Title: Numrelasticity

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Abstract: Nonlinear numerical relativistic elasticity may be necessary for simulations including neutron star crusts. Basic simulations of large deformations in relativistic elastic matter will be detailed, and issues necessary for more realistic simulations covered. This work is in collaboration with Carsten Gundlach.

MICRA - Numrelasticity

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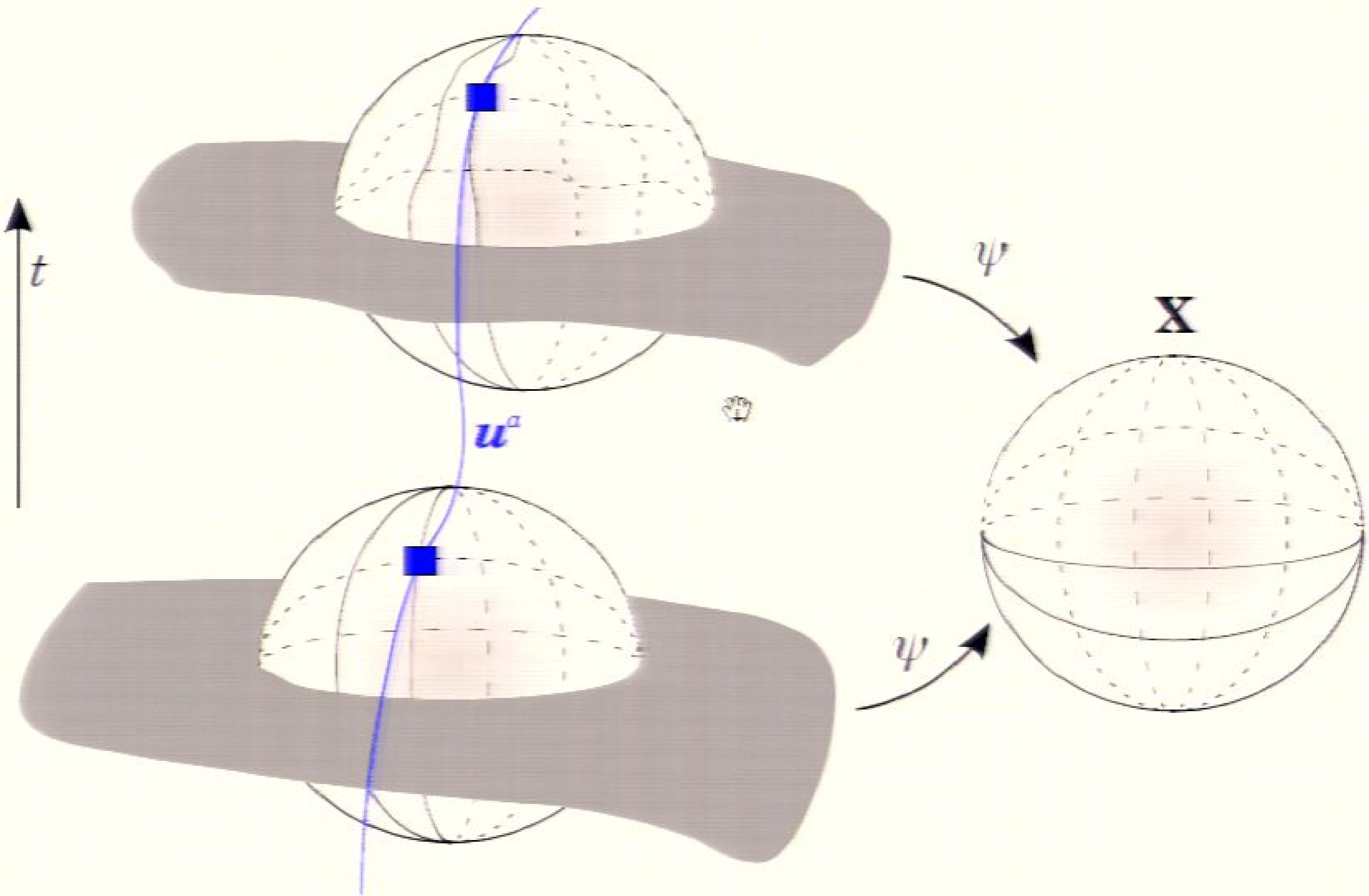
MICRA, June 2011

Outline

1. Continuum mechanics
   - Matter space
   - Dynamics
   - EOS
2. Numerical implementation
   - Equations
   - Evolution
3. Results
   - Newtonian results
   - SR results
4. Going further
   - Multi-D
   - Coupling
   - Conclusions

Matter space



A *body* is given by a *reference configuration* X, and its deformation computed from the map ψ.

Dynamics

The stress-energy tensor is that of hydro, plus anisotropic terms π^{ab}:

$$T^{ab} = (e + p) u^a u^b + p\, g^{ab} + \pi^{ab}.$$

This gives the balance laws

$$(\sqrt{\gamma_x}\mathcal{U})_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms.}$$

with (introducing $\pi = v^i v^j \pi_{ij} = \gamma^{ij}\pi_{ij}$, and ignoring gauge terms)

$$\mathcal{U} = \begin{pmatrix} D \\ S_j \\ \tau \end{pmatrix} = \begin{pmatrix} nW \\ nhW^2 v_j + \pi_{ij} v^i \\ nhW^2 - p - D - \pi \end{pmatrix}, \quad \mathcal{F}^i \sim \begin{pmatrix} D\hat{v}^i \\ nhW^2 v_j \hat{v}^i + p\delta^i{}_j + \pi^i{}_j \\ (nhW^2 - D)\hat{v}^i + \pi^{0i} \end{pmatrix}$$

EOS

The EOS depends on the strain g^{AB} compared to the reference k_{AB} and e.g. the entropy, in addition to any polarizing effects.

Simplify in two ways:

1. **Homogeneous**: $\epsilon \equiv \epsilon(g^{AB}, k_{AB}, s)$
2. **Isotropic**: $\epsilon \equiv \epsilon(\rho, I^{1,2}, s)$ – the strain dependence is encoded in the invariants of $k^{A}{}_{B}$.

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Simple tests here use toy EOS using gamma-law fluid plus term proportional to *a* shear scalar. Existence and uniqueness of weak solutions not clear.

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Equations

For completeness we note the full system:

$$k_{AB,t} + \hat{v}^j k_{AB,j} = 0,$$
$$\psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 2\hat{v}^j \psi^A{}_{[i,j]}.$$

and, as given earlier

$$(\sqrt{\gamma_x}\mathcal{U})_{,t} + \left(\alpha\sqrt{\gamma_x}\mathcal{F}^i\right)_{,i} = \text{source terms.}$$

We also have constraints

$$\psi^A_{[i,j]} = 0.$$

and an EOS $\epsilon \equiv \epsilon(n, I^1, I^2, s)$ where $n, I^{1,2}$ are scalar invariants of $k^A{}_B$

Evolution

We note that by assumption k_{AB} is differentiable. We can thus evolve k_{AB} using naive central differencing.

We then choose primitive variables $(\psi^A{}_i, v^i, p)$ and evolve the remaining equations using standard HRSC methods:

- MoL – typically RK3;
- Slope limiting RSA – typically van Leer MC;
- HLL flux – typically with excessive dissipation.

Equations

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Fix flat space and ignore hyperbolicity fix for now: the source terms ar[e] all trivial.

Con2Prim

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Converting $(k_{AB}, \psi^A{}_i, S_j, \tau) \to (v^i, p)$ is the only remaining task.

Standard iterative approach:

1. *Guess* four quantities: $\overline{p - \pi}$ and $\overline{\pi_{ij} v^j}$;
2. Compute all terms consistent with the guess; in particular, $\overline{n}, \overline{I^{1,2}},$ can be found;
3. Use the EOS to compute p and π_{ab} from the above;
4. Compute the residuals for the guesses.

Reduces to standard approach for hydro; *very* expensive (50% of computational time).

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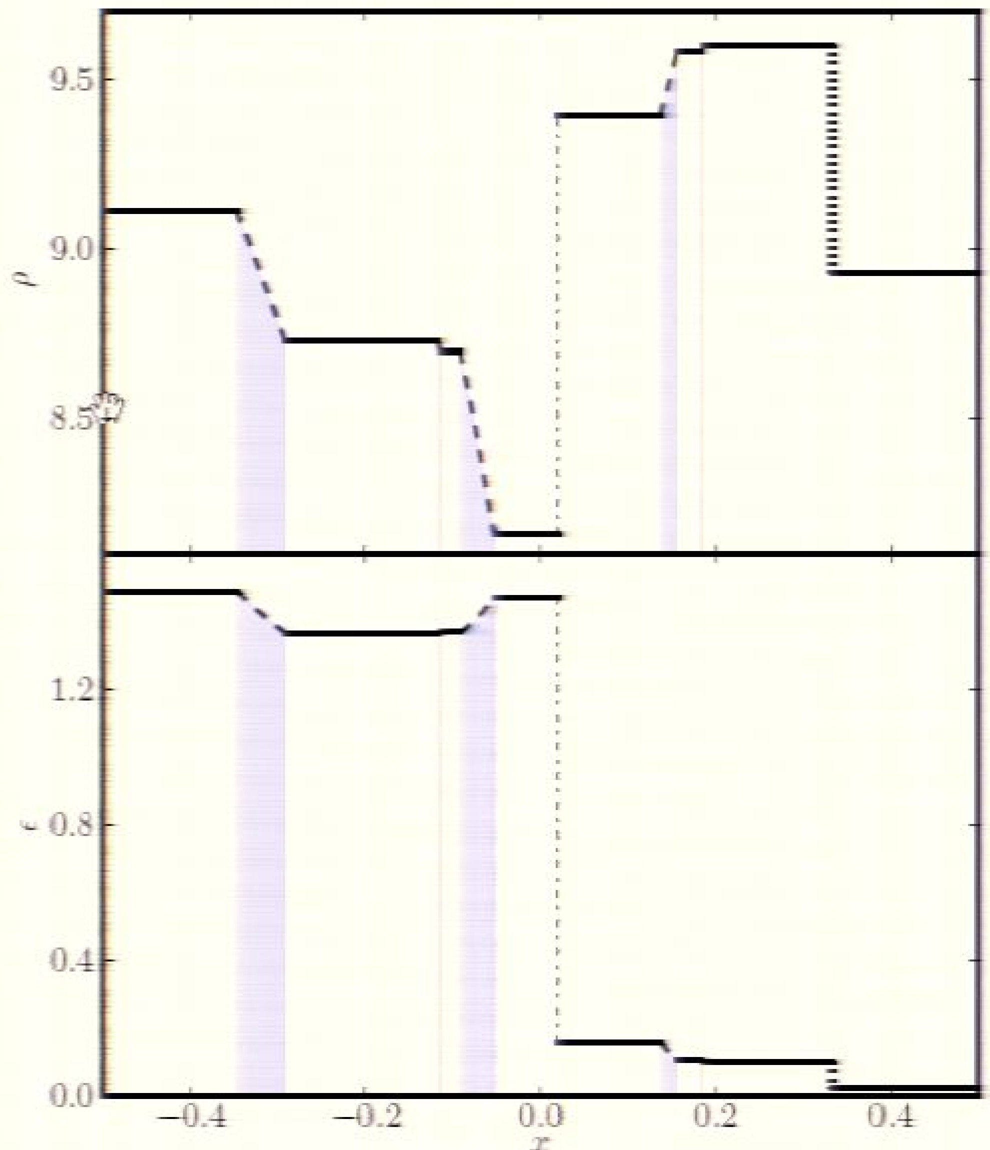
Newtonian – BDRT1 – structure

A Newtonian shock tube with all waves.

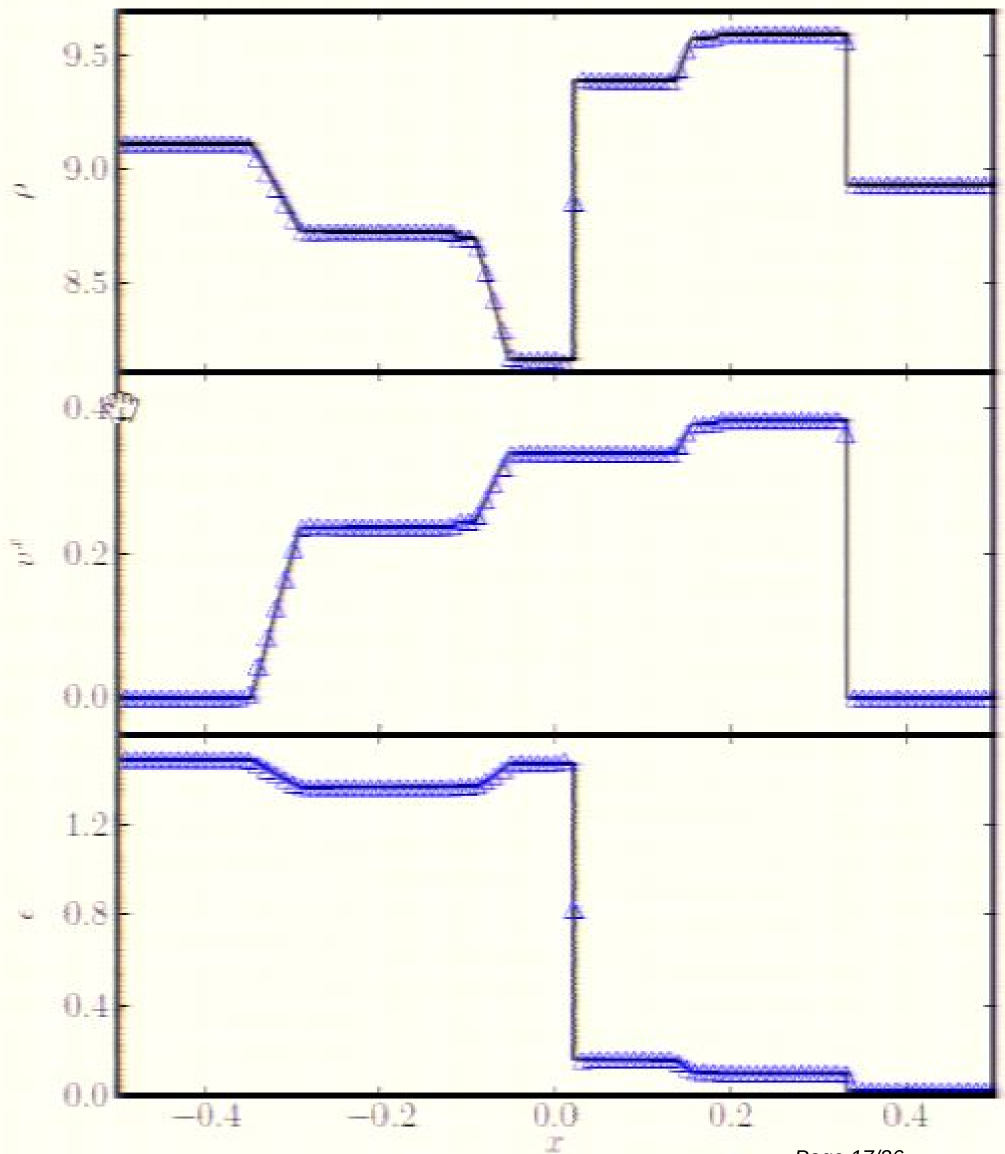
Only the right wave is a shock. Some rarefactions are very steep.



Newtonian – BDRT1 – results

Results using 1000 points (100 shown).

All features well captured. No oscillations. Minor under/over shoots.



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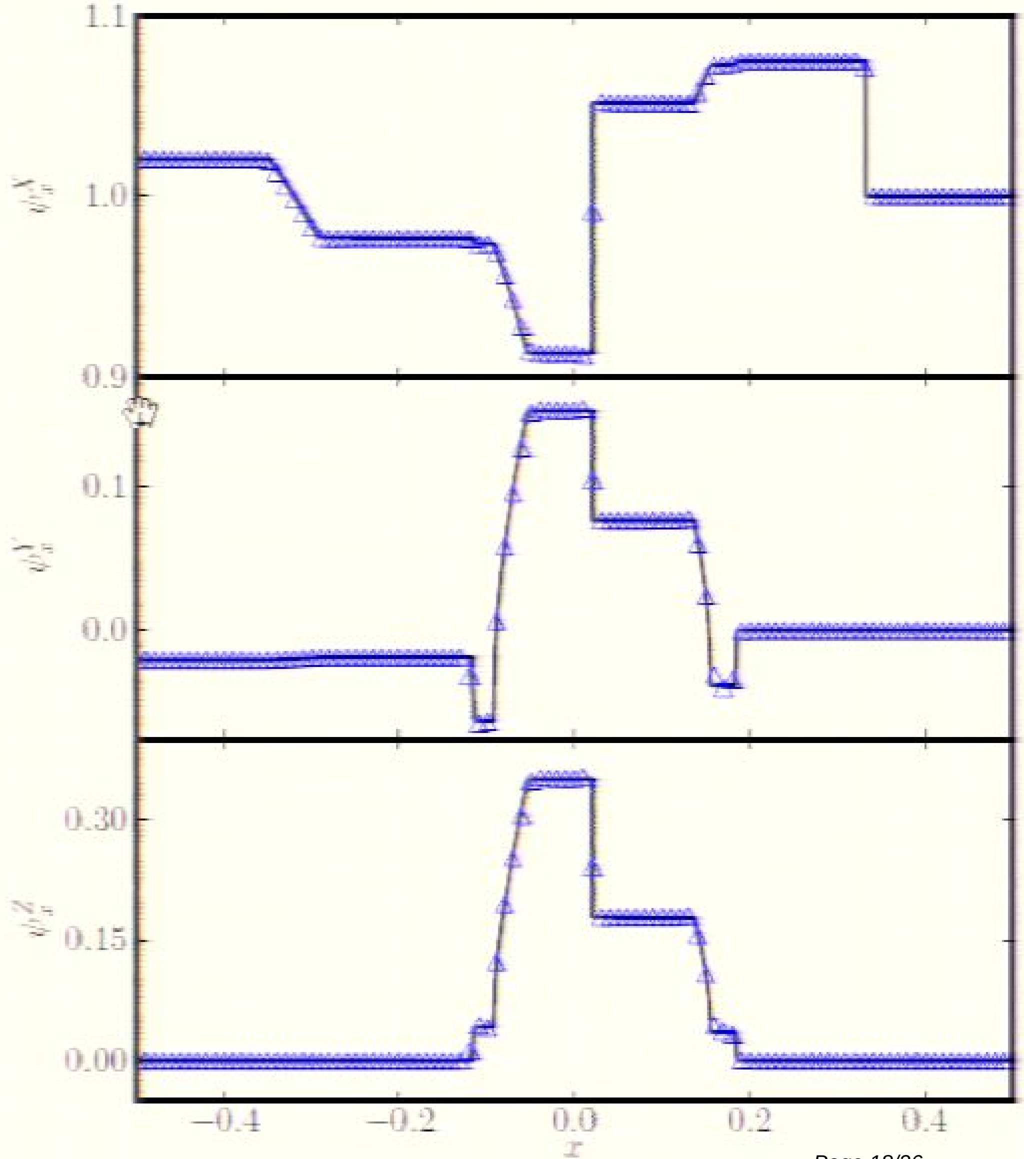
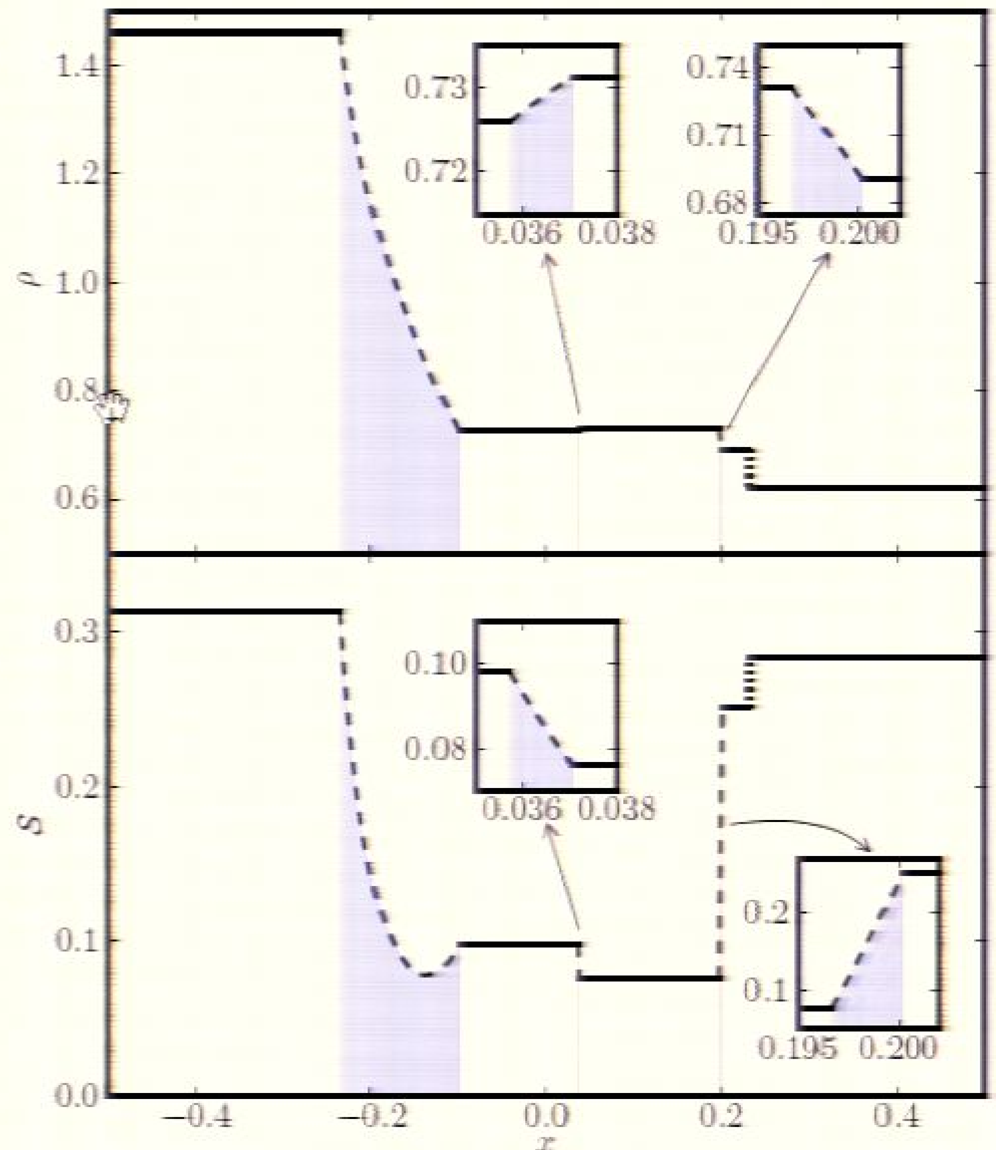
2- and 6-waves only clear in deformation components.



SR – 4-wave test – structure

A relativistic shock tube with 4 waves - no contact, 3- or 5-wave.

Only the right wave is a shock. Most rarefactions are very steep.



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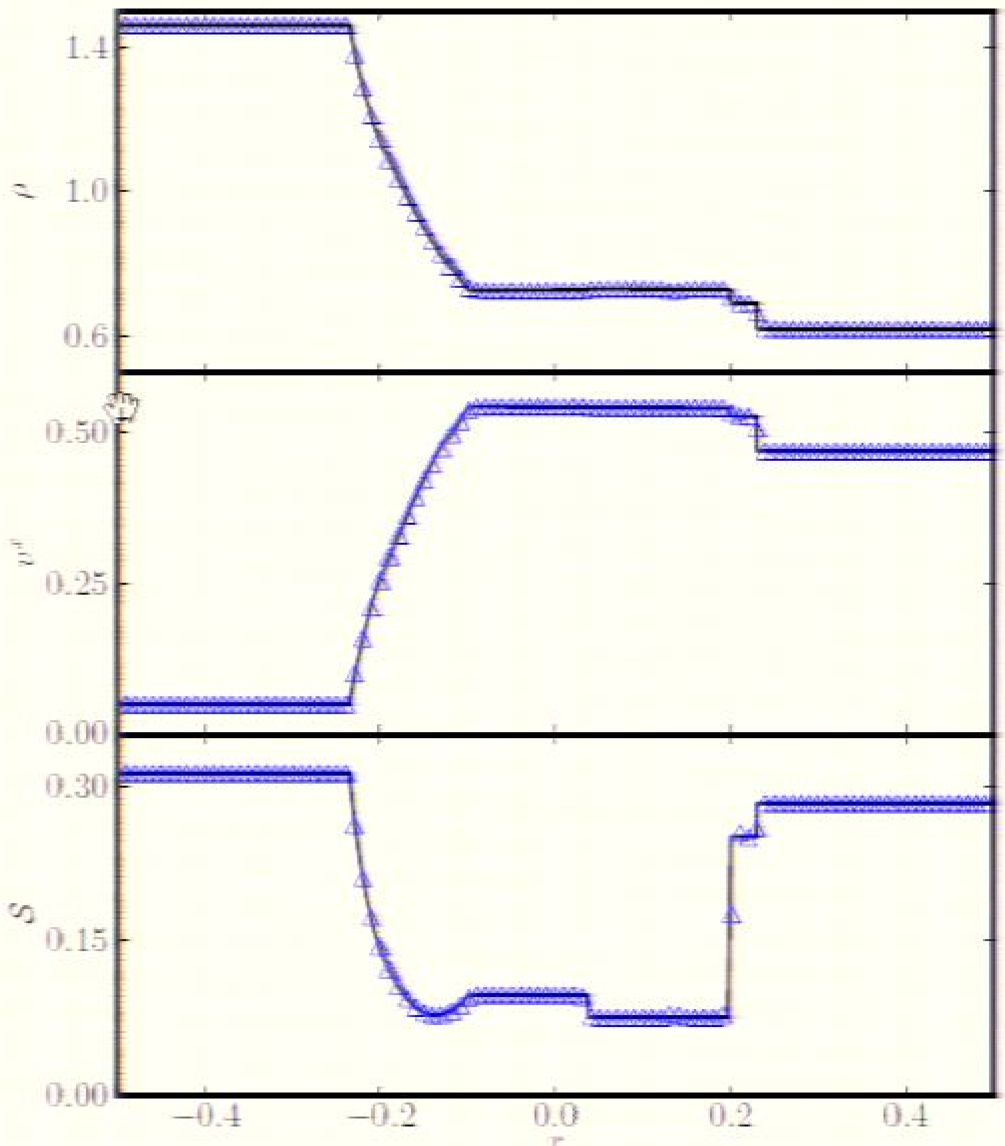
Noticeable glitch near trivial contact - converges away.



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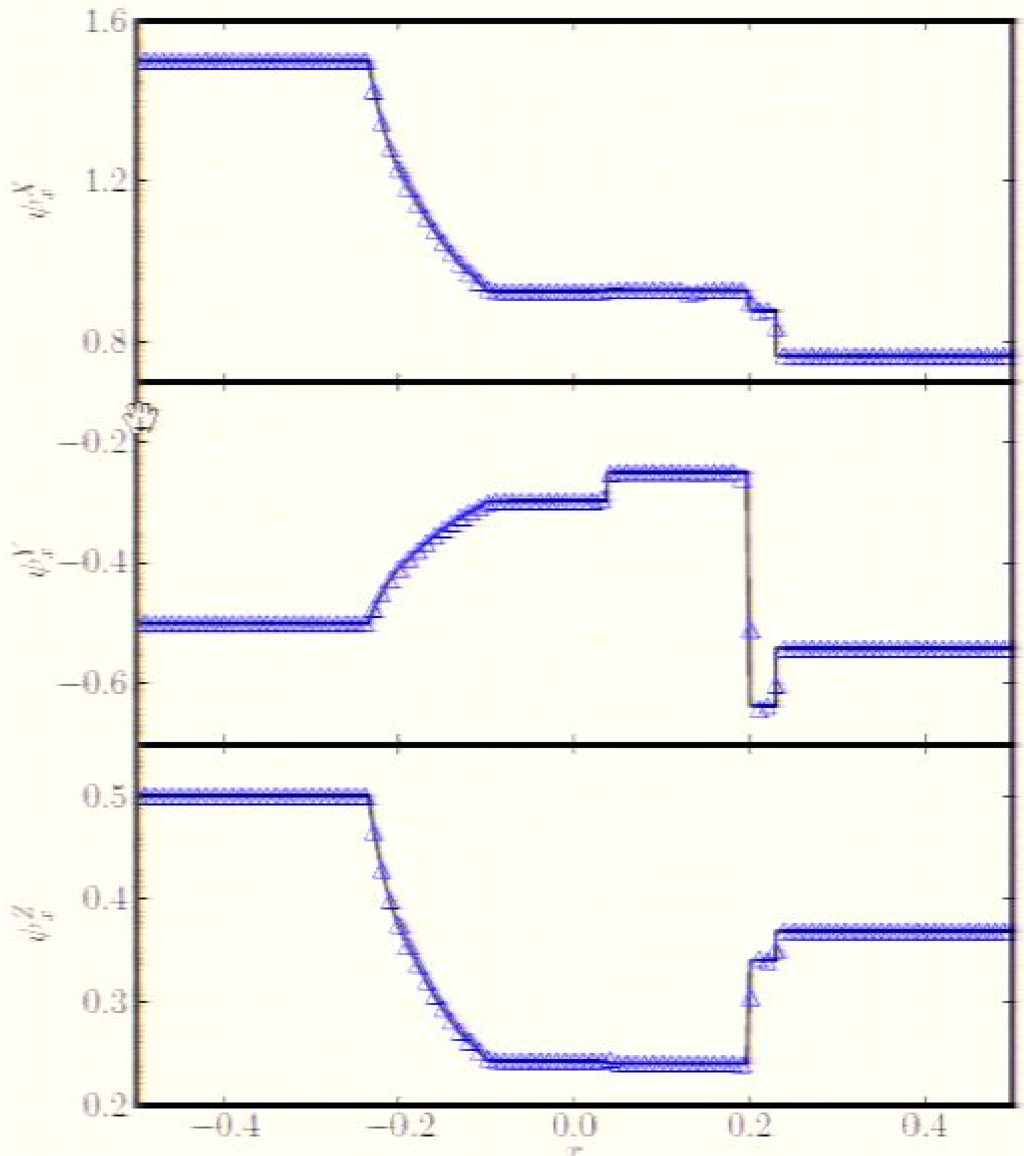
Strong deformation best seen in ψ_X^Y.



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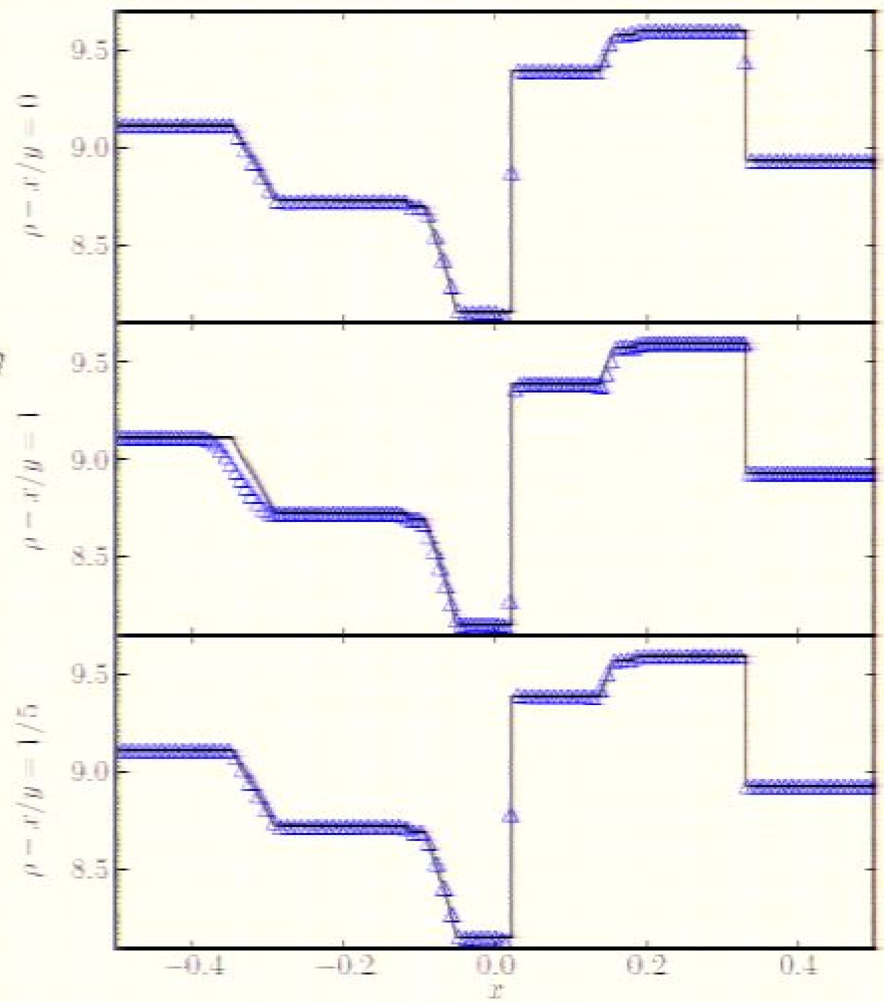
Multi-D

Newtonian literature suggests problems with naive evolution of ψ:

1. hyperbolicity issues explain this;
2. fixes can be implemented
   1. constraint addition in sources stabilizes it
   2. constraint damping used by some groups.



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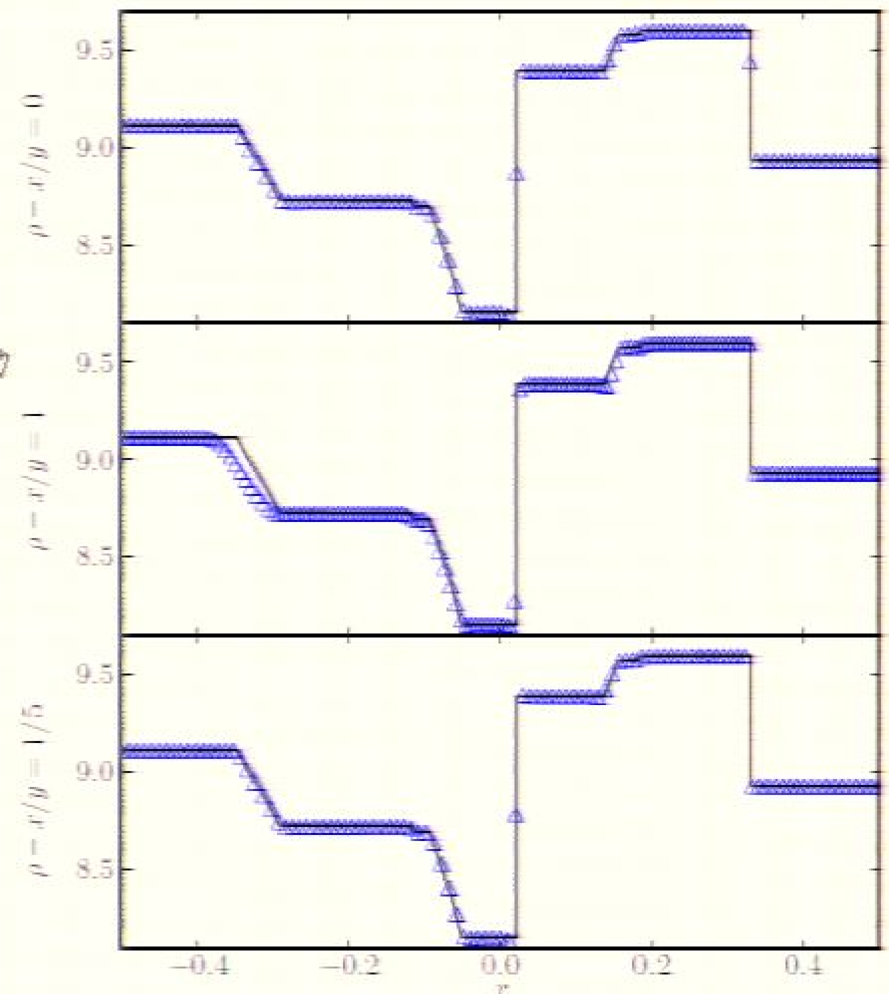

In our simulations we haven't seen problems (*yet*!). When we do

1. same fixes can be implemented;
2. Balsara-type constrained transport can be implemented.

Coupling

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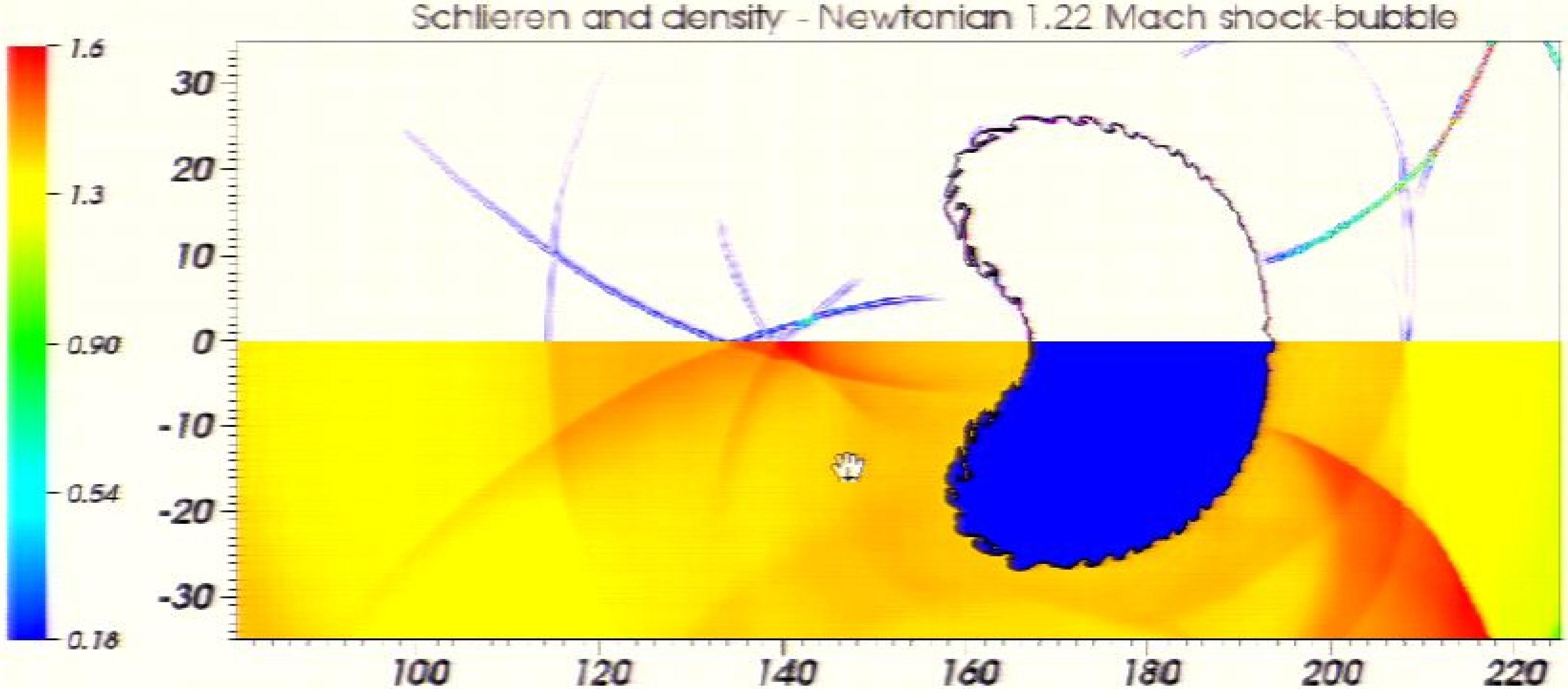
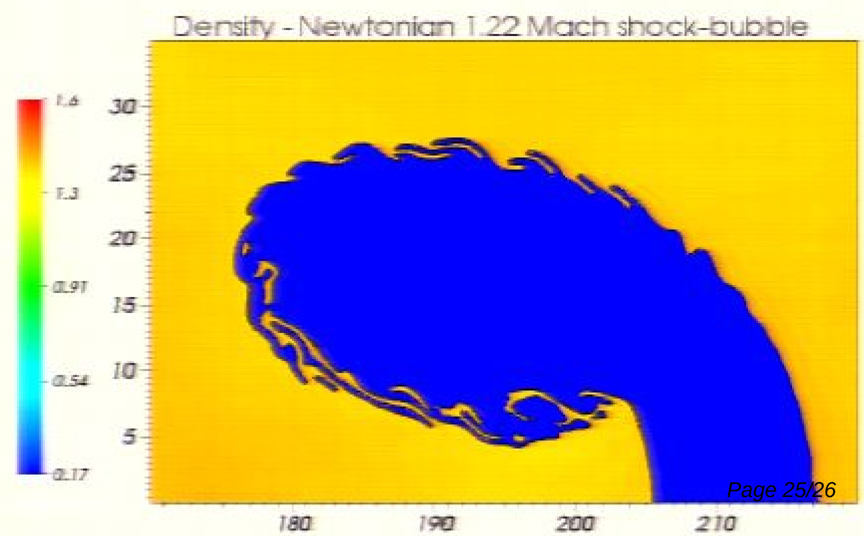


For e.g. NS crusts we need to extend techniques from Millmore & Hawke to couple to fluids; methods exist in Newtonian.



Conclusions

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- Elasticity isn't difficult to implement.
- "Real" EOS would be nice!
- Outstanding questions include
  1. Accurate numerics – characteristic structure really complex
  2. Multi-D issues – especially constraints
  3. Weak solution existence/uniqueness implies EOS constraints?
  4. Multi-material coupling, and melting/freezing.
- Elasticity as a toy model for true multifluids shows future issues...