Title: The Curious Nonexistence of Gaussian 2-designs

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Abstract: Continuous-variable SICPOVMS seem unlikely to exist, for a variety of reasons. But that doesn't rule out the possibility of other 2-designs for the continuous-variable Hilbert space L2(R). In particular, it would be nice if the coherent states -- which form a rather nice 1-design -- could be generalized in some way to get a 2-design comprising *Gaussian* states. So the question is: "Can we build a 2-design out of Gaussian states?". The answer is "No, but in a very surprising way!" Like coherent states, Gaussian states have a natural transitive symmetry group. For coherent states, it's the Heisenberg group. For Gaussian states, it's the affine symplectic group -- the Heisenberg group plus squeezings and rotations. And this group acts irreducibly on the symmetric subspace of L2(R) x L2(R)... which, by Schur's Lemma, implies that the Gaussian states *should* be a 2-design. Yet a very simple explicit calculation shows that they are not! The resolution is fascinating -- it turns out that the "symplectic twirl" involves an integral that does not quite converge, and this provides a loophole out of Schur's Lemma. So, in the end, we: (1) Show that Gaussian 2-designs do not exist, (2) Demonstrate a major stumbling block to *any* symplectic-covariant 2-designs for L2(R), (3) Gain a pretty complete understanding of *one* of the [formerly] mysterious discrepancies between discrete and continuous Hilbert spaces.

The Curious Non-Existence of Gaussian 2-Designs

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UNM

The Order of Events

1. Gaussian 2-designs *should* exist.
2. Gaussian 2-designs *don't* exist.

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-1. What are designs? especially 2-designs?

0. Why would we *want* a 2-design made of Gaussian states?

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3. Infinity is different from finite (in a newish way).

Designs

- State designs are sets of pure states that mimic (to some degree) the uniform ensemble of *all* pure states.

- A t-design $\mathcal{E} = \{|\psi_k\rangle\}$ for Hilbert space H satisfies:

  (i) Every t-th order polynomial in $|\psi\rangle\langle\psi|$ has the same average value over $\mathcal{E}$ as it does over the [unique] unitarily invariant ensemble of states (Haar measure).

  (ii) The t-copy mixed state for $\mathcal{E}$, $\rho^{(t)} = \mathrm{avg}_{\mathcal{E}}\left(|\psi\rangle\langle\psi|^{\otimes t}\right)$ is equal to the t-copy mixed state for Haar measure.

  (iii) $\rho^{(t)}$ is proportional to the projector on the symmetric subspace of t copies, $\Pi_{\mathrm{symm}}\left(H^{\otimes t}\right)$.

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Applications of Designs

- A **1-design** is a rank-1 POVM -- e.g., orthogonal basis.
  - Many boring uses, e.g. averaging expectation values.
- **3-**, **4-**, etc. designs do not have a lot of known uses.
- **2-designs** are the sweet spot:
  - SICPOVMs, MUBs, stabilizer states (overkill)...
  - Optimal tomographic measurements
  - Optimal *process*-tomographic input ensembles
  - Can average quadratic functions over all states:
  * variances, e.g. $\Delta x^2 = \langle \mathbf{x}^2 \rangle - \langle \mathbf{x} \rangle^2$
  * gate fidelity, $\langle \psi | \mathcal{N}\left[|\psi\rangle\langle\psi| \right] |\psi\rangle$

Generating Designs

- *Unitary* design = a set of unitary operators $\{U_k\}$ that mimic the Haar ensemble of unitaries.
  - Necessary & sufficient condition:
    $\{U_k |\psi_0\rangle\}$ is a *state* t-design for all $|\psi_0\rangle$.
  - So unitary t-designs generate state t-designs.

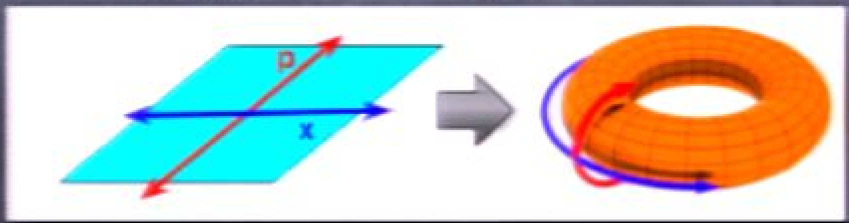
- If $\{U_k\}$ represent a group, then a nice condition is:
  The representation $\{U_k\}$ must be *irreducible* on the symmetric subspace of $H^{\otimes t}$.
  $\Rightarrow$ no invariant subspaces
  $\Rightarrow$ twirling takes *any* state to uniform mixture.

- So: irreducible representations generate state designs.

Heisenberg 1-Designs

- The *Heisenberg Group* ***H***: translations on flat phase space
  - position shifts,
  - momentum boosts,
  - Berry phases (ignore these to get *projective* representation)

- Case 1: Discrete $d \times d$ phase space:
  - ***H*** is represented on H_d. Irreducible. d^2 elements.
  - Generates (e.g.) $\boldsymbol{x}$ and $\boldsymbol{p}$ bases. Also SICPOVMs (!!).



- Case 2: Continuous x-p phase space.
  - ***H*** represented on $L^2(\mathbb{R})$. Irreducible. Uncountable.
  - Generates $\boldsymbol{x}$ / $\boldsymbol{p}$ bases. Also coherent states.

Measure Theory for Continuous Designs

- Averaging over *continuous* designs requires measure theory.

- $\{U_k\}$ is a representation $\Rightarrow$ group-invariant measure.
  - **Finite rep**: always a unique Haar measure.
  - **Compact infinite rep**: always a unique Haar measure.
  - **Noncompact**: *usually* left and right Haar measures.

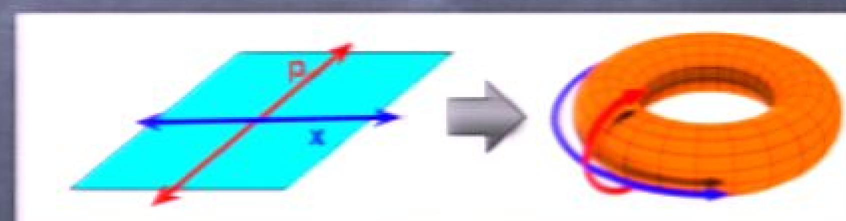
- Examples:

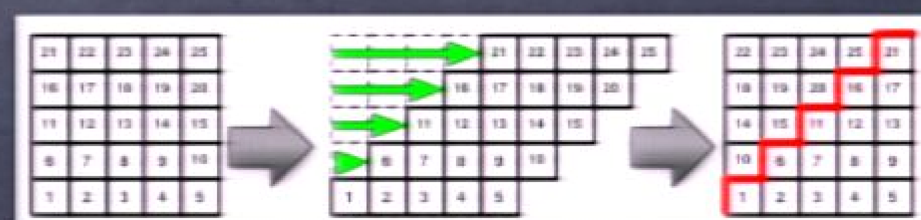
  - ***SU(2)***... compact... all reps have a Haar measure.

  - ***H*** on $L^2(\mathbb{R})$... noncompact... has a nice Haar measure equal to Lebesgue measure on $\mathbb{R}^2$.

Symplectic [Linear] Transforms

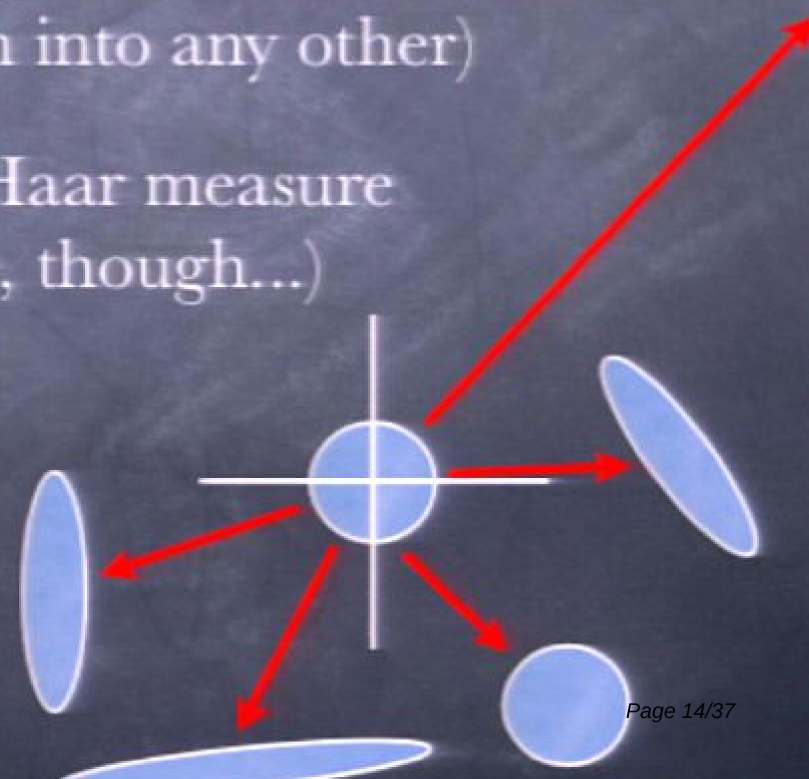
- Heisenberg group = linear *translations* on phase space.
- Other linear transforms: shears, rotations*, squeezes*.
  => *Symplectic group* $\boldsymbol{Sp} = \boldsymbol{SL_2}$ = 2x2 matrices w/det 1.
  - Noncompact [continuously infinite] Lie group.
  - Add in $\boldsymbol{H}$ to get $\boldsymbol{WSp}$, the *affine symplectic group*.
- Discrete $\boldsymbol{SL_2(Z_d)}$
  - Generates MUBs (2-design)
  - Pretty easy to show that it's a unitary 2-design.





Symplectic Transforms on $L^2(\mathbb{R})$

- ***Sp*** **=** ***SL*₂(R)**:
  rotations, shears, squeezes on continuous phase space
- ***WSp*** is a *transitive* symmetry group for Gaussian states
  (i.e., transforms every Gaussian into any other)
- Has a well-defined *left*-invariant Haar measure
  (left and right are not the same, though...)
- So: ***WSp***-invariant measure over Gaussian states is a candidate Gaussian 2-design.



Gaussian 2-Designs *Should* Exist

Why? ***WSp*** is irreducible on $\text{Symm}\left(L^2(\mathbb{R}) \otimes L^2(\mathbb{R})\right)$.

1. Refactor $L^2(x) \otimes L^2(y) \quad \leftrightarrow \quad L^2(x-y) \otimes L^2(x+y)$

$$\equiv \quad L^2(x_+) \otimes L^2(x_-)$$

2. ***H*** subgroup acts *only* on x_+ factor, & is irreducible.

3. ***SL*₂(R)** contains an ***SO(2)*** subroup
   = harmonic oscillator dynamics
   => irreps are 1-dimensional (HO basis states!)

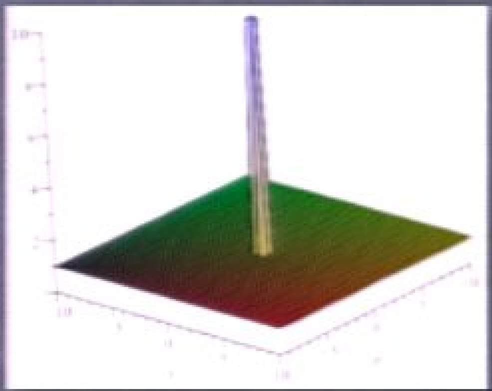
4. Squeezing mixes *all* even (odd) HO basis states.

=> ***WSp*** is irreducible on Symm/Antisymm subspaces

Gaussian 2-Designs *Don't* Exist

Why? Π_{symm} is not in convex hull of Gaussian $|\psi\rangle\langle\psi|^{\otimes 2}$.

1. $\Pi_{\text{symm}} = \mathbb{1}_{x_-} \otimes \frac{1}{2}(1+\mathbb{P})_{x_-}$. Wigner $W(x,p) \propto \delta(x)\delta(p) + 1$
   ...*Impossible* to build w/Gaussians!



   1. Refactor $|\psi\rangle\langle\psi|^{\otimes 2}$ in x_+ / x_- factorization
      $\Rightarrow$ tensor product of pure Gaussians, *and...*
      ... $\psi(x_-)$ is a squeezed vacuum state;
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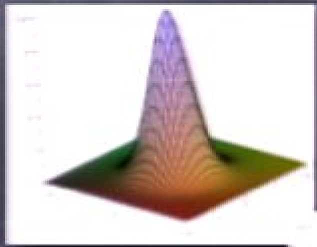
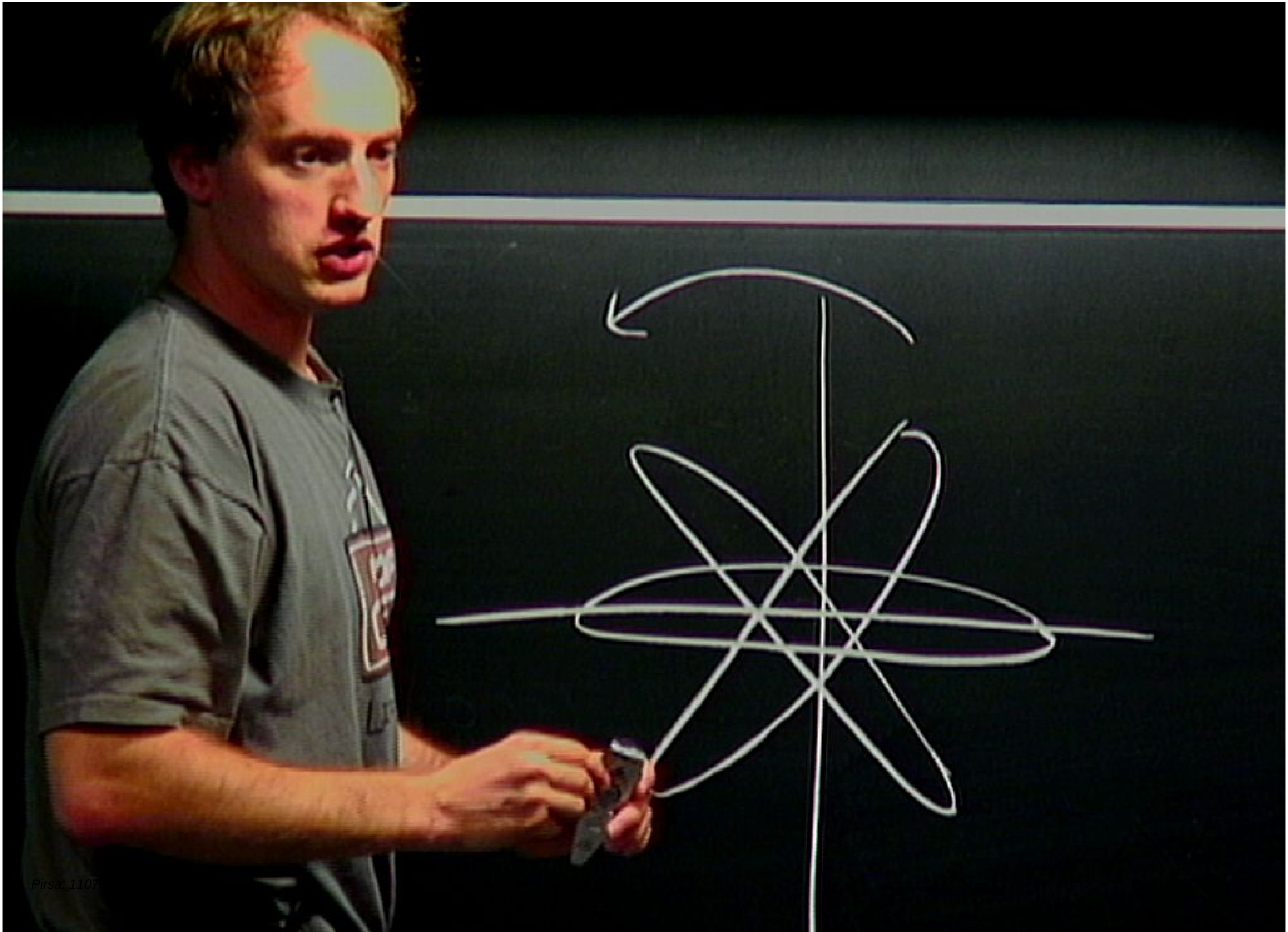
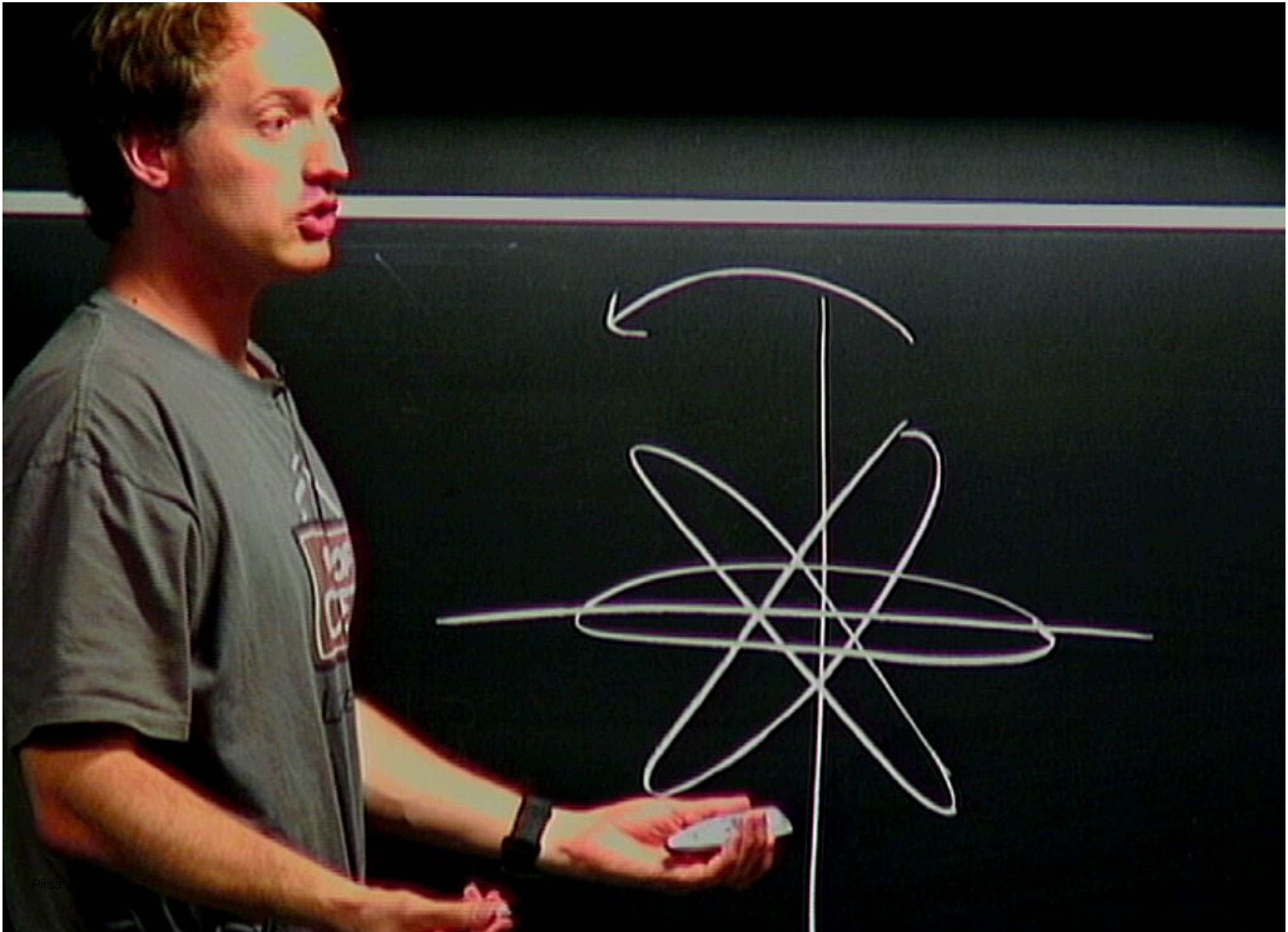
   2. ***H*** twirling depolarizes the x_+ factor, so ignore it.

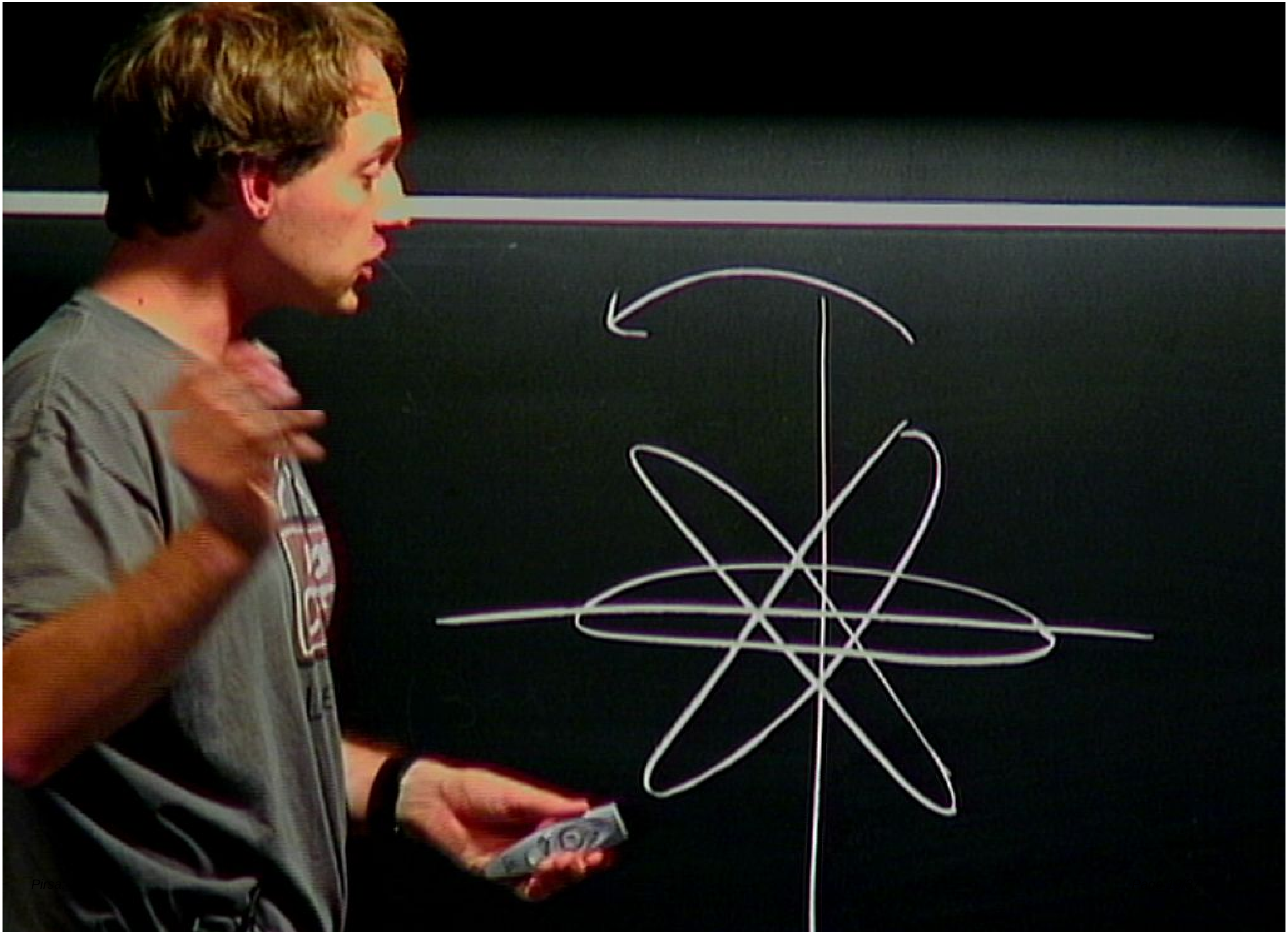


   3. Average $W(x_-, p_-)$ over ***SO*(2)** subgroup
   4. *Every* such W drops off between e^{-r^2} and $1/r$.

   5. So we cannot possibly build up $\delta(x)\delta(p) + 1$.





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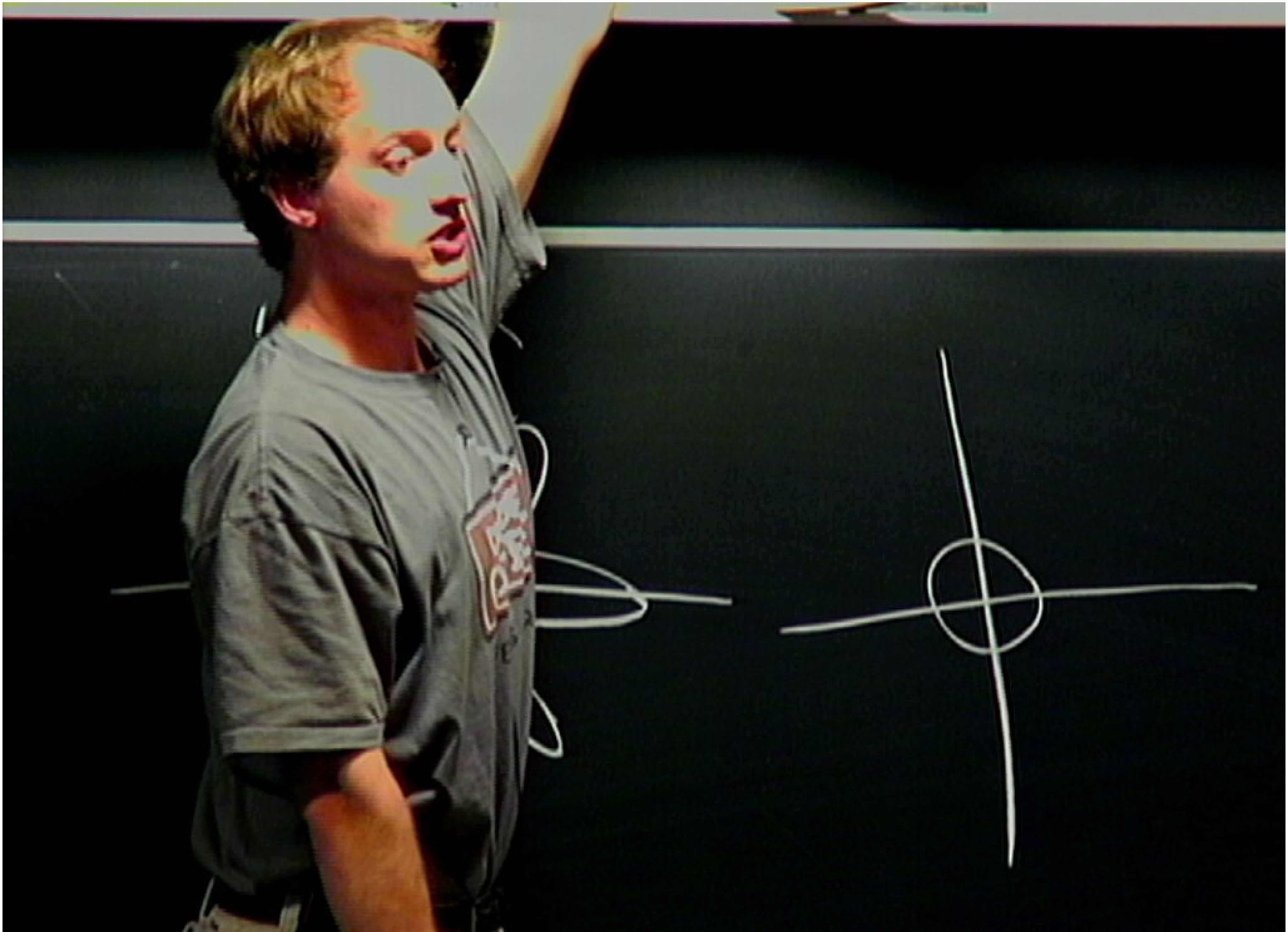

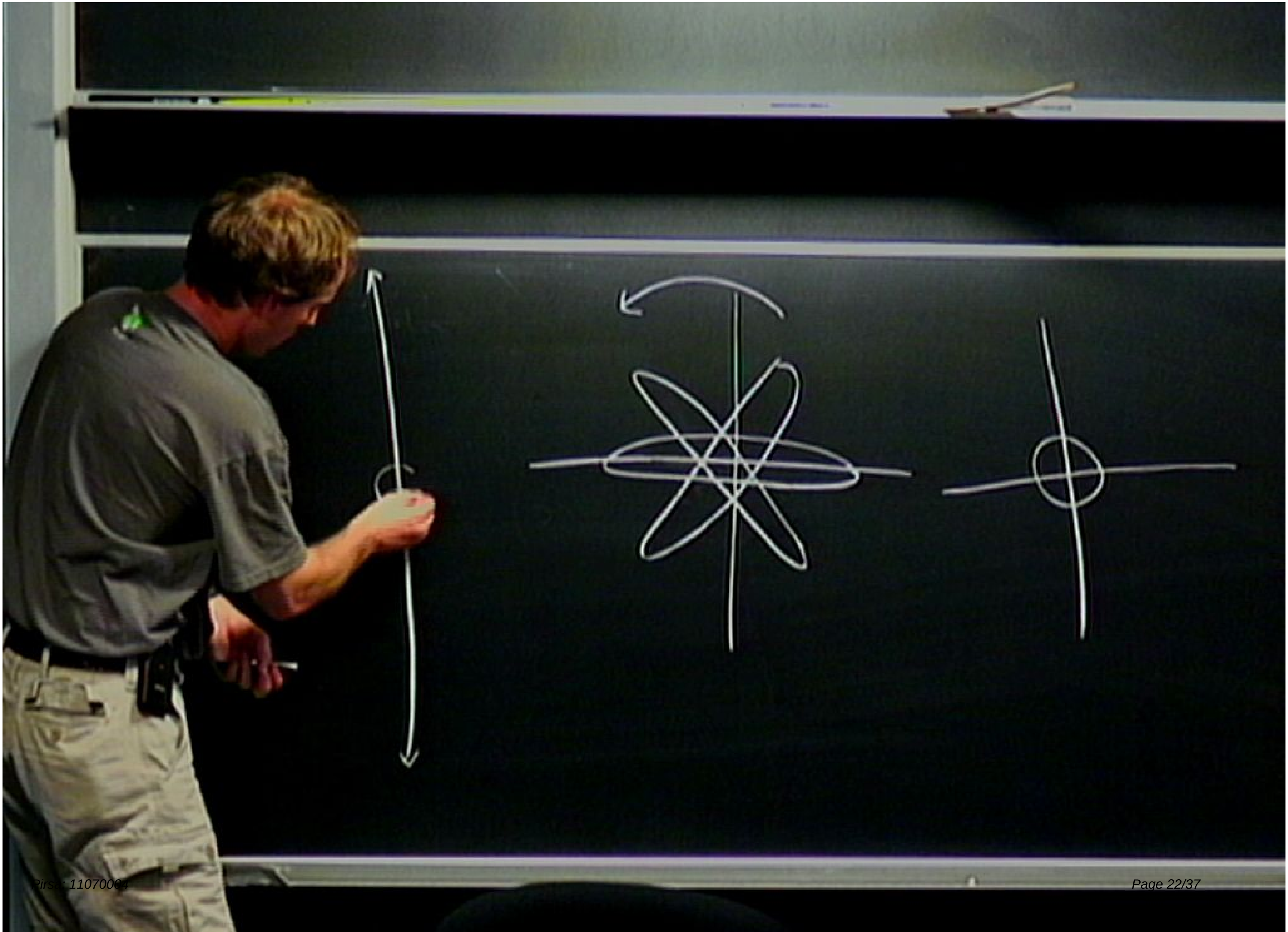

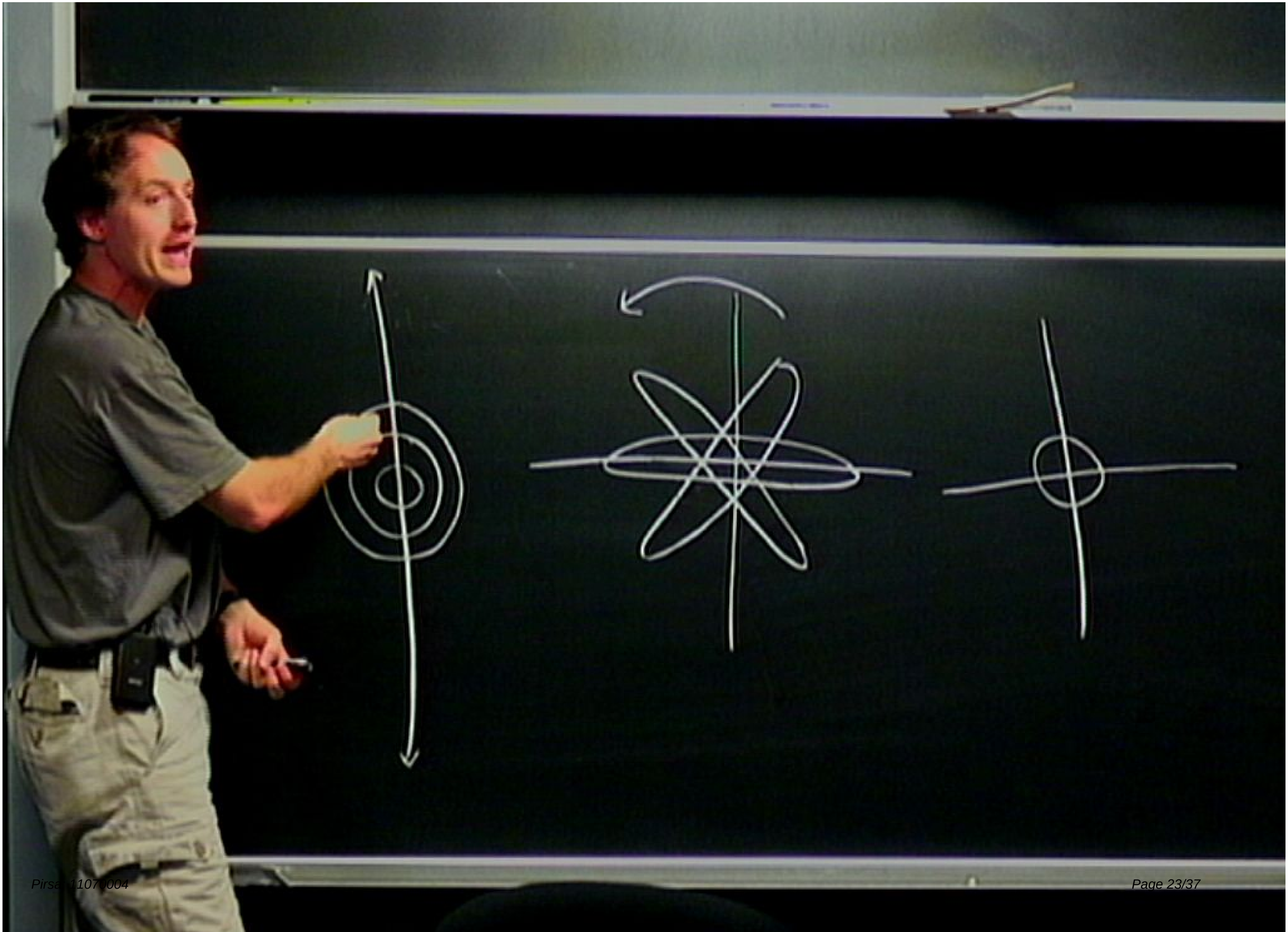

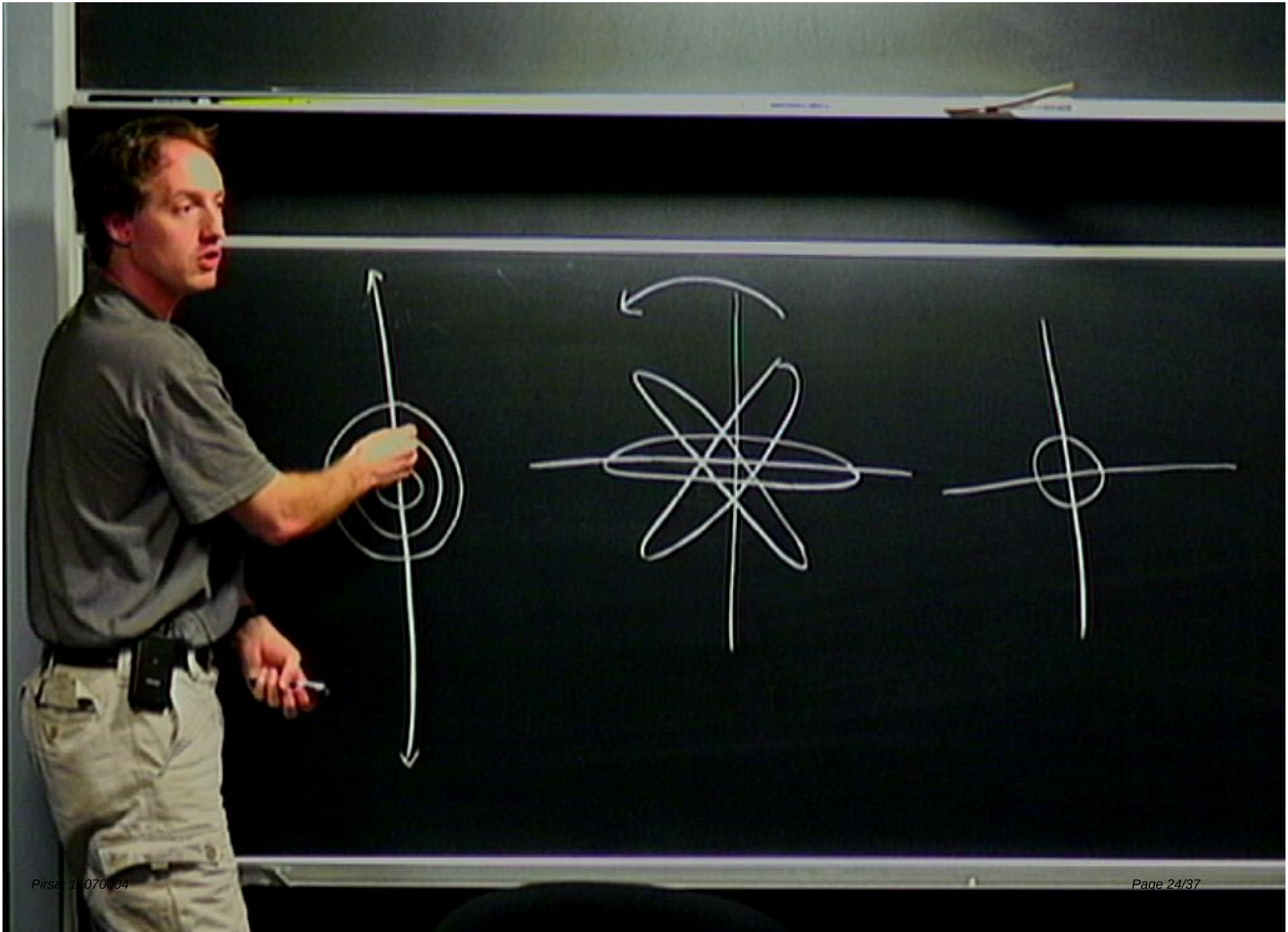

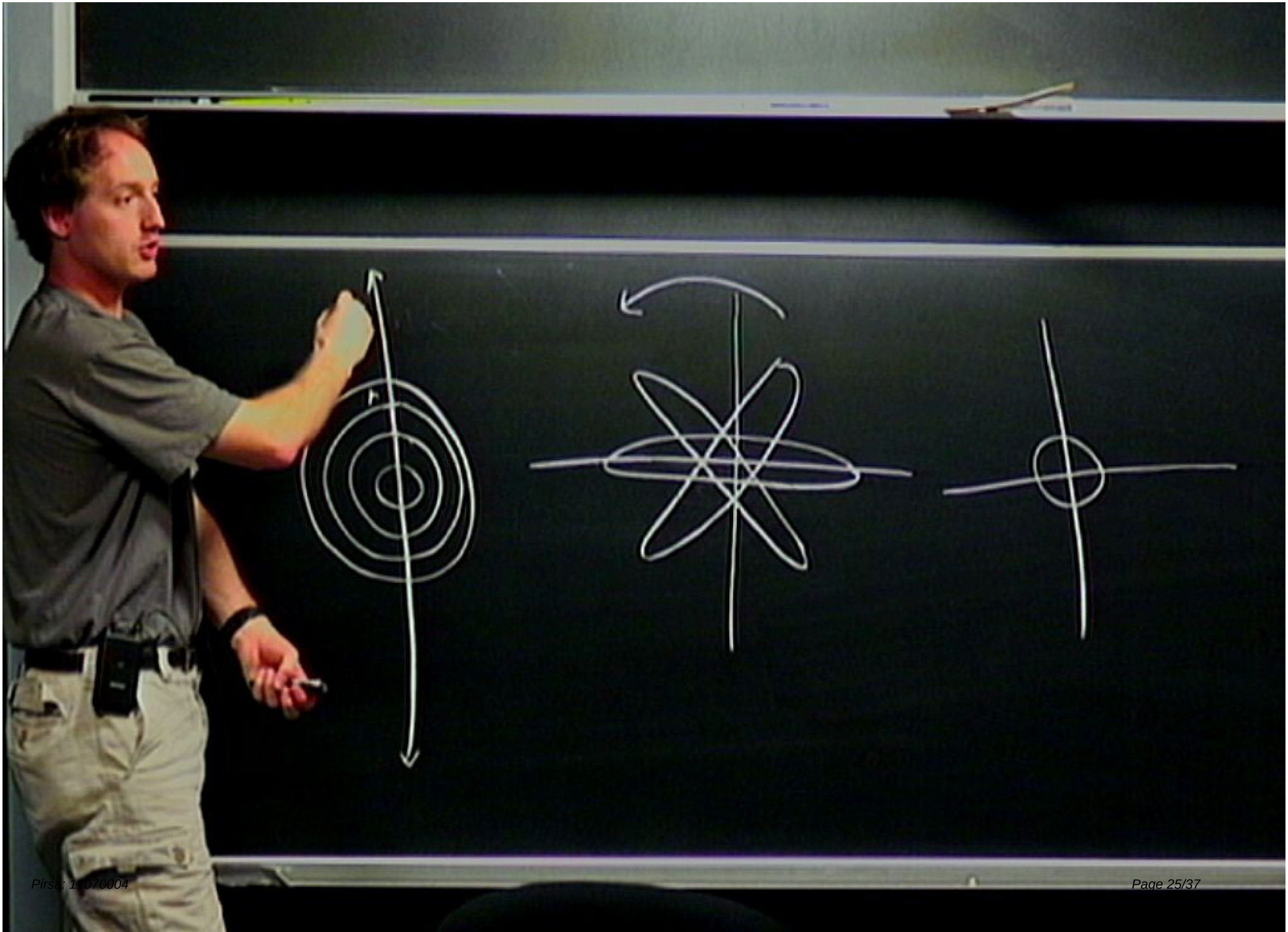

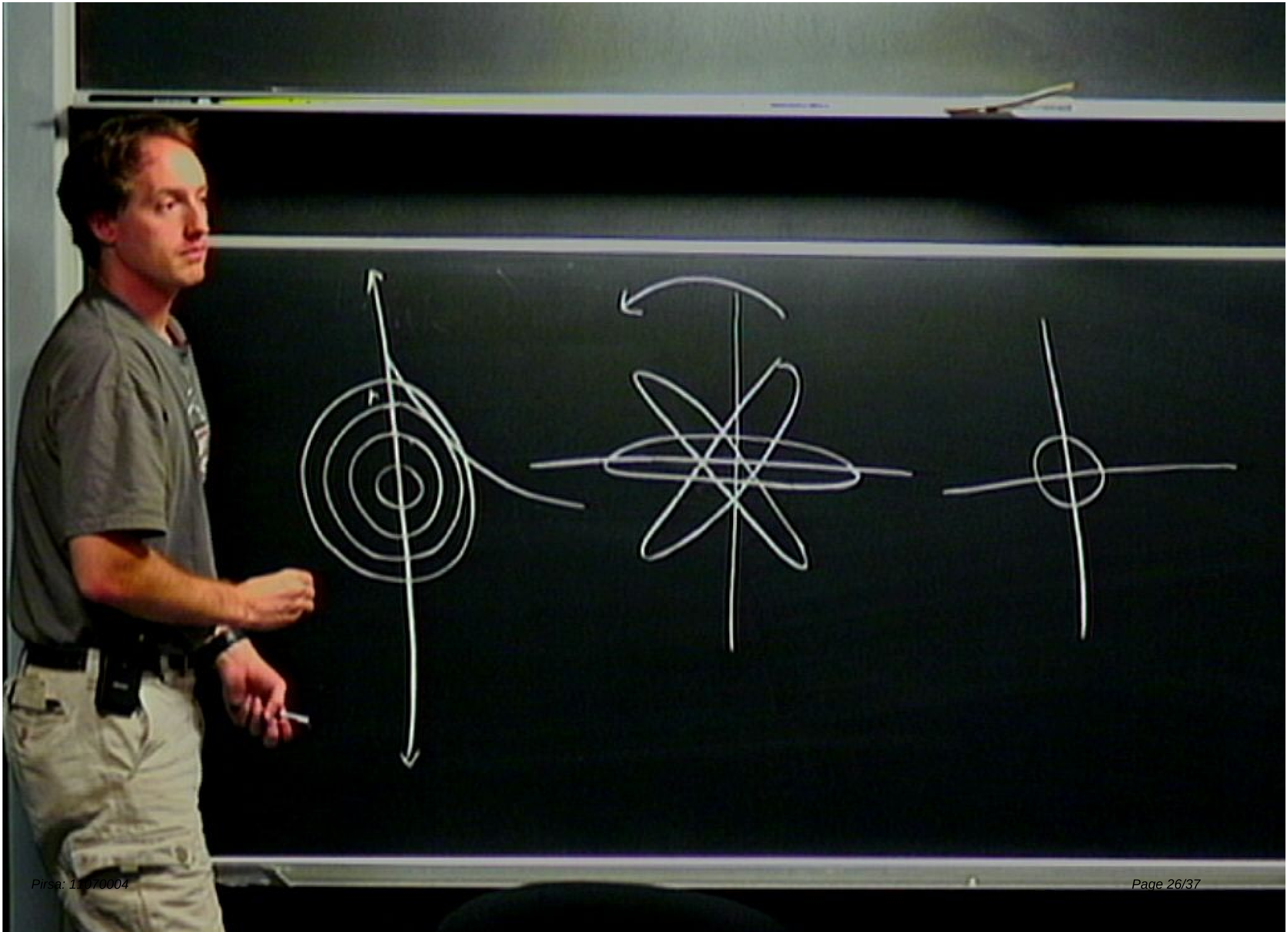

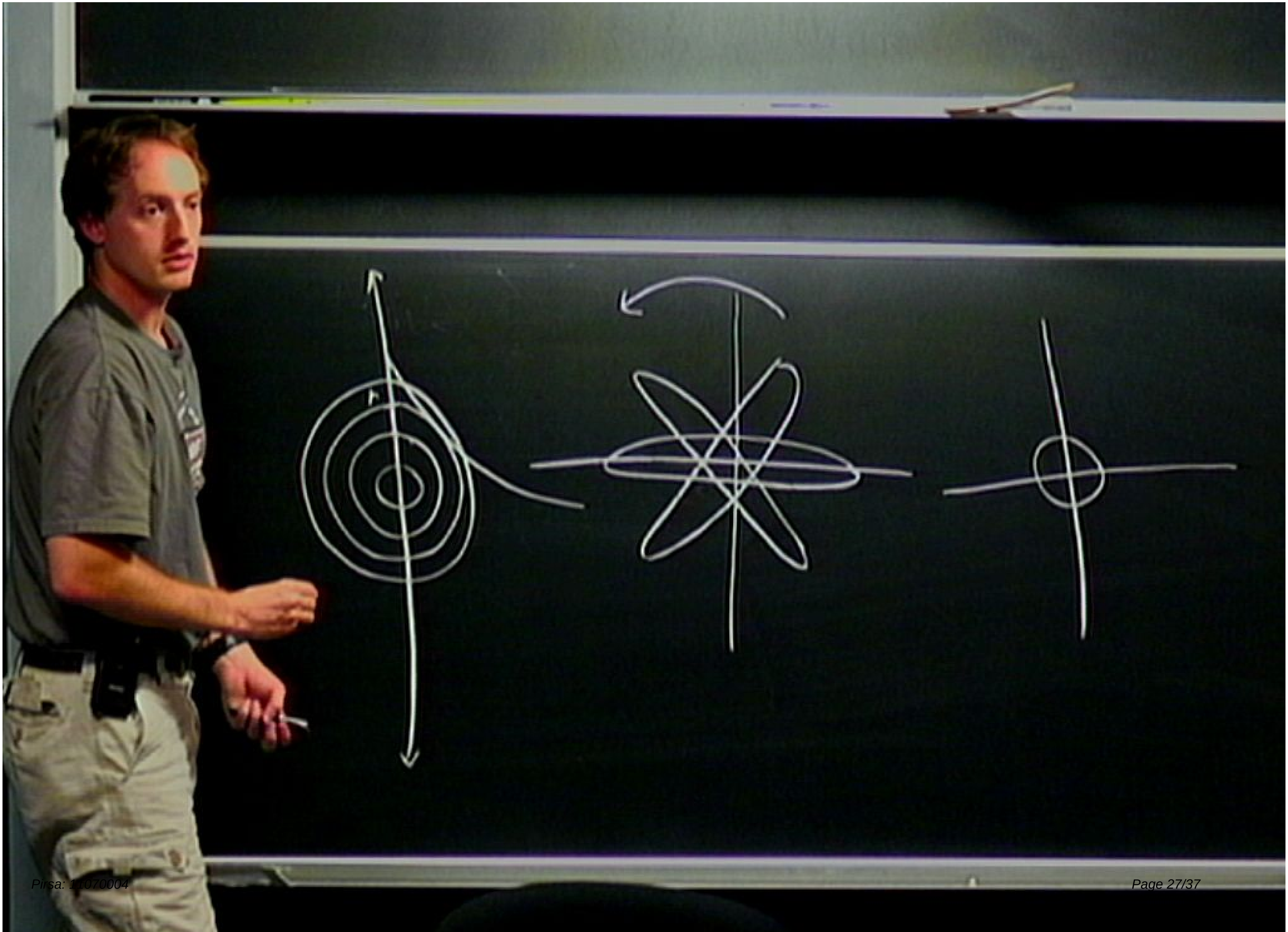

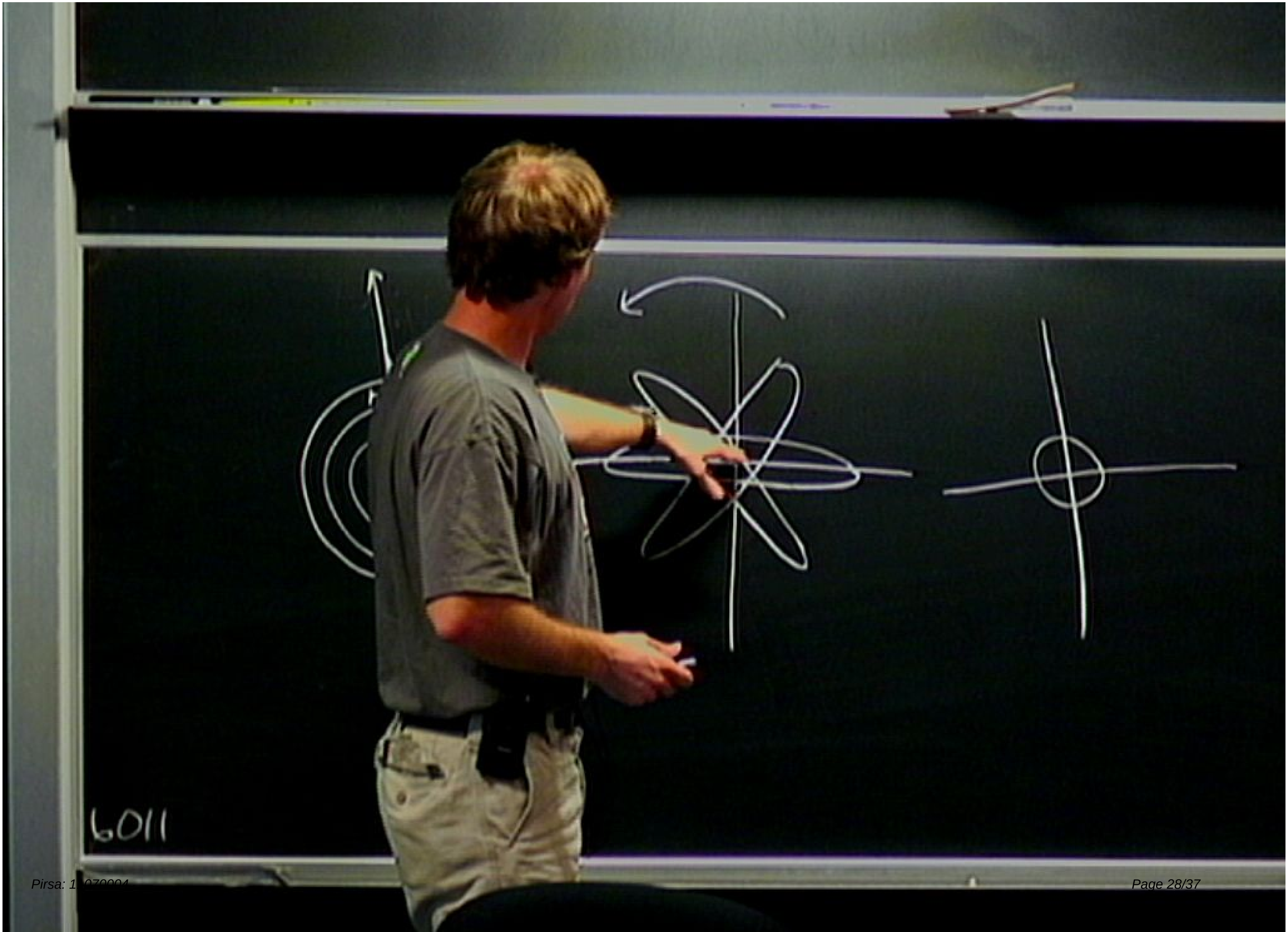
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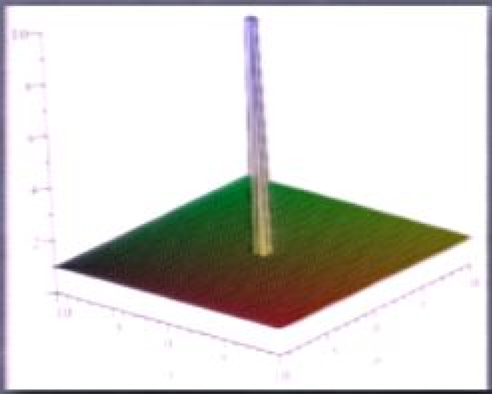

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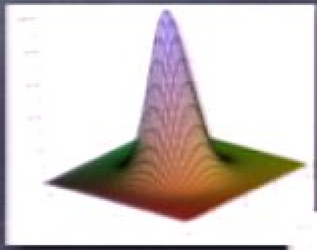

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WTF?

- Have we just proved a contradiction?
- ...Maybe Schur's Lemma doesn't hold?
  "If ρ is irrep-invariant, it must be $\propto \mathbb{1}_{\text{irrepspace}}$."
- No. Schur's Lemma still holds (the only ***SL*₂(R)**-invariant operator on $\text{Symm}(H^{\otimes 2})$ is Π_{symm}.
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  NOTHING. The twirling integral diverges.
- ***SL*₂(R)**-twirling of $|\psi\rangle\langle\psi|^{\otimes 2}$ diverges...
  ...so the "average" state isn't defined
  ...so we can derive contradictory properties for it.

A Study in Non-Convergence

- Problem is *not* that the measure doesn't exist.
  You just can't integrate $|\psi\rangle\langle\psi|^{\otimes 2}$ over it.

- Contrast with ***H***-twirling & coherent states
  - ***H***-measure is unbounded, so *technically* the integral of $|\psi\rangle\langle\psi|$ doesn't converge (non-Cauchy sequence).
  - But project $|\psi\rangle\langle\psi|$ onto any *bounded* subspace, and it does converge... to $\mathbb{1}$.
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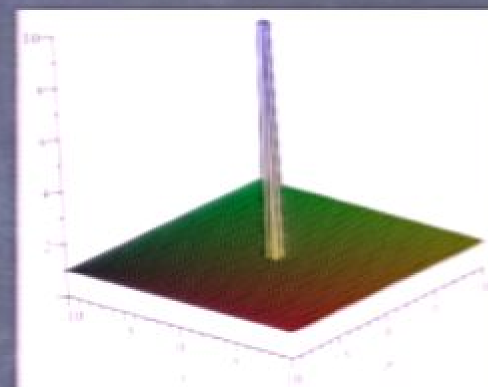

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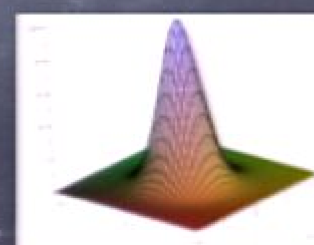

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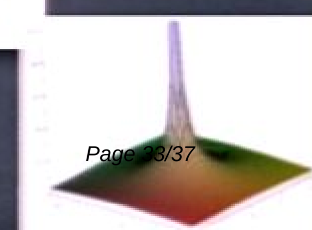

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Conclusions (Math)

- Even with a perfectly well-defined measure, some functions can't be integrated (and some can, of course).
  - e.g., $\boldsymbol{SL_2}(\mathbf{R})$-twirling is still well-defined!
- Sometimes, ill-defined integrals *really are* ill-defined!!!
- This issue probably rules out *any* well-behaved $\boldsymbol{SL_2}(\mathbf{R})$-covariant 2-designs for $L^2(\mathbb{R})$.
- SICPOVM-type solutions are still possible, but if they exist, they must be *really* nasty.
  (maximal entanglement between x_+ and x_- factors).

Conclusions (Physics)

- **There are no Gaussian 2-designs**.
- You can't even get very close.
- **You can't even get close on $E<E_0$ subspaces.**
  N.B. 2-designs are possible -- but not Gaussian ones.
- But... a *good* ensemble of squeezed states is a *lot* closer to being a 2-design than the coherent state ensemble.
  - 2-design is flat over E...
    squeezed states ~ $1/E$ decline...
    coherent states ~ e^{-E} decline.
- **First *practical* difference between infinite and finite Hilbert spaces** (for quantum info science) that I'm aware of.