Title: Discontinuous Galerkin methods for general relativistic hydrodynamics

Date: Jun 23, 2011 04:30 PM

URL: http://pirsa.org/11060036

Abstract: I will present the formalism needed for the application of discontinuous Galerkin methods to general relativistic hydrodynamics and the results obtained in the spherically symmetric case.

Outline

1. Introduction
2. Discontinuous Galerkin methods
3. Results and Conclusions

Why bother?

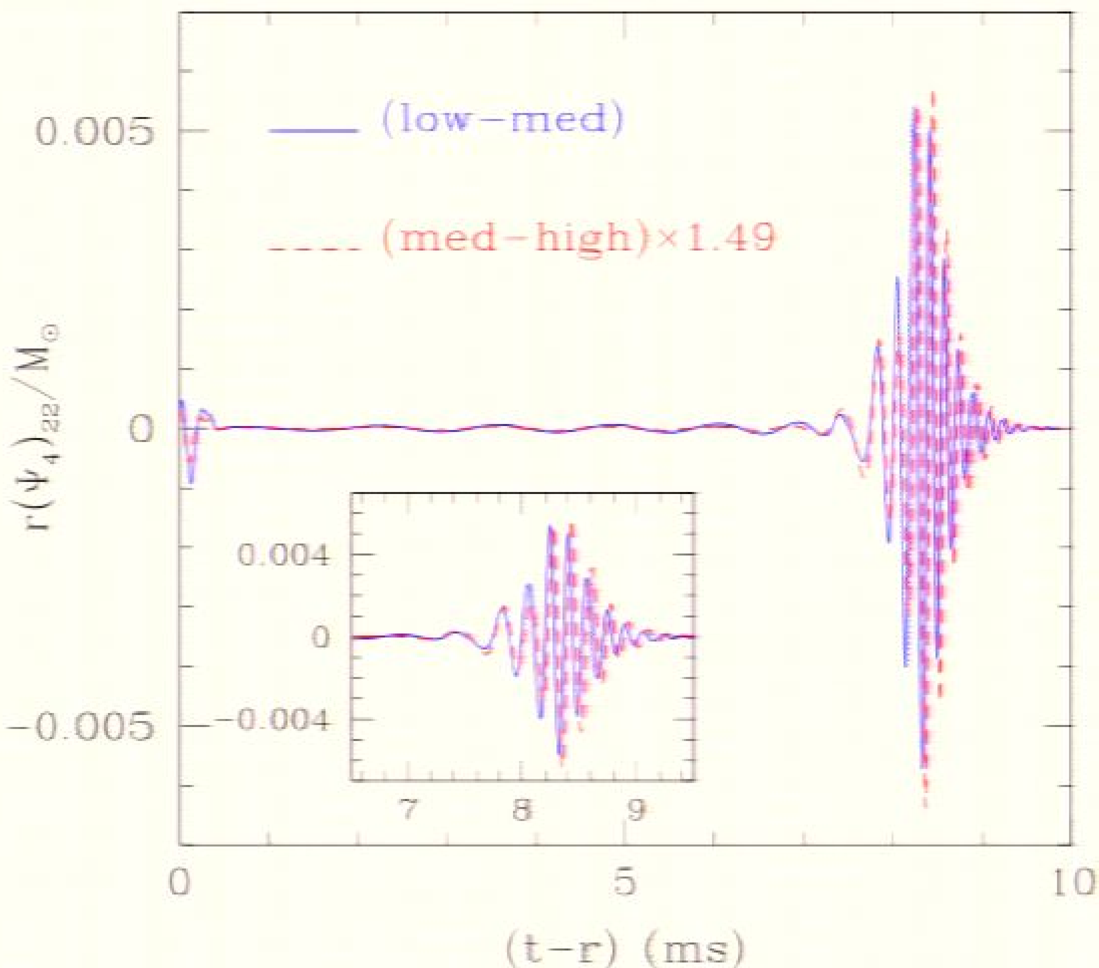


Figure: From Baiotti et al. 2009

Discontinuous Galerkin:

- are conservative and shock capturing;
- high order on smooth solutions
- unstructured grids;
- have nearly-linear scalability.

Discontinuous Galerkin methods for the vacuum Einstein equations: Zumbusch 2009, Field et al. 2010.

Discontinuous Galerkin vs finite volumes methods

Conservation law

$$\partial_t u + \partial_x F(u) = 0$$

Finite volumes

$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u \,\mathrm{d}x = \mathcal{F}^{j-1/2}(\bar{u}_{j-1}, \bar{u}_j) - \mathcal{F}^{j+1/2}(\bar{u}_j, \bar{u}_{j+1})$$

Discontinuous Galerkin vs finite volumes methods

Conservation law

$$\partial_t u + \partial_x F(u) = 0$$

Finite volumes

$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u \,\mathrm{d}x = \mathcal{F}^{j-1/2}(w^-_{j-1/2}, w^+_{j-1/2}) - \mathcal{F}^{j+1/2}(w^-_{j+1/2}, w^+_{j+1/2})$$

Discontinuous Galerkin vs finite volumes methods

Conservation law

$$\partial_t u + \partial_x F(u) = 0$$

Finite volumes

$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u \,\mathrm{d}x = \mathcal{F}^{j-1/2}(w^-_{j-1/2}, w^+_{j-1/2}) - \mathcal{F}^{j+1/2}(w^-_{j+1/2}, w^+_{j+1/2})$$

Discontinuous Galerkin

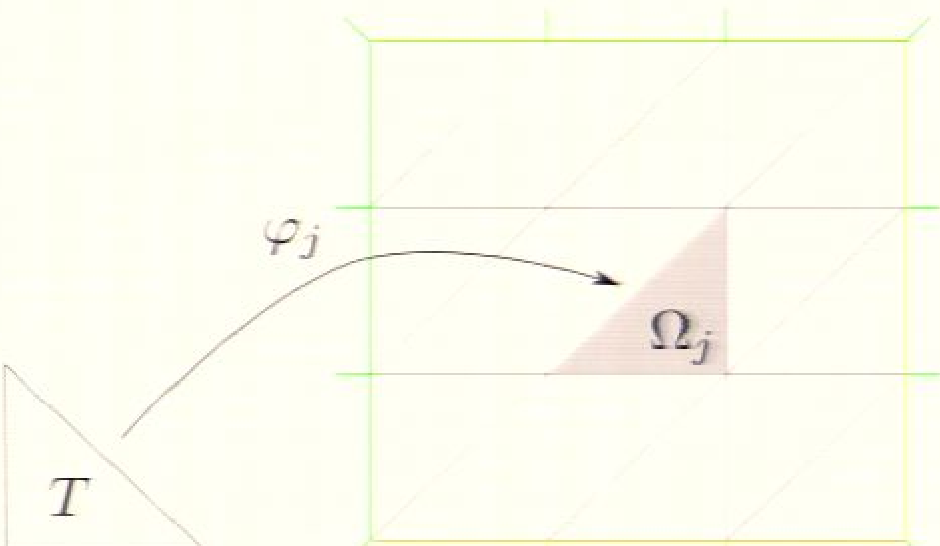
$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u\, \phi \,\mathrm{d}x = \phi_{j-1/2}\mathcal{F}^{j-1/2}(u^-_{j-1/2}, u^+_{j-1/2}) - \phi_{j+1/2}\mathcal{F}^{j+1/2}(u^-_{j+1/2}, u^+_{j+1/2}) + \int_{x_{j-1/2}}^{x_{j+1/2}} u\, \partial_x \phi \,\mathrm{d}x.$$

A model problem

0. Take a (scalar) conservation law

$$\partial_t u + \partial_i F^i(u) = 0, \qquad (t, x) \in [0, T] \times \Omega.$$

1. Generate a grid



2. Construct weak formulation

$$\int_{\Omega_j} \partial_t u \, \phi \, \mathrm{d}x = \int_{\Omega_j} F^i(u) \, \partial_i \phi \, \mathrm{d}x - \int_{\partial\Omega_j} \mathcal{F}^i(u) \, \nu_i \, \phi \, \mathrm{d}\sigma, \quad \forall \phi \in C_0^1(\Omega).$$

3. Discretize

Look for "polynomial solutions": $[\varphi_j]_* u \in \mathbb{P}_D[T]$.

4. Evolve

$$\partial_t \boldsymbol{u}_j = \boldsymbol{\Lambda}_j[\boldsymbol{u}_j] - \boldsymbol{F}_j[\boldsymbol{u}].$$

Discontinuous Galerkin vs finite volumes methods

Conservation law

$$\partial_t u + \partial_x F(u) = 0$$

Finite volumes

$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u \,\mathrm{d}x = \mathcal{F}^{j-1/2}(w^-_{j-1/2}, w^+_{j-1/2}) - \mathcal{F}^{j+1/2}(w^-_{j+1/2}, w^+_{j+1/2})$$

Discontinuous Galerkin

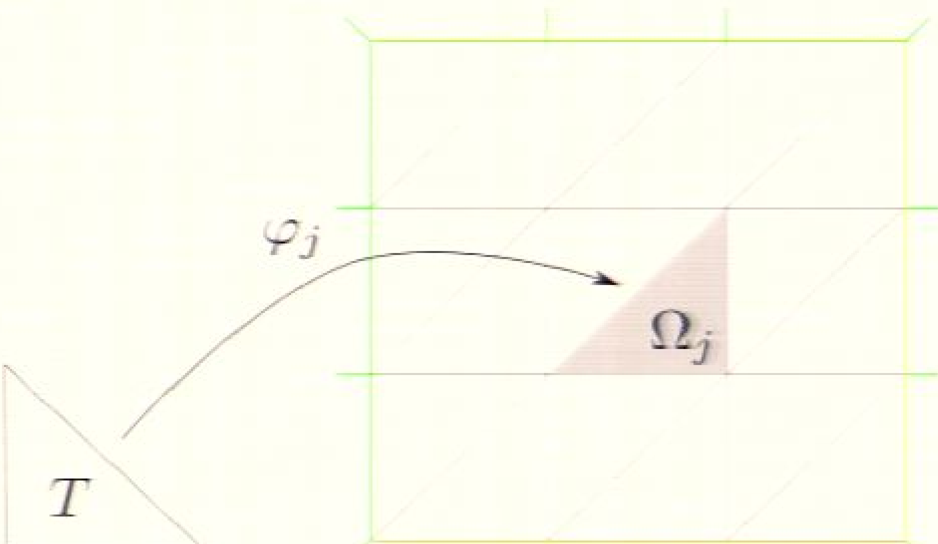
$$\partial_t \int_{x_{j-1/2}}^{x_{j+1/2}} u\,\phi \,\mathrm{d}x = \phi_{j-1/2}\mathcal{F}^{j-1/2}(u^-_{j-1/2}, u^+_{j-1/2}) - \phi_{j+1/2}\mathcal{F}^{j+1/2}(u^-_{j+1/2}, u^+_{j+1/2}) + \int_{x_{j-1/2}}^{x_{j+1/2}} u\,\partial_x\phi \,\mathrm{d}x.$$

A model problem

0. Take a (scalar) conservation law

$$\partial_t u + \partial_i F^i(u) = 0, \qquad (t, x) \in [0, T] \times \Omega.$$

1. Generate a grid



2. Construct weak formulation

$$\int_{\Omega_j} \partial_t u \, \phi \, \mathrm{d}x = \int_{\Omega_j} F^i(u) \, \partial_i \phi \, \mathrm{d}x - \int_{\partial\Omega_j} \mathcal{F}^i(u) \, \nu_i \, \phi \, \mathrm{d}\sigma, \quad \forall \phi \in C_0^1(\Omega).$$

3. Discretize

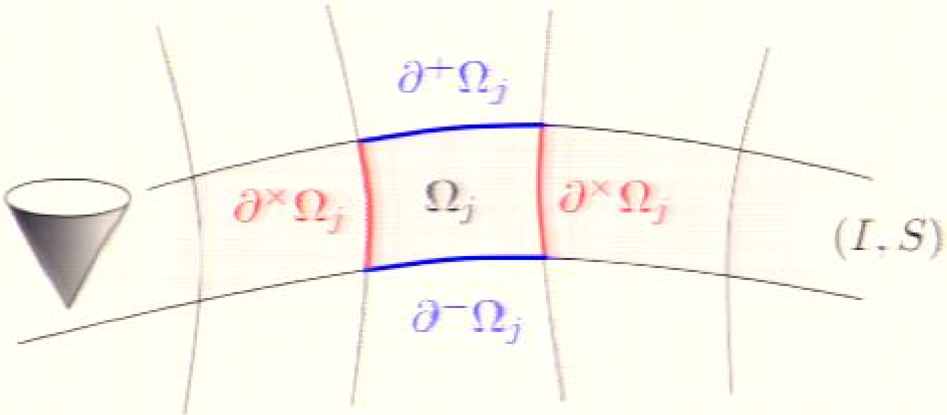
Look for “polynomial solutions”: $[\varphi_j]_* u \in \mathbb{P}_D[T]$.

4. Evolve

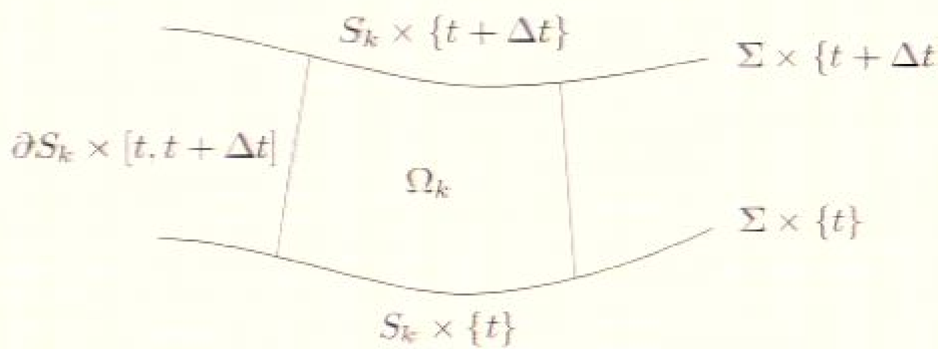
$$\partial_t \boldsymbol{u}_j = \boldsymbol{\Lambda}_j[\boldsymbol{u}_j] - \boldsymbol{F}_j[\boldsymbol{u}].$$

Discontinuous Galerkin methods for general relativistic hydrodynamics

Spacetime formulation



3+1 formulation



Spacetime discontinuous Galerkin

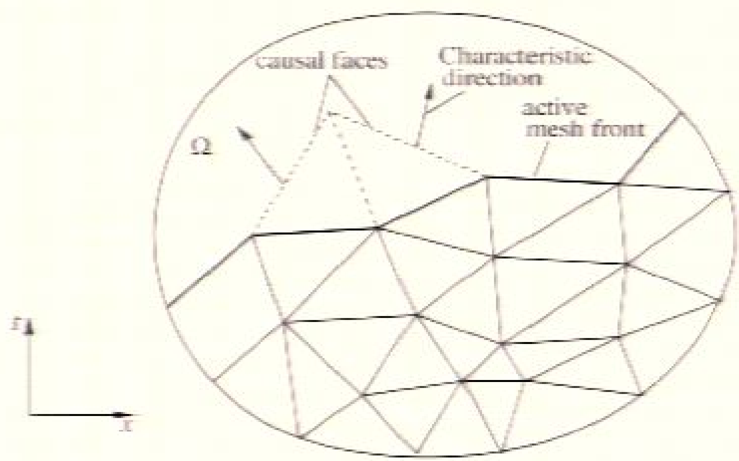
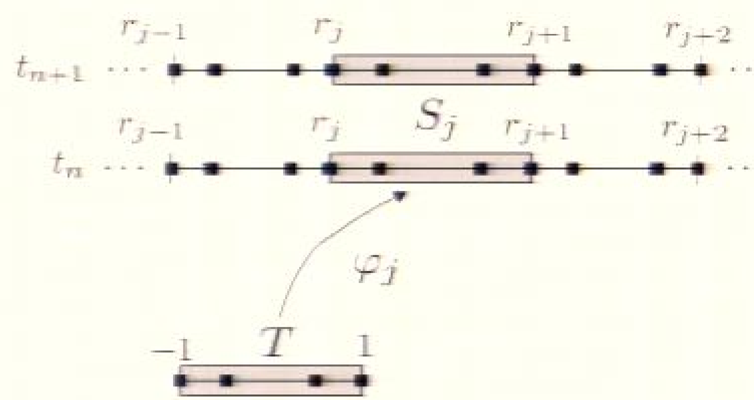


Figure: From Palaniappan 2004

Runge-Kutta discontinuous Galerkin



The EDGES code

Continuity equation

$$\sum_{j=1}^{P} \partial_t \int_{S_j} J^0 \, \phi \, \boldsymbol{\eta} = \sum_{j=1}^{P} \left[\int_{S_j} J^i \, \partial_i \phi \, \boldsymbol{\eta} - \int_{\partial S_j} \mathcal{J}^i \, \phi \, \eta_{i\alpha\beta} \right], \quad \forall \phi \in C_0^1(S)\,.$$

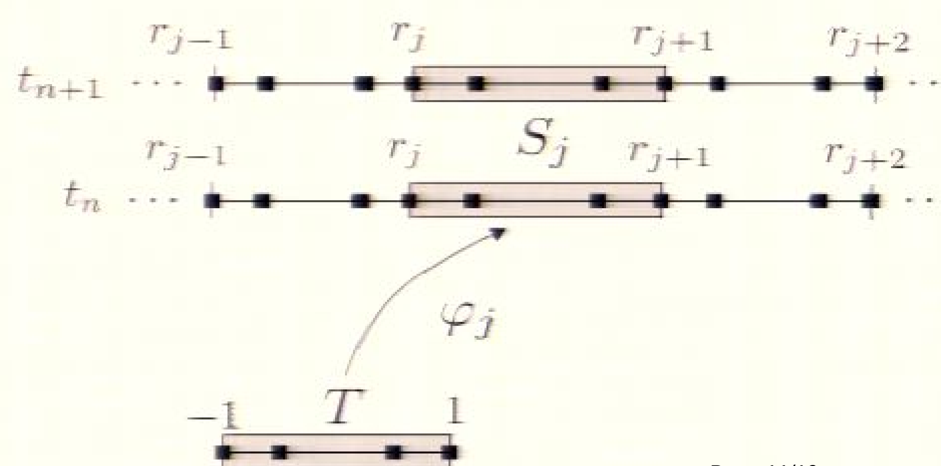
Momentum equation

$$\sum_{j=1}^{P} \partial_t \int_{S_j} T^{0\mu} \, \phi \, \boldsymbol{\eta} = \sum_{j=1}^{P} \left[\int_{S_j} T^{i\mu} \, \partial_i \phi \, \boldsymbol{\eta} - \int_{\partial S_j} \mathcal{T}^{i\mu} \, \phi \, \eta_{i\alpha\beta} - \int_{S_j} T^{\alpha\beta} \, \Gamma^{\mu}{}_{\beta\alpha} \, \phi \, \boldsymbol{\eta} \right], \quad \forall \phi \in C_0^1(S)\,.$$

Notes

- 1D/spherical symmetry in radial-polar gauge
- Legendre-Galerkin pseudospectral collocation of the equations in each element
- HLLE approximate Riemann solver
- 2nd order Runge-Kutta time integration
- Interior penalty discontinuous Galerkin scheme for the spacetime evolution

Grid structure



Spherical shock reflection

- Lorentz factor 1000.
- 200 elements, polynomials of degree 3.
- 4th order filter for stabilization

- The error is dominated by wall heating.
- The average relative error on the compression ratio is 1.39 %, comparable with the 2.2 % reported by Romero et al. (1996) using a standard Roe-FV 2nd order scheme.

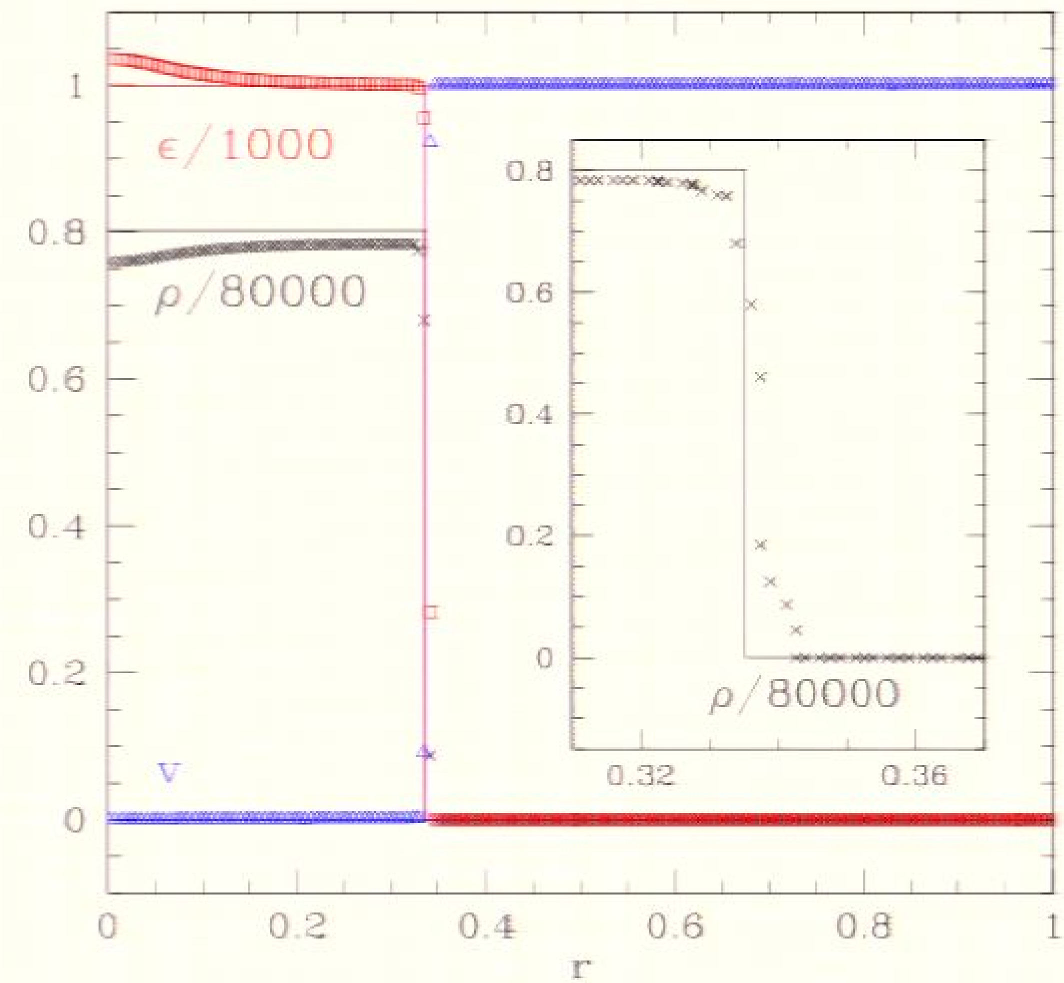
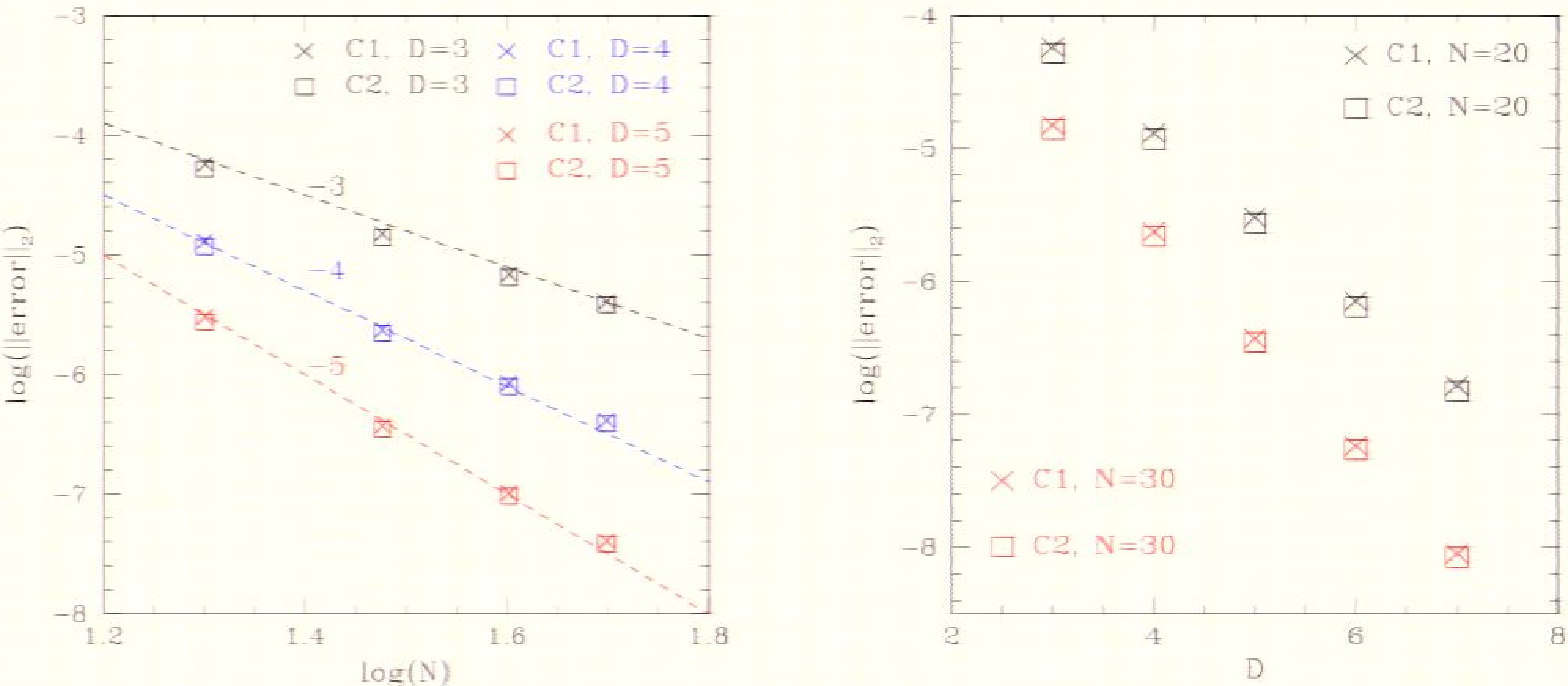


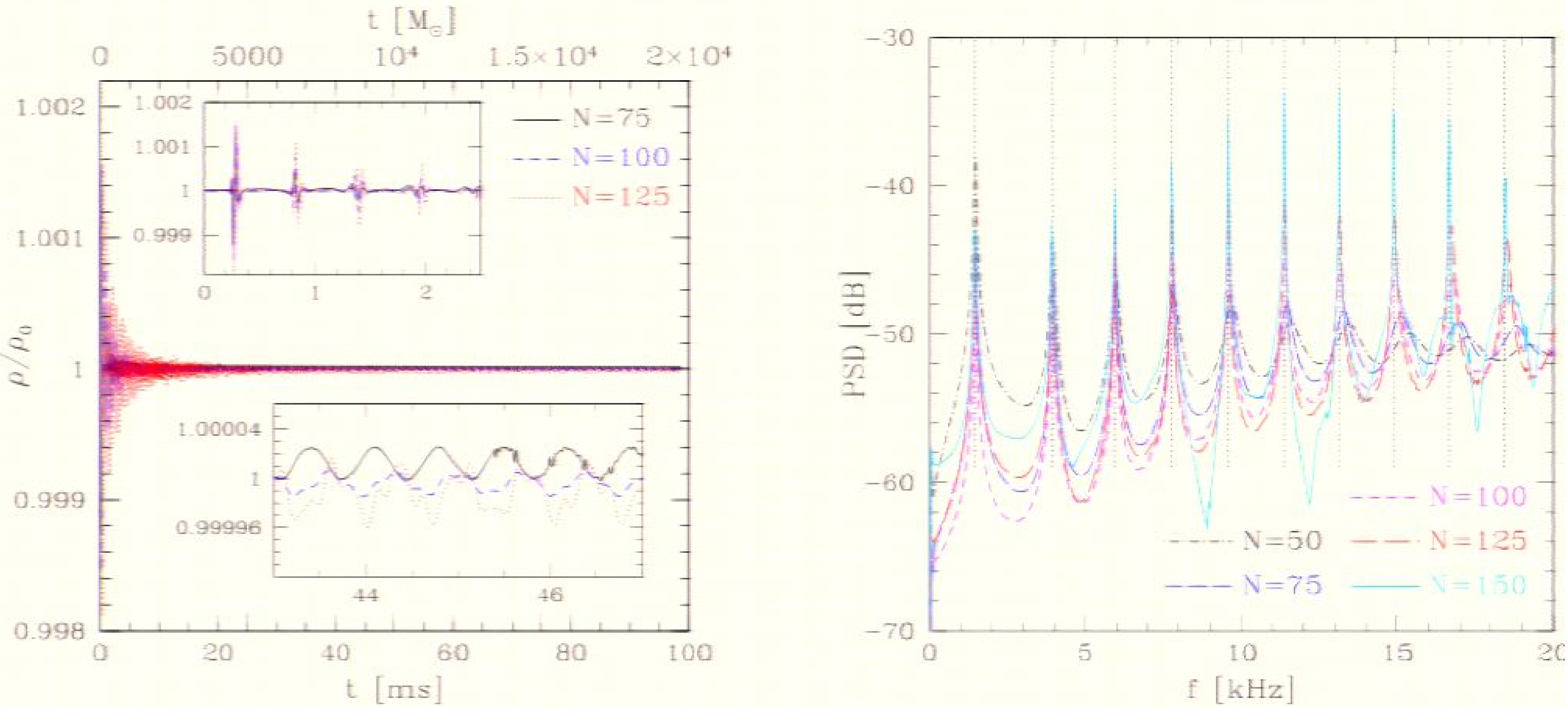
Figure: In the main frame we show a point every five.

Michel's spherical accretion



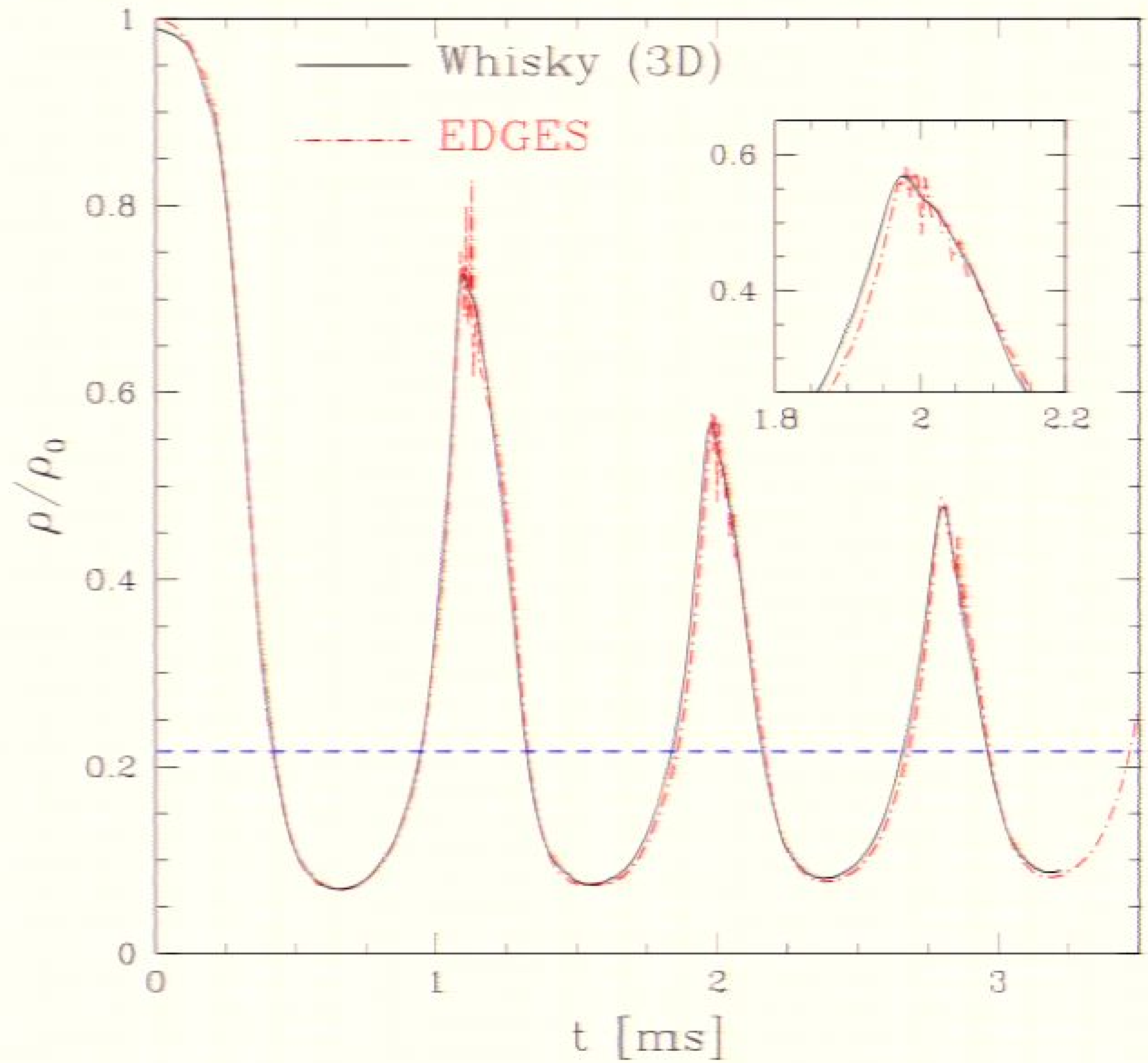
- Given in the implicit form $\sqrt{-g}J^r = C_1$, $\sqrt{-g}T^r_t = C_2$.
- C_1 and C_2 fixed by boundary conditions, can be measured to estimate the error.
- The measure is done while varying the number of elements, N, and the degree of the polynomial representation, D.

Linear oscillations of a spherical star in full-GR

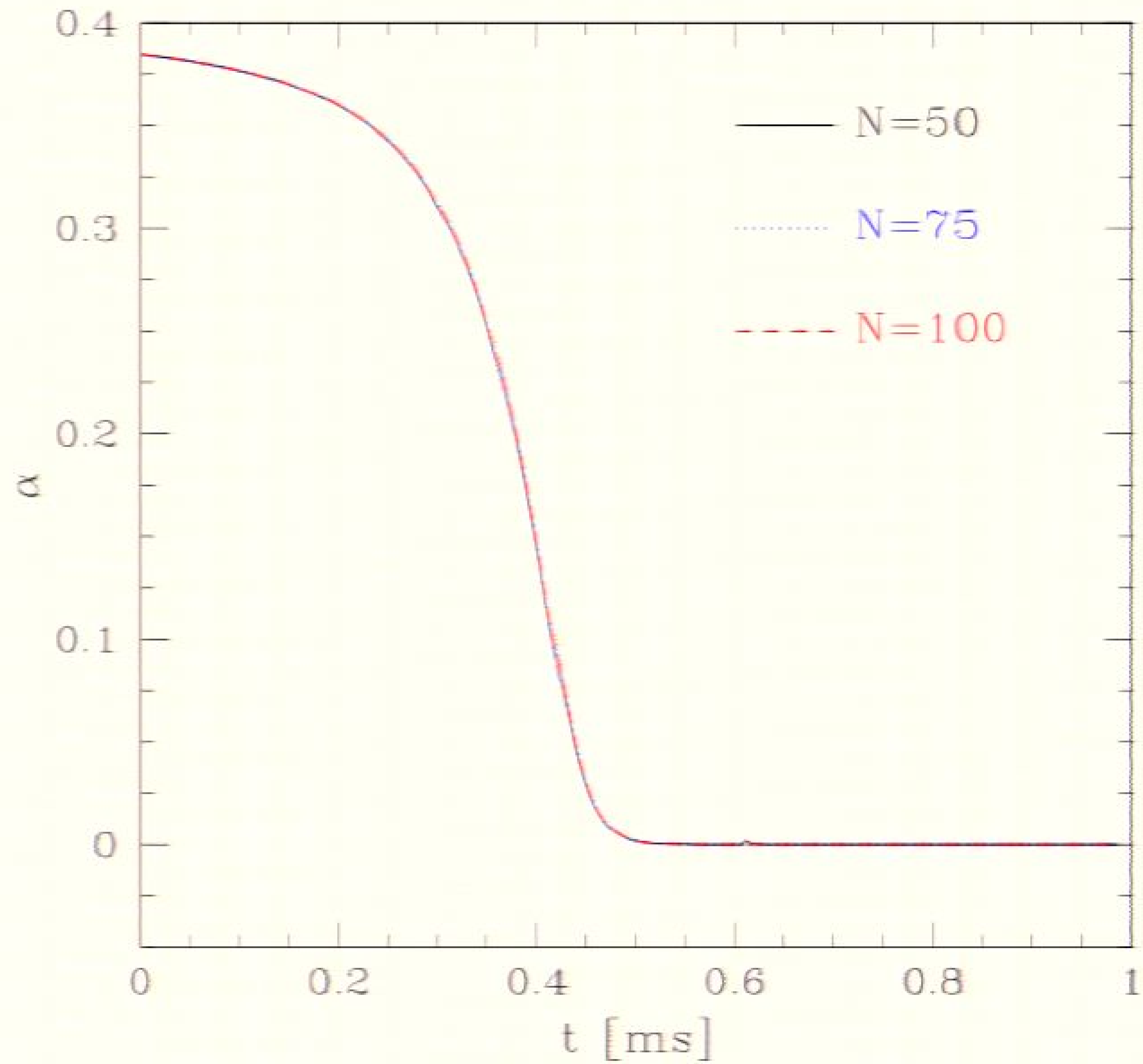


Evolution of the central density under the effects of the numerical perturbations.

Non-linear oscillations of a relativistic spherical star



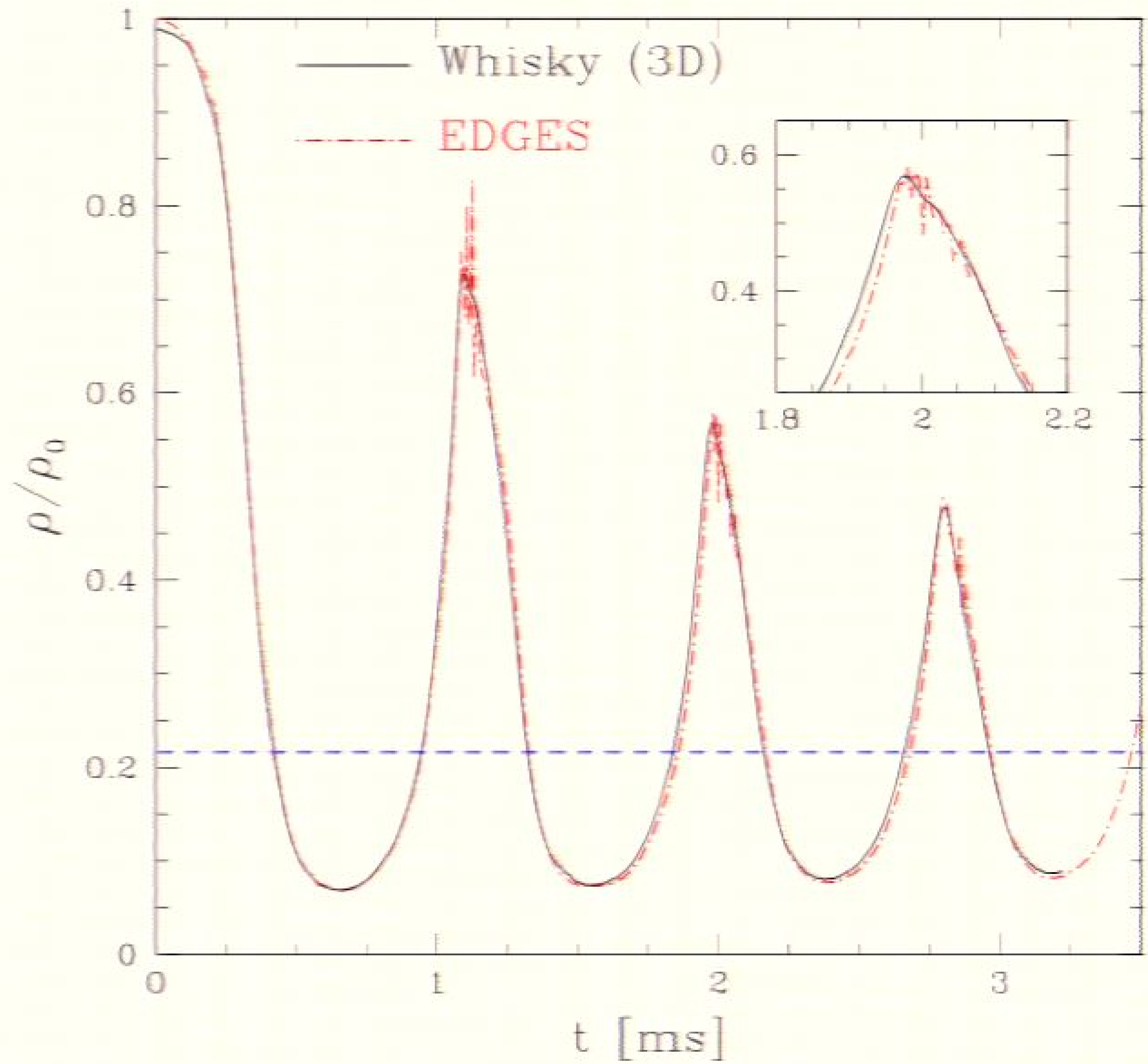
Gravitational collapse of a relativistic star



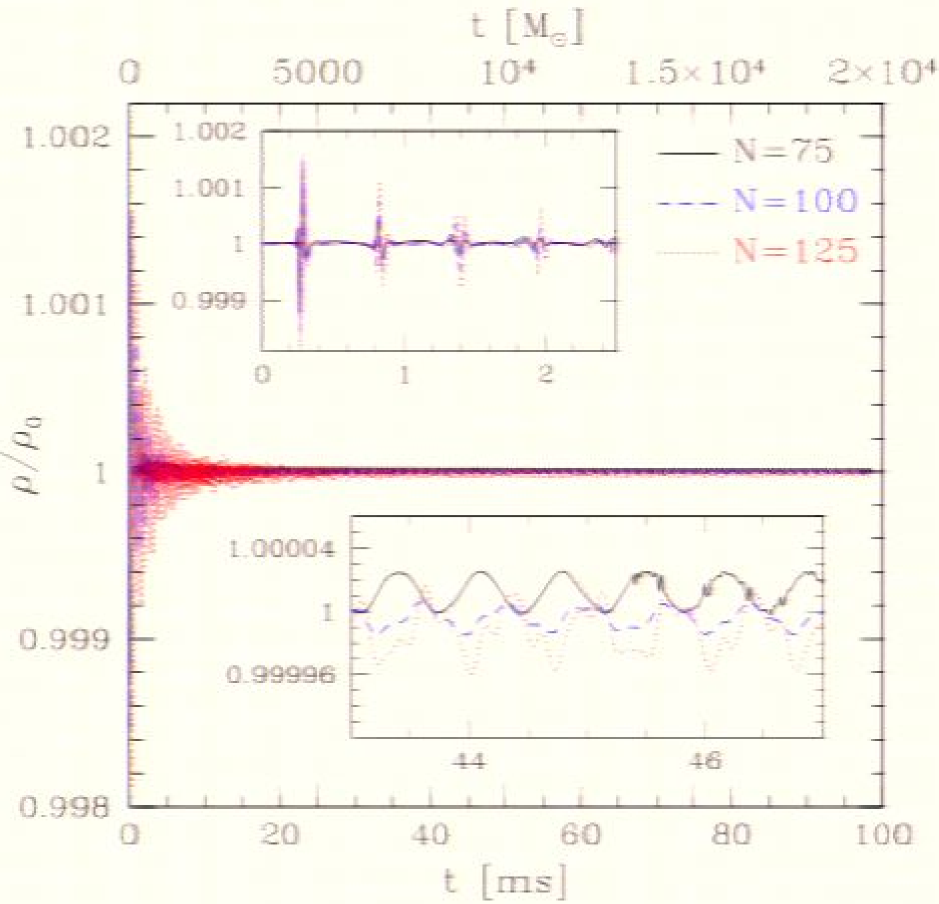
Conclusions

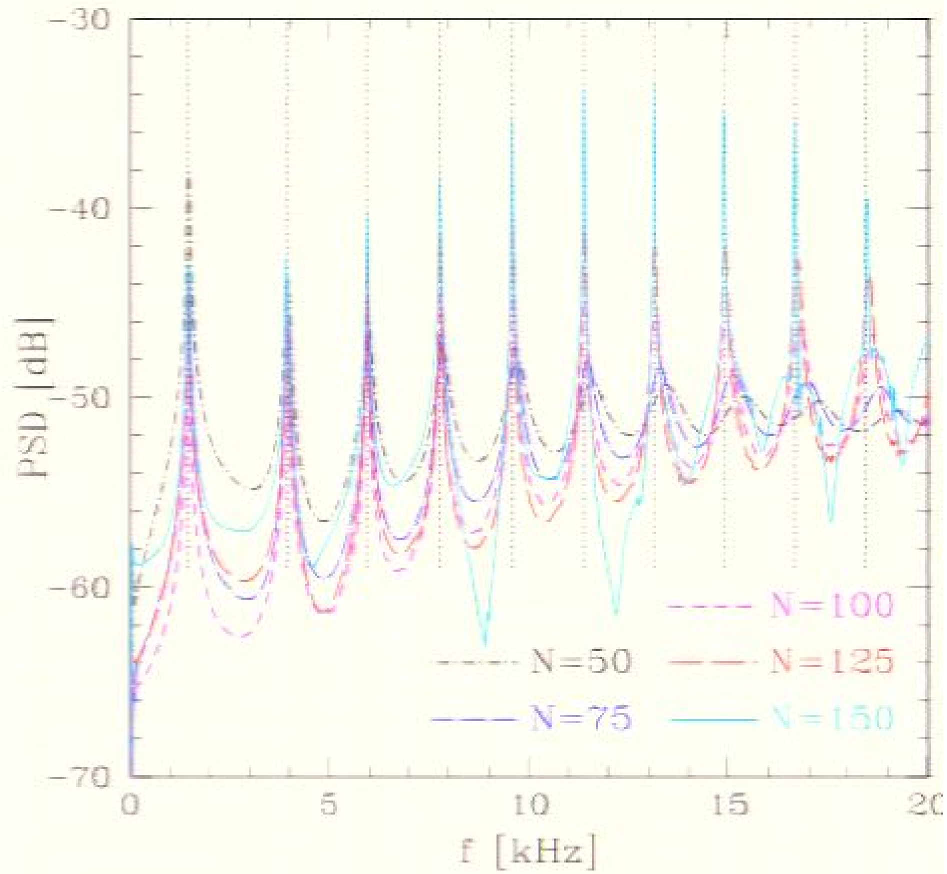
- Discontinuous Galerkin methods can be seen as a generalization of finite volumes and spectral methods: the hope is to get the best from both worlds.
- We derived a framework for the application of discontinuous Galerkin methods to relativistic hydrodynamics.
- Discontinuous Galerkin methods are able to handle strong relativistic shocks and, at the same time, attain high-order accuracy in smooth flow regions.
- Discontinuous Galerkin methods are a very attractive alternative to traditional finite volumes finite differences methods for general relativistic hydrodynamics.

Non-linear oscillations of a relativistic spherical star



Linear oscillations of a spherical star in full-GR





Evolution of the central density under the effects of the numerical perturbations.