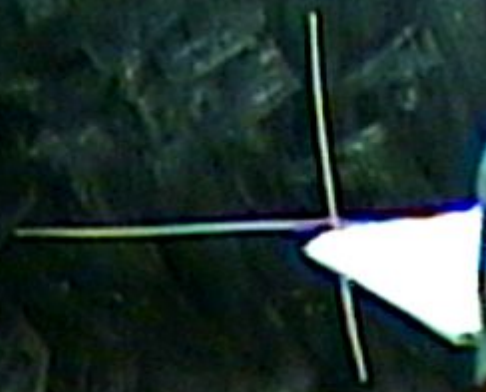
$$= \int_{-i}^{+i\infty+z_0} \frac{1}{b}$$

$+i\infty+z_0$

$$\frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)}$$

$$a^z b^{-\lambda-z}$$

$$\int_0^{\infty} \lambda \beta: \alpha \left(\sum_{j=1}^n \beta_j - 1 \right) \prod_j \beta_j^{\alpha_j - 1} = \frac{\prod_j \Gamma(\alpha_j)}{\Gamma(\sum \alpha_j)}$$



2. HB representation

Want to use

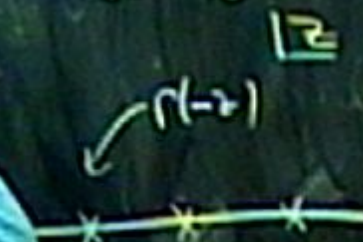
$$\int_0^\infty x^\lambda \prod_{j=1}^n (x + \beta_j)^{-1} \prod_{j=1}^m \beta_j^{\alpha_j - 1} dx = \frac{\prod_{j=1}^m \Gamma(\alpha_j)}{\Gamma(\sum \alpha_j)}$$

factorize w.r.t.

idea

$$\frac{1}{x}$$

$$= \int_{-i\infty - z_0}^{+i\infty - z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z}$$



2. HB representations

Want to use

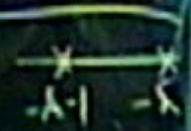
$$\int_0^\infty x^{\lambda-1} \alpha \left(\sum_{j=1}^n \beta_j - 1 \right) \prod_{j=1}^n \beta_j^{\alpha_j - 1} dx = \frac{\prod_{j=1}^n \Gamma(\alpha_j)}{\Gamma(\sum \alpha_j)}$$

factorize w.r.t

idea

$$= \int_{-i\infty - z_0}^{+i\infty - z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z}$$

$\Gamma(\lambda+z)$



$\Gamma(-z)$



2. MB representations

Want to use

$$\int_0^\infty x^{\lambda-1} \prod_{j=1}^n \Gamma(\beta_j - 1) \prod_{j=1}^m \Gamma(\alpha_j - 1) = \frac{\prod_{j=1}^m \Gamma(\alpha_j)}{\Gamma(\sum \alpha_j)}$$

factorize Γ

idea $\frac{1}{(a+z)}$

$$\int_{-i\infty+z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z}$$



Series eq:

$$\frac{a}{(a+b)^x} = b^{-x} \frac{a}{(1 + \frac{a}{b})^x} = b^{-x} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(x) \Gamma(n+1)}$$

$\frac{a}{b} < 1$



2. HB representation

Want to use

$$\int_0^\infty x^\lambda \prod_{j=1}^n \beta_j^{-x} = \frac{\prod_{j=1}^n \Gamma(\beta_j)}{\Gamma(\sum \beta_j)}$$

factorize Γ

idea

$$\frac{1}{(a+b)^k} = \int_{-i\infty - z_0}^{+i\infty - z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(k+z)}{\Gamma(k)} a^z b^{-k-z}$$



$$-k < z_0 < 0$$



2. HB representation

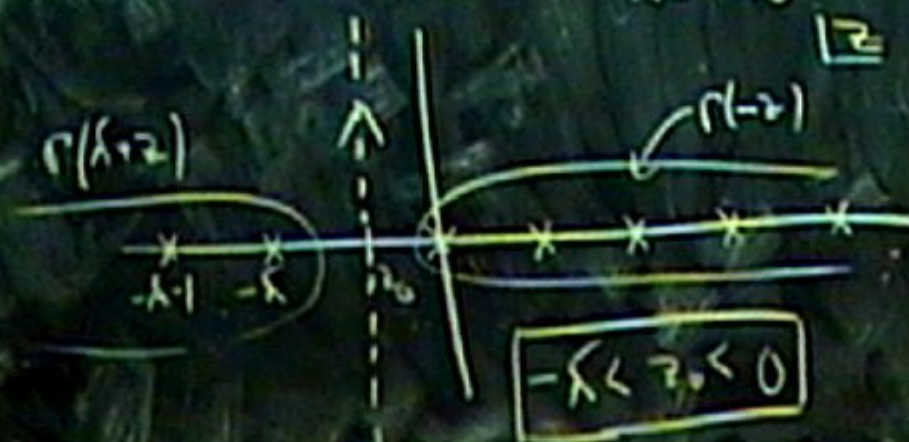
Want to use

$$\int_0^\infty x^\lambda \prod_{j=1}^n \beta_j^{-1} \prod_{j=1}^m \beta_j^{\alpha_j-1} = \frac{\prod_{j=1}^m \Gamma(\alpha_j)}{\Gamma(\sum \alpha_j)}$$

factorize w.r.t.

idea

$$\frac{1}{(a+b)^\lambda} = \int_{-i\infty-z_0}^{+i\infty+z_0} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} a^z b^{-\lambda-z} = b^{-\lambda} \left(\frac{a}{b}\right)^\lambda$$



$$-\lambda < z_0 < 0$$



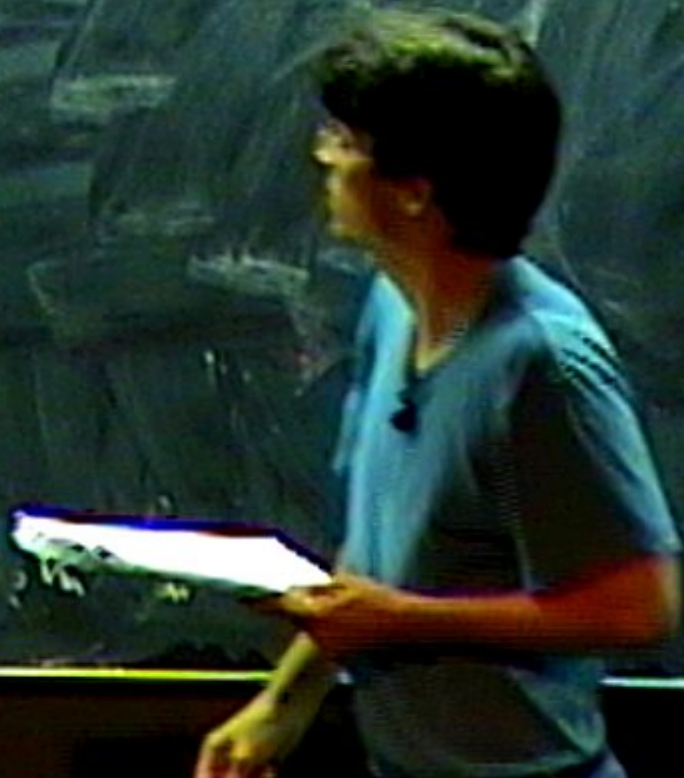
Series eq:

$$\frac{a}{(a+b)^x} = b^{-x} \frac{a}{\left(1 + \frac{a}{b}\right)^x} = b^{-x} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(x) \Gamma(n+1)}$$

$$\frac{a}{b} \ll 1$$

MB

$$\frac{a}{b} \ll 1$$



Series eq:

$$\frac{1}{(a+b)^x} = b^{-x} \frac{1}{(1+\frac{a}{b})^x} = b^{-x} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(x+1)}{\Gamma(x)\Gamma(n+1)}$$

$\frac{a}{b} \ll 1$

MB

0.

close to



Series eq:

$$\frac{a}{(a+b)^x} = b^{-x} \frac{a}{(1 + \frac{a}{b})^x} = b^{-x} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(x)\Gamma(x-n)}$$

$\frac{a}{b} \ll 1$

MB $\frac{a}{b} \ll 1$ close the integration on the right

Series eq:

$$\frac{a}{(a+b)^r} = b^{-r} \frac{a}{(1+\frac{a}{b})^r} = b^{-r} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(r) \Gamma(n+1)}$$

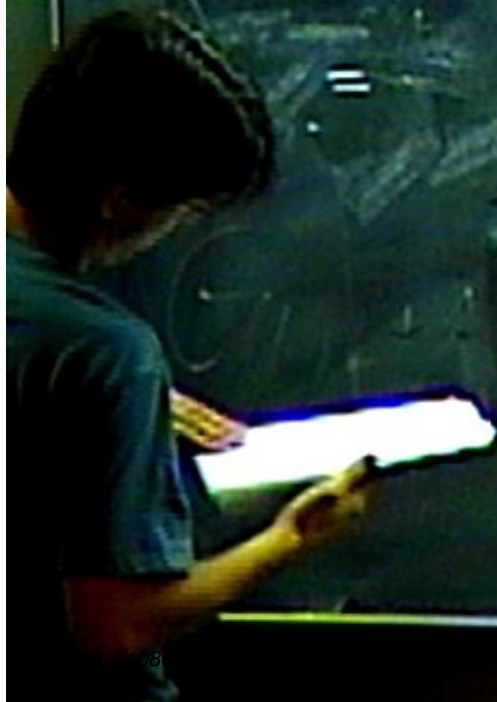
$\frac{a}{b} < 1$

MB

$\frac{a}{b} < 1$

due the interval on the right.

=



Series eq:

$$\frac{1}{(a+b)^s} = b^{-s} \frac{1}{\left(1 + \frac{a}{b}\right)^s} = b^{-s} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(s)\Gamma(n+1)}$$

$\frac{a}{b} < 1$

MB

$\frac{a}{b} < 1$

due to in. contour on the right.

$$= \oint h = \sum_{n=0}^{\infty} \text{Res} \left[\frac{\Gamma(-s)\Gamma(s+1)}{\Gamma(s)} \left(\frac{a}{b}\right)^z b^{-s}, z=n \right]$$

Series eq:

$$\frac{a^x}{(a+b)^x} = b^{-x} \frac{a^x}{(1+\frac{a}{b})^x} = b^{-x} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+1)}{\Gamma(x)\Gamma(n+1)}$$

$$\frac{a}{b} < 1$$

MB

$$\frac{a}{b} < 1$$

due the in. contour on the right.

$$= \int h = \oint h = \left[\frac{\Gamma(-z)\Gamma(z+1)}{\Gamma(x)} \left(\frac{a}{b}\right)^z b^{-x}, z=n \right]$$

$$\sim \frac{(-1)^n}{\epsilon}$$

Series eq:

$$\frac{a^{\lambda}}{(a+b)^{\lambda}} = b^{-\lambda} \frac{a^{\lambda}}{(1 + \frac{a}{b})^{\lambda}} = b^{-\lambda} \sum_{n=0}^{\infty} \left(-\frac{a}{b}\right)^n \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)\Gamma(n+1)}$$

$$\frac{a}{b} \ll 1$$

MB

$$\frac{a}{b} \ll 1$$

close the contour on the right

$$= \int h = \oint h = \sum_{n=0}^{\infty} \text{Res} \left[\frac{\Gamma(-z)\Gamma(\lambda+z)}{\Gamma(\lambda)} \left(\frac{a}{b}\right)^z b^{-\lambda}, z=n \right]$$

$$\Gamma(-n+\epsilon) \sim \frac{(-1)^n}{\epsilon \Gamma(n+1)}$$

What are wedges?

- 1) numerical anal.
- 2) disc. cont.

1. 2

IS ZHAO

What are wedges?

1) numerical eval.

2) disc contour \rightarrow series exp.



What a web?

1) numerical eval.

2) close contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

ZHAO

What are we doing?

1) numerical eval.

2) close contour \rightarrow series exp.
 \rightarrow asymptotic exp.

$s \rightarrow t$

lin. reg. $D=4-2\epsilon$

ZHAO

What are we doing?

1) numerical eval.

2) chose contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dim. reg. $D=4-\epsilon$

out $\epsilon \rightarrow$

ZHAO

What are we doing?

1) numerical eval.

2) close contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

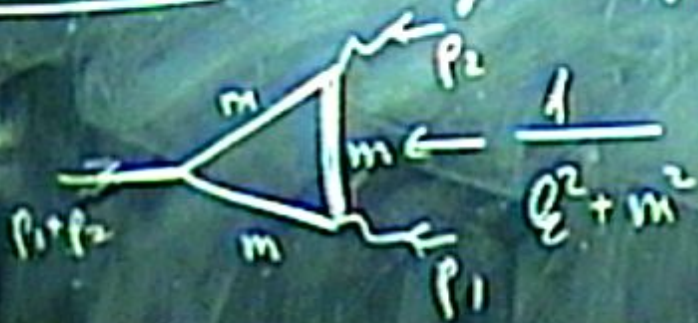
3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion

1)

Example

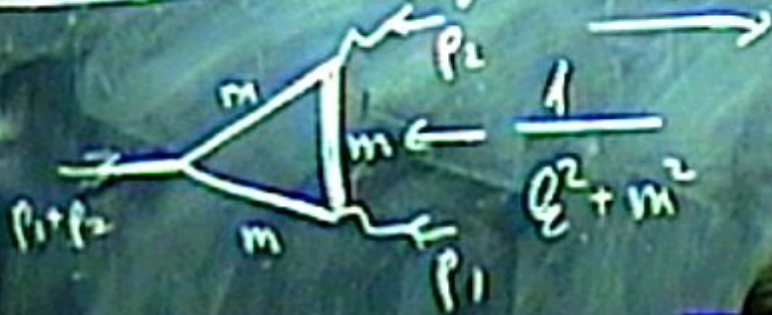


Example triangle integral



Example

triangle integral

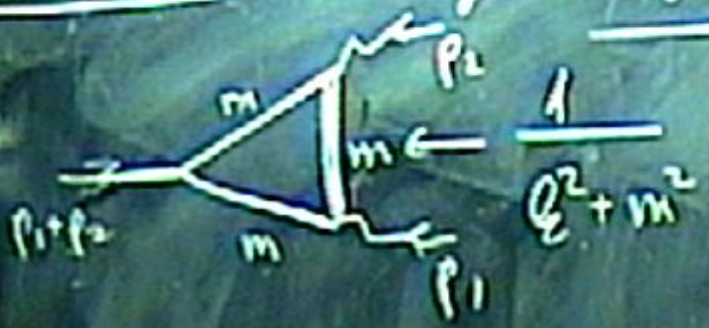


\int



Example

triangle integral

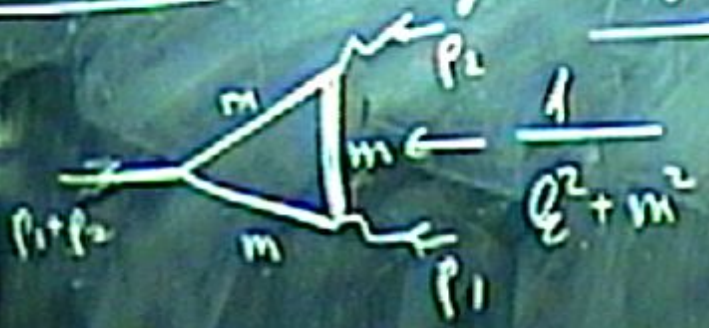


$$\int \frac{d^4 k}{i\pi^2} \frac{1}{(Q^2 + m^2) ((Q + p_1)^2 + m^2) ((Q - p_2)^2 + m^2)}$$



Example

triangle integral



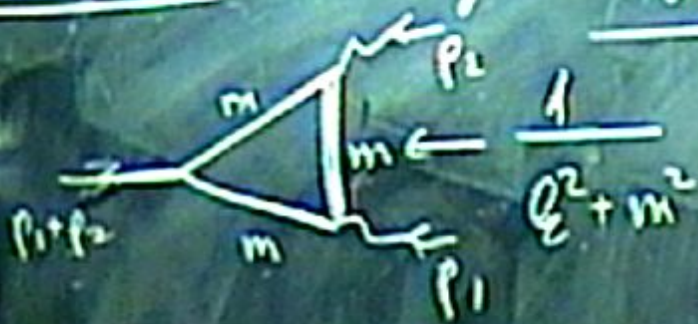
$$\int \frac{d^4 Q}{i\pi^2} \frac{1}{(Q^2 + m^2) ((Q + p_1)^2 + m^2) ((Q - p_2)^2 + m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$
 $S = (p_1 + p_2)^2 \quad m^2$



Example

triangle integral



$$\int \frac{d^4 k}{i\pi^2} \frac{1}{(k^2+m^2)((k+p_1)^2+m^2)((k-p_2)^2+m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$

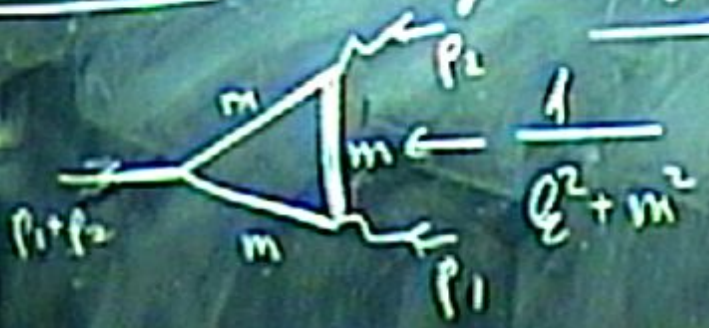
$$s = (p_1 - p_2)^2 = m^2$$

$$a_j = 1 \quad j=1, 2, 3$$

$$\mathcal{L} = \int_0^1 \int_0^{1-\beta_1} d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{\dots}$$

Example

triangle integral



$$\int \frac{d^4 k}{i\pi^2} \frac{1}{(k^2 + m^2) ((k+p_1)^2 + m^2) ((k-p_2)^2 + m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$ $p_1^2 = x_{12}^2$

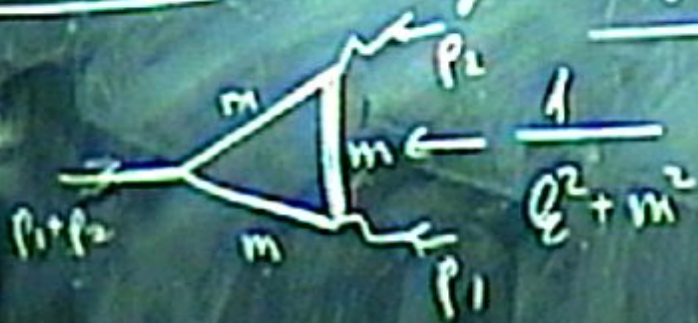
$$s = (p_1 + p_2)^2$$

$$a_j = 1 \quad j=1..3 \quad a = \sum$$

$$\hookrightarrow = \int_0^1 \int_0^{1-\beta_1} d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}$$

Example

triangle integral



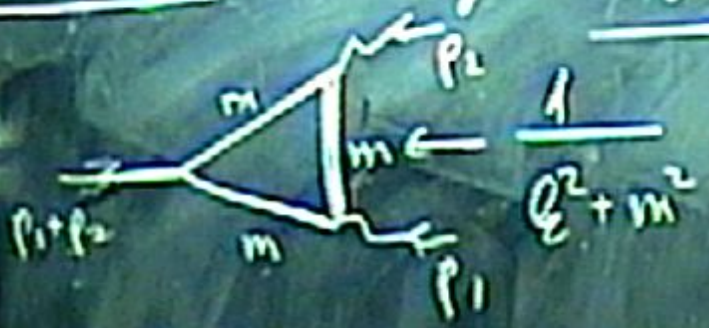
$$\int \frac{d^4 k}{i\pi^2} \frac{1}{(k^2 + m^2) ((k+p_1)^2 + m^2) ((k-p_2)^2 + m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$ $p_1^2 = x_{12}^2 = 0$
 $S = (p_1 + p_2)^2 = x_{13}^2 = m^2$ $x_{23}^2 = 0$
 $a_j = 1, 3$ $a = \sum a_j = 3$

$$\mathcal{L} = \int_0^1 \int_0^{1-x} \mathcal{B}_{j=1,2,3} \delta(x_3 - 1)$$

Example

triangle integral



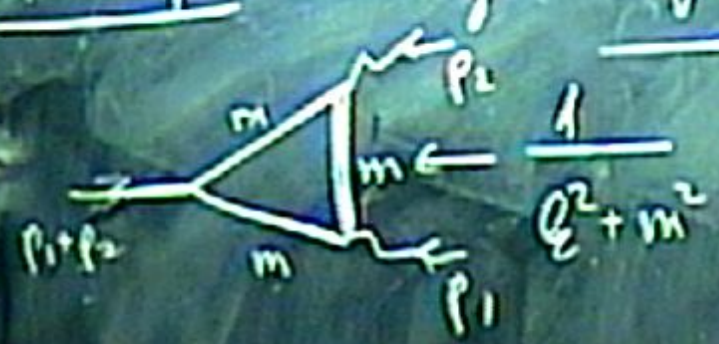
$$\int \frac{d^4 Q}{i\pi^2} \frac{1}{(Q^2 + m^2)((Q + p_1)^2 + m^2)((Q - p_2)^2 + m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$ $p_1^2 = x_{12}^2 = 0$
 $s = (p_1 + p_2)^2 = x_{13}^2 = m^2$ $x_{23}^2 = 0$
 $a_j = 1$ 3 $a = \sum a_j = 3$

$$\hookrightarrow = \int_0^1 \int_0^{1-\beta_1} d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{[\beta_1 \beta_2 \beta_3 s + m^2]}$$

Example

triangle integral



$$\int \frac{d^4 k}{i\pi^2} \frac{1}{(k^2 + m^2) ((k+p_1)^2 + m^2) ((k-p_2)^2 + m^2)}$$

kinematics, $p_1^2 = p_2^2 = 0$ $p_1^2 = x_{12}^2 = 0$

$$S = (p_1 + p_2)^2 = x_{13}^2 = m^2$$

$$a_j = 1 \quad j=1, 2, 3 \quad a = \sum a_j = 3$$

$$\hookrightarrow = \int_0^1 \int_0^{1-\beta_1} d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{[\beta_1 \beta_3 S + (\beta_1 + \beta_2 + \beta_3) m^2]}$$

$$s \int_{\Gamma} \delta(z) dz = s \int_{\Gamma} dz$$



$$S \text{ (triangle diagram)} = S \int d\beta_0 \frac{\delta(\sum \beta_{ij} = 1)}{\beta_1 \beta_3 S + m^2}$$

$$= \int_{-\infty}^{\infty} \frac{d^2 z}{2\pi} \rho(-z) \rho(1+z)$$



$$\begin{aligned}
 \mathcal{I} &= \mathcal{S} \int_{\mathcal{C}_1} dz \frac{\delta(z \beta_{ij} - 1)}{\beta_1 \beta_3 \mathcal{S} + m^2} \\
 &= \int_{-100}^{100} \frac{dz}{2\pi i} \frac{\Gamma(-z) \Gamma^3(1+z)}{\Gamma(3+2z)} \times \left(\frac{m^2}{\mathcal{S}}\right)^{-1-z} \\
 &\quad \uparrow \\
 &\quad -1 < \text{Re } z_0 < 0
 \end{aligned}$$



$$s \int_{-\infty}^{\infty} dt = s \int_{\mathcal{C}} dz \frac{\delta(z \beta_{ij} - 1)}{\beta_1 \beta_3 s + m^2}$$

$$= \int_{-100}^{100} \frac{dz}{2\pi i} \frac{\Gamma(-z) \Gamma^3(1+z)}{\Gamma(3+2z)} \times \left(\frac{m^2}{s}\right)^{-1-z}$$

$$-1 < \text{Re } z < 0$$

What are we doing?

1) numerical eval.

2) close contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion

1)

What are we doing?

1) numerical eval.

2) close contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion

1) assume $\frac{a^2}{s} > 0$. $\Gamma(a+ib) \sim e^{-|b| \frac{\pi}{2}}$ $a, b \in \mathbb{R}$
 $|b| \gg 1$

What are we doing?

1) numerical eval.

2) chose contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dir. reg. $D=4-2\epsilon$: carry out ϵ expansion

1)

$\frac{\epsilon^2}{s} > 0$

$\Gamma(a+ib) \sim e^{-|b| \frac{\pi}{2}}$ $a, b \in \mathbb{R}$
 $|b| \gg 1$

What are we doing?

1) numerical eval.

2) chose contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dir. reg. $D=4-2\epsilon$: carry out ϵ expansion

we $\frac{\epsilon^2}{s} > 0$

$$\Gamma(a+ib) \sim e^{-|b| \frac{\pi}{2}} \quad a, b \in \mathbb{R}$$
$$|b| \gg 1$$

$$z = z_0 + iy$$

What are we doing?

1) numerical eval.

2) chose contour \rightarrow series exp.

\rightarrow asymptotic exp. $s \rightarrow t$

3) in dim. reg. $D=4-2\epsilon$: carry out ϵ expansion

1) assume $\frac{a^2}{s} > 0$. $\Gamma(a+ib) \sim e^{-|b|\frac{\pi}{2}}$ $a, b \in \mathbb{R}$
 $|b| \gg 1$

$$z = z_0 + iy$$
$$dz = i dy$$

$$S \int_{\mathcal{D}} d\beta_j \frac{\delta(\sum \beta_j - 1)}{\beta_1 \beta_2 \beta_3 S + m^2}$$

$$I = \int_{-100}^{100} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(3+2z)} \times \left(\frac{m^2}{S}\right)^{-1-z}$$

$$-1 < \text{Re } z < 0$$



$$s \Delta = s \int d\beta_j \frac{\delta(\sum \beta_j - 1)}{\beta_1 \beta_2 \beta_3 s + m^2}$$

$$H = \int_{-100}^{+100} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma^3(1+z)}{\Gamma^3(1+z)} \times \left(\frac{m^2}{s}\right)^{-1-z}$$

$$-1 < \text{Re } z < 0$$

$$H = \int_{-\infty}^{\infty} \frac{idy}{2}$$

choose z

$$s \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j} \frac{\delta(\beta_j - 1)}{\beta_j \beta_3 s + m^2}$$

$$I = \int_{-\infty}^{\infty} \frac{dz}{2\pi i} \frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(3+2z)} \times \left(\frac{m^2}{s}\right)^{-1-z}$$

\uparrow
 $z < 0$

$Q(z)$

$$\frac{dy}{2\pi i} (z_0 + iy)$$

choose $z_0 = -\frac{1}{2}$

Integrate

$$s \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j} \frac{\delta(\beta_j - 1)}{\beta_j \beta_3 s + m^2}$$

$$I = \int_{-\infty}^{\infty} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\Gamma(1+z)^3}{\Gamma(3+2z)}}_{Q(z)} \times \left(\frac{m^2}{s}\right)^{-1-z}$$

\uparrow
 $-1 < \text{Re } z < 0$

$$I = \int_{-\infty}^{\infty} \frac{idy}{2\pi i} Q(z_0 + iy)$$

choose $z_0 = -\frac{1}{2}$.

NIntegrate

MB.m \rightarrow MBIntegrate

$$s \text{ (triangle)} = s \int d\beta_j \frac{\delta(\sum \beta_j - 1)}{\beta_1 \beta_2 s + m^2}$$

$$I = \int_{-100}^{100} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(3+2z)}}_{Q(z)} \times \left(\frac{s^2}{s}\right)^{-1-z}$$

$-1 < \text{Re } z < 0$

$$I = \int_{-5-i\infty}^{-5+i\infty} \frac{idy}{2\pi i} Q(z_0 + iy)$$

$$= \int_{-5-i\infty}^{-5+i\infty} \frac{idy}{2\pi i} Q\left(-\frac{1}{2} + iy\right)$$

choose $z_0 = -\frac{1}{2}$.

NIntegrate
 MB.m \rightarrow MBIntegrate

$$x = \frac{1}{\sqrt{1+z}}$$

$$z = \frac{1 - \sqrt{1+4x^2}}{1 + \sqrt{1+4x^2}}$$

Γ

$$\frac{1}{(q^2+m^2) ((q+p_1)^2+m^2) ((q-p_2)^2+m^2)}$$

$$\text{so, } p_1^2 = p_2^2 = 0 \quad p_1^2 = x_{12}^2 = 0$$

$$s = (p_1+p_2)^2 = x_{13}^2 = m^2 \quad x_{23}^2 = 0$$

$$a_j = 1 \quad j=1,2,3 \quad a = \sum a_j = 3$$

$$\Gamma = \int_0^1 \prod_{j=1,2,3} d\beta_j \frac{\delta(\sum \beta_j - 1)}{[\beta_1 \beta_3 s + \underbrace{(\beta_1 + \beta_2 + \beta_3)}_1 m^2]}$$

$$s \text{ (diagram)} = s \int d\beta_j \frac{\delta(\sum \beta_j - 1)}{\beta_1 \beta_3 s + m^2} = I(s)$$

$$I = \int_{-100}^{100} \frac{dz}{2\pi i} \underbrace{\frac{\Gamma(-z)\Gamma(1+z)^3}{\Gamma(3+2z)}}_{Q(z)} \times \left(\frac{m^2}{s}\right)^{-1-z}$$

$-1 < \text{Re } z < 0$

$$I = \int_{-5-i\infty}^{5-i\infty} \frac{idy}{2\pi i} Q(z_0 + iy)$$

$$\approx \int_{-5}^{5} \frac{idy}{2\pi i} Q\left(-\frac{1}{2} + iy\right)$$

choose $z_0 = -\frac{1}{2}$.

NIntegrate

MB.m \rightarrow MBIntegrate

2) close contour \rightarrow expansion

2) close contour \rightarrow expansion

$$x = \frac{z^2}{5} \quad |z| < 4$$

$$x = -1 - 2i$$



2) close contour \rightarrow expansion

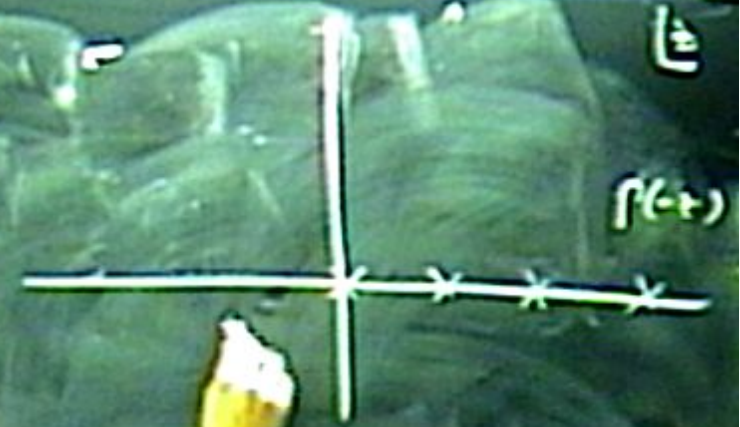
$$x = \frac{z^2}{5} \quad |z| < 1$$

$x^{-1/2} \rightarrow$ close contour

2) close contour \rightarrow expansion

$$x = \frac{a^2}{5} \quad r < 1$$

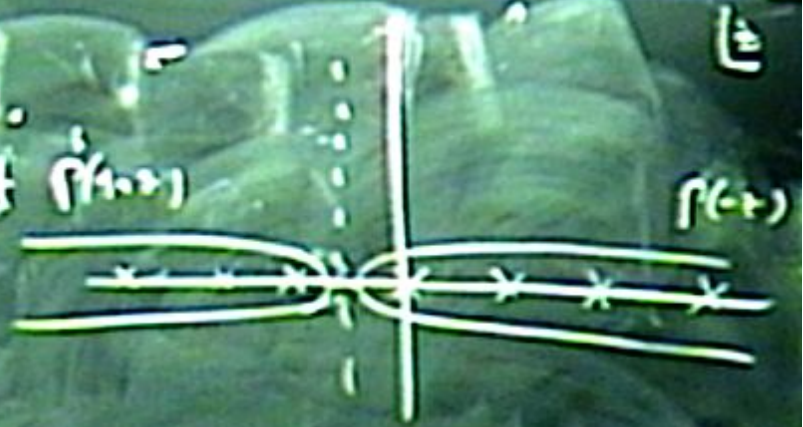
x^{-1-2} \rightarrow close contour on right



2) close contour \rightarrow expansion

$$x = \frac{a^2}{5} \quad r < 1$$

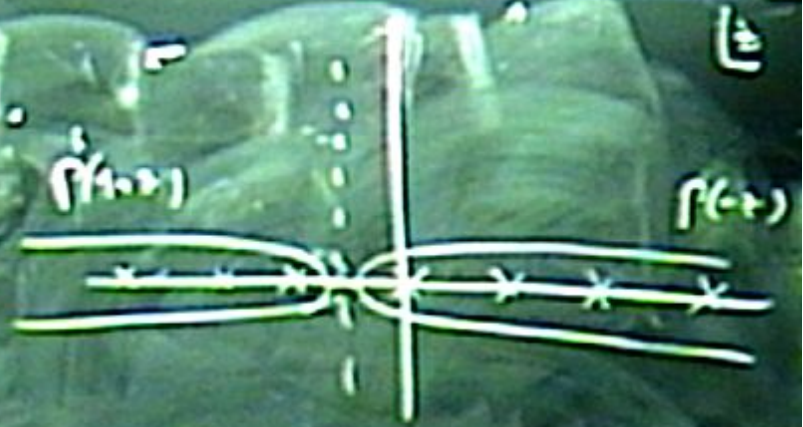
x^{-1-2} \rightarrow close contour on right ($\sigma > 0$)



2) close contour \rightarrow expansion

$$x = \frac{z^2}{5} \quad r < 4$$

x^{-1-2} \rightarrow close contour on left $f(z)$



2) close contour \rightarrow expansion

$$x = \frac{z^2}{5} \quad |z| < 1$$

$x^{-1-2} \rightarrow$ close contour on left side $f(z)$

$$\sum_{n=-1}^{-\infty} \text{Res}[G(z), z=0]$$

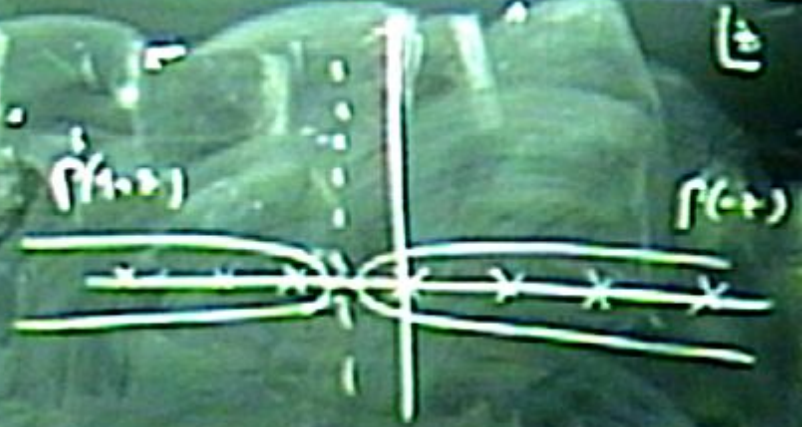


2) close contour \rightarrow expansion

$$x = \frac{z^2}{5} \quad |z| < \sqrt{5}$$

$x^{-1/2} \rightarrow$ close contour on left side $f(z)$

$$T(x) = \sum_{n=-1}^{-\infty} \text{Res}[G(z), z=0]$$

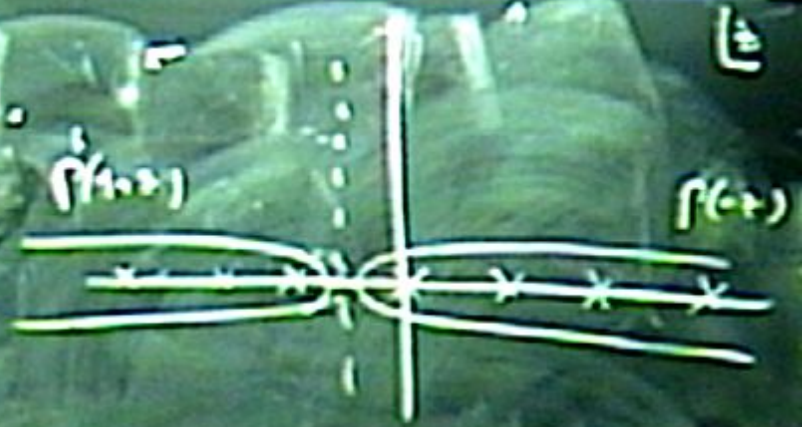


2) close contour \rightarrow expansion

$$x = \frac{a^2}{5} \quad x < 4$$

x^{-1-2} \rightarrow close contour on left $f(z)$

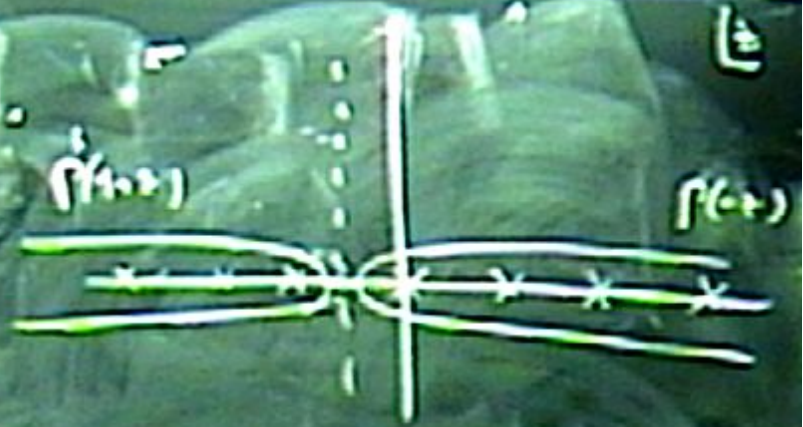
$$I(x) = \sum_{n=-\infty}^{\infty} \text{Res}[G(z), z=n]$$



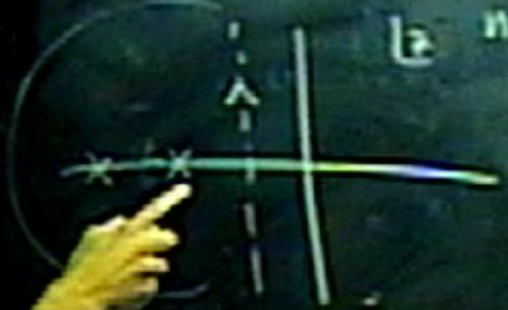
2) close contour \rightarrow expansion

$$x = \frac{v^2}{5} \quad v \ll 1$$

$x^{-1-2} \rightarrow$ close contour on left



$$I(x) = \sum_{n=-1}^{\infty} \text{Res}[G(z), z=0]$$



• could resum series

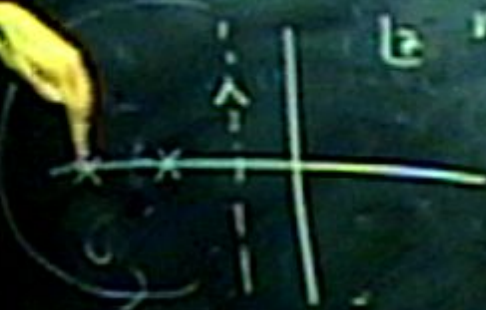
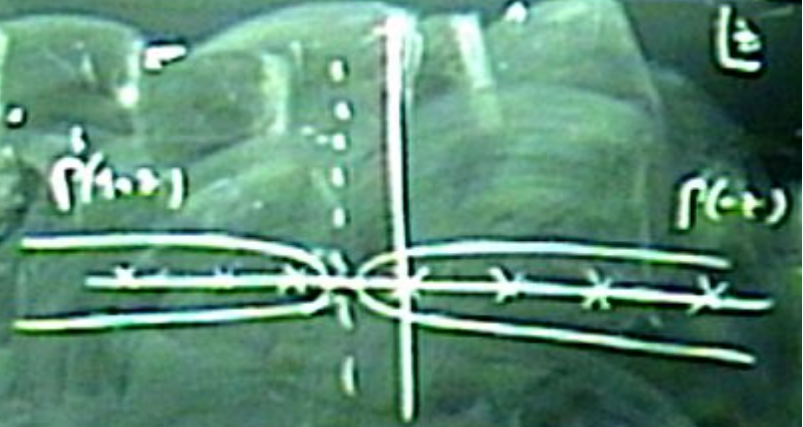
• obtain approx. formula for $v \ll 1$

2) close contour \rightarrow expansion

$$x = \frac{v^2}{5} \quad v \ll 1$$

$x^{-1/2} \rightarrow$ close contour on left

$$I(x) = \sum_{n=-1}^{\infty} \text{Res}[G(z), z=n]$$

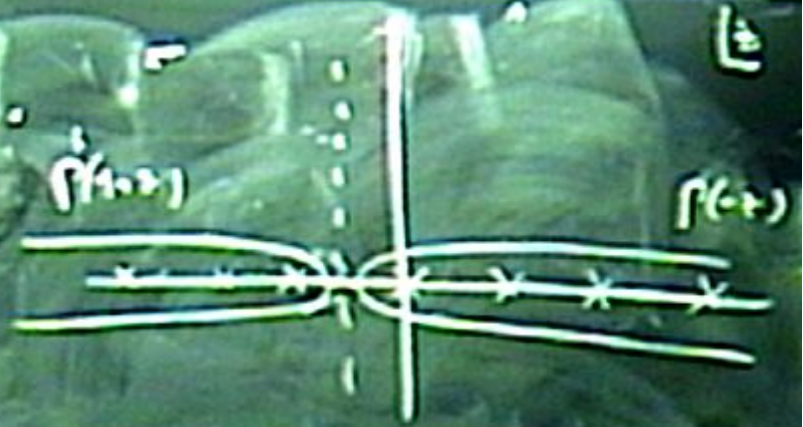


- could resum series
- obtain approx. formula for $v \ll 1$

2) close contour \rightarrow expansion

$$x = \frac{v^2}{5} \quad v \ll 1$$

$x^{-1/2} \rightarrow$ close contour on left



$$I(x) = \sum_{n=-1}^{\infty} \text{Res}[G(z), z=0]$$

- could resum series
- obtain approx. formula for $v \ll 1$

$$I(x) \approx \text{Res}[G(z), z=-1] + O(x)$$

$$x = \sqrt{z}$$

$$z = \frac{1 + \sqrt{1+4x}}{1 + \sqrt{1+4x}}$$

$$F(x) = \frac{1}{2} \log^2(z(x))$$

$$\frac{1}{(z^2+m^2) \left((z+p_1)^2+m^2 \right) \left((z-p_2)^2+m^2 \right)}$$

$$\text{is, } p_1^2 = p_2^2 = 0 \quad p_1^2 = x_{12}^2 = 0$$

$$s = (p_1 + p_2)^2 = x_{13}^2, \quad m^2 = x_{23}^2 = 0$$

$$a_j = 1 \quad j=1, 3 \quad a = \sum a_j = 3$$

$$L = \int_0^1 d\beta_{j=1,2,3} \frac{\delta(\sum \beta_j - 1)}{[\beta_1 \beta_3 s + \underbrace{(\beta_1 + \beta_2 + \beta_3)}_1 m^2]}$$