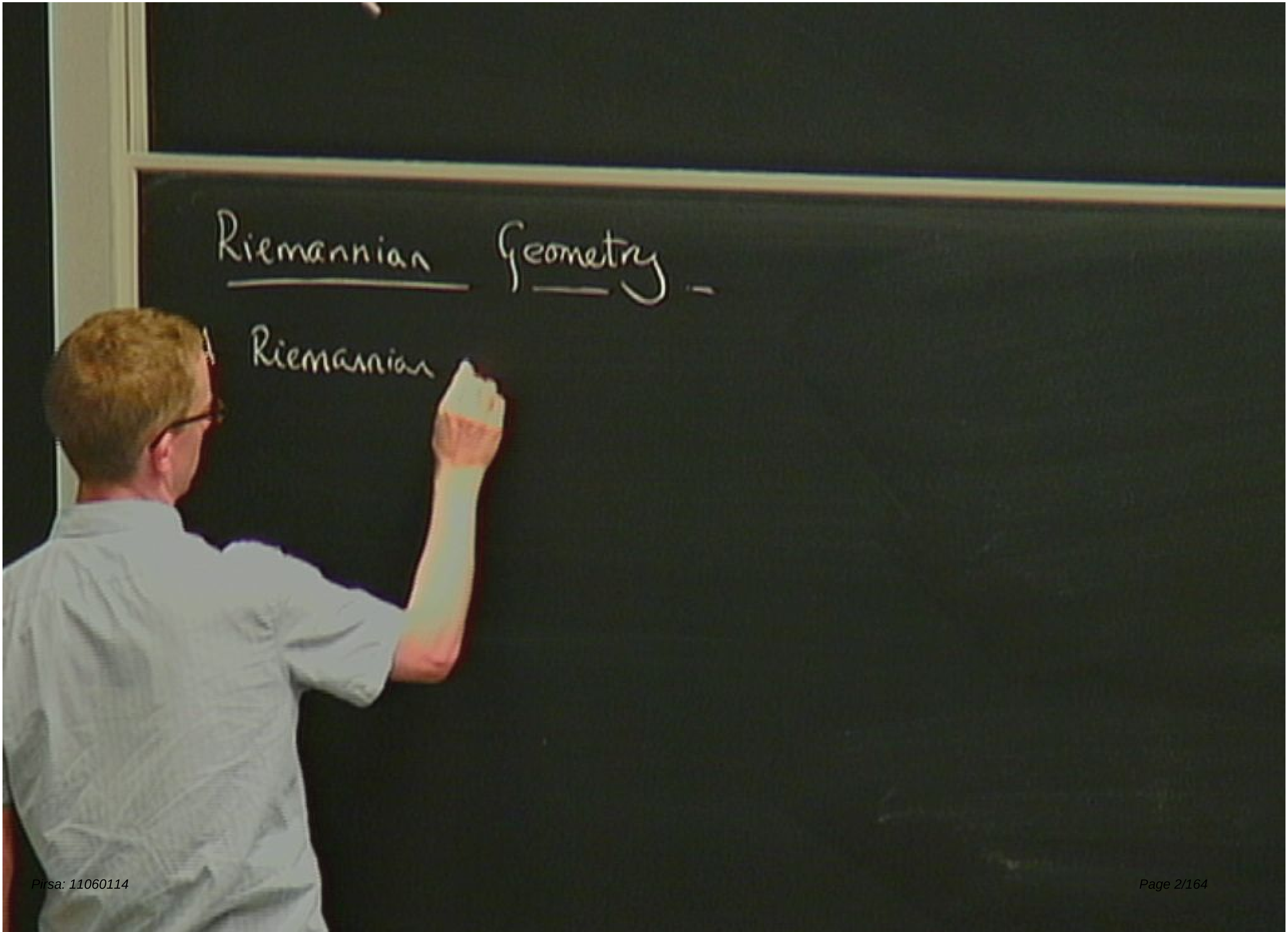
Title: Geometry & Topology for Physics - Lecture 4

Date: Jun 13, 2011 02:00 PM

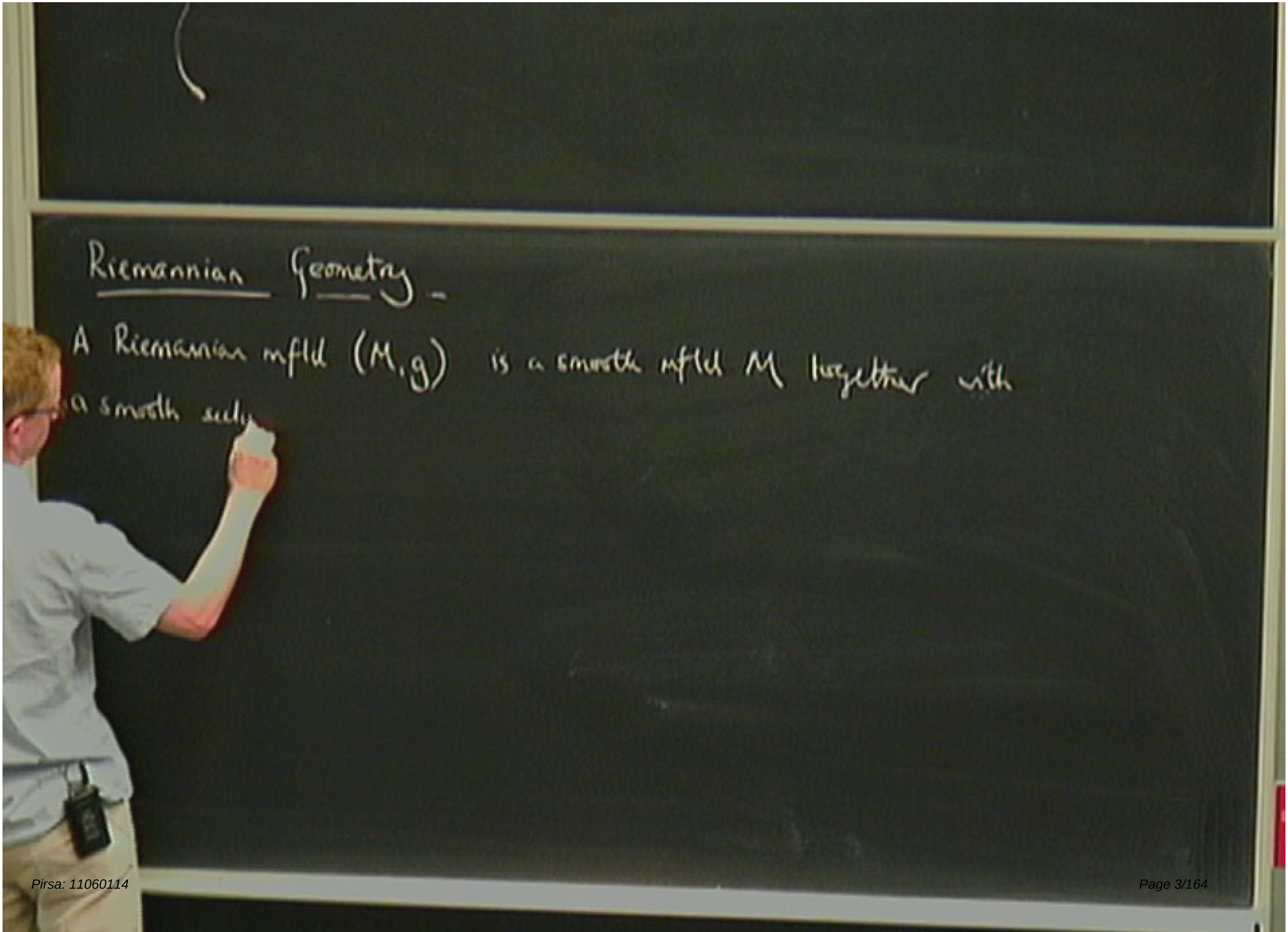
URL: http://pirsa.org/11060114

Abstract:



Riemannian Geometry

Riemannian



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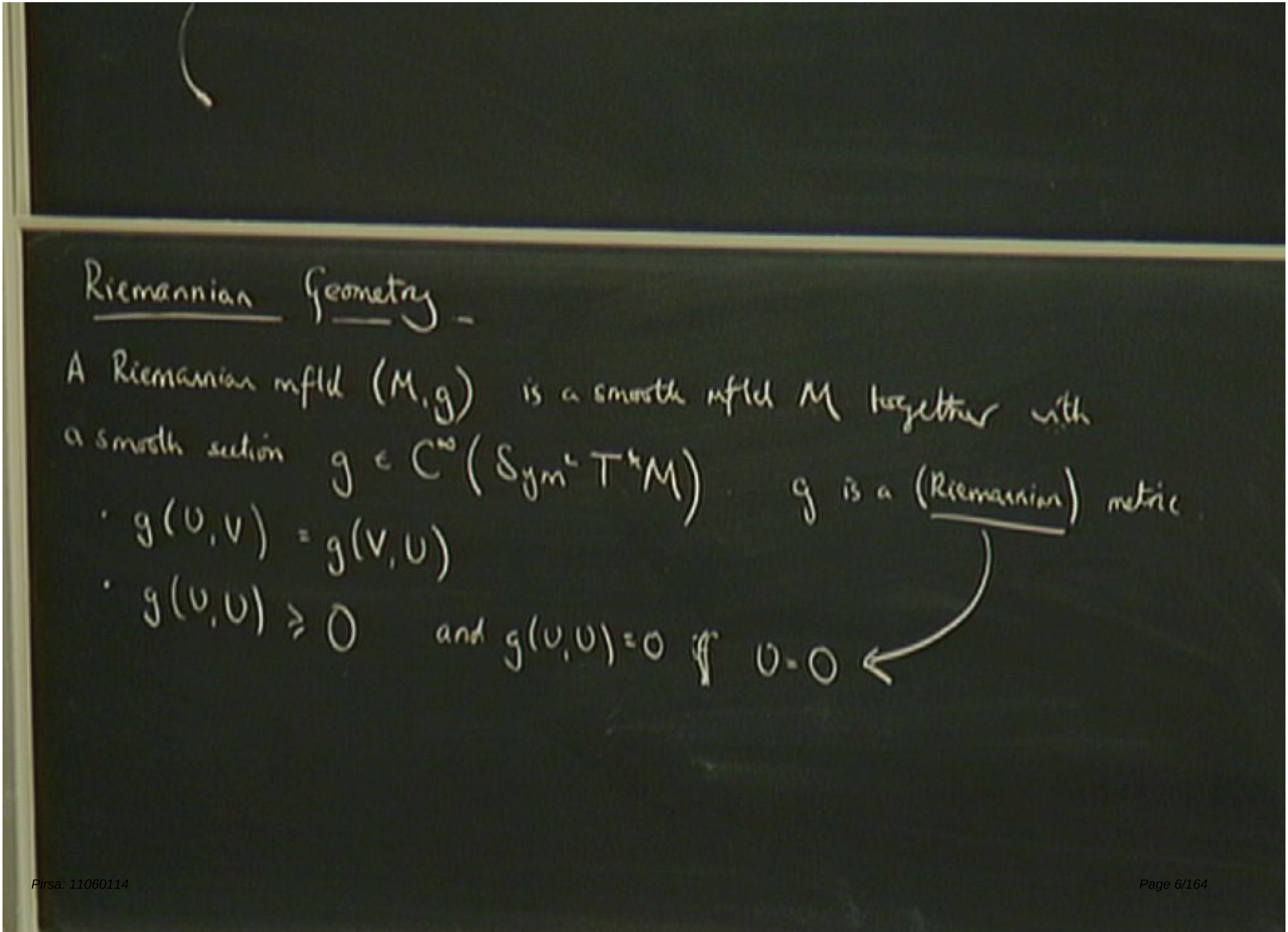

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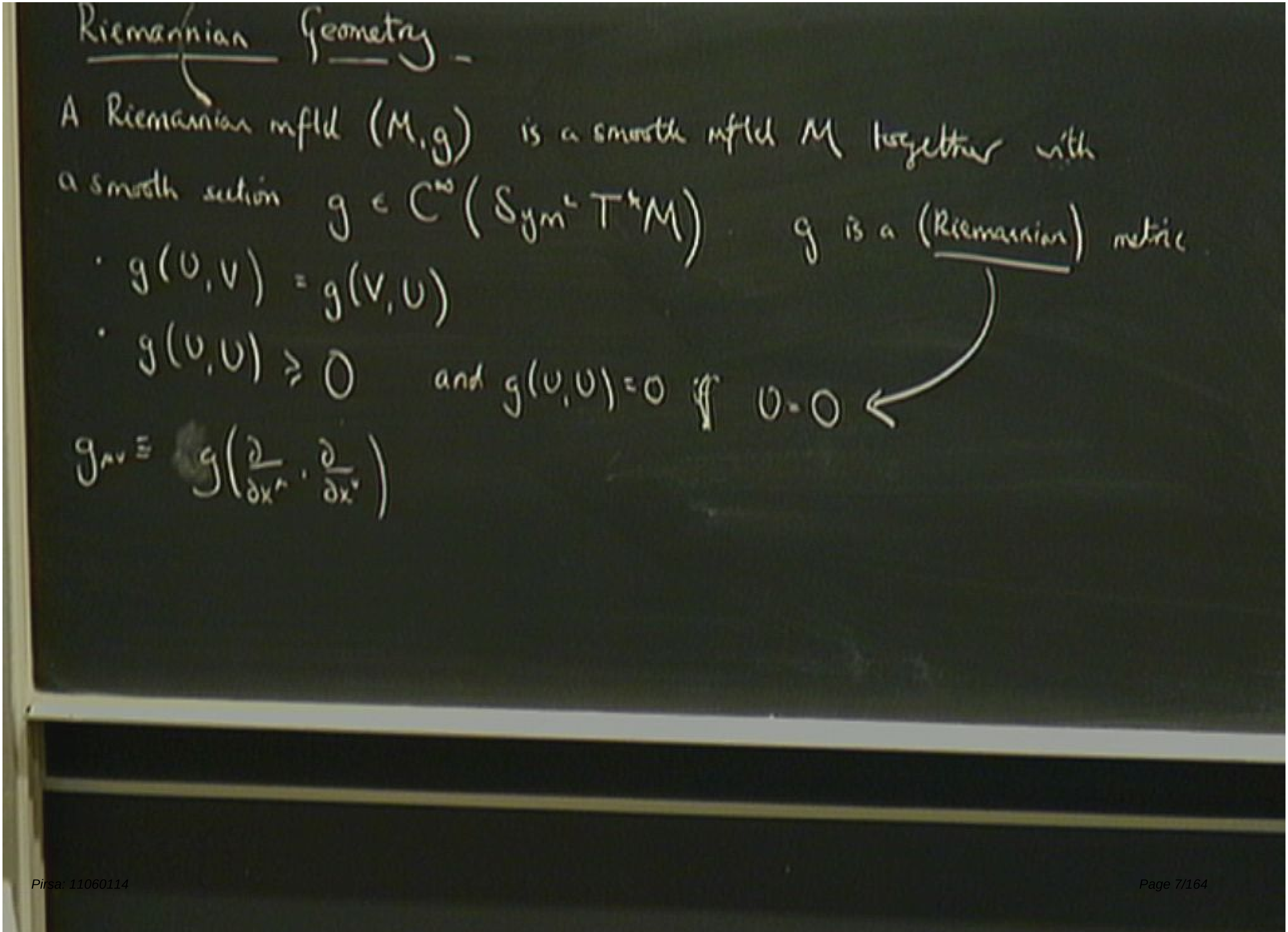

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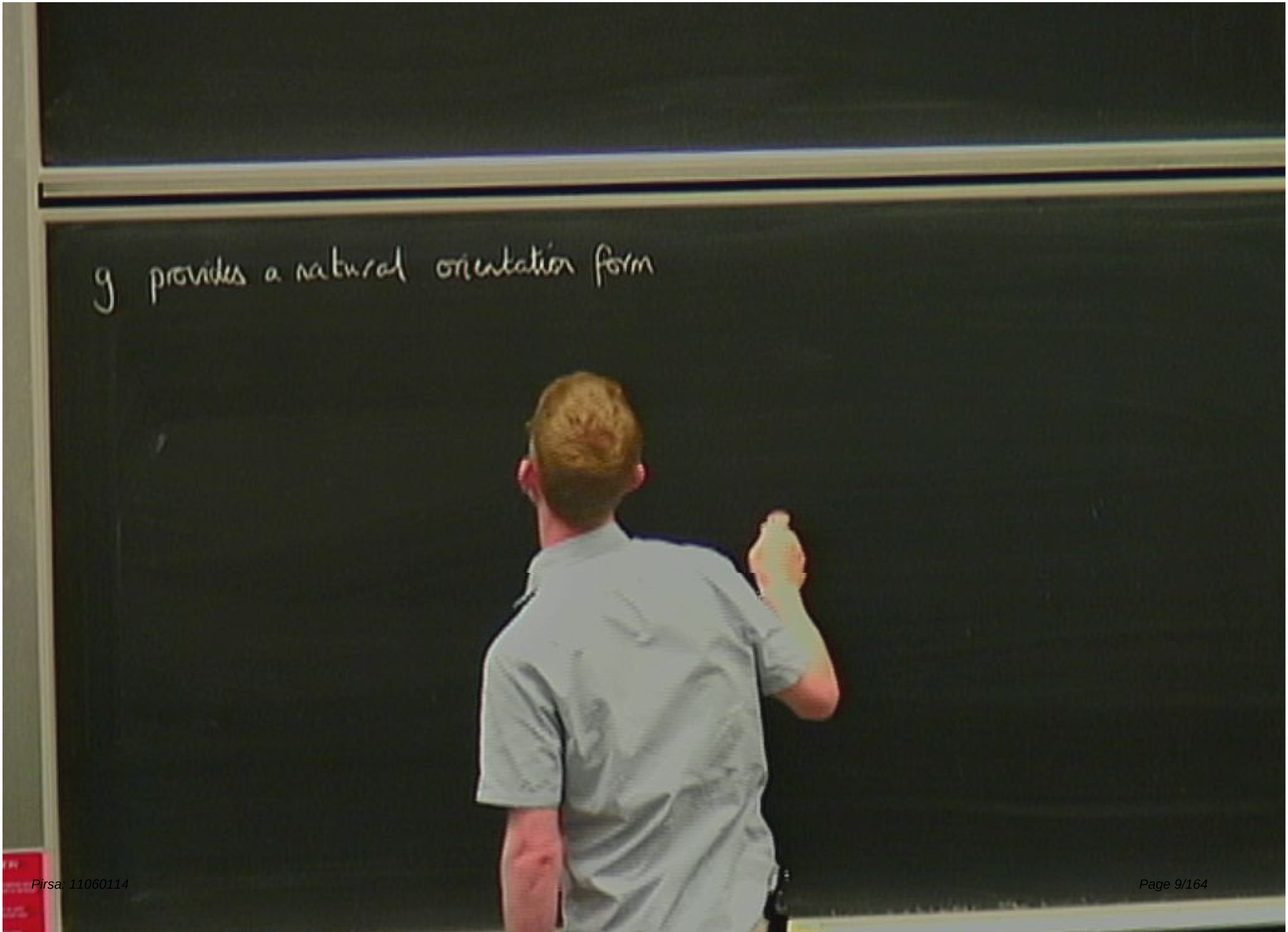
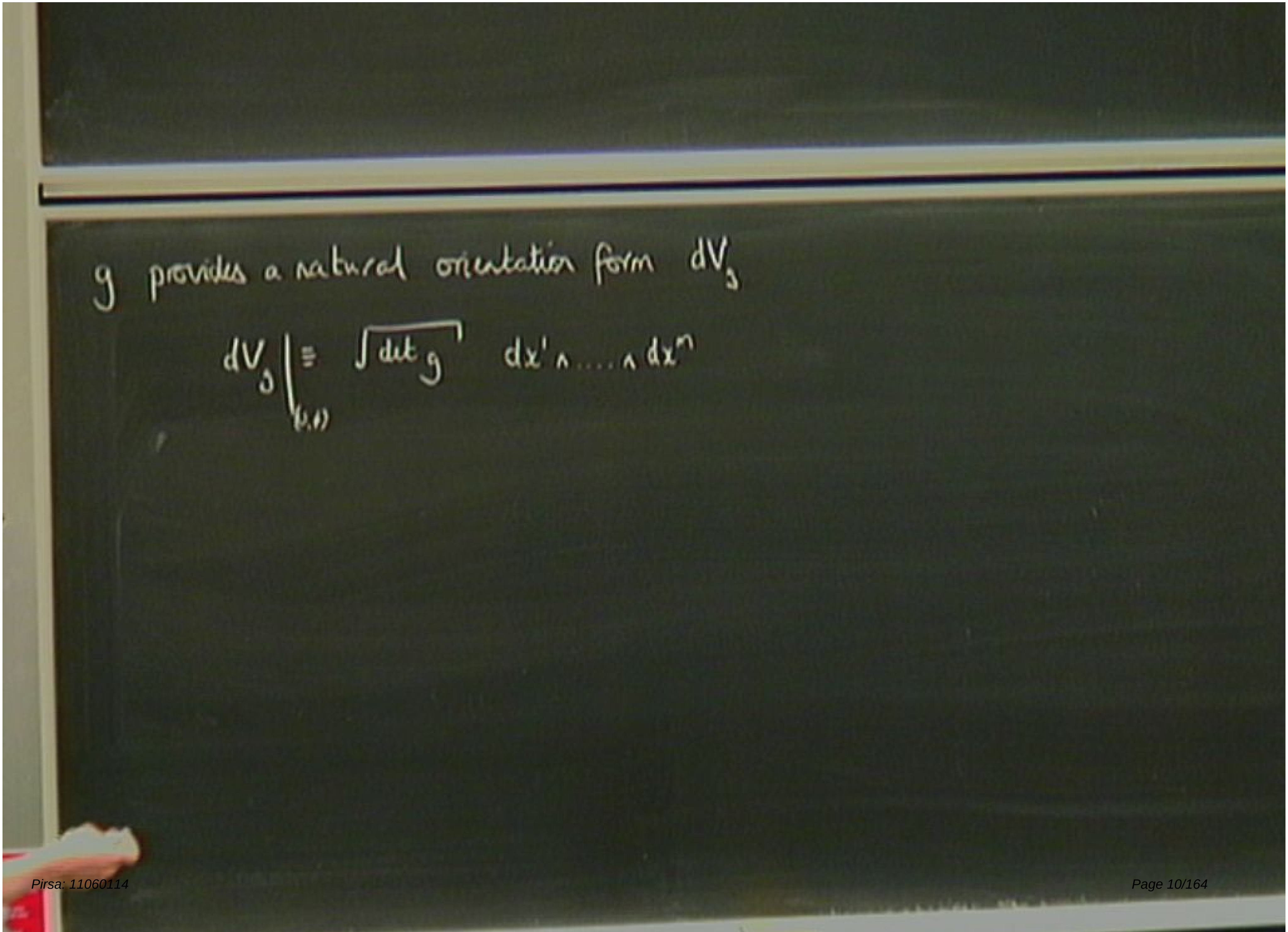
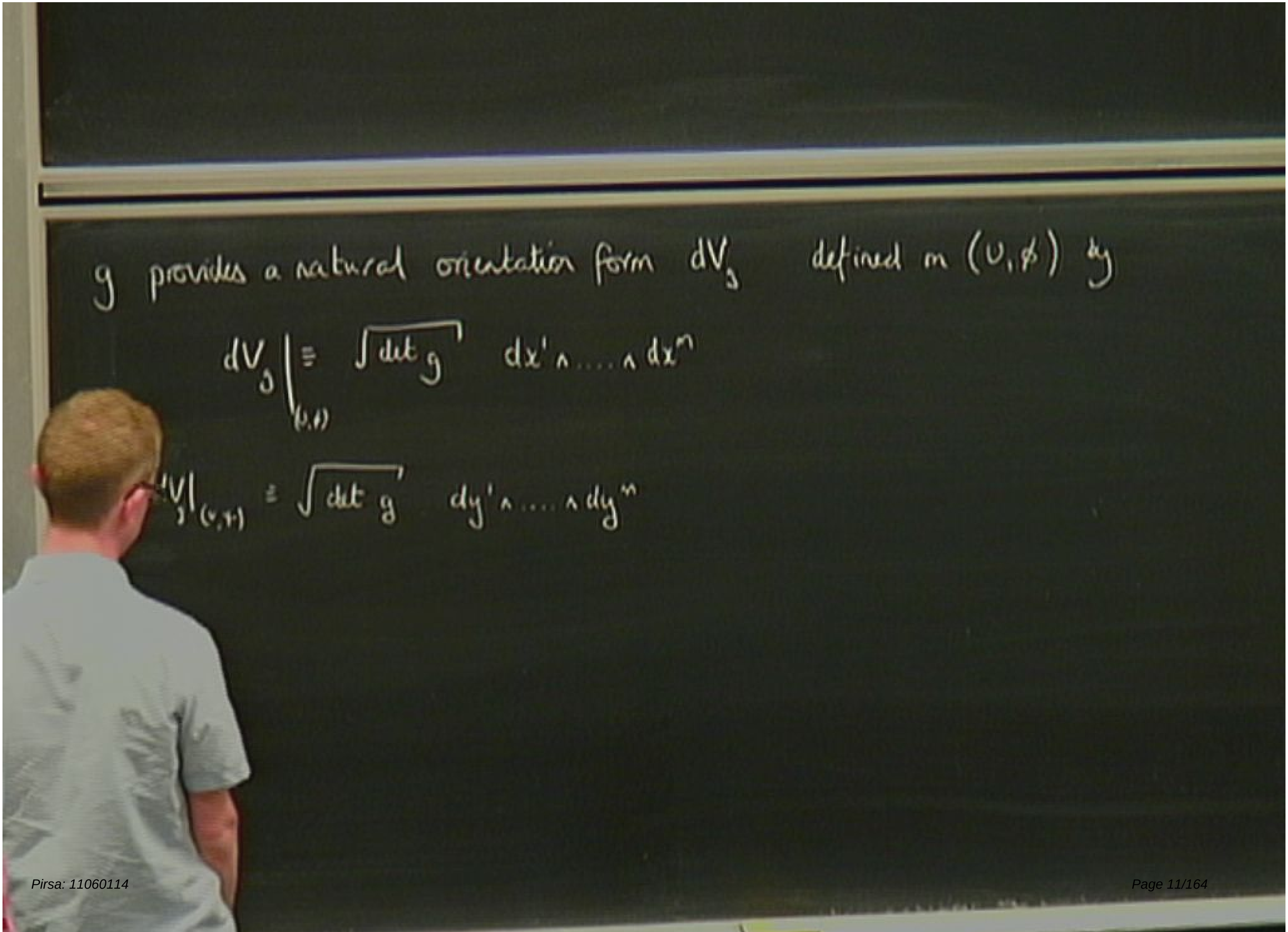


g provides a natural orientation form



g provides a natural orientation form dV_g

$$dV_g\Big|_{(U,x)} \equiv \sqrt{\det g}\; dx^1\wedge\ldots\wedge dx^n$$



g provides a natural orientation form dV_g defined on (U,ϕ) by

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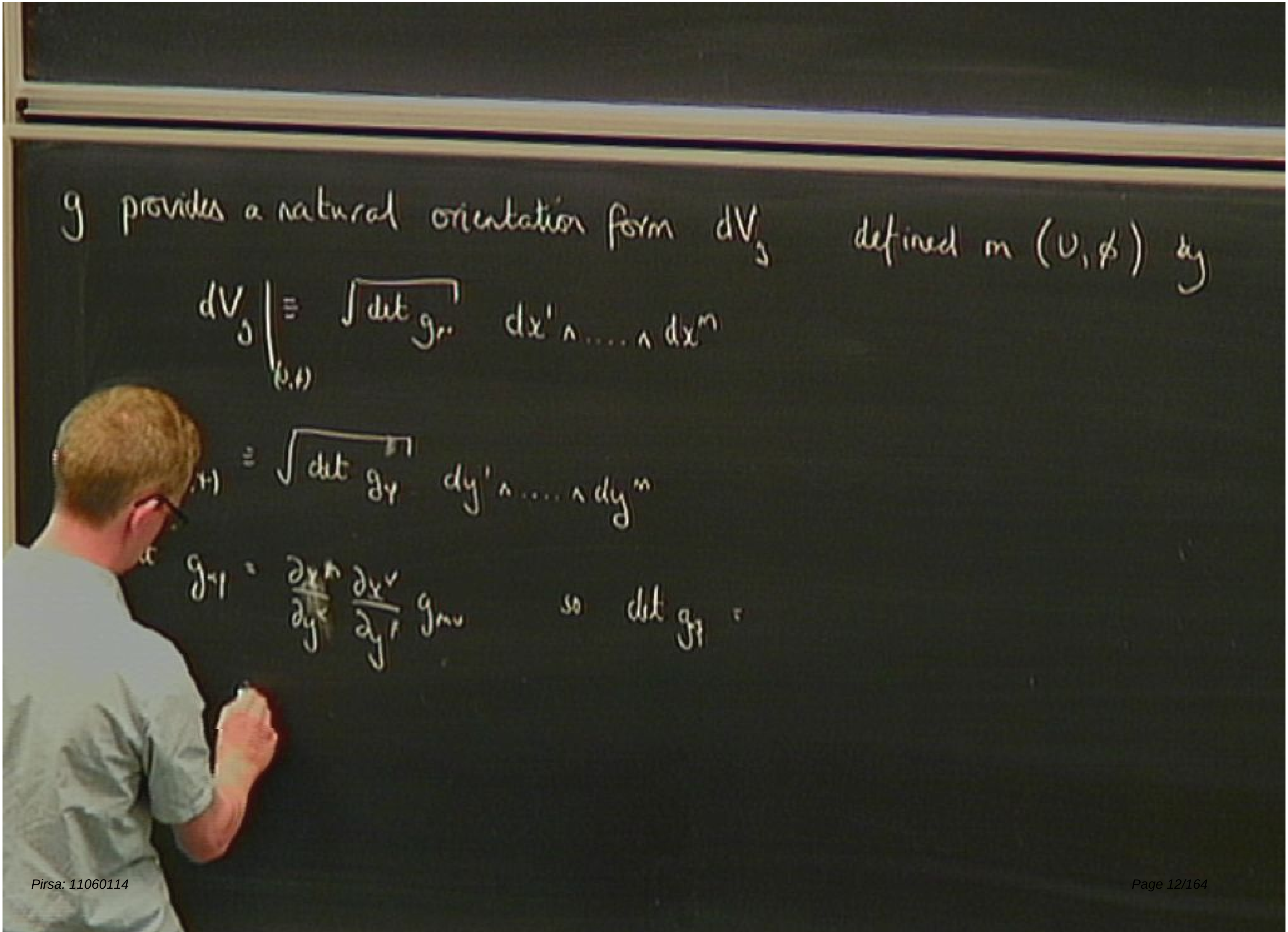

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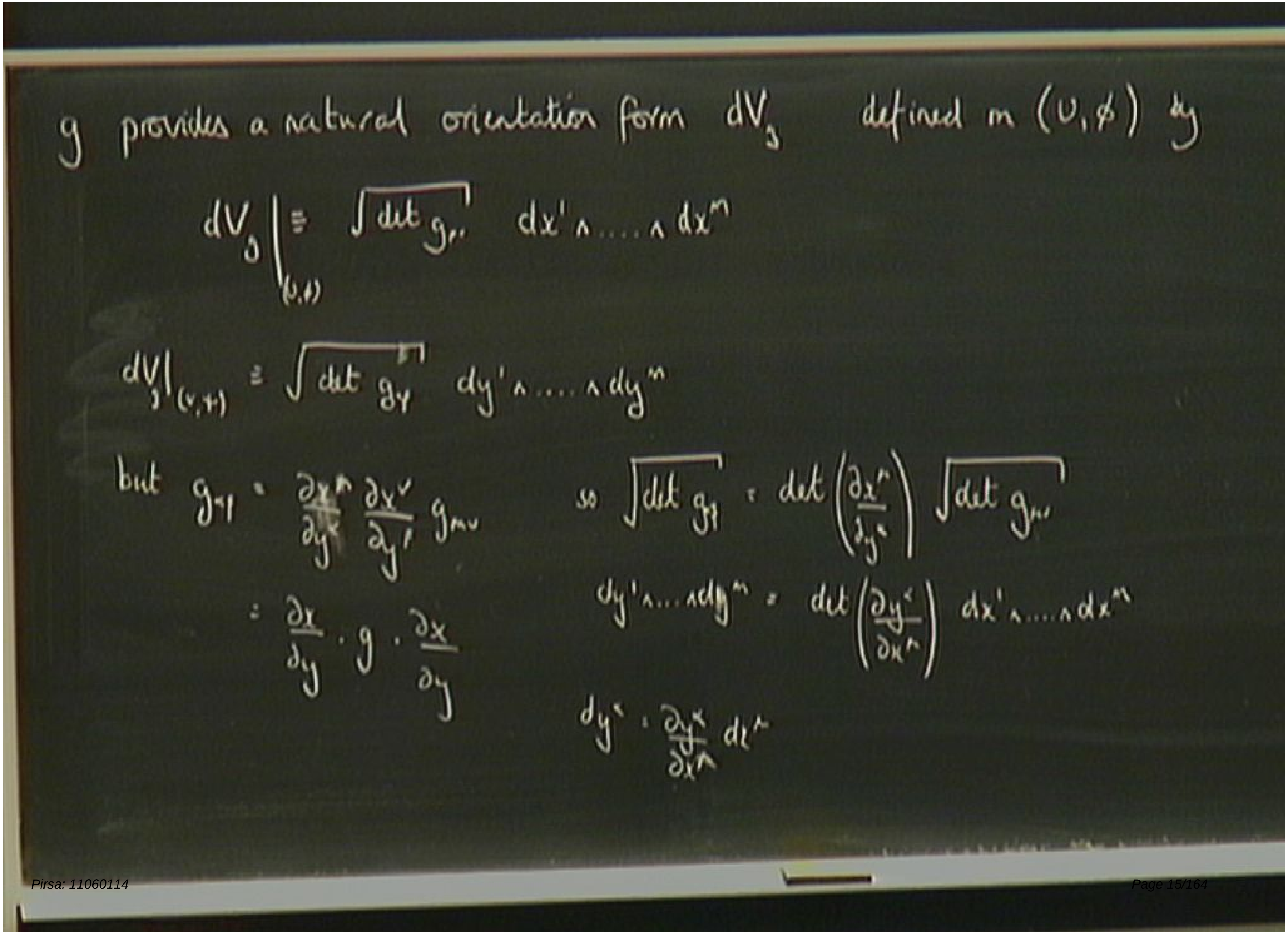

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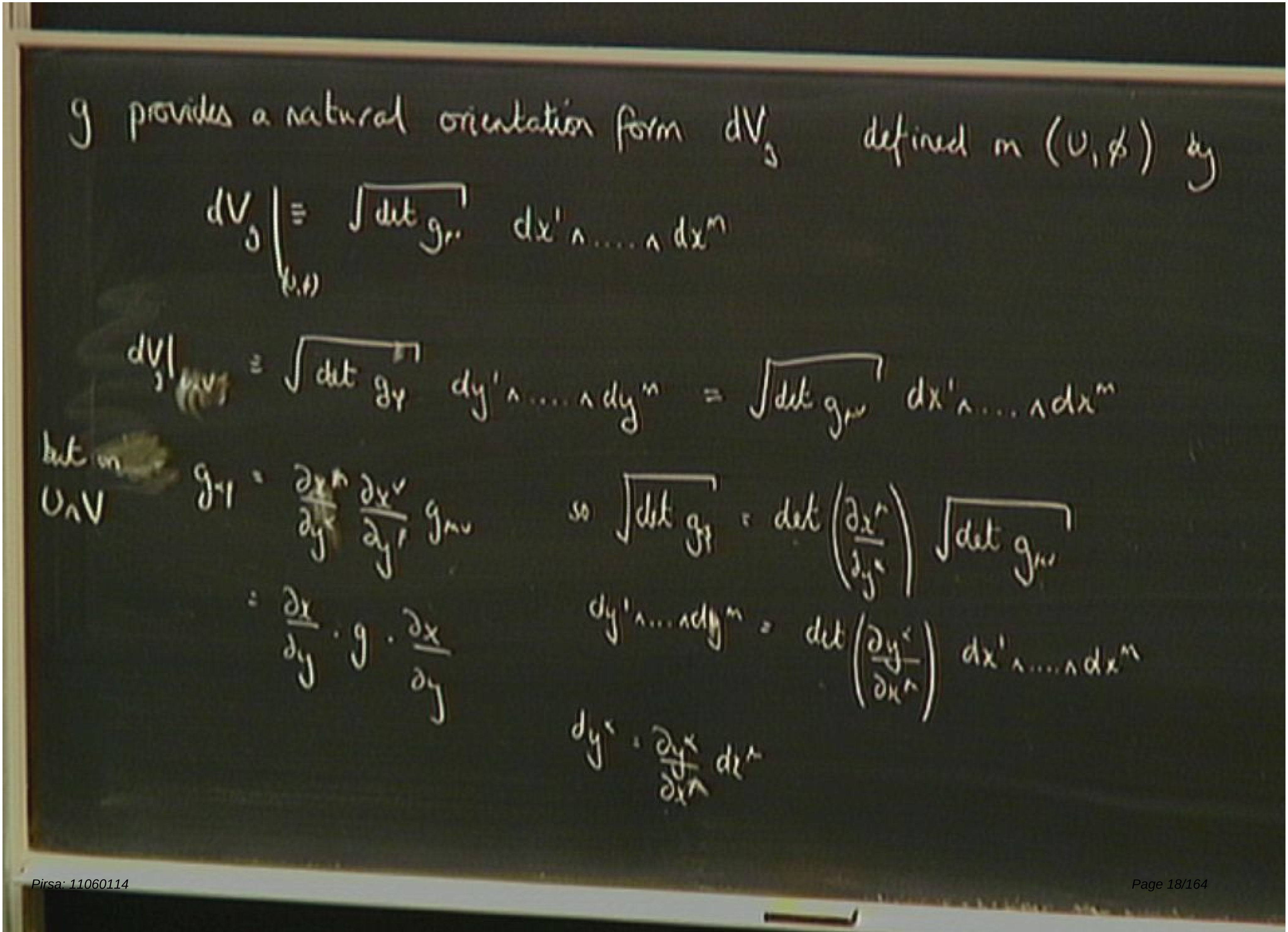

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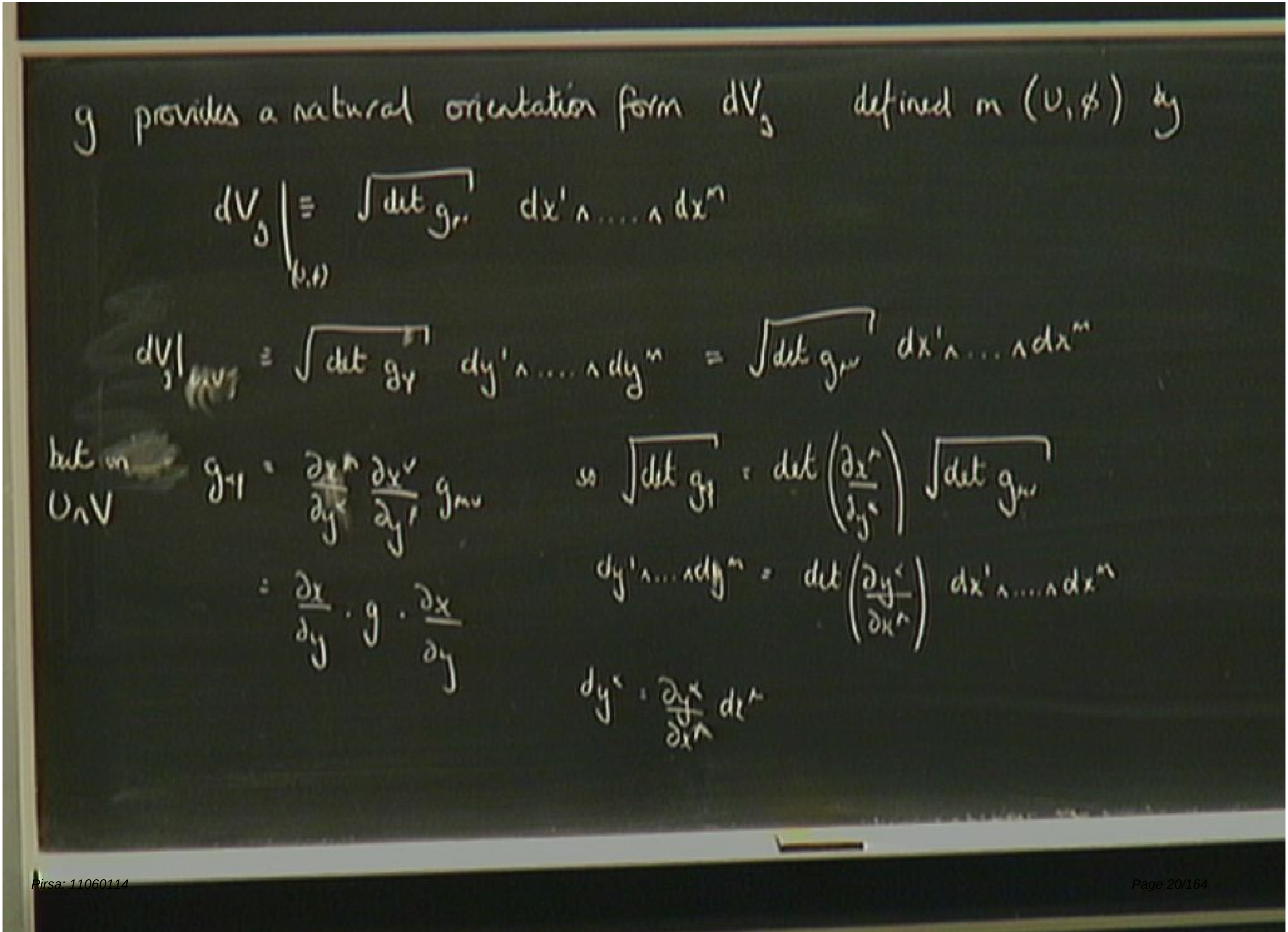

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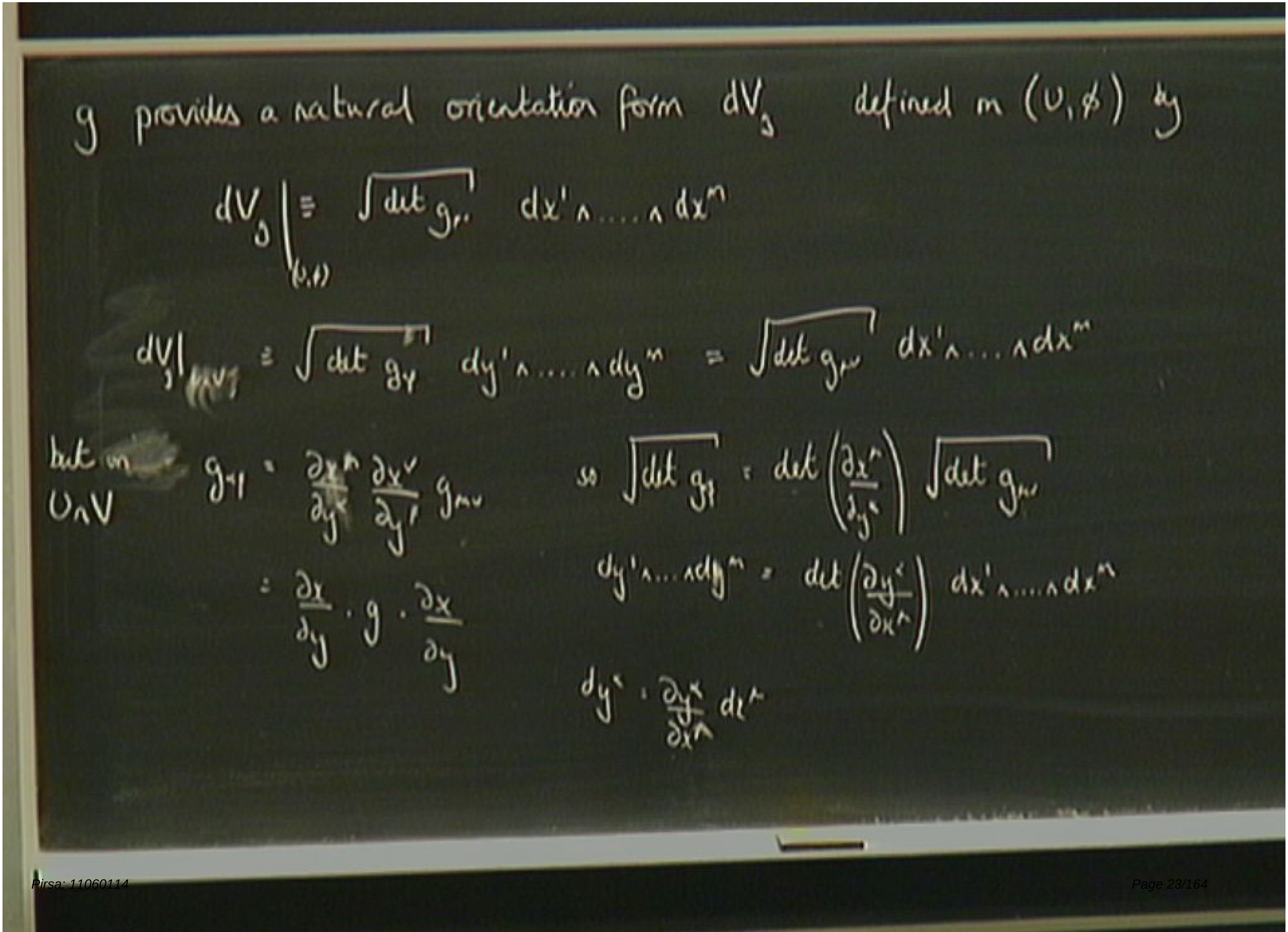

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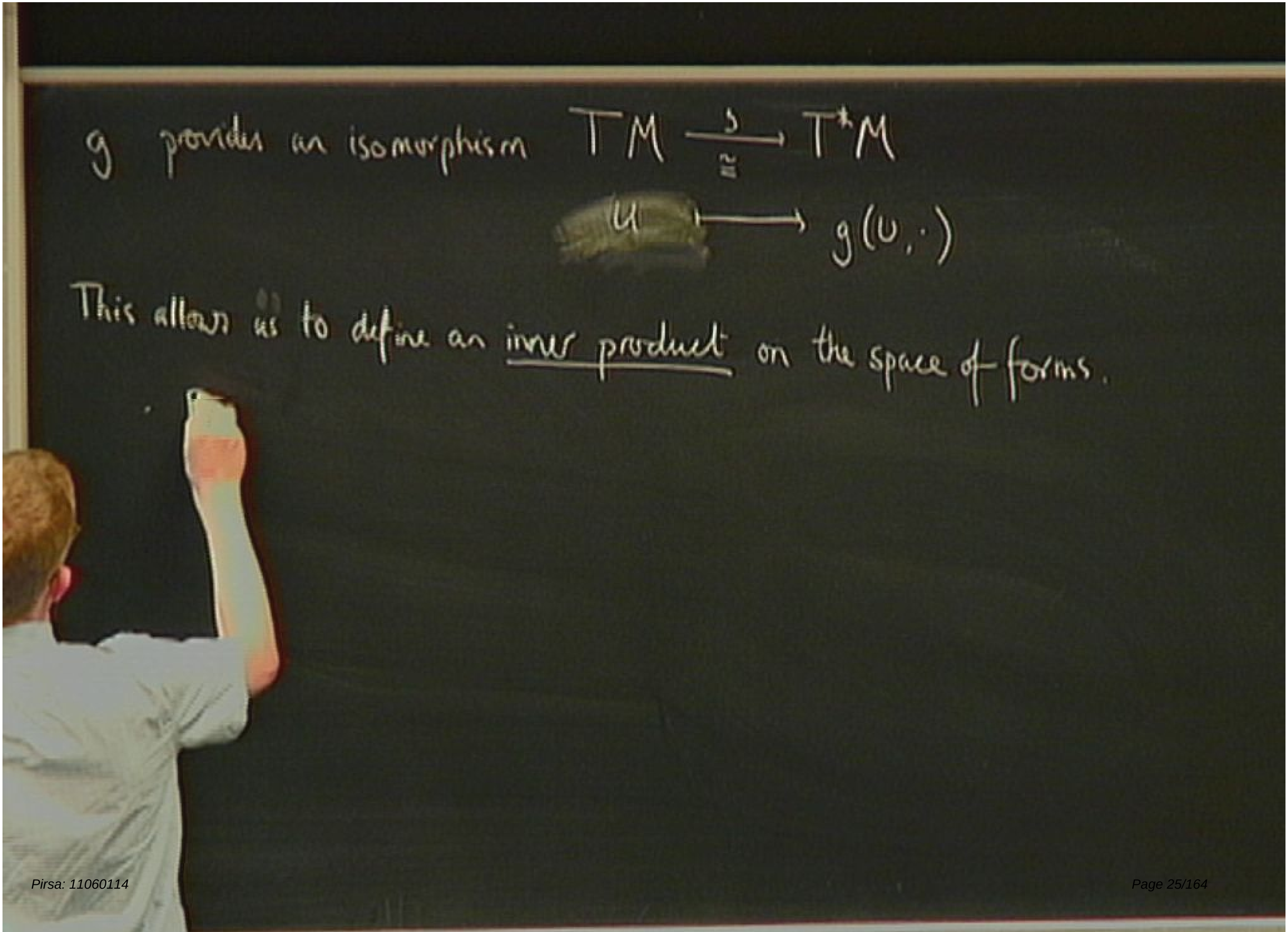
g provides an isomorphism $TM \xrightarrow[\cong]{\flat} T^*M$

$$u \longmapsto g(u, \cdot)$$



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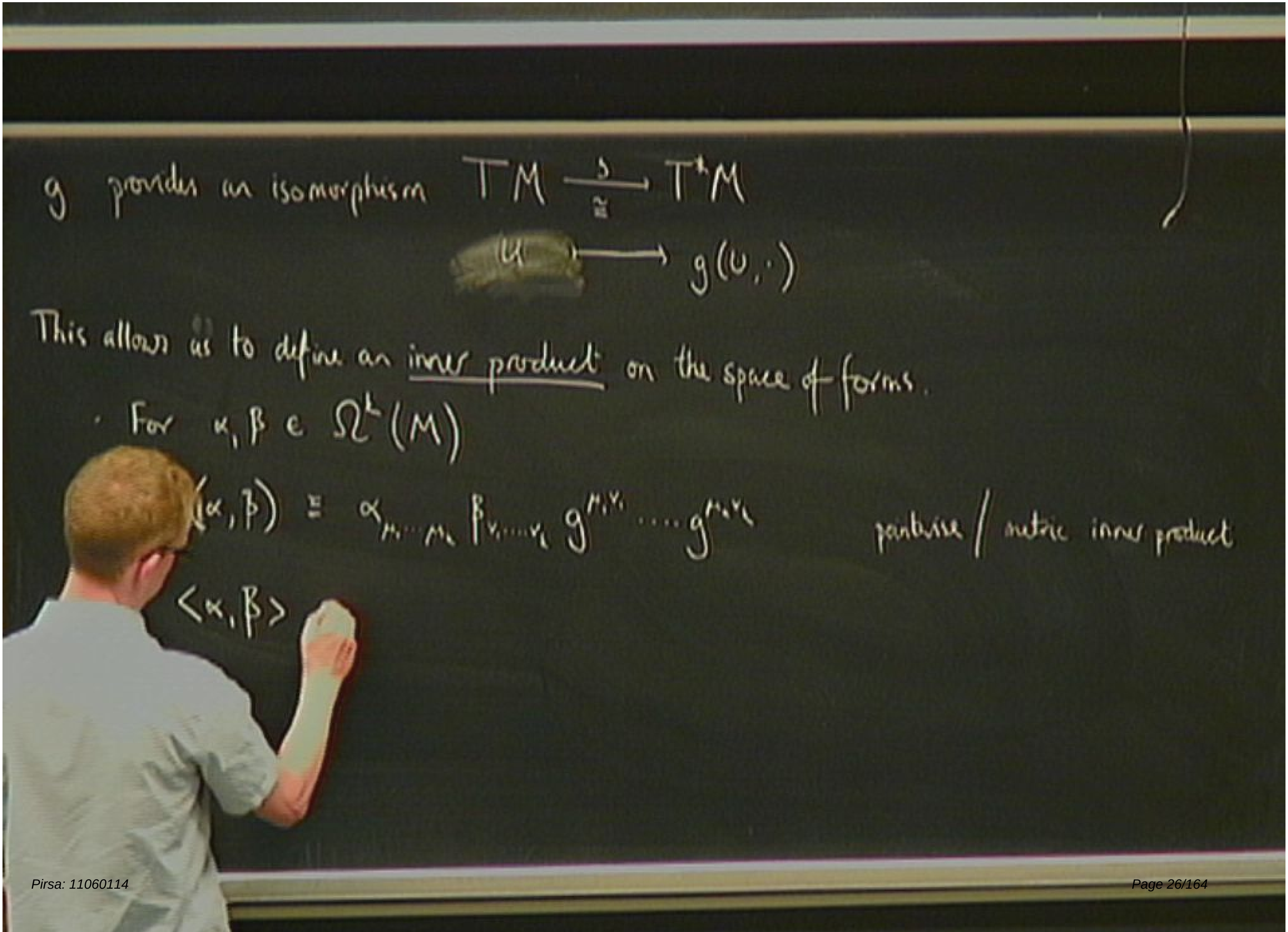
This allows us to define an inner product on the space of forms.



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$$(\alpha,\beta) \equiv \alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_k}\, g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}$$

pointwise / metric inner product

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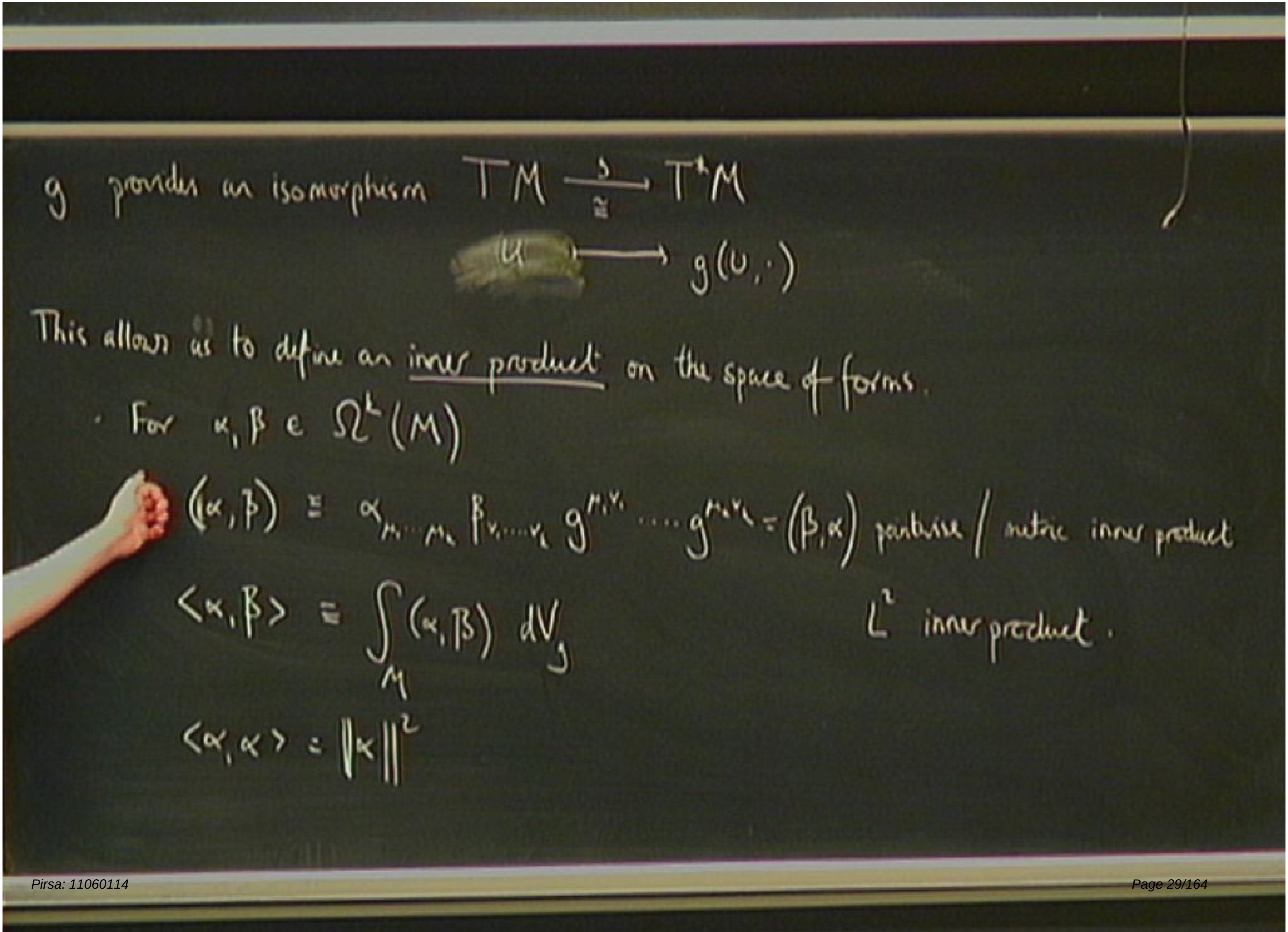

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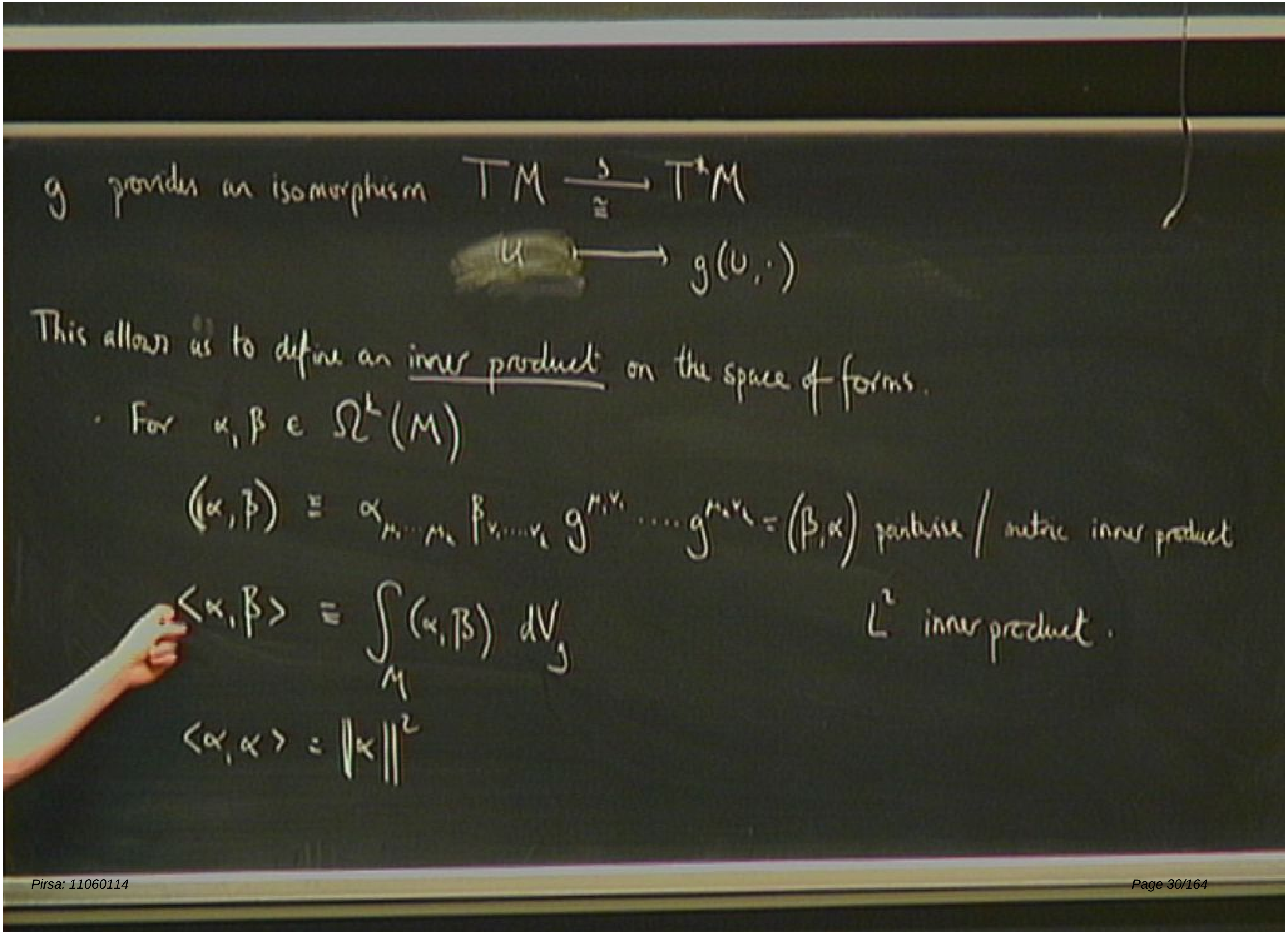

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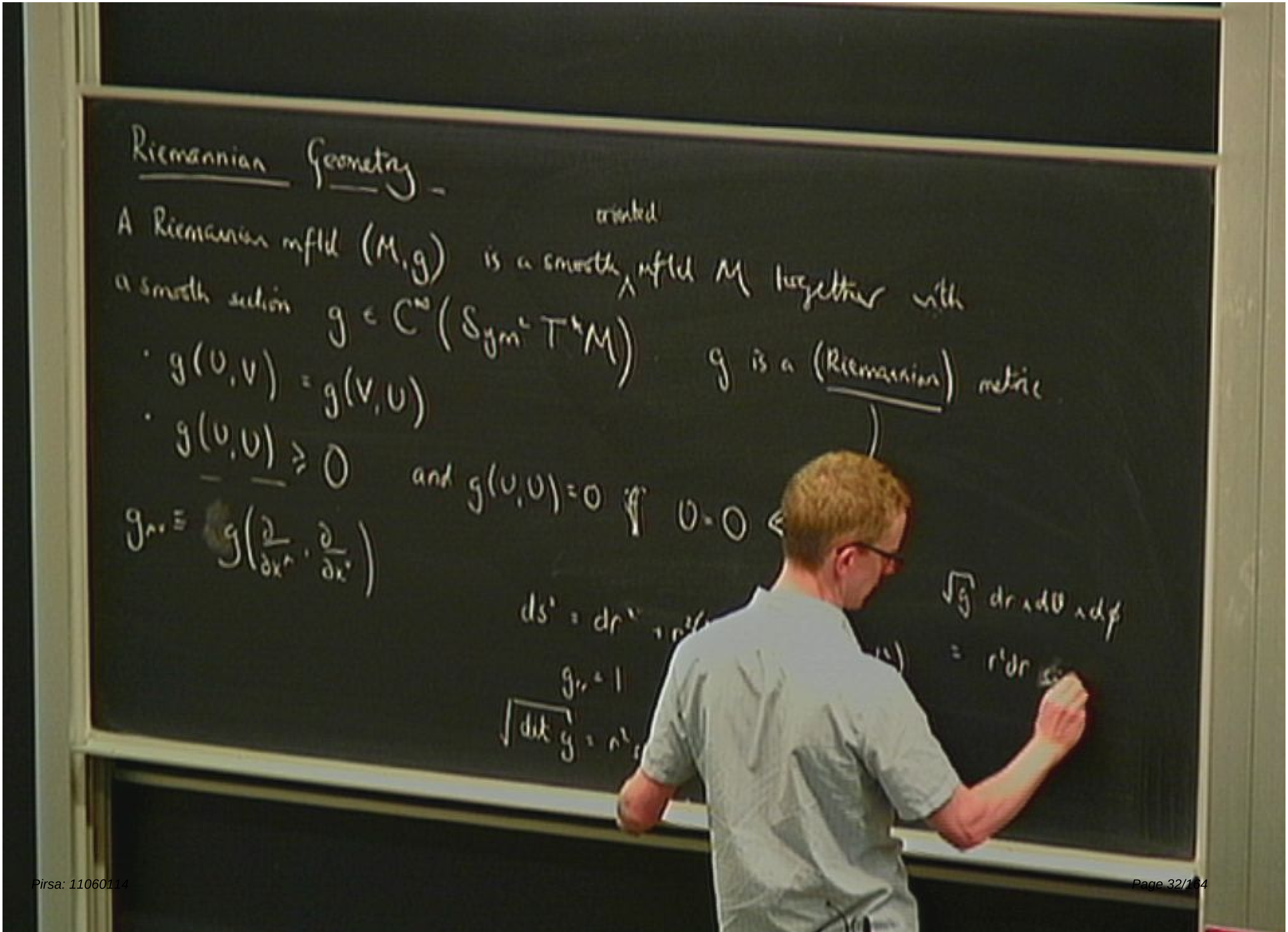

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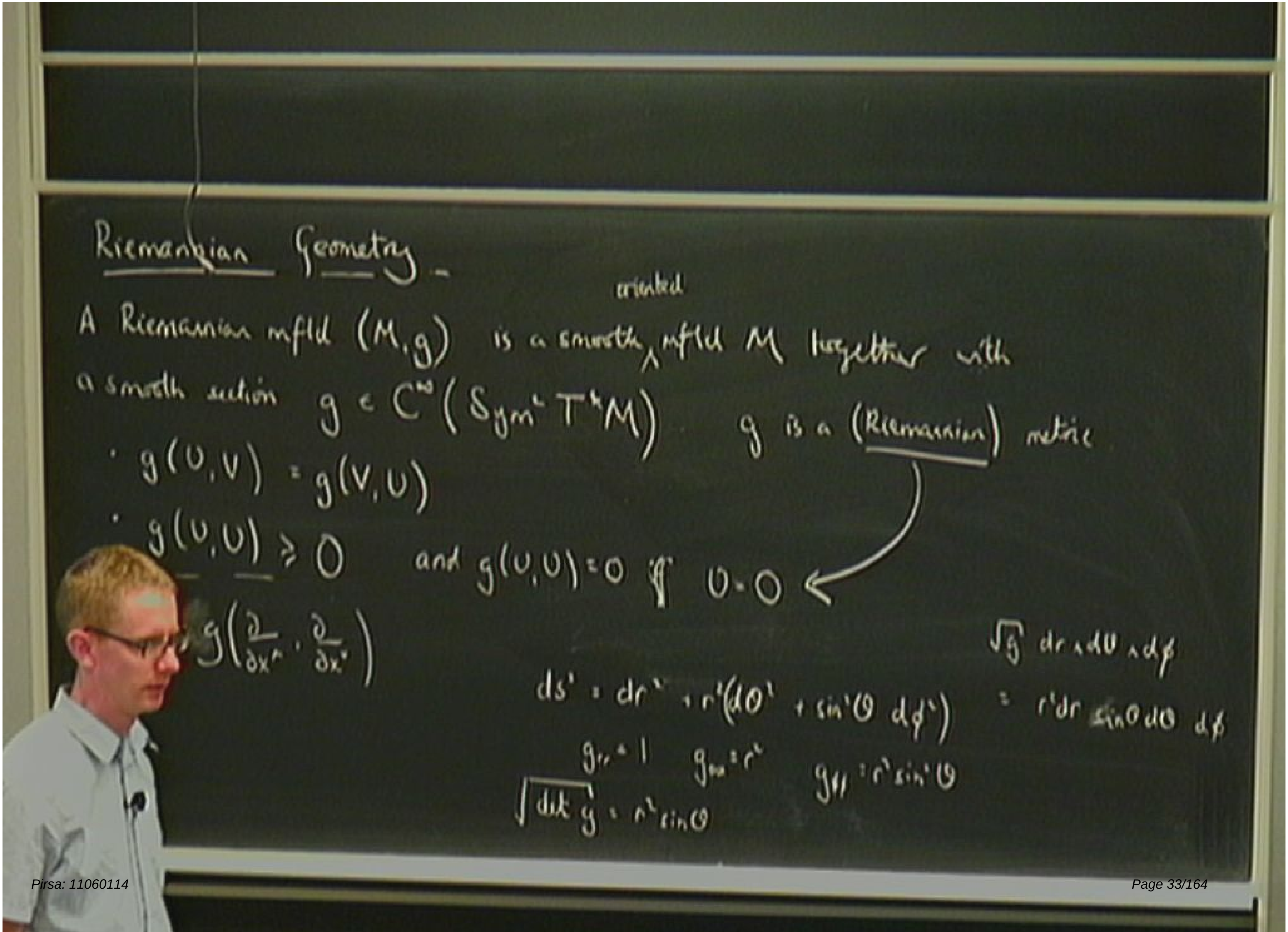
$$ds^2 = dr^2 + r^2($$

$$g_{rr} = 1$$

$$\sqrt{\det g} = r^2$$

$$\sqrt{g}\, dr \wedge d\theta \wedge d\phi$$

$$= r^2 dr$$



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$g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$

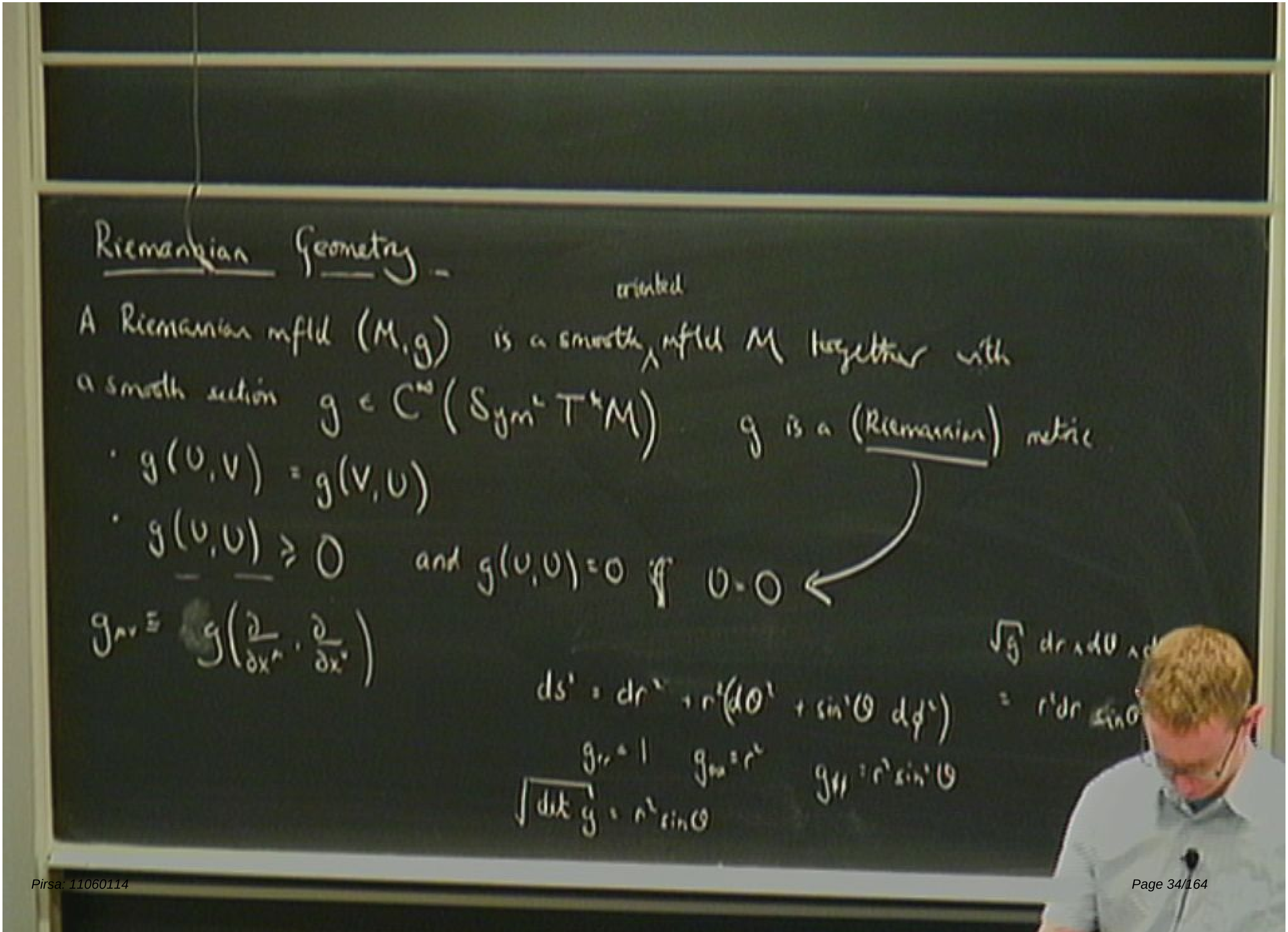
$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$g_{rr} = 1$ $g_{\theta\theta} = r^2$ $g_{\phi\phi} = r^2 \sin^2\theta$

$\sqrt{\det g} = r^2 \sin\theta$

$\sqrt{g} \, dr \wedge d\theta \wedge d\phi$

$= r^2 dr \sin\theta \, d\theta \, d\phi$



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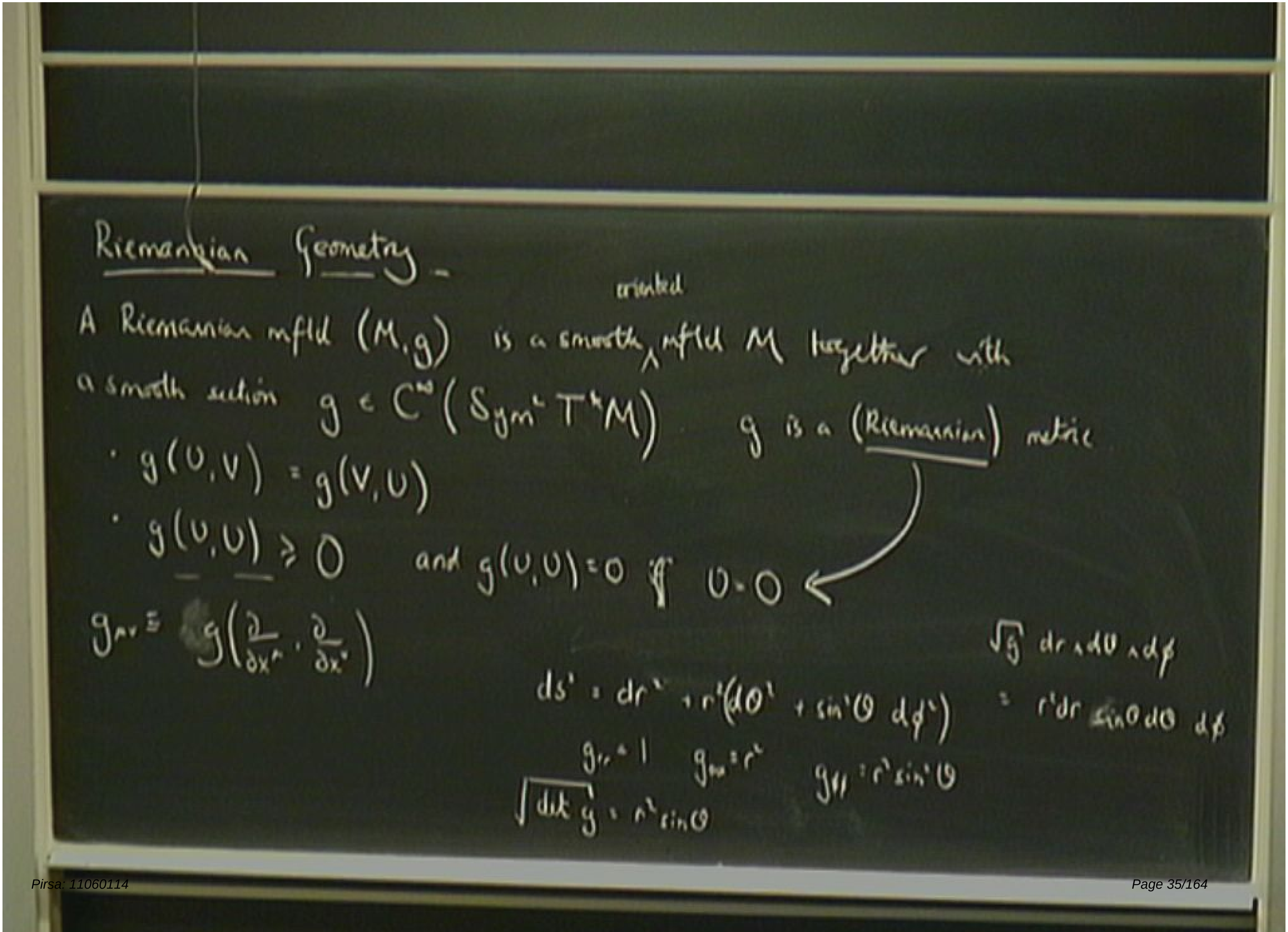


Riemannian Geometry –

A Riemannian mfld (M,g) is a smooth oriented mfld M together with a smooth section $g \in C^\infty(\mathrm{Sym}^2 T^*M)$ g is a (Riemannian) metric

- $g(U,V) = g(V,U)$
- $g(U,U) \geq 0$ and $g(U,U) = 0$ iff $U = 0$

$$g_{\mu\nu} \equiv g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)$$

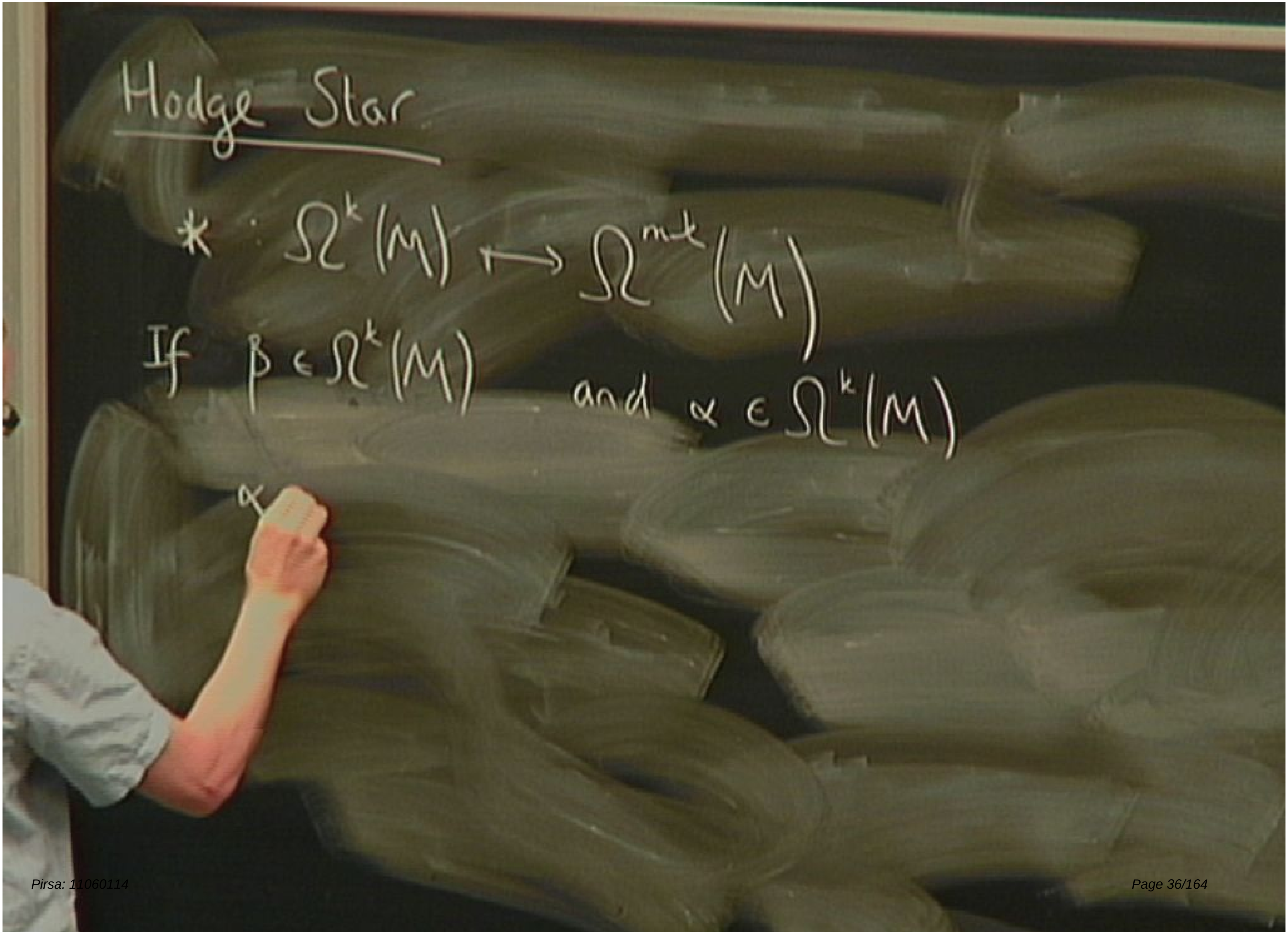
$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$g_{rr} = 1$ $g_{\theta\theta} = r^2$ $g_{\phi\phi} = r^2 \sin^2\theta$

$\sqrt{\det g} = r^2 \sin\theta$

$$\sqrt{g}\, dr \wedge d\theta \wedge d\phi$$

$$= r^2 dr \sin\theta \, d\theta \, d\phi$$



Hodge Star

$$* : \Omega^k(M) \longmapsto \Omega^{n-k}(M)$$

If $\beta \in \Omega^k(M)$ and $\alpha \in \Omega^k(M)$

α

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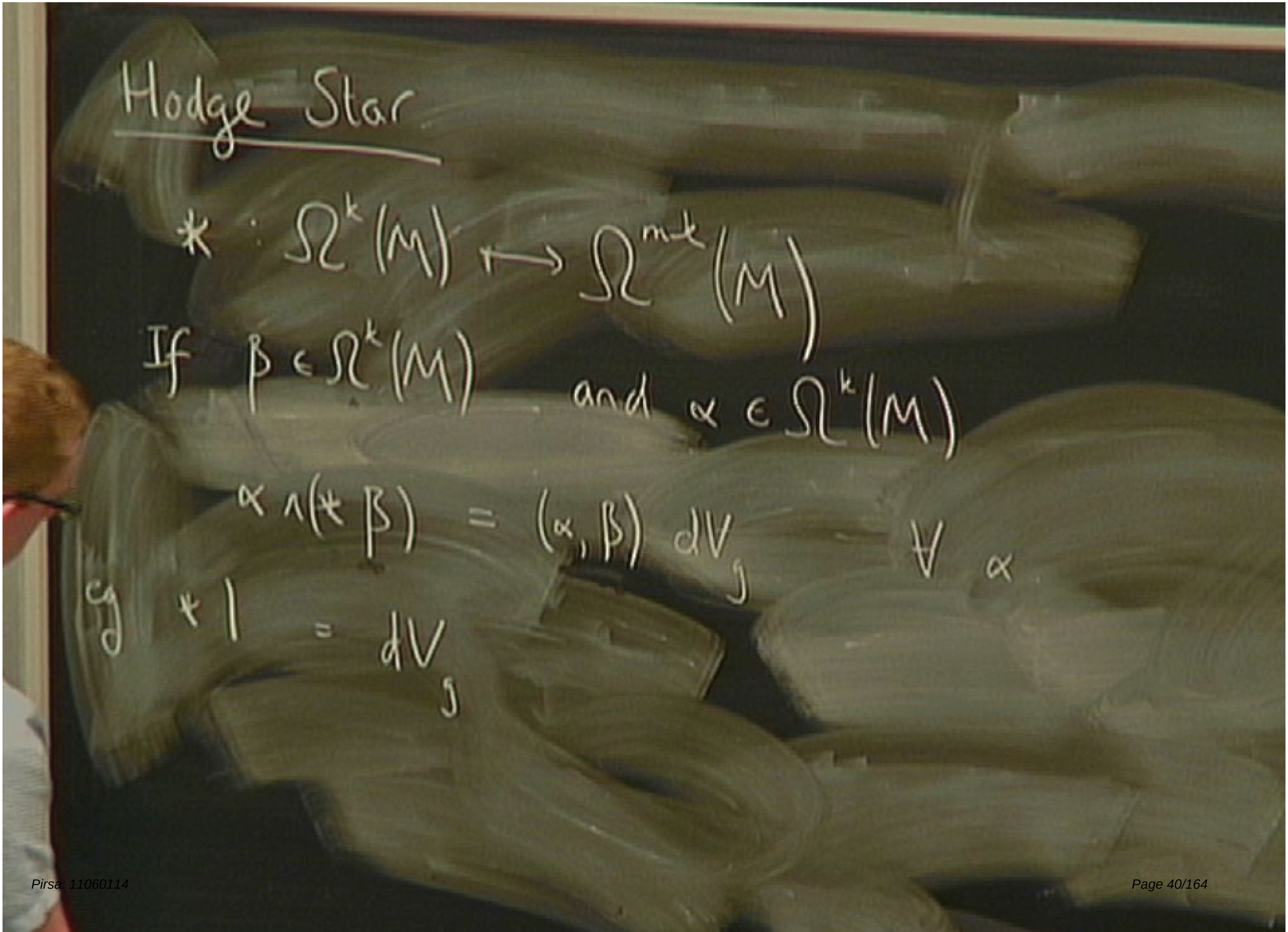

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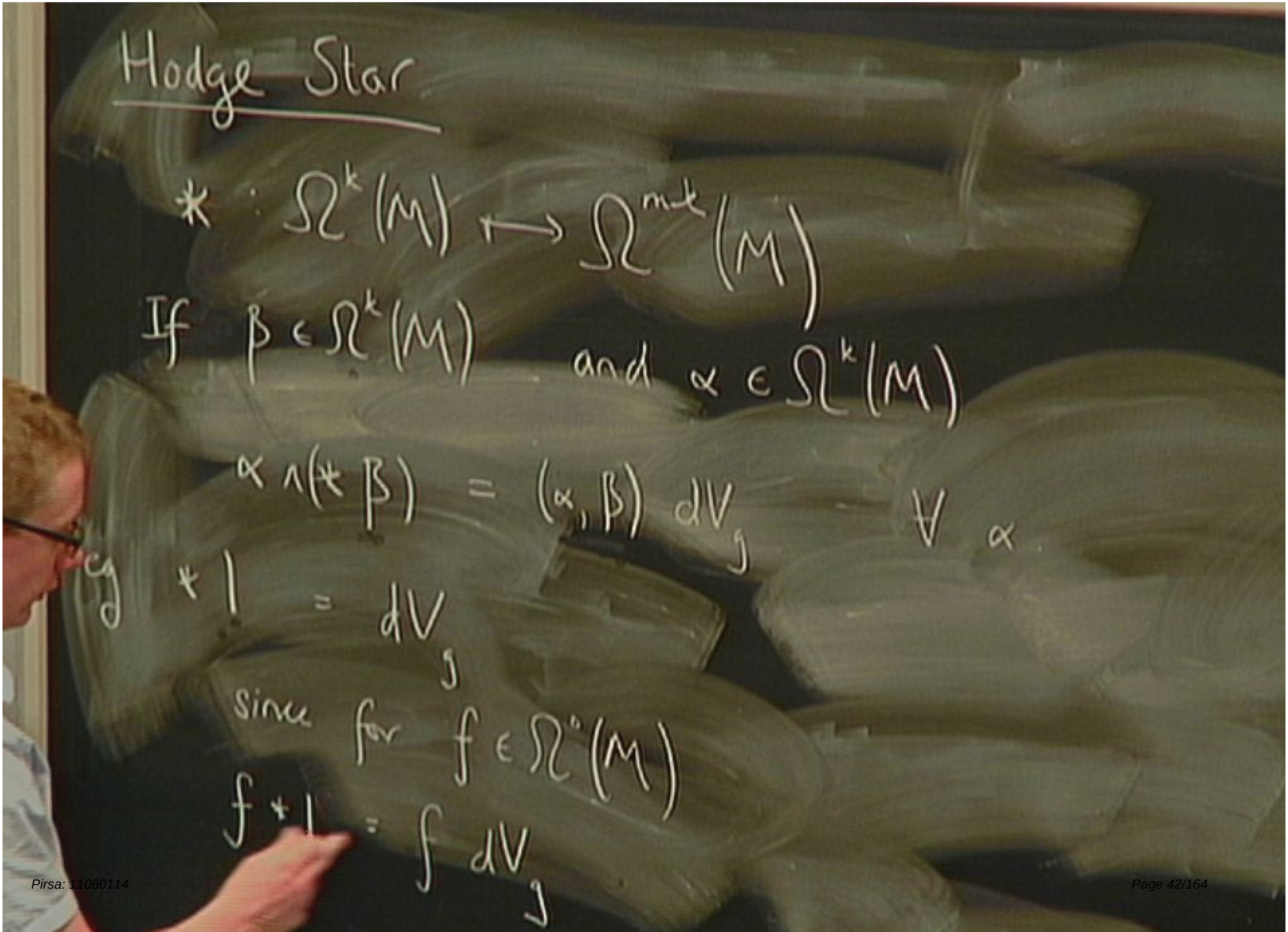

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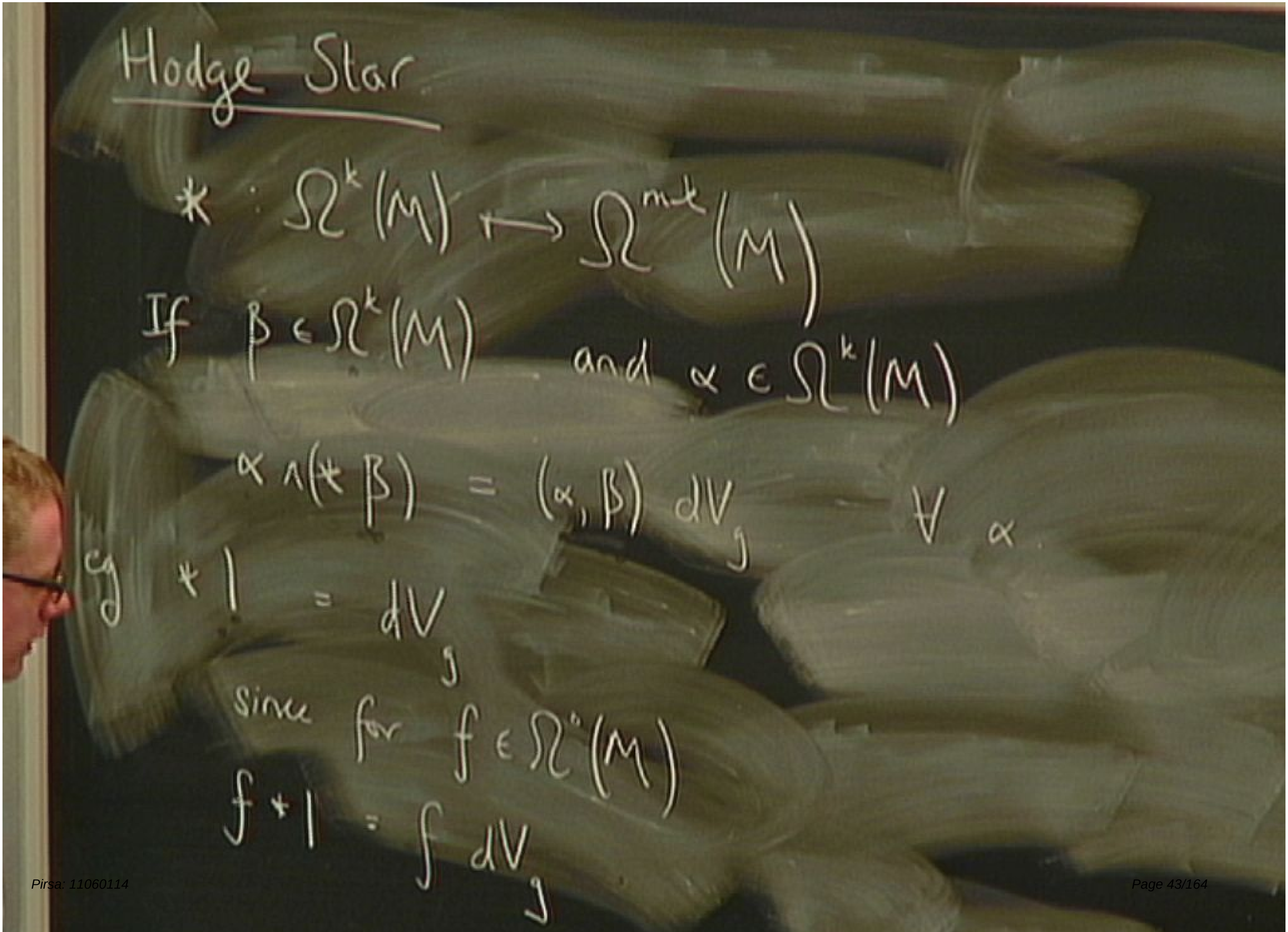

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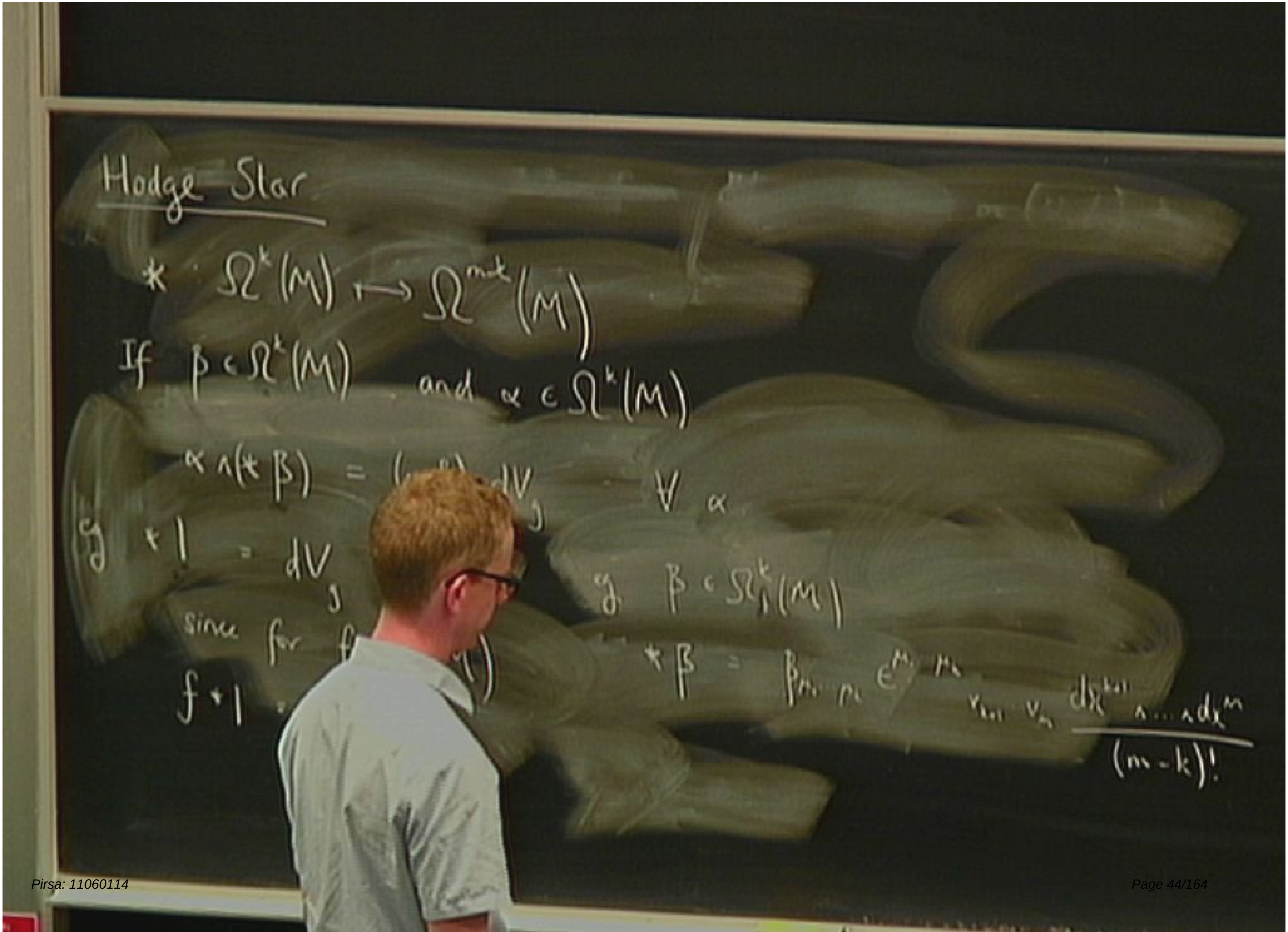

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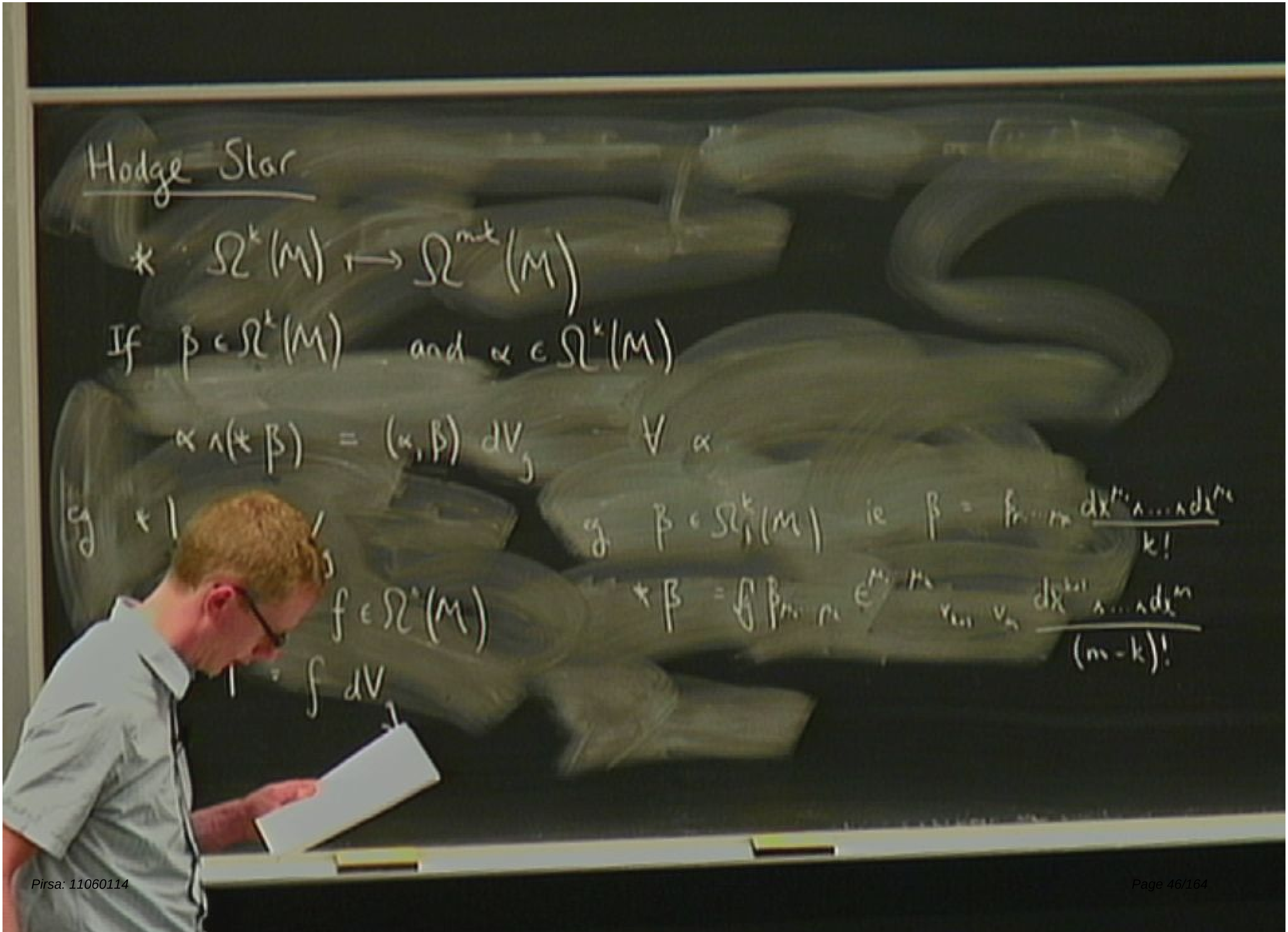

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eg $*1$ [illegible]

[illegible] $f \in \Omega^0(M)$

[illegible] $1 = \int dV_g$

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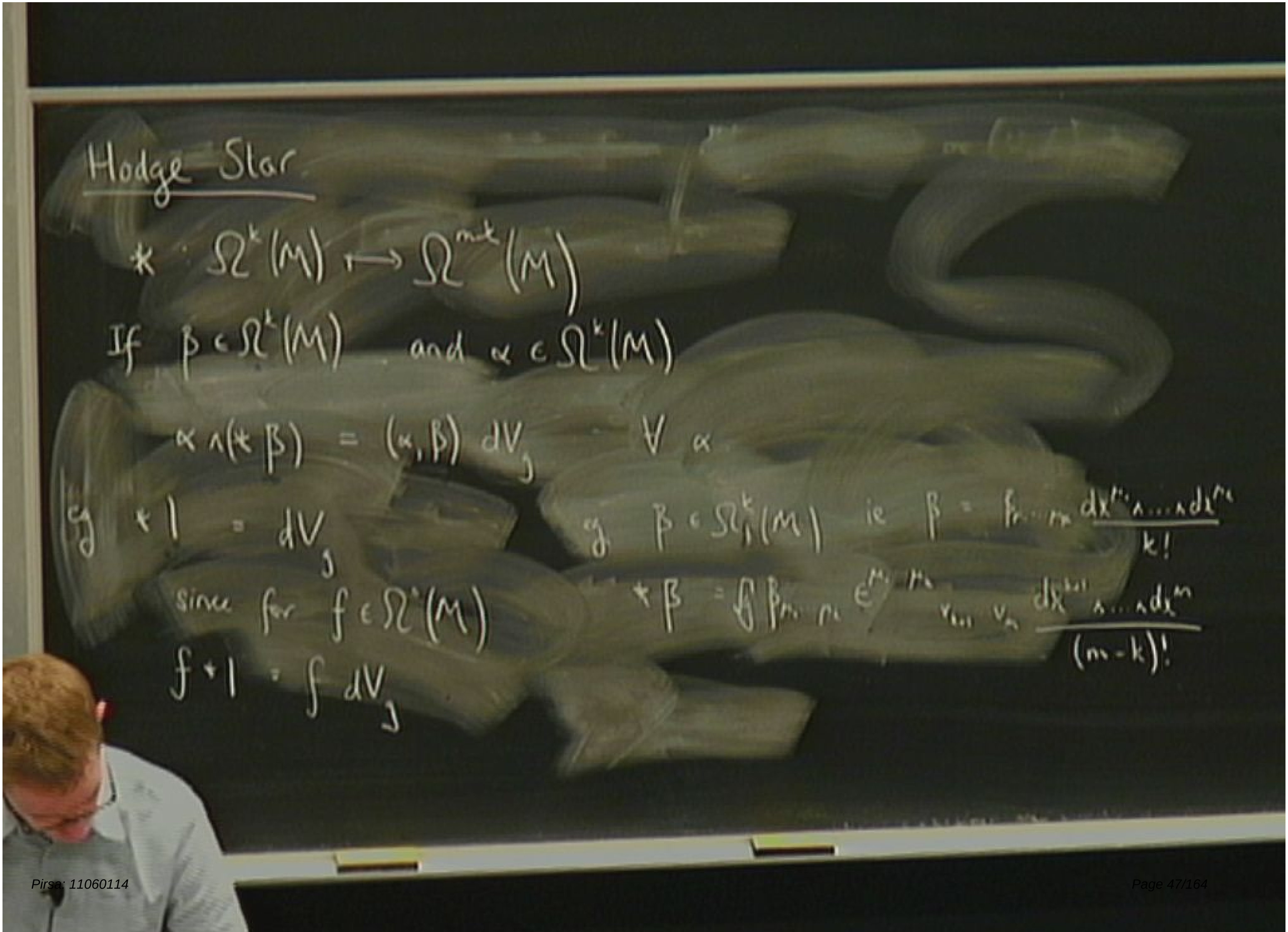

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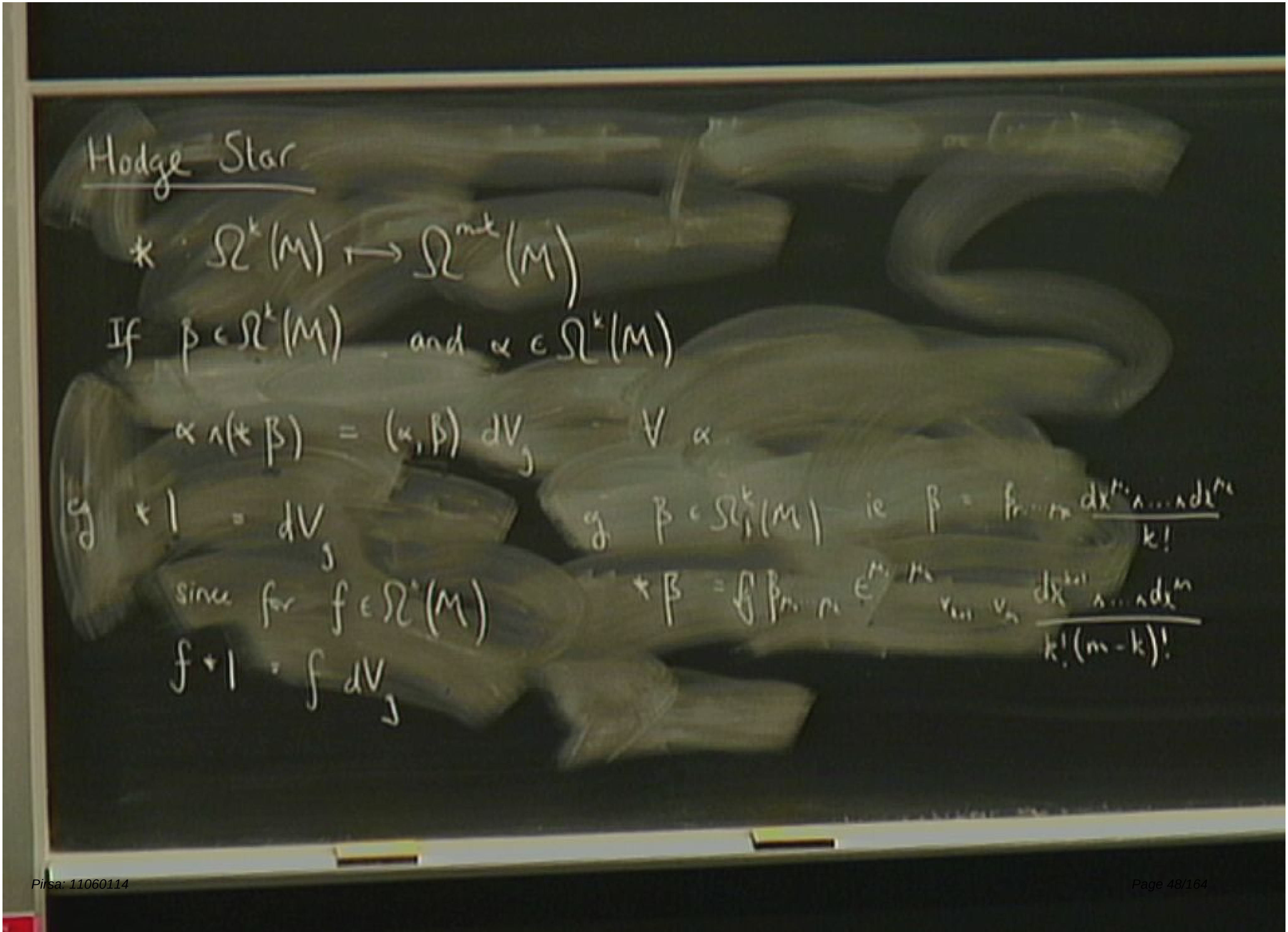

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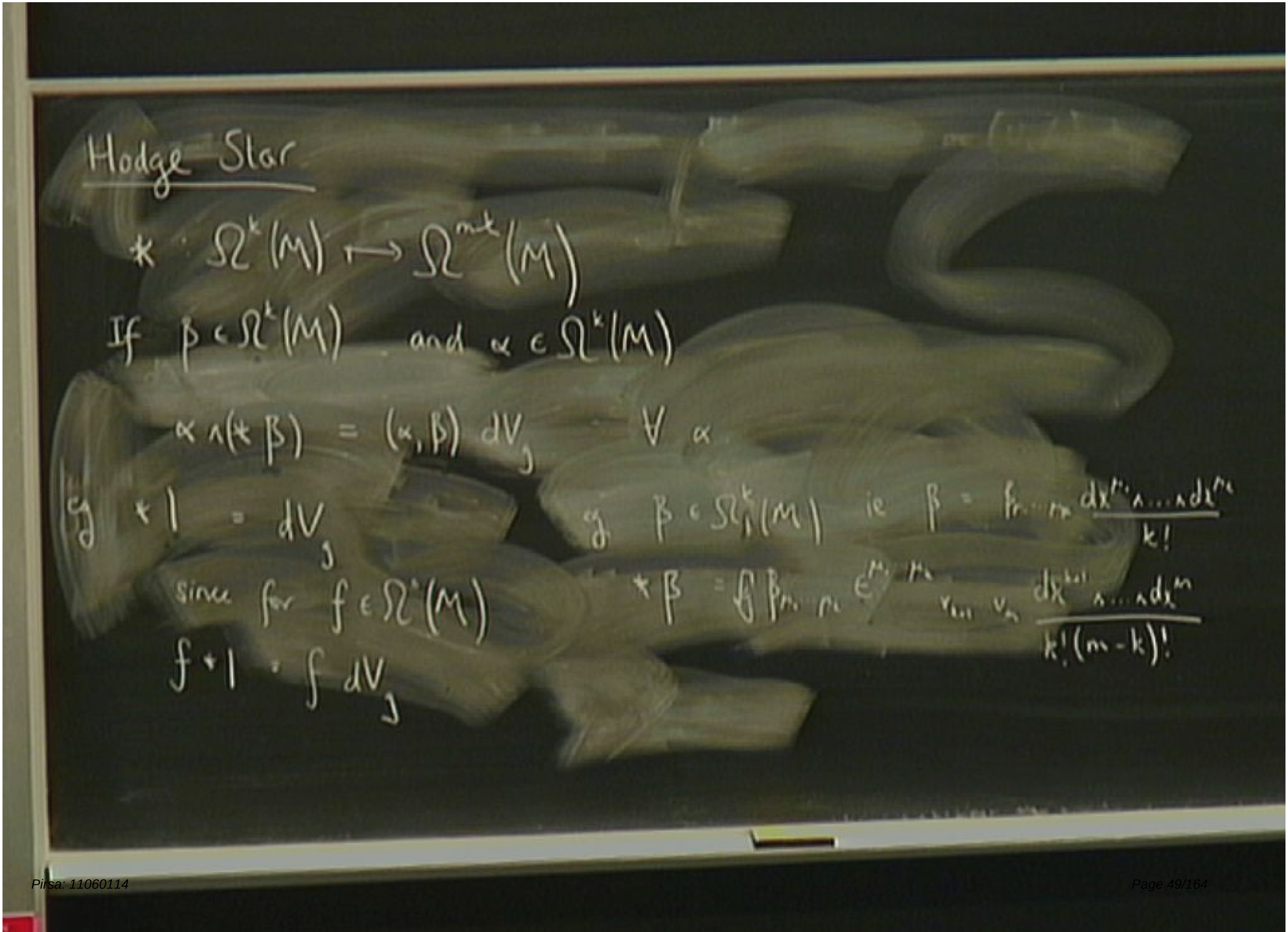

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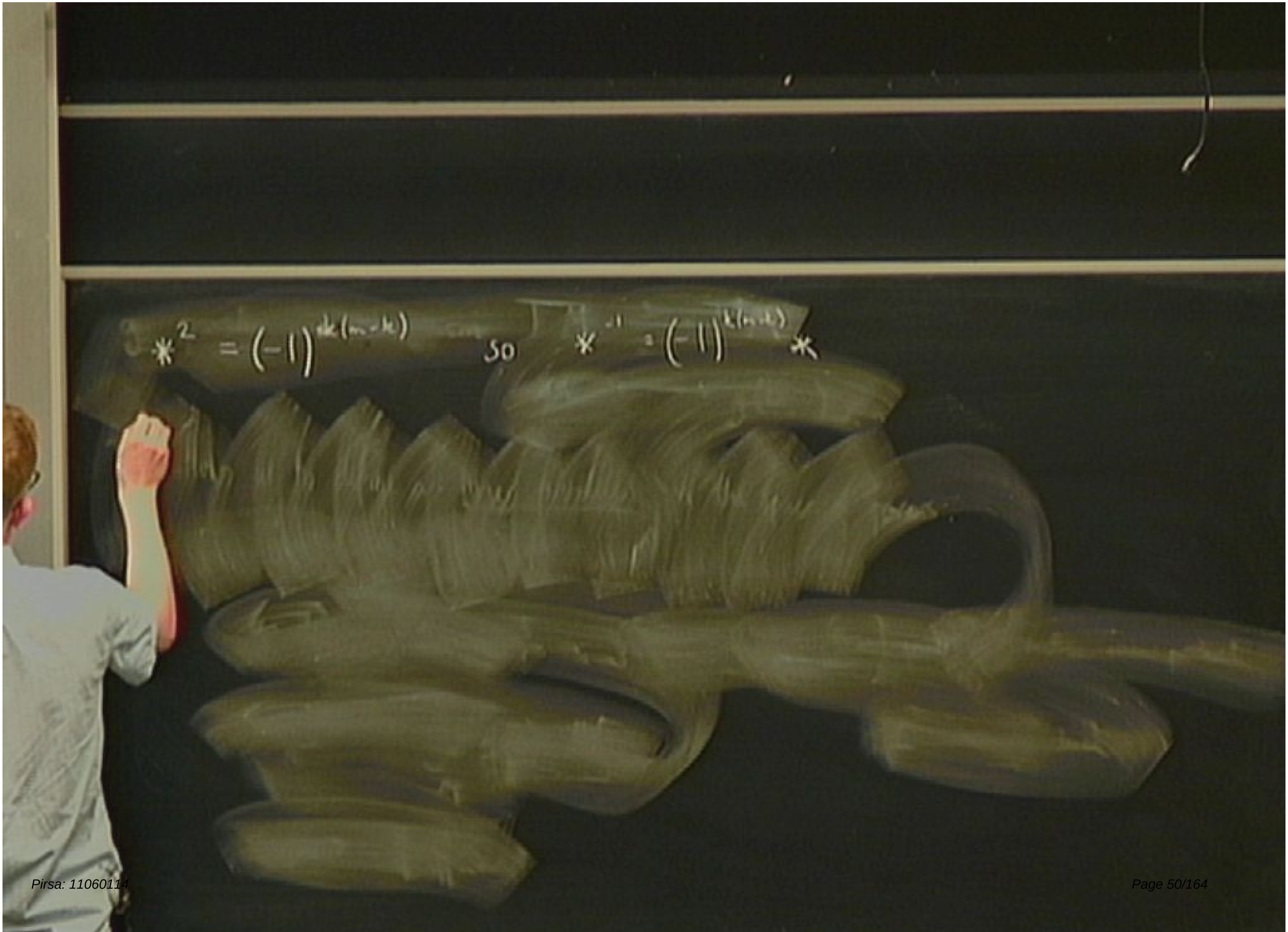

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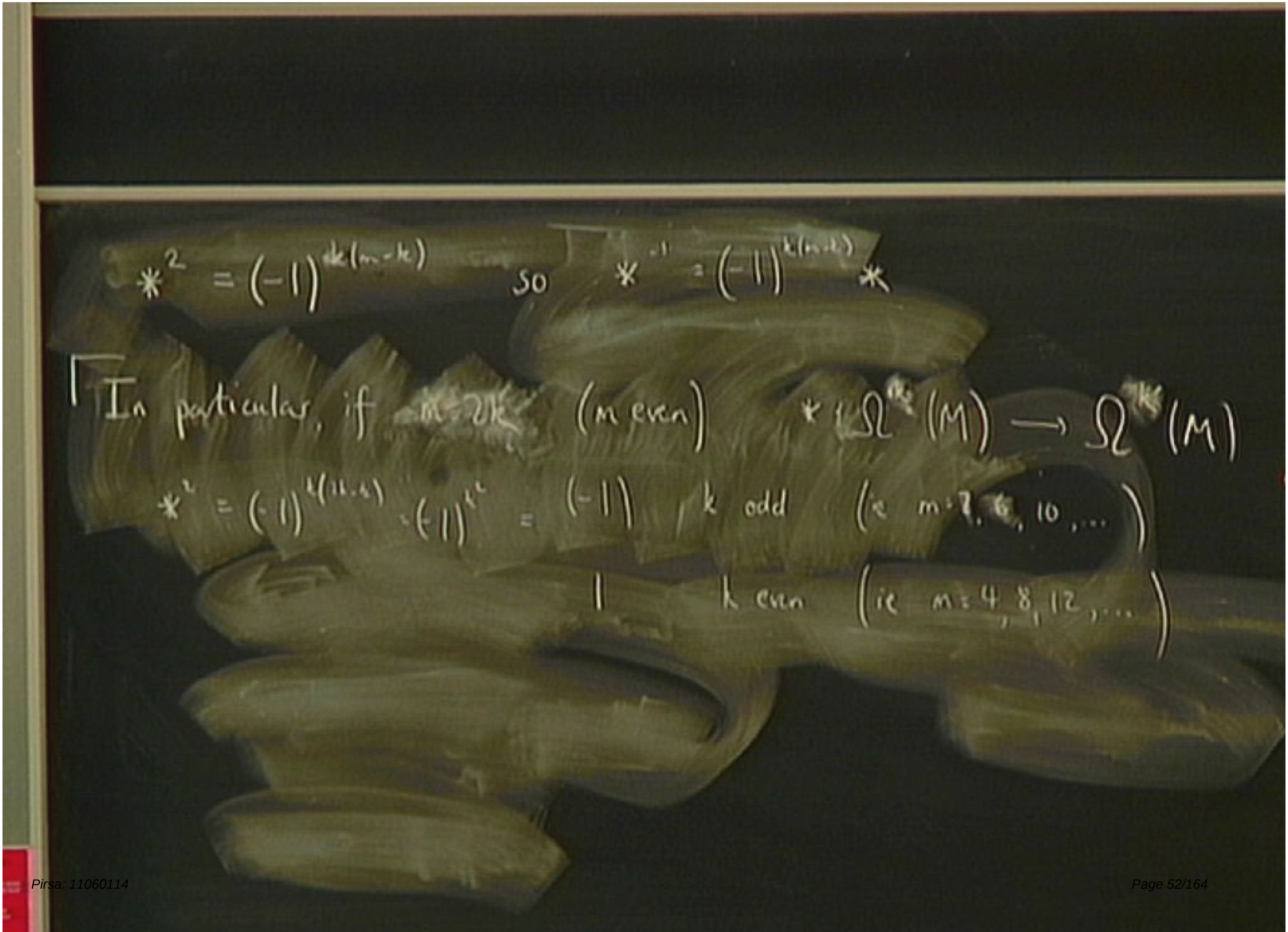

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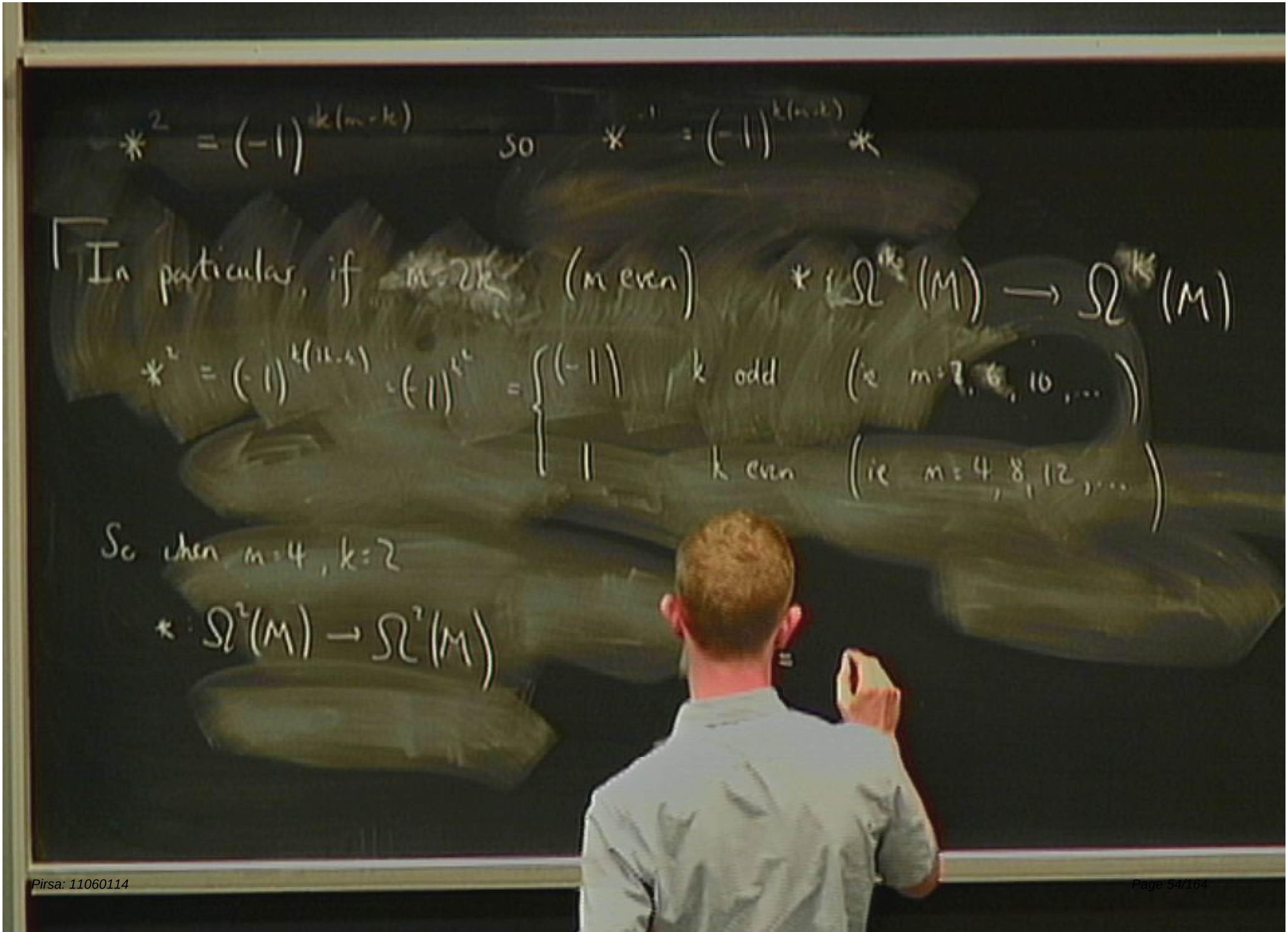

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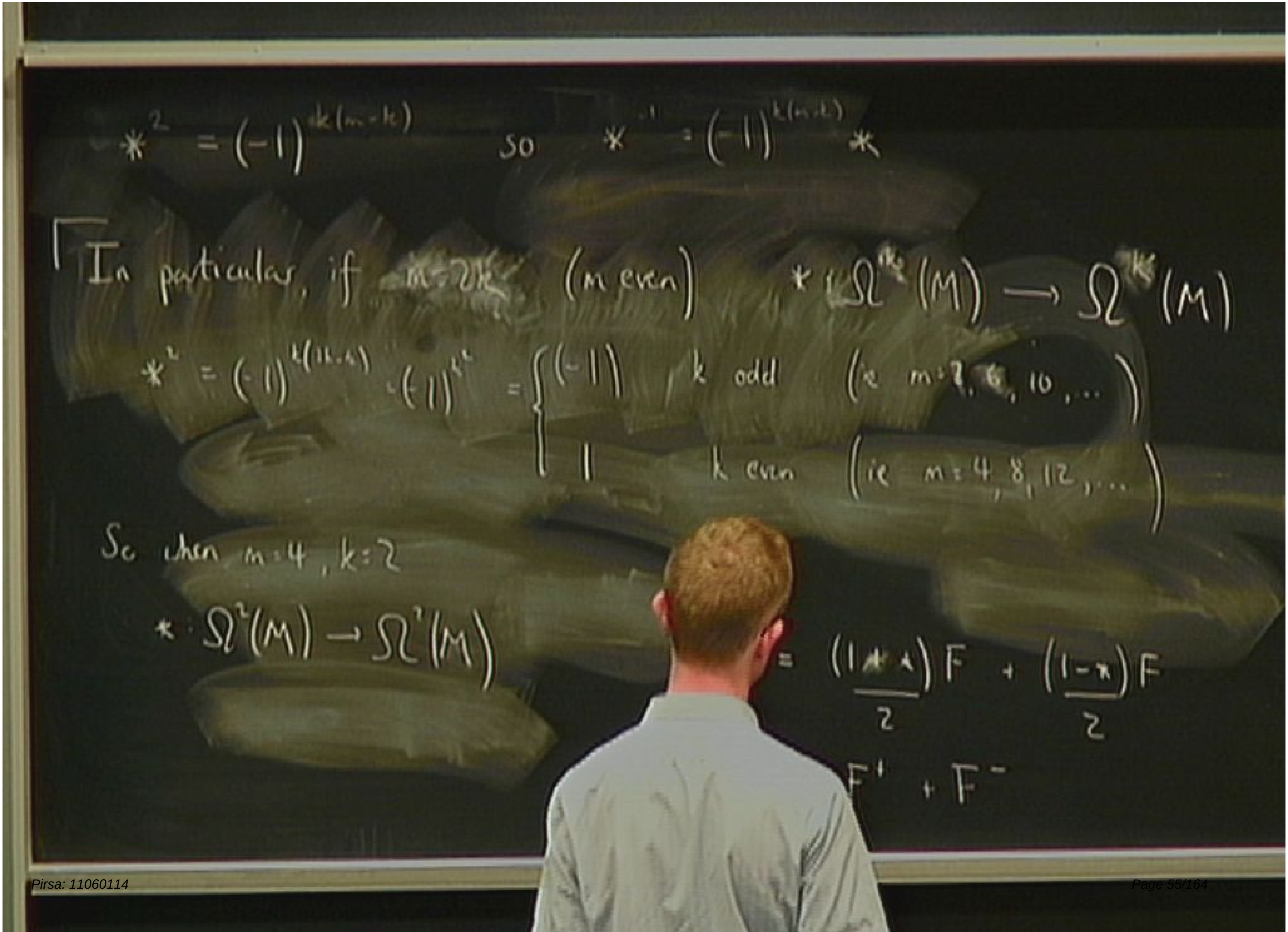

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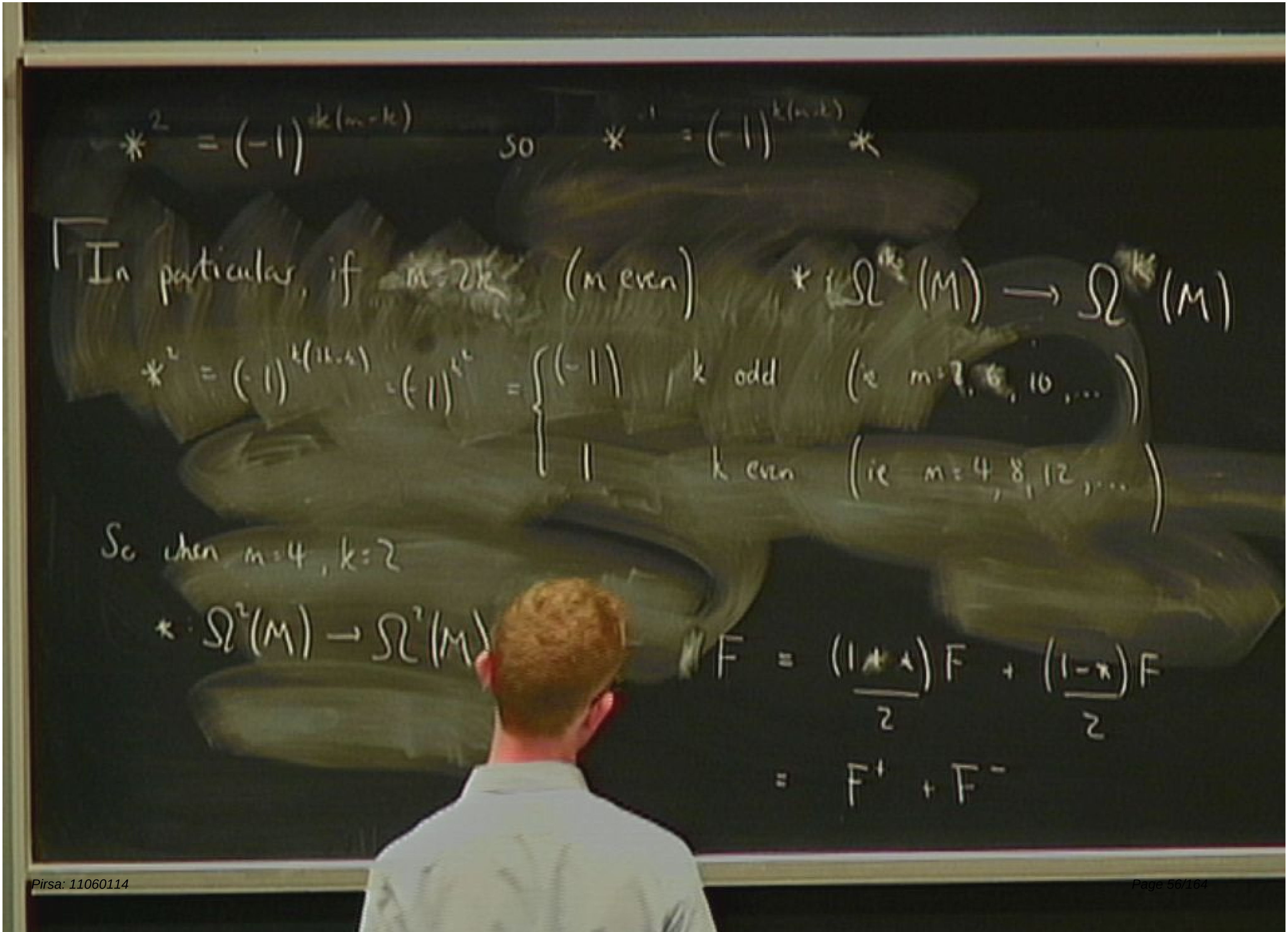
$$F^+ + F^-$$



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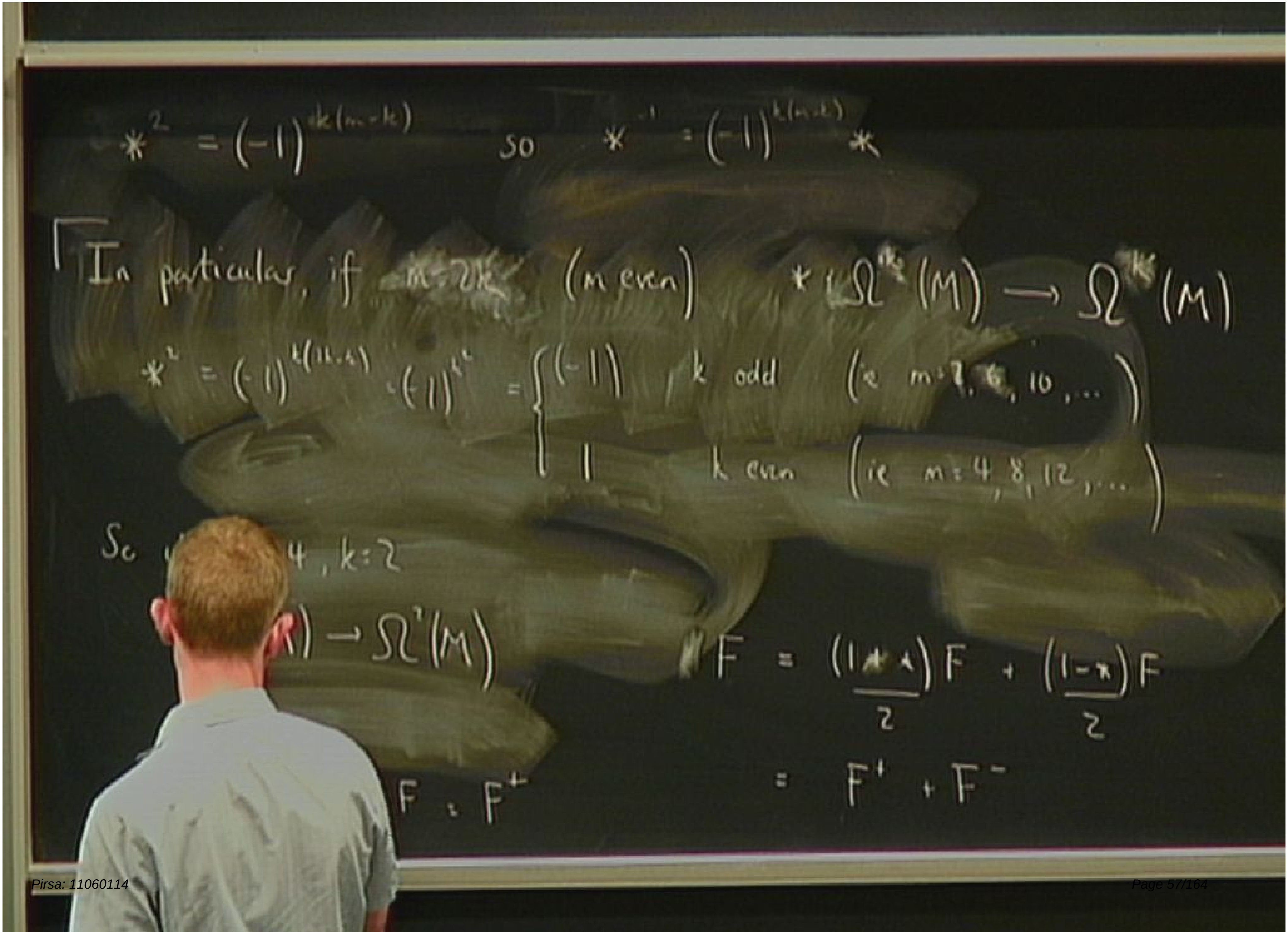

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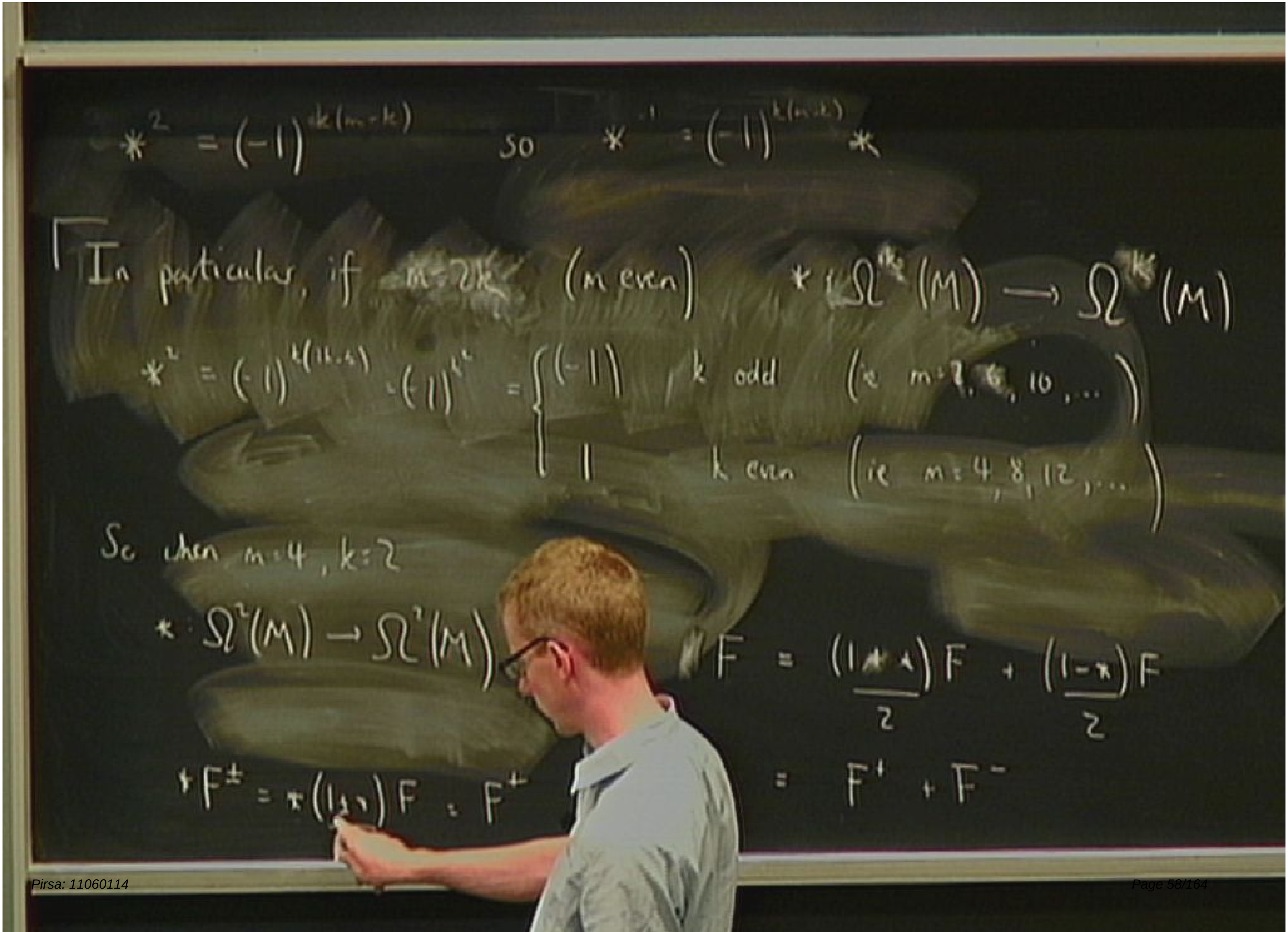

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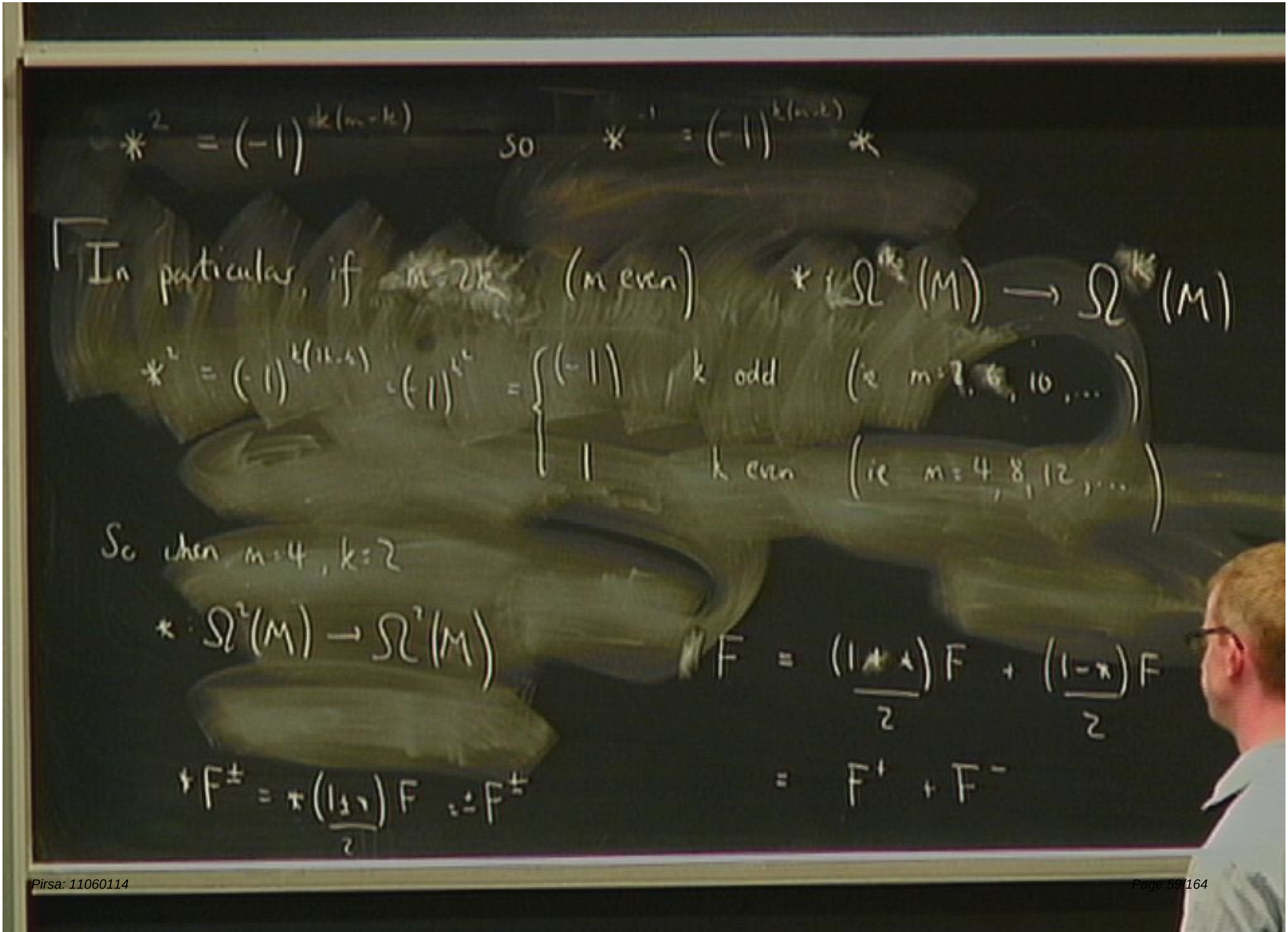

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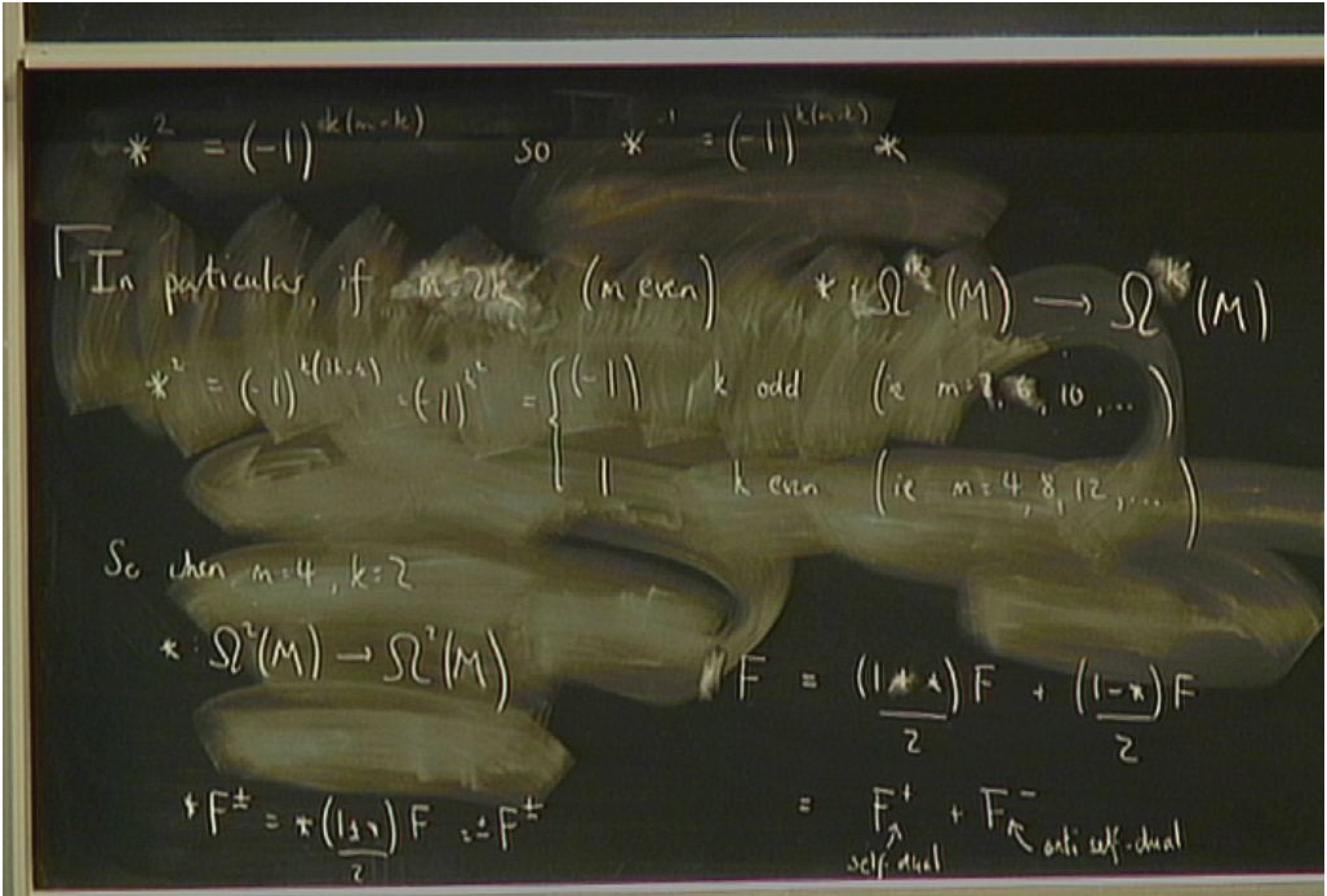

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$\int dx\; y = r^2 \sin\theta$

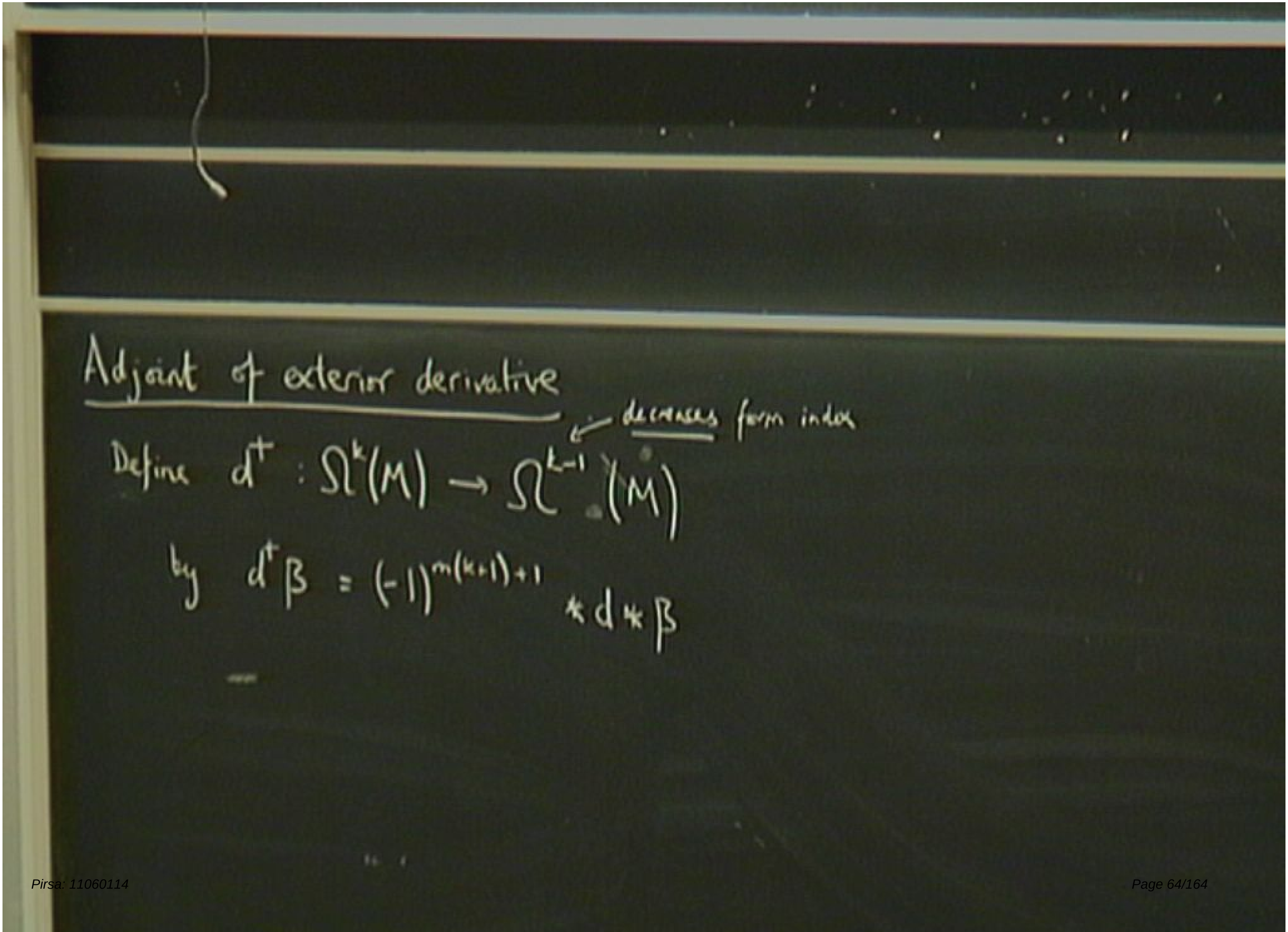
Adjoint of exterior derivative

Define $d^\dagger$

Adjoint of exterior derivative

decreases form index

Define $d^{\dagger} : \Omega^{k}(M) \rightarrow \Omega^{k-1}(M)$



Adjoint of exterior derivative

Define $d^\dagger : \Omega^k(M) \to \Omega^{k-1}(M)$ ← decreases form index

by $d^\dagger \beta = (-1)^{m(k+1)+1} * d * \beta$

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$$(\alpha, \beta) \equiv \alpha_{\mu_1 \cdots \mu_k} \beta_{\nu_1 \cdots \nu_k} g^{\mu_1 \nu_1} \cdots g^{\mu_k \nu_k}$$

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \, dV_g$$

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$d^\dagger$ is the adjoint of d under the L^2 norm:

Let (M,g) be a compact Riem mfld. Then

$\int$

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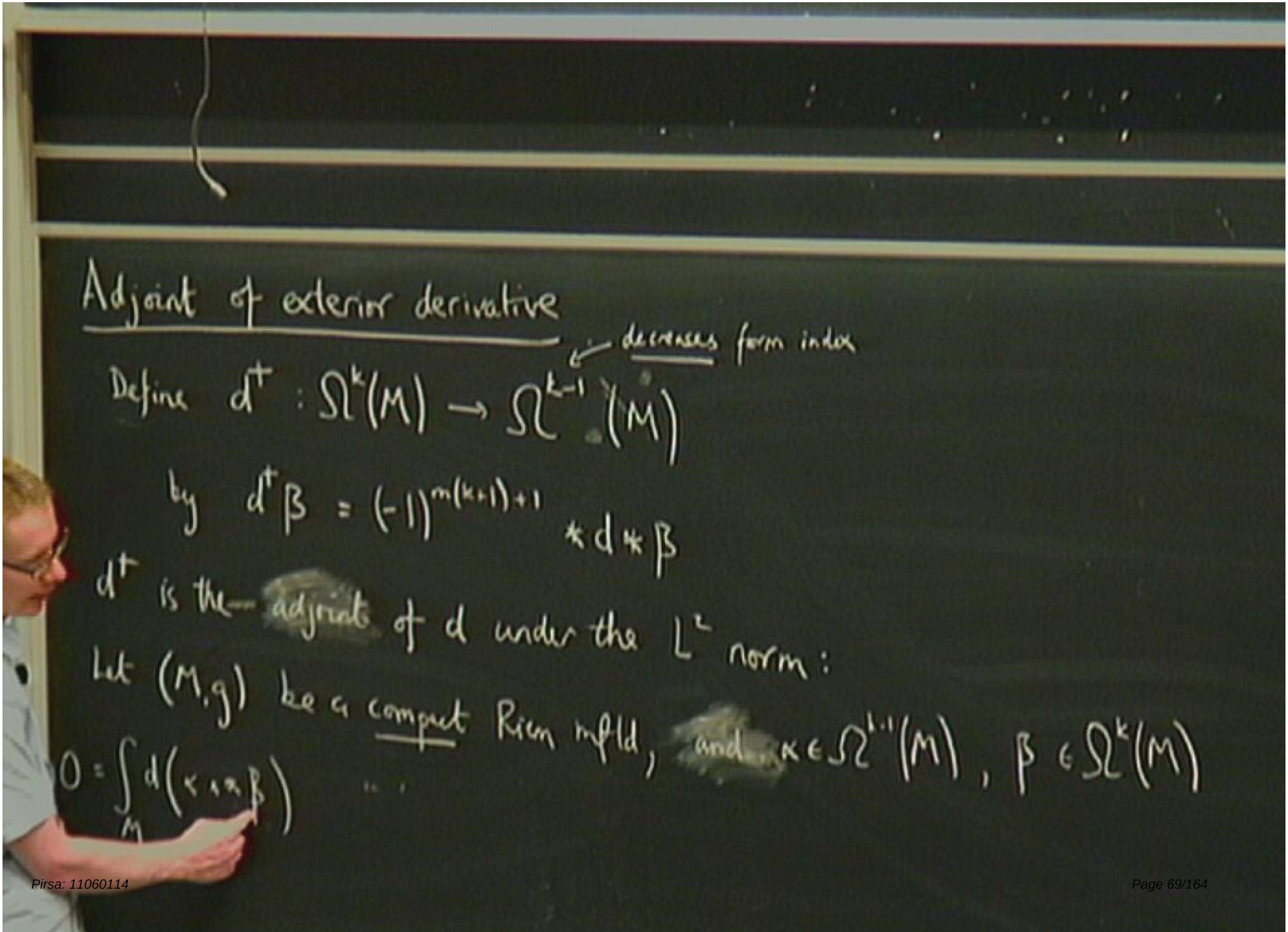

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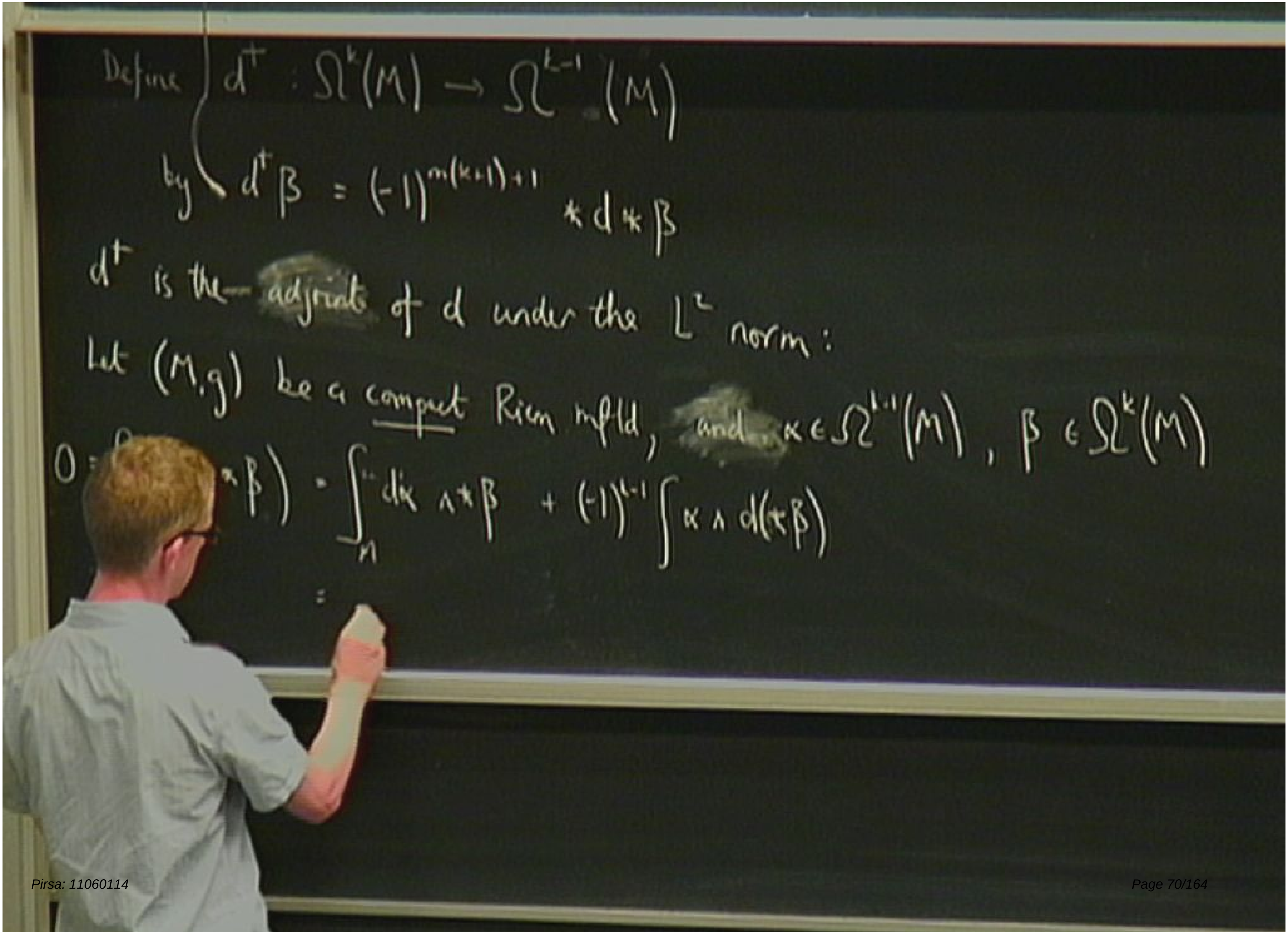

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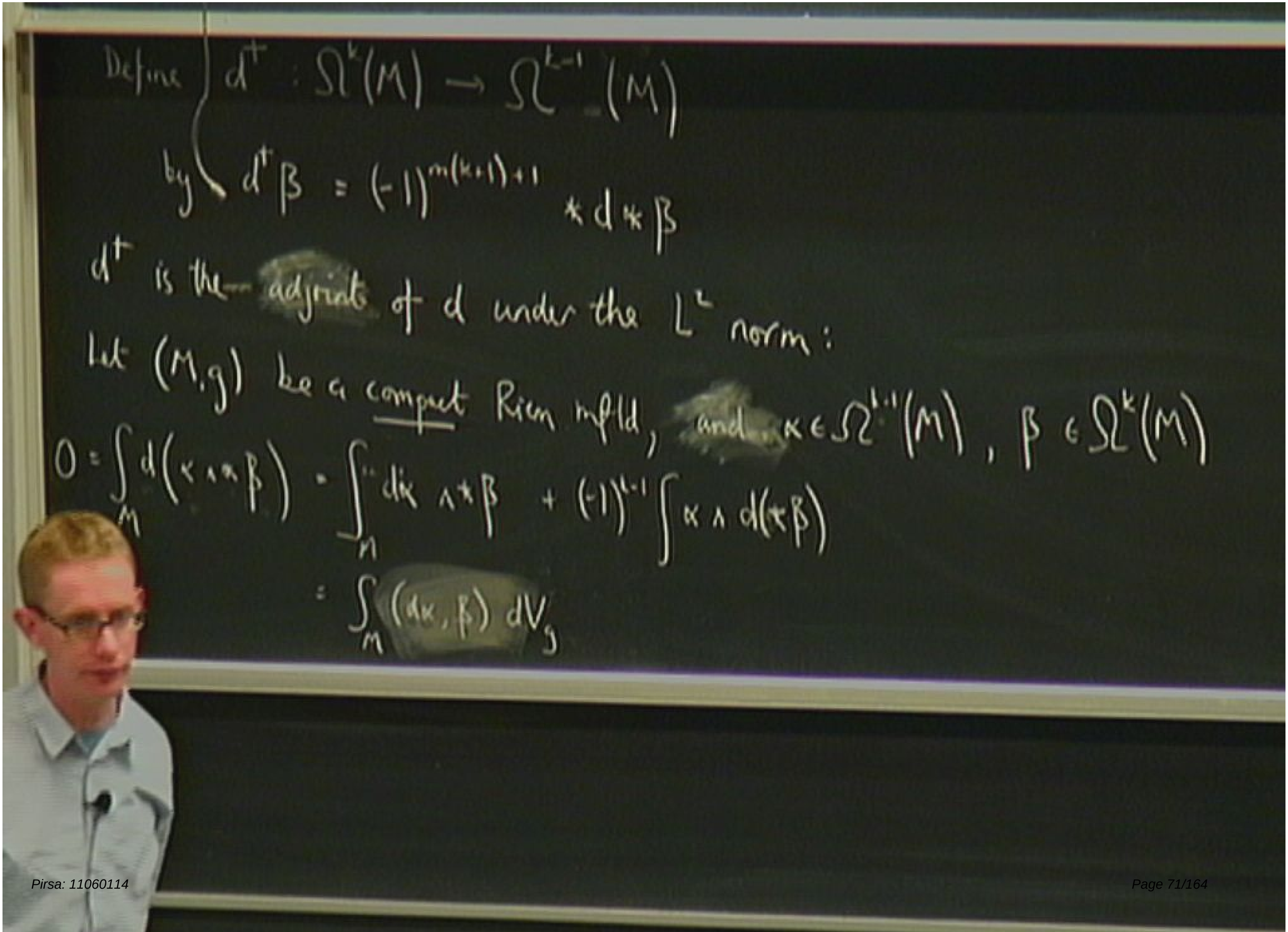
Let (M,g) be a $\underline{\text{compact}}$ Riem mfld, and $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$

$0 = \int$ [illegible] $*\beta) = \int_M d\alpha \wedge *\beta + (-1)^{k-1} \int \alpha \wedge d(*\beta)$

$=$



Define $\left\{ d^{\dagger} : \Omega^{k}(M) \to \Omega^{k-1}(M) \right.$

by $d^{\dagger}\beta = (-1)^{m(k+1)+1} \ast d \ast \beta$

$d^{\dagger}$ is the adjoint of d under the L^2 norm:

Let (M,g) be a <u>compact</u> Riem mfld, and $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^{k}(M)$

$$0 = \int_M d(\alpha \wedge \ast\beta) = \int_M d\alpha \wedge \ast\beta + (-1)^{k-1} \int \alpha \wedge d(\ast\beta)$$

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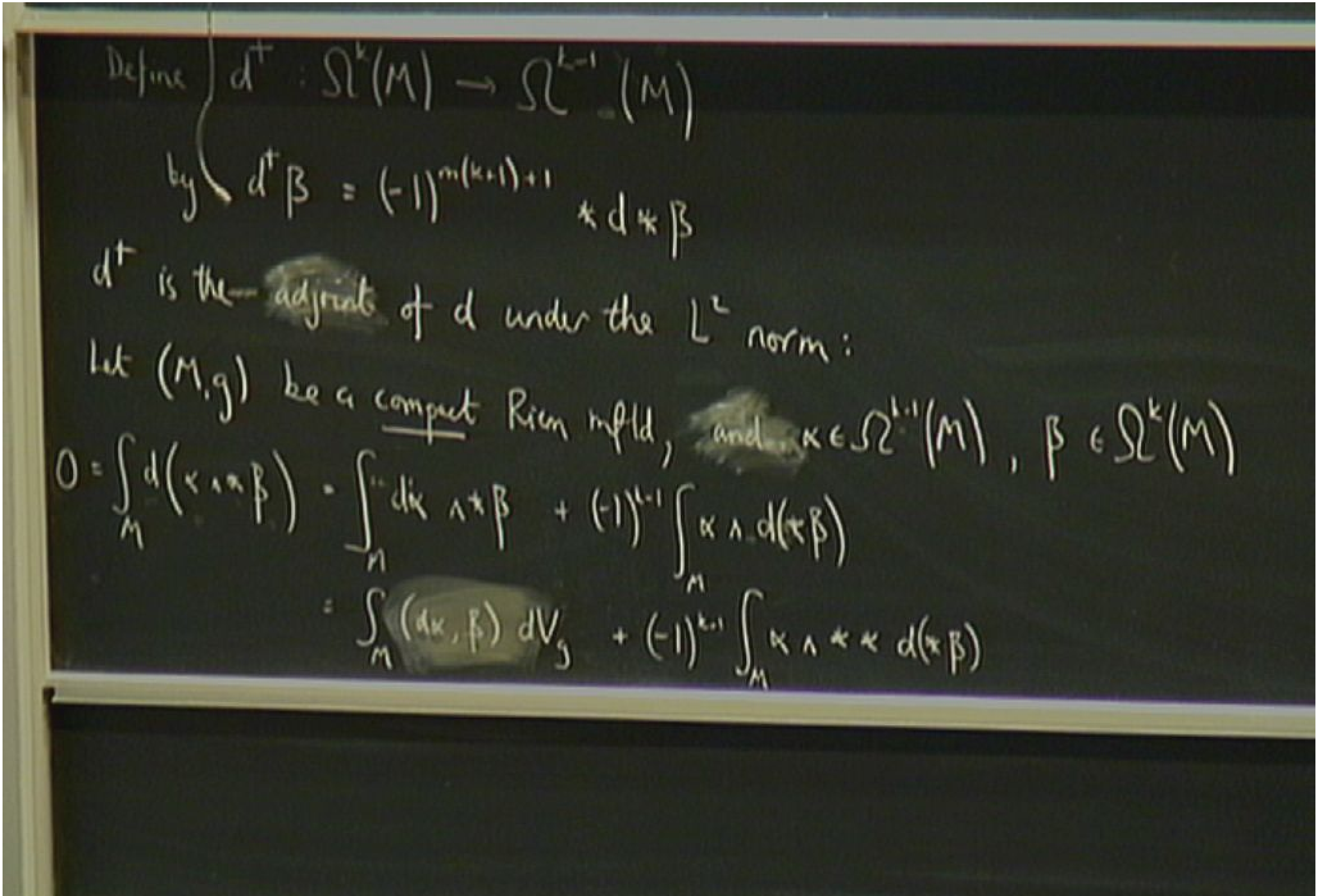

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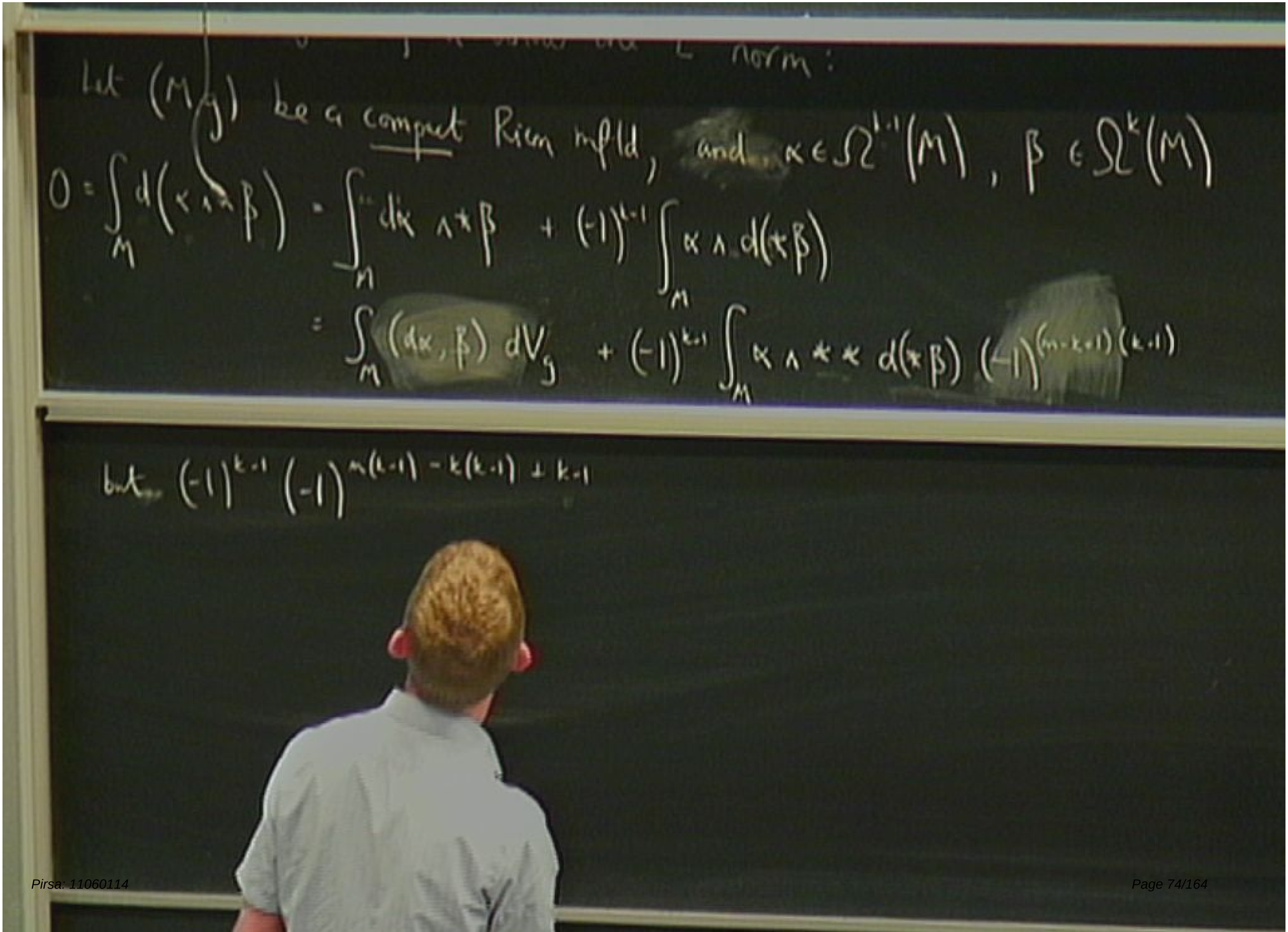

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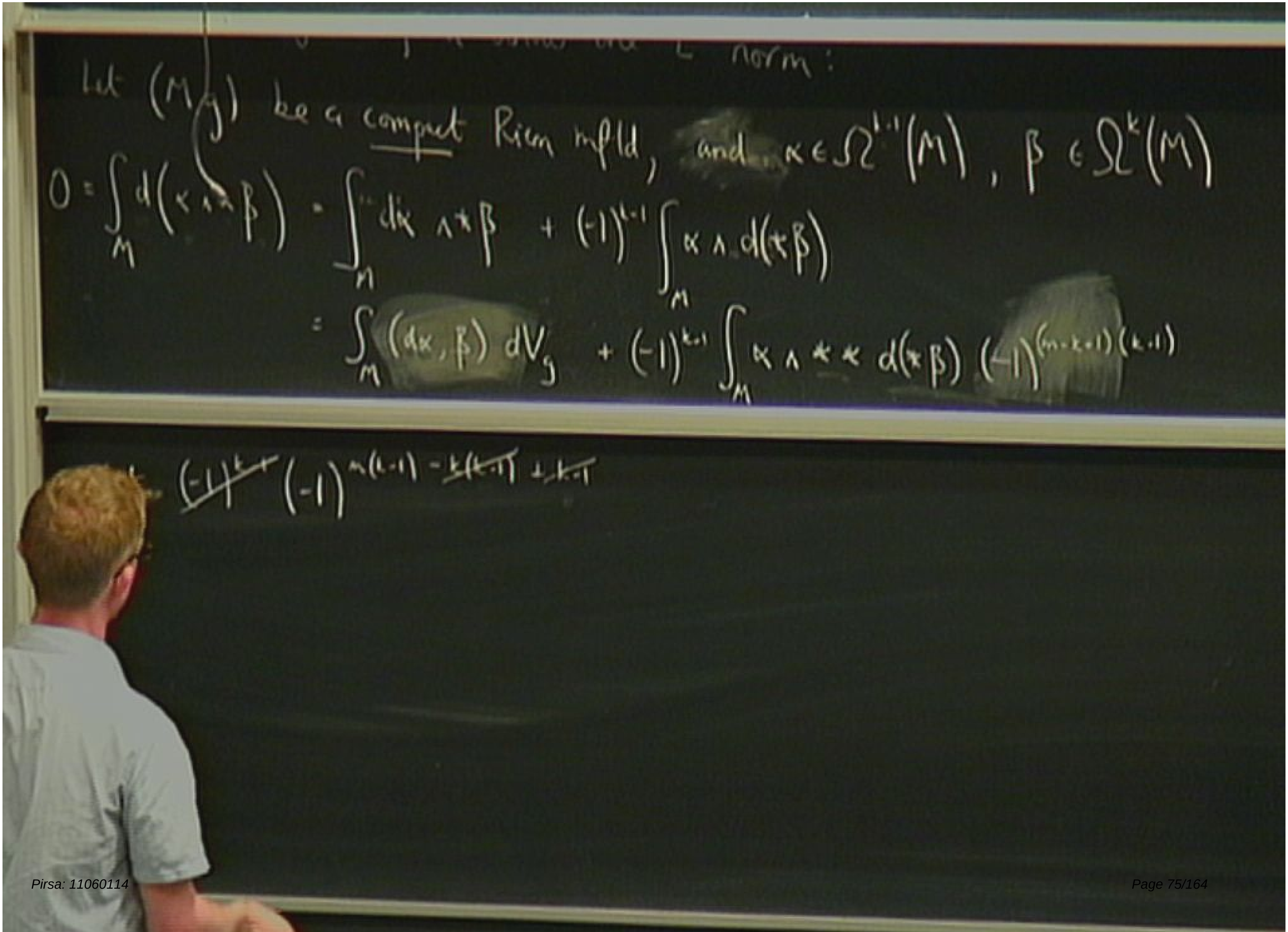
but $(-1)^{k-1} (-1)^{m(k-1) - k(k-1) \pm k-1}$



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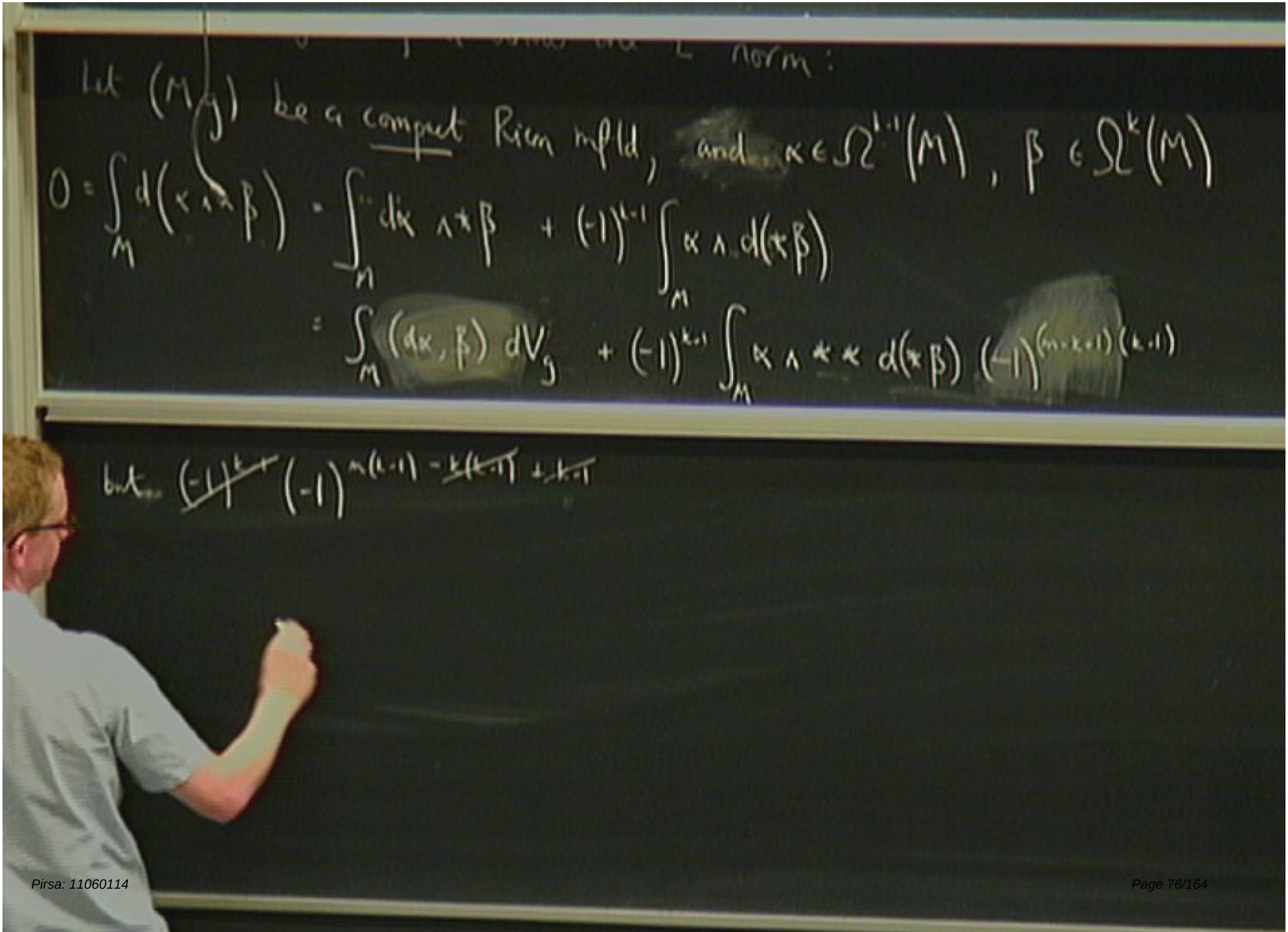

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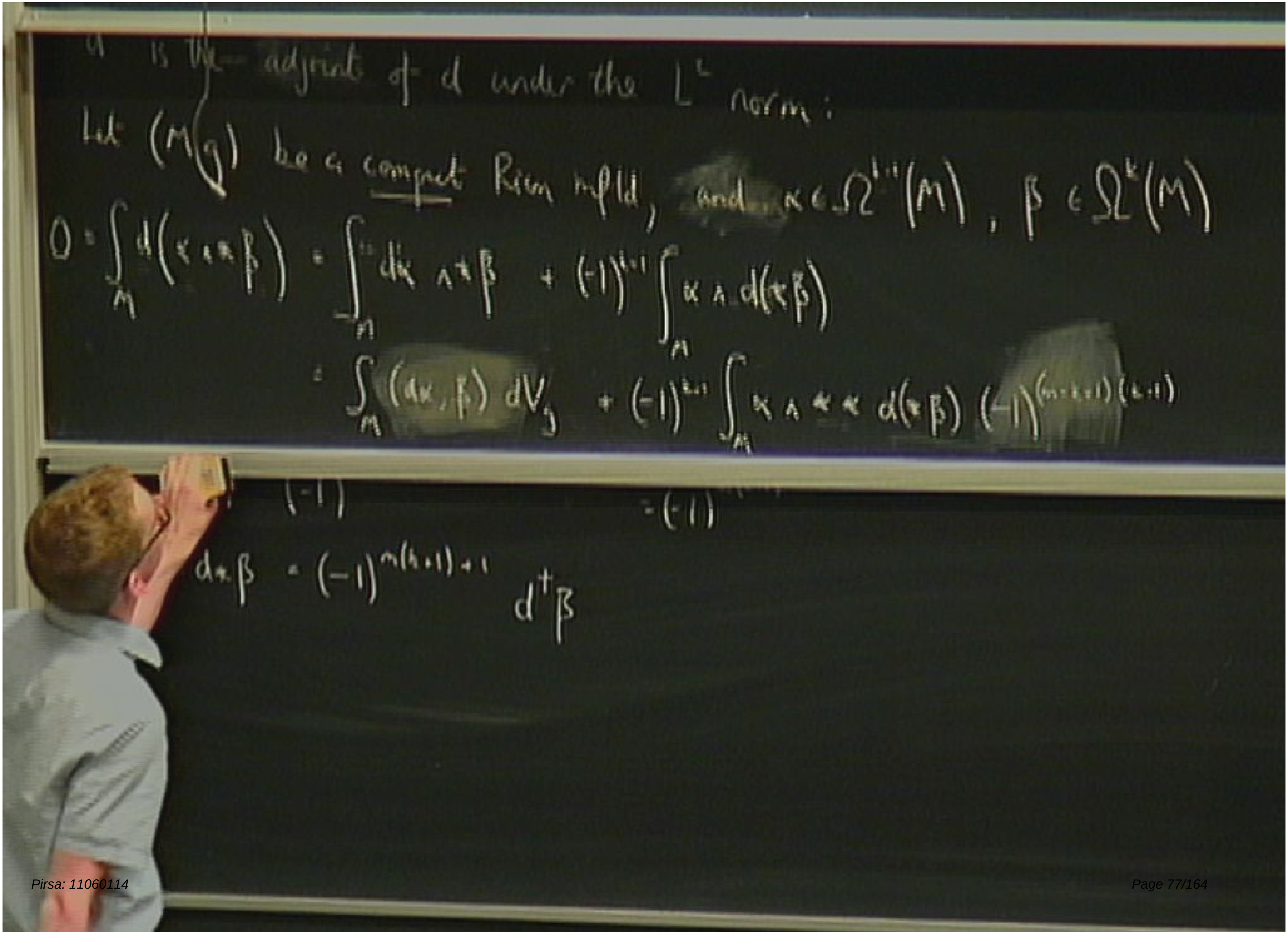

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(-1) $\quad = (-1)$

$$d*\beta = (-1)^{n(k+1)+1} \quad d^\dagger\beta$$

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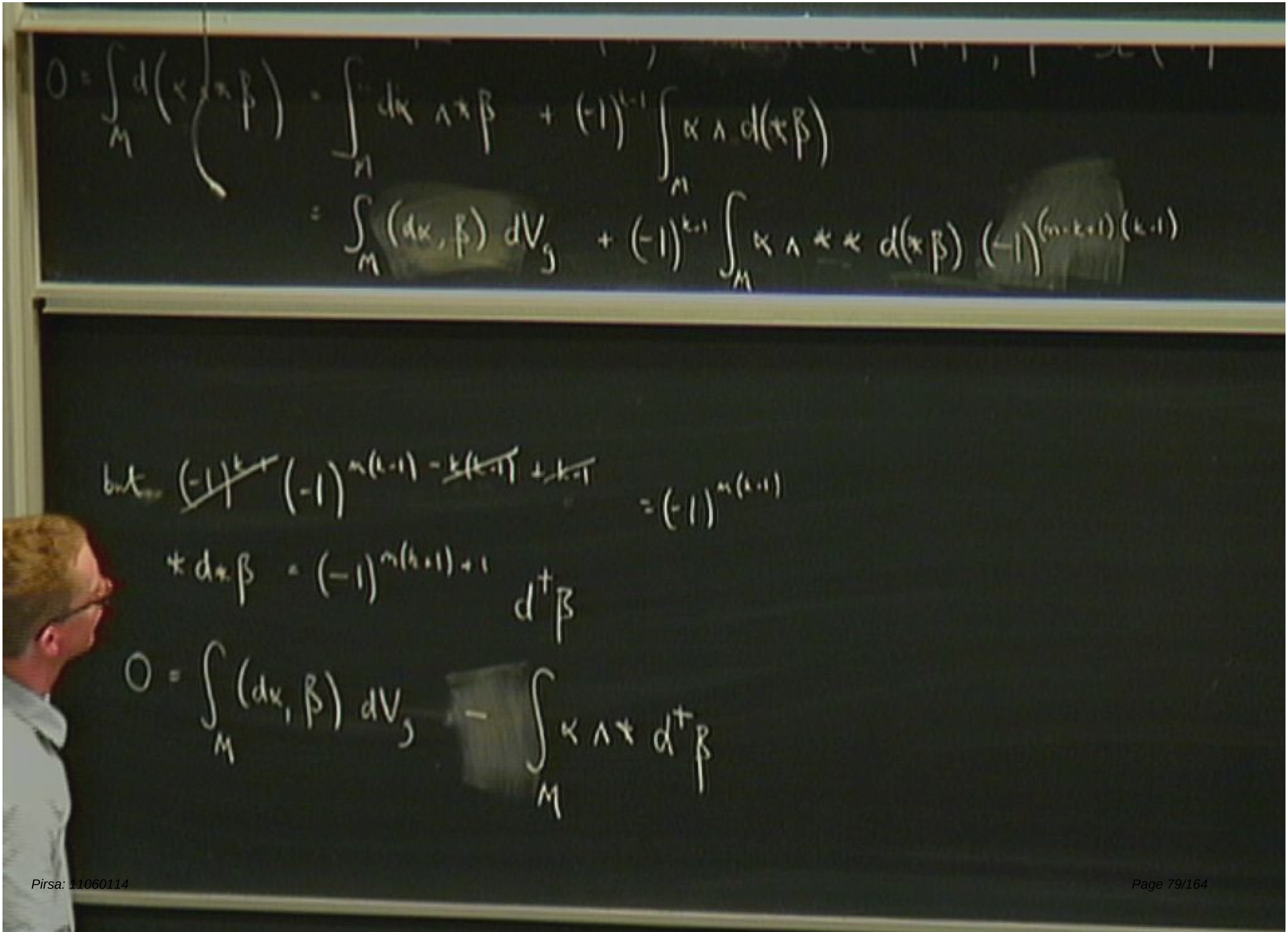

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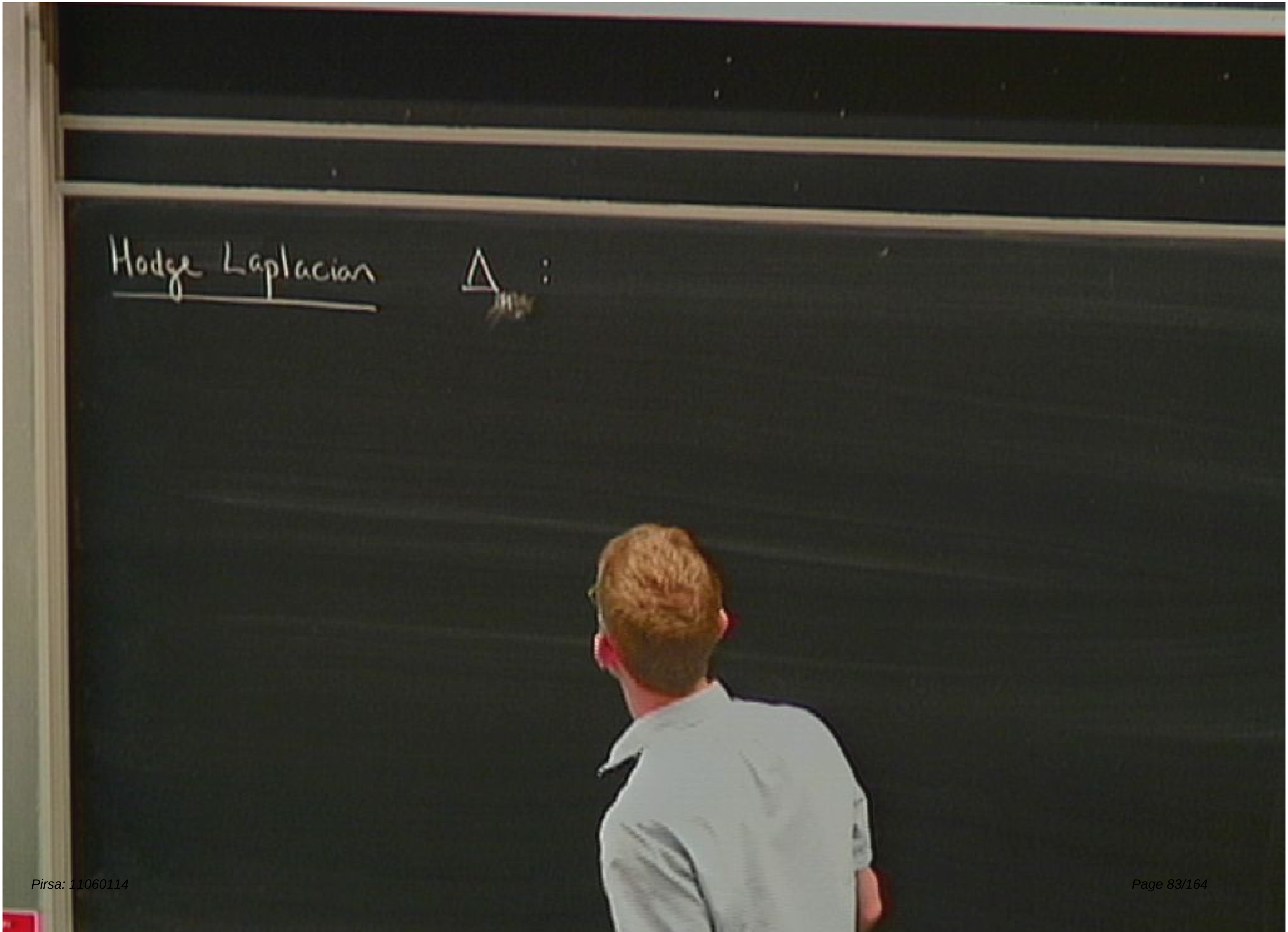
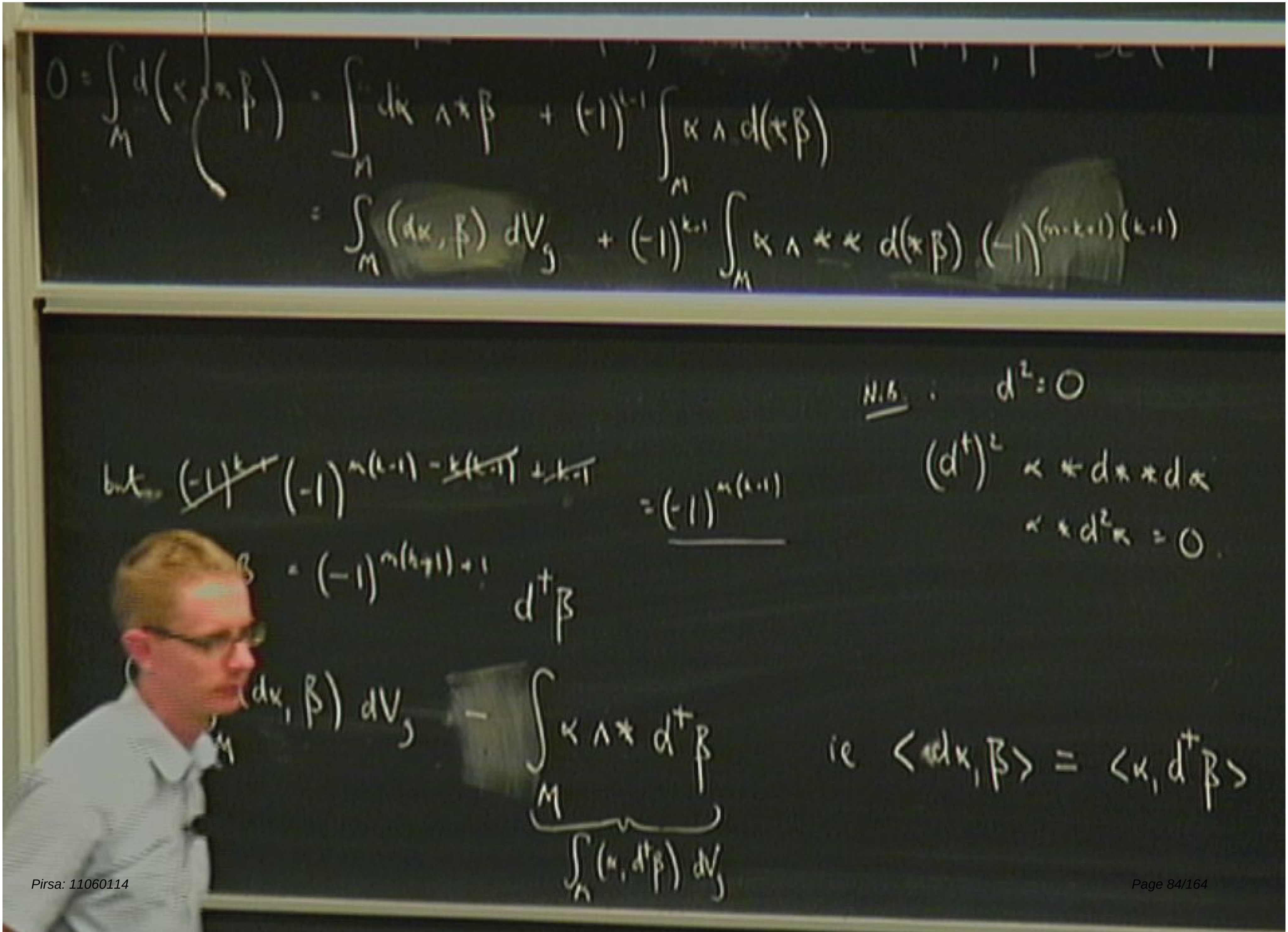


Hodge Laplacian Δ :



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N.B. : $d^2 = 0$

$(d^\dagger)^2 \propto * d * * d\alpha$

$\propto * d^2\alpha = 0$.

but $\cancel{(-1)^{k-1}}\,(-1)^{n(k-1) - \cancel{k(k-1)} + \cancel{k-1}}$ $= \underline{(-1)^{n(k-1)}}$

$\beta = (-1)^{n(k+1)+1}$ $d^\dagger\beta$

$$(d\alpha, \beta)\, dV_g - \underbrace{\int_M \alpha \wedge * d^\dagger\beta}_{\int_M (\alpha, d^\dagger\beta)\, dV_g}$$

ie $\langle d\alpha, \beta\rangle = \langle \alpha, d^\dagger\beta\rangle$

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$$(d^\dagger)^2 \alpha \propto *d**d\alpha$$

$$\propto *d^2\alpha = 0$$

but $\cancel{(-1)^{k-1}}\,(-1)^{n(k-1) - \cancel{k(k-1)} + \cancel{k-1}} = \underline{(-1)^{n(k-1)}}$

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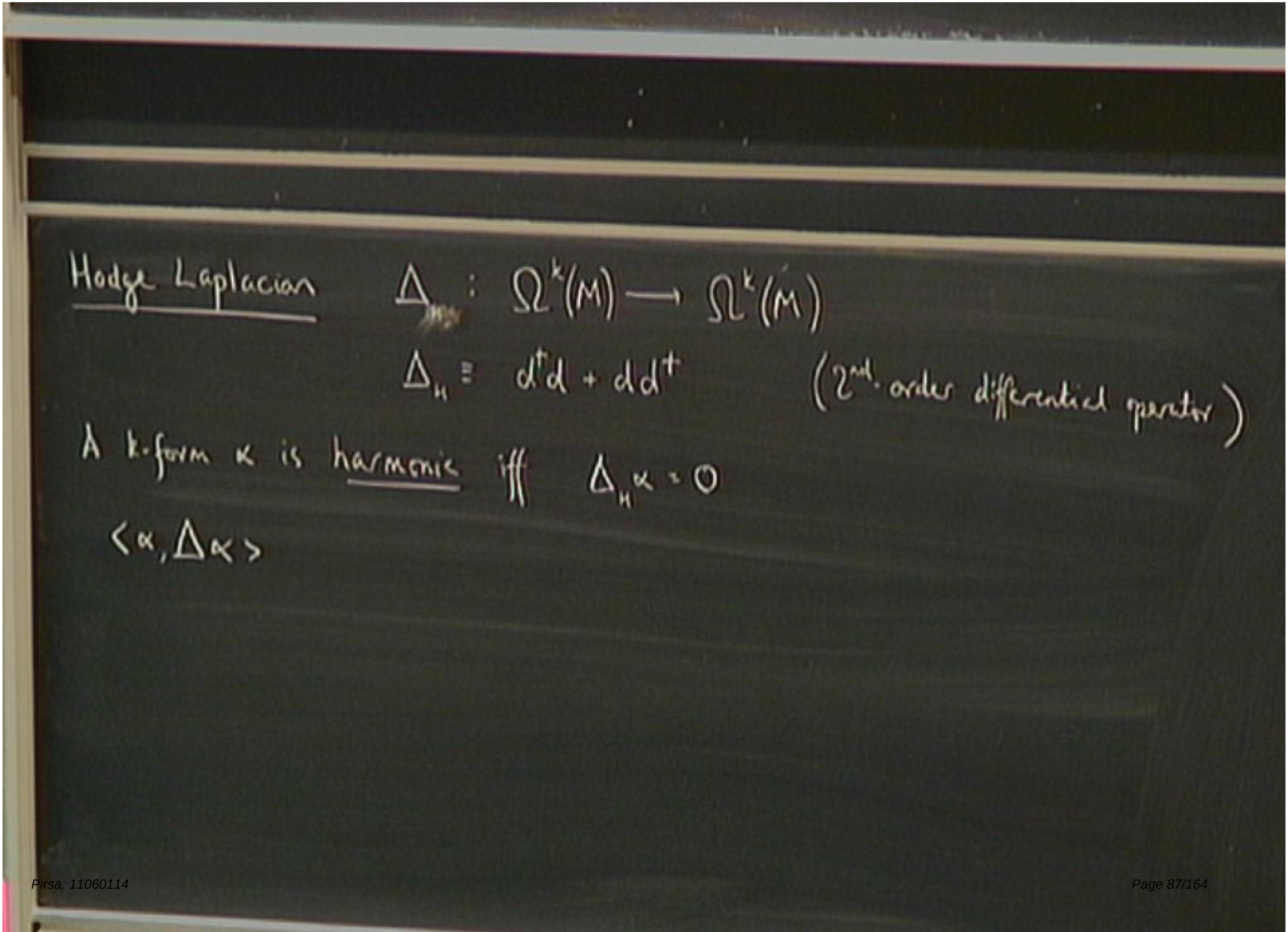

<u>Hodge Laplacian</u> $\Delta : \Omega^k(M) \longrightarrow \Omega^k(M)$

$\Delta_H \equiv d^\dagger d + d d^\dagger$ (2nd. order differential operator)

A k-form α is <u>harmonic</u> iff $\Delta_H \alpha = 0$

$\langle \alpha, \Delta \alpha \rangle$

Hodge Laplacian

$$\Delta_{H} : \Omega^k(M) \longrightarrow \Omega^k(M)$$

$$\Delta_H := d^\dagger d + d d^\dagger \qquad (2^{nd}\text{. order differential operator})$$

A k-form α is harmonic iff $\Delta_H \alpha = 0$

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, d d^\dagger \alpha \rangle + \langle \alpha, d^\dagger d \alpha \rangle$$

$$= \langle$$

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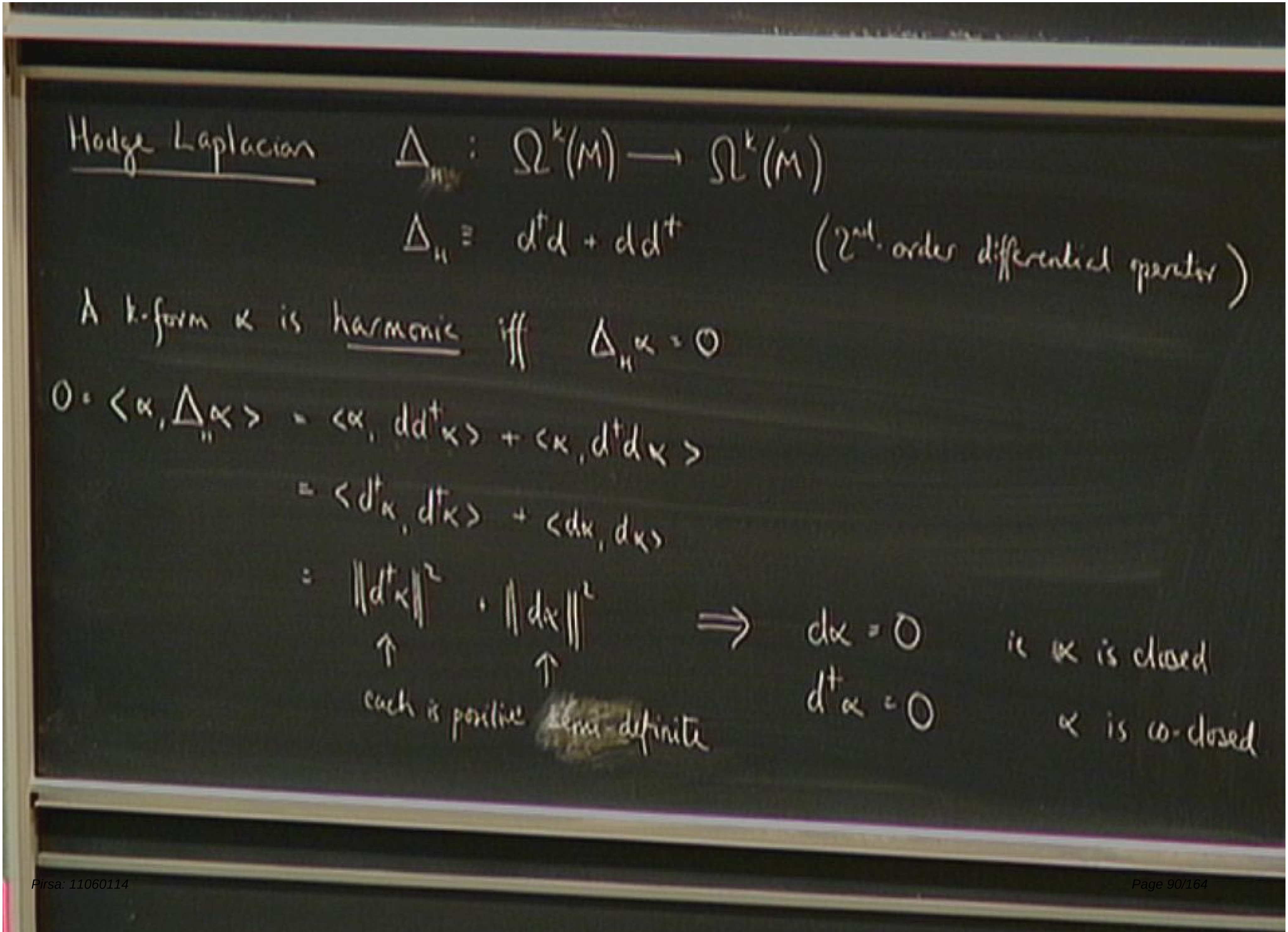
$$= \underset{\uparrow}{\|d^\dagger \alpha\|^2} + \underset{\uparrow}{\|d\alpha\|^2}$$

each is positive definite



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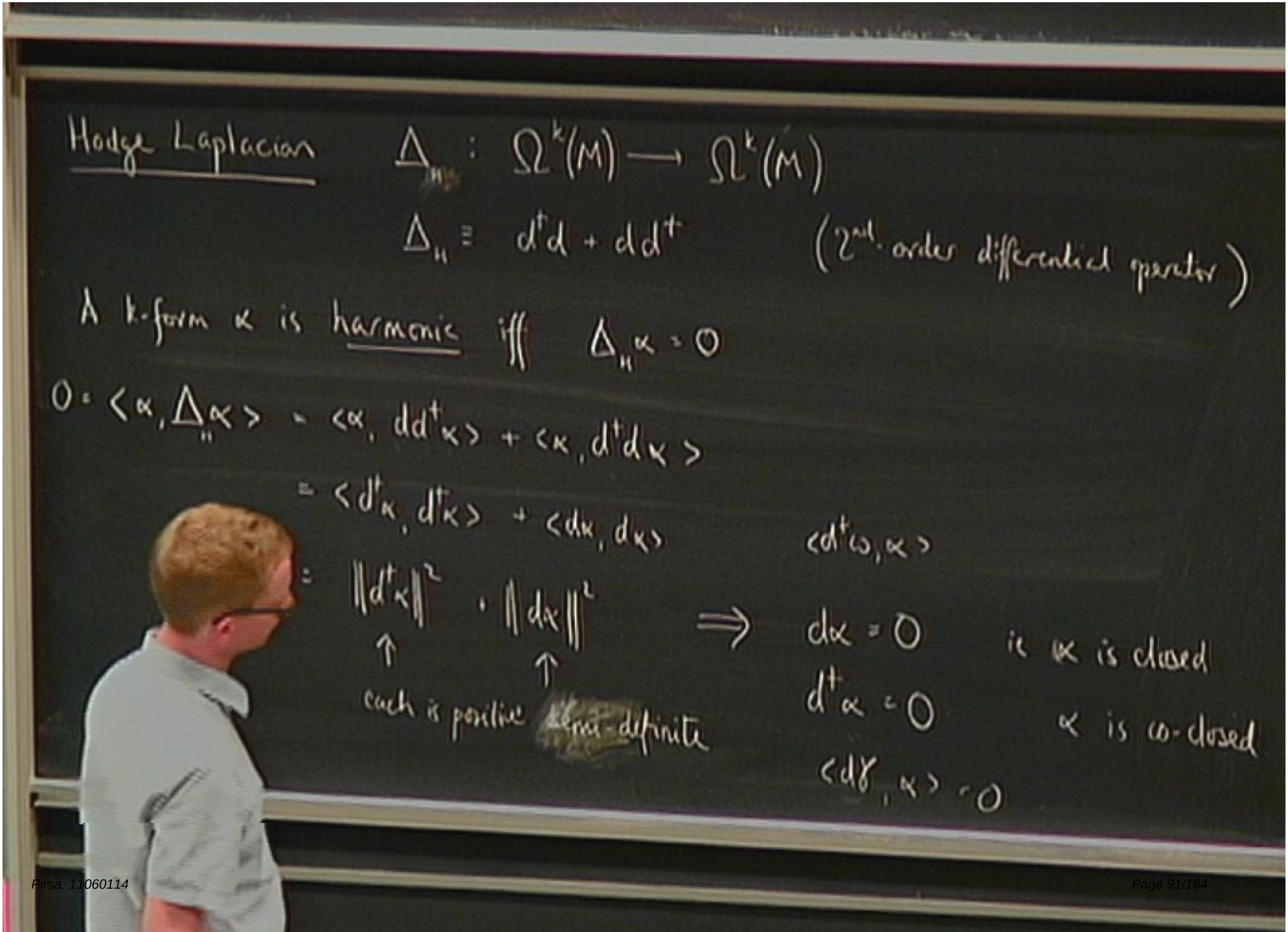
$$= \underset{\uparrow}{\|d^\dagger \alpha\|^2} + \underset{\uparrow}{\|d\alpha\|^2} \Longrightarrow \begin{array}{ll} d\alpha = 0 & \text{i.e. } \alpha \text{ is closed} \\ d^\dagger \alpha = 0 & \alpha \text{ is co-closed} \end{array}$$

each is positive semi-definite



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$= \langle d^\dagger \alpha, d^\dagger \alpha \rangle + \langle d\alpha, d\alpha \rangle$

$\langle d^\dagger \omega, \alpha \rangle$

$$= \underset{\uparrow}{\|d^\dagger \alpha\|^2} + \underset{\uparrow}{\|d\alpha\|^2} \Longrightarrow \begin{array}{l} d\alpha = 0 \\ d^\dagger \alpha = 0 \end{array}$$

each is positive semi-definite

i.e. α is closed

α is co-closed

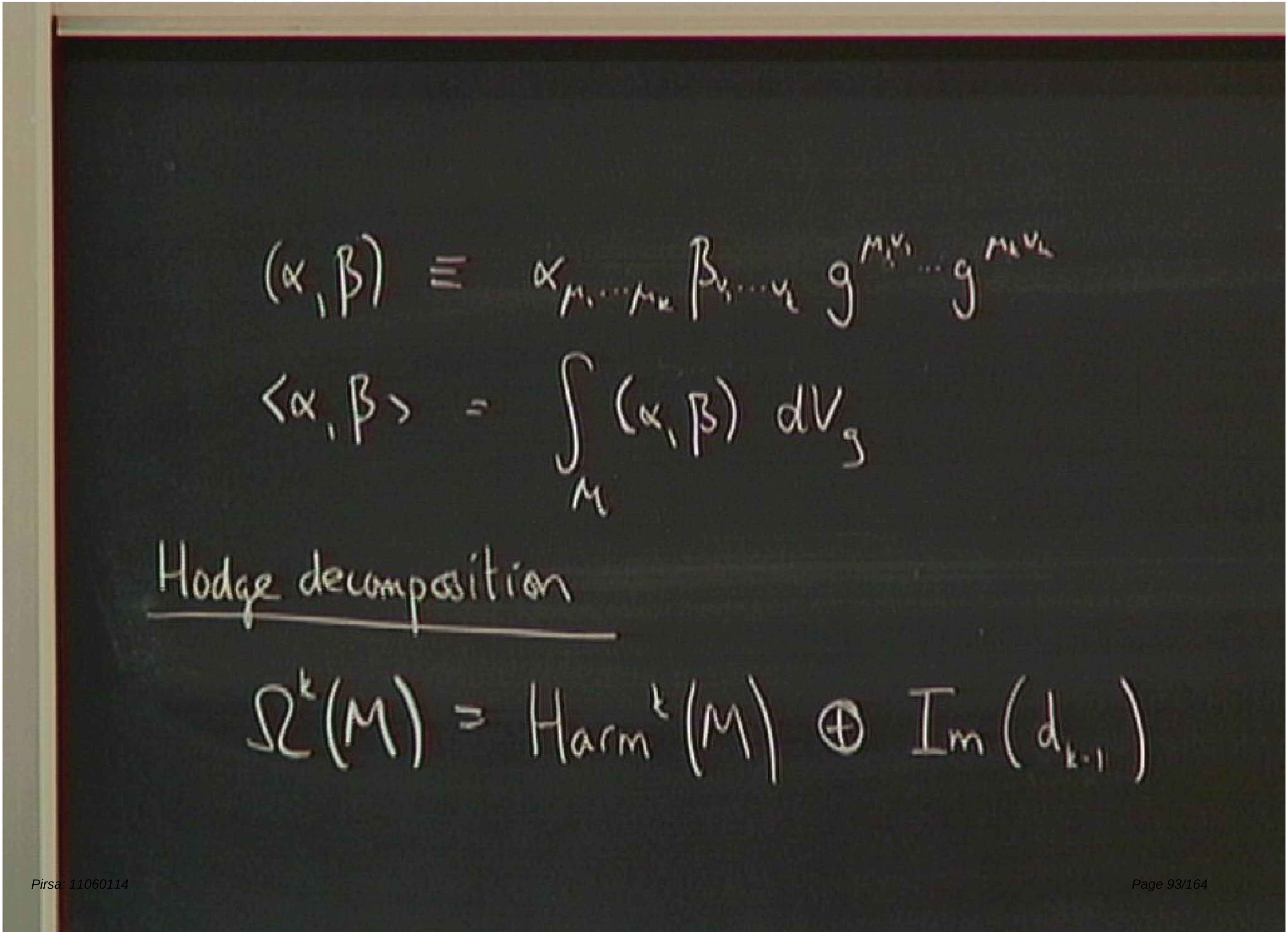
$\langle d\gamma, \alpha \rangle = 0$

$$(\alpha, \beta) \equiv \alpha_{\mu_1 \cdots \mu_k} \beta_{\nu_1 \cdots \nu_k} g^{\mu_1 \nu_1} \cdots g^{\mu_k \nu_k}$$

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \, dV_g$$

<u>Hodge decomposition</u>

$$\Omega^k(M) = \mathrm{Harm}^k(M) \oplus \mathrm{Im}\left(d_{k-1}\right)$$



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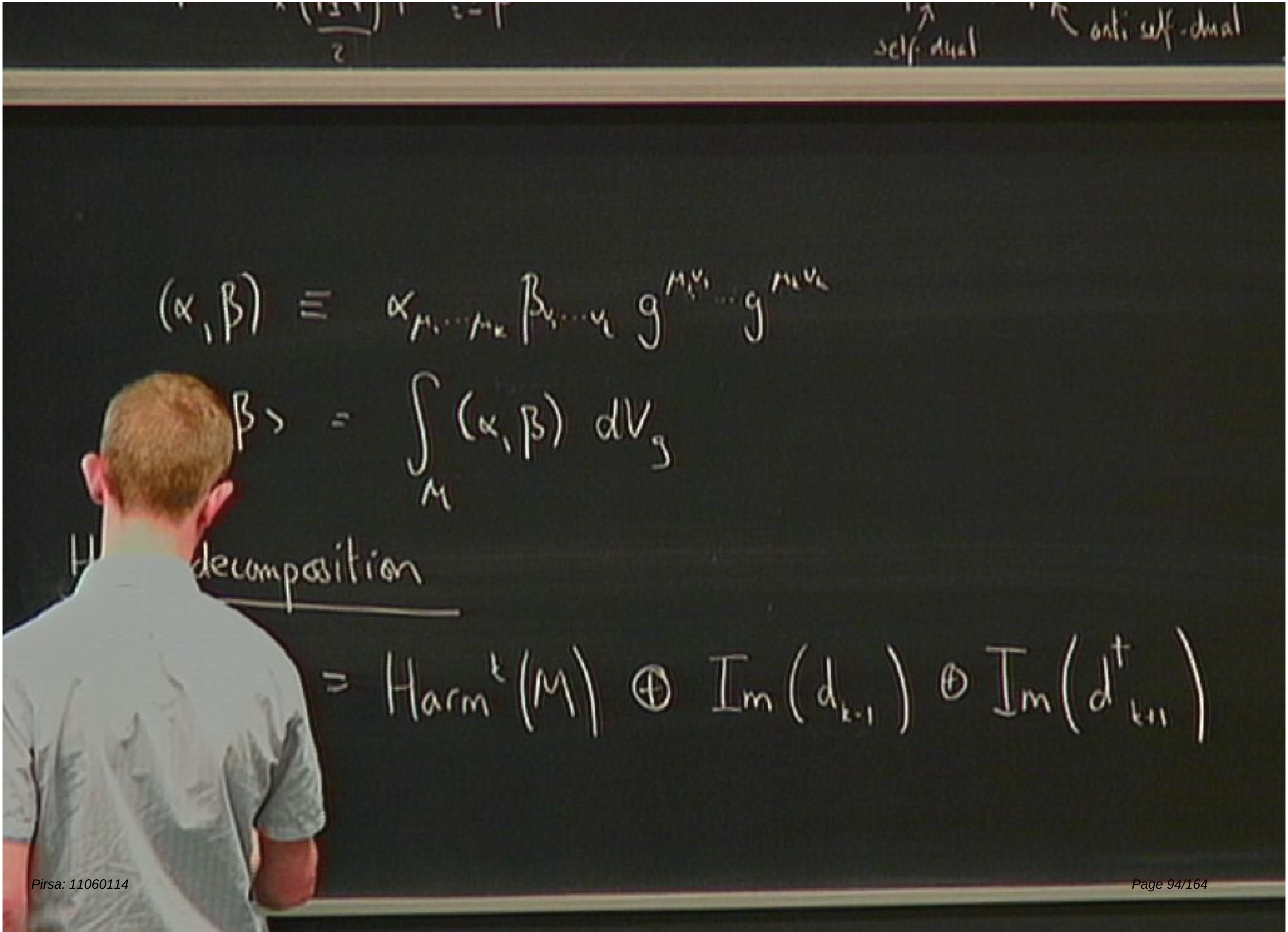

self-dual

anti self-dual

$$(\alpha,\beta) \equiv \alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_k}\, g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}$$

$$\beta\rangle = \int_M (\alpha,\beta)\, dV_g$$

H decomposition

$$= \mathrm{Harm}^k(M) \oplus \mathrm{Im}(d_{k-1}) \oplus \mathrm{Im}(d^\dagger_{k+1})$$

self-dual anti self-dual

$$(\alpha,\beta) \equiv \alpha_{\mu_1\cdots\mu_k}\beta_{\nu_1\cdots\nu_k}\, g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k}$$

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Hodge decomposition

$$\Omega^k(M) = \mathrm{Harm}^k(M) \oplus \mathrm{Im}(d_{k-1}) \oplus \mathrm{Im}(d^\dagger_{k+1})$$

This decomposition is orthogonal wrt L^2 inner product $\langle\, ,\, \rangle$.

$\langle d\gamma , \alpha$

$$\mathrm{Ker}(d_k) = \mathrm{Im}(d_{k-1}) \oplus H^k_{dR}(M ; \mathbb{R})$$

semi-definite

$\langle d\gamma , \alpha$

$$\mathrm{Ker}(d_k) = \mathrm{Im}\,(d_{k-1}) \oplus H^k_{dR}(M;\mathbb{R})$$

So Hodge decomposition provides an isomorphism

$$H^k_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^k(M)$$

positive semi-definite

$\langle d\gamma , \alpha$

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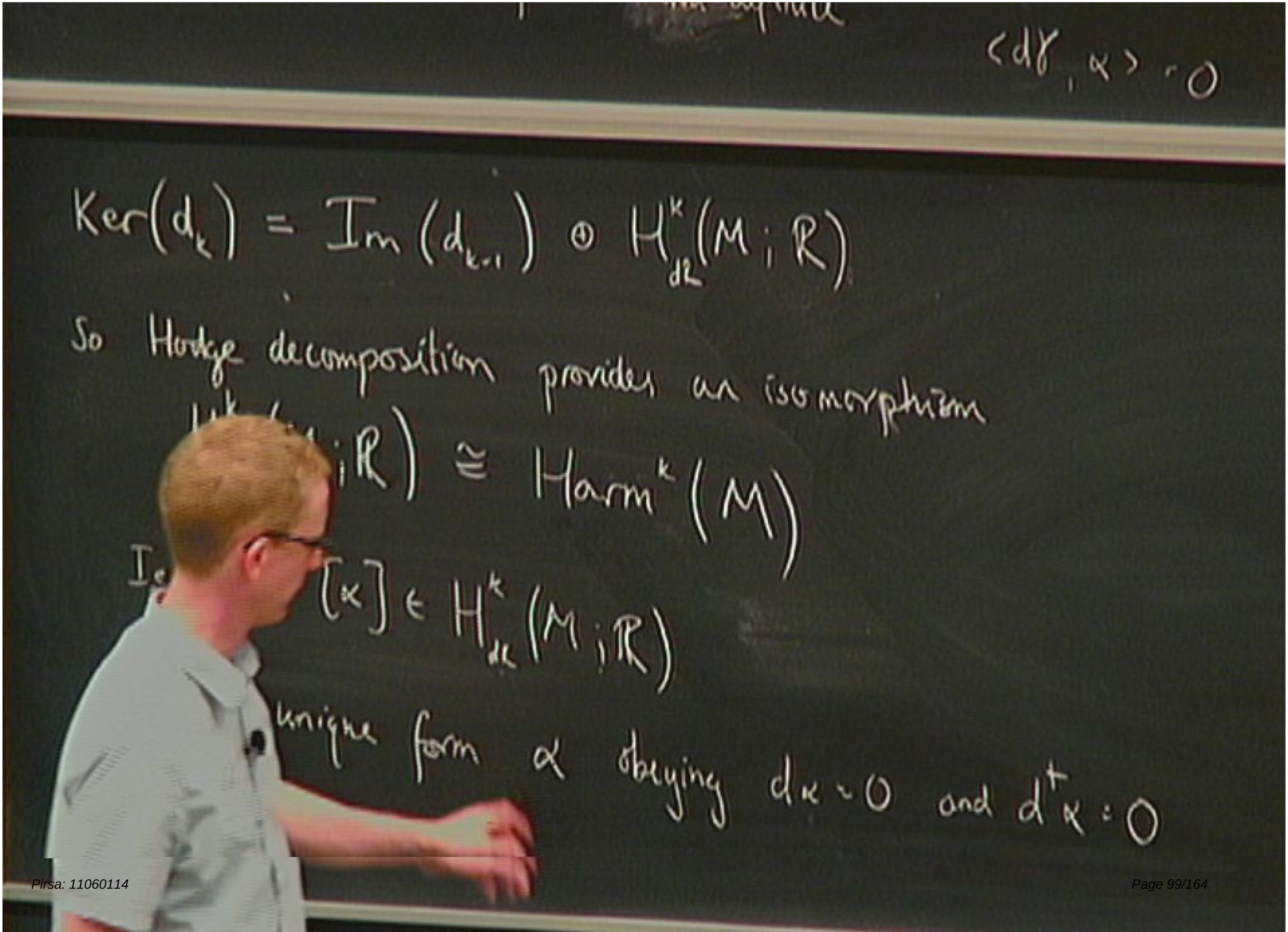

$\langle d\gamma, \alpha \rangle = 0$

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$$H^k(M;\mathbb{R}) \cong \mathrm{Harm}^k(M)$$

Ie $[\alpha] \in H^k_{dR}(M;\mathbb{R})$

unique form α obeying $d\alpha = 0$ and $d^{\dagger}\alpha = 0$

$\langle d\gamma, \alpha \rangle = 0$

$$\mathrm{Ker}(d_k) = \mathrm{Im}(d_{k-1}) \oplus H^k_{dR}(M;\mathbb{R})$$

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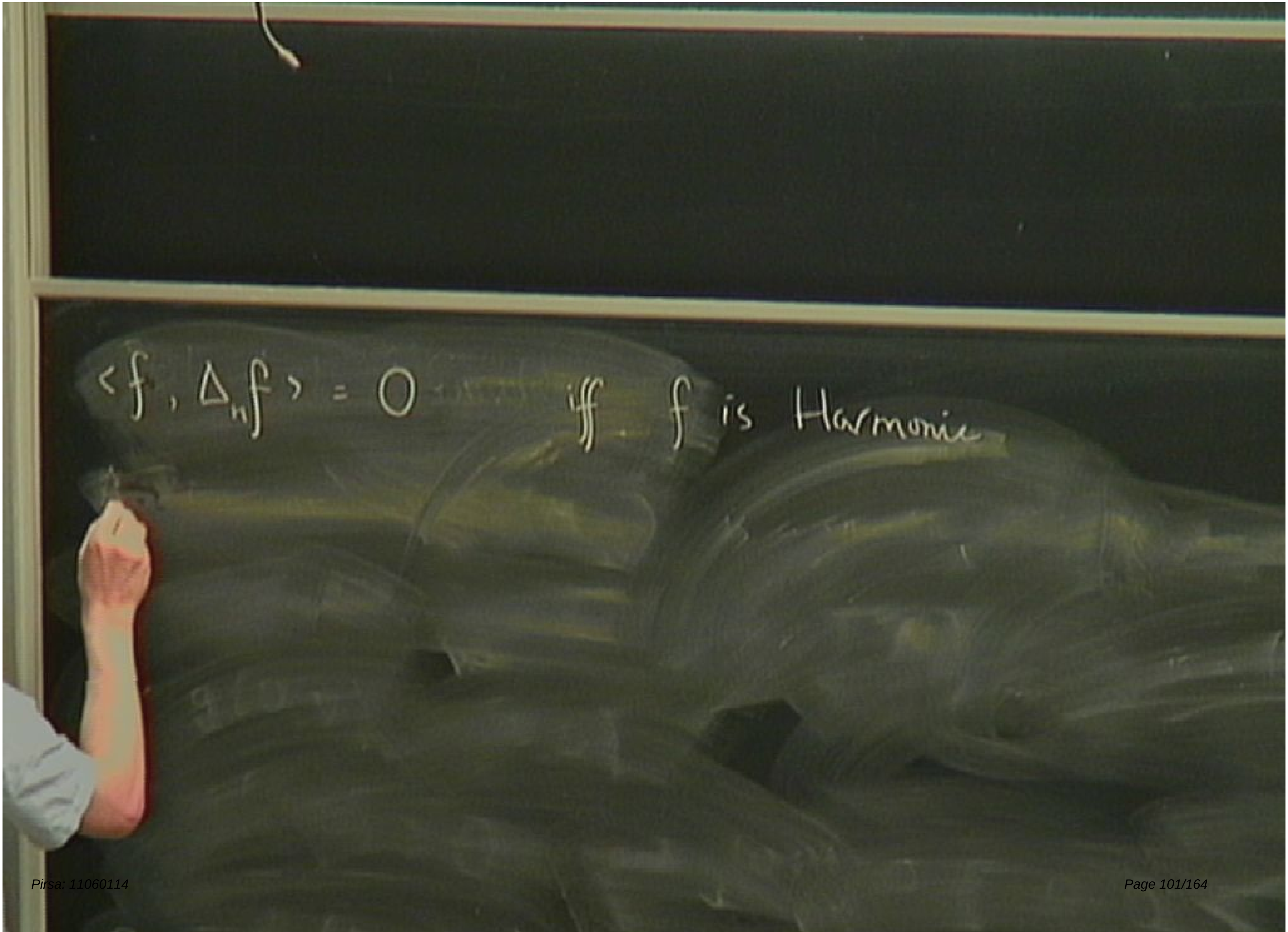
$$H^k_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^k(M)$$

I.e. if $[\alpha] \in H^k_{dR}(M;\mathbb{R})$

$\exists$ unique form α obeying $d\alpha = 0$ and $d^\dagger \alpha = 0$



$\langle f, \Delta_n f \rangle = 0$ iff f is Harmonic

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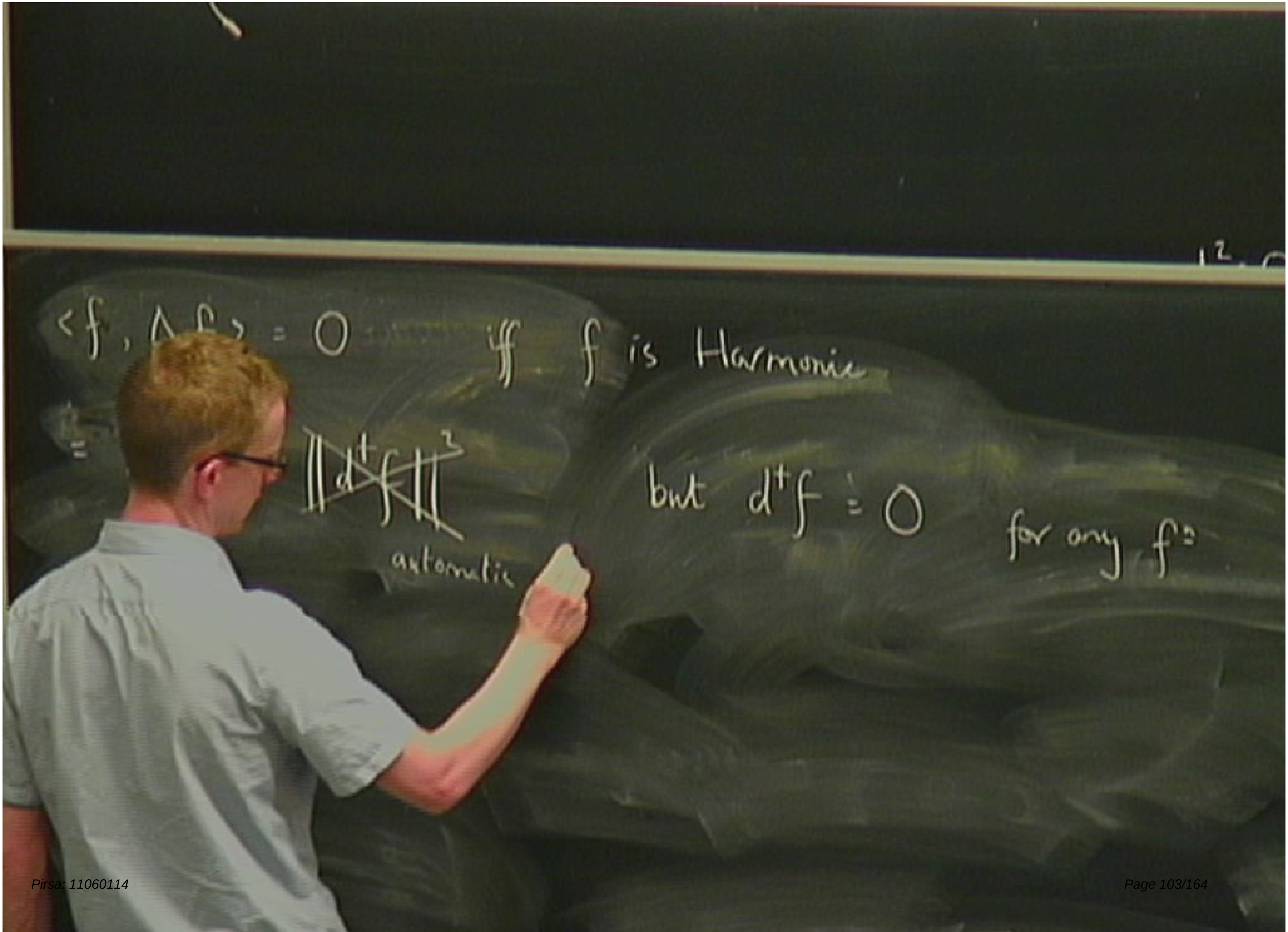
$= \|df\|^2 + \|d^\dagger f\|^2$

but $d^\dagger f = 0$ for any f:

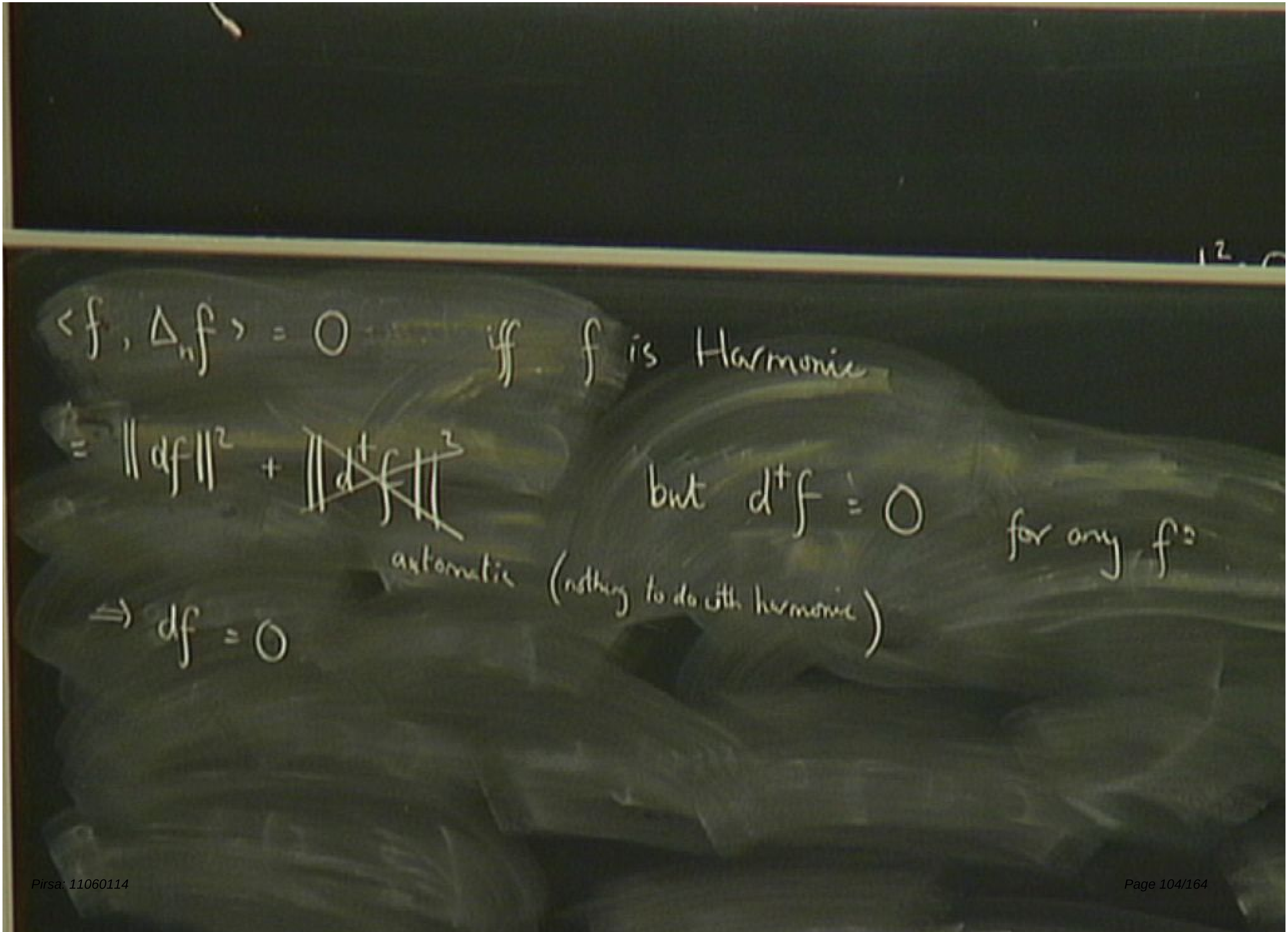


$\langle f, \Delta f \rangle = 0$ iff f is Harmonic

$= \cancel{|||d^\dagger f|||^2}$

automatic

but $d^\dagger f \overset{!}{=} 0$ for any f?



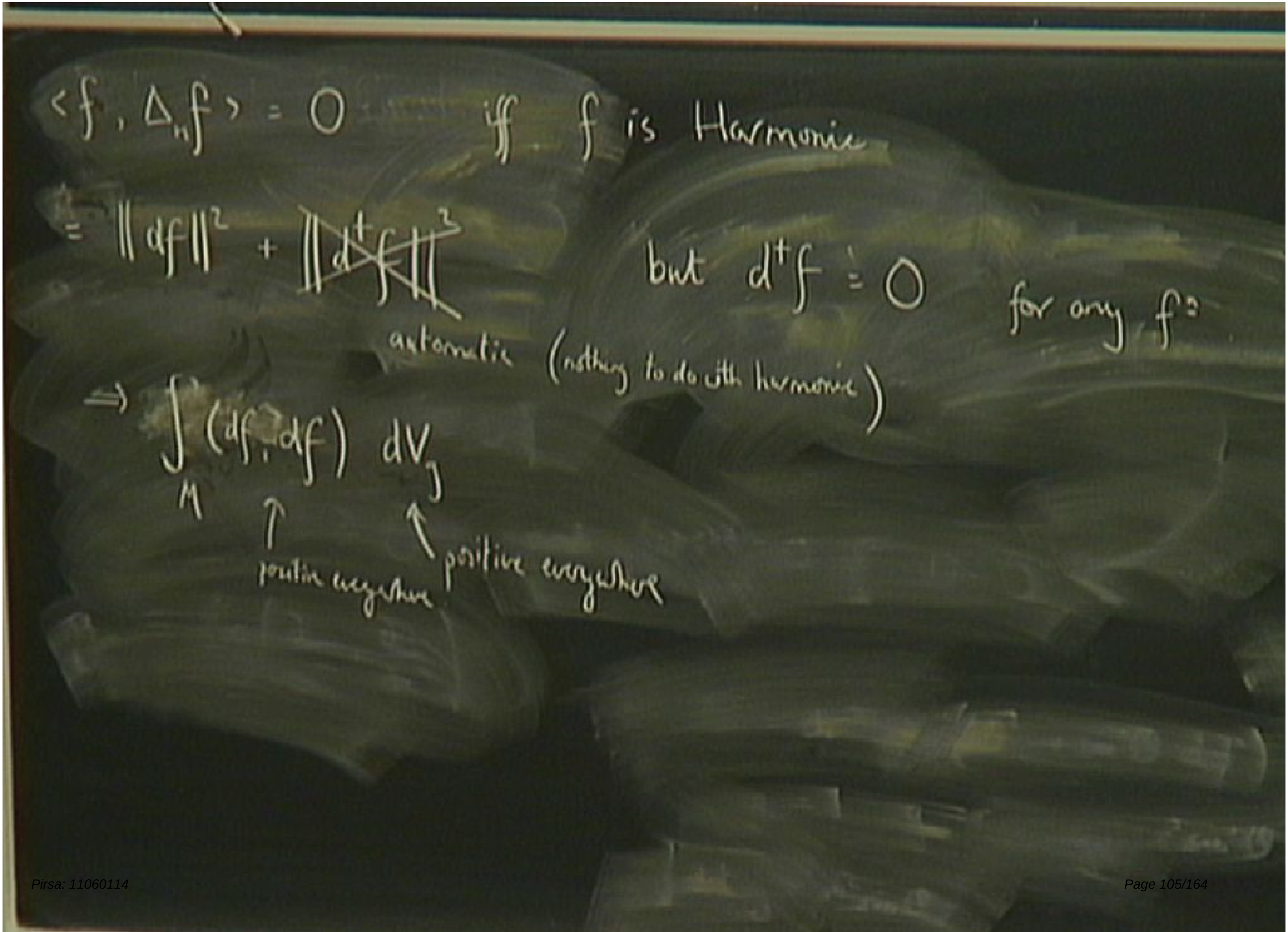
$\langle f, \Delta_n f \rangle = 0$ iff f is Harmonic

$= \|df\|^2 + \cancel{\|d^\dagger f\|^2}$

automatic (nothing to do with harmonic)

but $d^\dagger f = 0$ for any f:

$\Rightarrow df = 0$



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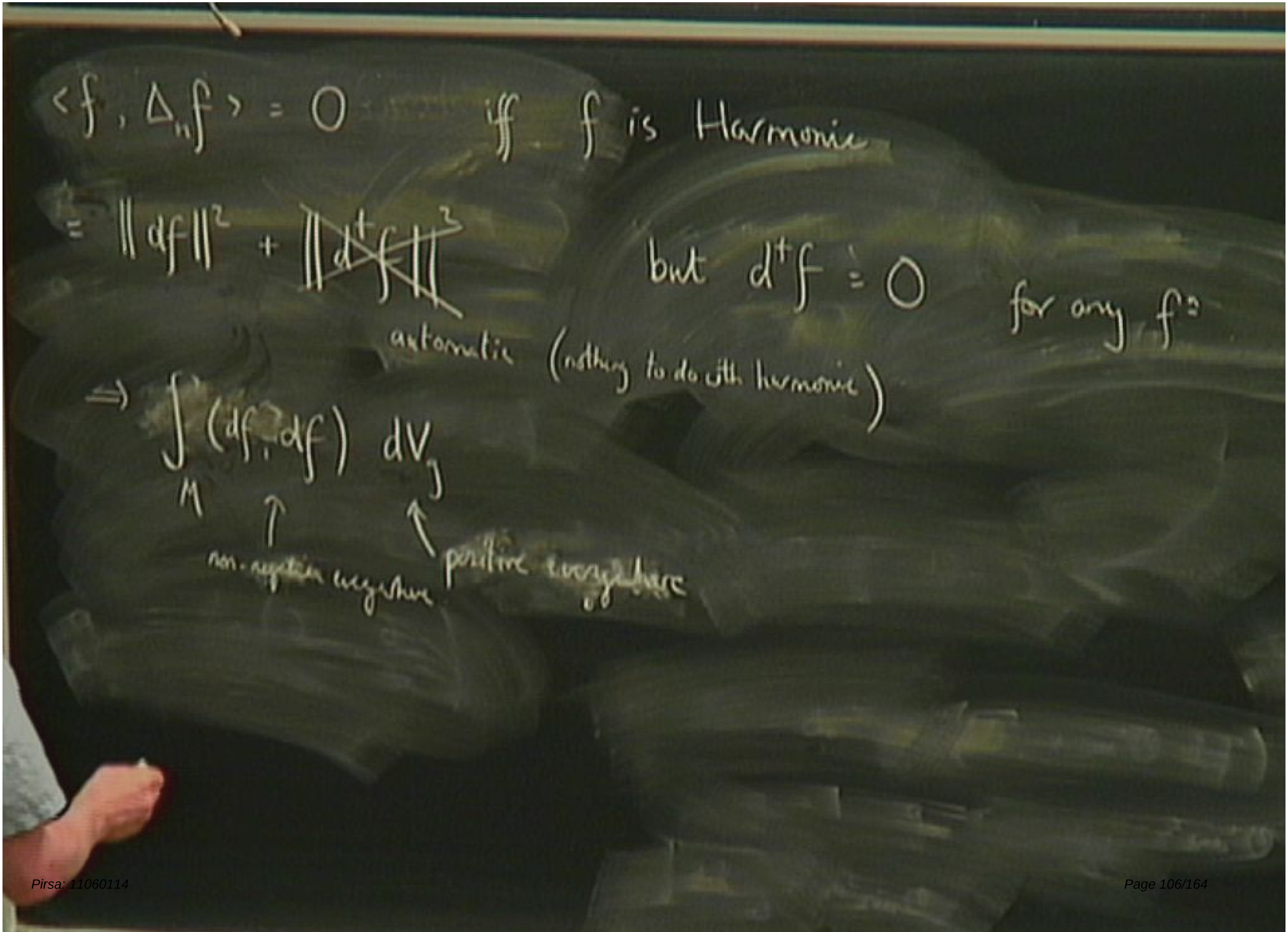
automatic (nothing to do with harmonic)

$$\Rightarrow \int_M (df, df)\, dV_g$$

positive everywhere

positive everywhere



$\langle f, \Delta_H f \rangle = 0$ iff f is Harmonic

$= \|df\|^2 + \cancel{\|d^\dagger f\|^3}$ but $d^\dagger f \overset{!}{=} 0$ for any f ?

automatic (nothing to do with harmonic)

$$\Rightarrow \int_M (df, df)\, dV_g$$

non-negative everywhere   positive everywhere

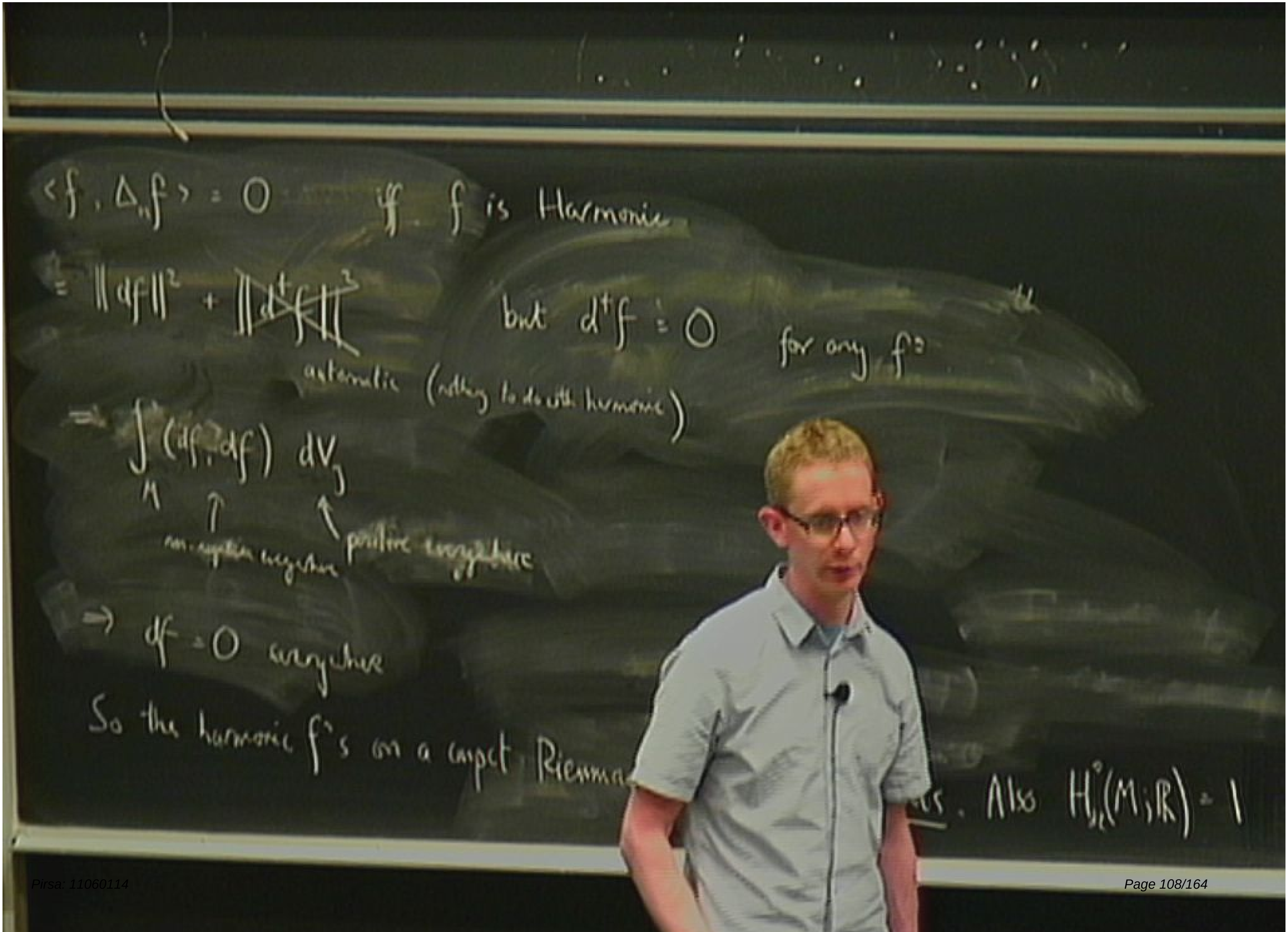
$\langle f, \Delta_H f \rangle = 0$ iff f is Harmonic

$= \|df\|^2 + \cancel{\|d^\dagger f\|^3}$ but $d^\dagger f = 0$ for any f:

automatic (nothing to do with harmonic)

$$= \int_M (df, df)\, dV_g$$

non-negative integrand   positive everywhere



$\langle f, \Delta_H f \rangle = 0$ iff. f is Harmonic

$= \|df\|^2 + \cancel{\|d^\dagger f\|^2}$ but $d^\dagger f \overset{!}{=} 0$ for any f?

automatic (nothing to do with harmonic)

$\Rightarrow \int_M (df, df)\, dV_g$

non-negative everywhere

positive everywhere

$\Rightarrow df = 0$ everywhere

So the harmonic f's on a cmpct Rieman

... Also $H^0_{dR}(M;\mathbb{R}) = 1$

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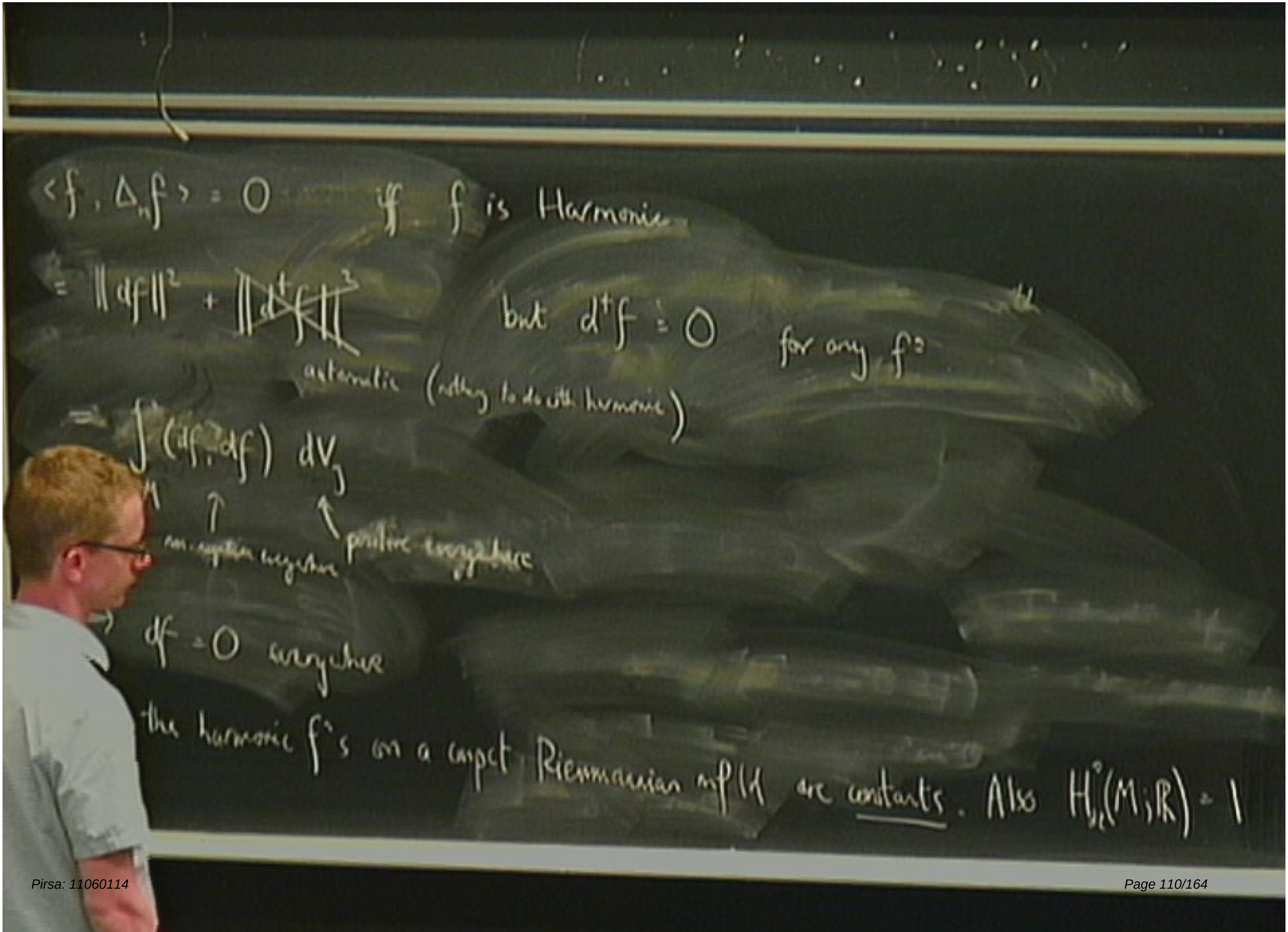
ic f's on a compct Riemannian mfld are constants. Also $H^0_{dR}(M;\mathbb{R}) = 1$



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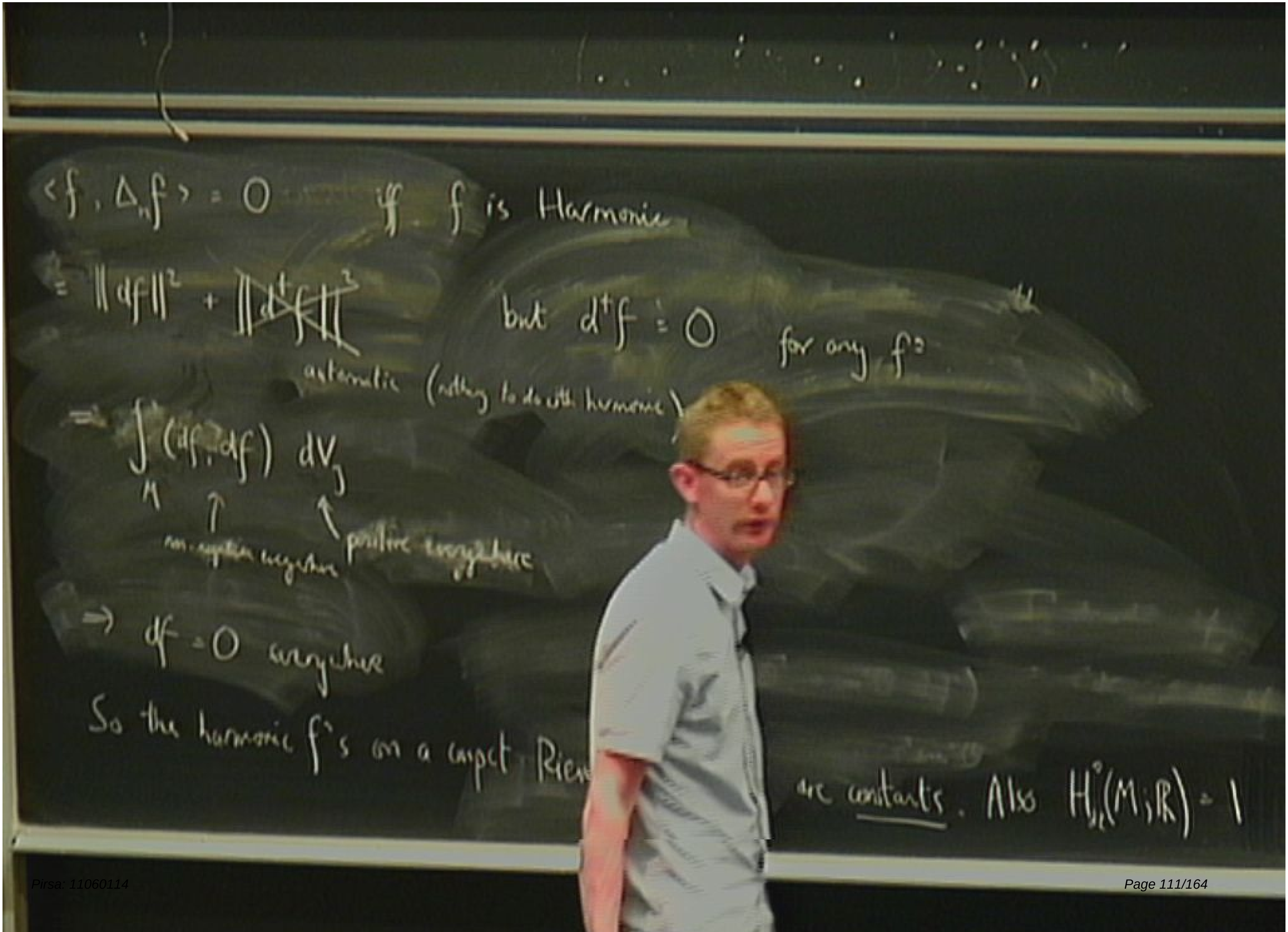

$\langle f, \Delta_n f \rangle = 0$ iff. f is Harmonic

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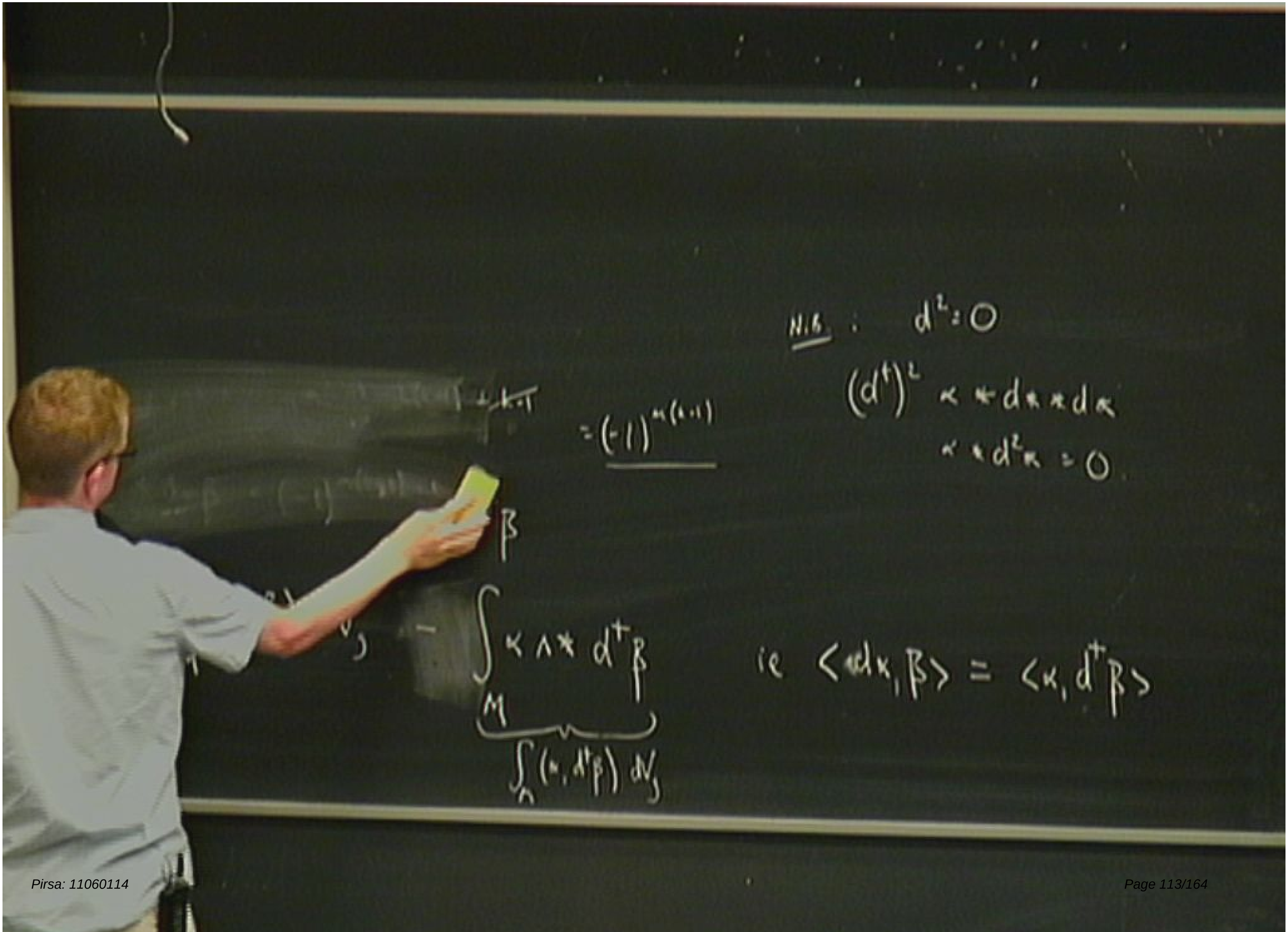


$\underline{N.B.}$: $d^2 = 0$

$(d^\dagger)^2 \propto *d**d\alpha$

$\propto *d^2\alpha = 0$

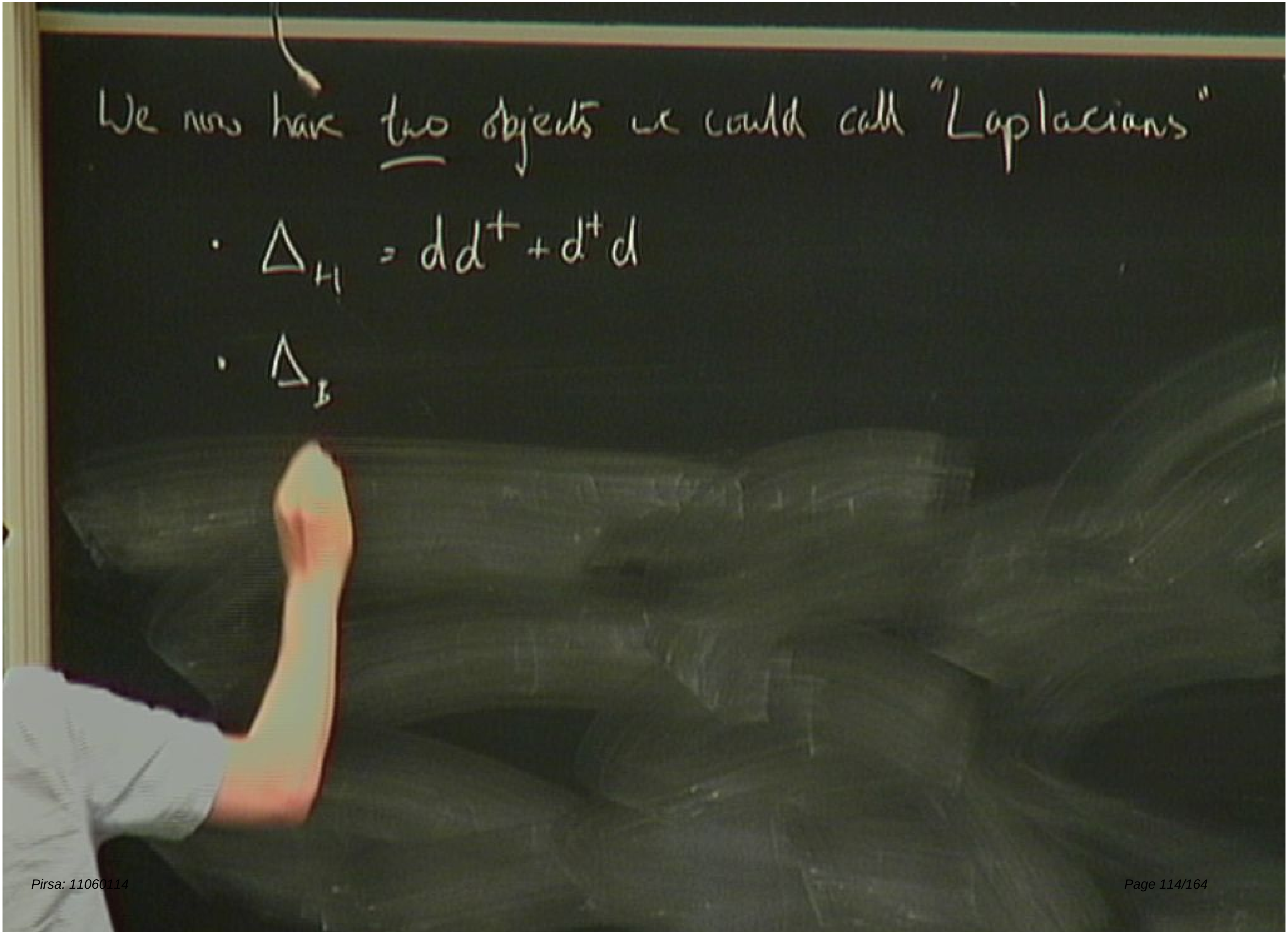
$+ \cancel{k+1}$

$\underline{= (-1)^{n(k+1)}}$

β

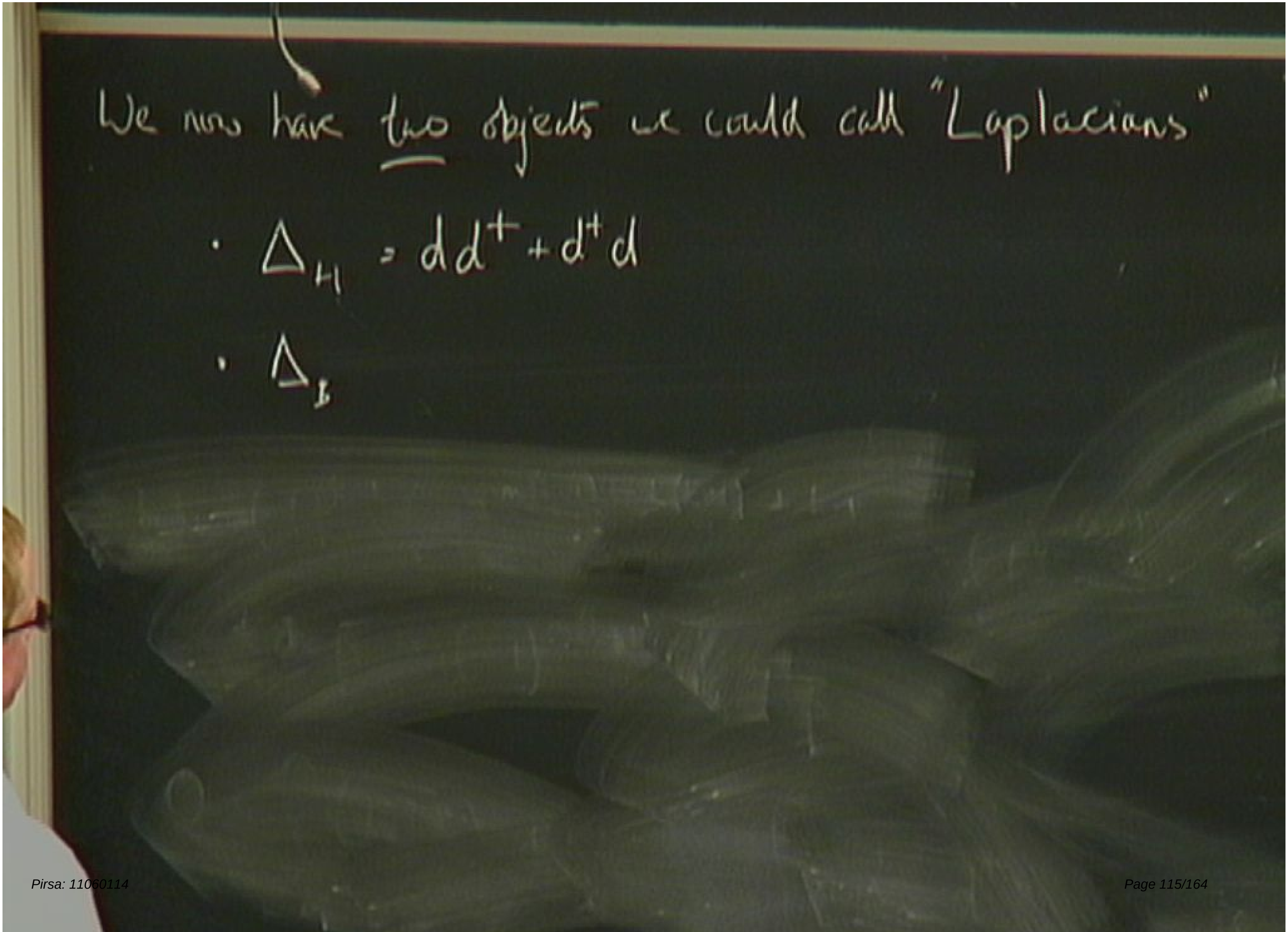
$dV_g \quad - \underbrace{\int_M \alpha \wedge * d^\dagger \beta}_{\int_M (\alpha, d^\dagger\beta)\, dV_g}$

ie $\langle d\alpha, \beta \rangle = \langle \alpha, d^\dagger \beta \rangle$



We now have <u>two</u> objects we could call "Laplacians"

- $\Delta_H = dd^+ + d^+ d$
- Δ_B



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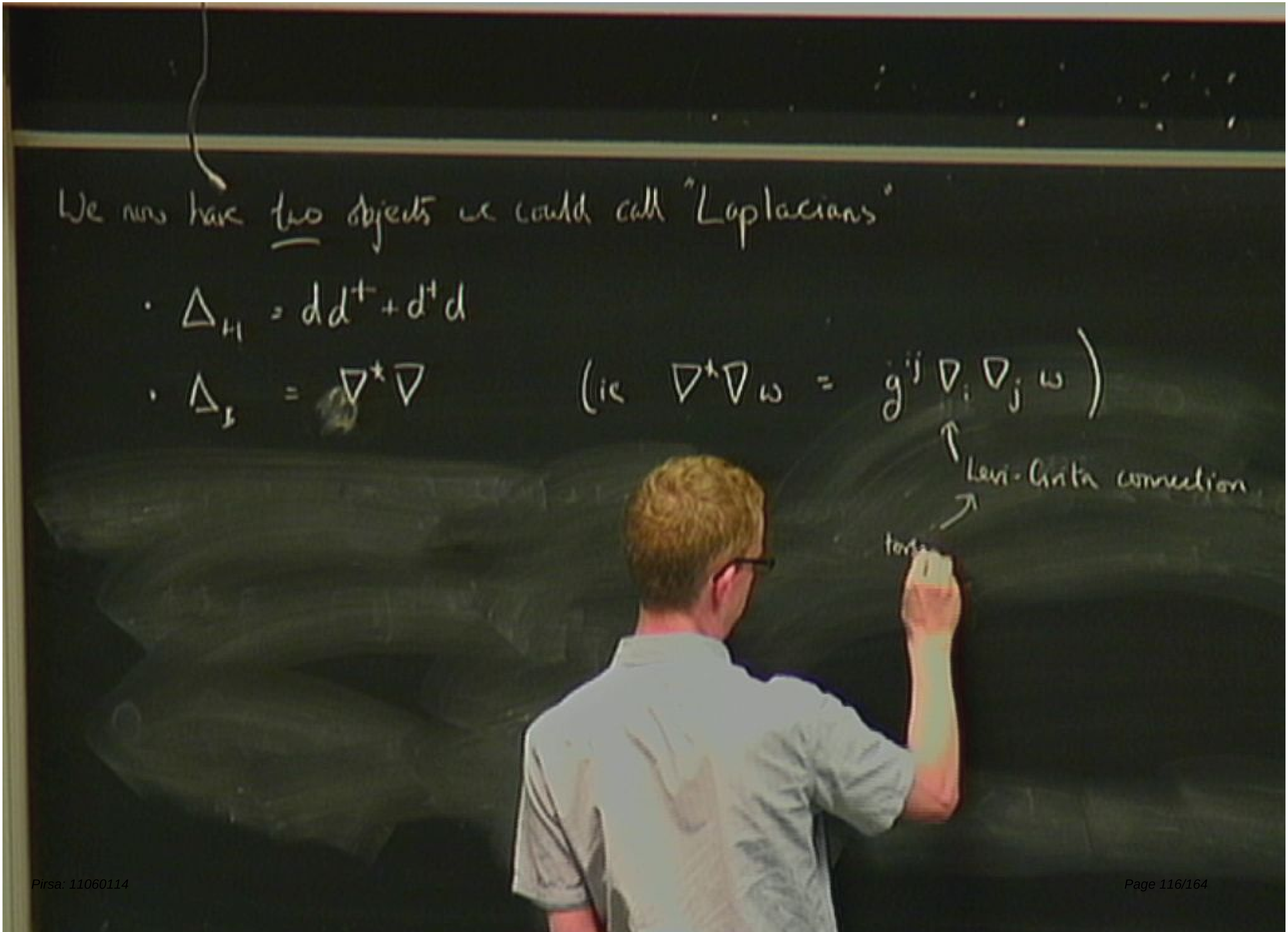

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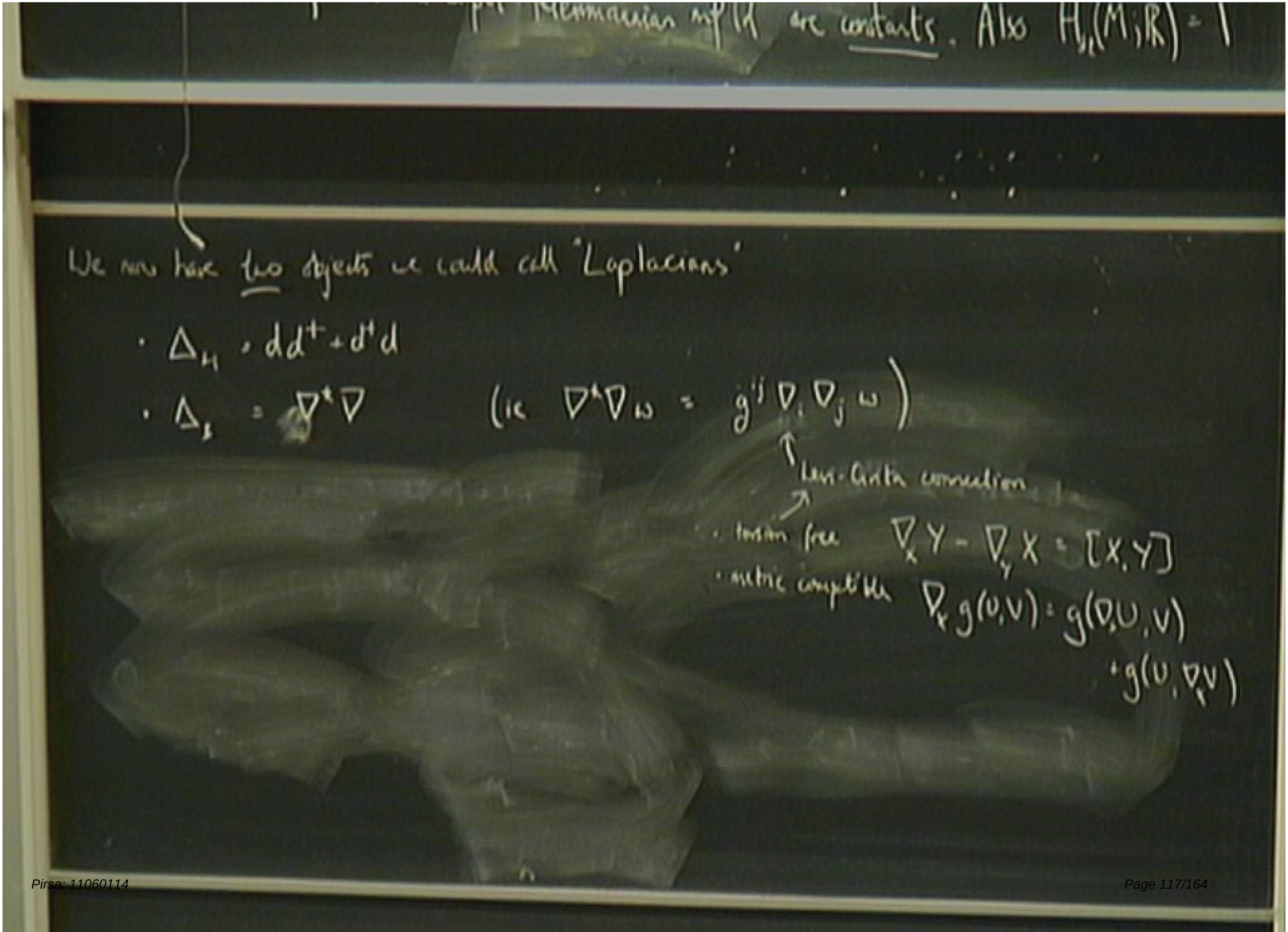
↑ Levi-Civita connection

tors ↗



Riemannian mfld are constants. Also $H_{dR}(M;\mathbb{R}) = $

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$\uparrow$ Levi-Civita connection

- torsion free $\nabla_X Y - \nabla_Y X = [X,Y]$
- metric compatible $\nabla_X g(U,V) = g(\nabla_X U, V) + g(U, \nabla_X V)$

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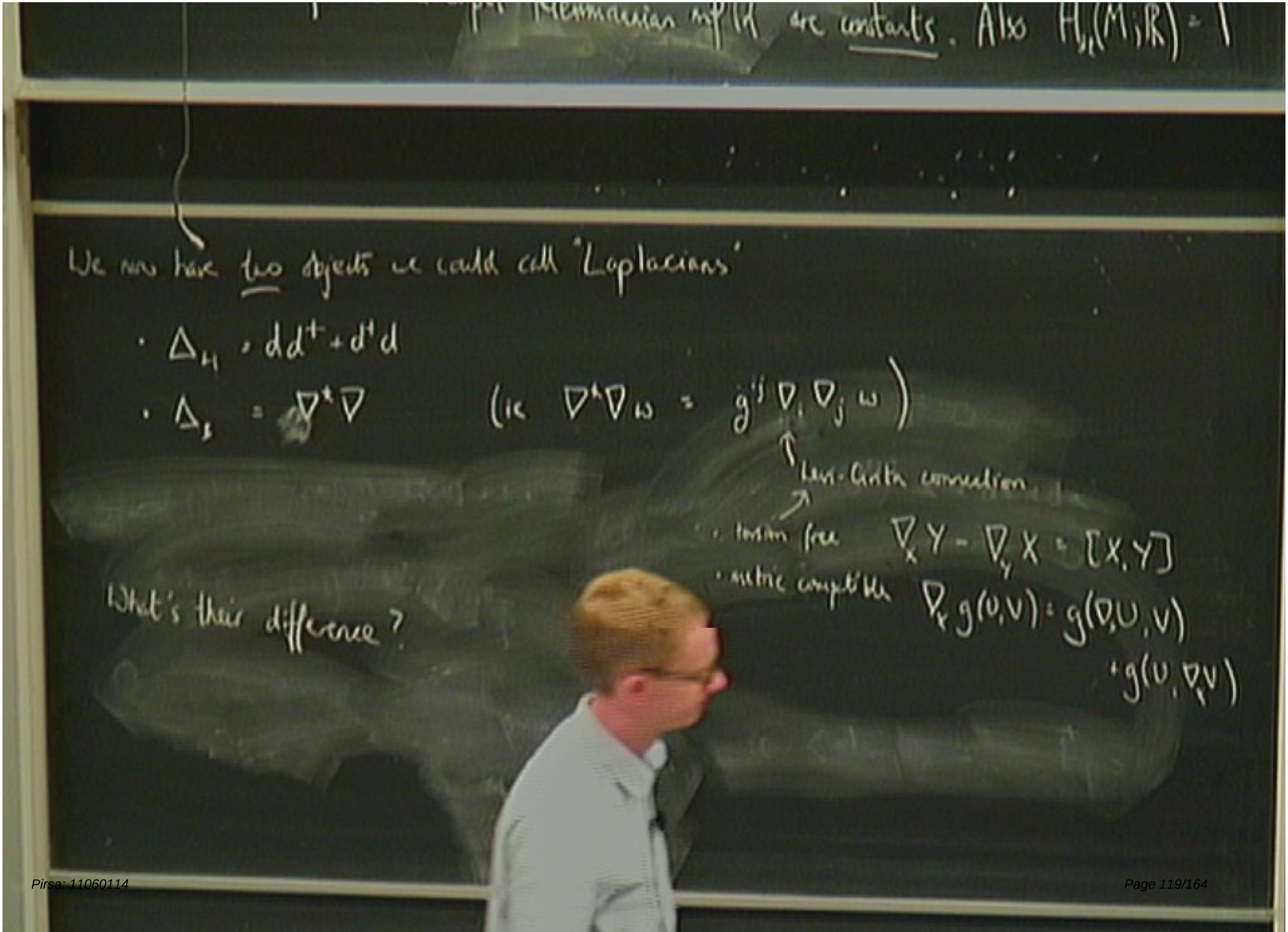

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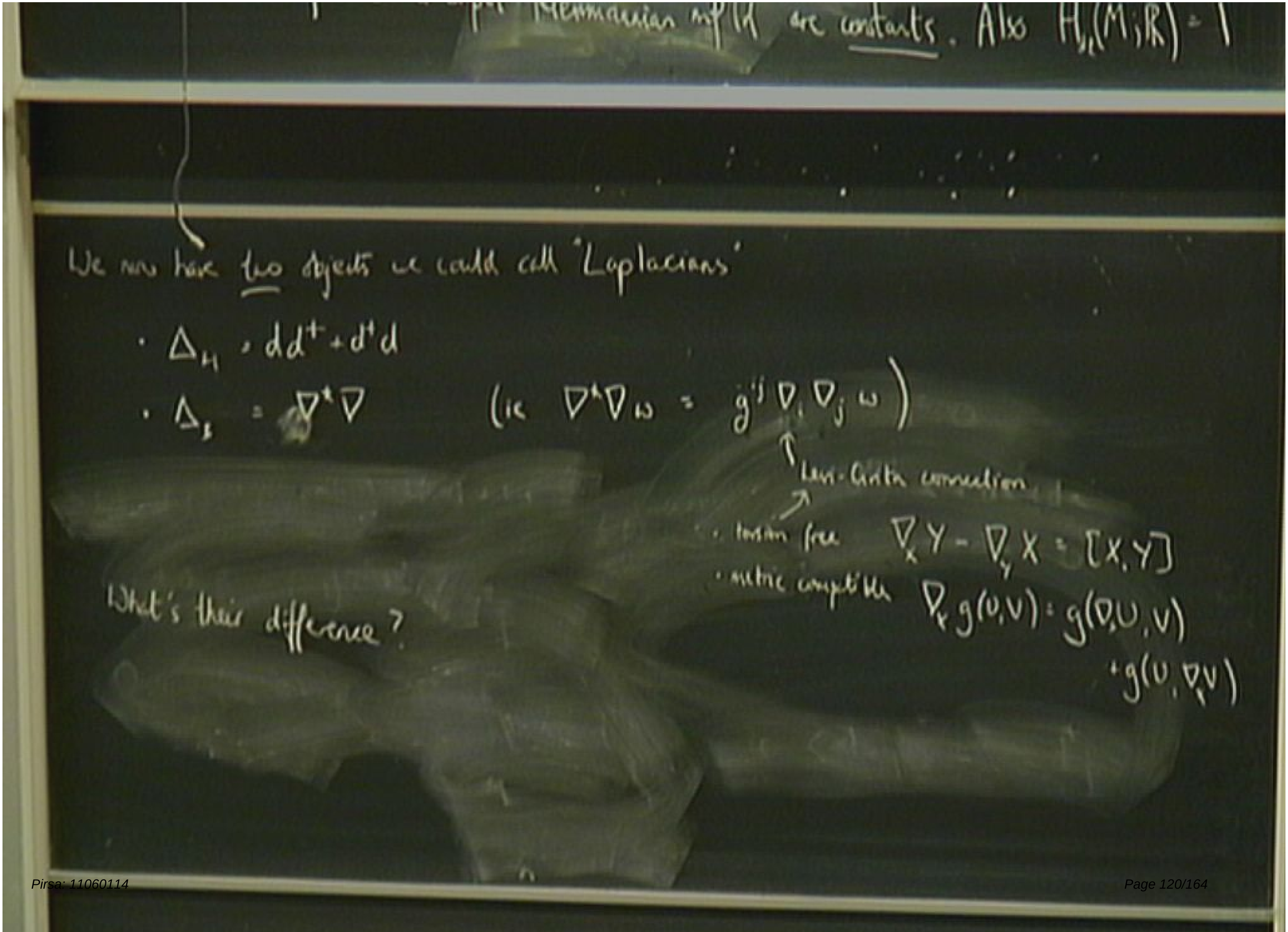
$\nearrow$

- torsion free $\nabla_X Y - \nabla_Y X = [X,Y]$
- metric compatible $\nabla_Y g(U,V) = g(\nabla_Y U, V) + g(U, \nabla_Y V)$

What's their difference?



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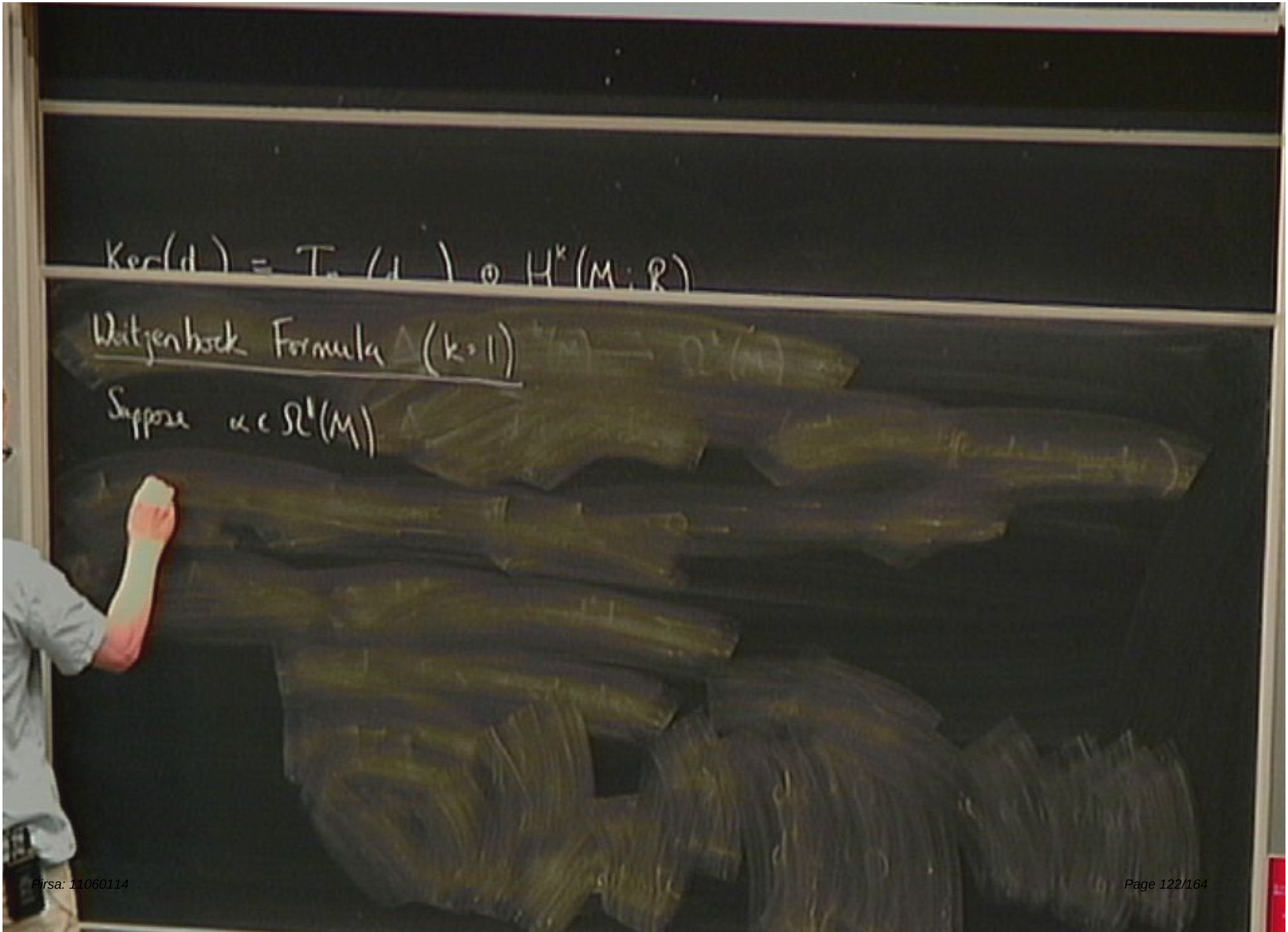
Weitzenbock Formula $\Delta\,(k=1)$



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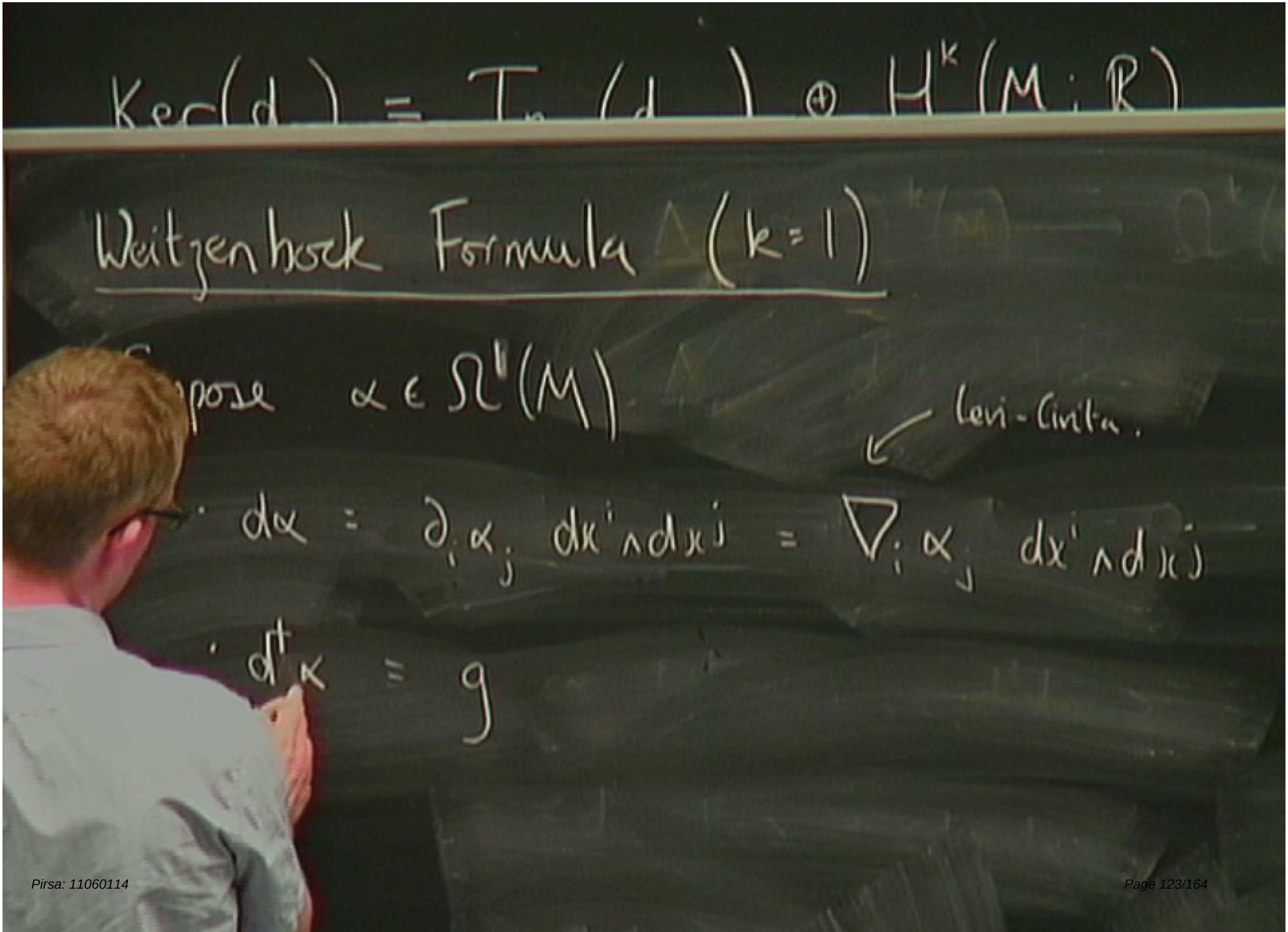
Weitzenbock Formula Δ $(k=1)$

Suppose $\alpha \in \Omega^1(M)$



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$$d\alpha = \partial_i \alpha_j \, dx^i \wedge dx^j = \nabla_i \alpha_j \, dx^i \wedge dx^j$$

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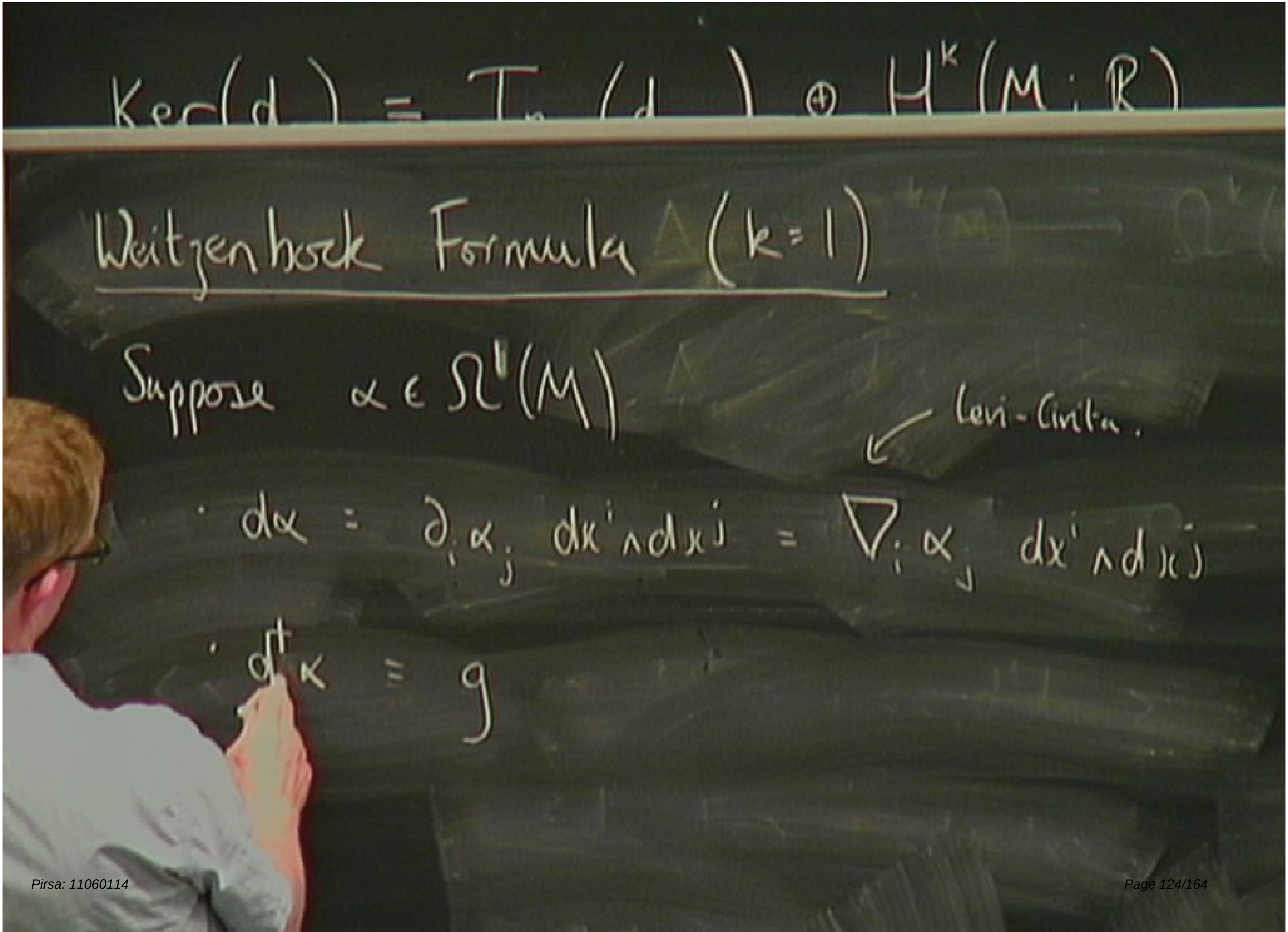

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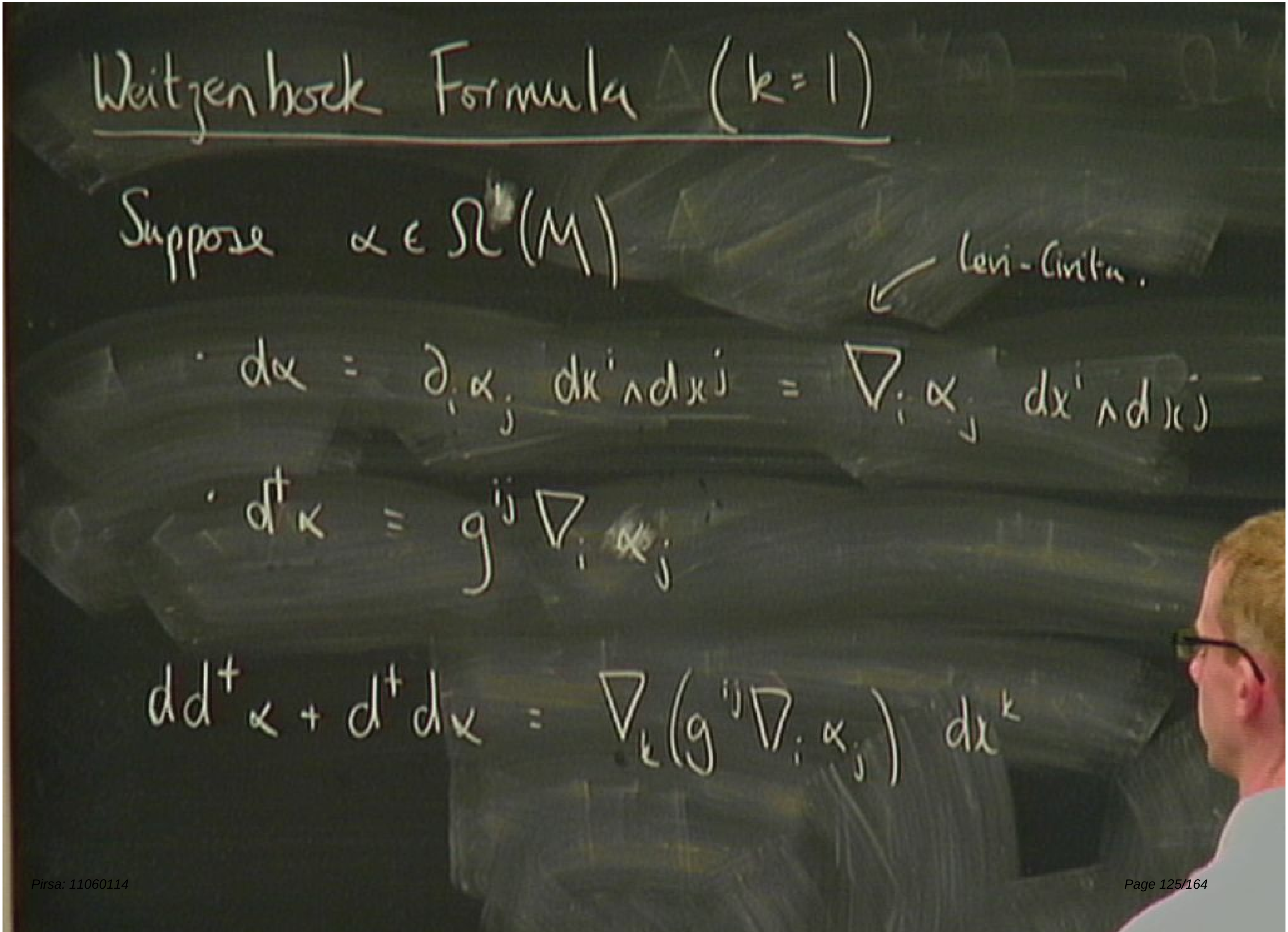

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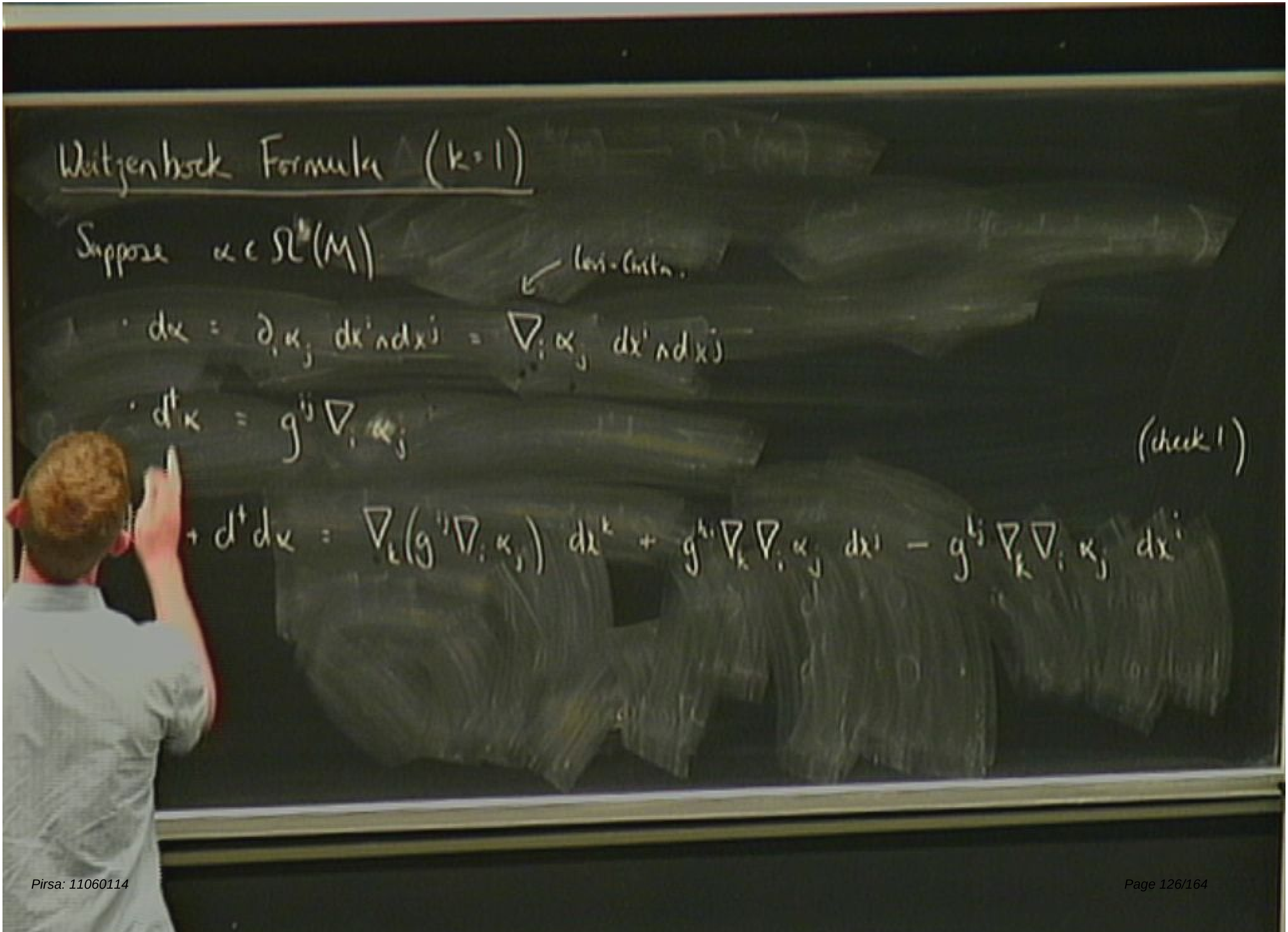

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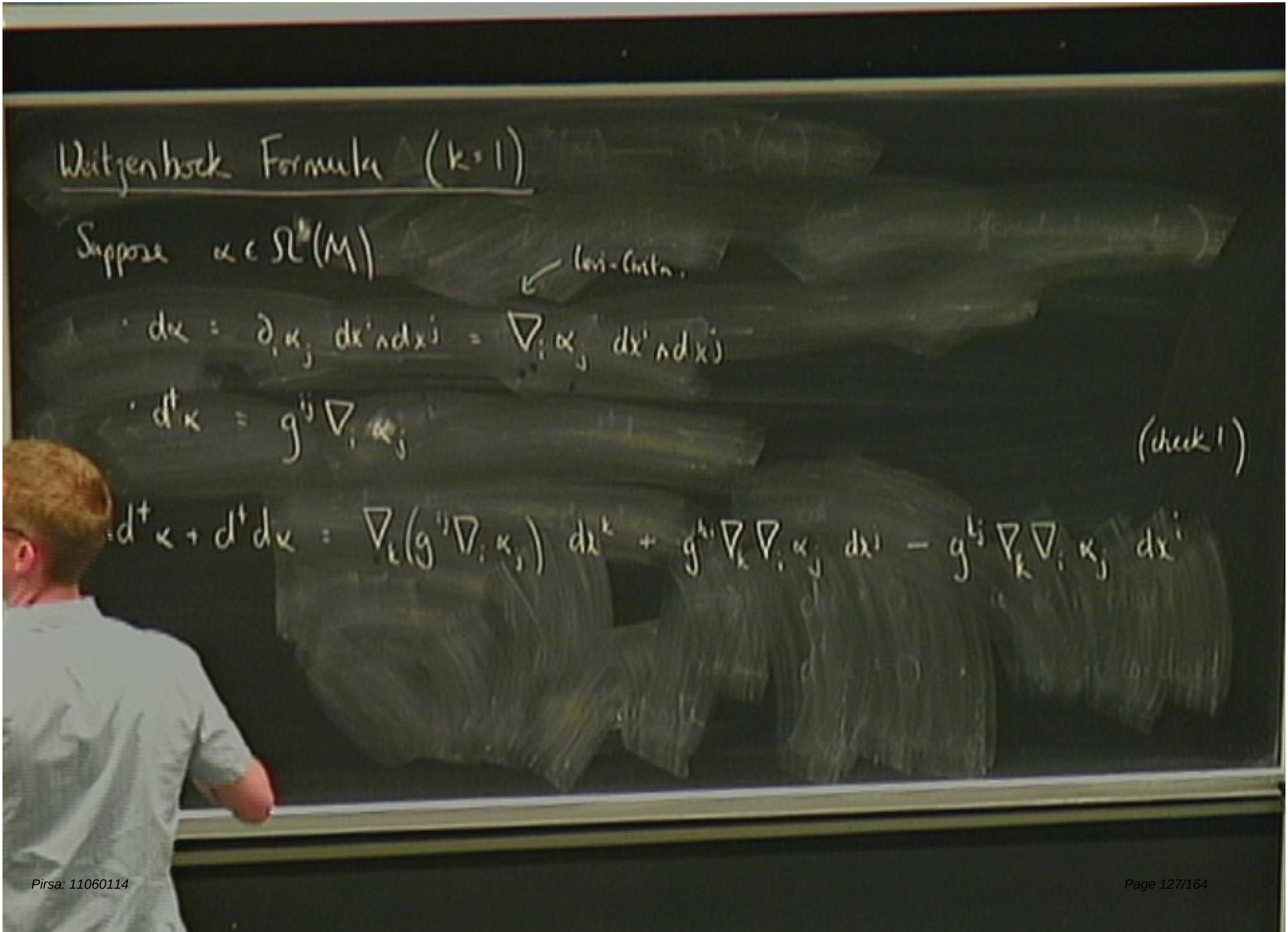
(check!)

$$+ d^\dagger d\alpha = \nabla_k (g^{ij} \nabla_i \alpha_j) \, dx^k + g^{ki} \nabla_k \nabla_i \alpha_j \, dx^j - g^{kj} \nabla_k \nabla_i \alpha_j \, dx^i$$



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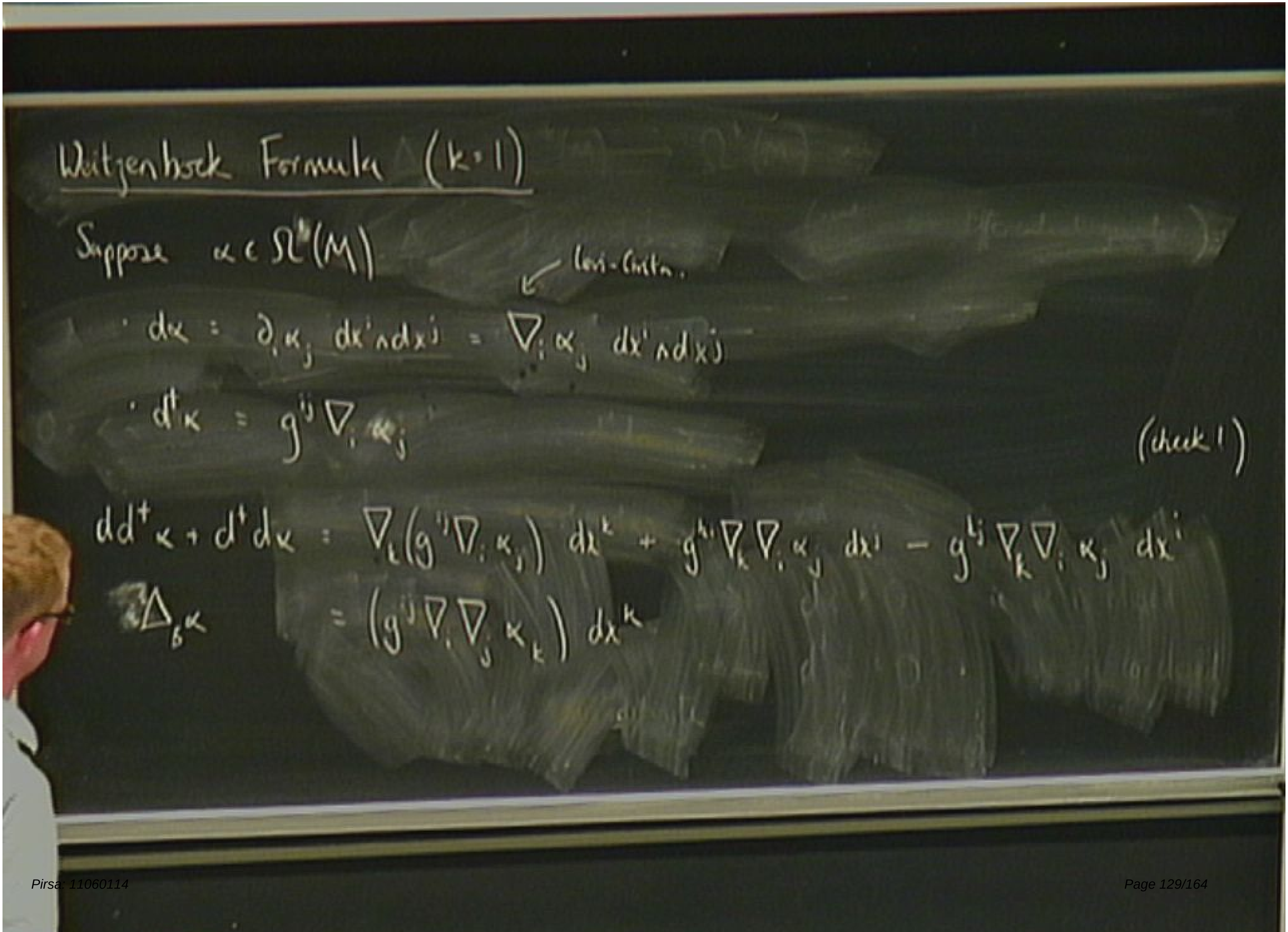

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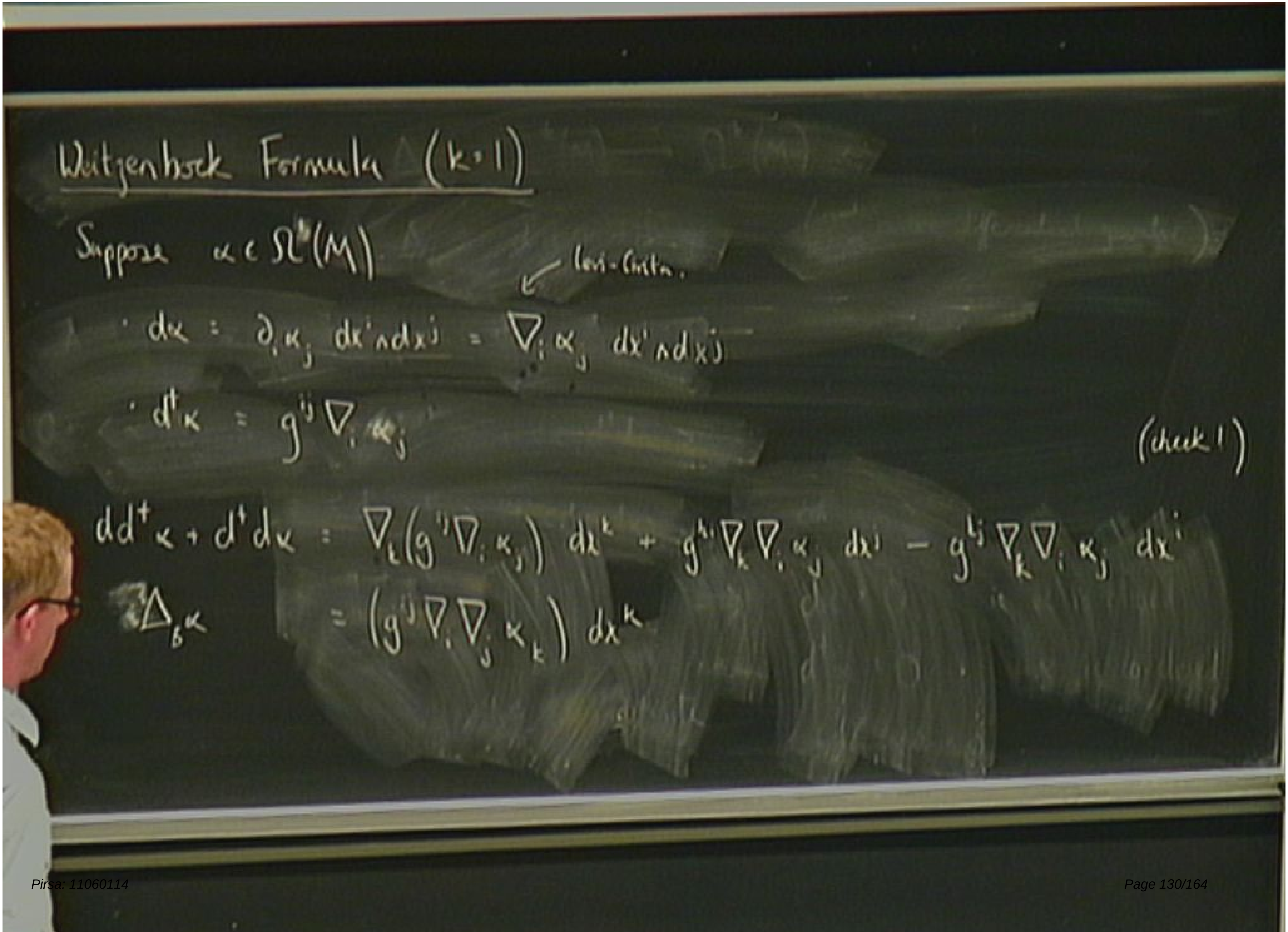

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$$\Delta_H \alpha - \Delta_B \alpha = g^{ij} (\nabla_k \nabla_i - \nabla_i \nabla_k) \alpha_j \, dx^k$$

j j $dx = 0$ and $d\alpha = 0$

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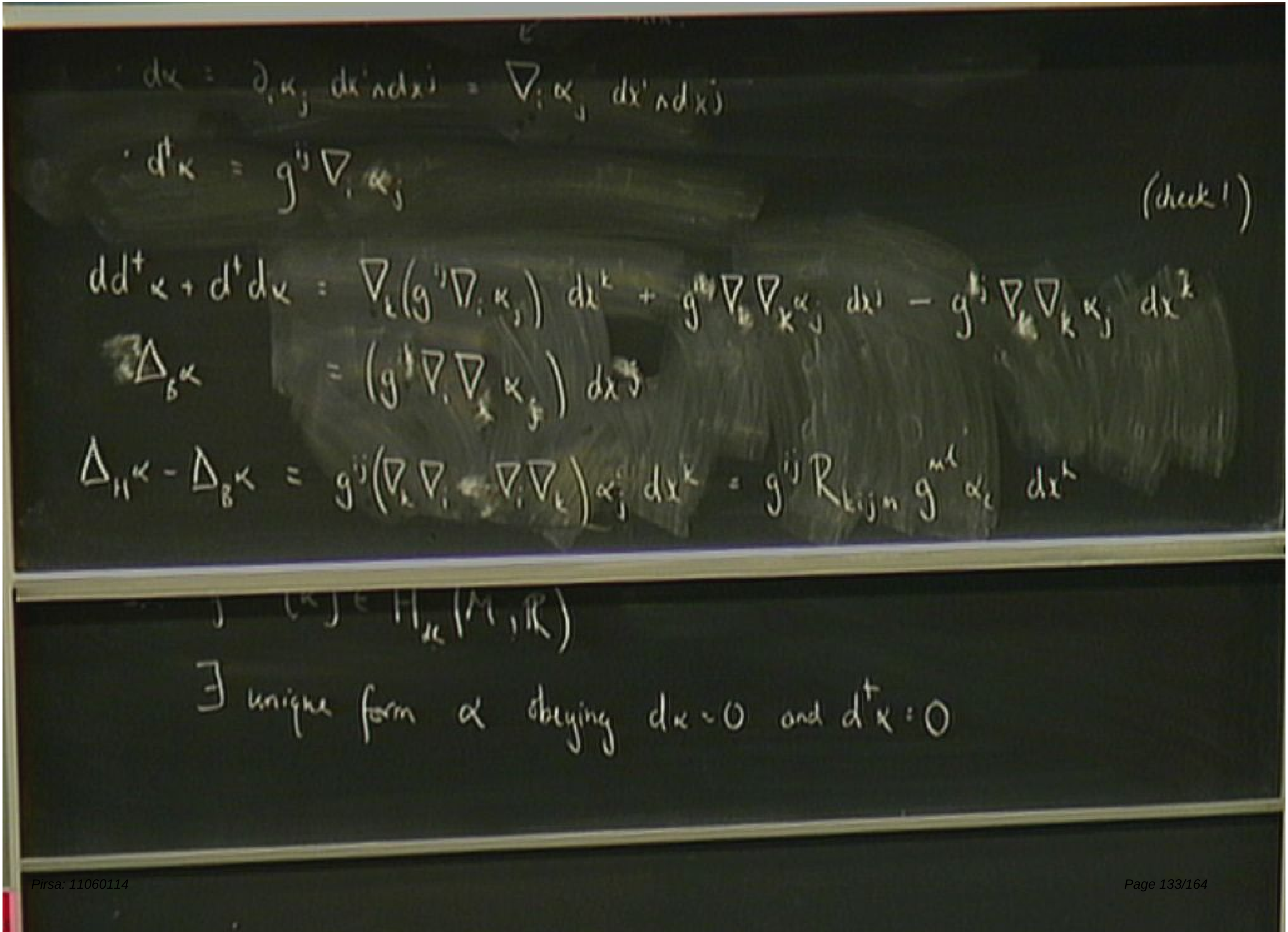
$$\Delta_H \alpha - \Delta \qquad g^{ij} (\nabla_k \nabla_i - \nabla_i \nabla_k) \alpha_j \, dx^k = g^{ij} R_{kijm} \, g^{ml} \alpha_l \, dx^k$$

form α obeying $d\alpha = 0$ and $d^\dagger \alpha = 0$



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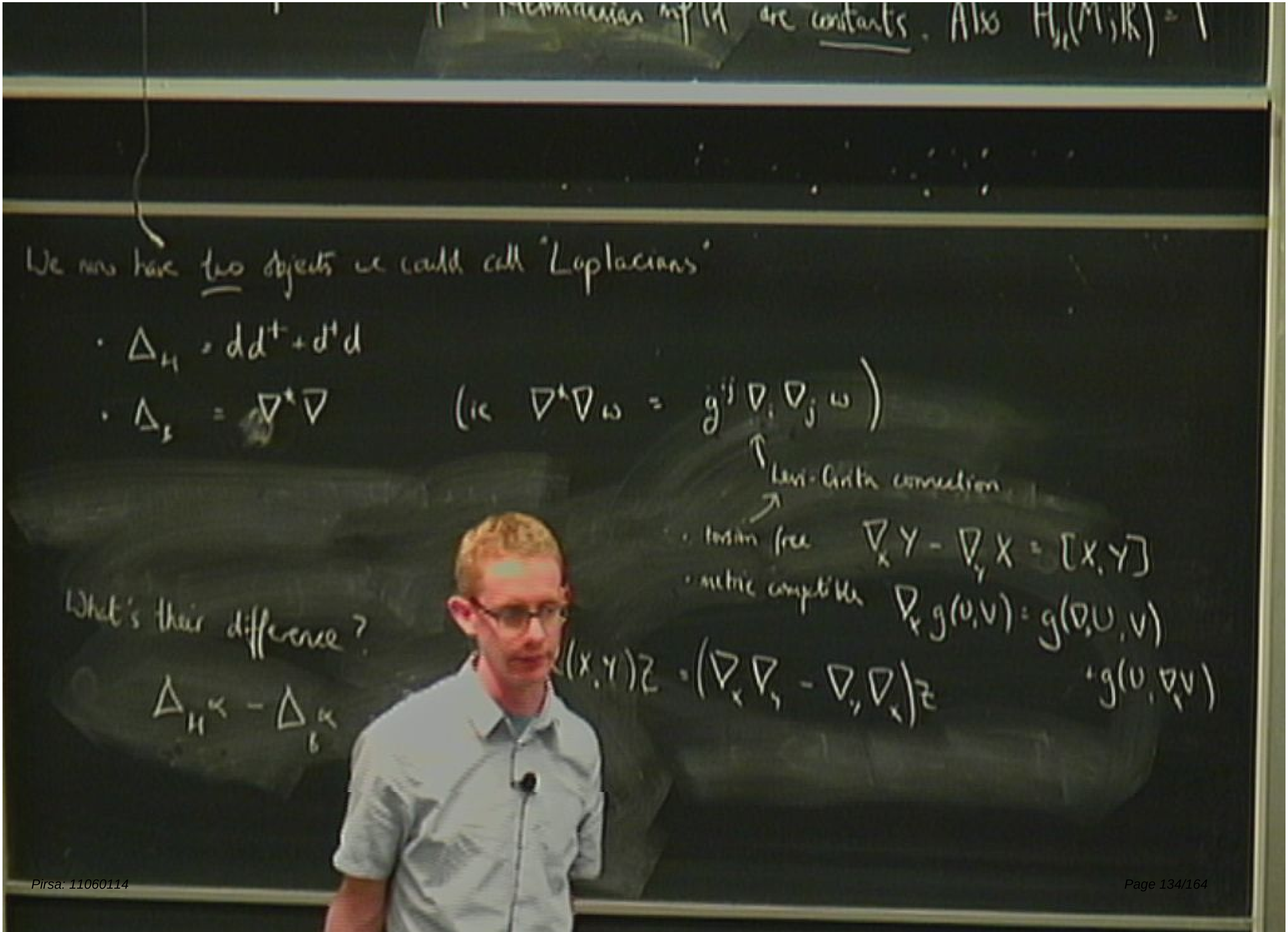
$[\alpha] \in H_{dR}(M, \mathbb{R})$

$\exists$ unique form α obeying $d\alpha = 0$ and $d^\dagger \alpha = 0$



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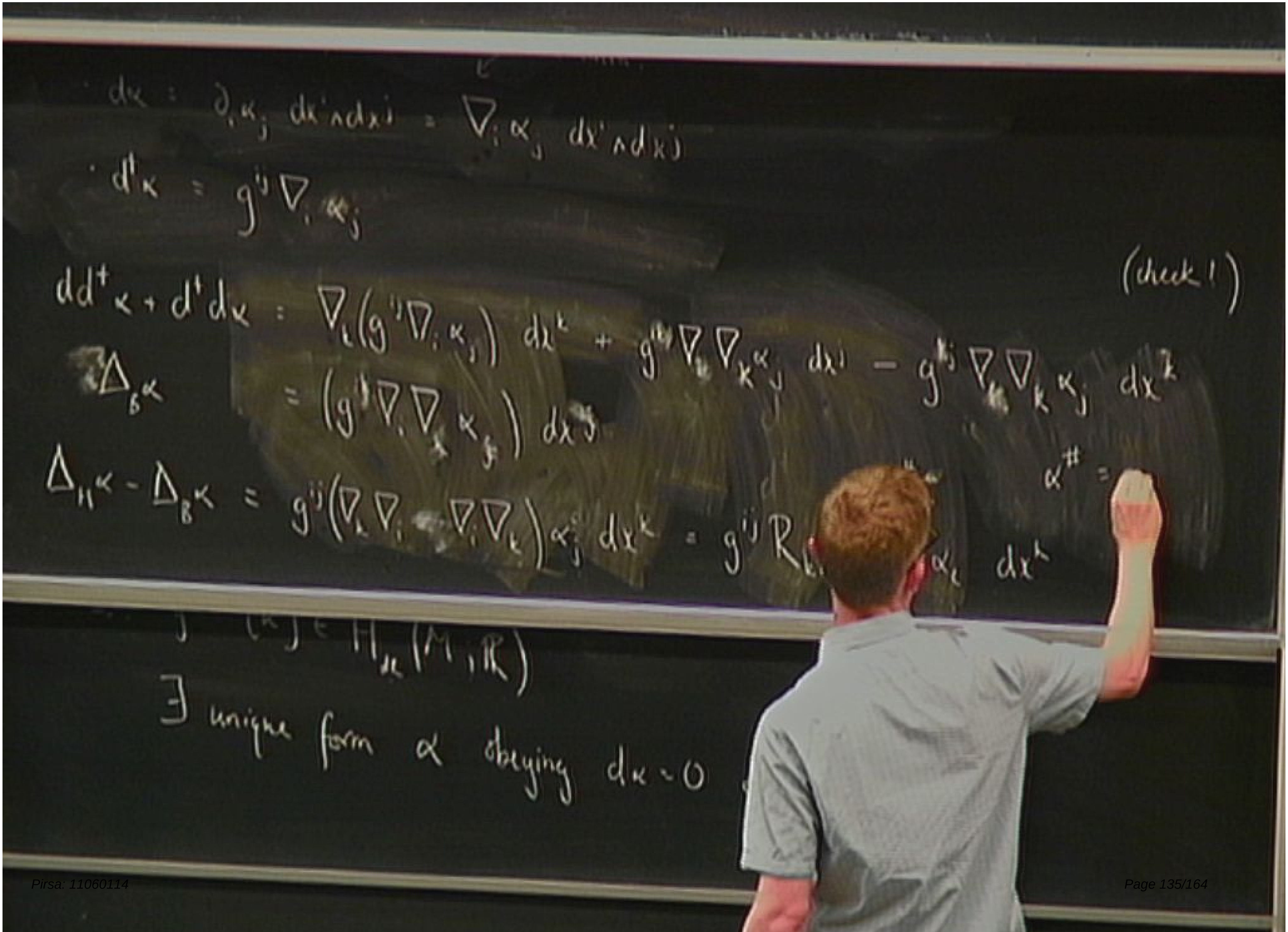
What's their difference?

$\Delta_H\alpha - \Delta_B\alpha$

$(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X)Z$



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$\alpha^\# =$

$[\alpha] \in H_{dR}(M, \mathbb{R})$

$\exists$ unique form α obeying $d\alpha = 0$

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(check!)

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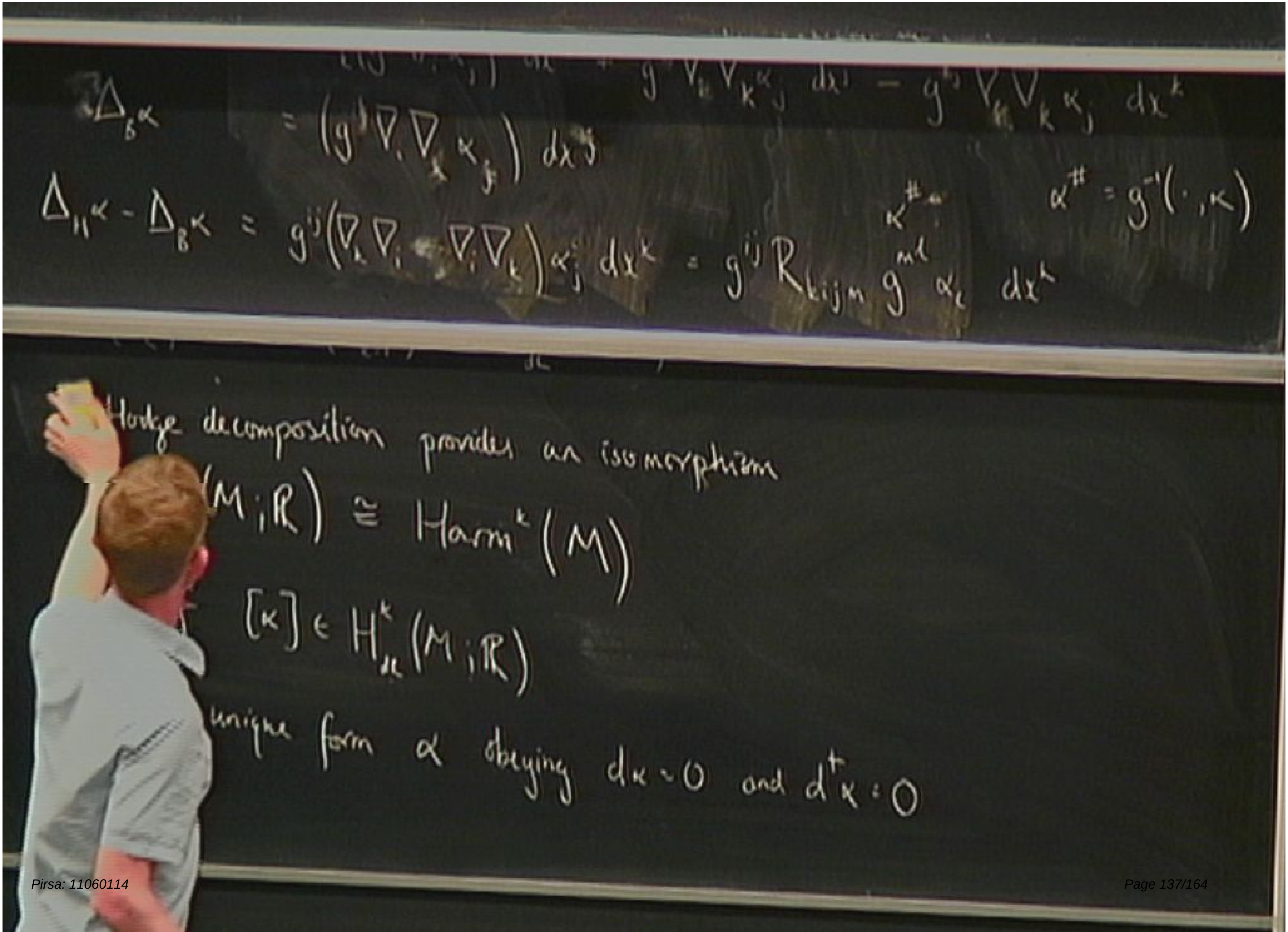
$\Delta_B\alpha = (g^{ij}\nabla_i\nabla_k\alpha_j)\, dx^j$

$\alpha^\# = g^{-1}(\cdot, \alpha)$

$\Delta_H\alpha - \Delta_B\alpha = g^{ij}(\nabla_k\nabla_i - \nabla_i\nabla_k)\alpha_j \, dx^k = g^{ij}R_{kijm}\underbrace{g^{m\ell}\alpha_\ell}_{}\, dx^k$ $\quad \alpha^{\#}_{i}$

$[\alpha] \in H_{dR}(M, \mathbb{R})$

$\exists$ unique form α obeying $d\alpha = 0$ and $d^\dagger\alpha = 0$



$g^{kj} \nabla_k \nabla_k \alpha_j \, dx^j - g^{kj} \nabla_k \nabla_k \alpha_j \, dx^k$

$$\Delta_B \alpha = \left(g^{ij} \nabla_i \nabla_k \alpha_j \right) dx^j$$

$\alpha^\#$ $\quad \alpha^\# = g^{-1}(\cdot, \alpha)$

$$\Delta_H \alpha - \Delta_B \alpha = g^{ij} \left(\nabla_k \nabla_i - \nabla_i \nabla_k \right) \alpha_j \, dx^k = g^{ij} R_{kijm} \, g^{ml} \alpha_l \, dx^k$$

Hodge decomposition provides an isomorphism

$$(M;\mathbb{R}) \cong \mathrm{Harm}^k(M)$$

$$[\alpha] \in H^k_{dR}(M;\mathbb{R})$$

unique form α obeying $d\alpha = 0$ and $d^\dagger \alpha = 0$

$$\Delta_B \alpha = \left(g^{ij} \nabla_i \nabla_k \alpha_j\right) dx^j$$

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$$\Delta_H \alpha - \Delta_B \alpha$$

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$$\Delta_H\alpha - \Delta_B\alpha = Ric(\cdot,\alpha^\#)$$

$$Ric \in Sym^2\,T^*M$$

$$Ric_{ij} = g^{k\ell}R_{ik\ell j}$$

$$\Delta_B\alpha = \left(g^{ij}\nabla_i\nabla_k\alpha_j\right)dx^j$$

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$$\alpha^{\#}$$

$$\alpha^{\#} = g^{-1}(\cdot,\alpha)$$

$$\Delta_H\alpha = \Delta_B\alpha \quad Ric(\cdot,\alpha^{\#})$$

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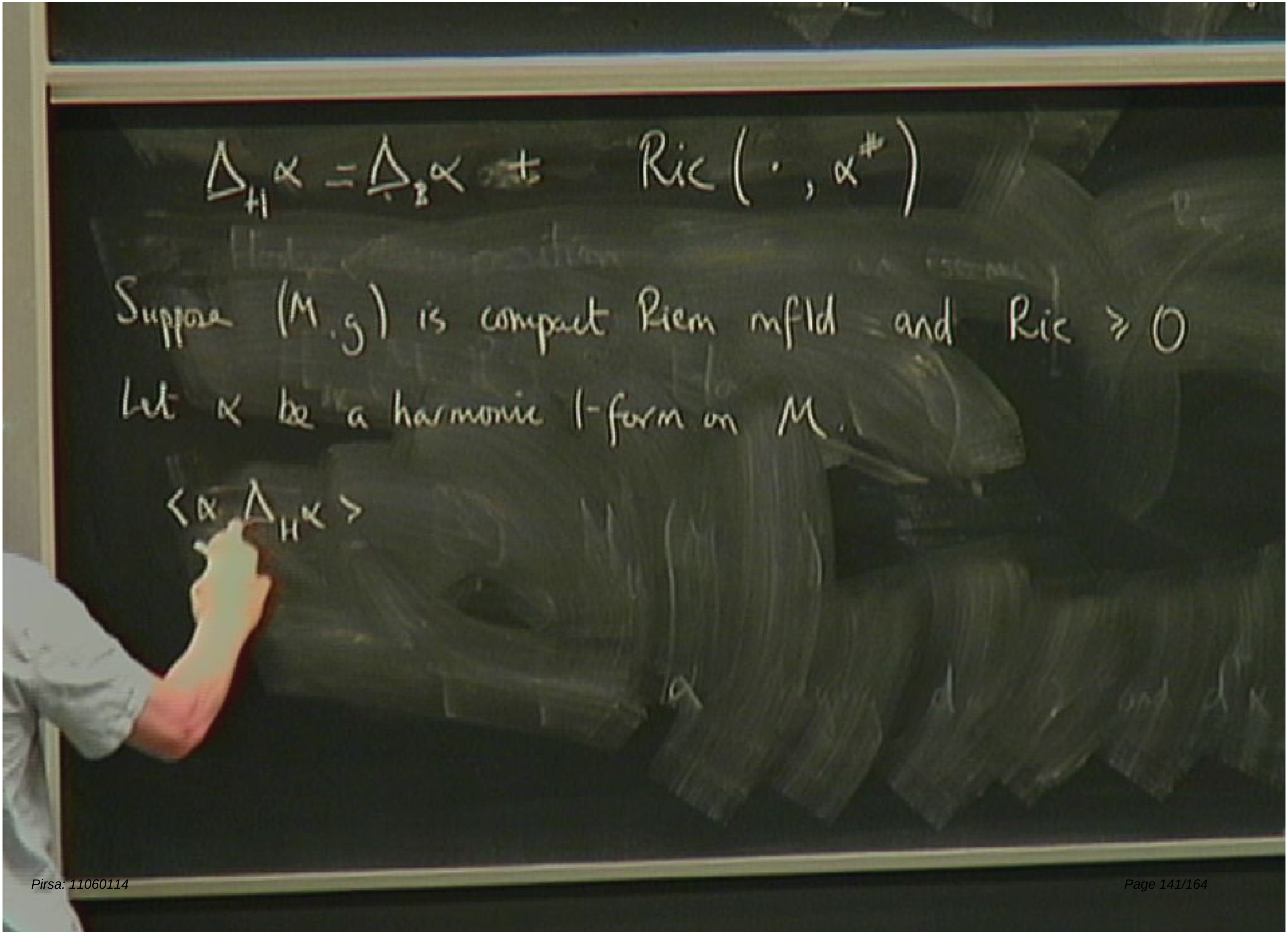
$$Ric_{ij} = g^{kl}R_{iklj}$$

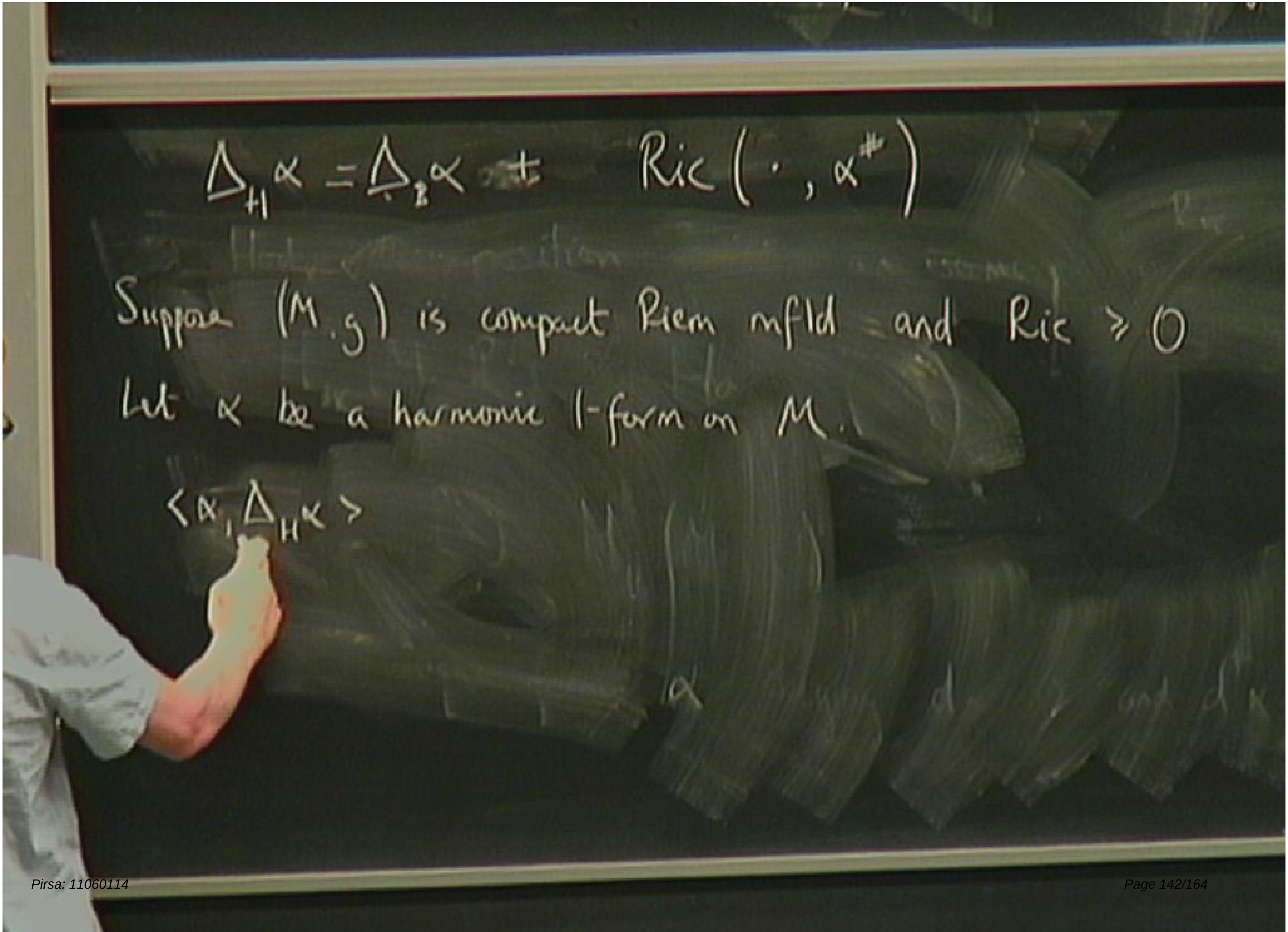


$$\Delta_H \alpha = \Delta_B \alpha + \mathrm{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α be a harmonic 1-form on M.

$$\langle \alpha \ \Delta_H \alpha \rangle$$



$$\Delta_H \alpha = \Delta_B \alpha + Ric(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $Ric \geq 0$

Let α be a harmonic 1-form on M.

$$\langle \alpha, \Delta_H \alpha \rangle$$

$$\Delta_H \alpha = \Delta_B \alpha \pm \mathrm{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geqslant 0$

Let α be a harmonic 1-form on M.

$$\langle \alpha, \Delta_H \alpha \rangle$$

$$\Delta_H \alpha = \Delta_B \alpha + \mathrm{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α be a harmonic 1-form on M.

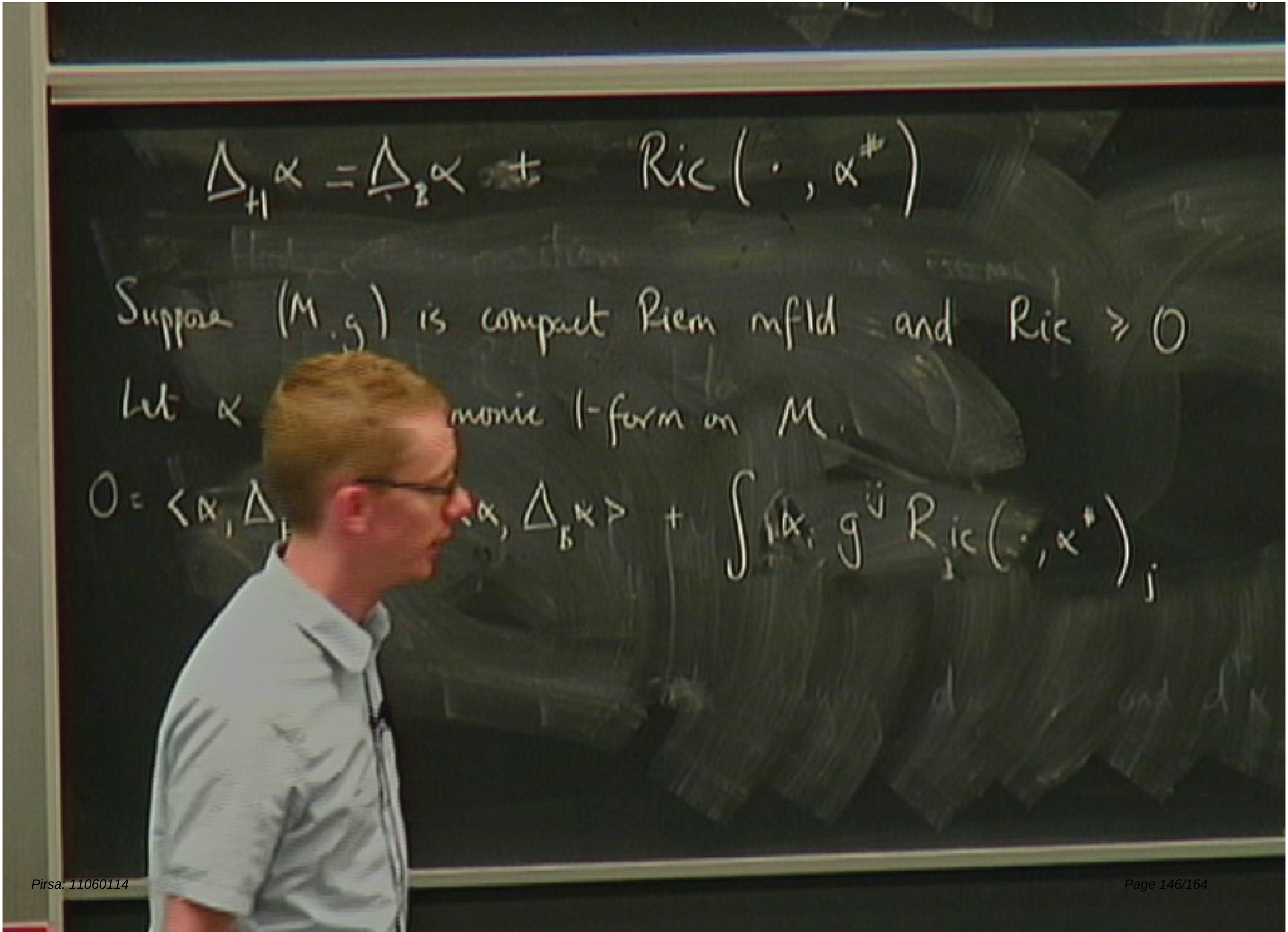
$0 = \langle \alpha, \Delta_H \alpha \rangle$

$$\Delta_H \alpha = \Delta_B \alpha \pm \mathrm{Ric}(\cdot, \alpha^{\#})$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int$$

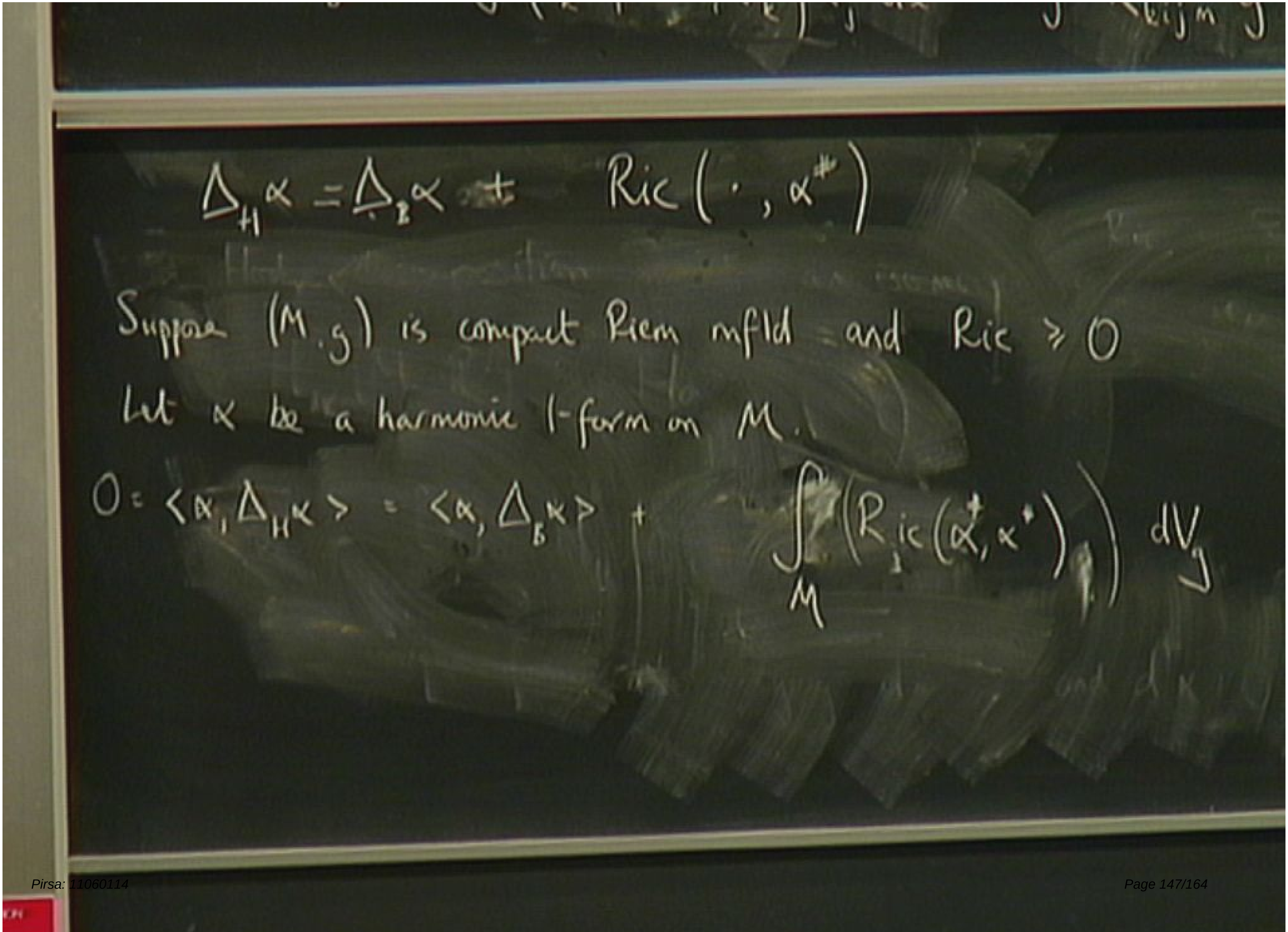


$$\Delta_H \alpha = \Delta_B \alpha \pm \mathrm{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α [illegible] monic 1-form on M.

$0 = \langle \alpha, \Delta$ [illegible] $\alpha, \Delta_B \alpha \rangle + \int \alpha_i \, g^{ij} \, R_i ic(\cdot, \alpha^\#)_j$



$$\Delta_H \alpha = \Delta_B \alpha + \operatorname{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\operatorname{Ric} \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int_M \left(\operatorname{Ric}(\alpha^\#, \alpha^\#) \right) dV_g$$

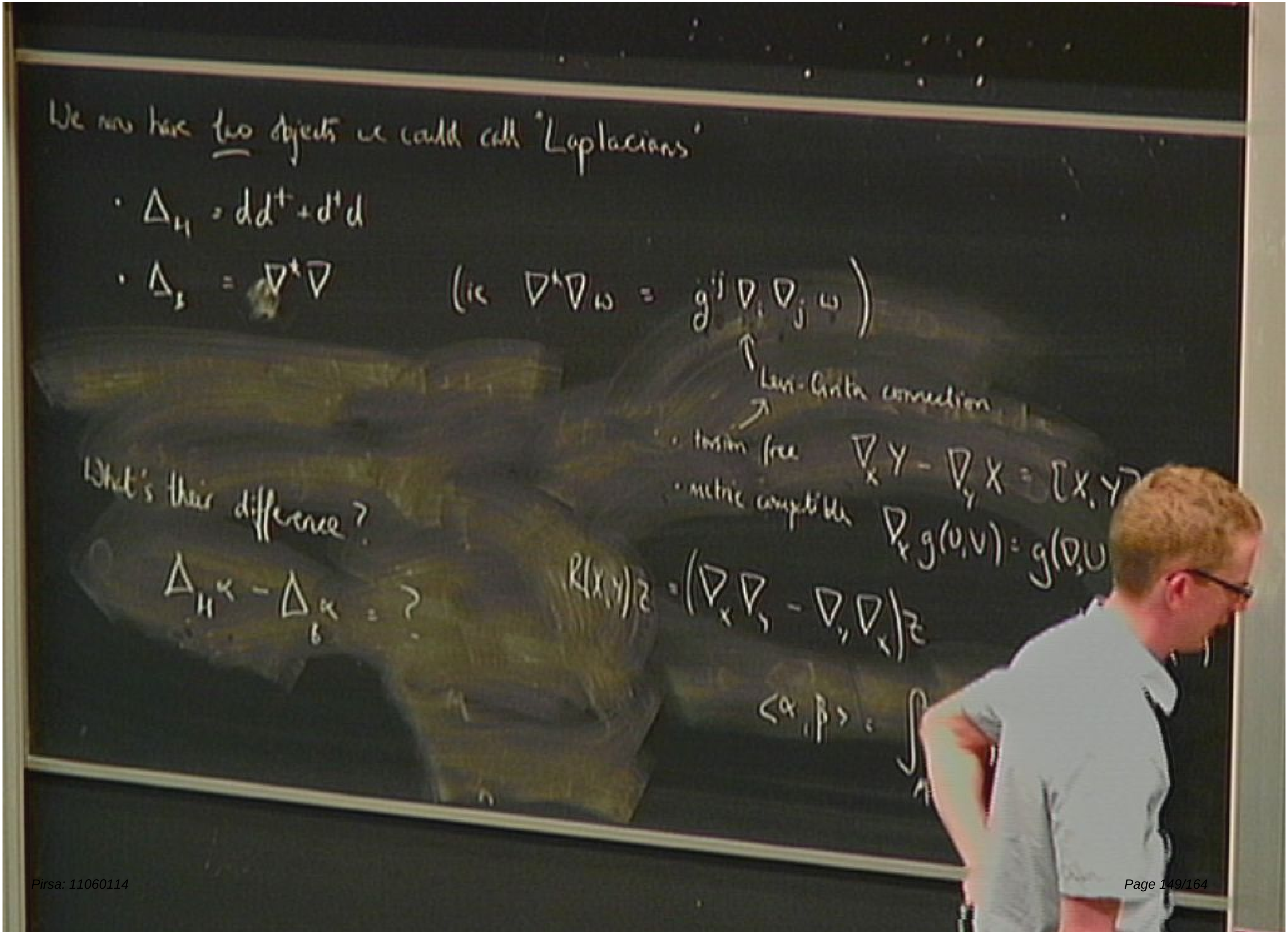
$$\Delta_H \alpha = \Delta_B \alpha \pm Ric(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $Ric \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int_M \left(Ric_g(\alpha^\#, \alpha^\#) \right) dV_g$$

$$= \int_M (\nabla \alpha, \nabla \alpha) \, dV_g + \int_M Ric(\alpha^\#, \alpha^\#) \, dV_g$$



We now have <u>two</u> objects we could call 'Laplacians'

- $\Delta_H = dd^\dagger + d^\dagger d$
- $\Delta_B = \nabla^\dagger \nabla$ (ie $\nabla^\dagger \nabla \omega = g^{ij} \nabla_i \nabla_j \omega$)

$\uparrow$ Levi-Civita connection

- torsion free $\nabla_X Y - \nabla_Y X = [X, Y]$
- metric compatible $\nabla_X g(U,V) = g(\nabla_X U$

What's their difference?

$$\Delta_H \alpha - \Delta_B \alpha = ?$$

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z$$

$$\langle \alpha , \beta \rangle = \int_M$$

$$\Delta_H \alpha = \Delta_B \alpha + \mathrm{Ric}(\cdot\,, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int_M \left(\mathrm{Ric}(\alpha^\#, \alpha^\#) \right) dV_g$$

$$= \int_M (\nabla \alpha, \nabla \alpha)\, dV_g + \int_M \mathrm{Ric}(\alpha^\#, \alpha^\#)\, dV_g$$

$$0 = \|\nabla \alpha\|^2 + \int_M \mathrm{Ric}(\alpha^\#, \alpha^\#)\, dV_g$$

$$\Delta_H \alpha = \Delta_B \alpha + \mathrm{Ric}(\cdot, \alpha^\sharp)$$

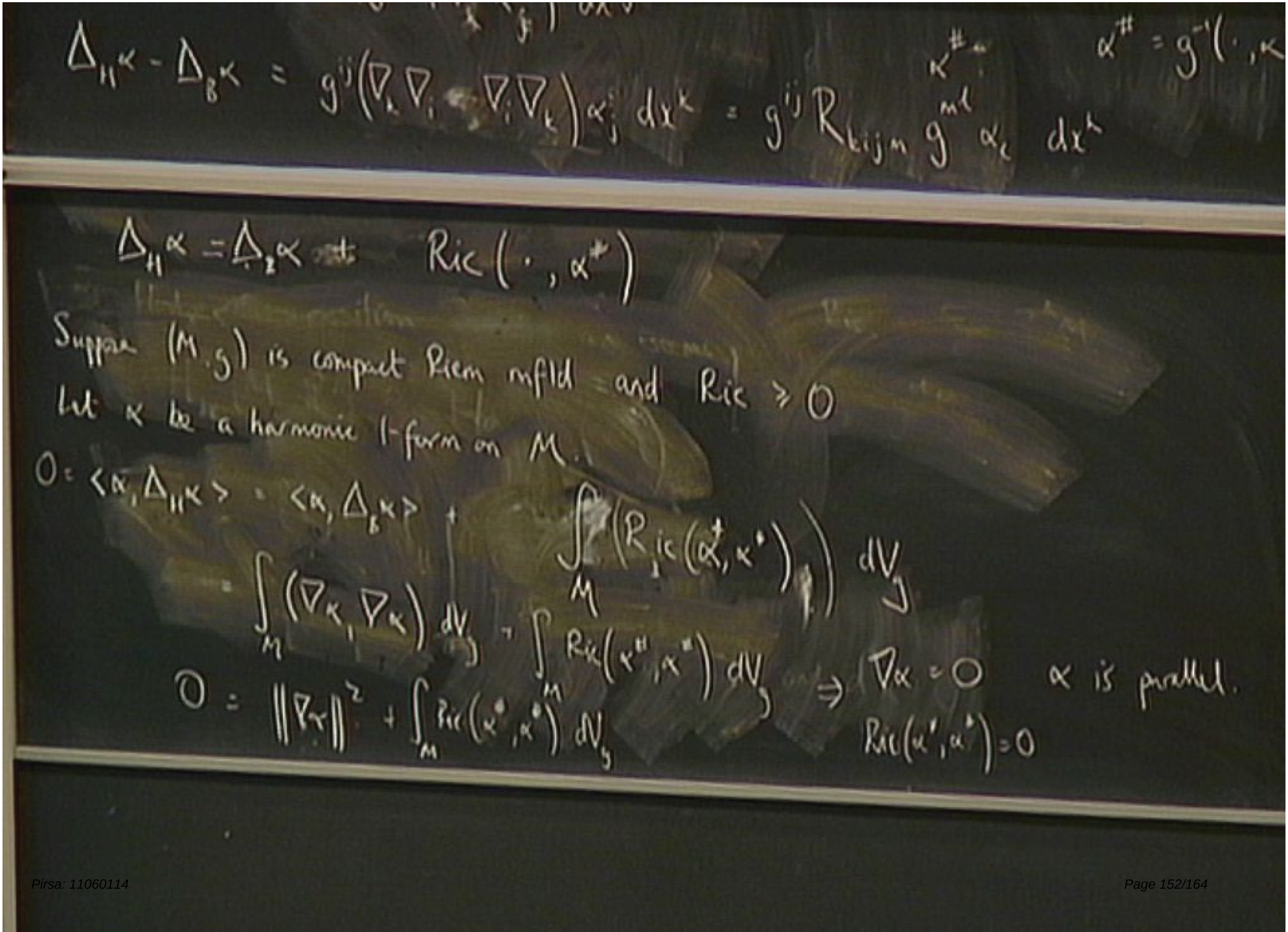
Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int_M \left(R_j ic(\alpha^\sharp, \alpha^\sharp) \right) dV_g$$

$$= \int_M (\nabla \alpha, \nabla \alpha)\, dV_g + \int_M \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp)\, dV_g \Rightarrow \nabla \alpha =$$

$$0 = \|\nabla \alpha\|^2 + \int_M \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp)\, dV_g$$



$$\Delta_H \alpha - \Delta_B \alpha = g^{ij}(\nabla_k \nabla_i - \nabla_i \nabla_k)\alpha_j \, dx^k = g^{ij} R_{kijm} g^{m\ell} \alpha_\ell \, dx^k$$

$\alpha^\# =$ $\alpha^\# = g^{-1}(\cdot, \alpha$

$$\Delta_H \alpha = \Delta_B \alpha + \mathrm{Ric}(\cdot, \alpha^\#)$$

Suppose (M, g) is compact Riem mfld and $\mathrm{Ric} \geq 0$

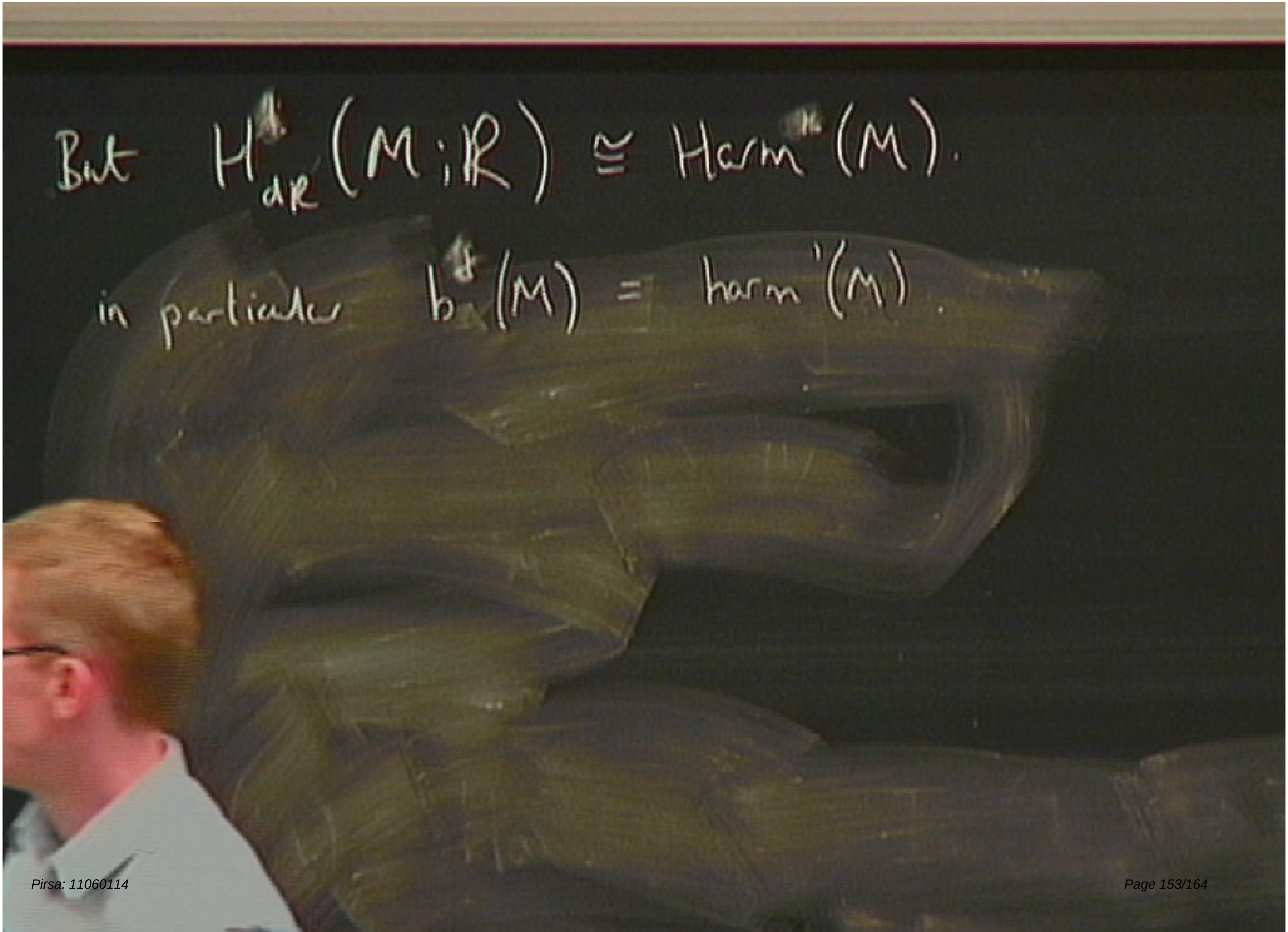
Let α be a harmonic 1-form on M.

$$0 = \langle \alpha, \Delta_H \alpha \rangle = \langle \alpha, \Delta_B \alpha \rangle + \int_M \left(\mathrm{Ric}(\alpha^\#, \alpha^\#)\right) dV_g$$

$$= \int_M (\nabla\alpha, \nabla\alpha) \, dV_g + \int_M \mathrm{Ric}(\alpha^\#, \alpha^\#) \, dV_g$$

$$0 = \|\nabla\alpha\|^2 + \int_M \mathrm{Ric}(\alpha^\#, \alpha^\#) \, dV_g \Rightarrow \nabla\alpha = 0 \quad \alpha \text{ is parallel.}$$

$$\mathrm{Ric}(\alpha^\#, \alpha^\#) = 0$$



But $H^k_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^k(M)$.

in particular $b^k(M) = \mathrm{harm}^k(M)$.

But $H^1_{dR}(M;\mathbb{R}) \cong \text{Harm}^1(M)$.

in particular $b^1(M) = \text{harm}^1(M)$.

so if $\text{Ric} \geq 0$ (& not $\text{Ric} = 0$ everywhere), then

2 self dual

But $H^{*}_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^{*}(M)$.

in particular $b^{1}(M) = \mathrm{harm}^{1}(M)$.

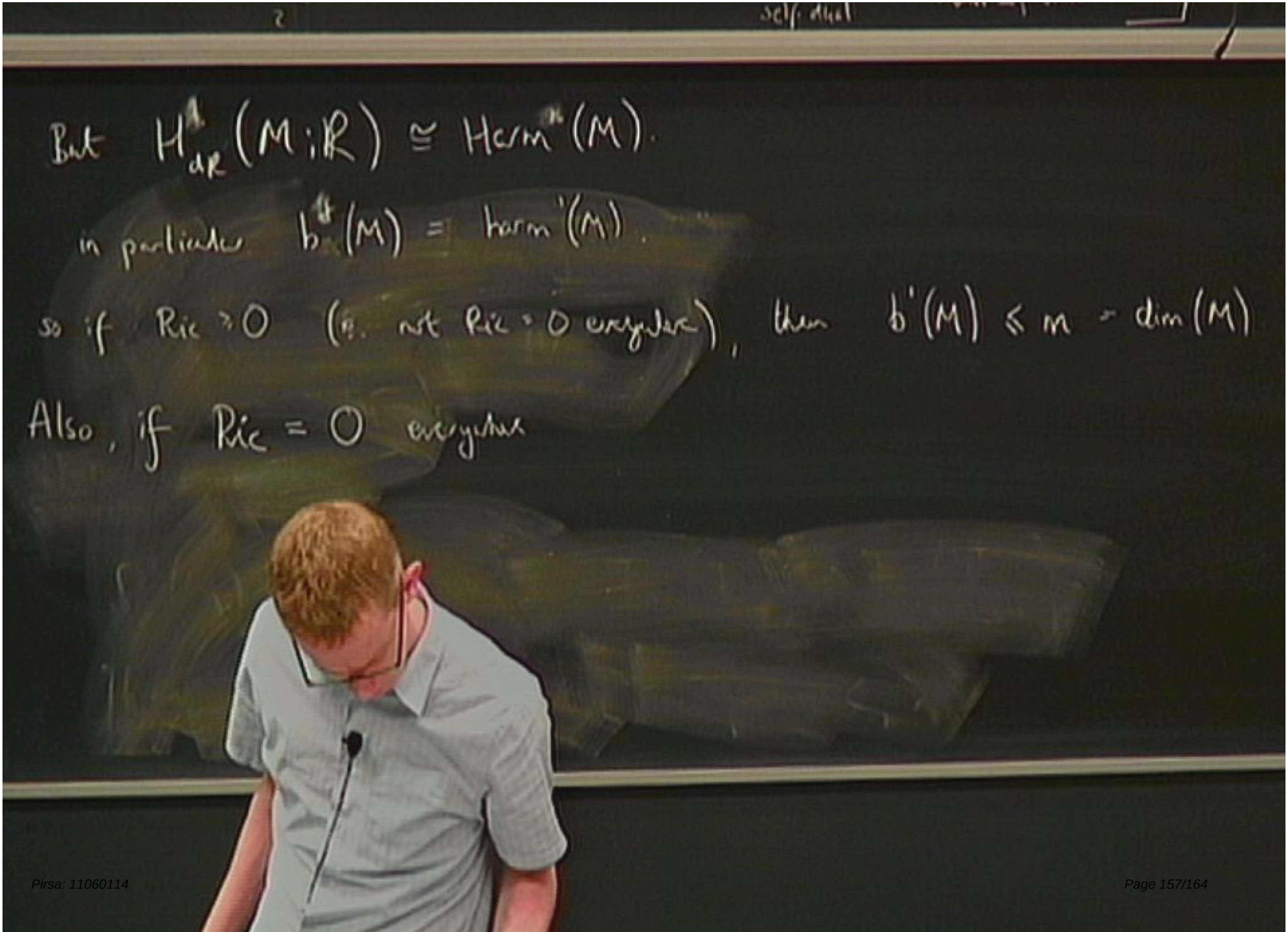
so if $\mathrm{Ric} \geq 0$ (& not $\mathrm{Ric} = 0$ everywhere), then $b^{1}(M) \leq m = \dim(M)$

2 self dual

But $H^*_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^*(M)$.

in particular $b^*(M) = \mathrm{harm}^1(M)$.

so if $\mathrm{Ric} \geq 0$ (i.e. not $\mathrm{Ric} = 0$ everywhere), then $b^1(M) \leq m = \dim(M)$

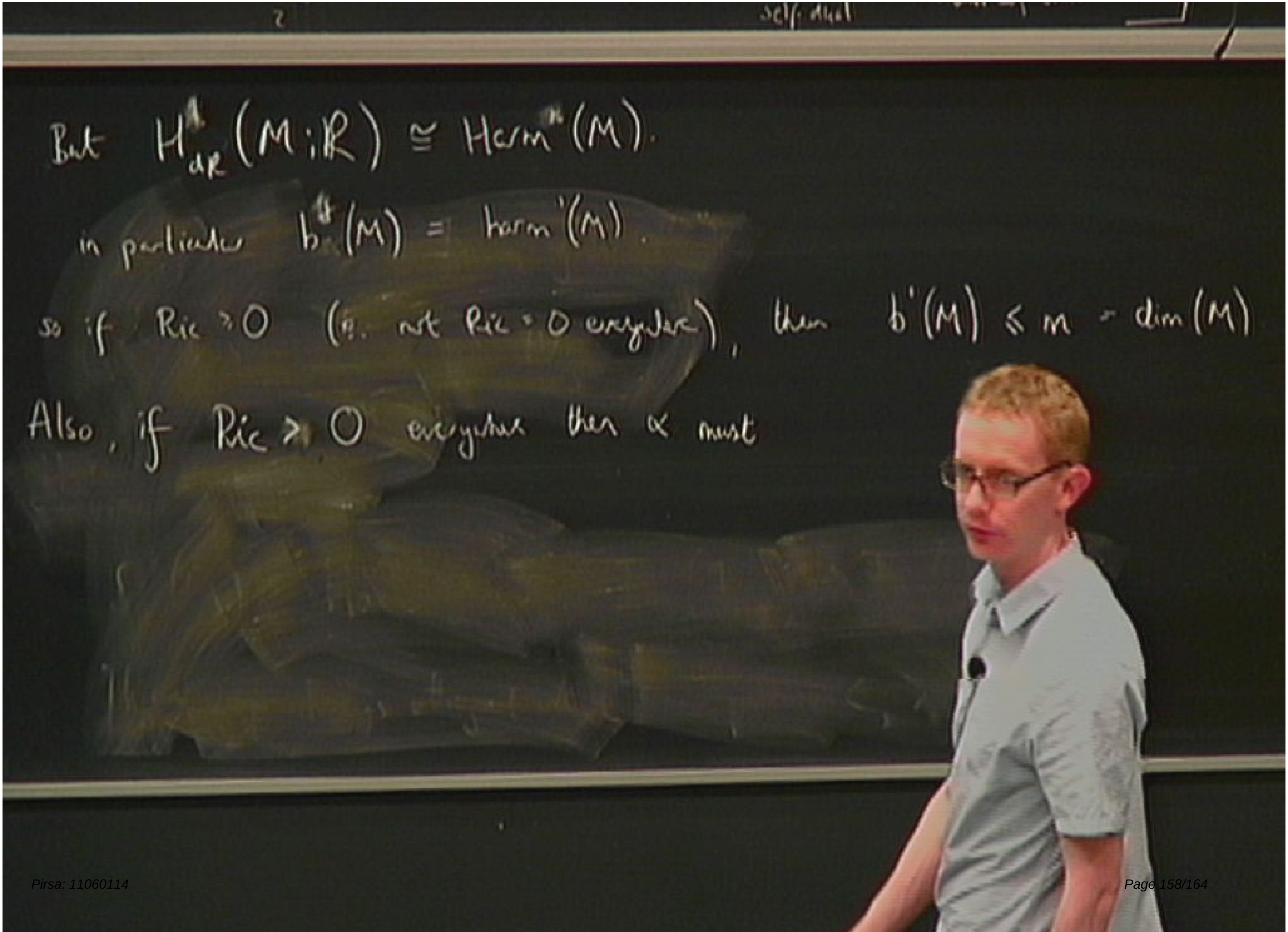


But $H^1_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^1(M)$.

in particular $b^1(M) = \mathrm{harm}^1(M)$.

so if $\mathrm{Ric} \geq 0$ (& not $\mathrm{Ric} = 0$ everywhere), then $b^1(M) \leq n = \dim(M)$

Also, if $\mathrm{Ric} = 0$ everywhere



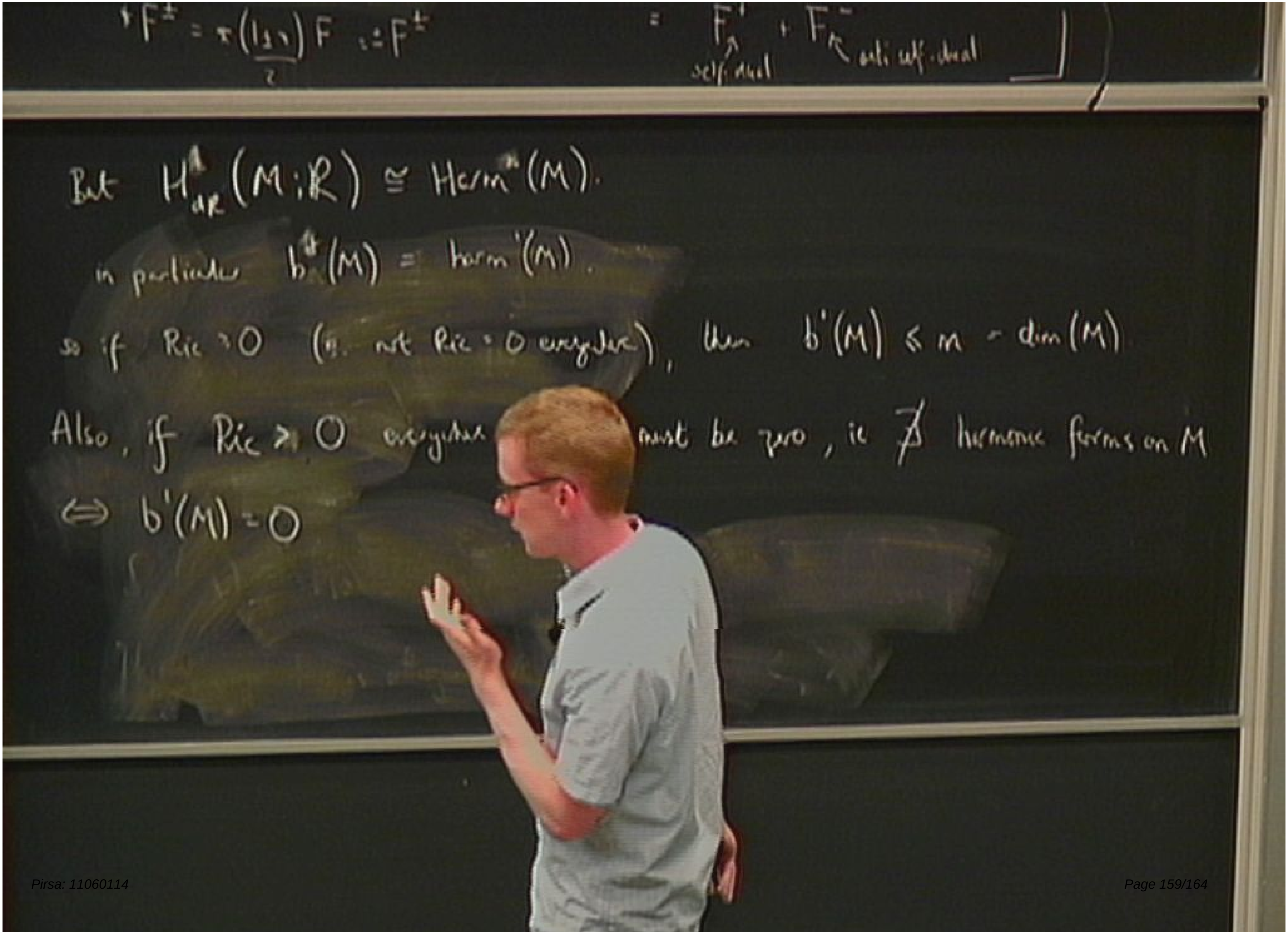
2 self-dual

But $H^1_{dR}(M;\mathbb{R}) \cong Harm^1(M)$.

in particular $b^1(M) = harm^1(M)$.

so if $Ric \geq 0$ (i.e. not $Ric = 0$ everywhere), then $b^1(M) \leq m = \dim(M)$

Also, if $Ric > 0$ everywhere then α must



$*F^{\pm} = \pm\left(\frac{1 \pm *}{2}\right)F =: F^{\pm}$ $= \underset{\text{self-dual}}{F^{+}} + F^{-}$ anti self-dual

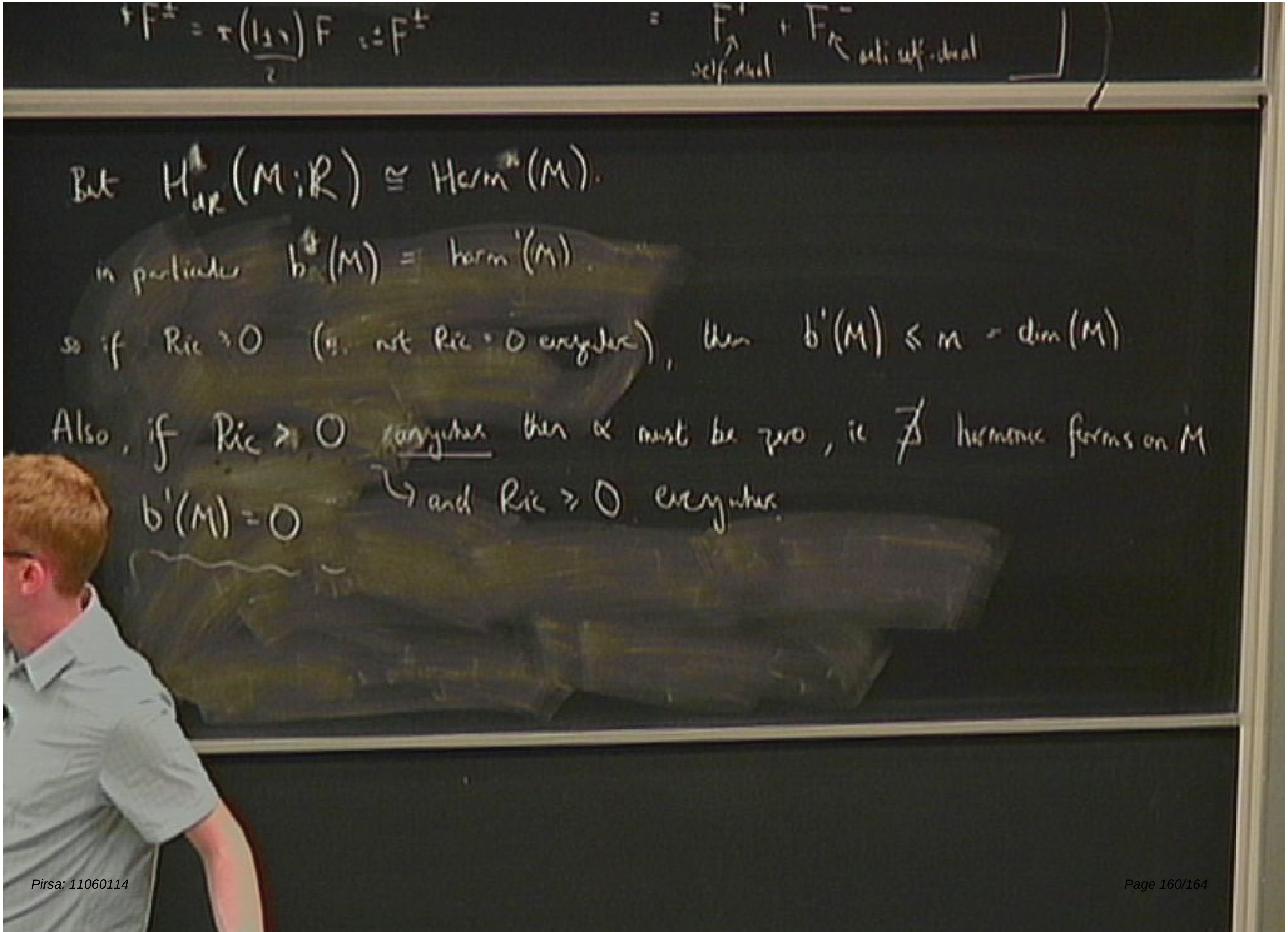
But $H^{*}_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^{*}(M)$.

in particular $b^{1}(M) = \mathrm{harm}^{1}(M)$.

so if $\mathrm{Ric} \geq 0$ (i.e. not $\mathrm{Ric} = 0$ everywhere), then $b^{1}(M) \leq n = \dim(M)$

Also, if $\mathrm{Ric} \geq 0$ everywhere must be zero, ie $\nexists$ harmonic forms on M

$\Leftrightarrow b^{1}(M) = 0$



$*F^\pm = \pm\left(\frac{1\pm *}{2}\right)F =: F^\pm$ $= \underset{\text{self-dual}}{F^+} + \underset{\text{anti self-dual}}{F^-}$

But $H^1_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^1(M)$.

in particular $b^1(M) = \mathrm{harm}^1(M)$.

so if $\mathrm{Ric} \geq 0$ (i.e. not $\mathrm{Ric} > 0$ everywhere), then $b^1(M) \leq n = \dim(M)$

Also, if $\mathrm{Ric} \geq 0$ everywhere then α must be zero, i.e. $\nexists$ harmonic forms on M

↳ and $\mathrm{Ric} > 0$ everywhere

$\underset{\sim\sim\sim}{b^1(M) = 0}$

$*F^\pm = *\left(\frac{1 \pm *}{2}\right)F = \pm F^\pm$ $\quad = \underset{\text{self-dual}}{F^+} + \underset{\text{anti self-dual}}{F^-}$

But $H^1_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^1(M)$.

in particular $b^1(M) = \mathrm{harm}^1(M)$.

so if $\mathrm{Ric} \geq 0$ (i.e. not $\mathrm{Ric} = 0$ everywhere), then $b^1(M) \leq m = \dim(M)$

Also, if $\mathrm{Ric} \geq 0$ everywhere then α must be zero, i.e. $\nexists$ harmonic forms on M

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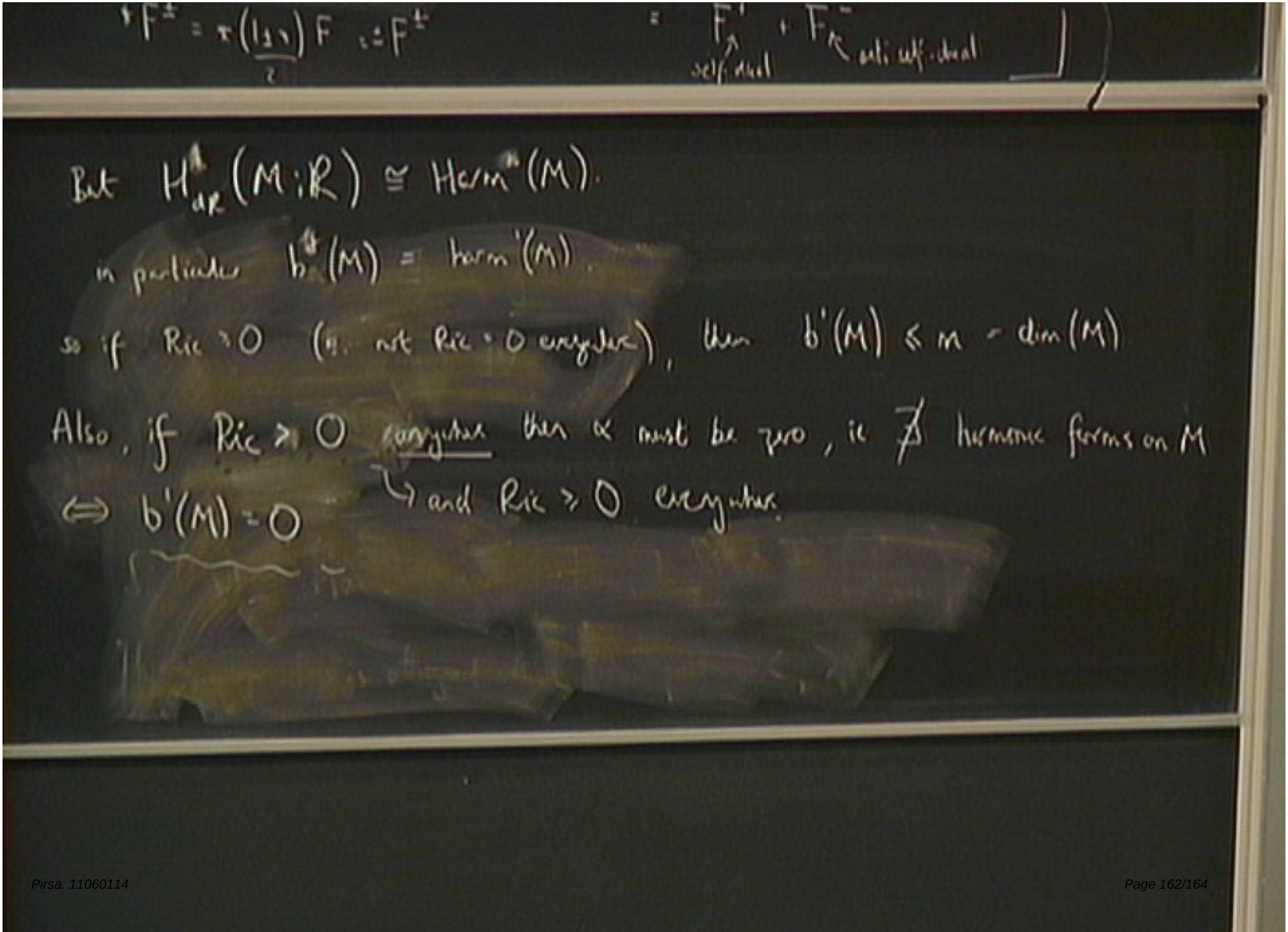

$*F^{\pm} = \pm\left(\frac{1\pm *}{2}\right)F =: F^{\pm}$ $= \underset{\text{self-dual}}{F^+} + \underset{\text{anti self-dual}}{F^-}$

But $H^1_{dR}(M;\mathbb{R}) \cong \mathrm{Harm}^1(M)$.

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so if $\mathrm{Ric} \geq 0$ (i.e. not $\mathrm{Ric} = 0$ everywhere), then $b^1(M) \leq n = \dim(M)$

Also, if $\mathrm{Ric} \geq 0$ $\underline{\text{everywhere}}$ then α must be zero, ie $\nexists$ harmonic forms on M

$\hookrightarrow$ and $\mathrm{Ric} > 0$ everywhere.

$\Leftrightarrow b^1(M) = 0$

$$\Delta_H\alpha - \Delta_B\alpha = g^{ij}(\nabla_k\nabla_i - \nabla_i\nabla_k)\alpha_j\, dx^k = g^{ij}R_{kijm}\, g^{ml}\alpha_l\, dx^k$$

$$\boxed{\Delta_H\alpha = \Delta_B\alpha + Ric(\cdot, \alpha^\#)}$$

Suppose (M, g) is compact Riem mfld and $Ric \geq 0$

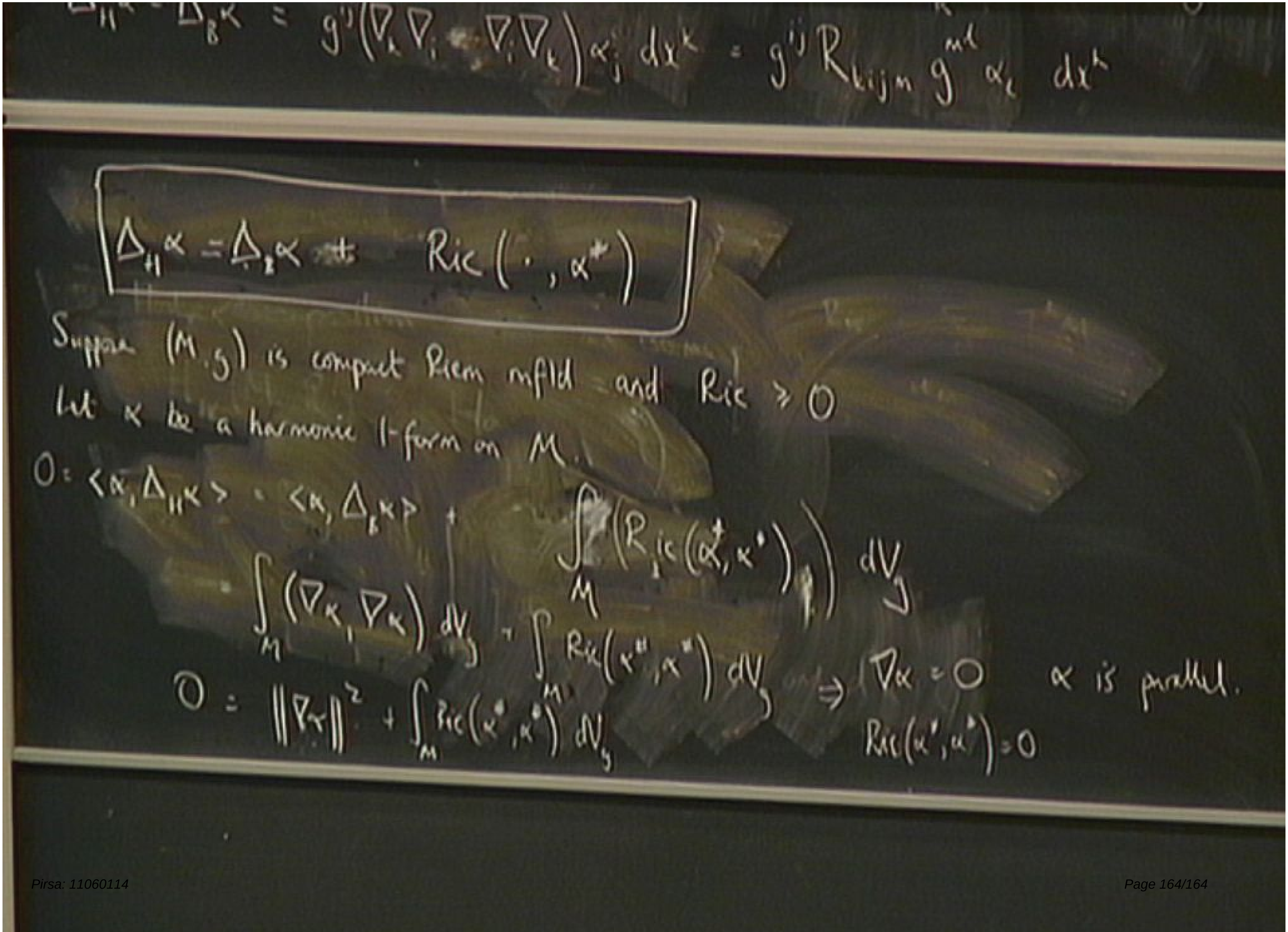
Let α be a harmonic 1-form on M.

$$0 = \langle\alpha, \Delta_H\alpha\rangle = \langle\alpha, \Delta_B\alpha\rangle + \int_M \left(Ric(\alpha^\#, \alpha^\#)\right) dV_g$$

$$= \int_M (\nabla\alpha, \nabla\alpha)\, dV_g + \int_M Ric(\alpha^\#, \alpha^\#)\, dV_g$$

$$0 = \|\nabla\alpha\|^2 + \int_M Ric(\alpha^\#, \alpha^\#)\, dV_g \Rightarrow \begin{matrix} \nabla\alpha = 0 \\ Ric(\alpha^\#, \alpha^\#) = 0 \end{matrix}$$

α is parallel.



$$\Delta_H\alpha - \Delta_B\alpha = g^{ij}(\nabla_k\nabla_i - \nabla_i\nabla_k)\alpha_j\,dx^k = g^{ij}R_{kijm}\,g^{ml}\alpha_l\,dx^k$$

$$\boxed{\Delta_H\alpha = \Delta_B\alpha + Ric(\cdot,\alpha^\#)}$$

Suppose (M,g) is compact Riem mfld and $Ric \geq 0$

Let α be a harmonic 1-form on M.

$$0 = \langle\alpha,\Delta_H\alpha\rangle = \langle\alpha,\Delta_B\alpha\rangle + \int_M \left(Ric(\alpha^\#,\alpha^\#)\right) dV_g$$

$$= \int_M (\nabla\alpha,\nabla\alpha)\,dV_g + \int_M Ric(\alpha^\#,\alpha^\#)\,dV_g$$

$$0 = \|\nabla\alpha\|^2 + \int_M Ric(\alpha^\#,\alpha^\#)\,dV_g \Rightarrow \begin{array}{l}\nabla\alpha = 0 \quad \alpha \text{ is parallel.}\\ Ric(\alpha^\#,\alpha^\#) = 0\end{array}$$