Title: Combining MHD and microphysics

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Abstract: I will discuss the techniques that can be used to include arbitrary equations of state in MHD simulations, particularly the ways in which one may perform conservative to primitive variable conversion numerically for such simulations.

MHD and Microphysics

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Overview

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- The interaction of MHD and Microphysics
- GRMHD code elements
- Con2Prim conversion with and w/o B-fields
- Defining a well-behaved EOS

MHD, Microphysics

- As noted before, many numerical relativity codes now include MHD and/or microphysics. Much of the interaction occurs via the equation of state.
- MHD evolution in low density regions where $B^2 \gg P_{gas}$ requires special handling, with the force-free limit proving extremely useful for handling magnetospheres (see Carlos Palenzuela's talk).
- NS crusts may have fantastically interesting MHD-related behavior, but are not actually *hydro*dynamic in the sense the NR community is used to.
- For the rest of this brief talk, I'll focus on how adopting a more physically motivated EOS affects a numerical relativity MHD code.

A day in the life of a GRMHD code

As mentioned in Matt Duez's talk, GRMHD codes based on Conservative schemes generally have the same set of steps:

- (Smooth) field evolution
- (Shockable) matter evolution, including
  - Conservative $\leftrightarrow$ Primitive variable conversion
  - Reconstruction at cell faces
  - Solving the Riemann problem, including calculation of sound speeds.
- Incorporation of microphysical/radiative effects
- Analysis calculations
- Grid-architecture calculations

The steps in red are typically where the EOS enters, and where MHD affects the structure of the code relative to pure hydro evolution.

The EinsteinToolkit: A GR(M)HD code

- As summarized in Erik Schnetter's talk, the EinsteinToolkit is a *public* codebase that can be used to perform numerical relativity simulations, and many other tasks as well.
- The hydrodynamics module, GRHydro, is based on the `Whisky` code (Baiotti et al. 2003/10), and has been evolving since the initial release.
- A more general EOS treatment (`EOS_Omni`) has been added by Christian Ott et al. to allow for the seamless use of many different types of EOS models.
- MHD functionality is currently being added, including divergence cleaning to maintain the "no-monopole" constraint.

Conservative to primitive conversion in GR(M)HD

- Separate conservative and primitive variable sets are nearly inevitable – they arise in both relativistic Eulerian and Lagrangian SPH codes (Rosswog 2010)
- The typical conservative variables are relativistic variants of:
  1. Density: $D = \rho\gamma = \frac{\rho}{\sqrt{1-v^2}}$
  2. Momentum (×3): $\vec{S} = \alpha T^0_i$
  3. Energy: $\tau = \alpha^2 T^{00} - D$
  4. B-field (×3): $\vec{B}$
- We generally assume $T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu} \left[+b^2 u^\mu u^\nu + b^2 g^{\mu\nu}/2 - b^\mu b^\nu\right]$
- Determining the Lorentz factor or some quantity that depends monotonically on it generally disentangles the system.
- This is often done by first determining the enthalpy $h = 1 + \epsilon + P/\rho$ - which is not the world's most convenient variable since many codes actually implement $P = P(\rho, \epsilon)$.

Conservative to primitive conversion in general

- C2P is the most obvious non-explicit set of equations in GRMHD – numerical methods have to be **efficient** when calculating the inversion at every point on a grid (or SPH particle position)
- The solution method has to be **extremely robust** – Failure is not an option (....except inside a BH event Horizon [Faber et al. 2007]). Any errors will propagate throughout the computational volume.
- Unphysical conservative sets are possible due to differencing errors – need to be corrected as carefully as possible. In practice, the primitives **<u>must</u> remain smooth**.

C2P w/o B-fields

Without B-fields, we may iteratively solve for the pressure given values $D \equiv \rho\gamma$, τ, and $S \cdot S$:

$$
\begin{aligned}
D^2\gamma^2 &= D^2 + \frac{|S \cdot S|}{h^2} \\
Dh\gamma &= \tau + D + P
\end{aligned}
$$

which yields

$$
\begin{aligned}
\gamma &= \frac{Dh\gamma}{Dh} = \frac{\tau + D + P}{\sqrt{(\tau + D + P)^2 - |S \cdot S|}} \\
\epsilon &= \frac{Dh\gamma - P\gamma - D}{D} \\
P_{new} &= P(\rho, \epsilon)
\end{aligned}
$$

Subject to the conditions:

$$
\begin{aligned}
\text{Positive energies} &: \quad \tau \geq 0 \\
\text{Enthalpy} \geq 1 &: \quad |S \cdot S| \leq \tau(\tau + 2D)
\end{aligned}
$$

C2P with B-fields

With B-fields, everything gets much more complicated. With $W \equiv \rho h \gamma^2 = D h \gamma$, we have

$$v^2 = \frac{|S \cdot S| W^2 + (S \cdot B)^2 (B^2 + 2W)}{(B^2 + W)^2 W^2} \quad (1)$$

$$W - \frac{(S \cdot B)^2}{2W^2} = \tau + D + P - \frac{B^2}{2}(1 + v^2) \quad (2)$$

and it is very difficult to lay out analytic criteria guaranteed to yield a solution.

Noble et al (2006) laid out a number of ways to solve the system above, esp. for polytropes. The two most efficient are to:

1. 1-d Newton-Raphson on Eq. 2 to find W, and evaluate $v^2(W)$
2. 2-d Newton-Raphson for both Eqs. to determine v^2 and w

The latter was more stable in tests, and failures are NOT allowed

Refining the approach

- Mignone and McKinney (2007) worked out an approach for more general EOS that assumes we can quickly evaluate

$$P(\rho, h) \quad [= P(\rho, \rho\epsilon + P)]$$

  Unfortunately, the last equation is implicit for many tabulated EOS, requiring nested N-R iterations.

- What we want is a scheme in terms of $P(\rho, \epsilon)$, possibly including derivatives

$$\left(\frac{\partial P}{\partial \rho}\right)_{\epsilon} \;\&\; \left(\frac{\partial P}{\partial \epsilon}\right)_{\rho}$$

- Similar to Matt Duez's point, the method will work terribly unless P and the derivatives are sufficiently smooth to allow us to apply Newton-Raphson.

Voila

Let $u = \rho\epsilon$, and use W, u (and v^2) as the variables for a 2-d (3-d) Newton-Raphson treatment.

$$0 = v^2 W + D\sqrt{1-v^2} + u - \frac{(S\cdot B)^2}{2W^2} - \tau - D + \frac{B^2}{2}(1+v^2)$$

$$0 = u + P\left(\rho \equiv D\sqrt{1-v^2}, \epsilon \equiv \frac{u}{D\sqrt{1-v^2}}\right) - W(1-v^2) + D\sqrt{1-$$

and either

$$0 = |S\cdot S| + \frac{(S\cdot B)^2(B^2+2W)}{W^2} - v^2(B^2+W)^2$$

or

$$v^2 = \frac{|S\cdot S|W^2 + (S\cdot B)^2(B^2+2W)}{(B^2+W)^2 W^2}$$

Conclusions

- Since 3-d N-R is significantly more numerically involved than 2-d, the ideal method is almost certainly to use 2-d first. If it fails, use 3-d, If that fails, determine if a solution is possible and only then use a bracketed, involved, 1-d search for the primitives.
- Con2Prim works for any well-defined EOS model, though smooth ones certainly are more friendly in the numerical sense.
- Add'l parameters, e.g., temperature are easily included so long as they are known along with the conservative set prior to inversion.
- Since the EinsteinToolkit is public, the method from the previous slide will show up sometime very shortly in case people want to implement it or modify it.

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