Title: On the Existence of a Residue Entangled State in eLOCC Transformations

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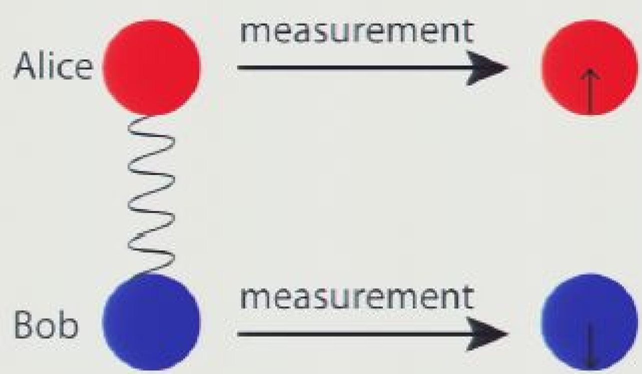
Abstract: Quantum entanglement is a valuable resource in the field of quantum information science and allows one to accomplish many information processing tasks. In quantum transformations an entangled state A can be converted to another state B through local operations assisted by classical communication (LOCC). It has also been demonstrated that there exist entangled states A, B, C such that state A cannot be converted to a state B, but A otimes C can be converted to B otimes C by LOCC, where C is a suitably chosen entangled state acting as the catalyst. This is known as entanglement assisted LOCC or eLOCC. I will show that for certain A and B it is possible to obtain an extra entangled state R, called the residue entangled state in an eLOCC transformation. That is to say A otimes C can be converted to B otimes R otimes C even though A cannot be converted to B by LOCC. I will discuss the necessary and sufficient conditions for such a transformation to occur.

Introduction: Quantum Entanglement as a Resource

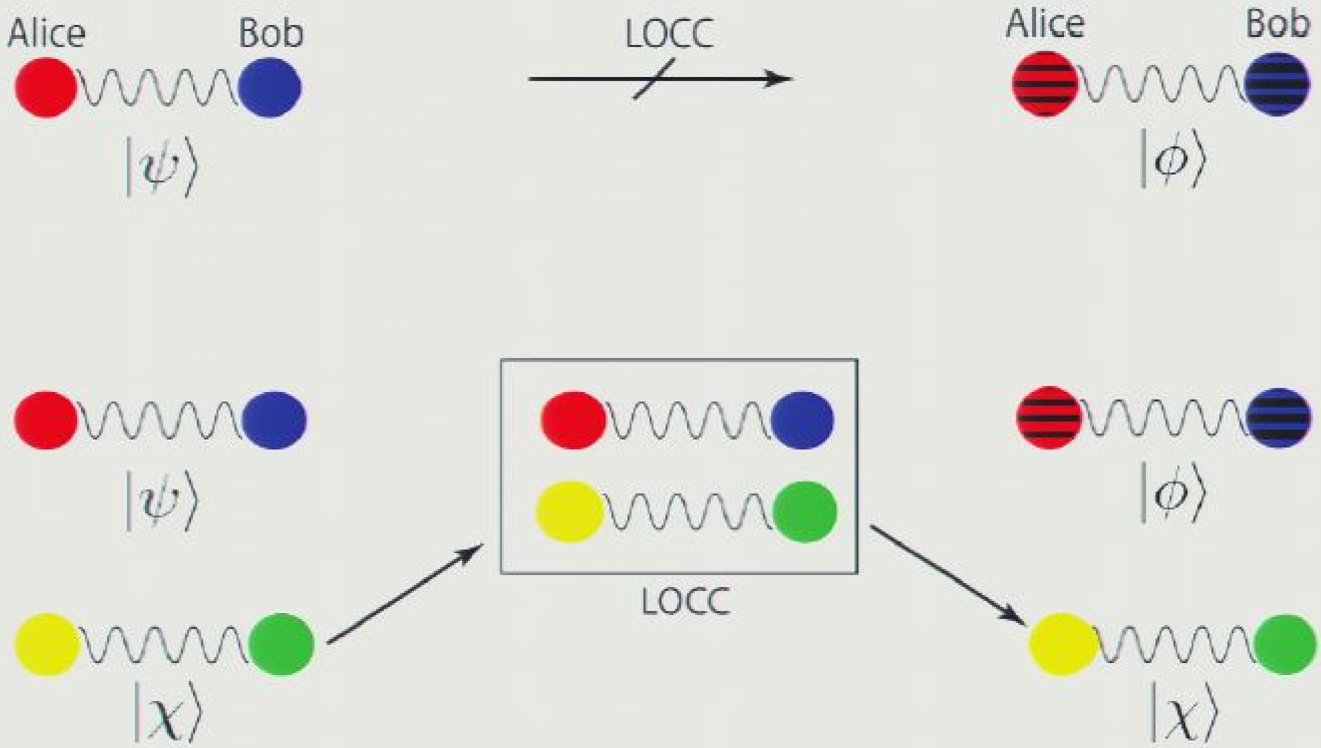
- Quantum entanglement is a valuable physics resource in the field of quantum info science.
- Local operations assisted by classical communication (LOCC).
- To accomplish many quantum info tasks, the entanglement is "consumed".



- An entangled state can be used as a catalyst.

Entanglement Catalysis

- entanglement assisted LOCC (eLOCC).



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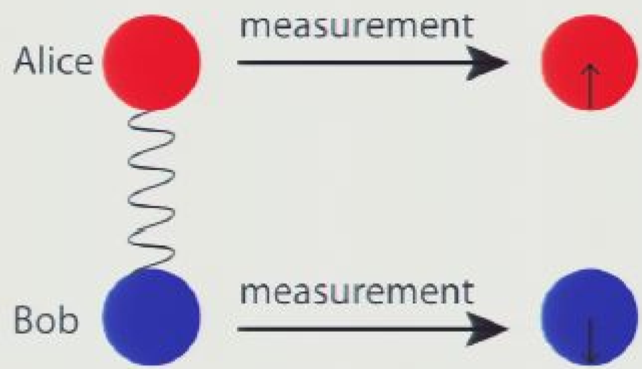

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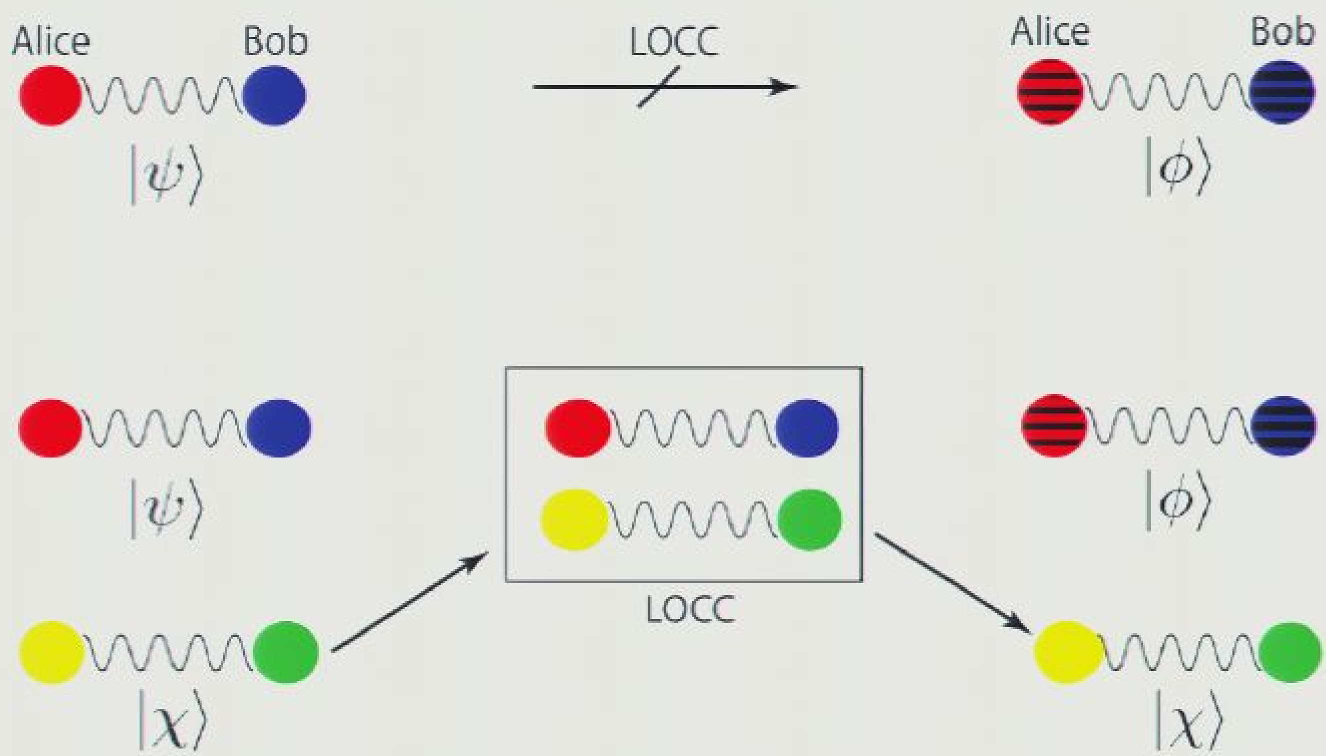
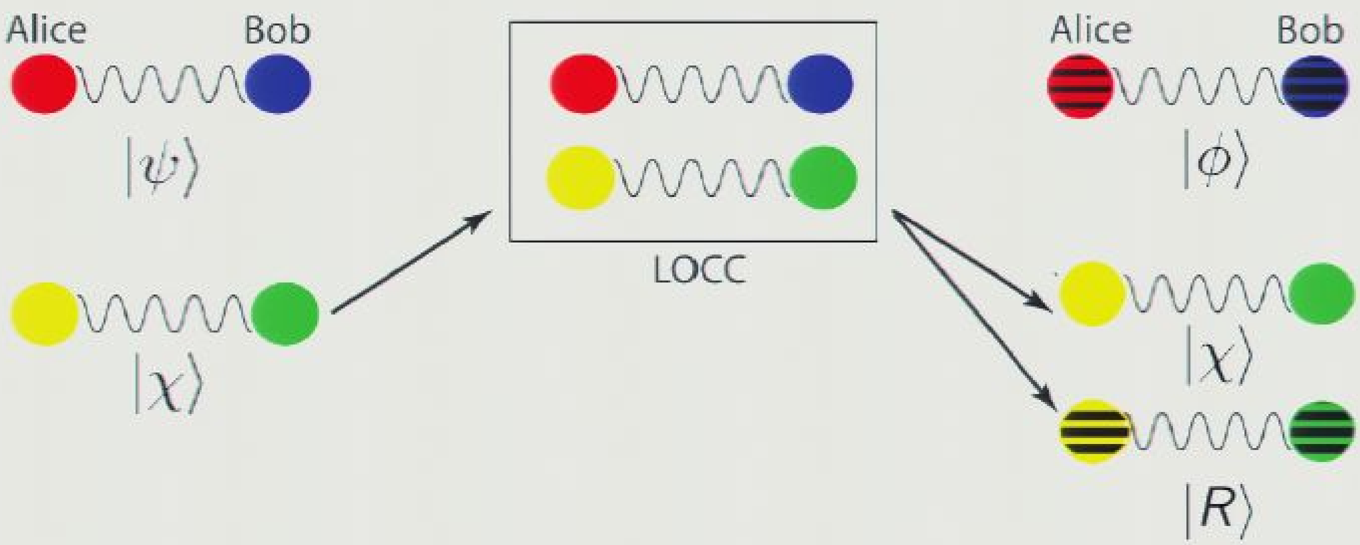


$$|\psi\rangle \otimes |\chi\rangle \longrightarrow |\phi\rangle \otimes |\chi\rangle$$

Entanglement Catalysis: Residue Entangled State

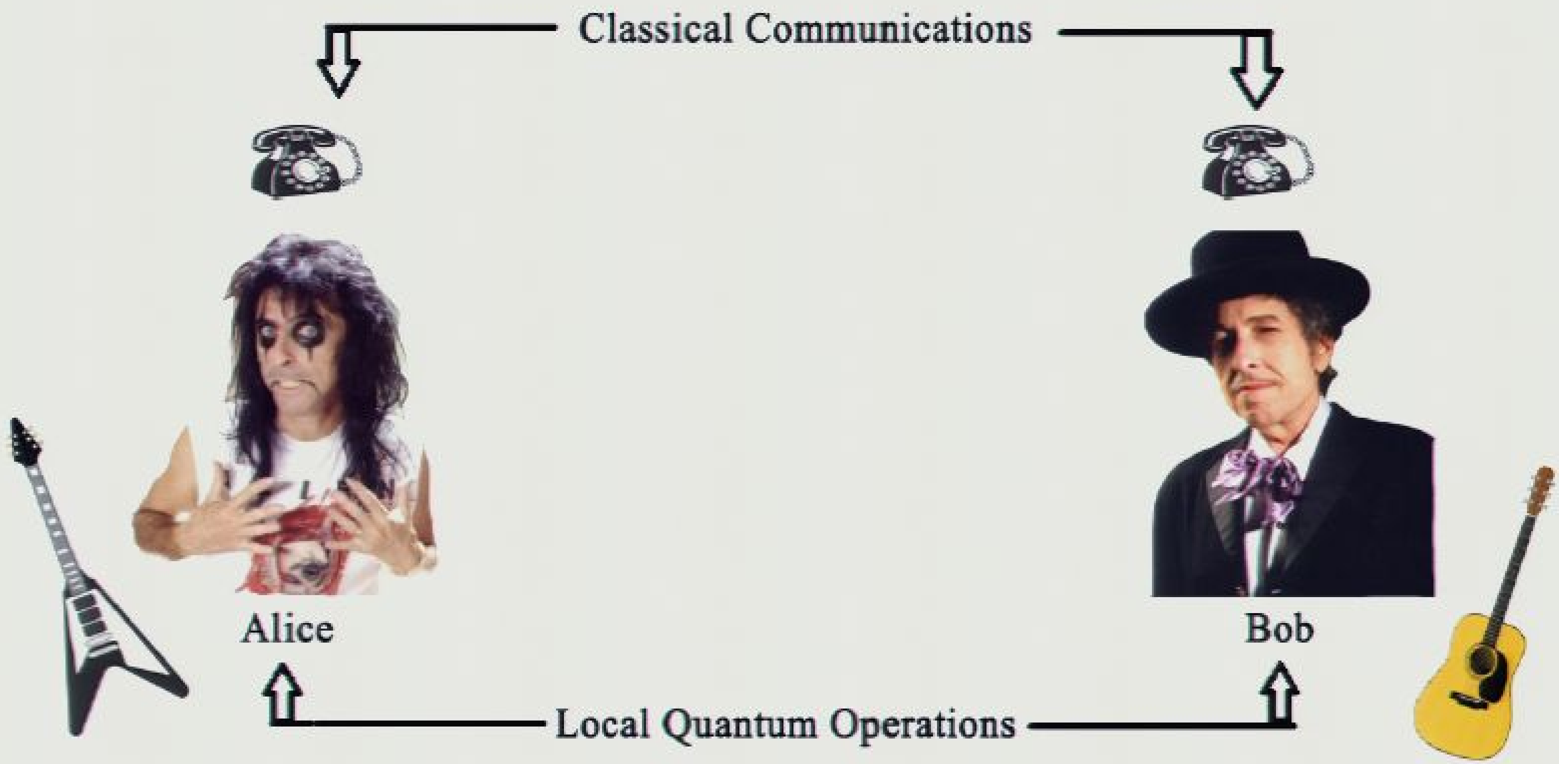
- In some cases more resources can be obtained in the product



$$|\psi\rangle \otimes |\chi\rangle \longrightarrow |\phi\rangle \otimes |R\rangle \otimes |\chi\rangle$$

- How do we know when this possible?

LOCC



- Shared entangled bipartite state: $|\psi\rangle_{AB} \in \mathbb{C}^n \otimes \mathbb{C}^n$
- Possible operations on $|\psi\rangle_{AB}$:
  - Unitary operations, projective measurements, generalized measurements.

Quantifying Entanglement: Entanglement Monotones

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- Entanglement monotones are defined as functions that do not increase under LOCC.

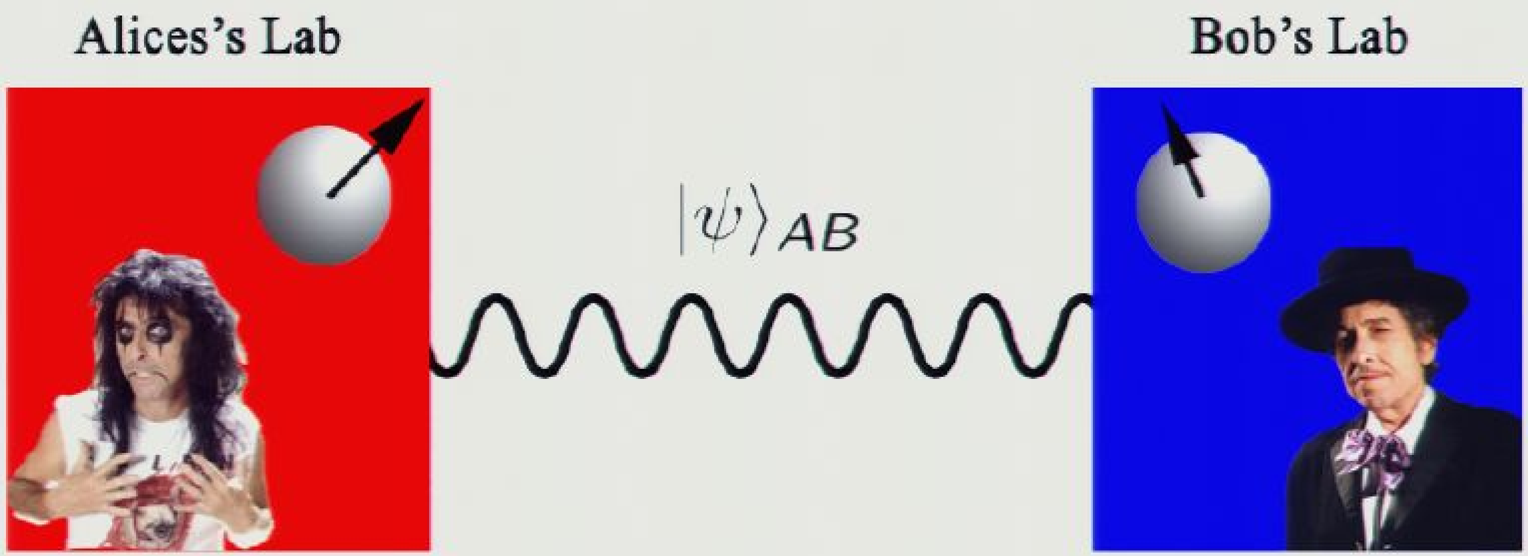
$$|\tilde{\psi}\rangle_{AB} = \sum_{i,j} a_{i,j} |i\rangle_A |j\rangle_B = \sum_{i=1}^{n} \sqrt{p_i} |u_i\rangle_A |v_i\rangle_B$$

- $(U^\dagger \otimes V^\dagger) |\tilde{\psi}\rangle_{AB} = |\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i} |i_A\rangle |i_B\rangle$
- For $|\tilde{\psi}\rangle_{AB} \overset{LOCC}{\longleftrightarrow} |\psi\rangle_{AB}$,

$$E\left(|\tilde{\psi}\rangle_{AB}\right) = E\left(|\psi\rangle_{AB}\right) = f\left(p_1, p_2, \ldots, p_n\right)$$

- Used entanglement monotones to characterize LOCC and eLOCC transformations.

Quantifying Entanglement



$\rho^A = Tr_B\left(|\psi\rangle_{AB}\langle\psi|\right)$ $\qquad$ $\rho^B = Tr_A\left(|\psi\rangle_{AB}\langle\psi|\right)$

- If $|\psi\rangle_{AB} \overset{LOCC}{\rightarrow} |\phi\rangle_{AB}$ then $E(|\psi\rangle_{AB}) \geq E(|\phi\rangle_{AB})$.
- eg: Von Neumann Entropy:

$$E\left(|\psi\rangle\right) = -\sum_i p_i \log p_i = -Tr\left(\rho_r \log \rho_r\right)$$

Nielsen's Theorem

- Two pure bipartite states:

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i}|i_A\rangle|i_B\rangle \text{ and } |\phi\rangle = \sum_{i=1}^{m} \sqrt{q_i}|i_A\rangle|i_B\rangle$$

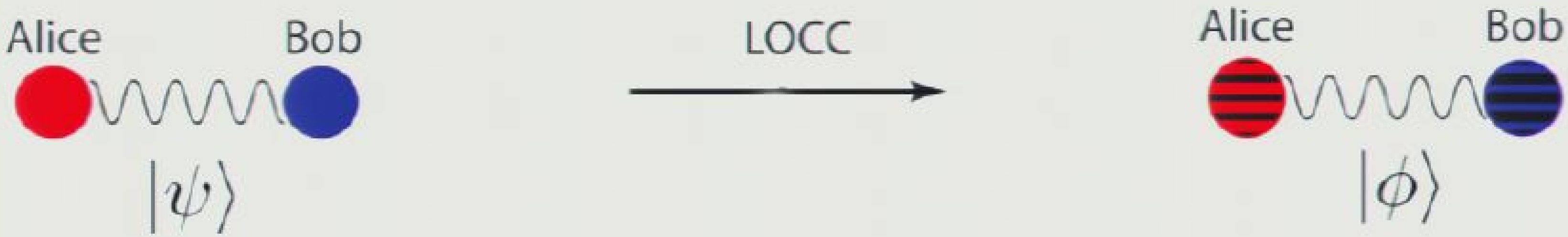
  - ordered Schmidt coefficients (OSC's): $p_1 \geq \ldots \geq p_n$ and $q_1 \geq \ldots \geq q_m$, $(n \geq m)$

- **Nielsen's Theorem**: The transformation $|\psi\rangle \rightarrow |\phi\rangle$ is possible by LOCC iff,

$$E_k(|\psi\rangle) \geq E_k(|\phi\rangle) \text{ for all } k = 1, 2 \ldots n.$$

where, $E_k(|\psi\rangle) = \sum_{i=k}^{n} p_i$

Majorization



$$|\psi\rangle \prec |\phi\rangle$$

if

$$\begin{aligned} p_1 &\leq q_1 \\ p_1 + p_2 &\leq q_1 + q_2 \\ &\vdots \\ p_1 + p_2 + \cdots + p_n &\leq q_1 + q_2 + \cdots + q_n \end{aligned}$$

- Used majorization to find examples so that $|\psi\rangle \overset{LOCC}{\nrightarrow} |\phi\rangle$

Necessary and Sufficient Conditions for eLOCC

- Rényi entropy of order α is given by,

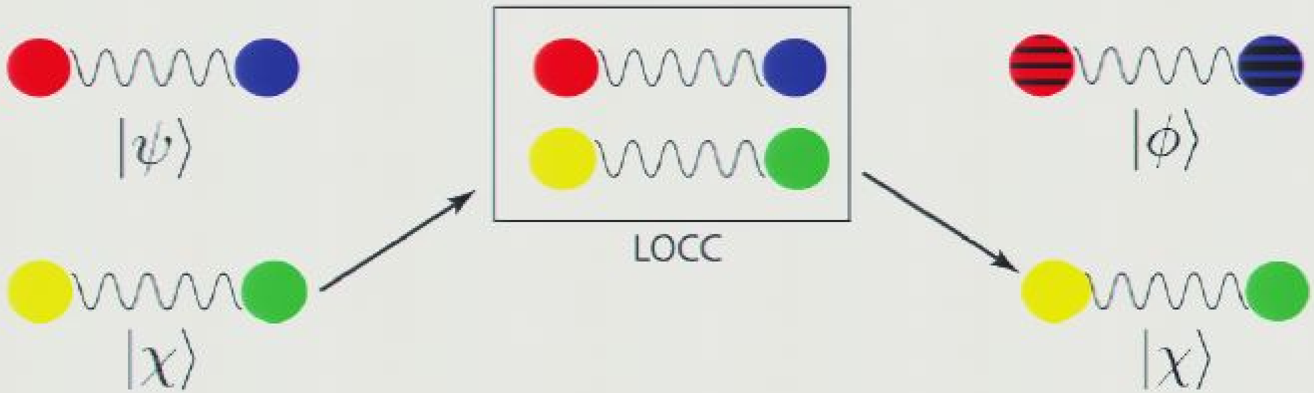
$$E_\alpha(|\psi\rangle) = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^{n} p_i^\alpha \right). \quad \text{for all } \alpha \geq 0.$$

- In the limit $\alpha \to 1$, the Rényi entropy becomes the Shannon entropy,

$$E_1(|\psi\rangle) = -\sum_{i-1}^{n} p_i \ln p_i.$$

- Rényi entropies are entanglement monotones and additive.

Additivity



$$
\begin{aligned}
|\psi\rangle \otimes |\chi\rangle &\xrightarrow{LOCC} |\phi\rangle \otimes |\chi\rangle \\
E_\alpha\left(|\psi\rangle \otimes |\chi\rangle\right) &\geq E_\alpha\left(|\phi\rangle \otimes |\chi\rangle\right) \\
E_\alpha\left(|\psi\rangle\right) + E_\alpha\left(|\chi\rangle\right) &\geq E_\alpha\left(|\phi\rangle\right) + E_\alpha\left(|\psi\rangle\right)
\end{aligned}
$$

- If there exists a catalyst, $E_\alpha\left(|\psi\rangle\right) \geq E_\alpha\left(|\phi\rangle\right)$

Turgut's Results

- Turgut provided a method for determining if an eLOCC transformation can occur.
- Extended definition of Rényi entropies to include negative values of α.
- The transformation $|\psi\rangle \rightarrow |\phi\rangle$ by eLOCC is equivalent to the strict positivity of,

$$\Delta R_\alpha = \frac{E_\alpha\left(|\psi\rangle\right) - E_\alpha\left(|\phi\rangle\right)}{\alpha}, \text{ for all } \alpha.$$

- With a residue entangled state:

$$\Delta R_\alpha = \frac{E_\alpha\left(|\psi\rangle\right) - E_\alpha\left(|\phi\rangle\right) - E_\alpha\left(|R\rangle\right)}{\alpha}, \text{ for all } \alpha.$$

Concurrence

- The kth concurrence of an n-dimensional vector x is given by,

$$C_k(x) \equiv \left(\frac{e_k(x)}{e_k(\iota_n)}\right)^{\frac{1}{k}},$$

where $\iota_n = \left(\frac{1}{n} \dots \frac{1}{n}\right) \in \mathbb{R}^n$, and $e_k(x) \equiv \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k}$.

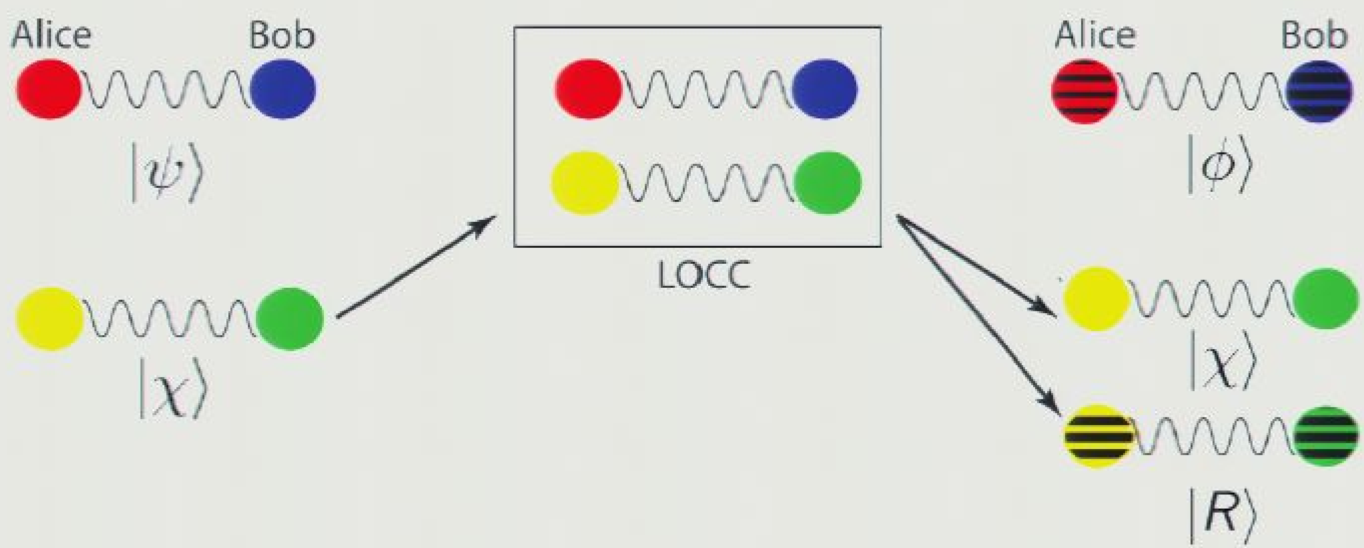
- G concurrence (k=n): $G(|\psi\rangle) = n(p_1 \cdot p_2 \cdots p_n)^{\frac{1}{n}}$

$$G(|\psi\rangle \otimes |\chi\rangle) \geq G(|\phi\rangle \otimes |R\rangle \otimes |\chi\rangle) \implies$$

$$G(|\psi\rangle)\,G(|\chi\rangle) \geq G(|\phi\rangle)\,G(|R\rangle)\,G(|\chi\rangle) \implies G(|R\rangle) \leq \frac{G(|\psi\rangle)}{G(|\phi\rangle)}$$

Definition: Residue Entangled State

- Let $|\psi\rangle$ and $|\phi\rangle$ be two pure bipartite states in H^{AB} and define the dimension of each vector to be n and m respectively. Here we assume the dimensions satisfy $m \geq 2$ and $n/m = s$ is an integer greater than one. We call $|R\rangle \in H^{A'B'}$, with dimension s, the residue entangled state if (i) the transformation $|\psi\rangle \rightarrow |\phi\rangle$ is not possible by LOCC and (ii) the transformation $|\psi\rangle \rightarrow |\phi\rangle \otimes |R\rangle$ is possible by eLOCC.



Thereom: Existence of the Residue Entangled State

- **Theorem 1:** Let $|\psi\rangle$ and $|\phi\rangle$ be unit vectors such that $|\psi\rangle, |\phi\rangle \in H^{AB}$ with dimensions n and m respectively. The dimensions satisfy $m \geq 2$ and n/m is an integer greater than one. If $|\psi\rangle \to |\phi\rangle$ is not possible by LOCC then there exists a residue entangled state $|R\rangle$ such that $|\psi\rangle \to |\phi\rangle \otimes |R\rangle$ is possible by eLOCC if and only if

$$
\begin{aligned}
(1) \quad & \Delta_\alpha = E_\alpha(|\psi\rangle) - E_\alpha(|\phi\rangle) > 0, \quad \text{for all} \quad \alpha \geq 0, \\
(2) \quad & p_{max} < q_{max}.
\end{aligned}
$$

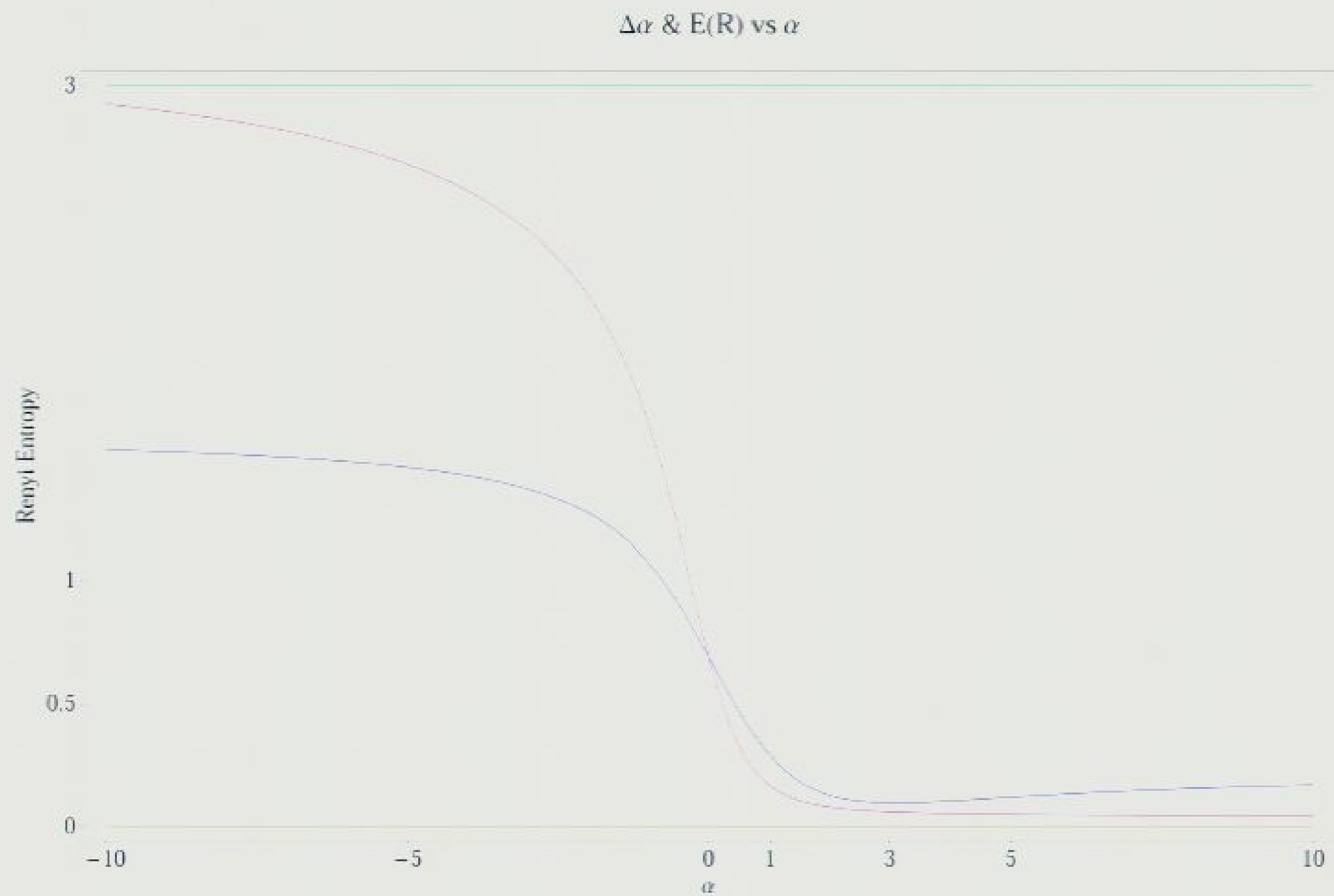
- Moreover the residue entangled state must satisfy,

$$
\begin{aligned}
0 &< E_\alpha(|R\rangle) \leq \Delta_\alpha, \quad \text{for} \quad \alpha \geq 0, \\
G(|R\rangle) &< \frac{G(|\psi\rangle)}{G(|\phi\rangle)}.
\end{aligned}
$$

Example: n=6, m=3

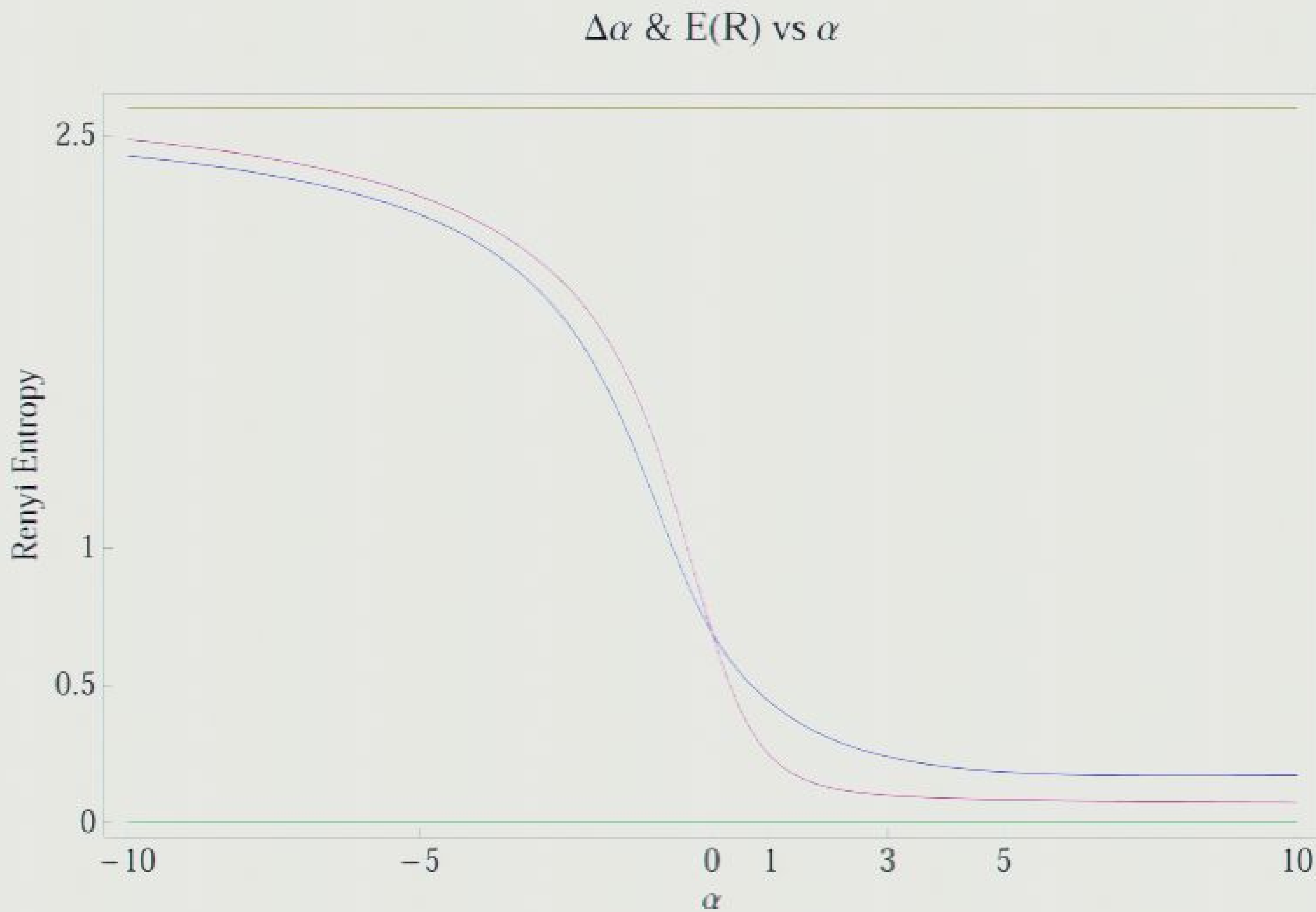
$$|\psi\rangle = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}, \frac{1}{20}\right),\ |\phi\rangle = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right),\ |R\rangle = \left(\frac{24}{25}, \frac{1}{25}\right).$$

- $|\psi\rangle \nrightarrow |\phi\rangle$ by LOCC since $\frac{2}{5} < \frac{1}{2}$ but $\frac{2}{5} + \frac{2}{5} > \frac{1}{2} + \frac{1}{4}$



Example: n=8, m=4

$$|\psi\rangle = \frac{1}{337}(97, 96, 37, 34, 31, 30, 9, 3),$$
$$|\phi\rangle = \frac{1}{2974}(891, 751, 711, 621),\ |R\rangle = \frac{1}{1000}(935, 65).$$



Conclusion

- Main problem studied: if two states are chosen so that $|\psi\rangle \rightarrow |\phi\rangle$ is not possible by LOCC then does there exist a state $|R\rangle$, such that $|\psi\rangle \rightarrow |\phi\rangle \otimes |R\rangle$ is possible by eLOCC?
- Provided necessary and sufficient conditions for the existence of $|R\rangle$.
- Shown examples of states $|\psi\rangle$ and $|\phi\rangle$ where a residue entangled state can be found.

Future Work

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- Bounds on $|R\rangle$ and how much entanglement this state can have must be further investigated
- These types of transformations are important because they provide extra conversion power for quantum states and in the case of the residue entangled state, provide more resources.