Title: A formalism for weak solutions of GR elasticity

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Abstract: I review different approaches to the kinematics and dynamics of (hyper)elasticity in GR, and describe one that is now being implemented in joint work with Ian Hawke.

A framework for weak solutions of GR elasticity

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- Motivation
- Kinematics and weak solutions
- Dynamics and equation of state
- Hyperbolicity

Motivation

- Old neutron stars have elastic crusts
- Elasticity, like fluids, admits weak solutions
- Modern fluid codes are HRSC, based on Riemann problems
- Newtonian weak solution codes (Miller and Colella)
- GR codes only linear and smooth (Cerdá-Durán et al)
- **Hyper**elasticity: memory of unsheared state, but not of history

Matter variables

Carter & Quintana 72: where are the particles? "Configuration" map

$$
\begin{aligned}
M_4 &\to X_3 \\
x^a &\mapsto \xi^A \\
\psi^A{}_a &:= \frac{\partial \xi^A}{\partial x^a}
\end{aligned}
$$

defines the matter 4-velocity u^a by

$$u^a \psi^A{}_a = 0$$

Primitive variables are 3-velocity $\hat{v}^i$ and configuration gradient $\psi^A i$, with

$$u^a = \alpha^{-1} W(1, \hat{v}^i) \quad \Rightarrow \quad \psi^A{}_t = -\psi^A{}_i \hat{v}^i$$

Kinematics

Integrability conditions

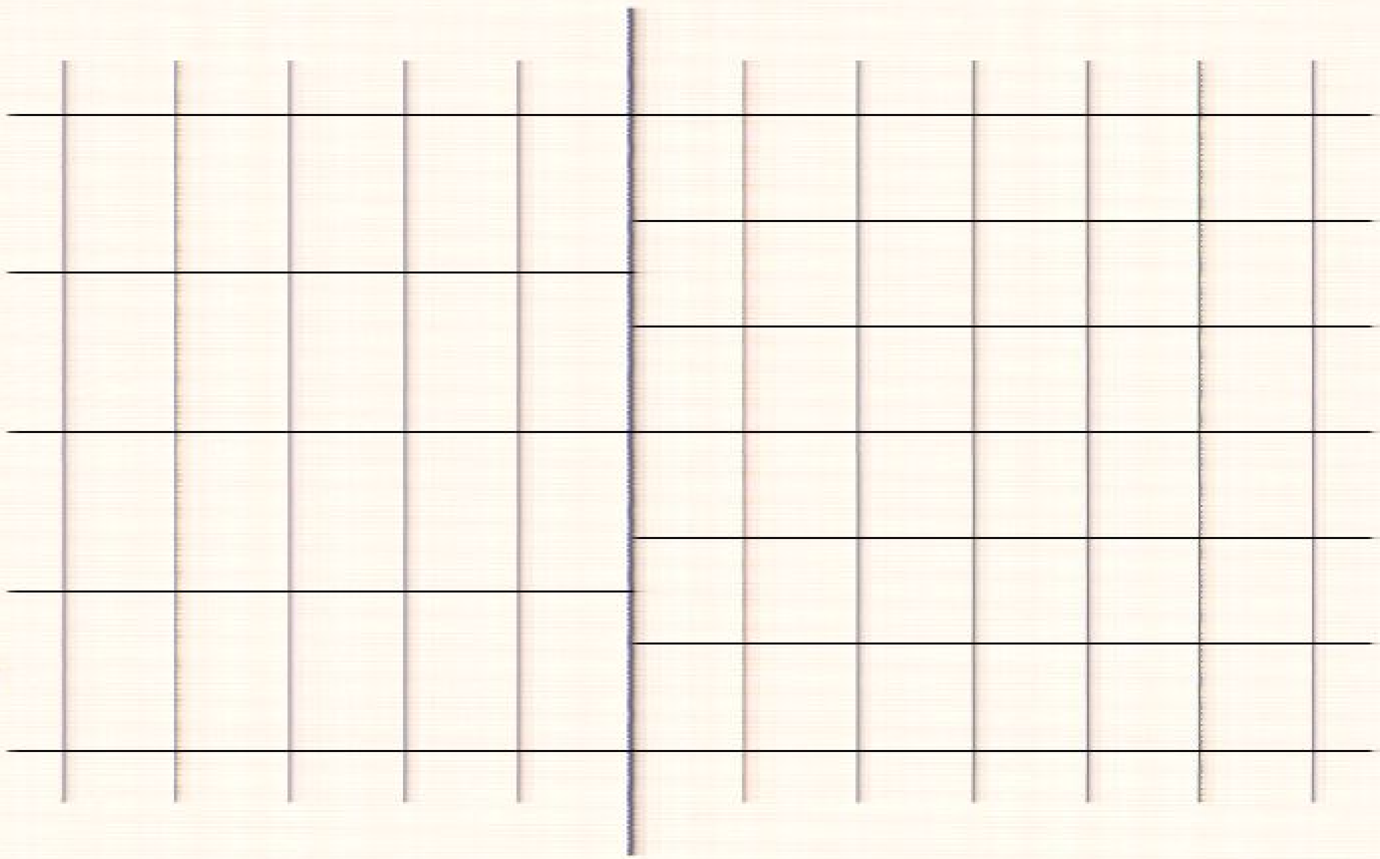
$$C^A{}_{ab} := \psi^A{}_{[a,b]} = 0$$

In a 3+1 split, these become

$$\begin{aligned} C^A{}_{ij} &= \psi^A{}_{[i,j]} = 0 \\ C^A{}_{it} &= \psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 0 \end{aligned}$$

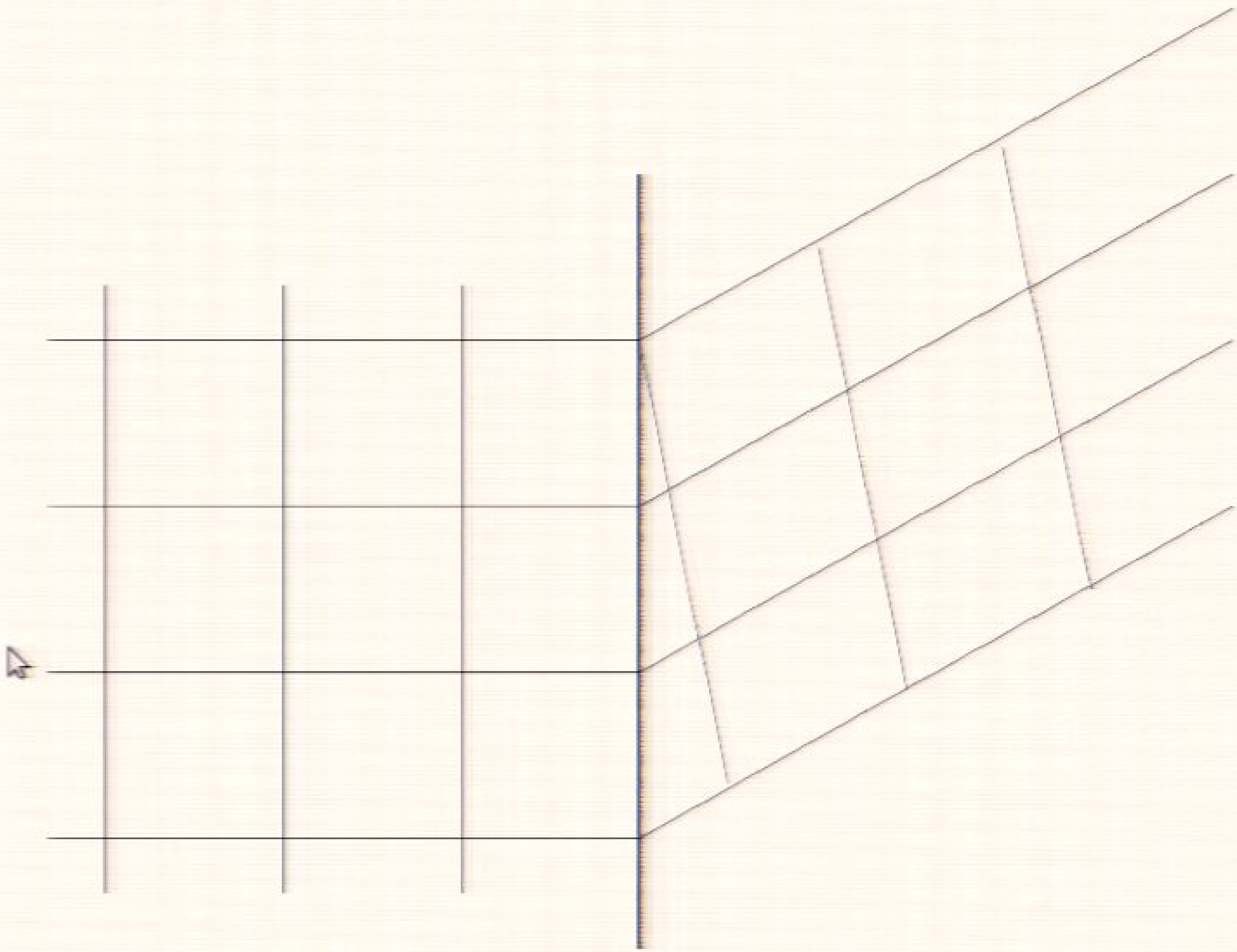
- Constraints are conserved by the evolution equations
- Constraints and evolution equations are in conservation law form
- Their jump conditions guarantee continuity of matter world lines and crystal lines

Kinematic conservation laws 1: no surgery



This is not allowed by $[\psi^{Y}{}_{y}] = 0$

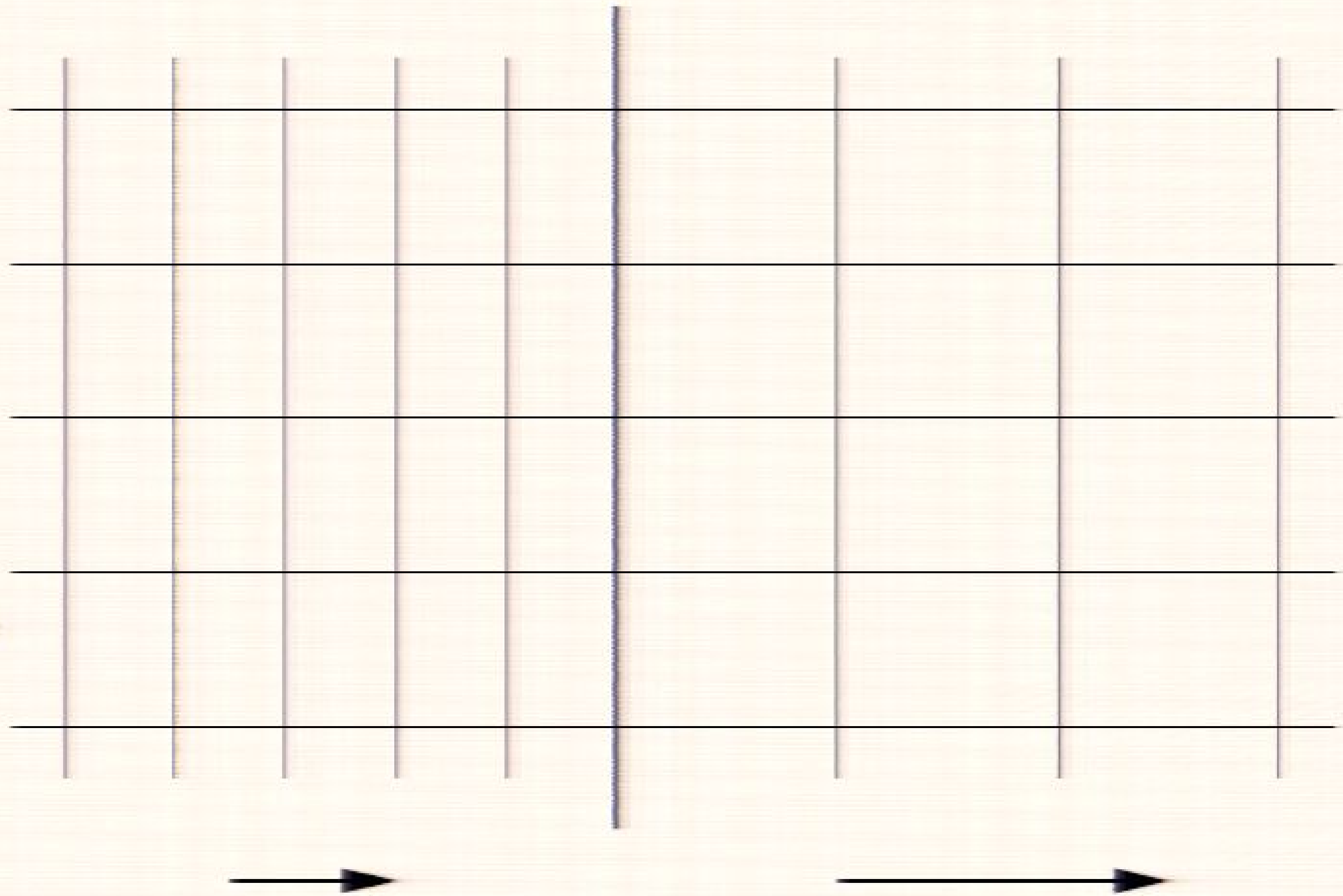
Kinematic conservation laws 2: no surgery



This is not allowed by $[\psi^X{}_y] = 0$

Kinematic conservation laws 3: density shocks

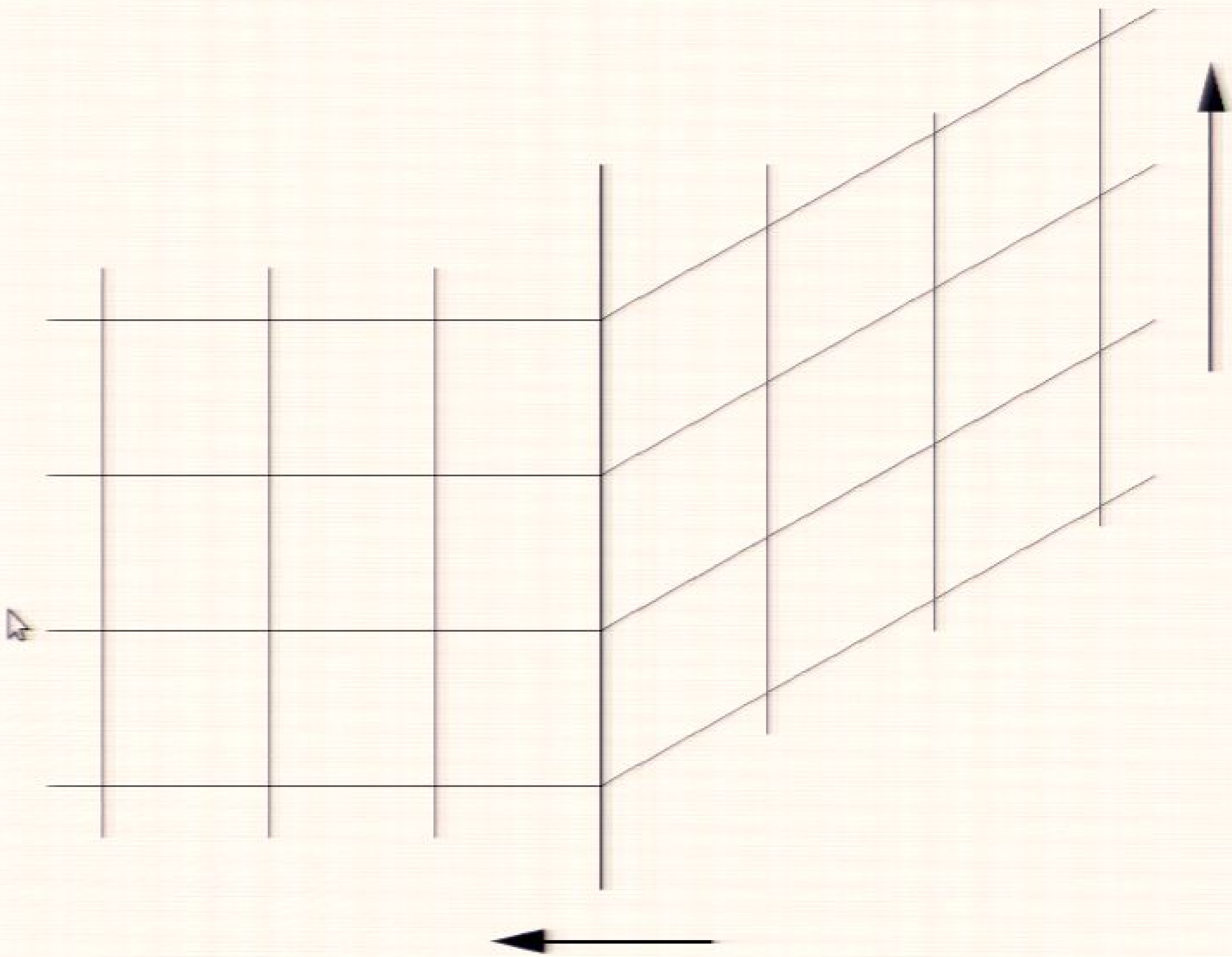
In the rest frame of the shock,



"density" and velocity are related by $[\psi^X{}_x v^x] = 0$

Kinematic conservation laws 4: shear shocks

In the rest frame of the left state,



shock speed and shear speed are related by $s(\psi^Y{}_x)_R = (v^y)_R$

Geometric structure on matter space X_3

- Volume form n_{ABC} counts particles (rest mass)
- Conformal metric η_{AB} measures "angles" (hence deformations)
- Together they determine a full metric k_{AB}
- In Newtonian literature $k_{AB} = \delta_{AB}$, but flat is too restrictive
- Entropy s per particle is a scalar on matter space (in smooth solutions)

Particle number flux

Pull back n_{ABC} to spacetime

$$n_{abc} := \psi^A{}_a \psi^B{}_b \psi^C{}_c n_{ABC}$$

and hence get the particle current

$$\begin{aligned} j^a &:= \frac{1}{3!}\epsilon^{abcd} n_{bcd} =: n u^a \\ \nabla_a j^a &= 0 \end{aligned}$$

In a 3+1 split

$$n = \frac{\sqrt{\det k_\xi}\sqrt{\det \psi_{x\xi}}}{W\sqrt{\det \gamma_x}}$$

Matter action and stress-energy

$$S := \int e\left(g^{ab}, \psi^{A}{}_{a}, k_{AB}, s\right)\sqrt{g_x}\, d^4x$$

Covariance on M_4 **and** on X_3 requires that

$$e(g^{ab}, \psi^{A}{}_{a}, \ldots) = e(g^{AB}, \ldots)$$

where

$$g^{AB} := \psi^{A}{}_{a}\psi^{B}{}_{b}g^{ab}$$

After some calculation,

$$\begin{aligned} T_{ab} &= e\, u_a u_b + p_{ab} \\ e &=: n(1+\epsilon) \\ p_{ab} &= 2n\,\psi^{A}{}_{a}\,\psi^{B}{}_{b}\,\frac{\partial \epsilon}{\partial g^{AB}} \end{aligned}$$

Isotropic matter

Only matter space tensor is the matter metric k_{AB}. Hence

$$e = e(g^{AB}, k_{AB}, s)$$

can only depend on s and the eigenvalues of the **matrix** ("deformation tensor") (Karlovini & Samuelsson CQG 2003)

$$k^A{}_B := g^{AC} k_{BC}$$

Choose

$$n = (\det k^A{}_B)^{1/2}$$

and two other invariants
First law of thermodynamics on a per particle basis

$$d\epsilon = T\, ds - p\, d\left(\frac{1}{n}\right) + f_1\, dI^1 + f_2\, dI^2$$

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Freezing and afterwards

- The unsheared state is defined as

$$k^A{}_B = n^{2/3} \delta^A{}_B$$

- Use this to initialize

$$k_{AB} = n^{2/3} g_{AB}$$

  at freezing
- Note k_{AB} is not flat
- After freezing

$$u^a k_{AB,a} = 0$$

  because k_{AB} is a tensor on matter space

Hyperbolicity

- Adding constraints to kinematic evolution equations

$$\psi^A{}_{i,t} + \left(\hat{v}^j \psi^A{}_j\right)_{,i} = 2\hat{v}^j \psi^A{}_{[j,i]}$$

  we get strong hyperbolicity (known in Newtonian literature)
- Also adding constraints to stress-energy conservation

$$\nabla_a T^{ab} = \text{constraints}$$

  gives symmetric hyperbolicity (proof based on Beig & Schmidt CQG 2003)

Questions for the audience

- What are interesting simulations to carry out next?
- Equation of state $\epsilon = \epsilon(n, s, \mathbf{I}^1, \mathbf{I}^2)$?
- Breaking, melting and freezing?
- Initial data for starquakes?
- What physics should we add next?
  - Elastic/fluid and elastic vacuum interfaces: does the ocean matter?
  - Magnetic fields: ideal MHD?
  - Superfluidity: Equations? Weak solutions?
  - Anisotropic crust matter?