Title: A Perturbation Solution Of The Mechanical Bidomain Model

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Abstract: This research focuses on finding analytical solutions to the mechanical bidomain model of cardiac tissue. In particular, a perturbation expansion is used to analyze the equations, with the perturbation parameter being inversely proportional to the spring constant coupling the intracellular and extracellular spaces. The results indicate that the intracellular and extracellular pressures are not equal, and that the two spaces move relative to each other. This calculation is complicated enough to illustrate the implications of the mechanical bidomain model, but is nevertheless simple enough to solve analytically. The zeroth-order of the perturbation expansion reveals that the intracellular and extracellular displacements are equal, thus making it unnecessary to account for either space on an individual basis. Yet, in the first-order of the expansion we see a shift and the intracellular and extracellular displacements are unequal. One application of the calculation is to the mechanical behavior of active cardiac tissue surrounding an ischemic region. Also, a hypothesis for the physical meaning of the pressure inequality is if this inequality is held for an extended period of time it may cause fluid to flow across the cell membrane and in the tissue.

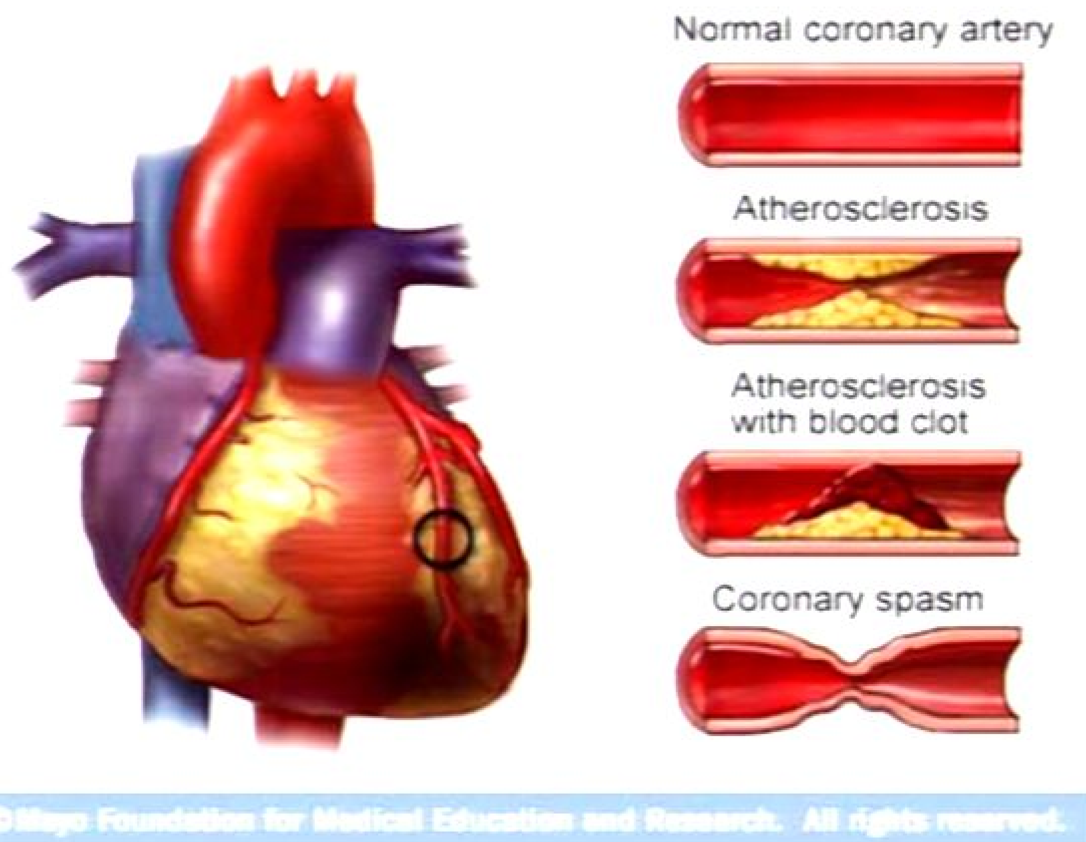
Introduction

- Mechanical bidomain model for cardiac tissue (similar to electrical bidomain model).
- Macroscopic continua.
- Accounts for the intracellular and extracellular spaces individually with a coupling constant K.
- Generalization of monodomain model - three contributions to the stress:
  - hydrostatic pressure,
  - active fiber tension,
  - and an extracellular collagen matrix.

Introduction (cont'd)

- Two qualitative predictions:
  - 1) Pressures are not equal, and
  - 2) ε (small as K becomes large)
- Our goal is to look at a more complicated two dimensional example, which better illustrates the implications of the model but nevertheless is simple enough to solve analytically.

Introduction (cont'd)



A region of ischemia surrounded by healthy tissue is one application of our work.

Methods

- We derive two-dimensional mechanical bidomain equations in terms of stream functions:

$$\frac{\partial T}{\partial x} - \frac{\partial p}{\partial x} + \gamma \frac{\partial^3 \phi}{\partial x^2 \partial y} = K(u_x - w_x)\,, \qquad (1)$$

$$-\frac{\partial p}{\partial y} = K(u_y - w_y), \qquad (2)$$

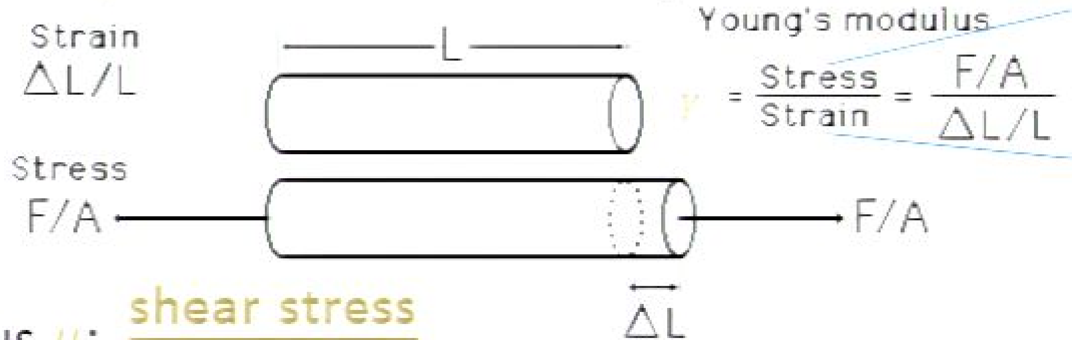
$$-\frac{\partial q}{\partial x} + \mu \left(\frac{\partial^3 \eta}{\partial x^2 \partial y} + \frac{\partial^3 \eta}{\partial y^3}\right) = -K(u_x - w_x), \qquad (3)$$

$$-\frac{\partial q}{\partial y} - \mu \left(\frac{\partial^3 \eta}{\partial x \partial y^2} + \frac{\partial^3 \eta}{\partial x^3}\right) = -K(u_y - w_y). \qquad (4)$$

- p and q are the intracellular and extracellular pressures, respectively.
- ϕ and η are the intracellular and extracellular stream functions, respectively.
- $T(x, y)$ is the active tension of the fibers
- γ is the intracellular Young's modulus along the fiber length.
- μ is the shear modulus of the isotropic extracellular space.
- K is a spring constant coupling the intracellular and extracellular spaces.

Background

- Young's modulus γ: a measure of elasticity -



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- Shear modulus μ: $\frac{\text{shear stress}}{\text{shear strain}}$
  - the component of stress that causes parallel layers of a material to move relative to each other in their own planes
  - strain that is applied parallel or tangential to the surface of a material
- Spring constant K: the force necessary to stretch (or compress) a spring a unit length.
- Stream functions ϕ and η: used to plot streamlines, which represent the trajectories of particles in a steady flow (streamlines are perpendicular to equipotential lines). The difference between the stream function values at any two points gives the volumetric flow rate (or volumetric flux) through a line connecting the two points

Methods (cont'd)

- By using stream functions, we ensure that the intracellular and extracellular displacements, $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$, are divergenceless, implying that each space is incompressible.

$$\left(u_x = \frac{\partial \phi}{\partial y}, u_y = -\frac{\partial \phi}{\partial x}, w_x = \frac{\partial \eta}{\partial y}, w_y = -\frac{\partial \eta}{\partial x}\right)$$

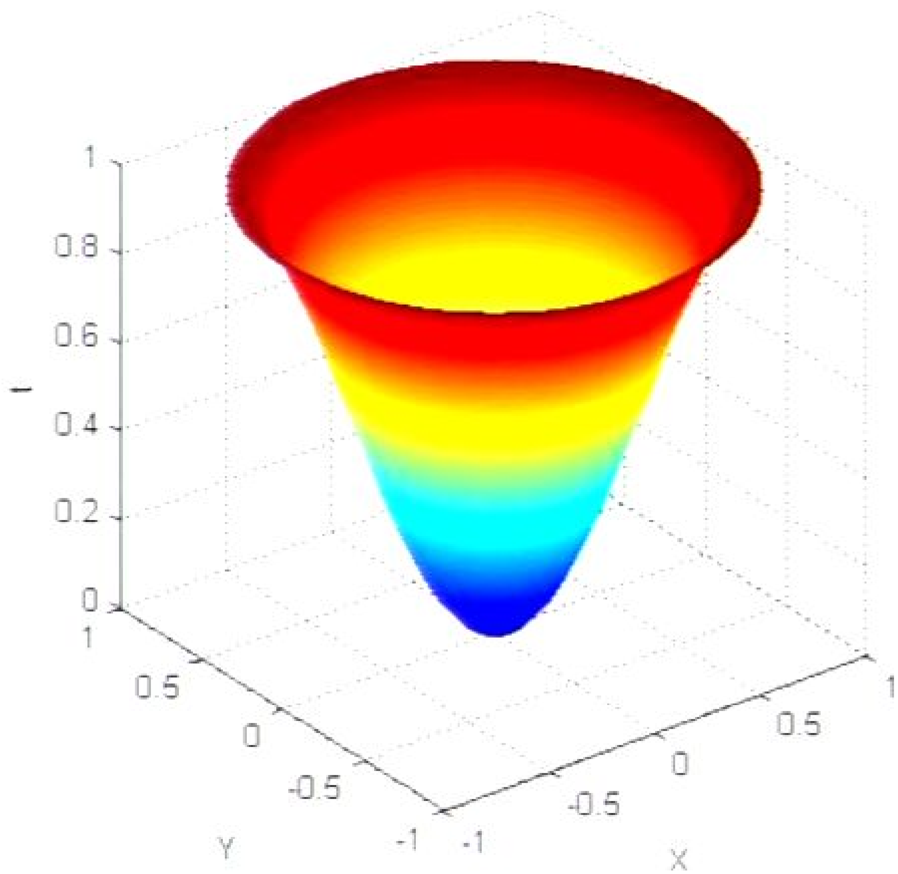
- We assume the fibers are uniform and in the x-direction.
- The equation we used for tension is:

$$T = T_o \frac{(x^2+y^2)}{a^2}\left[2 - \frac{(x^2+y^2)}{a^2}\right] = T_o(X^2+Y^2)[2-(X^2+Y^2)].$$

- As you can see, we scaled our equation by setting $X = \frac{x}{a}$ and Y$= \frac{y}{a}$, where a is a characteristic length.

Methods (cont'd)

- The figure below is representative of the tension.
- Physical meaning



Methods (cont'd)

- We introduced the dimensionless parameters:

$$\varepsilon = \frac{\mu}{Ka^2} \text{ and } \zeta = \frac{\gamma}{\mu}$$

- After some derivations and manipulations, we found these relationships amongst our original equations:
  - $\frac{a^2}{\mu}\nabla^2 p - \frac{a^2}{\mu}\frac{\partial^2 T}{\partial X^2} - \zeta\frac{\partial^4 \phi}{\partial X^3 \partial Y} = 0,$ (5)
  - $\nabla^2 q = 0,$ (6)
  - $\varepsilon\frac{a^2}{\mu}\frac{\partial^2 T}{\partial X \partial Y} + \varepsilon\zeta\frac{\partial^4 \phi}{\partial X^2 \partial Y^2} = \nabla^2 \lambda,$ (7)
  - $\varepsilon\nabla^4(\phi - \lambda) = -\nabla^2 \lambda.$ (8)
- Monodomain Equation (addition of Eqs. 7 and 8).

Methods (cont'd)

- With these equations, we performed a perturbation expansion, where:

$$\phi = \phi_o + \varepsilon\phi_1 + \cdots + \varepsilon^n\phi_n + \cdots, \quad (9)$$
$$\lambda = \lambda_o + \varepsilon\lambda_1 + \cdots + \varepsilon^n\lambda_n + \cdots, \quad (10)$$
$$p = p_o + \varepsilon p_1 + \cdots + \varepsilon^n p_n + \cdots, \quad (11)$$
$$q = q_o + \varepsilon q_1 + \cdots + \varepsilon^n q_n + \cdots. \quad (12)$$

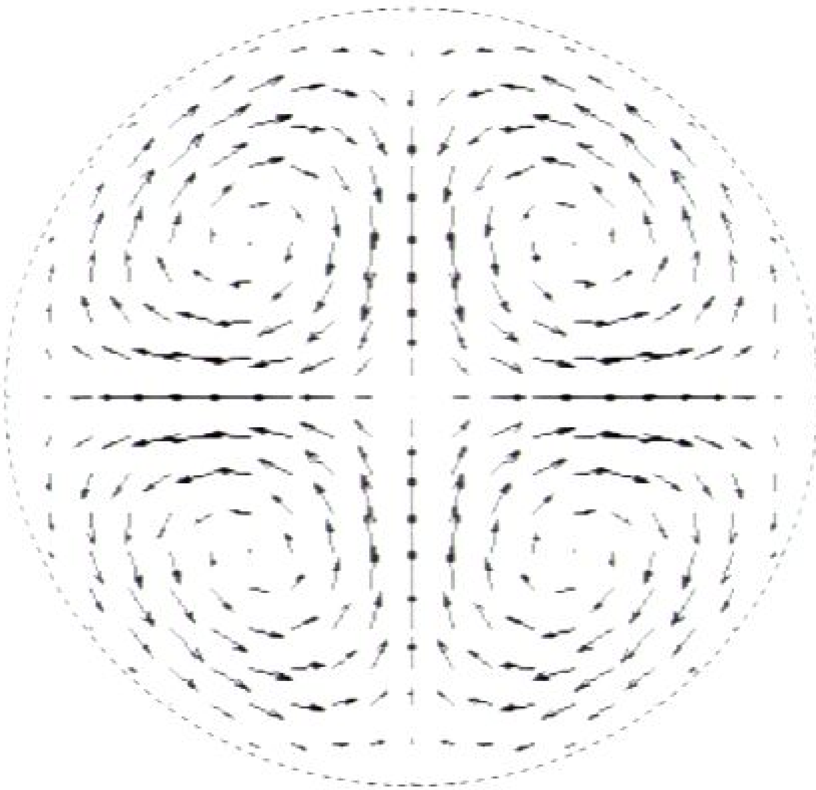
- What is perturbation expansion?

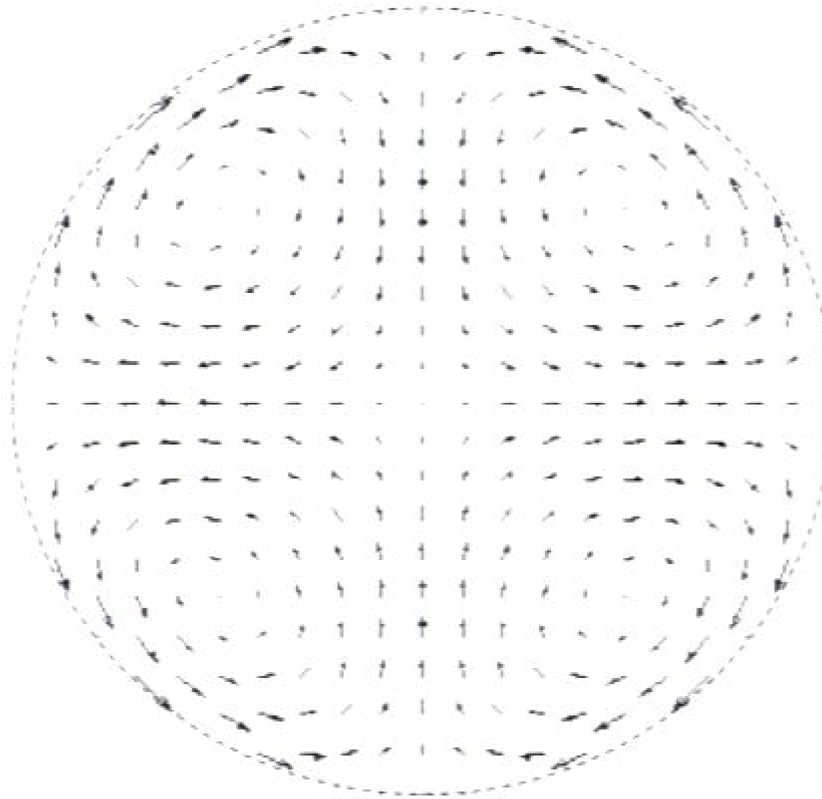
  Mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly, by starting from the exact solution of a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

Results

1. $\phi_o = \frac{a^2 T_o}{3(3\zeta+16)\mu} XY(X^2+Y^2-1)^2$
2. $\phi_1 = 0$
3. $\lambda_o = 0$
4. $\lambda_1 = -\frac{32a^2 T_o}{3(3\zeta-16)\mu} XY(X^2+Y^2-1)$
5. $\lambda = \phi - \eta$ implies that $\eta_o = \phi_o$ and $\eta_1 = -\lambda_1$
6. $p_o = -\frac{4T_o}{3(3\zeta+16)}[(\zeta+10)X^4 - (3\zeta+20)X^2 + 12X^2Y^2 - 4Y^2 + 2Y$
7. $q_o = -\frac{4T_o}{3(3\zeta+16)}(Y^2 - X^2)$

Results (cont'd)





- The figure on the left shows the displacement $\vec{\mathbf{u}}$ to zeroth order (which is the same as $\vec{\mathbf{w}}$) as a vector field plot.
- The figure on the right contains a vector plot of the difference between the intracellular and extracellular displacements, $\vec{\mathbf{u}} - \vec{\mathbf{w}}$, which is only non-zero in the first order.

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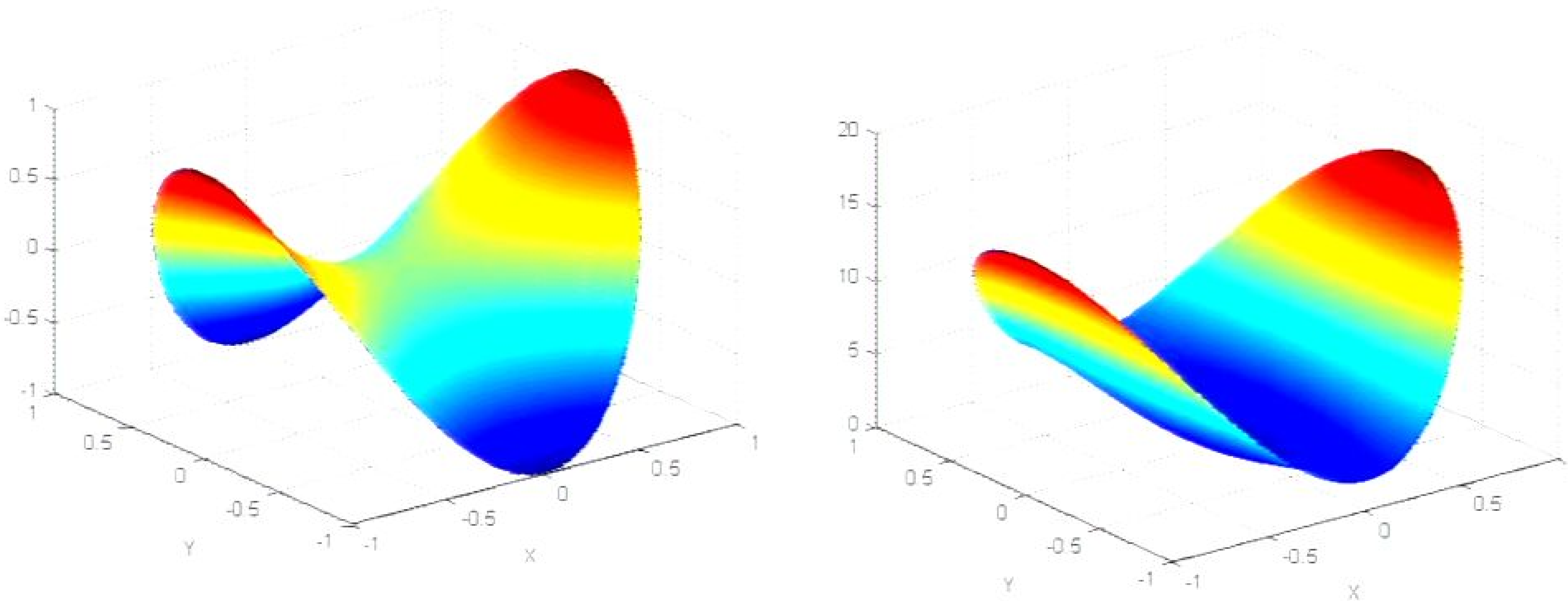
Results (cont'd)



- The left and right hand figures show the intracellular and extracellular pressures, respectively, as a function of position.
- Note that the two pressures have very different spatial distributions.
- Physical meaning

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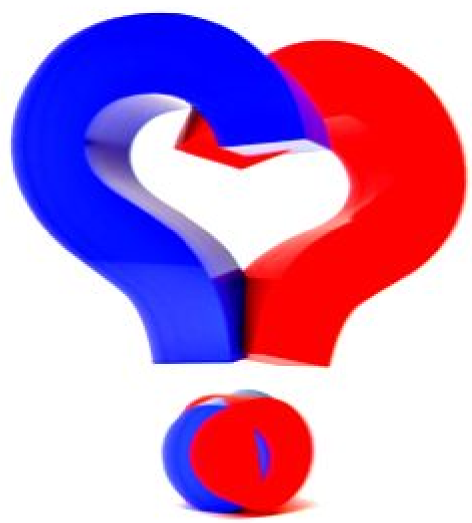

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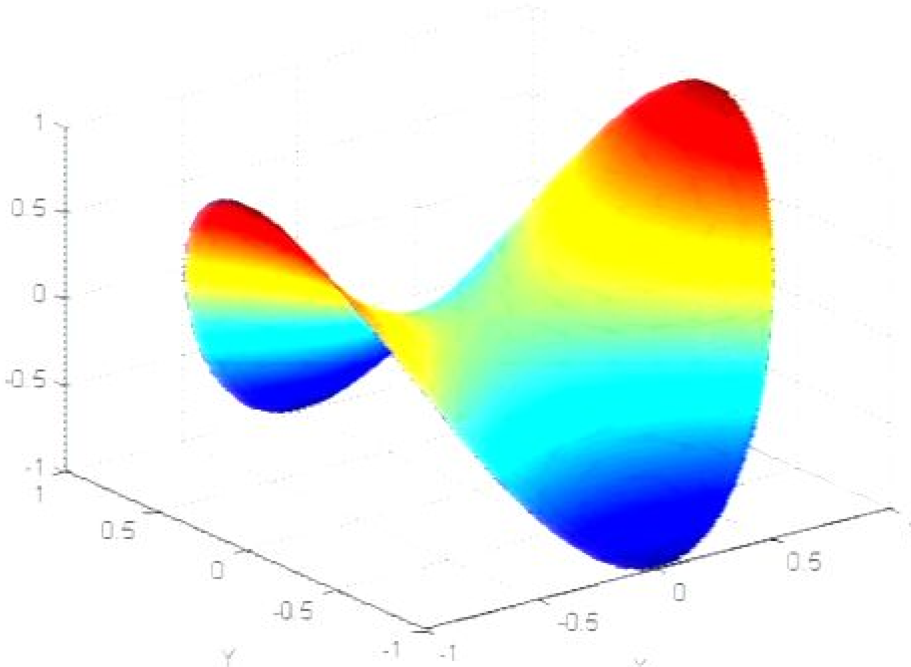

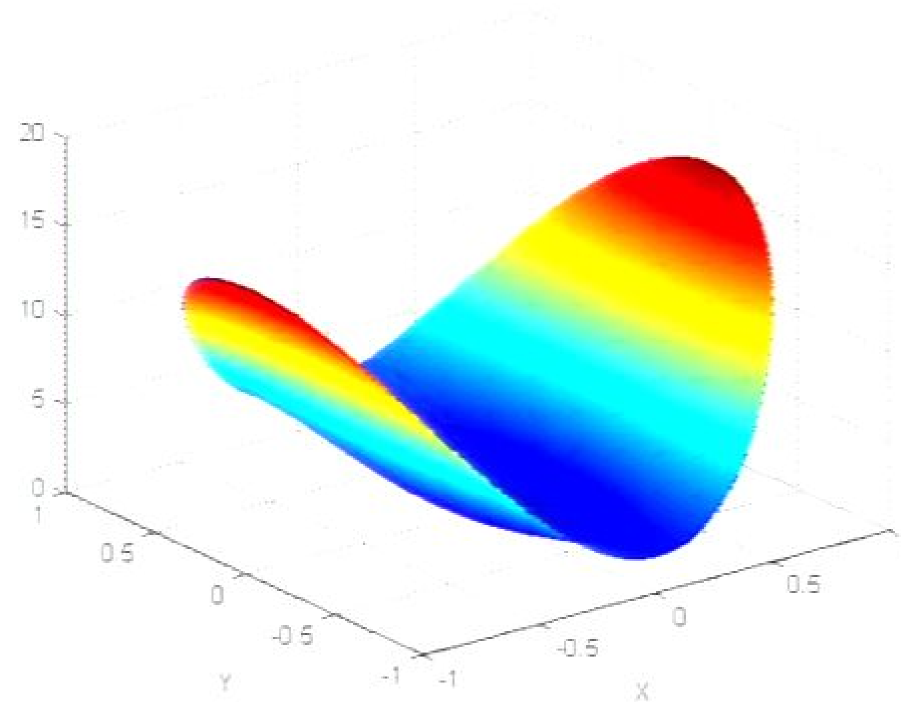

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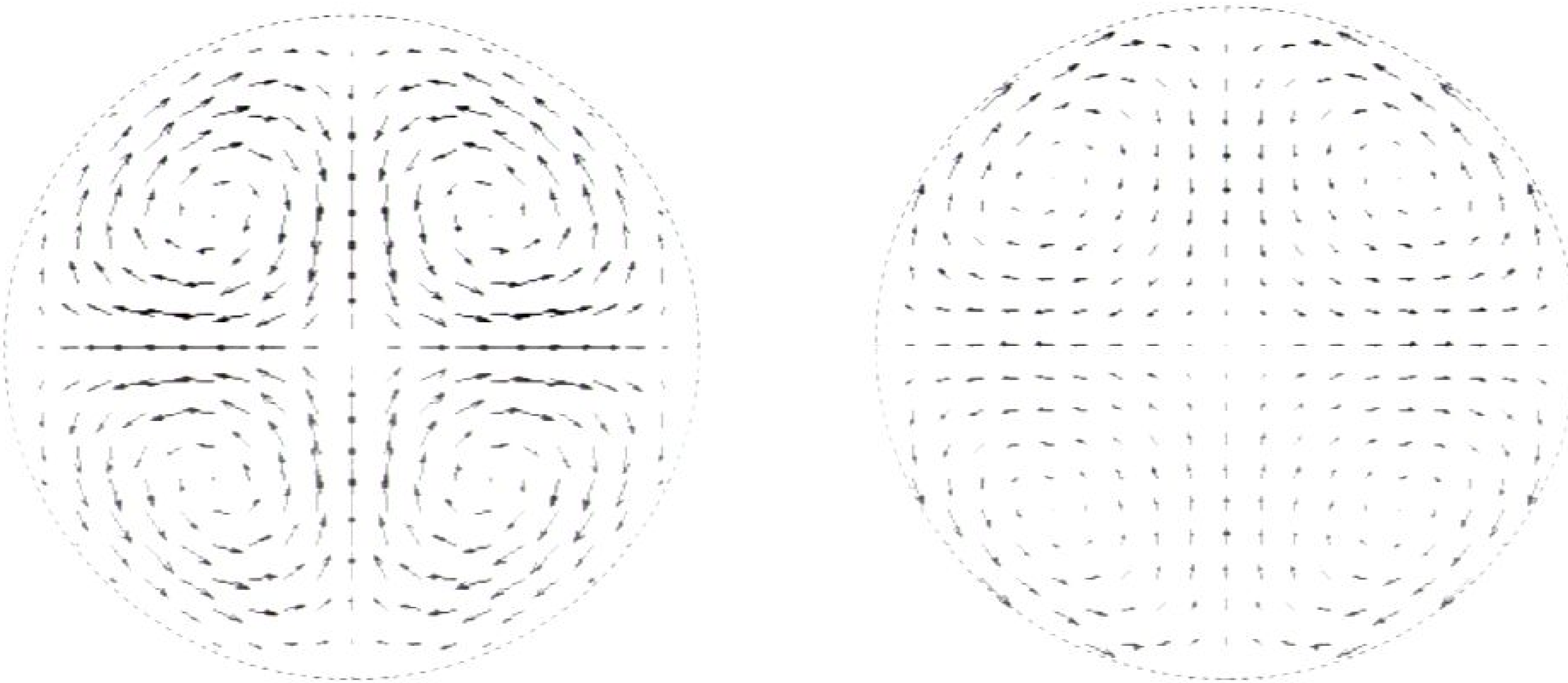

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