Title: Spinfoam Fermions

Date: Jul 06, 2011 04:00 PM

URL: http://pirsa.org/11070005

Abstract: A serious shortcoming of spinfoam loop gravity is the absence of matter.
I present a minimal and surprisingly simple coupling of a chiral fermion field in the framework of spinfoam quantum gravity.
This result resonates with similar ones in early canonical loop theory: the naive fermion hamiltonian was found to be just the extension of the simple

Spinfoam fermion

Elena Magliaro

Institute for Gravitation and the Cosmos, Penn State University

July, 2011 - Perimeter Institute

arXiv:1012.4719 in collaboration with M.Han, C.Perini, C.Rovelli, W.Wieland

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Plan of the talk

- LQG and SF in a nutshell
- Fermions on a 2-complex
- Coupling fermions to SF Quantum Gravity
- Fermions states
- Yang-Mills fields
- Conclusions and outlooks

LQG in a nutshell: kinematics

KINEMATICS: spin-networks

An abstract graph is defined by a set of L oriented links l, and a set of N nodes n. The Hilbert space $\mathcal{H}$ on which the theory is defined decomposes into the direct sum of Hilbert spaces associated to abstract graphs

$$\mathcal{H} = \oplus_\Gamma \mathcal{H}_\Gamma \quad \text{does not depend on a lattice, 'sums' over all of them}$$

$$\mathcal{H}_\Gamma = L^2(SU(2)^{\#links}/SU(2)^{\#nodes}) \quad \text{looks like quantum lattice YM}$$

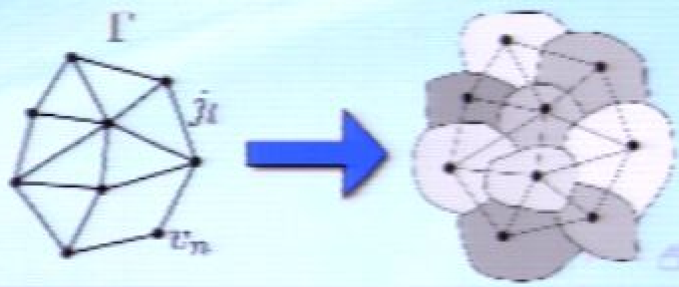
Elements of H_Γ are gauge-invariant wave-functions of the A.B. holonomy along the links of the graph

$$\psi(h_l) = \psi(g_{s(l)} h_l g_{t(l)}^{-1})$$

An orthonormal basis which diagonalizes the geometric operators (area, volume) is given by spin-networks: they are labeled by *spins* j, and invariant tensors v, *intertwiners*, associated to the links and nodes of the graph $|\Gamma, j_l, v_n\rangle$.

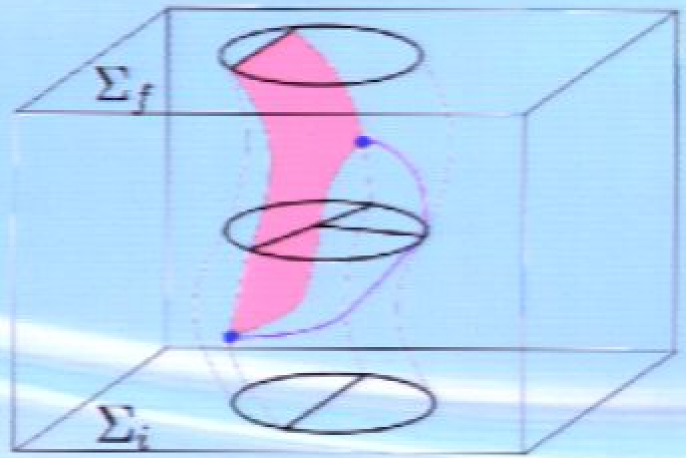
j_l area quantum numbers $\widehat{A}_l\, \psi_{j,v} = \gamma\sqrt{j_l(j_l+1)}\, \psi_{j,v}$

v_n volume quantum numbers



LQG in a nutshell: dynamics

DYNAMICS: spin-foams (2-complex, t-evolution of spin-network with branching points)



vertices v, edges e, faces f (e bounded by $s(e)/t(e)$)

Transition amplitude for the boundary spin-network:

$$W(j_{ext}, v_{ext}) = \sum_{j,v} \prod_f F(j) \prod_e E(j,v) \prod_v V(j,v)$$

Sort of Feynman rules for the vertices and propagators of the 2-complex, or Misner-Hawking path-integral over virtual geometries interpolating j_{ext}, v_{ext}

Spinfoams

Spinfoams are tentative covariant quantization of general relativity based on Holst action

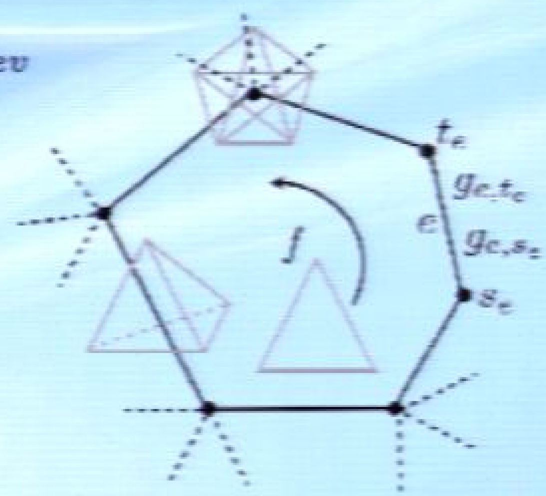
$$S=\int (e\wedge e)^*\wedge F(\omega)+\frac{1}{\gamma}\int (e\wedge e)\wedge F(\omega)$$

SF models are obtain

- starting from BF theory $S=\int B\wedge F(\omega)$ first order formalism
- imposing simplicity constraints on B: $B=(e\wedge e)^*+\frac{1}{\gamma}(e\wedge e)$
- discretizing on a 2-complex (triangulation)

$$\omega \longrightarrow g=P\exp\int_e \omega \xrightarrow{\text{discretize}} g_{ev}$$

$$B \xrightarrow{\text{discretize}} \int_\triangle B = B_{ef}$$



$\boxed{\text{Simplicity constraints:}}$

$$\vec{B}^{\pm}=j_f^{\pm}\vec{n}_{ef}$$

$$j_f^{\pm}=\frac{1\pm\gamma}{2}j$$

ermions on a 2-complex

$\boxed{\text{Dirac action}}$ $S_D = i\int d^4x\, \overline{\psi}_D\, \partial\!\!\!/ \psi_D + c.c.$

For a chiral spinor $\psi \in \mathbb{C}^2$ with two complex components $\psi^a, a = 0, 1$.

$$S = i\int d^4x\, \overline{\psi}\sigma^I \partial_I \psi + c.c. \qquad \sigma^I = (1, \sigma)$$

- Fix a coordinate system $x = (x^I)$ and discretize spacetime (union of 4-cells v)
- Consider the dual complex (v vertices e edges)
- Approximate the field $\psi(x)$ by its values $\psi_v = \psi(x_v)$ at each vertex
- Discretize the derivative on each edge as

$$\psi_{t_e} - \psi_{s_e} \sim (x^I_{t_e} - x^I_{s_e})\partial_I \psi$$

- The action can be discretized as a sum over 4-cells $S \to \sum_v S_v$

$$S_v \sim \frac{4i\mathrm{V}_v}{|v|} \sum_{e\in v} \overline{\psi}_v u^I_e \sigma_I \frac{\psi_{v+e} - \psi_v}{l_e}, \longrightarrow \boxed{\text{cancels with c.c}}$$

V_v volume 4-cell, $l_e = |x^I_{t_e} - x^I_{s_e}|$, $u^I_e = (x^I_{t_e} - x^I_{s_e})/l_e$ unit vector parallel to e,

$|v|$: # of edges bounded by v.

Consider 3-cell τ_e dual to e. The 4-cell v can be partitioned into the union of $|v|$ pyramids with base τ_e and height $h_e = l_e/2$.
Pyramid 4-volume= $\frac{1}{4} h_e v_e$ (v_e 3-volume of τ_e).

Vertex $\rightarrow$ edge variables

Proposal

$$S \sim \sum_v \frac{1}{4|v|} \underbrace{\sum_{e,e' \in v} \overline{\psi}_{e'} v_e \sigma_e \psi_e}_{S_v}$$

- Follows intuitively from the previous standard discretization in the regular case (but can be generalized)
- $\sigma_e = \sigma^I u_{eI}$ $\quad \sigma-$matrix in the direction of the edge e
- $\psi_e = \psi(x_e)$ $\quad x_e$ intersection between e and τ_e
- $\psi_e \sim \frac{1}{2}(\psi_{s_e} + \psi_{t_e})$ $\quad$ If the 2nd derivative of the field can be neglected

Quantum fermion field on a 2-complex

Partition function (Berezin integral) $Z = \int D\psi_e e^{iS}, \quad D\psi_e = \frac{1}{\mathrm{v}_e^2} d\psi_e d\overline{\psi}_e e^{-v_e \overline{\psi}_e \psi_e}$

This measure realize the scalar product at each edge seen as a bounding between two 4-cells . ψ_e are anticommuting variables. v_e needed for dimensional reasons (in the classical theory $\langle\psi|\psi'\rangle = \int d^3x \sqrt{q}\, \overline{\psi}(x)\psi'(x)$)

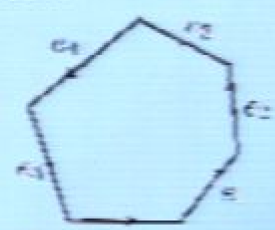
Recall: Berezin non vanishing integral: $\int d\psi d\overline{\psi} \psi^a \overline{\psi}^b \psi^c \overline{\psi}^d = \epsilon^{ac}\epsilon^{bd} = \delta^{ab}\delta^{cd} - \delta^{ad}\delta^{bc}$

Expand the vertex action in Taylor series

$$e^{iS_v} = 1 + \sum_{ee'} \overline{\psi}_e \mathrm{v}_e \sigma_e \psi_e + \sum_{e_1 e_2 e_3 e_4} (\overline{\psi}_{e_1} \mathrm{v}_{e_2} \sigma_{e_2} \psi_{e_2})(\overline{\psi}_{e_3} \mathrm{v}_{e_4} \sigma_{e_4} \psi_{e_4}) + \ldots$$

The series stops because there can be at most 4 fermions (2 ψ and 2 $\overline{\psi}$) per edge. On each edge can live 0,2,4 fermions. The volume factor cancels with vertex action.

$$Z = \sum_{\{c\}} \prod_c A_c, \qquad A_c = (-1)^{|c|} Tr[\sigma_{e_1} \ldots \sigma_{e_N}]$$



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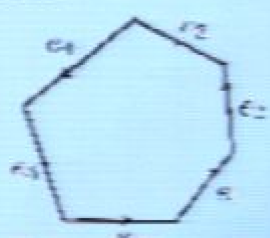

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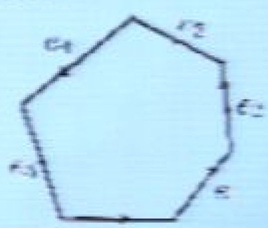

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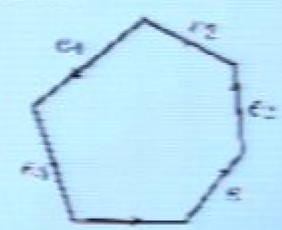

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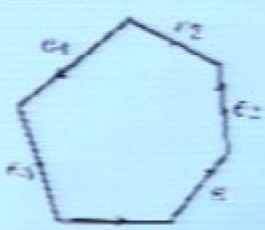

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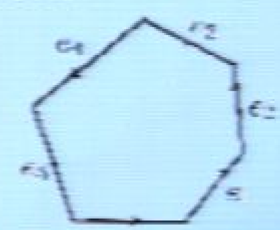

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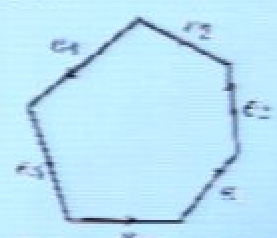

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Generalization to curved spacetimes

Holonomy of the spin connection can be $\neq 1$ (curvature is confined on 2-cells). If the parallel transport $s_e \to e \neq t_e \to e$:

$$S = i \sum_{e,e'} \mathbf{v}_e \, \overline{\psi}_{e'} g^{\dagger}_{e s_e} g_{e t_e} \psi_e.$$

$$A_c = (-1)^{|c|} Tr[\sigma_{e_1} \dots \sigma_{e_N}] \longrightarrow A_c = (-1)^{|c|} \, Tr[g^{\dagger}_{e_1 v_1} g_{e_1 v_2} \cdots g^{\dagger}_{e_n v_n} g_{e_n v_1}]$$

where $(v_1, e_1, v_2, e_2, ..., v_n, e_n)$ is the sequence of vertices and oriented edges crossed by the cycle c.
defining $g^* = (g^{-1})^{\dagger} = -\epsilon g^{\dagger} \epsilon$

$$A_c = (-1)^{|c|} \, Tr[g^*_{v_1 e_1} g_{e_1 v_2} \cdots g^*_{v_n e_n} g_{e_n v_1}]$$

$$Z = \sum_{\{c\}} \prod_c (-1)^{|c|} \, \chi^{\frac{1}{2}} \Big(\prod_{e \in c} (g^*_{s_e e} g_{e t_e})^{\epsilon_{ec}} \Big)$$

- $\chi^{\frac{1}{2}}(g)$ character in the fundamental representation of $SL(2, \mathbb{C})$,
- $\epsilon_{ec} = \pm 1$ according to whether the orientations of the edge and the cycle match.

Coupling to Quantum Gravity

Spinfoam partition function:

$$Z_{QG} = \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \sum_{j_f} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{ef}} \Big) \prod_{e\in\partial f} \chi^{j_f}(h_{ef})$$

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- They have the same geometrical interpretation: parallel transport $e \to v$
- Asymptotics (saddle point): g rotates the arbitrary Lorentz frame at the 3-cell where the time direction is aligned with the normal of the 3-cell.
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$$Z_{QGF} = \sum_{\{c\}} \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{ef}} \Big)$$

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$\{c\}$ worldlines running along the edges of the foam that do not overlap more than once

ermions states

Fermionic Fock space: By Pauli principle only antisymmetric states. There are at most two orthogonal states in $H^{\frac{1}{2}}$ (at most two particles with all other quantum numbers equal).
Two-particle state: singlet (spin 0) of $H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}$.

$$F = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus A(H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}) = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus \mathbb{C}$$

A basis in this state is given by $|c\rangle$ where $c = \emptyset, +, -, 2$.
States in F can be conveniently represented as holomorphic Berezin functions $f(\psi)$ of an anticommuting Grassmann variable in $H^{\frac{1}{2}}$. Taylor expanding the state gives

$$f(\psi) = c_\emptyset + c_a \psi^a + c_2\, \epsilon_{ab} \psi^a \psi^b$$

$$|f|^2 := |c_\emptyset|^2 + \bar{c}_a c_a + |c_2|^2, \qquad (c_\emptyset, c_a, c_2) \in \mathbb{C}^4 \simeq F$$

The Berezin integral used before is simply a way to write this Fock space and its scalar product.

ermions states and Spinfoam

Pure gravity partition function on a 2-complex in the SN basis

$$Z = \sum_{j_f, \mathrm{v}_e} \prod_f d_{j_f} \prod_v A_v(j_f, \mathrm{v}_e) \qquad A_v(j_f, \mathrm{v}_e) = \int dg_{ev}\; A_v(j_f, \mathrm{v}_e, g_{ev})$$

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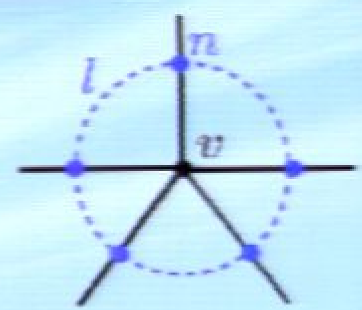

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no fermions on the boundary	single or odd fermions	$\psi, \overline{\psi}$ on different nodes

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$$\Rightarrow \quad Z = \sum_{j_f, v_e, \{c\}} \int dg_{ev} \prod_f d_{j_f} \prod_v A_v(j_f, v_e, g_{ev}) \prod_c A_c(g_{ev})$$

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The net effect of the fermions is simply to add fermion world-lines over the foam. The weight of each world-line is a contraction of spin-connection group elements along the world-line, taken in the fundamental representation. The worldlines than run over the foam carry a $j = \frac{1}{2}$ representation, and couple to the intertwiners at the edges. They overlap at most twice and where they overlap, they run in the $j = 0$ representation.

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Now spinfoams carry representations of $SL(2, \mathbb{C})$ and intertwiners at the nodes have a possible extra leg representing fermions in (antisymmetric products of) the fundamental representation of $SL(2, \mathbb{C}) \times G$.

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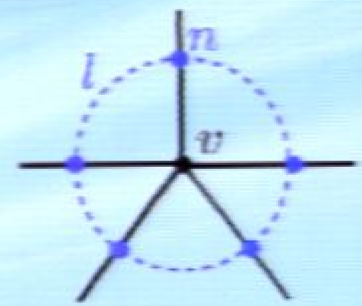

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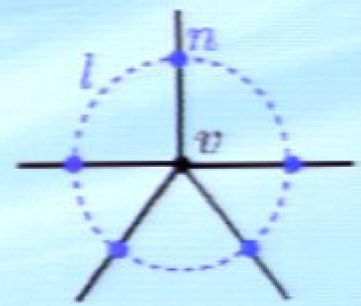

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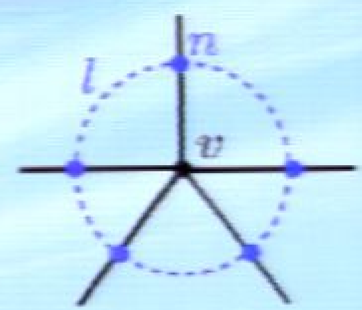

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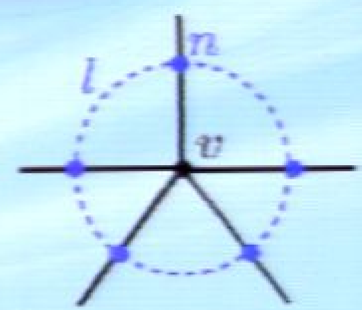

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$$Z = \sum_{j_f, \mathrm{v}_e} \prod_f d_{j_f} \prod_v A_v(j_f, \mathrm{v}_e) \qquad A_v(j_f, \mathrm{v}_e) = \int dg_{ev}\, A_v(j_f, \mathrm{v}_e, g_{ev})$$

$$A_v(j_l, \mathrm{v}_n, g_n) = \int dk_l\, \Psi_{j_l, v_n}(k_l) \prod_l \chi^{\gamma j_l, j_l}(k_l\, g_{s_l} g_{t_l}^{-1})$$

- v_e quantum numbers and eingevalues of the basis that diagonalizes the volume.
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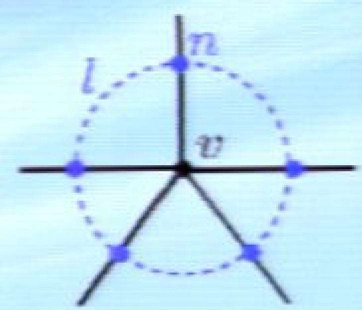

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no fermions on the boundary	single or odd fermions	$\psi, \overline{\psi}$ on different nodes

Fermionic Fock space + kinematics of quantum gravity

- ⋆ Fermions reside on the nodes n of a graph, (like LGT) $\Rightarrow$ a copy of F to each node.
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⇓

the intertwiner couples the spinor to the gravitational magnetic indices

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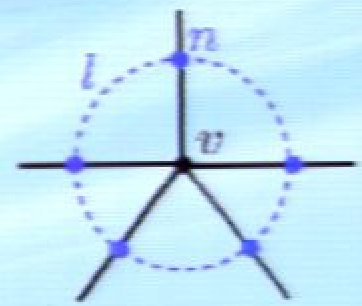

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Fermionic Fock space: By Pauli principle only antisymmetric states. There are at most two orthogonal states in $H^{\frac{1}{2}}$ (at most two particles with all other quantum numbers equal).
Two-particle state: singlet (spin 0) of $H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}$.

$$F = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus A(H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}) = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus \mathbb{C}$$

A basis in this state is given by $|c\rangle$ where $c = \emptyset, +, -, 2$.
States in F can be conveniently represented as holomorphic Berezin functions $f(\psi)$ of an anticommuting Grassmann variable in $H^{\frac{1}{2}}$. Taylor expanding the state gives

$$f(\psi) = c_\emptyset + c_a \psi^a + c_2\, \epsilon_{ab} \psi^a \psi^b$$

$$|f|^2 := |c_\emptyset|^2 + \bar{c}_a c_a + |c_2|^2, \qquad (c_\emptyset, c_a, c_2) \in \mathbb{C}^4 \simeq F$$

The Berezin integral used before is simply a way to write this Fock space and its scalar product.

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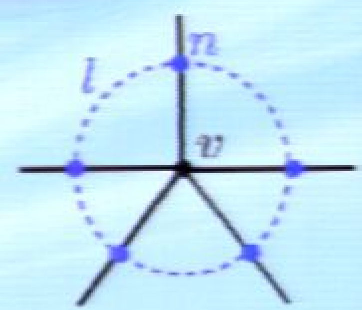

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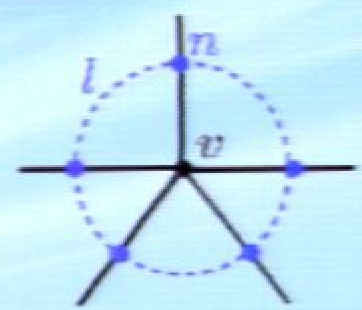

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The Berezin integration has the only effect of connecting the free indices of the cell-propagators and to cancel the volume factors.

$$\Rightarrow \quad Z = \sum_{j_f, v_e, \{c\}} \int dg_{ev} \prod_f d_{j_f} \prod_v A_v(j_f, v_e, g_{ev}) \prod_c A_c(g_{ev})$$

which is equivalent to to the previous expression. The dependence of the Lorentz frame of the boundary drops in gluing the individual group elements. A local Lorentz transformation $g_{ve} \to \Lambda_v g_{ve}$ leaves the amplitude invariant.

Summarizing

The net effect of the fermions is simply to add fermion world-lines over the foam. The weight of each world-line is a contraction of spin-connection group elements along the world-line, taken in the fundamental representation. The worldlines than run over the foam carry a $j = \frac{1}{2}$ representation, and couple to the intertwiners at the edges. They overlap at most twice and where they overlap, they run in the $j = 0$ representation.

Yang-Mills fields

If fermions live in fundamental representation of a compact group $G \Rightarrow$ the theory is invariant under global G transformations. To make it invariant under local gauge transformations introduce $U_{ve} \in G$ associated to each wedge (v, e), and consider

$$S = i \sum_e \overline{\psi}_{s_e} U^{\dagger}_{s_e e} \mathbf{v}_e \sigma_e U_{t_e e} \psi_{t_e}$$

Now spinfoams carry representations of $SL(2, \mathbb{C})$ and intertwiners at the nodes have a possible extra leg representing fermions in (antisymmetric products of) the fundamental representation of $SL(2, \mathbb{C}) \times G$.

Dynamics The Yang-Mills action can be generated by the one-loop radiative corrections to the fermion action in the Yang-Mills field, as suggested by Zel'dovich.

$$Z = \sum_{j_f \mathbf{v}_e} \int d\psi_e \int dU_e \prod_f d_{j_f} \prod_v A_v(j_f, \mathbf{v}_e, \psi_e, U_e)$$

where the gravity+Yang-Mills+fermion vertex amplitude is

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Conclusions

- We have defined a tentative quantum theory of gravity, fermions and Yang Mills fields with the following strategy:

1. A spinfoam amplitude can be seen as a definition of the gravitational path integral which keeps into account the quantization of geometry
2. we have found a discretization of the fermion action which remains valid on a curved spacetime and which is expressed in terms of the same variables that appear in the spinfoam amplitudes
3. this calculation is not a "derivation" or a "quantization", but rather as a heuristic hint, yielding an ansatz for a definition of the coupled gravity-fermion theory.
4. The theory appears to have the correct degrees of freedom and the proper symmetries (CPT theorem by Han-Rovelli)

- A fermion, is essentially an extra "face" of spin $\frac{1}{2}$, which is non-local over the 2-complex. At fixed time, it can be seen as a "non-local" loop that disappears outside spacetime, to reappear far away, like a Wheeler-Smolin "Kerr-Newman fermion"

)utlooks

- To understand the mechanism for the generation of Yang-Mills kinetic term by the fermionic loops
- The mass term has been included in a variation of this model (Han-Rovelli) but there are still many issues (is it fundamental or can be obtained in an effective way?)
- The semiclassical analysis is still missing: can we obtain in the semiclassical limit the classical action of the system gravity + fermions?
- The derivation of the model is very tentative, is there a more rigorous derivation?
- ...

Thank you!

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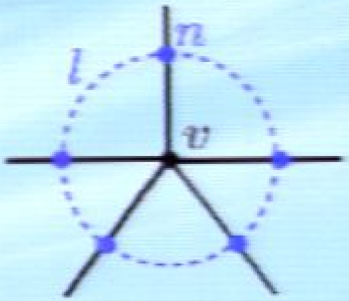

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Coupling to Quantum Gravity

Spinfoam partition function:

$$Z_{QG} = \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \sum_{j_f} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{lf}}\Big) \prod_{e\in\partial f} \chi^{j_f}(h_{ef})$$

g_{e_s}, g_{e_t} can be identified with the $SL(2,\mathbb{C})$ matrices in the fermion action

- They have the same geometrical interpretation: parallel transport $e \to v$
- Asymptotics (saddle point): g rotates the arbitrary Lorentz frame at the 3-cell where the time direction is aligned with the normal of the 3-cell.
  For $l_P \ll 1$, g takes precisely the value needed to yield the fermion action.

Proposal for SF dynamics + fermions

$$Z_{QGF} = \sum_{\{c\}} \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{lf}}\Big)$$

$$\prod_{e\in\partial f} \chi^{j_f}(h_{ef}) \prod_c (-1)^{|c|} \chi^{\frac{1}{2}}\Big(\prod_{e\in c_n} (g_{es_e} g_{et_e}^{\dagger})^{\epsilon_{ec}}\Big)$$

$\{c\}$ worldlines running along the edges of the foam that do not overlap more than once

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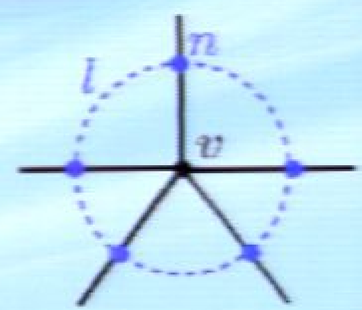

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- ⋆ States gravity+fermion live on $(\otimes_l L_2[SU(2)]) \otimes (\otimes_n F)/SU(2)^N$.
- ⋆ <u>States</u>: $\Psi(h_l, \psi_n)$. <u>Basis</u>: simple generalization of the pure gravity spin networks

- Choose an intertwiner basis that diagonalizes the volume of n (v_n vol eigenvalue). $|j_l, \mathrm{v}_n\rangle \longrightarrow |j_l, \mathrm{v}_n, c_n\rangle$, c_n labels the basis $|c\rangle$ in the Fock space at the node.
- At each v-valent node n bounded by links with spins $j_1, ..., j_v$

$$\mathrm{v}_n \text{ invariant tensor in the tensor product of} \begin{cases} j_1, ..., j_v & \text{if } c_n = \emptyset, 2 \\ j_1, ..., j_v, \frac{1}{2} & \text{if } c_n = \pm \end{cases}$$

⇓

the intertwiner couples the spinor to the gravitational magnetic indices

- $|j_l, \mathrm{v}_n\rangle$ Fock vacuum $|j_l, \mathrm{v}_n, c_n = \emptyset\rangle$
- $|j_l, \mathrm{v}_n, (n, a)\rangle$ all nodes in the vacuum except for a single node n with a one-particle state with spin a
- $|j_l, \mathrm{v}_n, (n_i, \pm)\rangle$ $i = 1, ..., N$ $\quad N$ particles in the nodes $n_1, ..., n_N$ with spins $\pm_n$
- $|j_l, \mathrm{v}_n, (n, +), (n, -)\rangle$ two-particle singlets

ermions states

Fermionic Fock space: By Pauli principle only antisymmetric states. There are at most two orthogonal states in $H^{\frac{1}{2}}$ (at most two particles with all other quantum numbers equal).
Two-particle state: singlet (spin 0) of $H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}$.

$$F = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus A(H^{\frac{1}{2}} \otimes H^{\frac{1}{2}}) = \mathbb{C} \oplus H^{\frac{1}{2}} \oplus \mathbb{C}$$

A basis in this state is given by $|c\rangle$ where $c = \emptyset, +, -, 2$.
States in F can be conveniently represented as holomorphic Berezin functions $f(\psi)$ of an anticommuting Grassmann variable in $H^{\frac{1}{2}}$. Taylor expanding the state gives

$$f(\psi) = c_\emptyset + c_a \psi^a + c_2\, \epsilon_{ab} \psi^a \psi^b$$

$$|f|^2 := |c_\emptyset|^2 + \bar{c}_a c_a + |c_2|^2, \qquad (c_\emptyset, c_a, c_2) \in \mathbb{C}^4 \simeq F$$

The Berezin integral used before is simply a way to write this Fock space and its scalar product.

Coupling to Quantum Gravity

Spinfoam partition function:

$$Z_{QG} = \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \sum_{j_f} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{ef}}\Big) \prod_{e\in\partial f} \chi^{j_f}(h_{ef})$$

g_{e_s}, g_{e_t} can be identified with the $SL(2,\mathbb{C})$ matrices in the fermion action

- They have the same geometrical interpretation: parallel transport $e \to v$
- Asymptotics (saddle point): g rotates the arbitrary Lorentz frame at the 3-cell where the time direction is aligned with the normal of the 3-cell.
  For $l_P \ll 1$, g takes precisely the value needed to yield the fermion action.

Proposal for SF dynamics + fermions

$$Z_{QGF} = \sum_{\{c\}} \sum_{j_f} \int_{SL(2,\mathbb{C})} dg_{ve} \int_{SU(2)} dh_{ef} \prod_f d_{j_f} \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} (g_{es_e} h_{ef} g_{et_e}^{-1})^{\epsilon_{ef}}\Big)$$

$$\prod_{e\in\partial f} \chi^{j_f}(h_{ef}) \prod_c (-1)^{|c|} \chi^{\frac{1}{2}}\Big(\prod_{e\in c_n} (g_{es_e} g_{et_e}^{\dagger})^{\epsilon_{ec}}\Big)$$

$\{c\}$ worldlines running along the edges of the foam that do not overlap more than once