Title: An effective theory for the neutron star inner crust

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Abstract: The inner crust of neutron stars has a remarkable property that it is crystalline as well as superfluid. I will describe the low energy theory of systems with this property in general, and describe how to relate the low energy constants of the theory to derivatives of the free energy with respect to lattice shape and chemical potentials. As an application, I will discuss the mixing of lattice and superfluid modes in the neutron star inner crust.

Summary

1. The neutron star inner crust is crystalline as well as superfluid
2. Discuss the low energy theory for such systems
3. Relation of the low energy coefficients (LECs) of the lagrangian to thermodynamic derivatives
4. Application to the neutron star crust

The neutron star inner crust

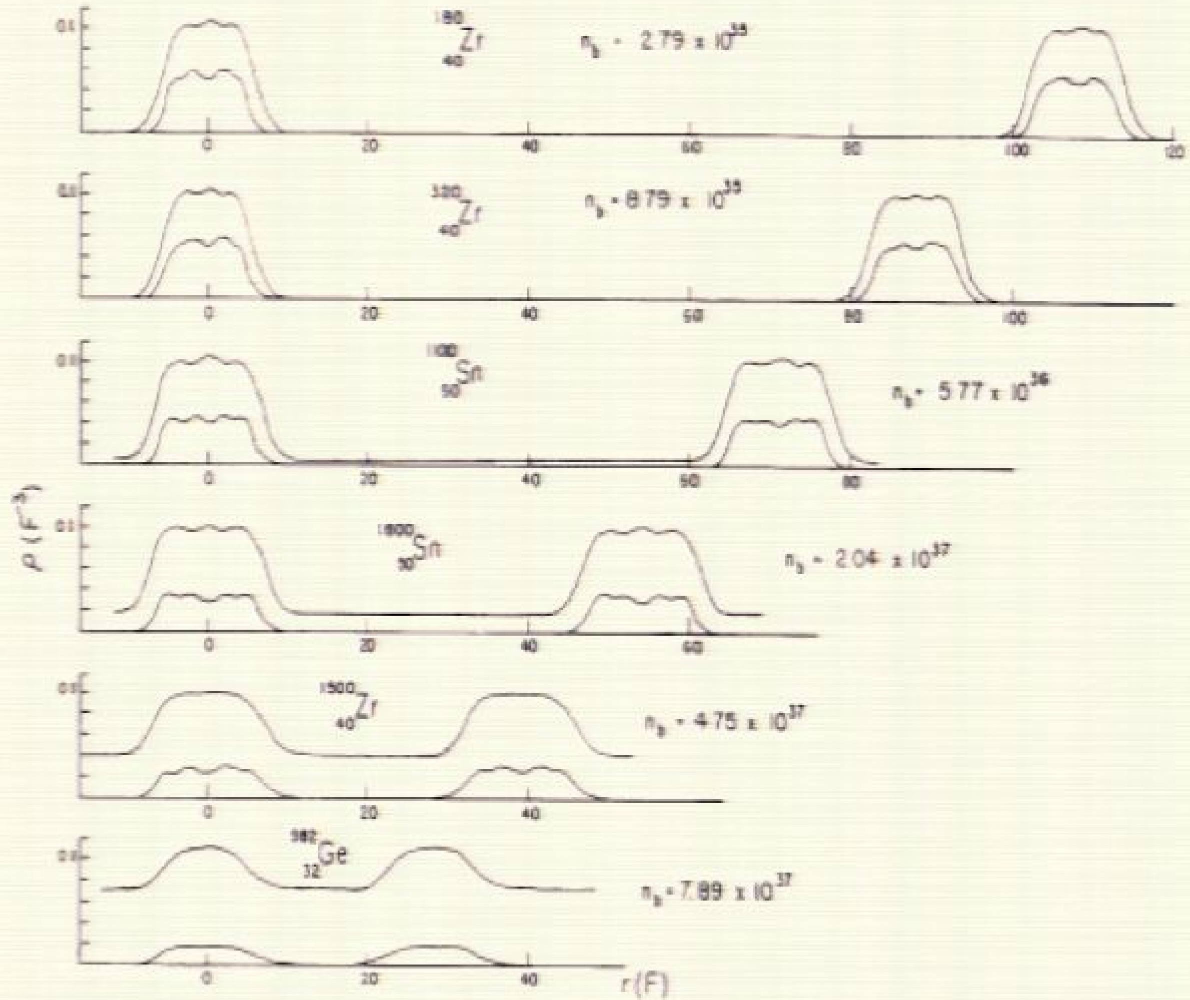
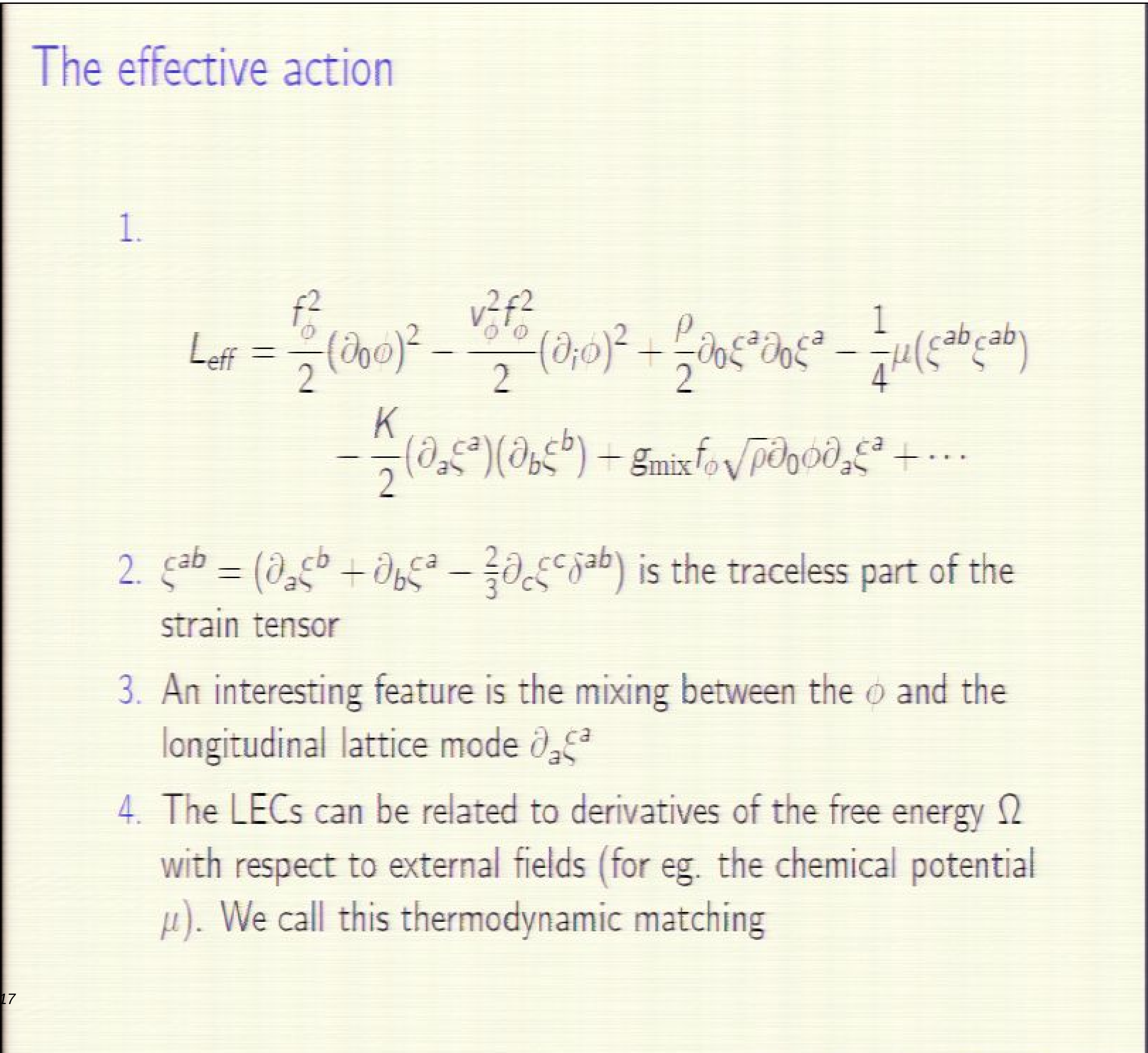


Fig. 3. Proton and neutron density distributions occurring along an axis joining the centers of two adjacent unit cells.

(Negele Vautherin (1973))

Low energy fields

1. One Goldstone mode is associated with the phase modulation of the condensate $\langle \psi_1 \psi_2 \rangle \propto |\Delta| e^{-2i\phi(x)}$
2. The second set of Goldstone modes is associated with translations and are the lattice phonons $\xi^a(x)$
3. Symmetries require invariance under constant shifts
    - $\phi(x) \to \phi(x) + \theta$
    - $\xi^a(x) \to \xi^a(x) + b^a$



The effective action

1. $$L_{eff} = \frac{f_\phi^2}{2}(\partial_0\phi)^2 - \frac{v_\phi^2 f_\phi^2}{2}(\partial_i\phi)^2 + \frac{\rho}{2}\partial_0\xi^a\partial_0\xi^a - \frac{1}{4}\mu(\xi^{ab}\xi^{ab})$$
$$- \frac{K}{2}(\partial_a\xi^a)(\partial_b\xi^b) + g_{\rm mix} f_\phi\sqrt{\rho}\partial_0\phi\partial_a\xi^a + \cdots$$
2. $\xi^{ab} = (\partial_a\xi^b + \partial_b\xi^a - \frac{2}{3}\partial_c\xi^c\delta^{ab})$ is the traceless part of the strain tensor
3. An interesting feature is the mixing between the ϕ and the longitudinal lattice mode $\partial_a\xi^a$
4. The LECs can be related to derivatives of the free energy Ω with respect to external fields (for eg. the chemical potential μ). We call this thermodynamic matching

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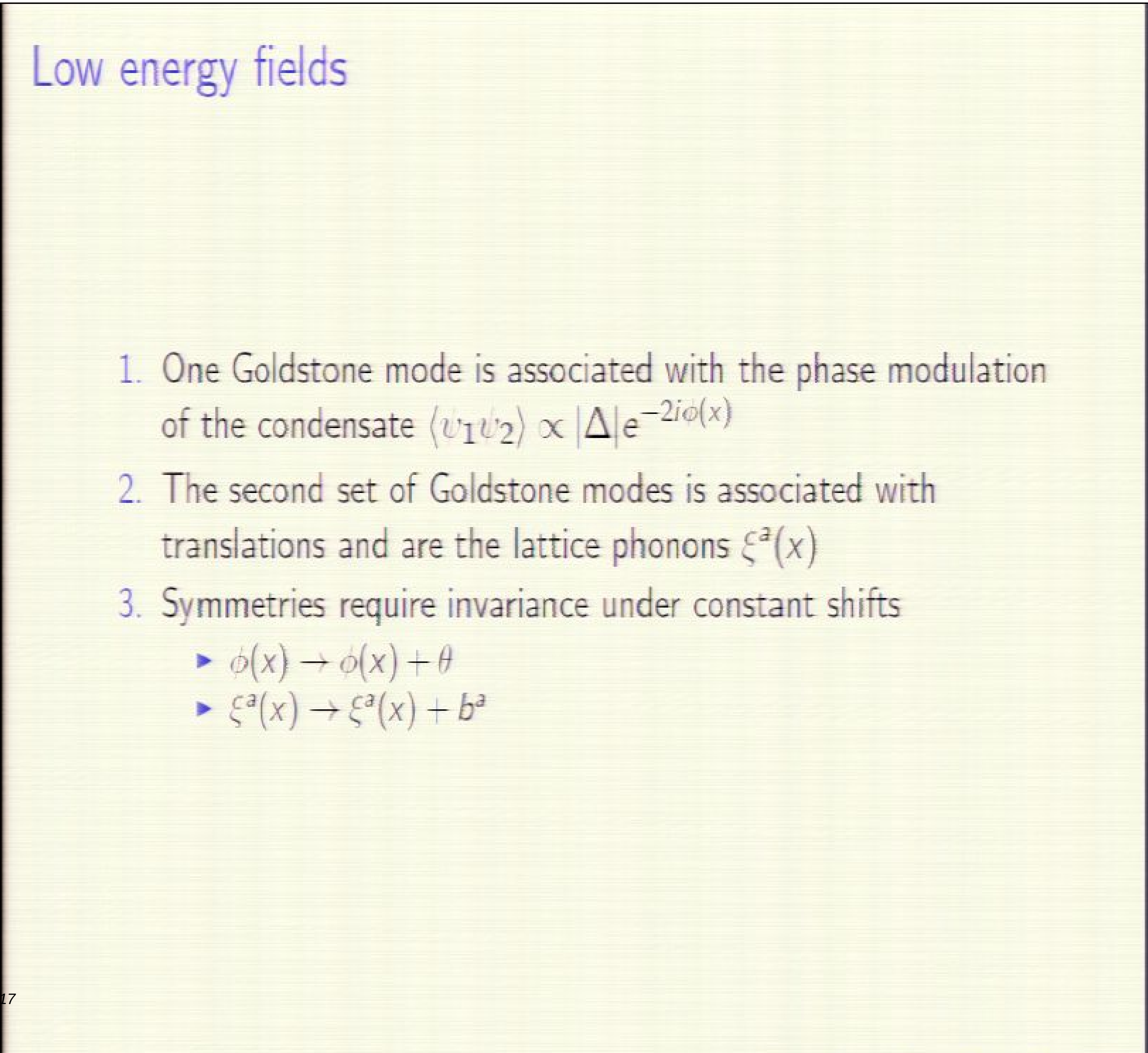

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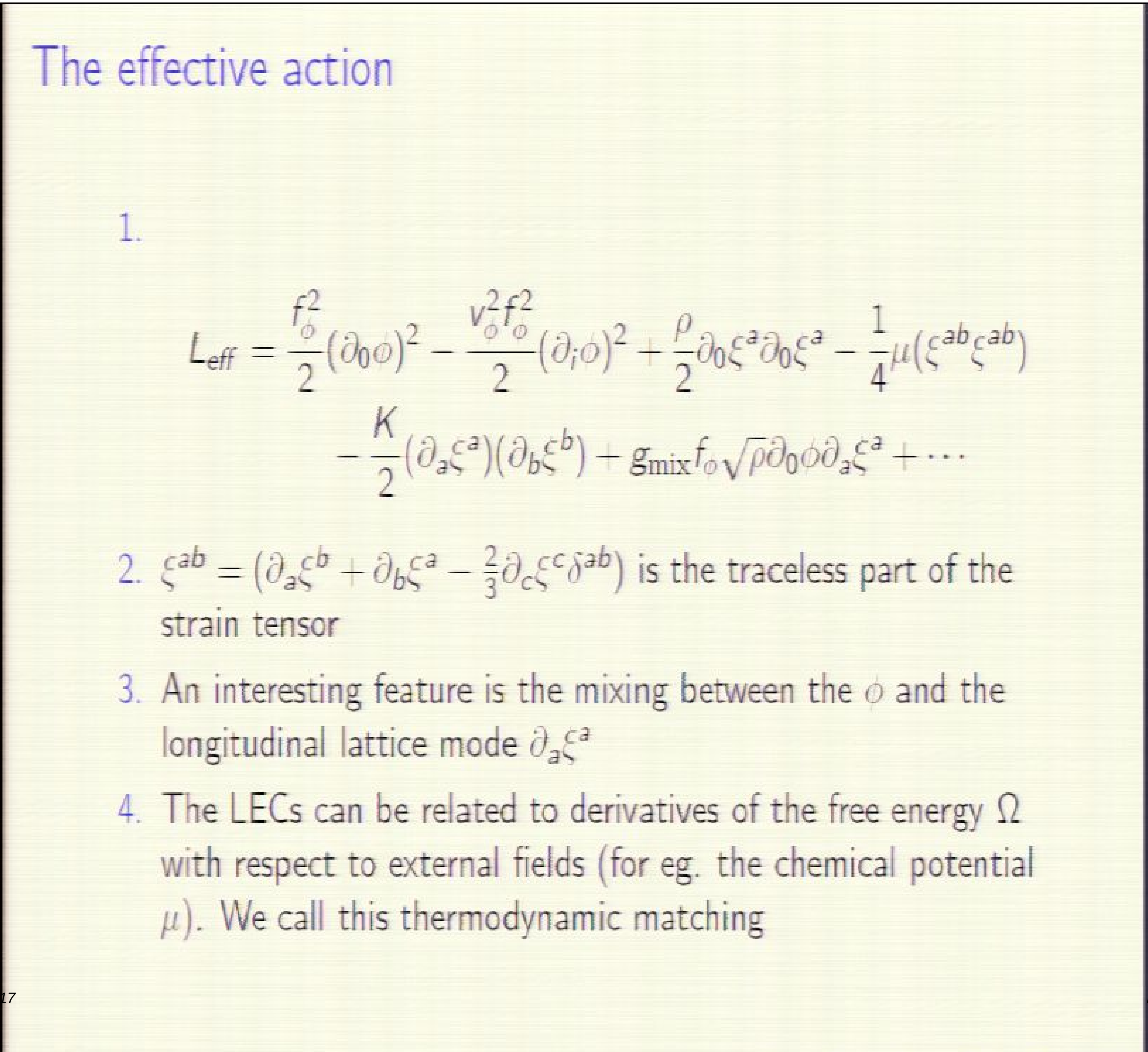

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3. Slightly more general form $\mathcal{L} = \mathcal{L}_v[\psi] + A_\mu(x)j^\mu$
4. For constant $A_\mu(x) = \bar{A}_\mu = (\mu.0)$ we get back the standard grand canonical picture, but in the intermediate stages we keep the external field general
5. Action invariant under gauge transformations $\psi \to \psi \exp(i\theta(x))$ which corresponds to the transformation of the phase field, $\phi \to \phi + \theta(x)$, and, $A_\mu \to A_\mu - \partial_\mu\theta$

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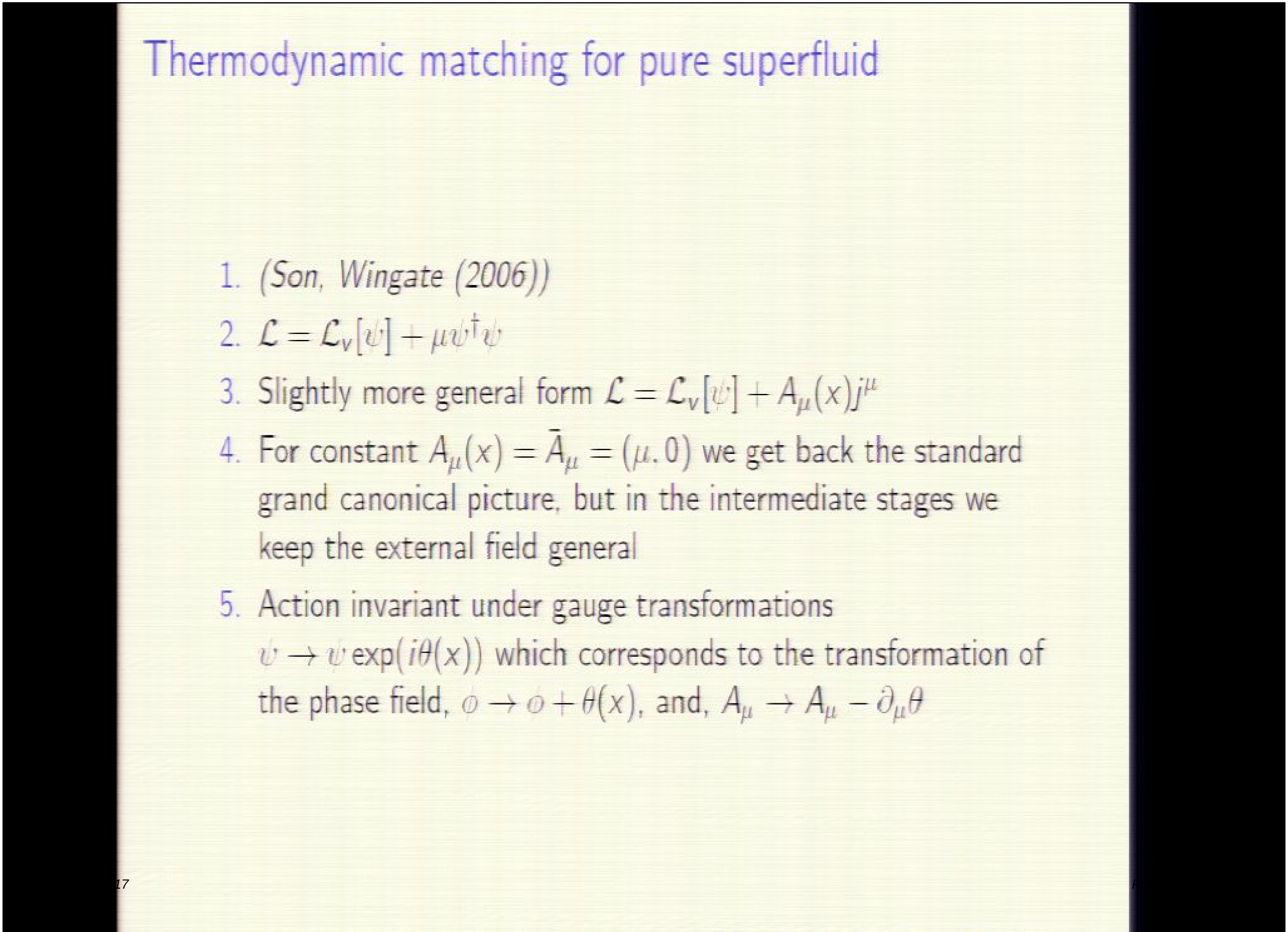

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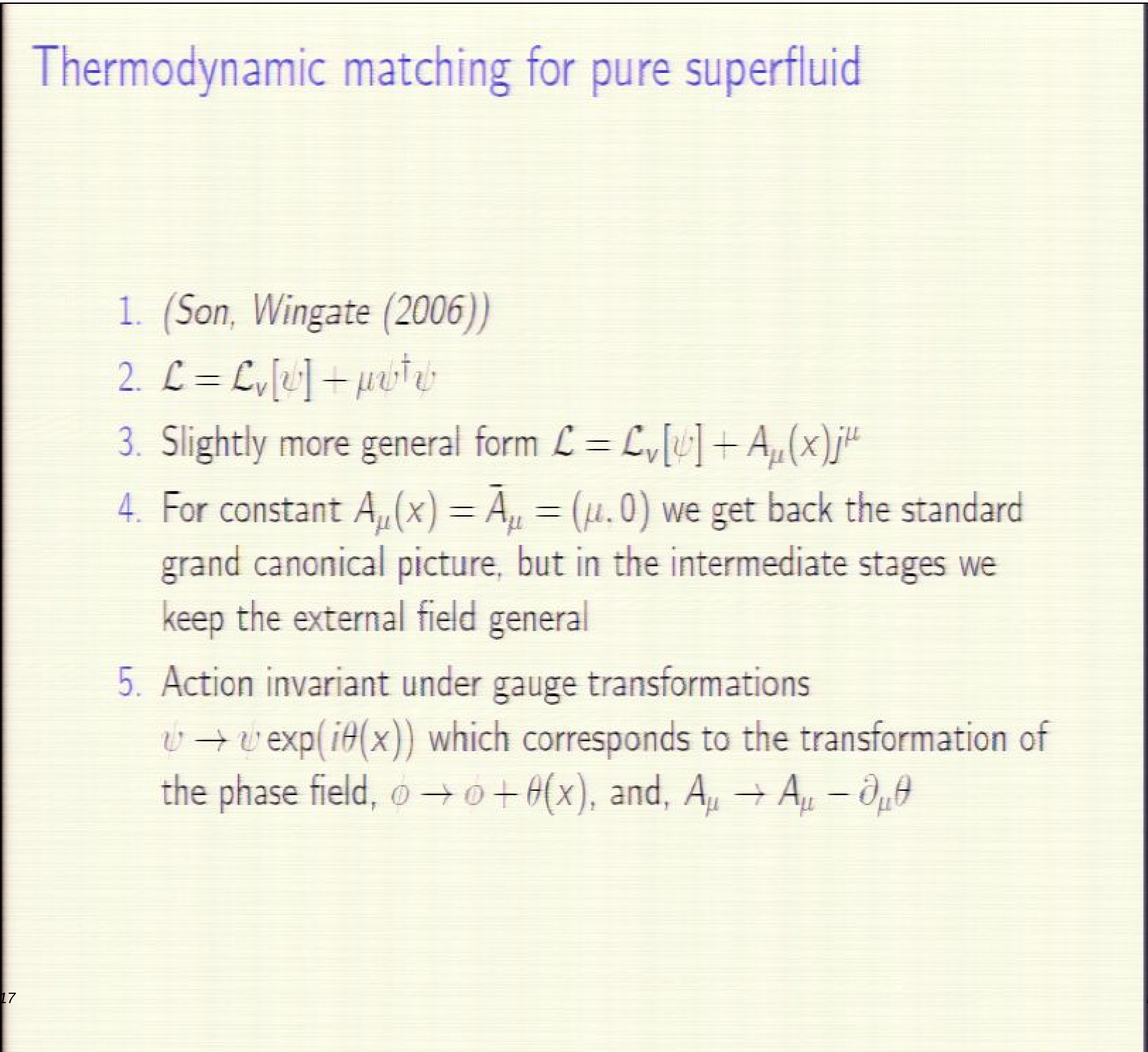

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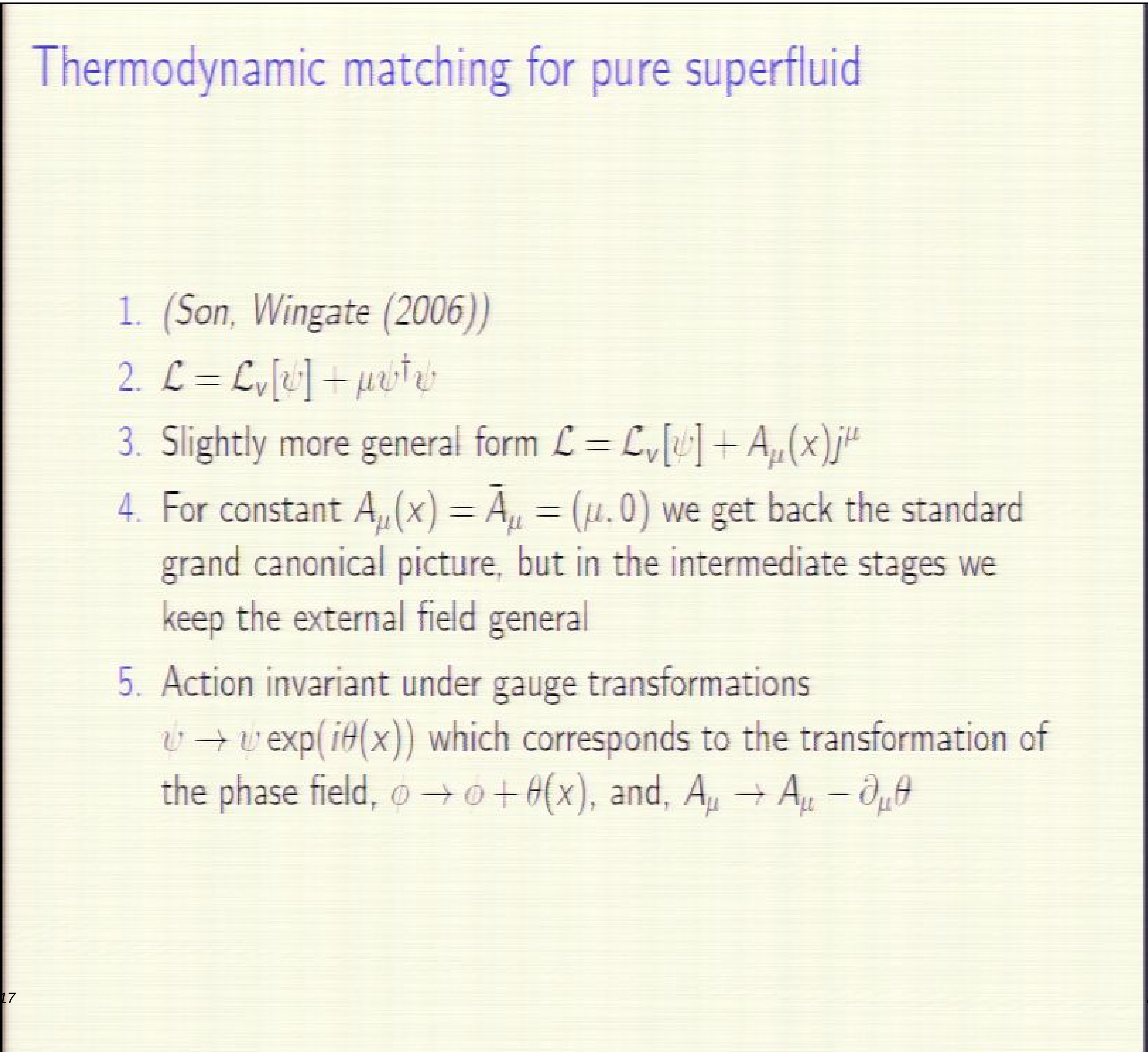

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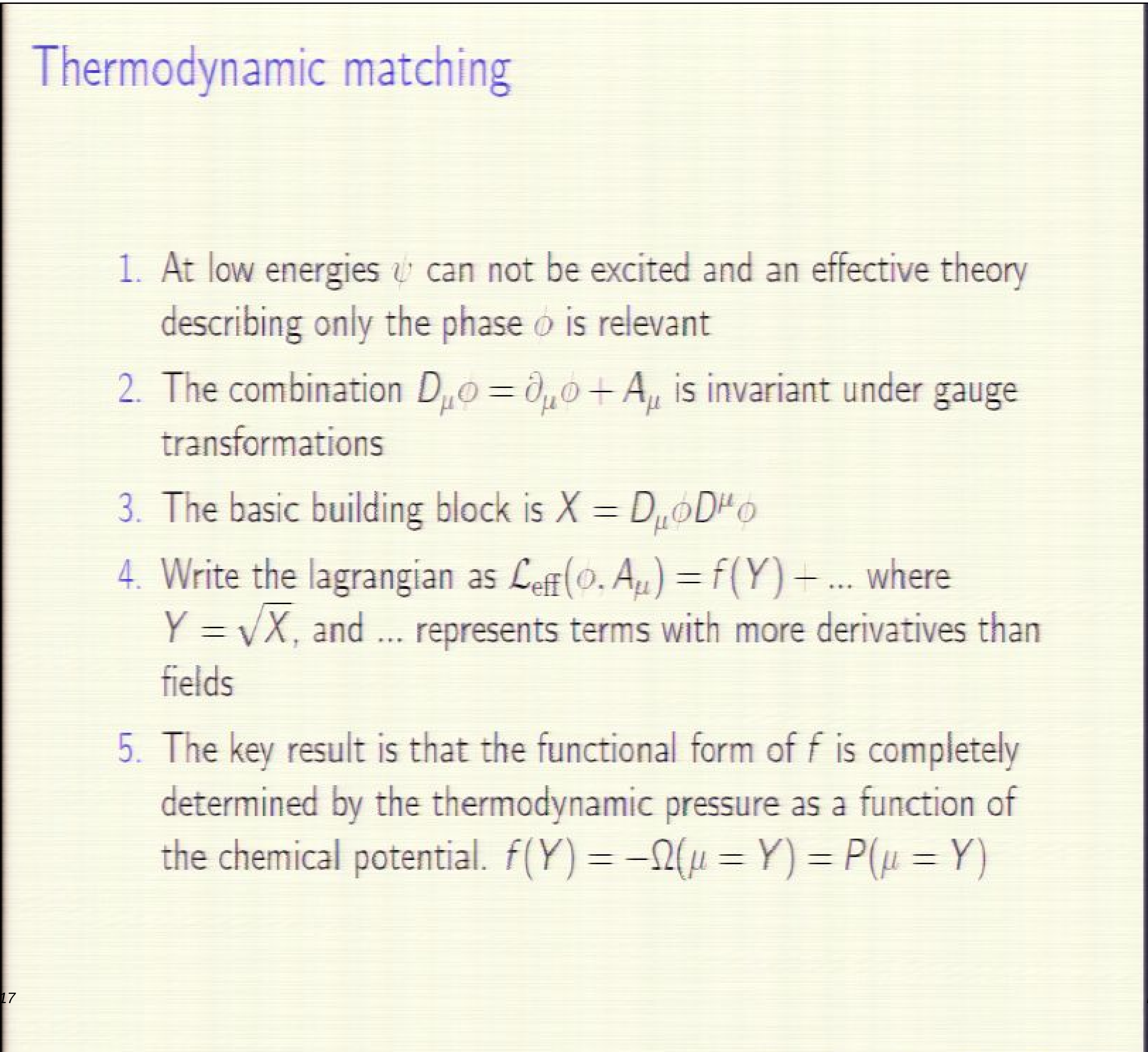
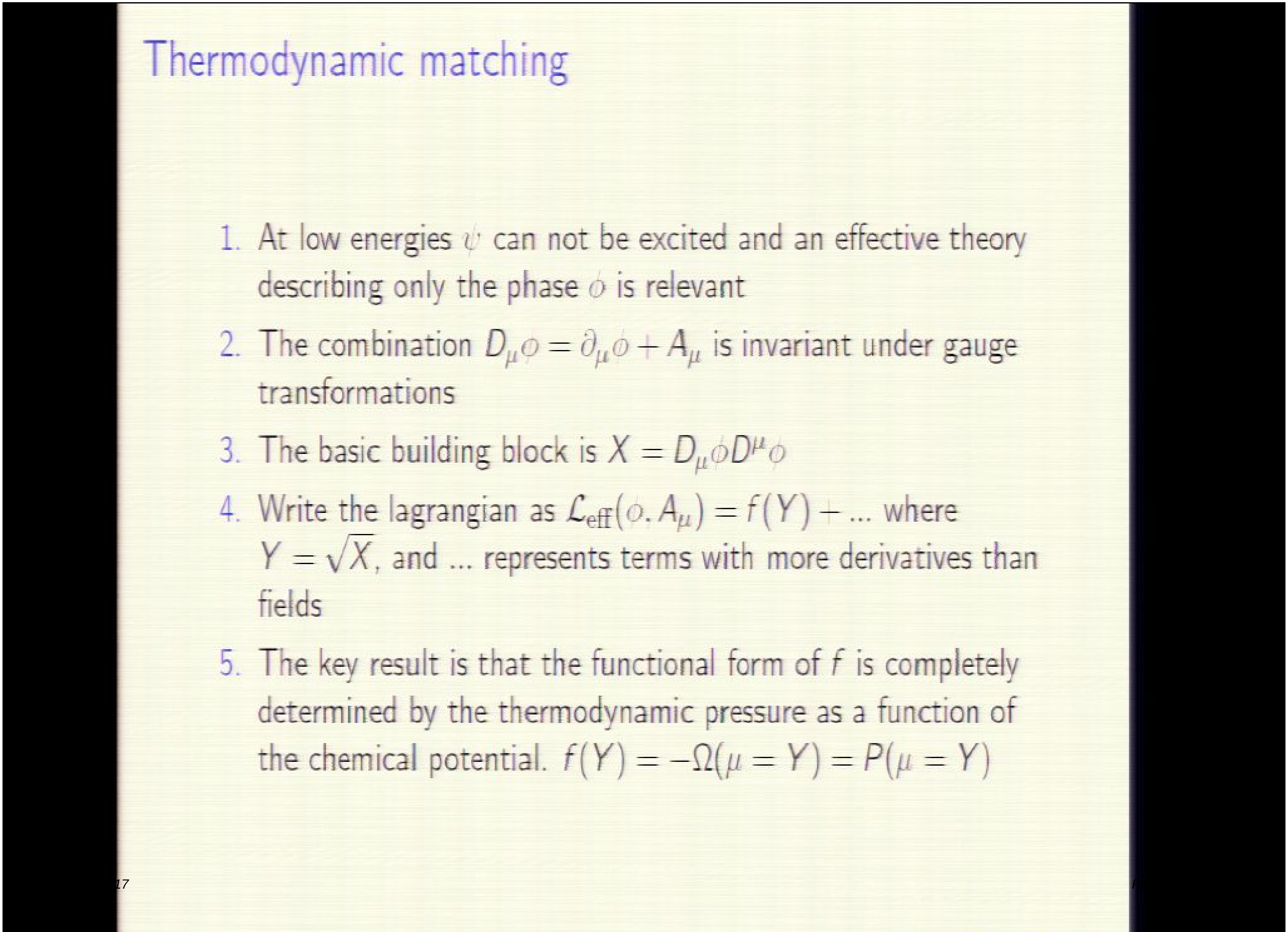


Thermodynamic matching

1. At low energies ψ can not be excited and an effective theory describing only the phase ϕ is relevant
2. The combination $D_\mu\phi = \partial_\mu\phi + A_\mu$ is invariant under gauge transformations
3. The basic building block is $X = D_\mu\phi D^\mu\phi$
4. Write the lagrangian as $\mathcal{L}_{\text{eff}}(\phi, A_\mu) = f(Y) + \dots$ where $Y = \sqrt{X}$, and ... represents terms with more derivatives than fields
5. The key result is that the functional form of f is completely determined by the thermodynamic pressure as a function of the chemical potential. $f(Y) = -\Omega(\mu = Y) = P(\mu = Y)$



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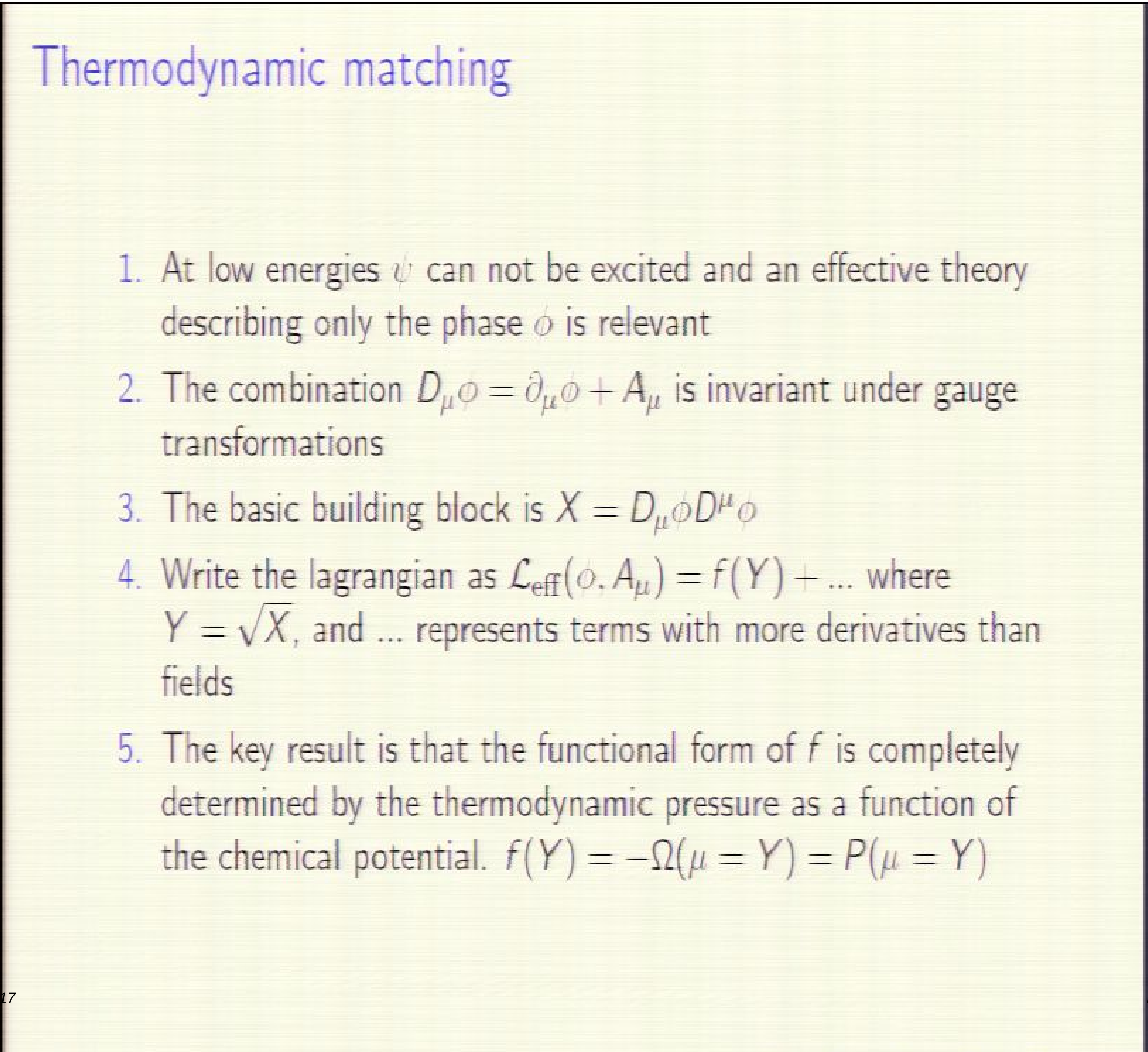

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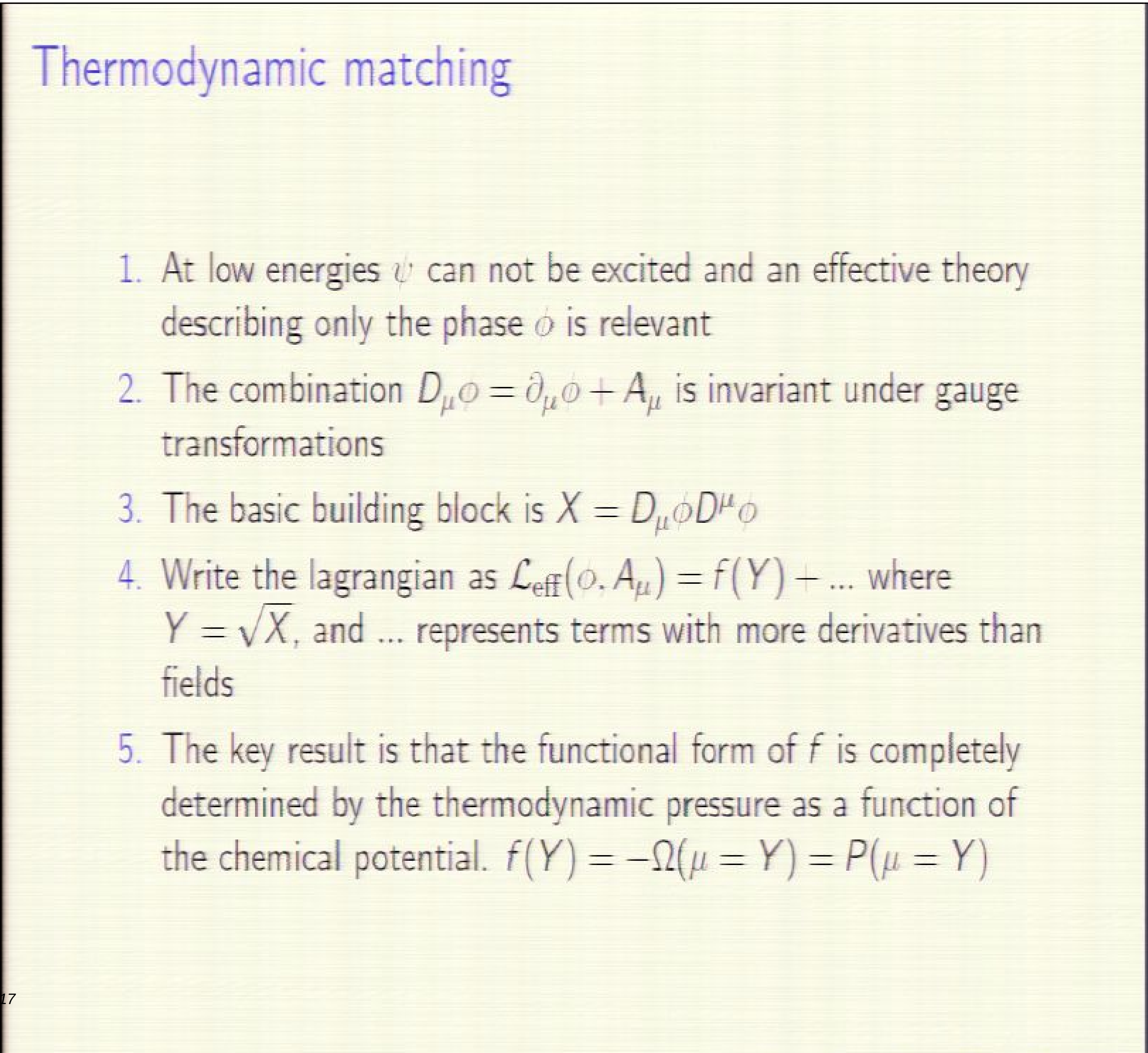

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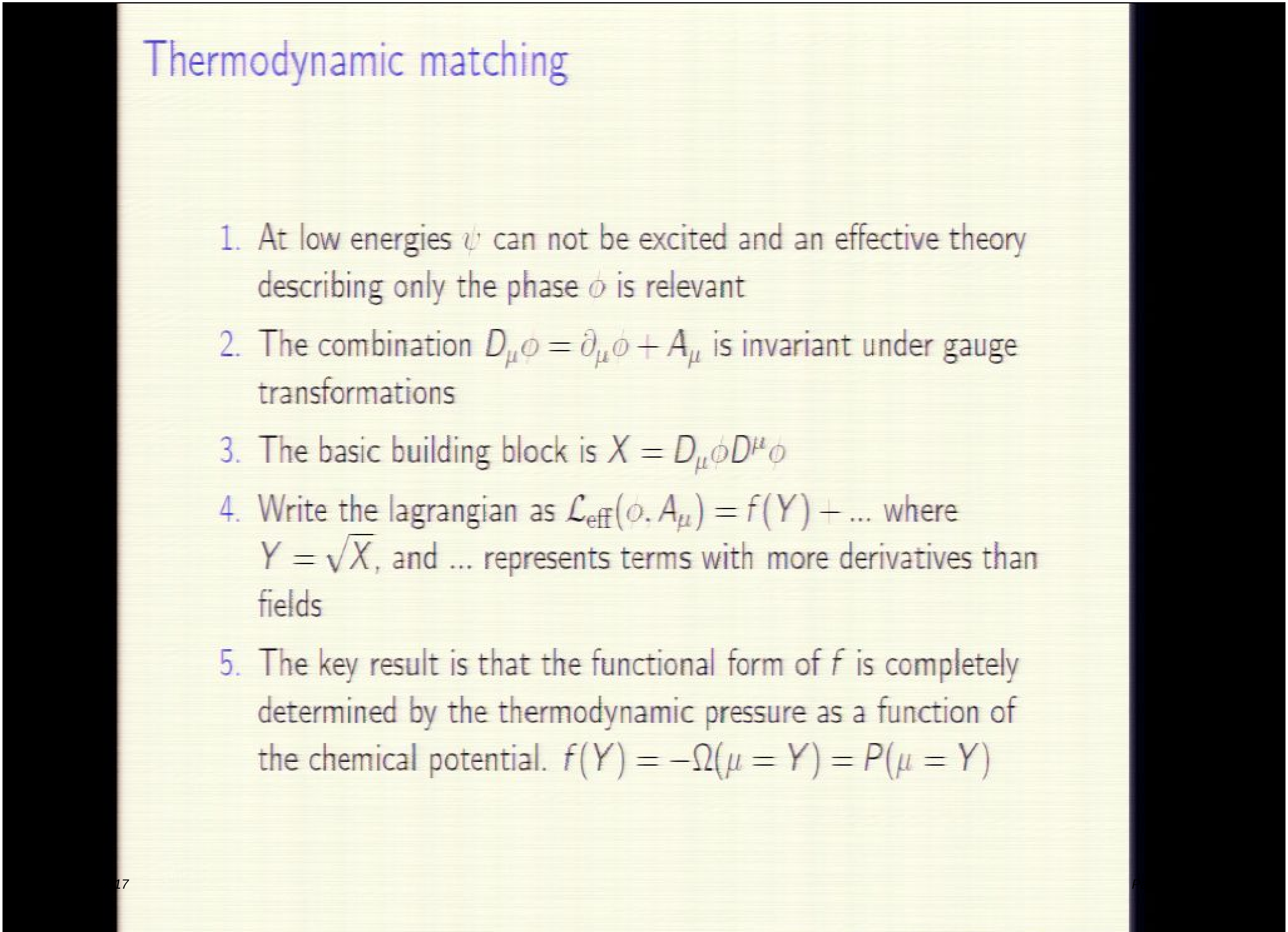

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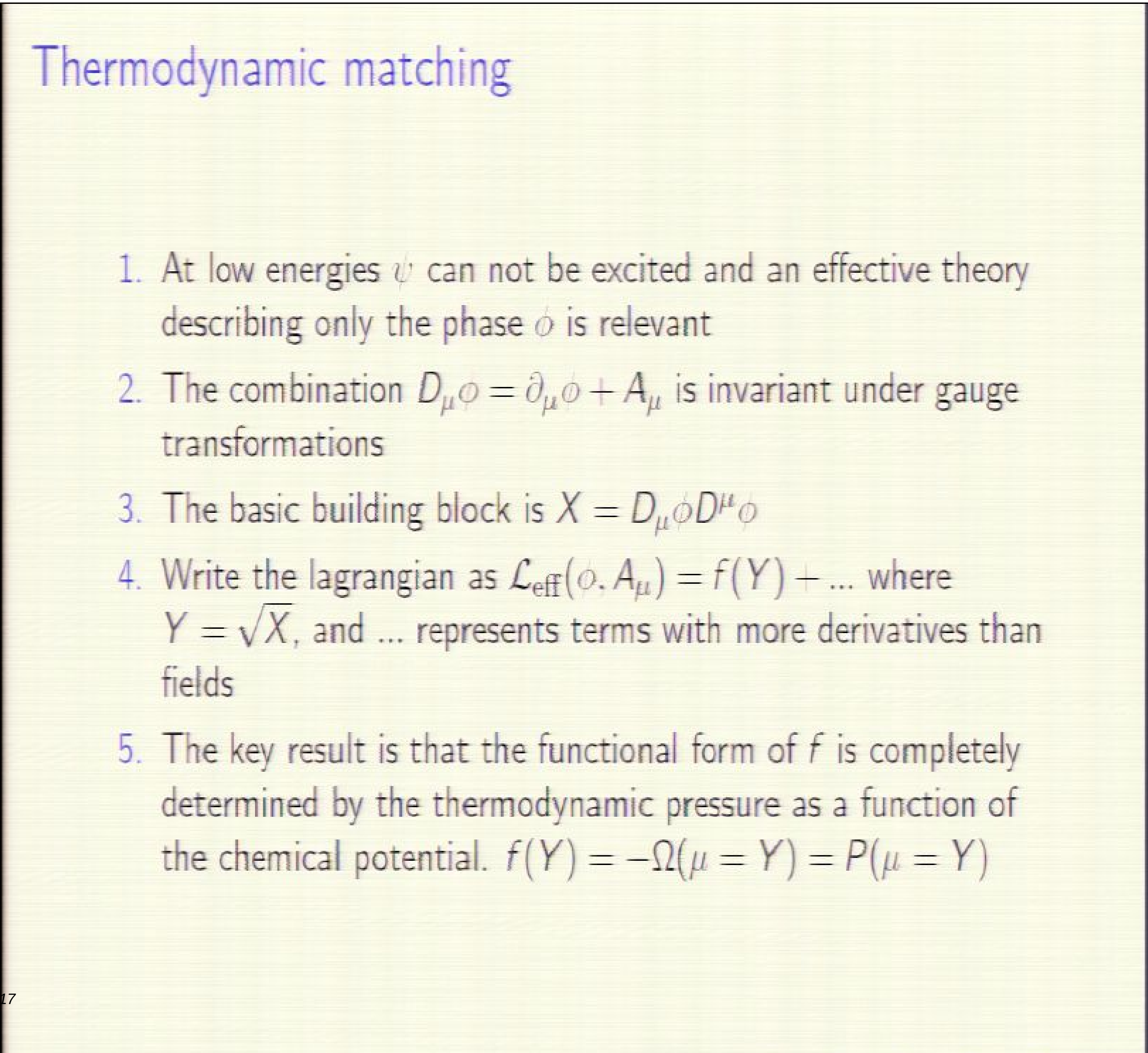

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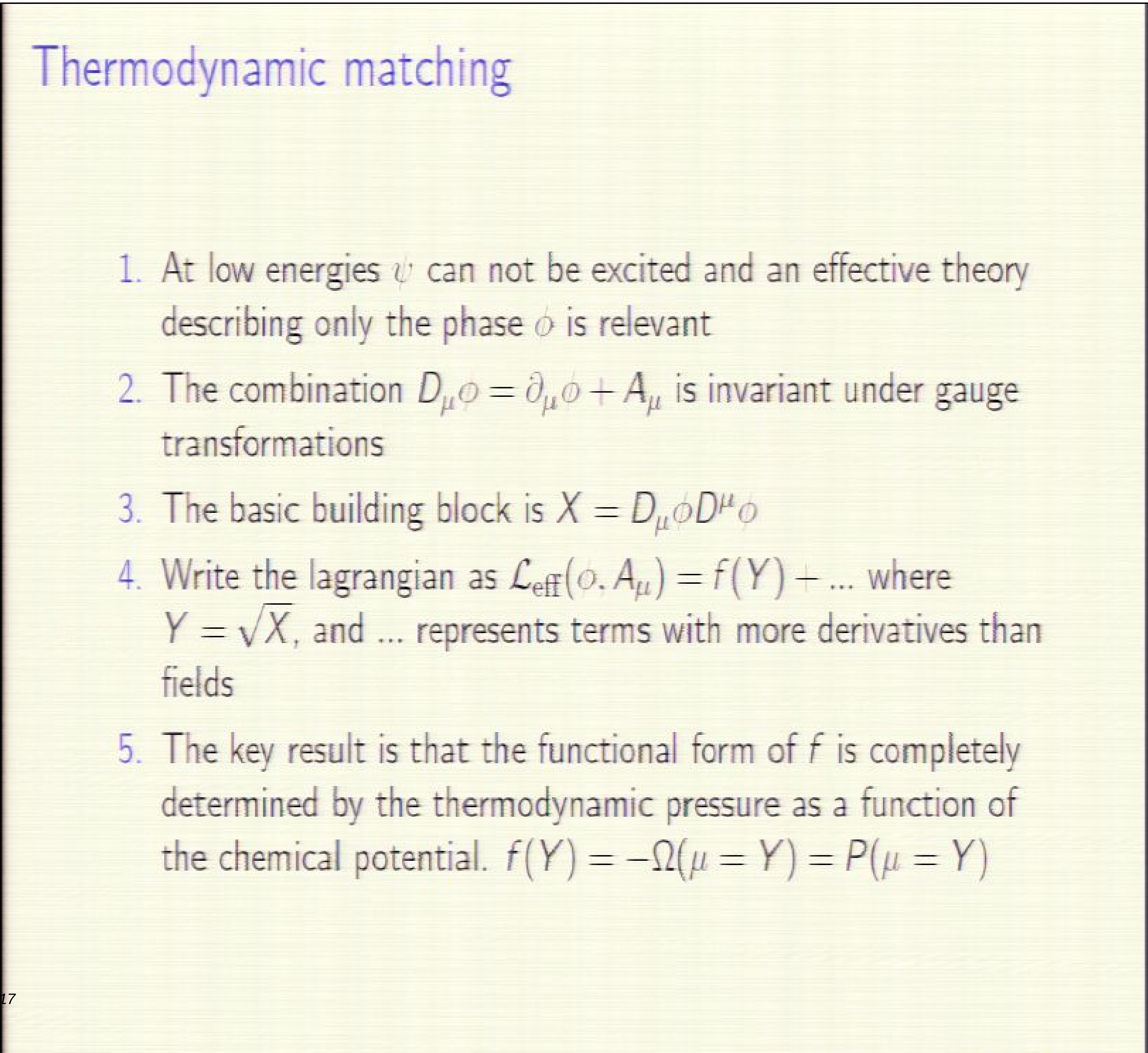

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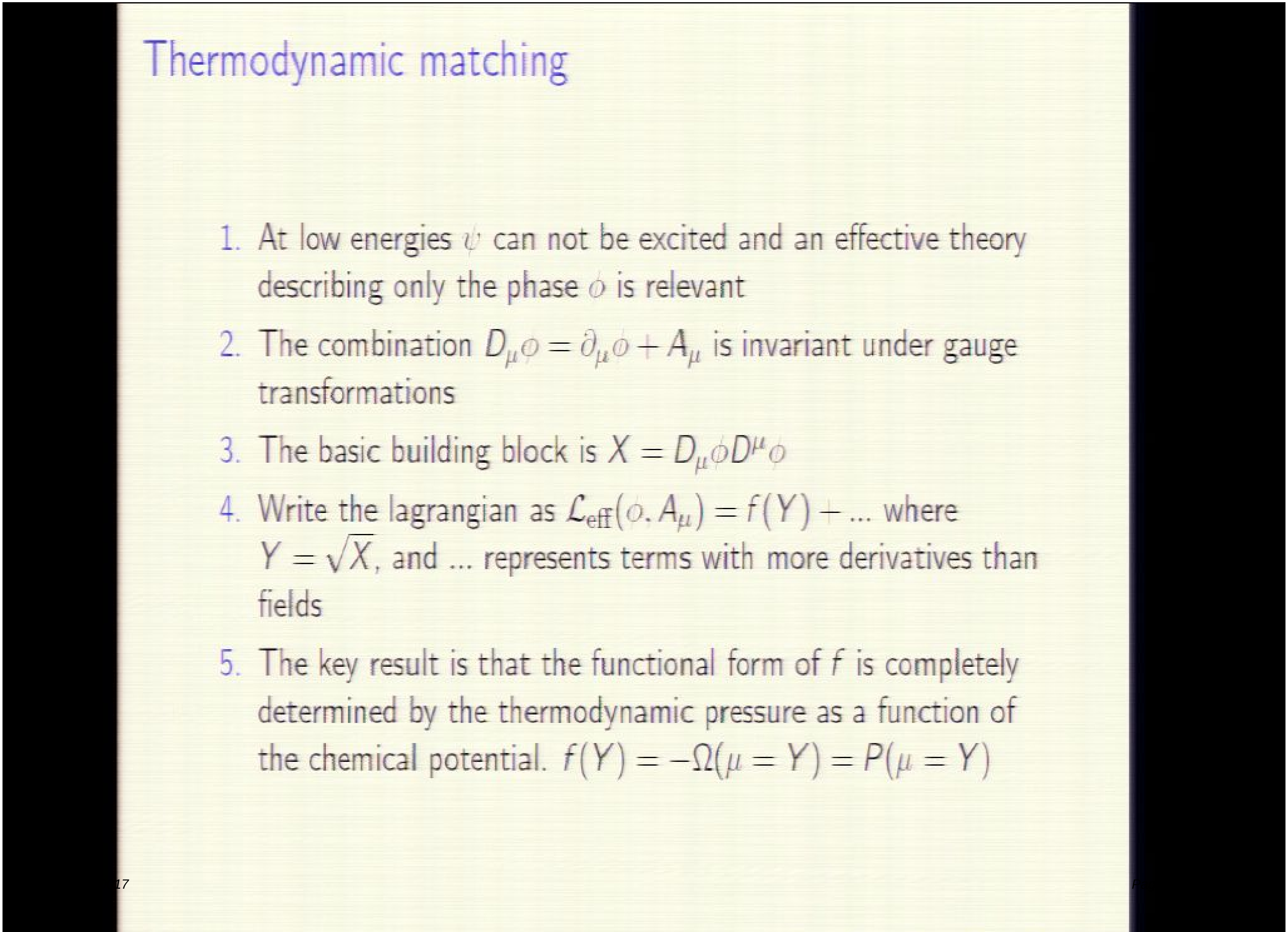

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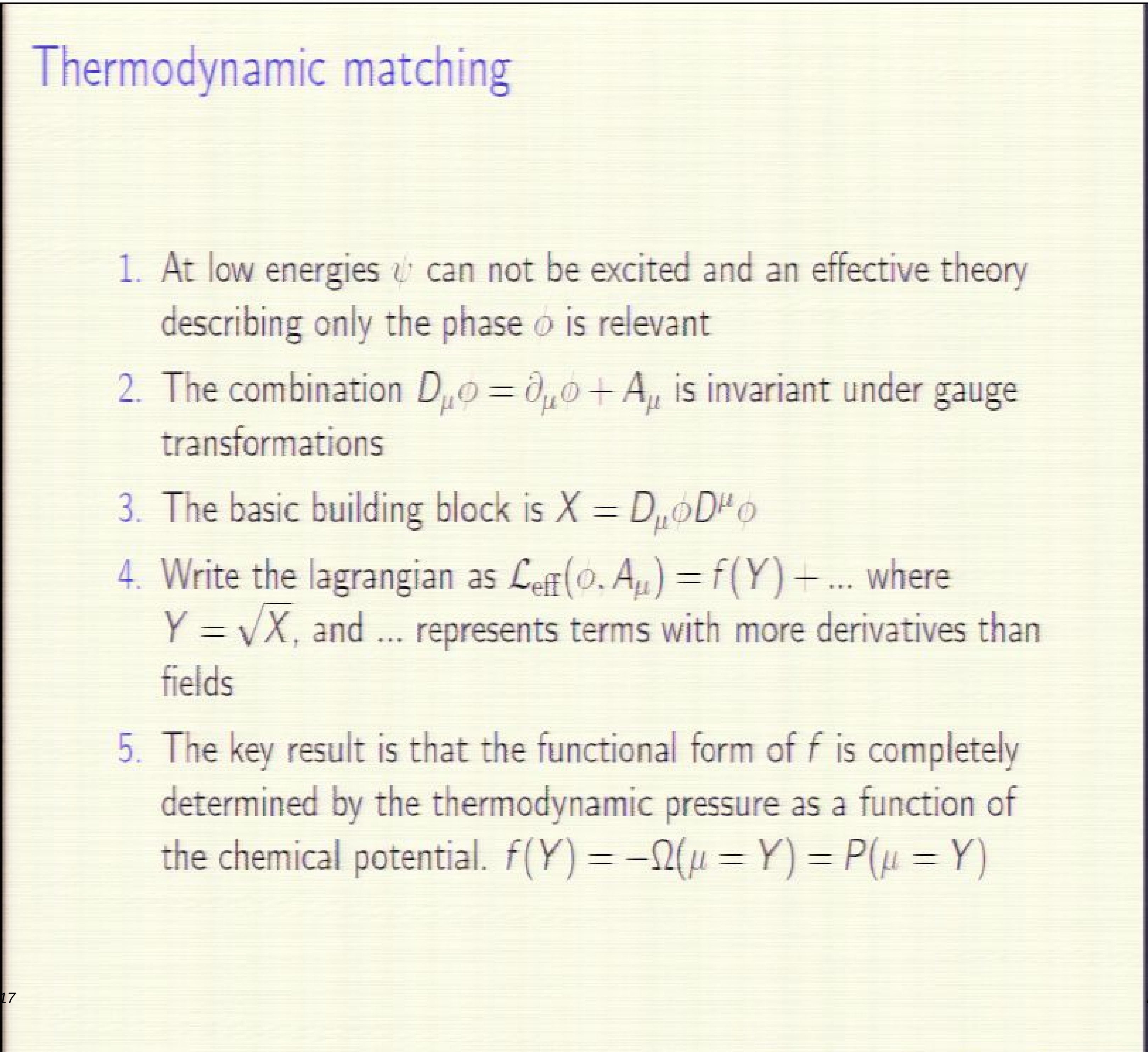

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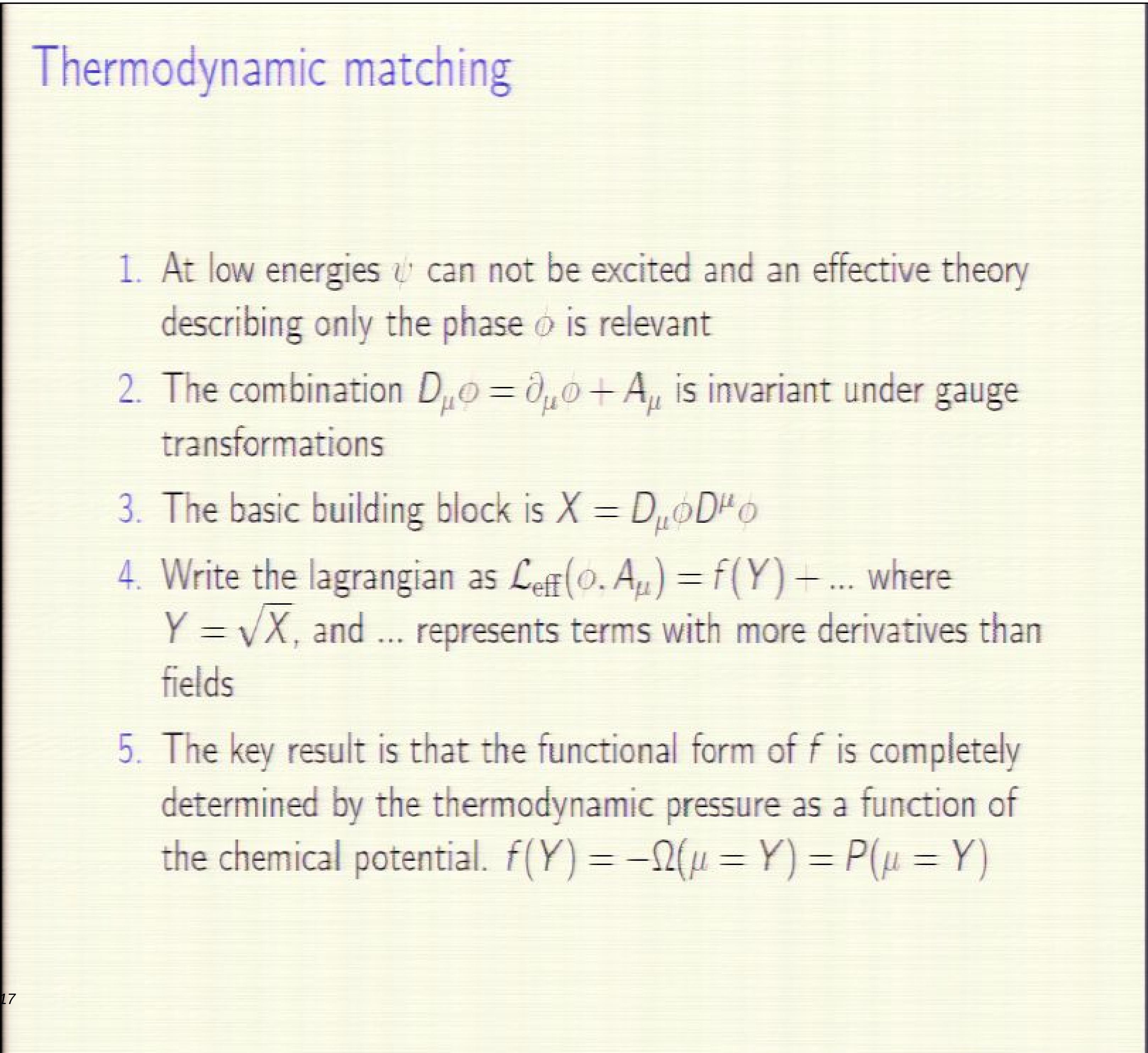

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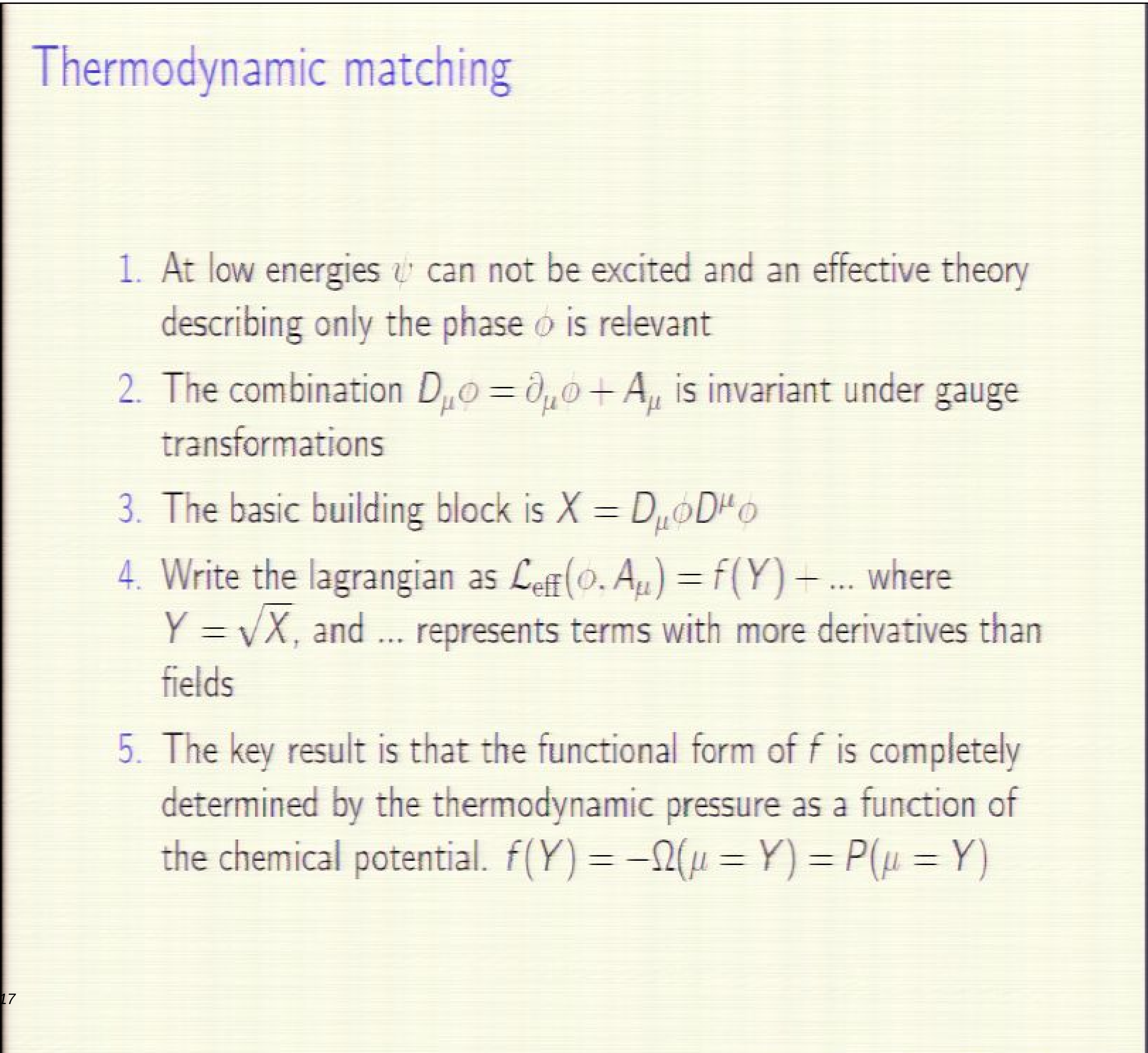

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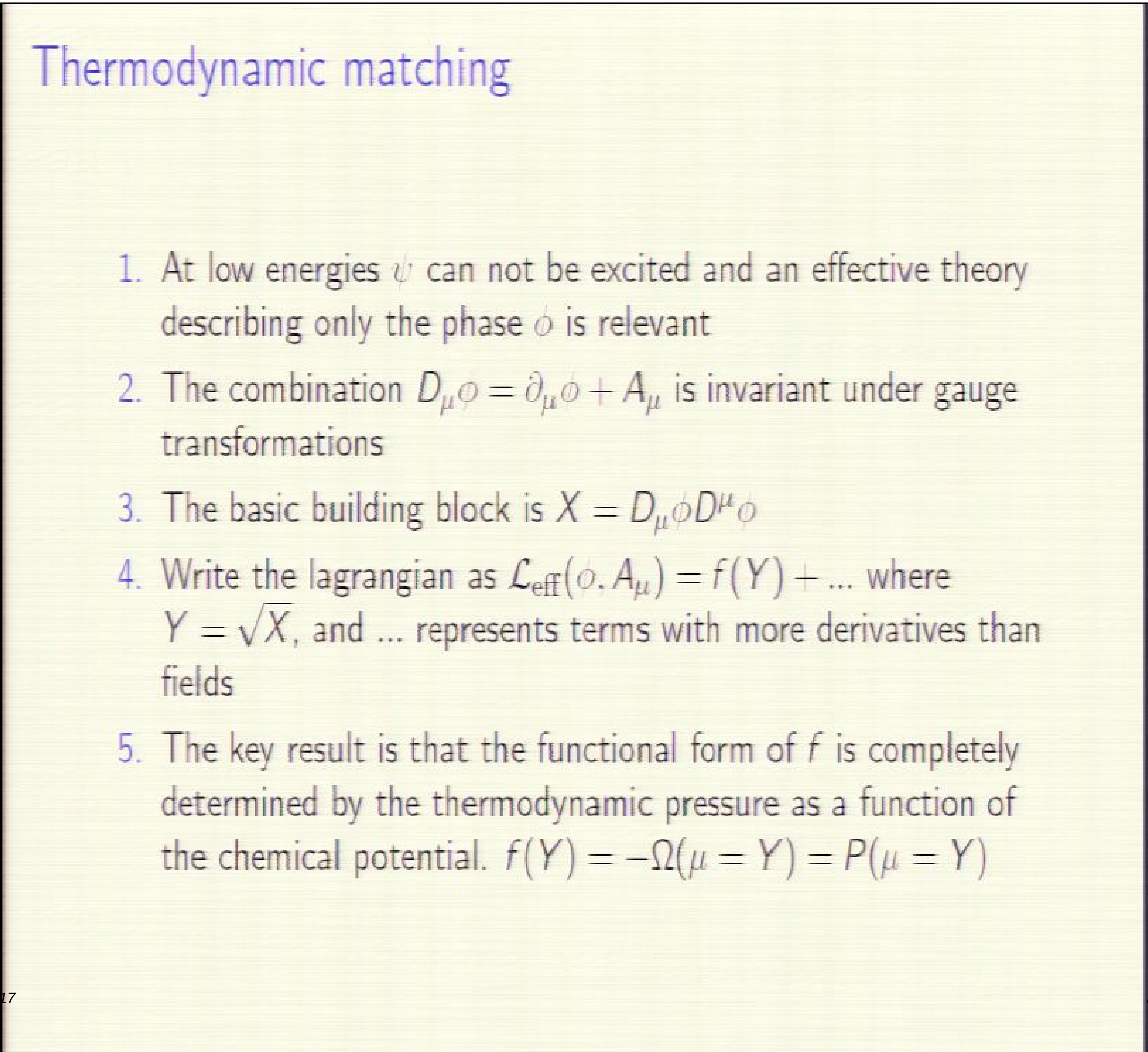

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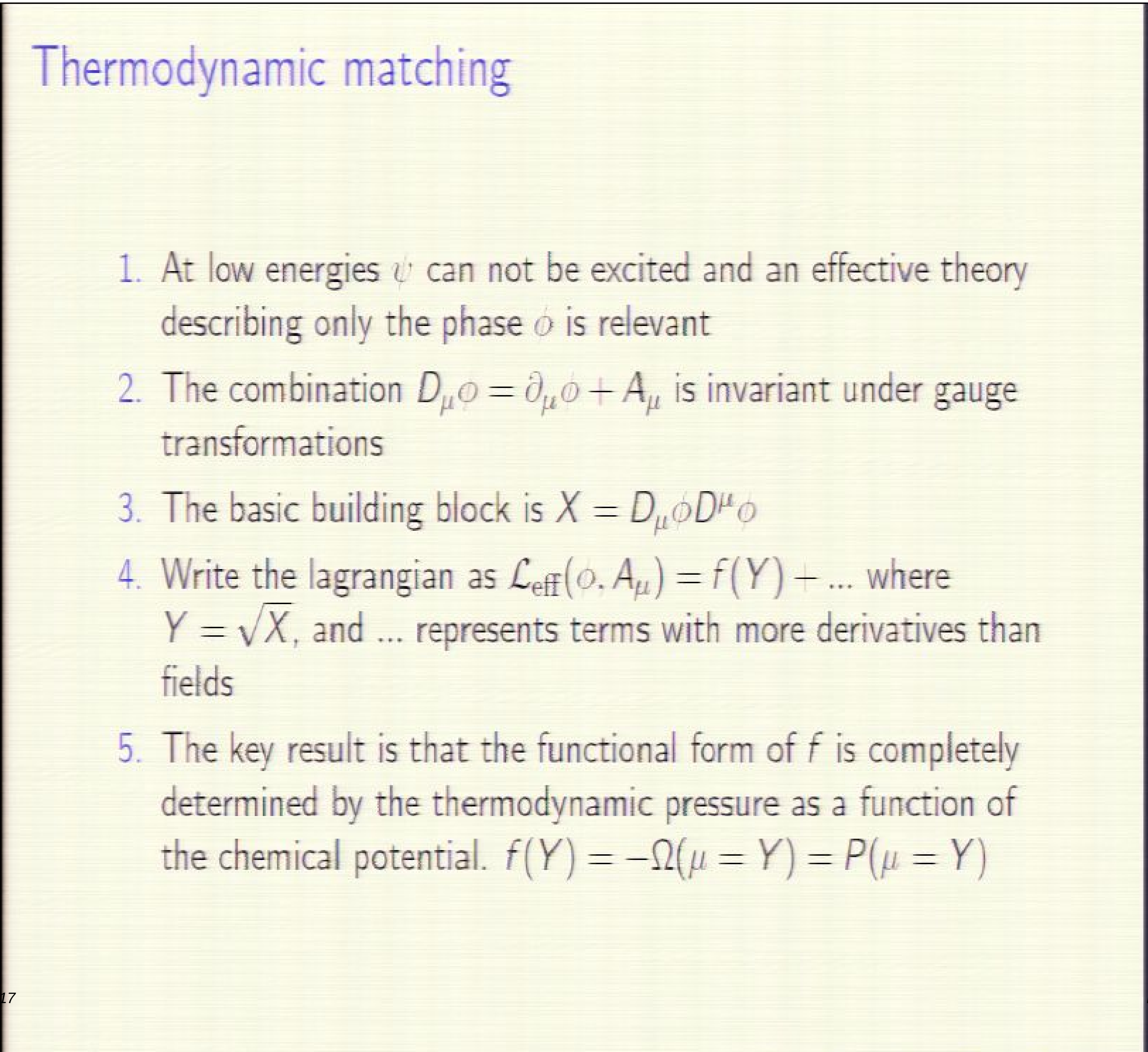

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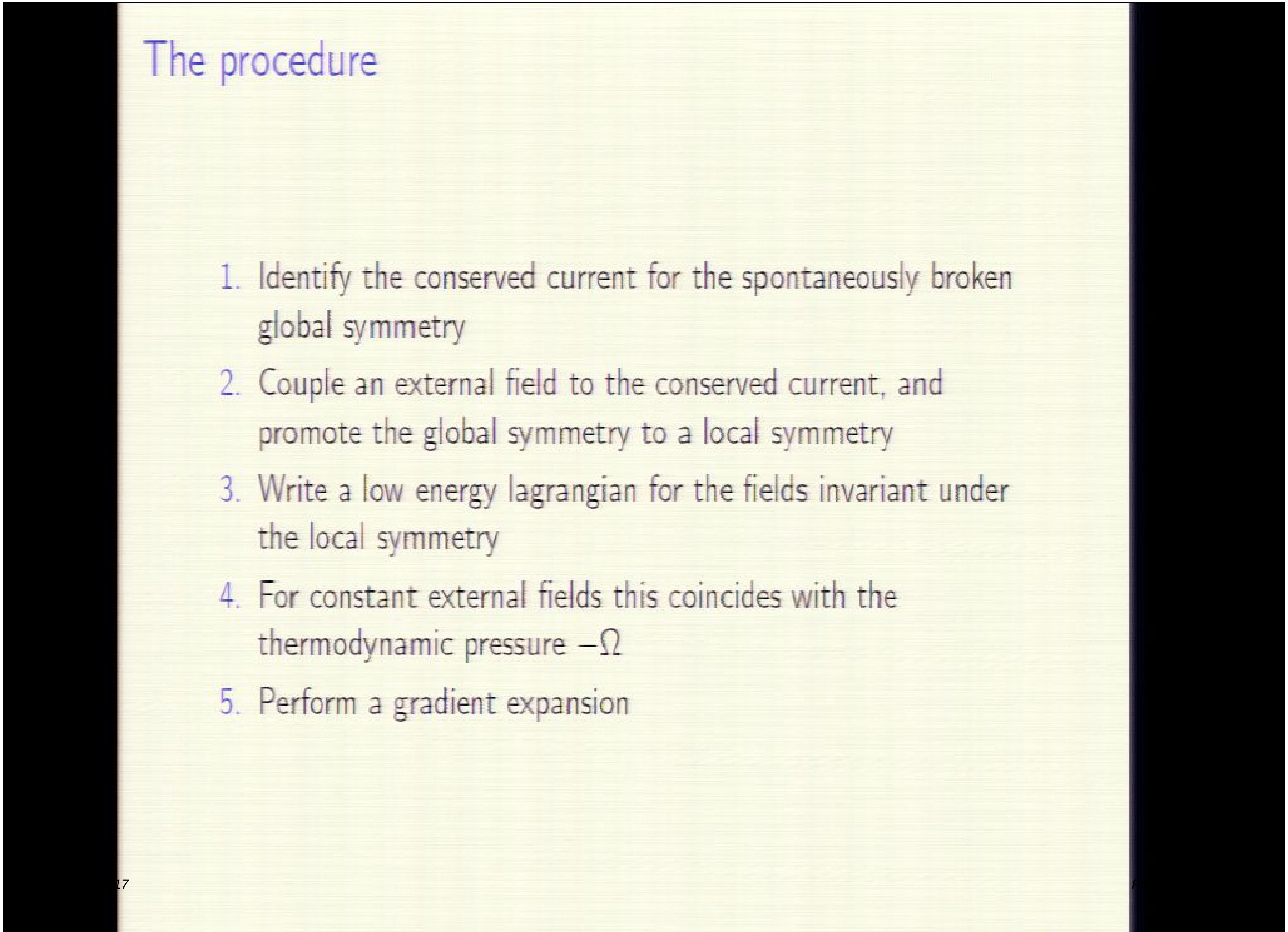
Hydrodynamics of the Goldstone mode

1. For a slowly varying field φ we can expand in $\partial\varphi$ about the equilibrium point
2. Taking the non-relativistic limit

$$\mathcal{L}_{\text{eff}}(\varphi) = P(\mu_n) + \frac{dP}{d\mu}(\partial_t\varphi) + \frac{1}{2}\frac{d^2P}{d\mu^2}(\partial_t\varphi)^2 - \frac{dP}{d\mu}\frac{(\partial_i\varphi)^2}{2m} + \ldots$$

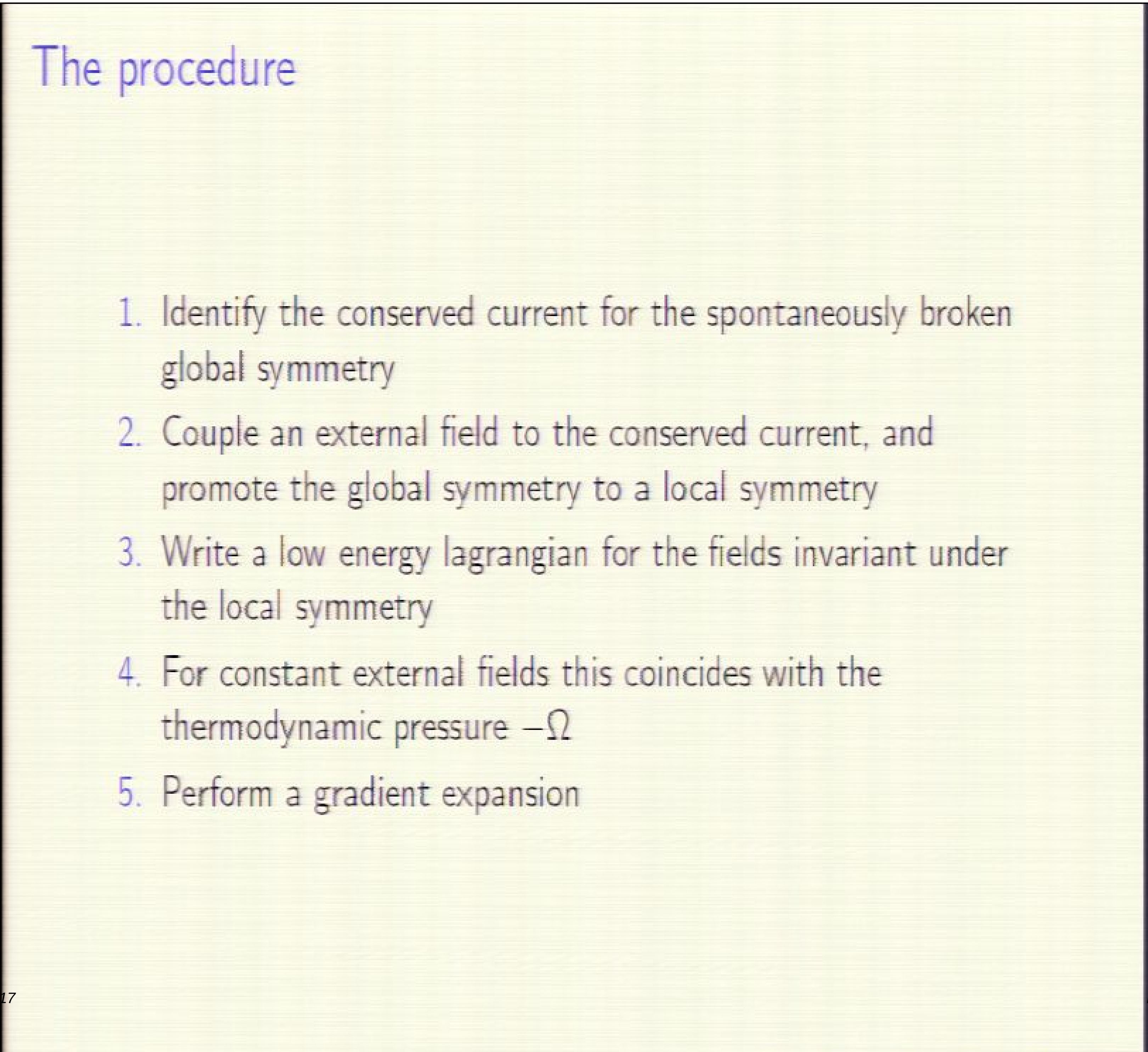
$$\sim \frac{f_\phi^2}{2}\left[(\partial_t\varphi)^2 - c_\phi^2(\partial_i\varphi)^2\right]$$

3. $f_\phi^2 = \frac{\partial^2 P}{\partial\mu^2} = \frac{dn}{d\mu}$, $c_\phi^2 = \frac{n}{mf_\phi^2}$



The procedure

1. Identify the conserved current for the spontaneously broken global symmetry
2. Couple an external field to the conserved current, and promote the global symmetry to a local symmetry
3. Write a low energy lagrangian for the fields invariant under the local symmetry
4. For constant external fields this coincides with the thermodynamic pressure $-\Omega$
5. Perform a gradient expansion



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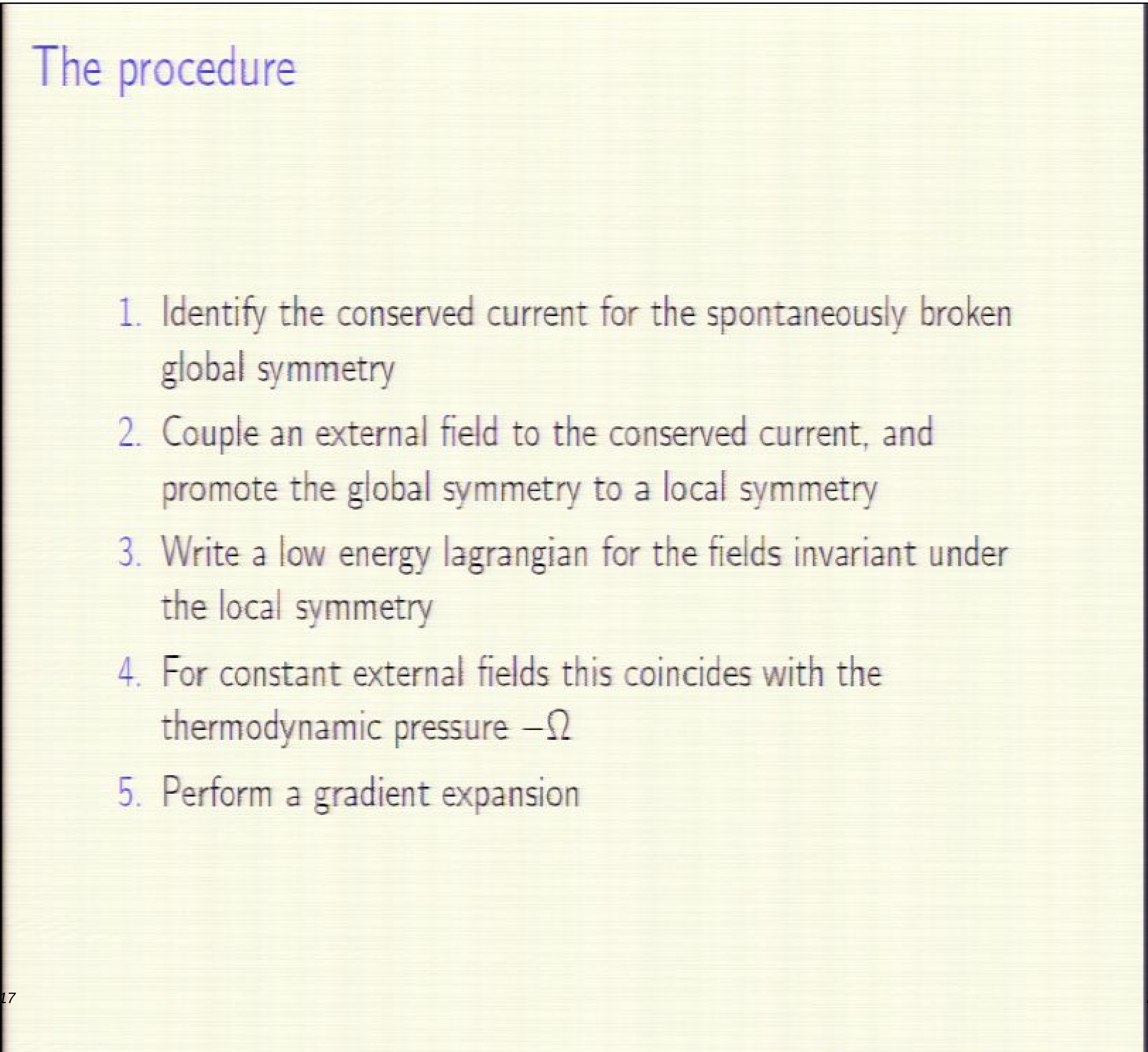

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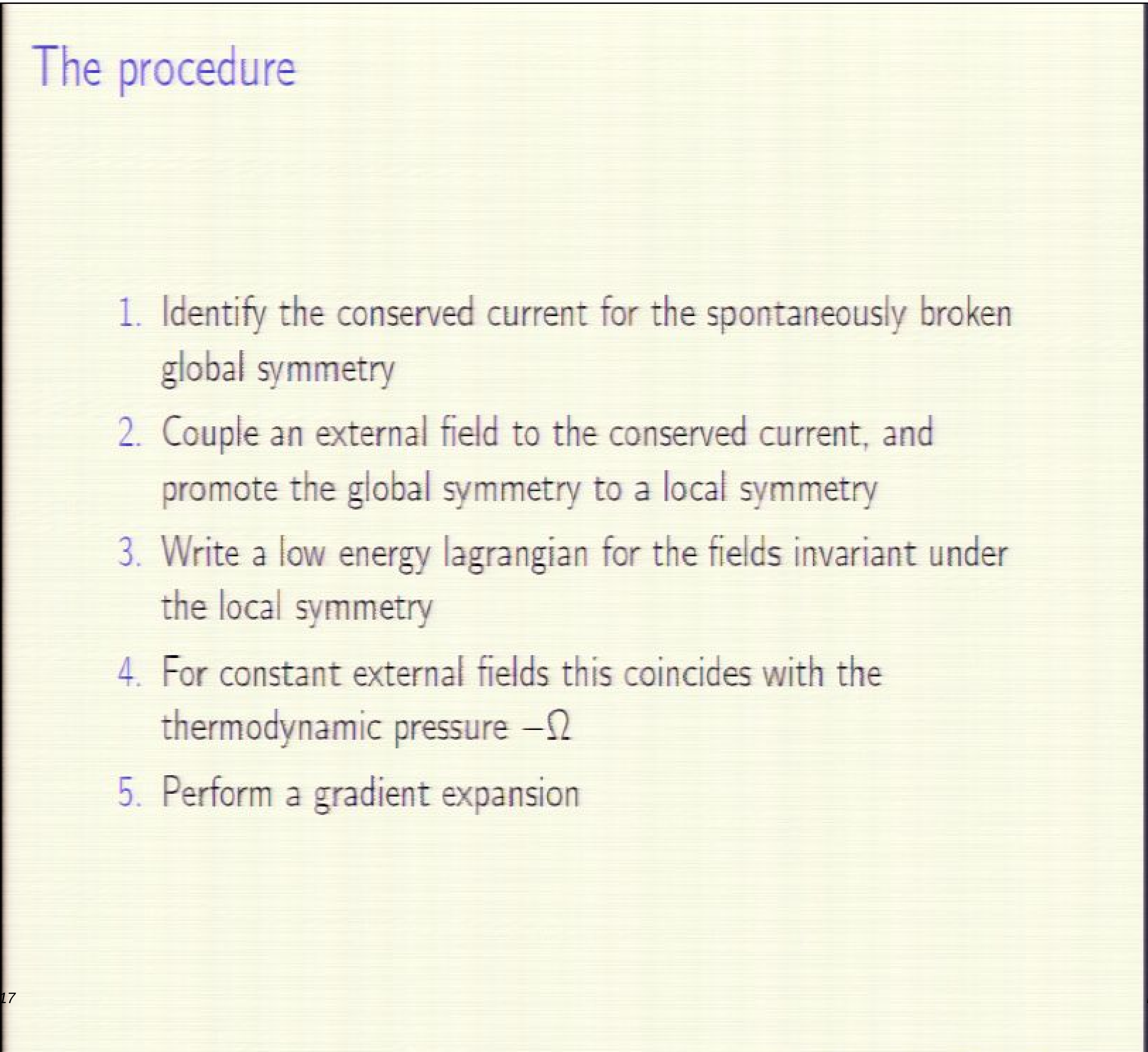

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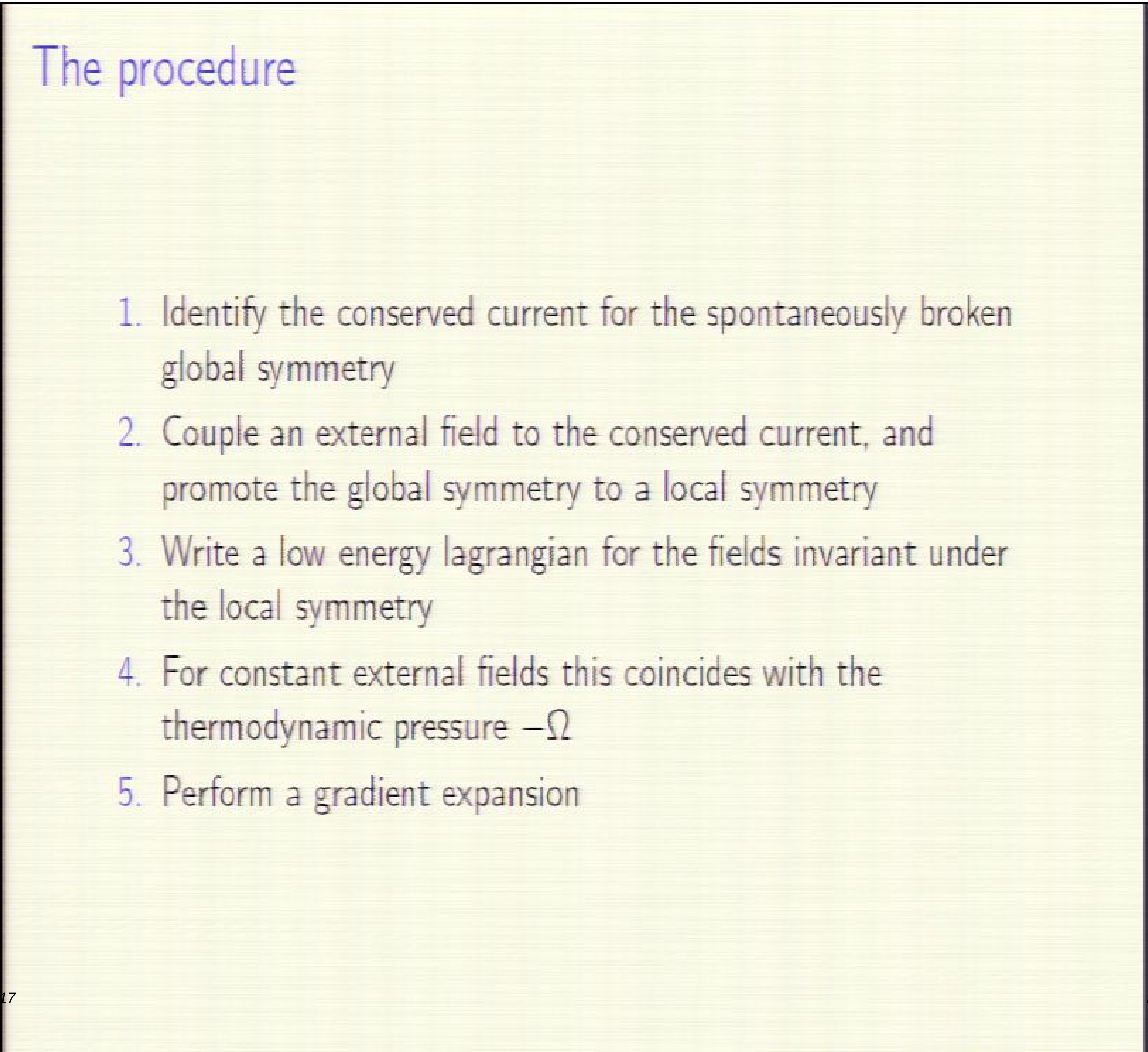

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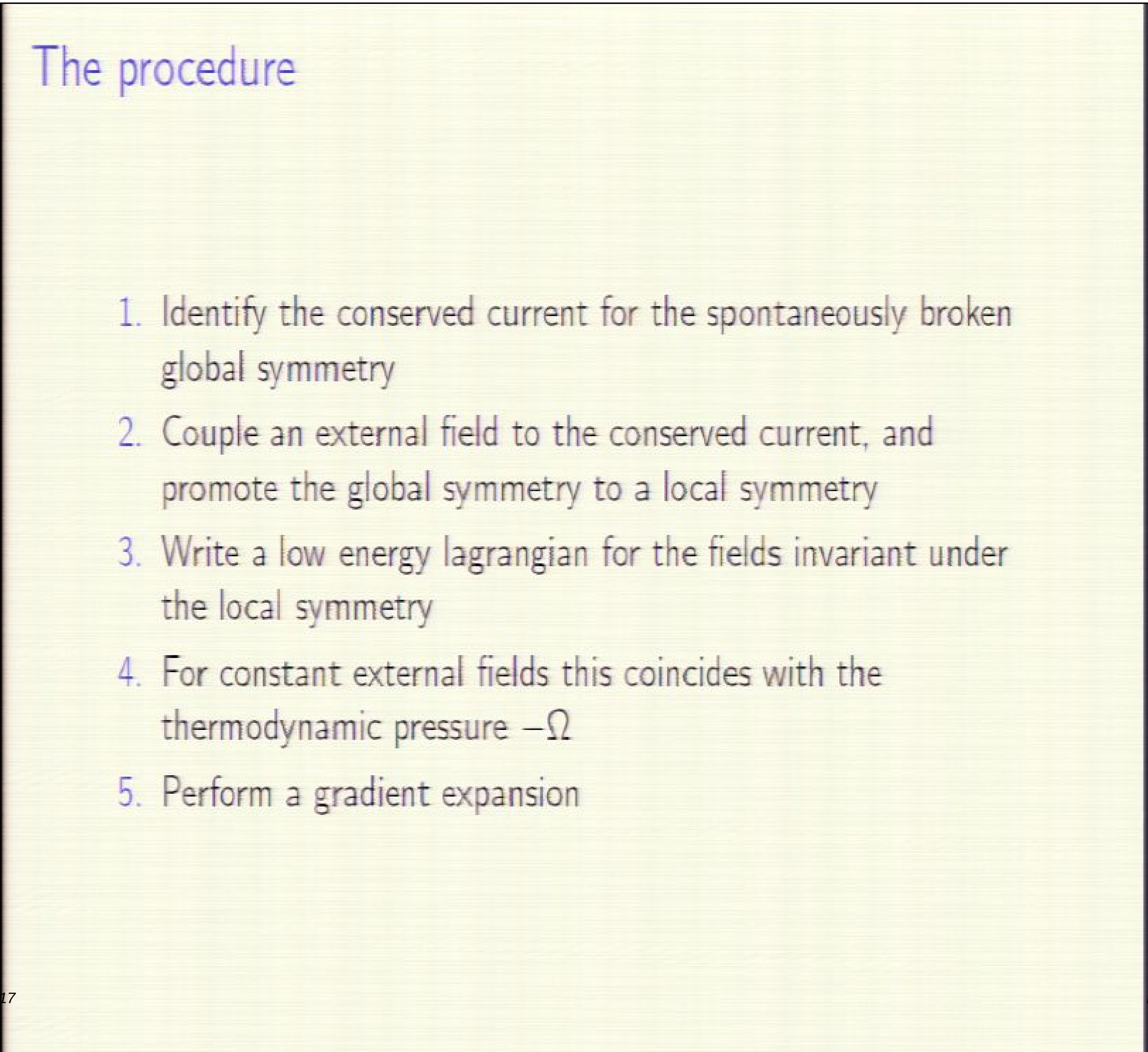

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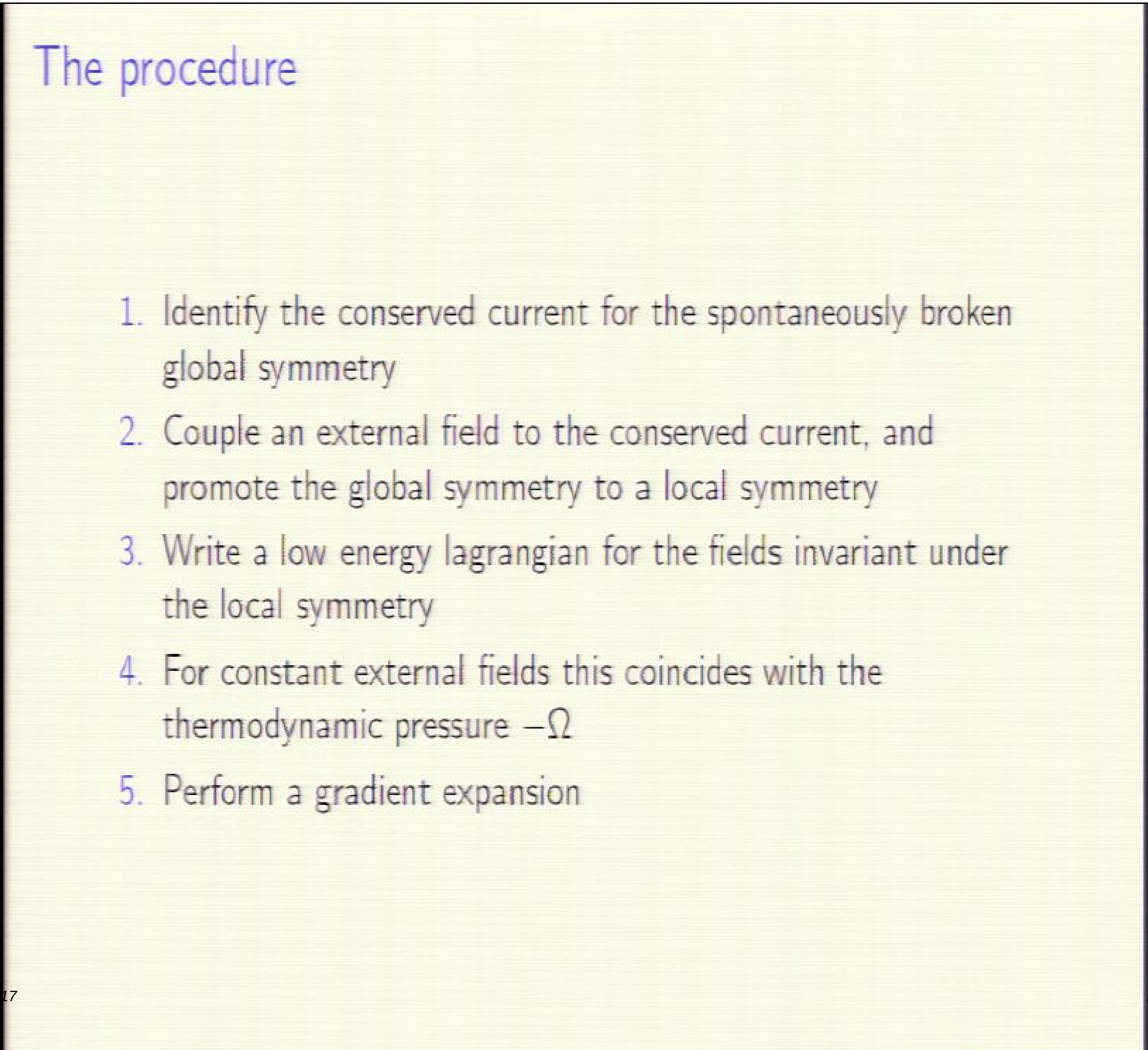

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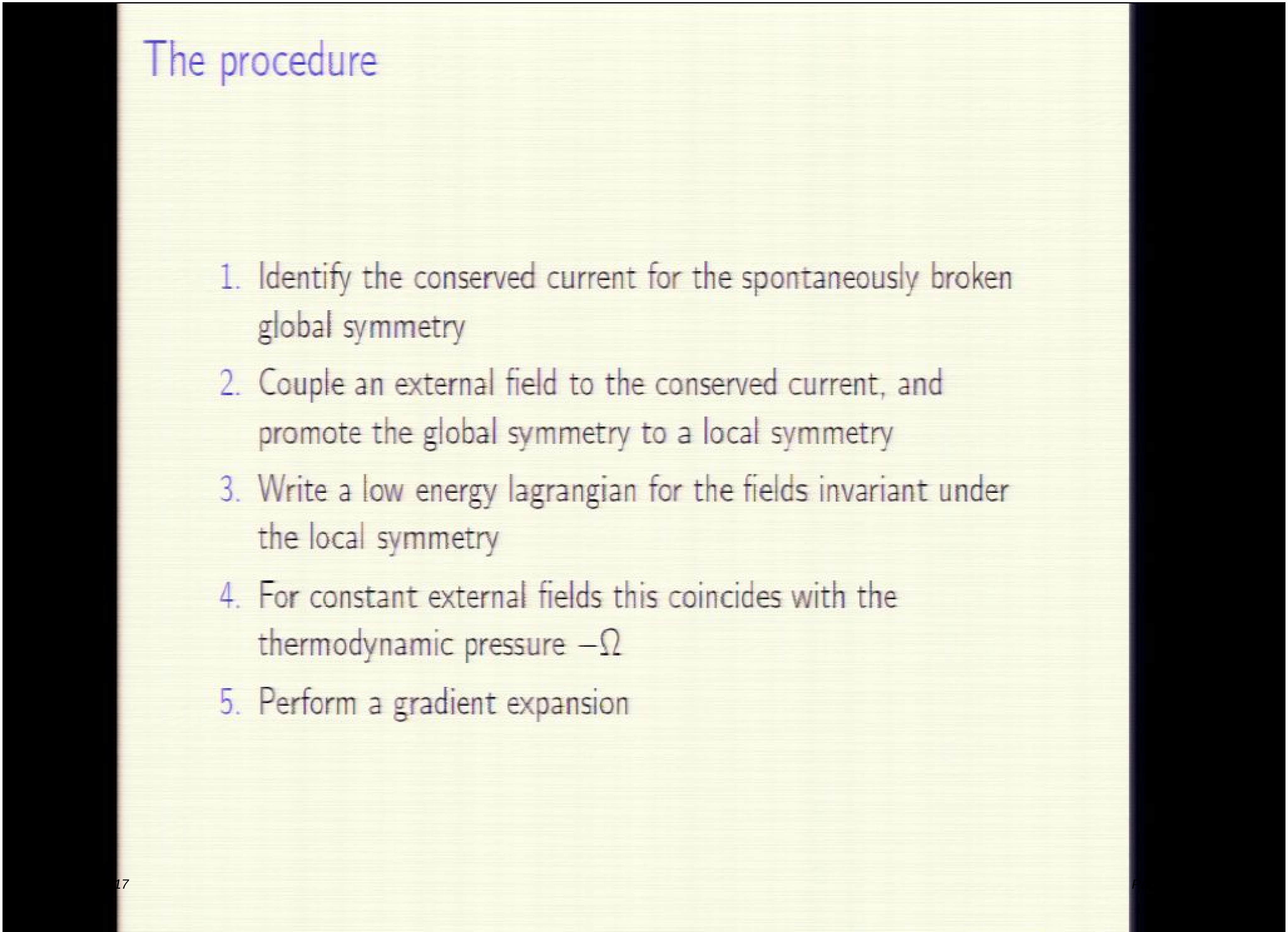

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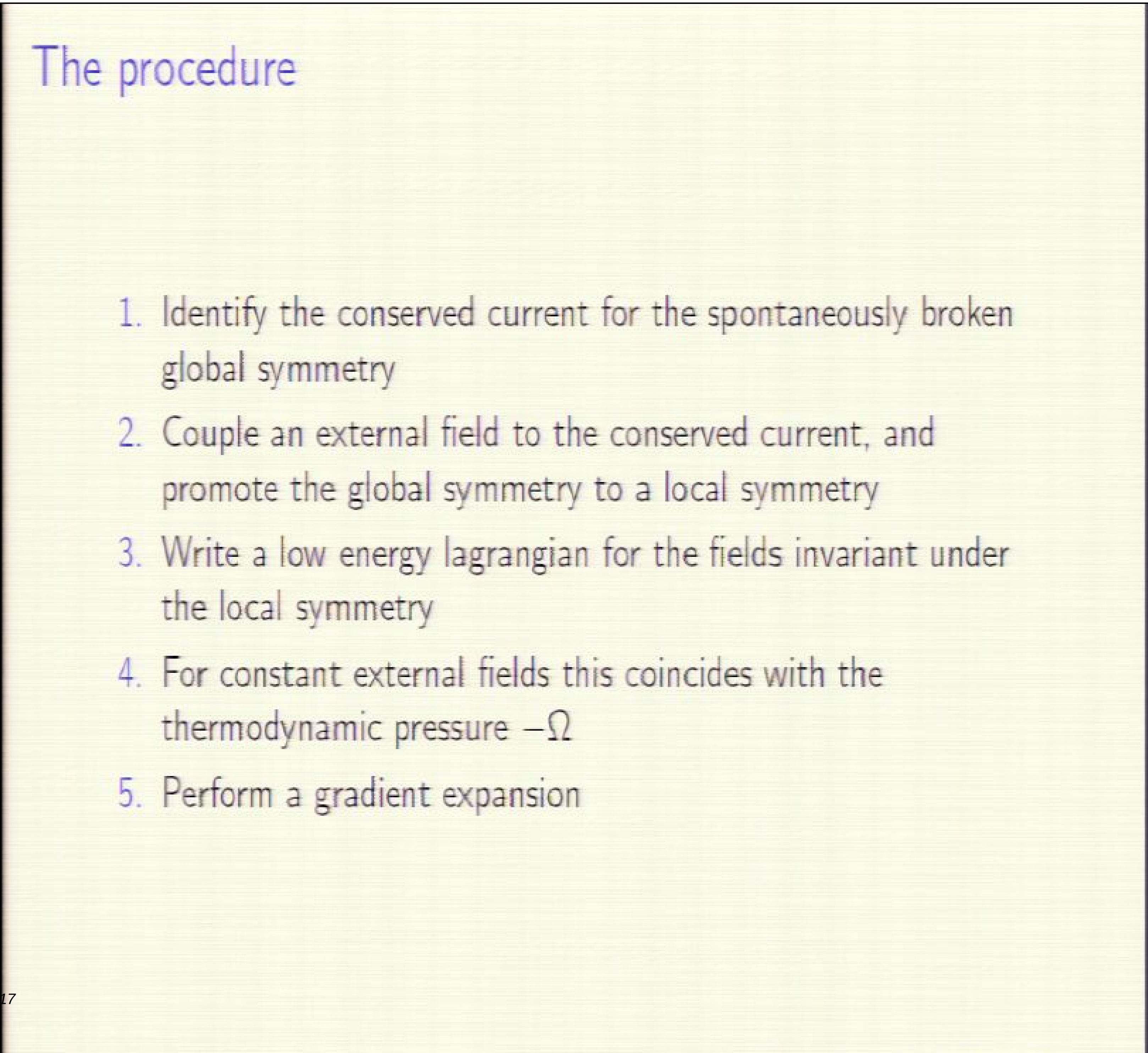

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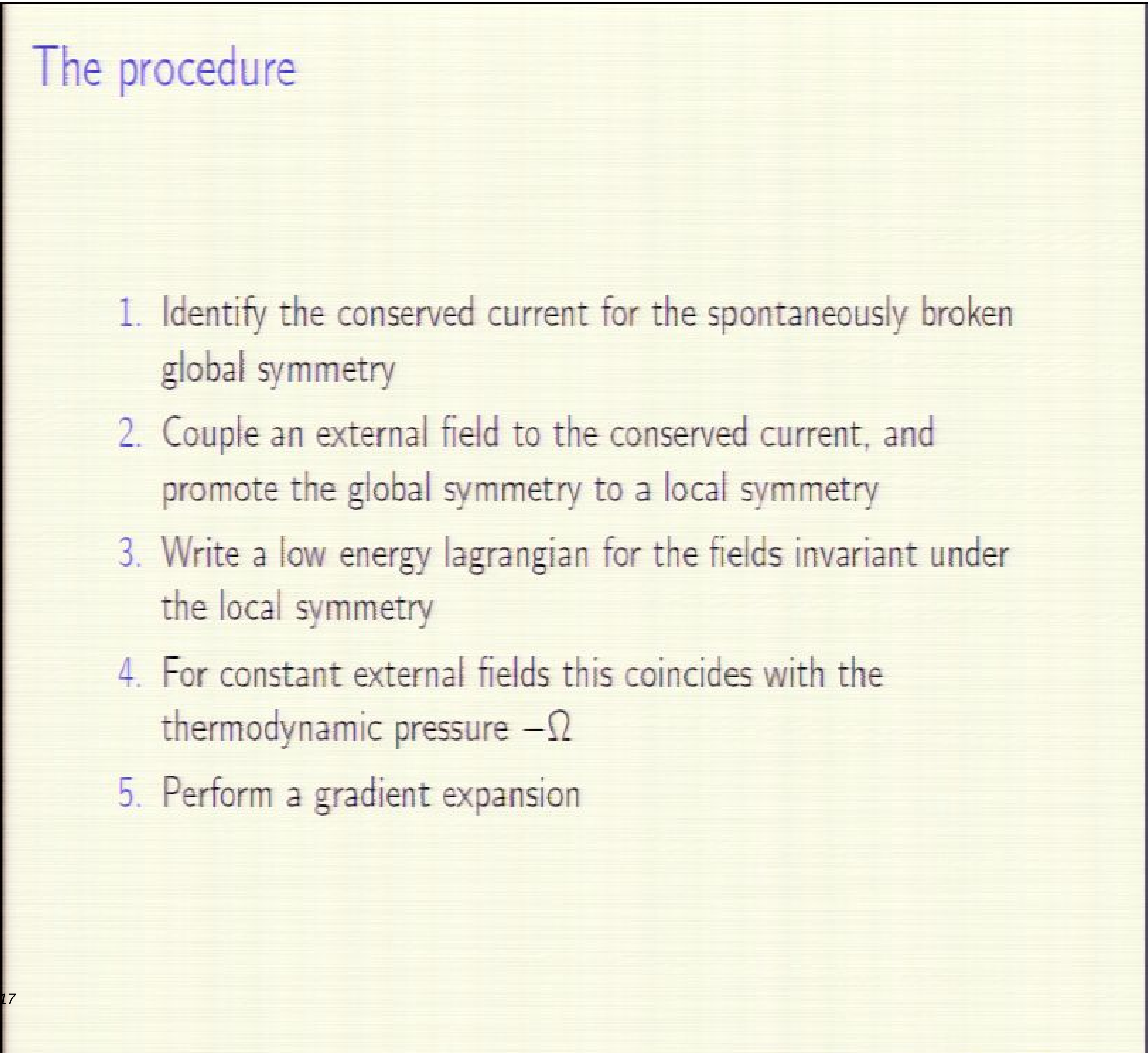

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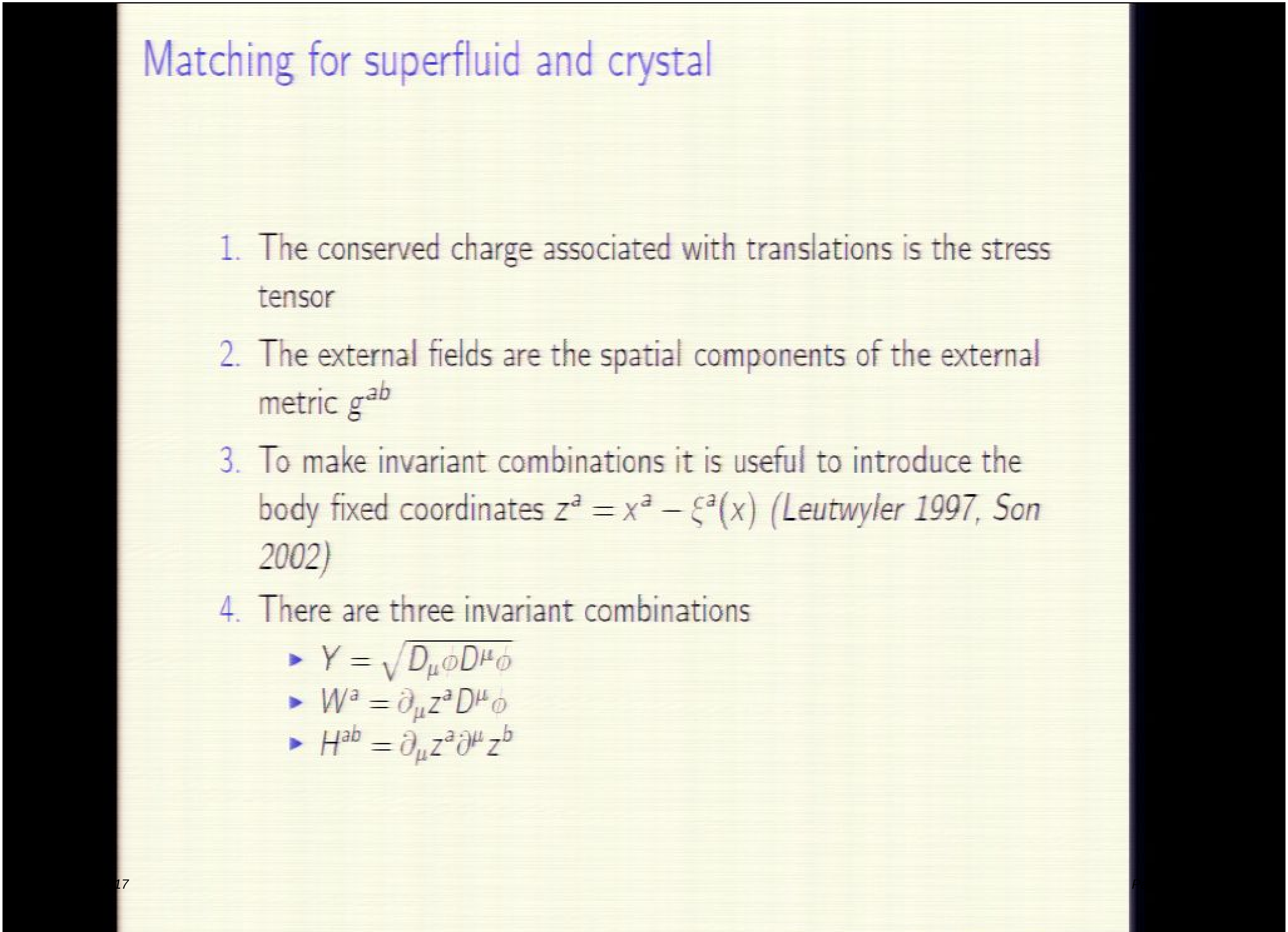
Matching for superfluid and crystal

1. Therefore the effective lagrangian is a function of Y, W^a and H^{ab}. $\mathcal{L}_{\text{eff}}(\phi, \xi^a, A_\mu, g_{\mu\nu}) = f(Y, W^a, H^{ab}) + ...$
2. The functional form of f is determined by the free energy function
3. $f(\tilde{A}_0, \mathbf{A}, \bar{g}_{ab}) = -\Omega(\tilde{A}_\mu, \bar{g}_{ab}) = -\mathcal{E}(\tilde{A}_\mu, \bar{g}_{ab}) + \tilde{A}_\mu j^\mu$



Matching for superfluid and crystal

1. The conserved charge associated with translations is the stress tensor
2. The external fields are the spatial components of the external metric g^{ab}
3. To make invariant combinations it is useful to introduce the body fixed coordinates $z^a = x^a - \xi^a(x)$ *(Leutwyler 1997, Son 2002)*
4. There are three invariant combinations
    - $Y = \sqrt{D_\mu \phi D^\mu \phi}$
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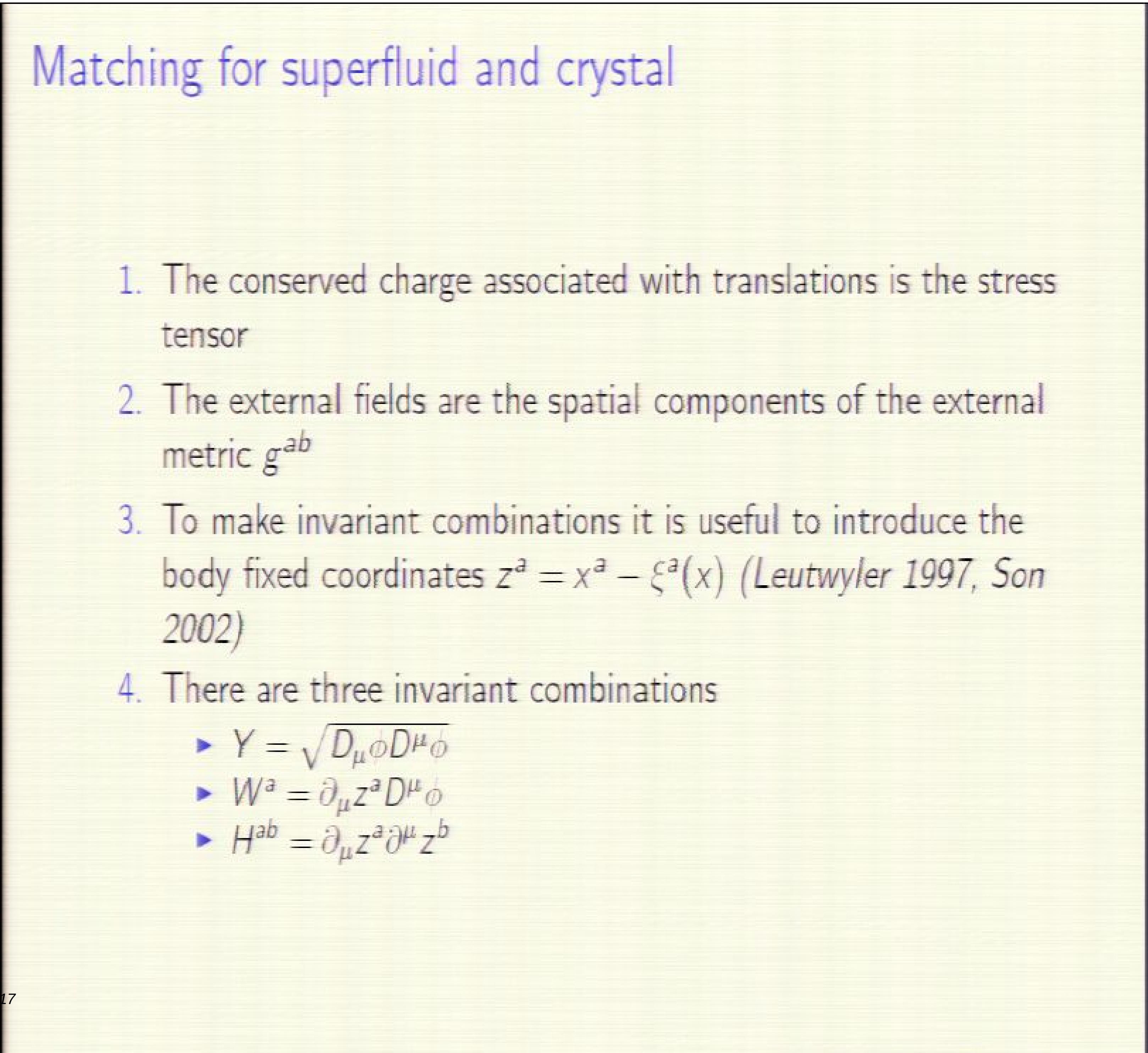

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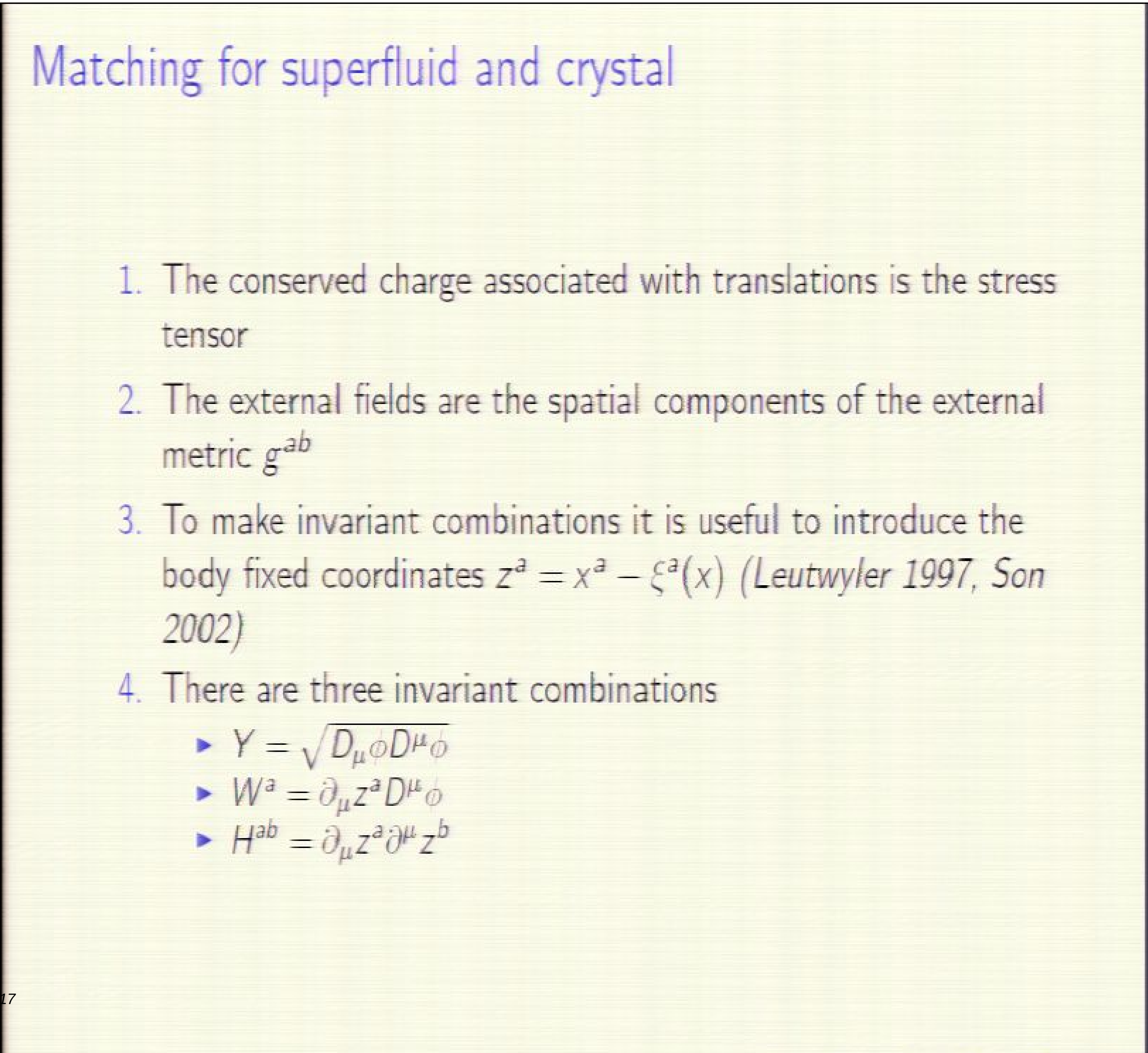

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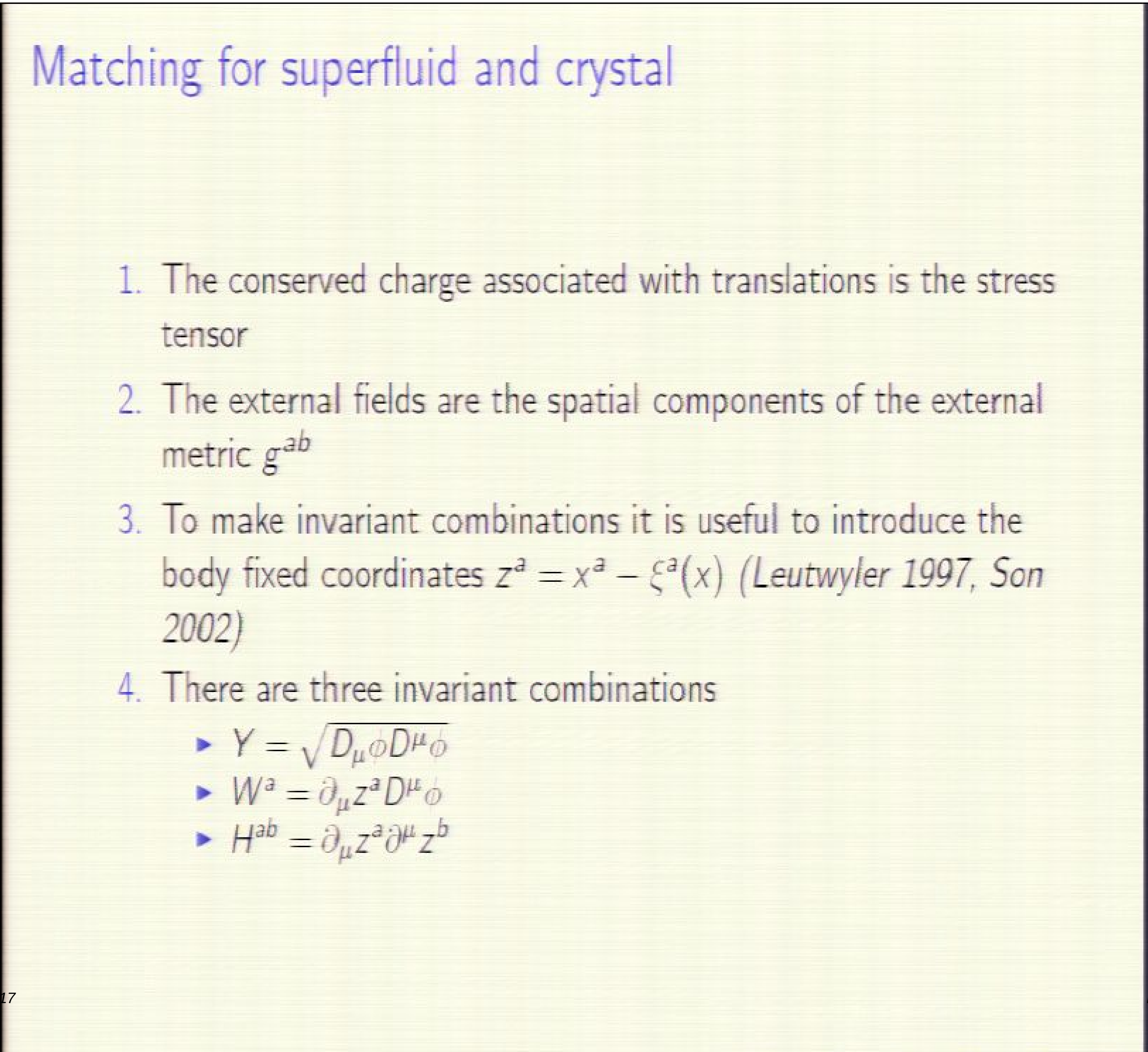

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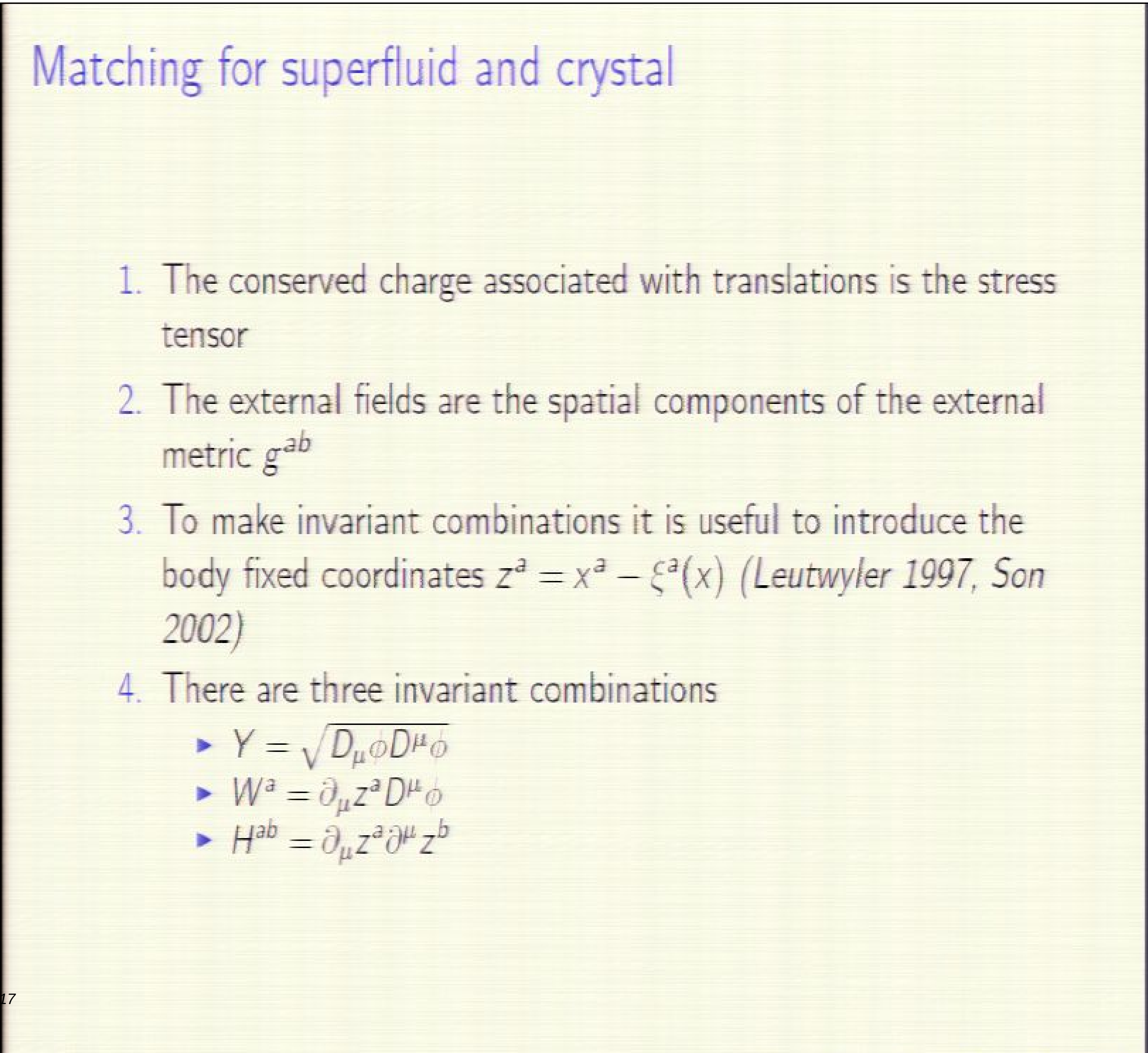

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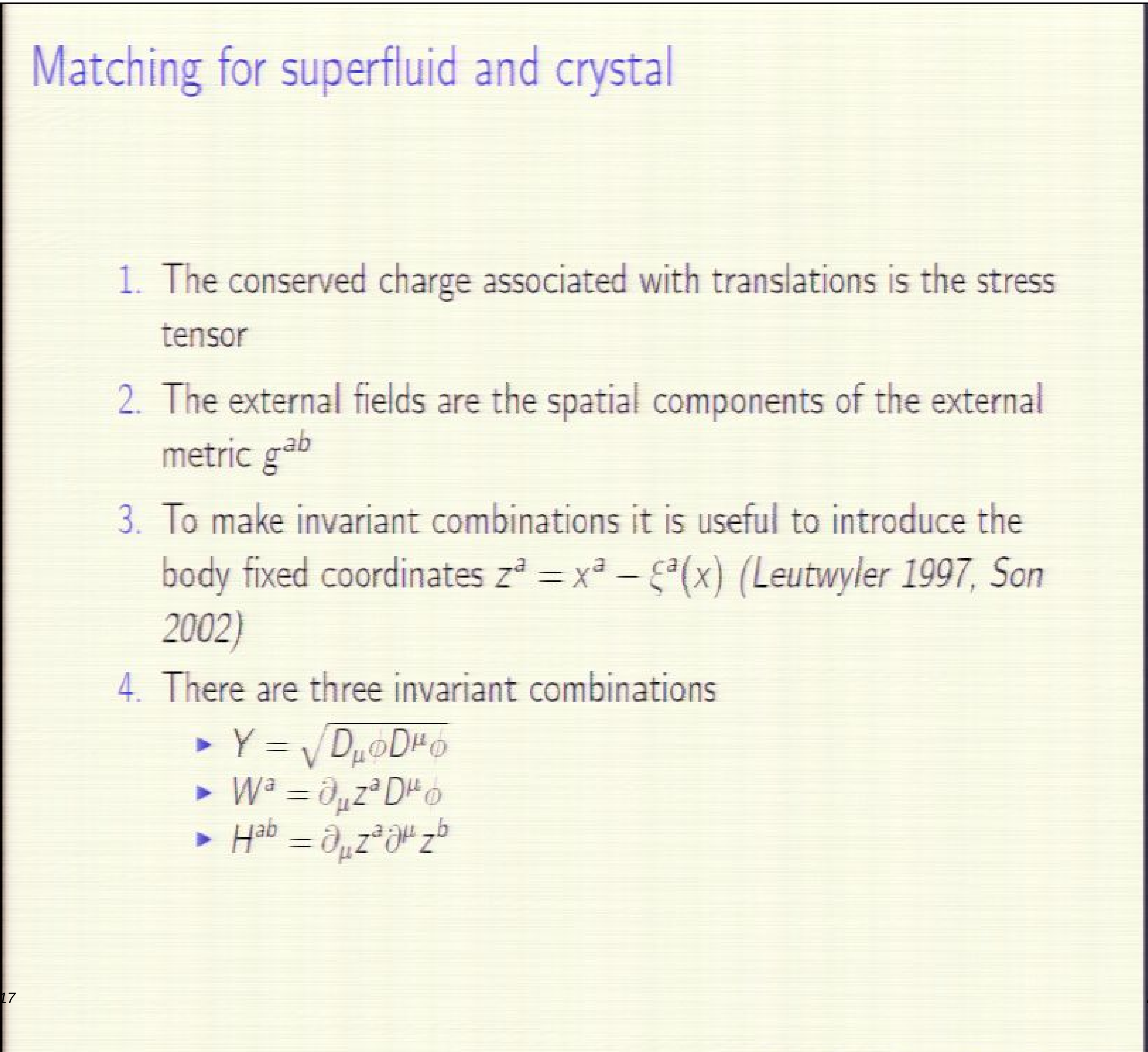

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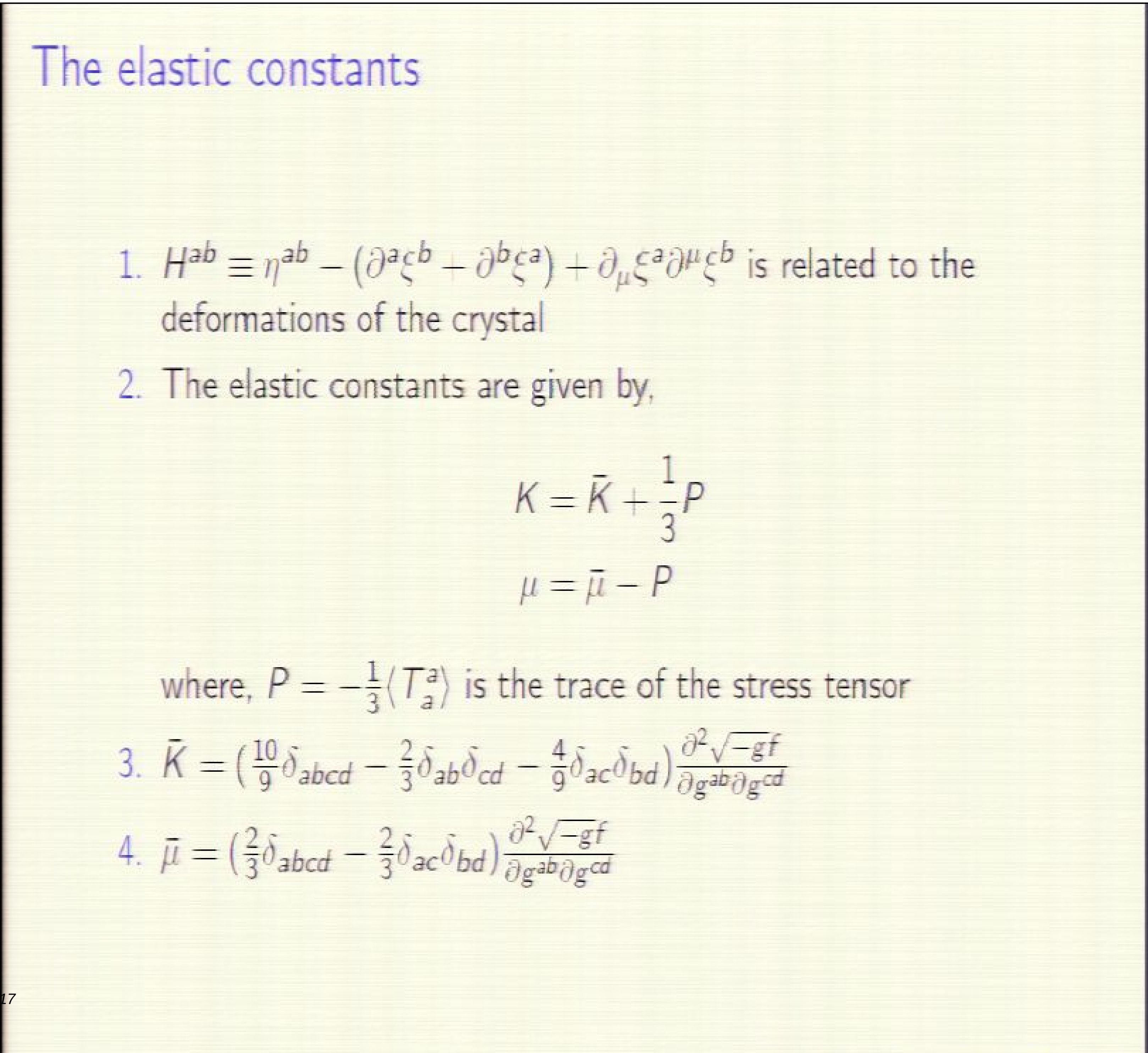
Matching for superfluid and crystal

1. Therefore the effective lagrangian is a function of Y, W^a and H^{ab}. $\mathcal{L}_{\text{eff}}(\phi, \xi^a, A_\mu, g_{\mu\nu}) = f(Y, W^a, H^{ab}) + ...$
2. The functional form of f is determined by the free energy function
3. $f(\tilde{A}_0, \mathbf{A}, \bar{g}_{ab}) = -\Omega(\tilde{A}_\mu, \bar{g}_{ab}) = -\mathcal{E}(\tilde{A}_\mu, \bar{g}_{ab}) + \tilde{A}_\mu j^\mu$

Quadratic lagrangian

1. Expanding near the equilibrium, $Y = \mu$, $W^a = 0$, $H^{ab} = -\delta^{ab}$ and keeping only the quadratic terms

$$\begin{aligned}
\mathcal{L}_0 = {} & \frac{1}{2}\Big[\frac{\partial^2 f}{\partial Y^2}\Big](\partial_0\phi)^2 - \frac{1}{2}\Big[\frac{1}{m}\frac{\partial f}{\partial Y} - \frac{\partial^2 f}{3\partial W^c \partial W^c}\Big](\partial_i\phi)^2 \\
& + \frac{1}{2}\Big[\frac{2}{3}\frac{\partial f}{\partial H^{cc}} + m^2\frac{\partial^2 f}{3\partial W^c \partial W^c}\Big]\dot{\xi}^a\dot{\xi}^a \\
& + \Big[\frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y} + m\frac{\partial^2 f}{3\partial W^c \partial W^c}\Big](\partial_c\xi^c)(\partial_0\phi) \\
& - \frac{1}{4}[\mu]\,\xi^{ab}\xi^{ab} - \frac{1}{2}[K](\partial_c\xi^c)^2
\end{aligned}$$



The elastic constants

1. $H^{ab} \equiv \eta^{ab} - (\partial^a\xi^b + \partial^b\xi^a) + \partial_\mu\xi^a\partial^\mu\xi^b$ is related to the deformations of the crystal
2. The elastic constants are given by,

$$K = \bar{K} + \frac{1}{3}P$$
$$\mu = \bar{\mu} - P$$

where, $P = -\frac{1}{3}\langle T^a_a\rangle$ is the trace of the stress tensor

3. $\bar{K} = (\frac{10}{9}\delta_{abcd} - \frac{2}{3}\delta_{ab}\delta_{cd} - \frac{4}{9}\delta_{ac}\delta_{bd})\frac{\partial^2\sqrt{-g}f}{\partial g^{ab}\partial g^{cd}}$
4. $\bar{\mu} = (\frac{2}{3}\delta_{abcd} - \frac{2}{3}\delta_{ac}\delta_{bd})\frac{\partial^2\sqrt{-g}f}{\partial g^{ab}\partial g^{cd}}$

Entrainment

1. $n_b = m\frac{\partial^2 f}{3\partial W^c \partial W^c}$ is the density of the superfluid that is entrained on the lattice
2. The density of neutrons that participate in transport is $n_c = \frac{1}{3}\langle j^i j^i\rangle(q=0) = n_{tot} - n_b$ *(Also see Chamel, Pethick, Reddy)*
3. Related to the change in energy associated with relative motion between the superfluid and the lattice. To see that, note that in the non-relativistic limit $W^a \sim m(-\frac{1}{m}\partial_a\phi - \partial_0\xi^a + \frac{1}{m}\partial_i\phi\partial_i\xi^a)$

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The mixing parameter

1. $g_{mix} = \frac{1}{f_\phi \rho}[m\frac{\partial^2 f}{3\partial W^c \partial W^c} + \frac{2}{3}\frac{\partial^2 f}{\partial H^{cc}\partial Y}]$
2. Interestingly, one contribution to the mixing parameter comes from the entrainment term
3. In the case where one conserved species (p) forms the lattice and the second species (n) is superfluid, $\delta H^{cc} = -\frac{1}{n_p}\delta n_p$.
4. Therefore $g_{\rm mix} = \frac{1}{f_\phi \sqrt{\rho}}[n_b - n_p\frac{\partial n_n}{\partial n_p}]$

Estimating n_b in the neutron star crust

1. Model the system as clusters of protons localized on lattice sites, with some neutrons (n_b) bound or entrained on the sites, and the rest ($n_f = n_n - n_b$) unbound
2. Can use mass tables to obtain the density profiles of neutrons and protons
3. We take n_b as the density of bound neutrons in the Wigner-Seitz approximation
4. The second contribution is estimated by noting that $n_p \frac{\partial n_n}{\partial n_p} \sim n_p f_\phi^2 \tilde{V}_{np}$. For typical values of $\tilde{V}_{np}$, the first term dominates over the second term

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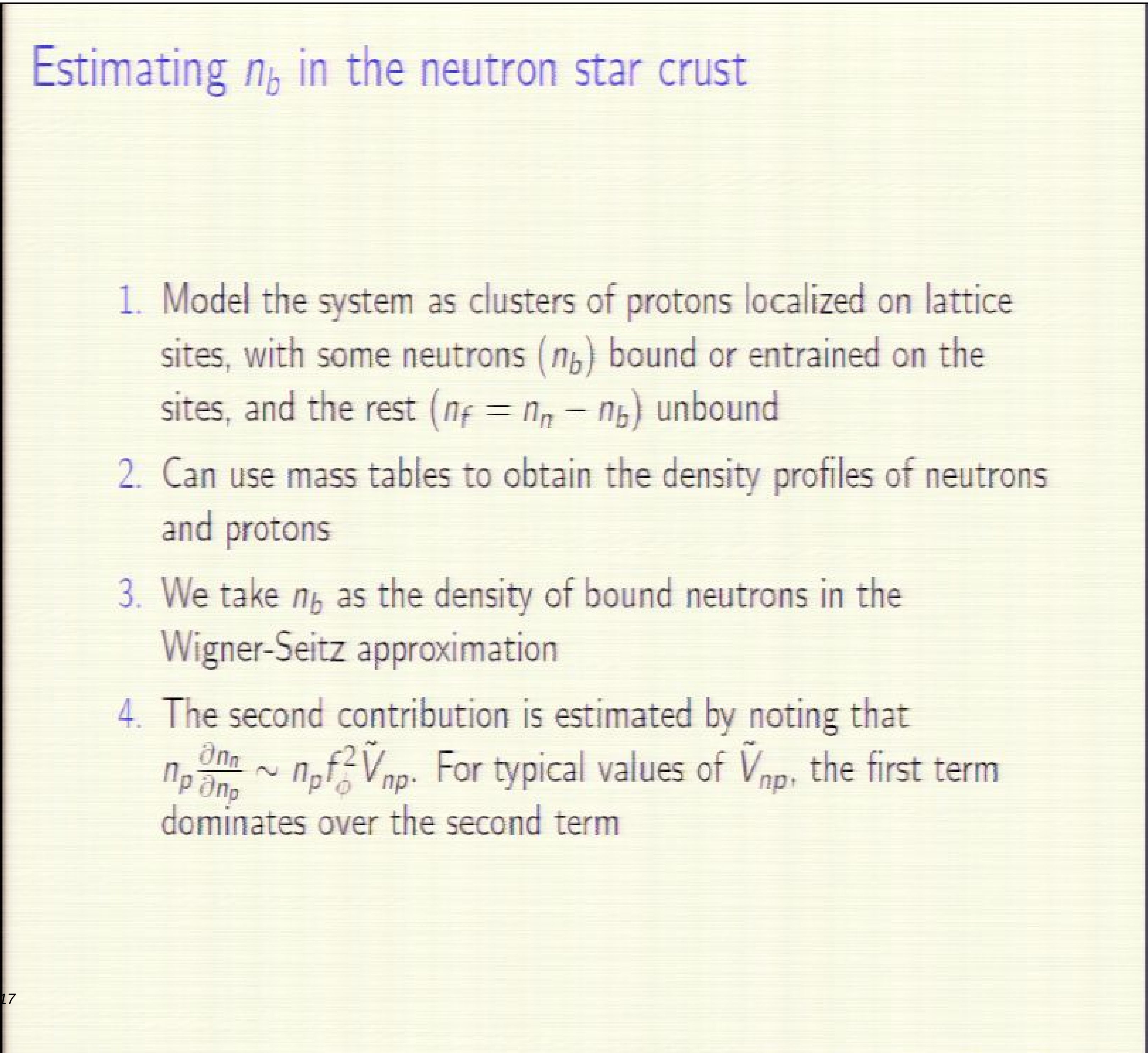

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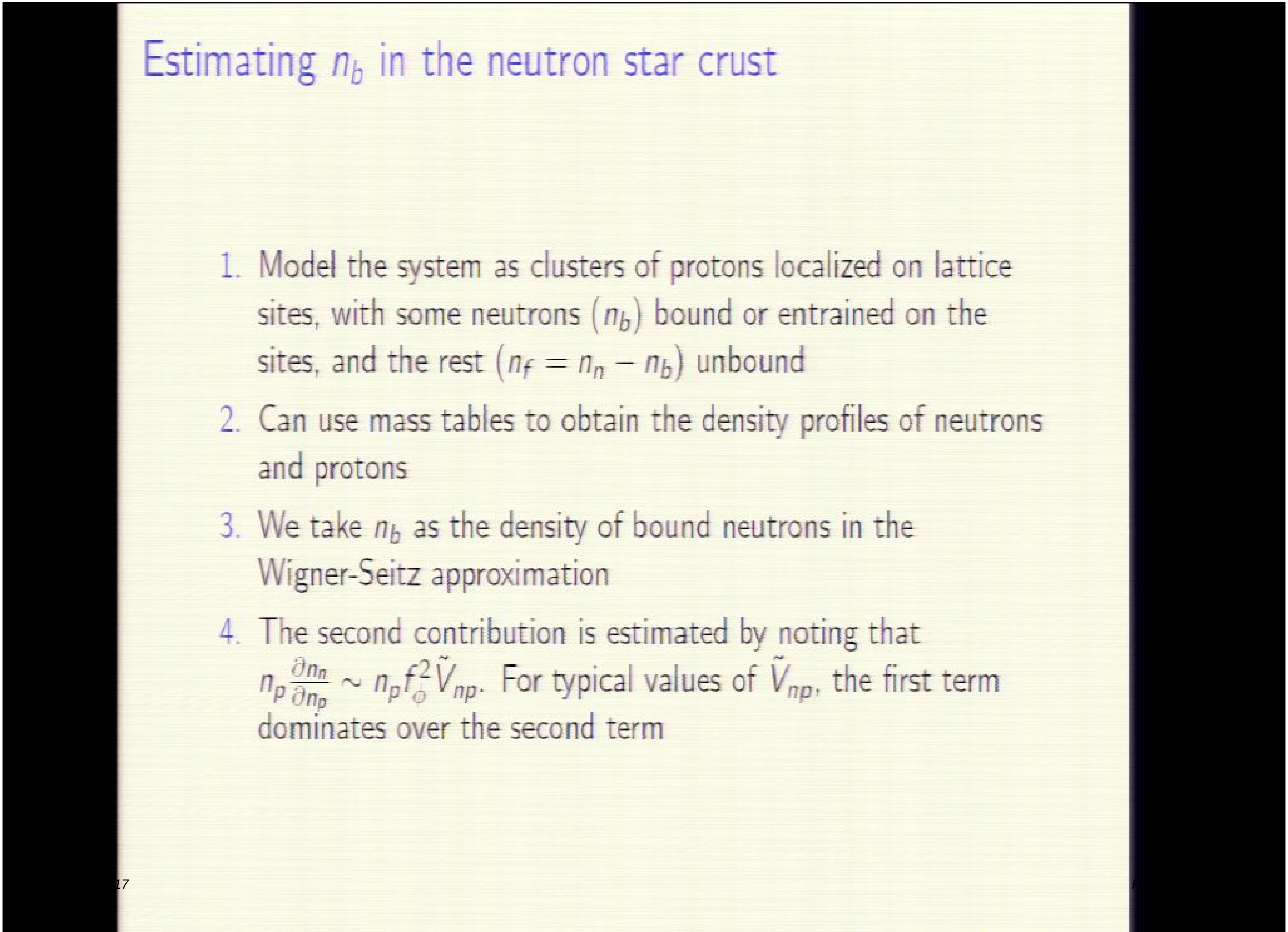

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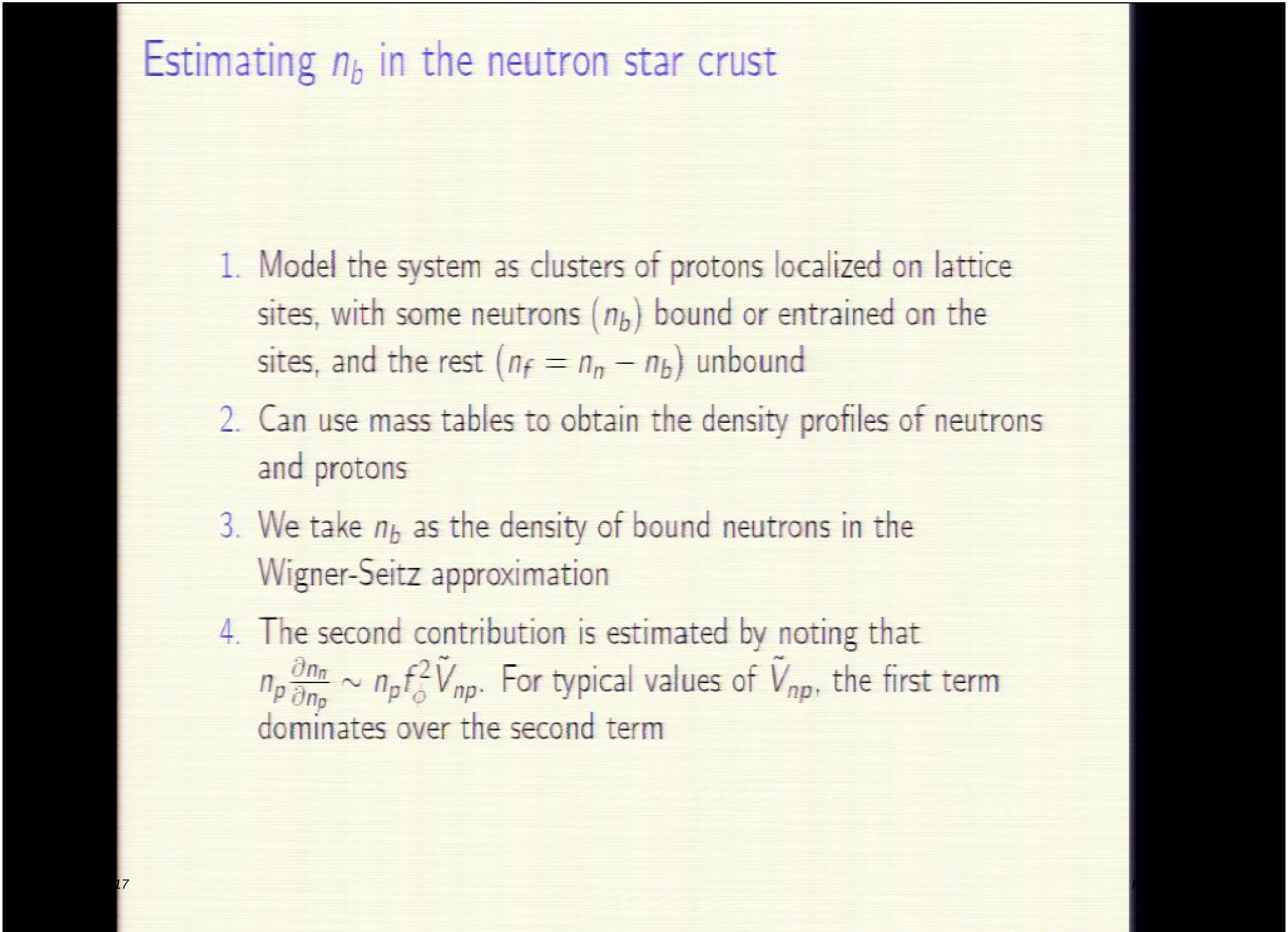

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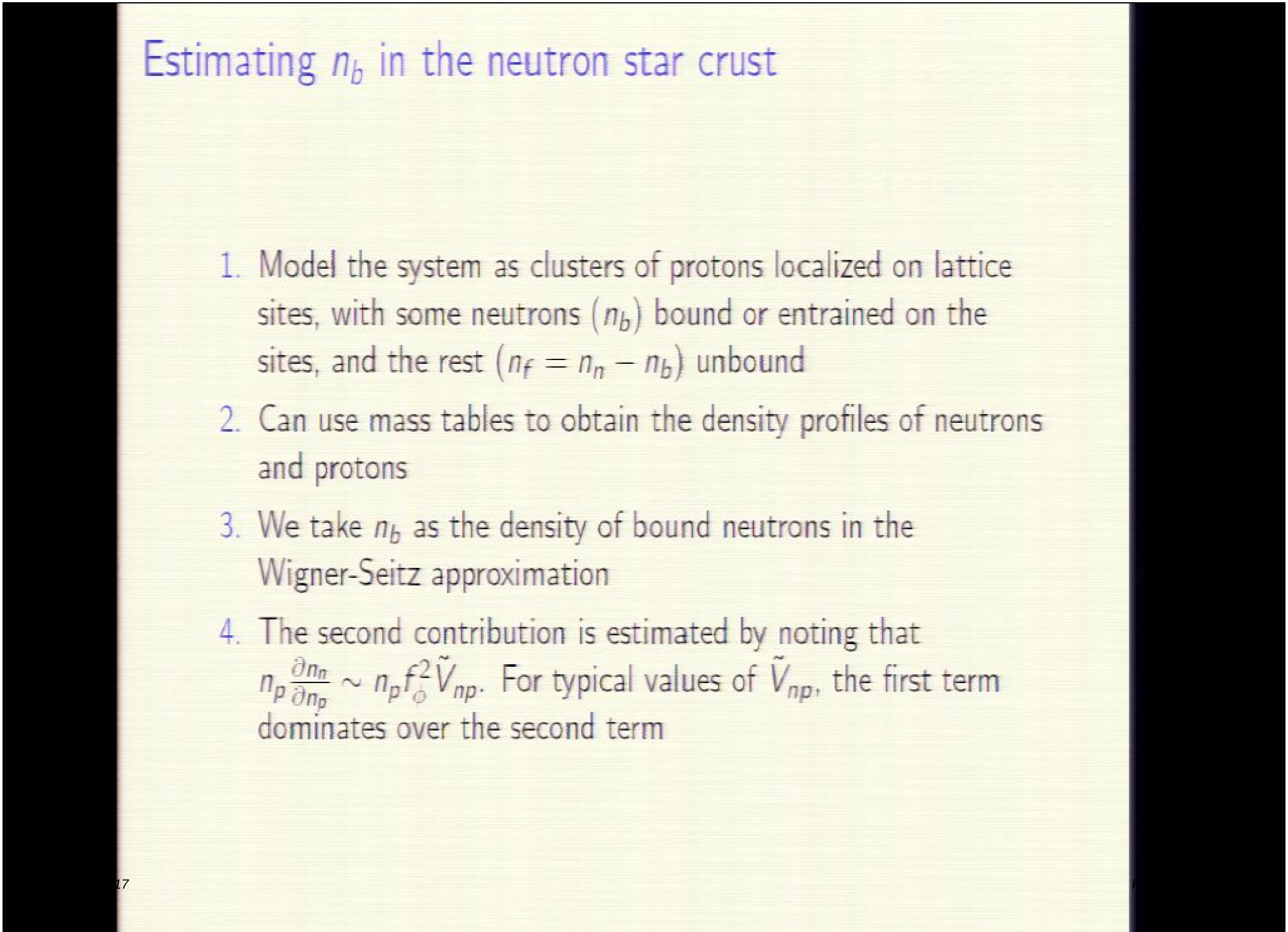

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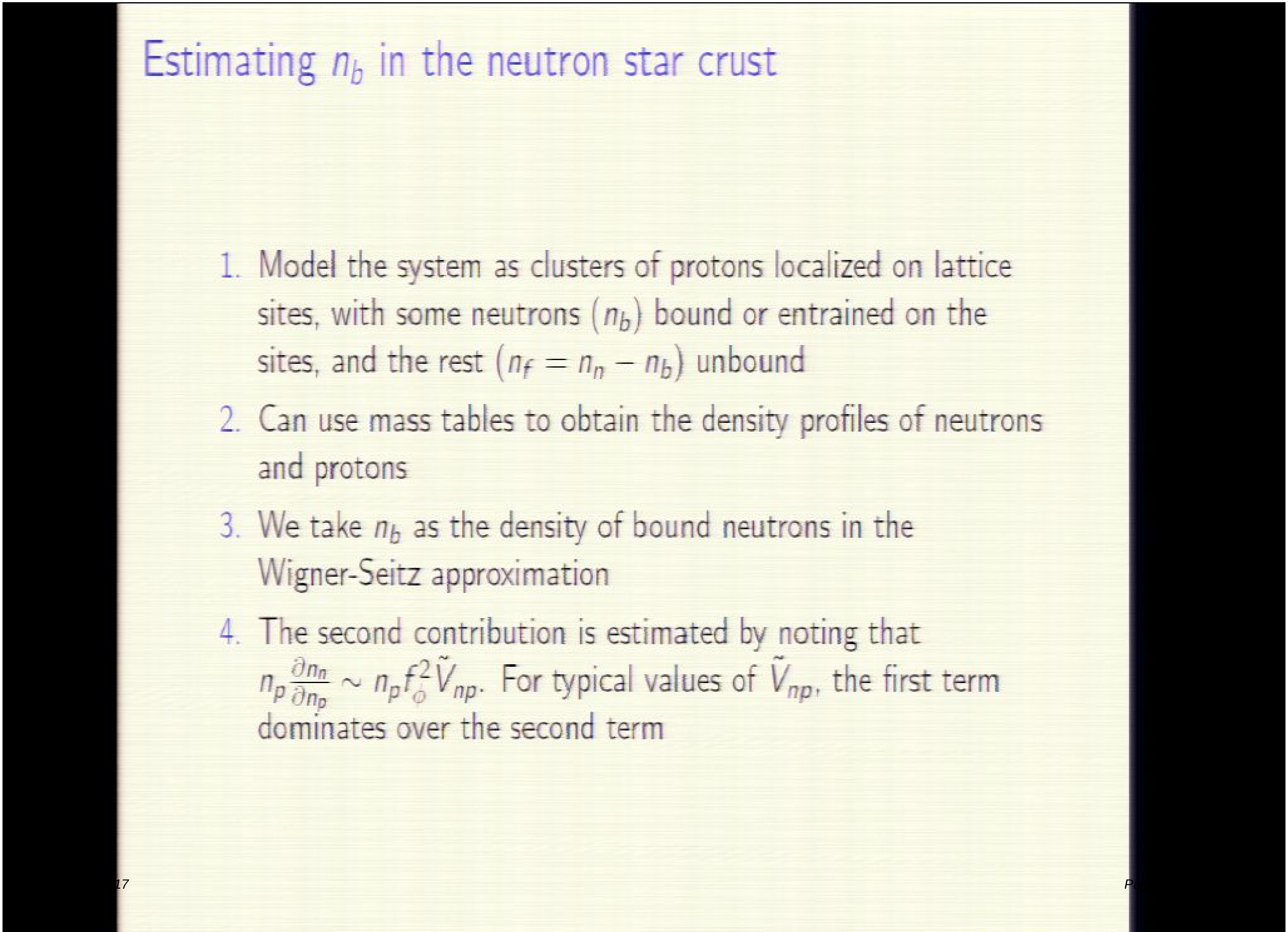

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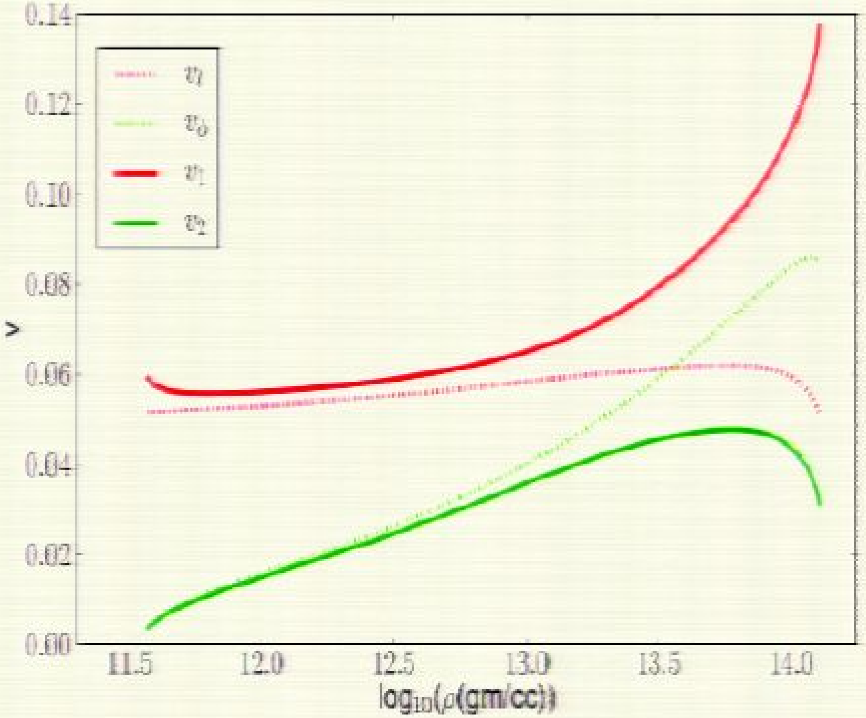
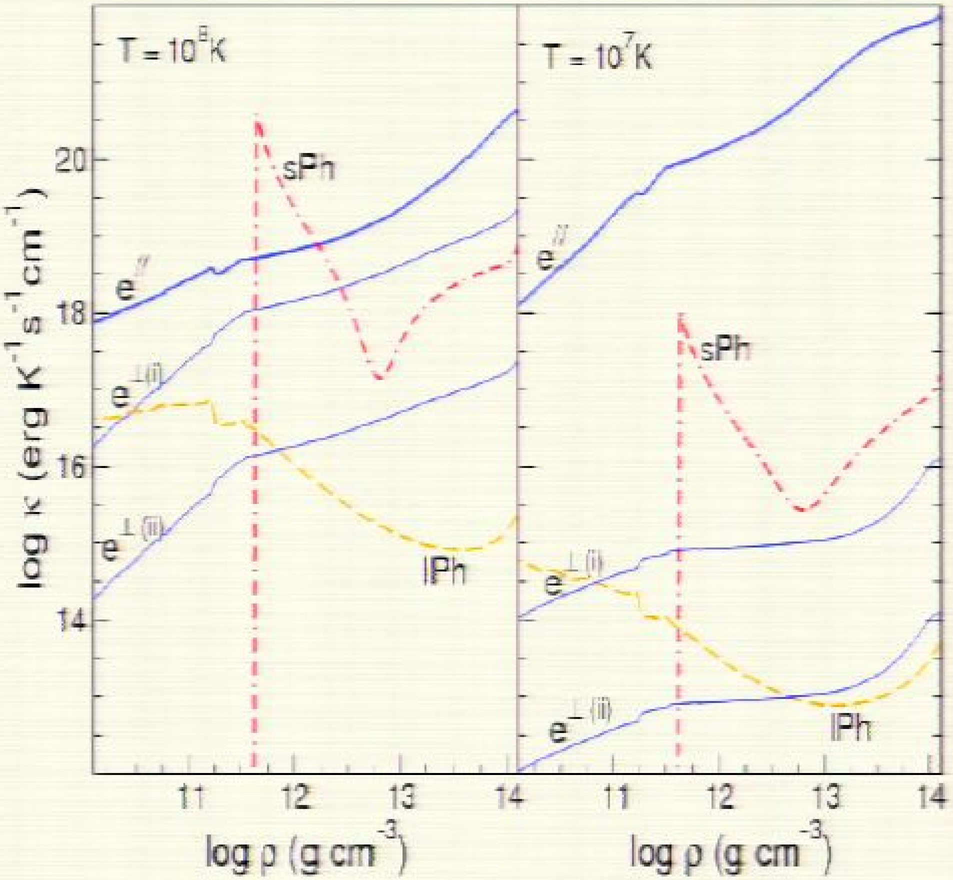
Mixing in the neutron star crust



Examples of other parameters

1. $C_{v\phi} = \frac{2\pi^2 T^3}{15}(\frac{1}{v_\phi^3}$
2. $\lambda_\phi(\omega) = \frac{v_\xi^2}{g_{\mathrm{mix}}^2}\left[\frac{1+(1-\alpha^2)^2(\omega\tau_\xi)^2}{\alpha(\omega\tau_\xi)^2}\right]\lambda_\xi(\omega)$
3. $\lambda_\xi(\omega) \sim \frac{2}{\pi\omega}$
4. $\alpha = \frac{v_\xi}{v_\phi}$
5. Thermal conductivity $\kappa = \frac{1}{3} C_{v\phi} v_\phi \lambda_\phi$

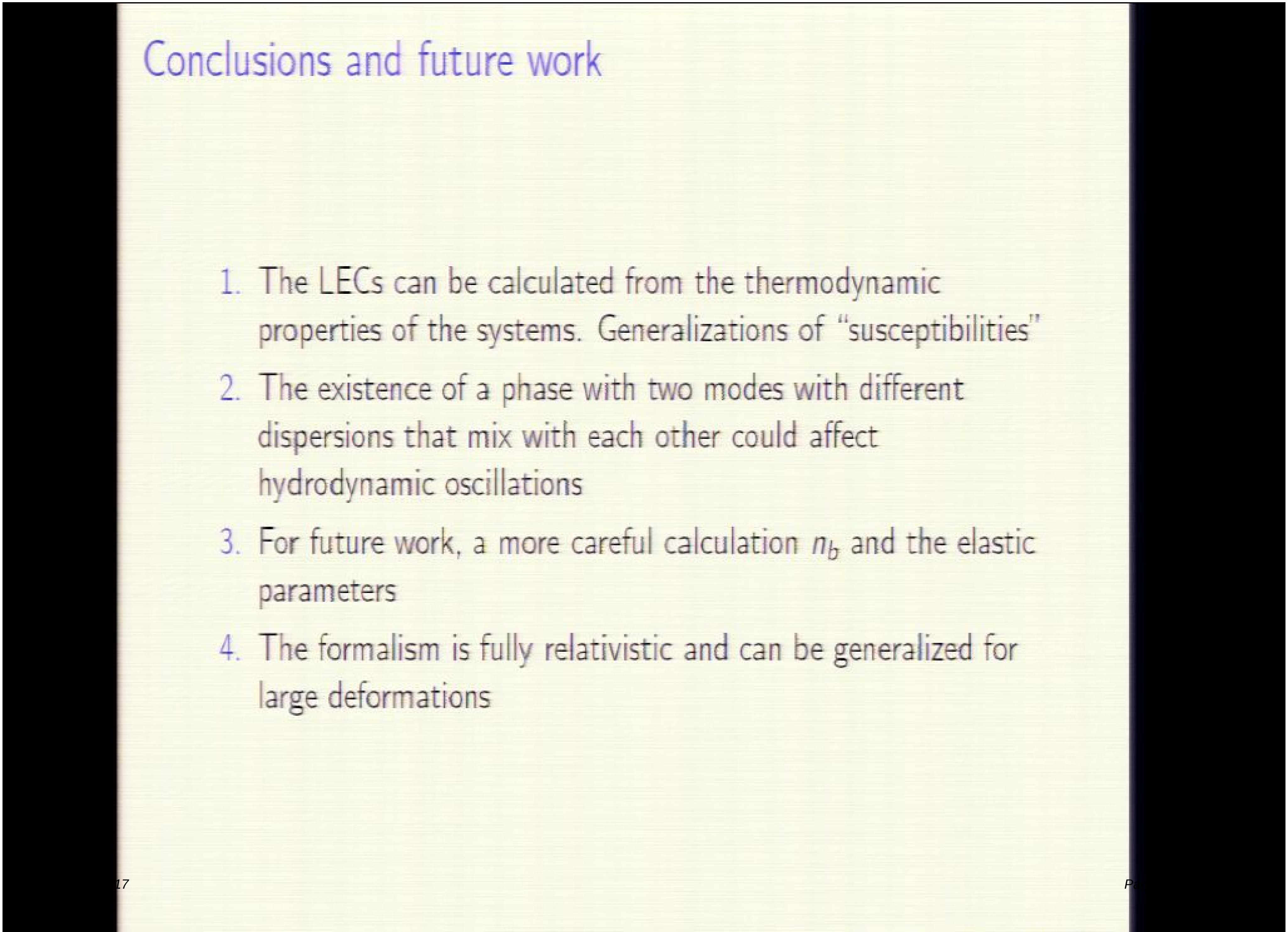
Conductivity



1. $B = 10^{13} - 10^{14}$
2. *Aguilera et. al*

Better calculation of n_b

1. The biggest uncertainty is n_b
2. A quantum calculation with the full band structure without making a Wigner Seitz approximation is desirable
3. One can then calculate the correlation function $n_c = \frac{1}{3}\langle j^i j^i \rangle (q=0)$ *(Chamel et. al.)*



Conclusions and future work

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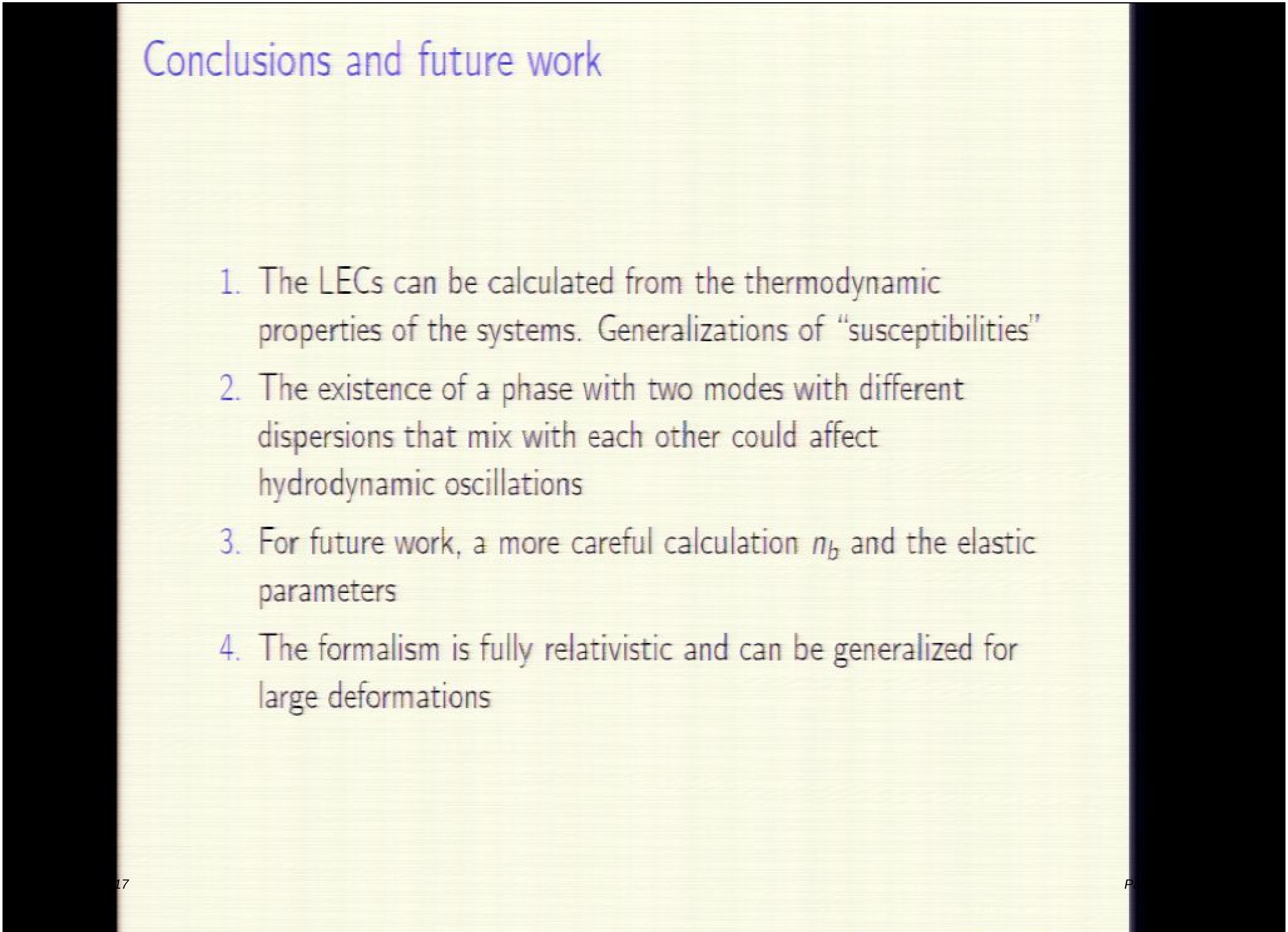
2. The existence of a phase with two modes with different dispersions that mix with each other could affect hydrodynamic oscillations
3. For future work, a more careful calculation n_b and the elastic parameters
4. The formalism is fully relativistic and can be generalized for large deformations



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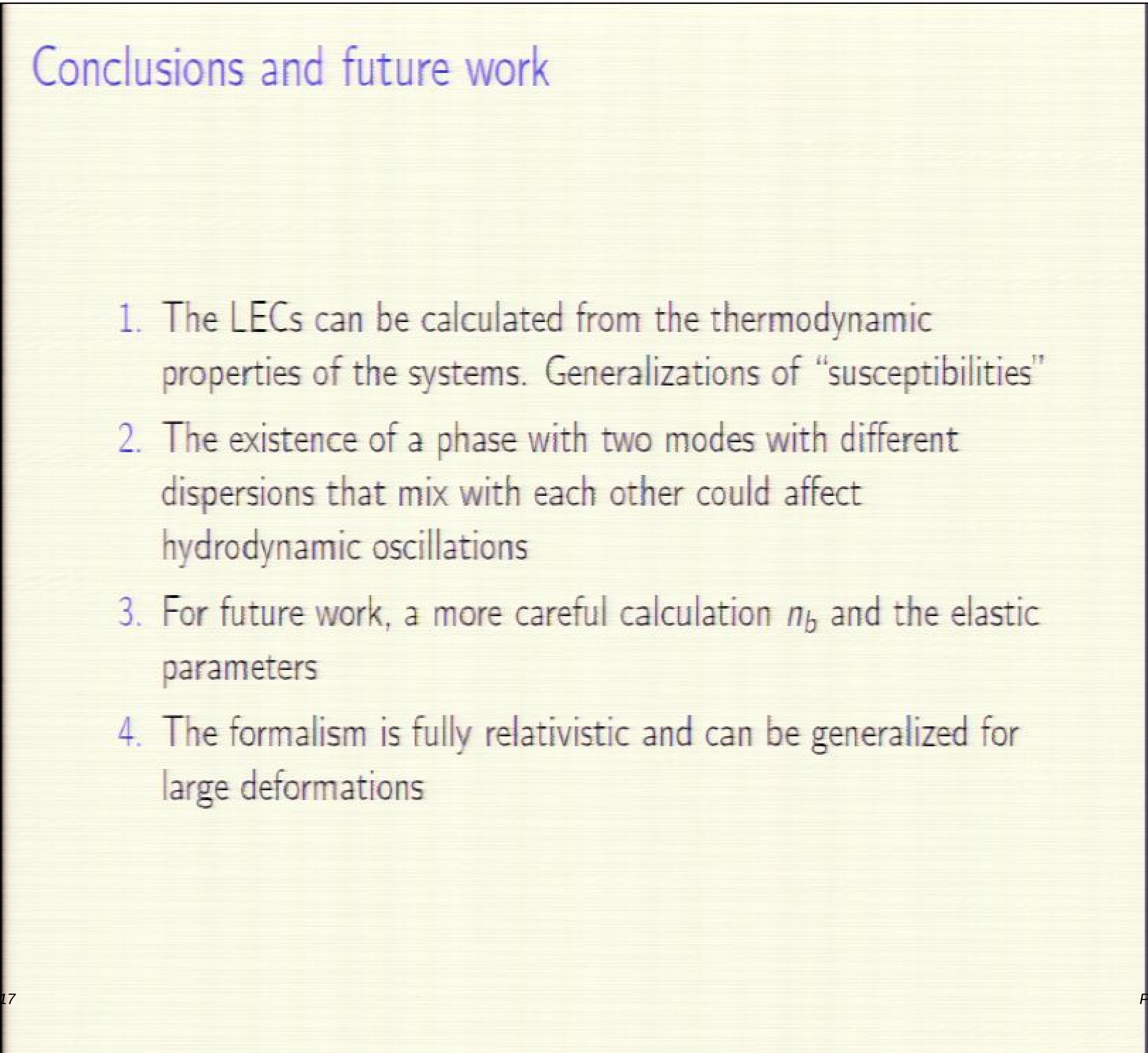

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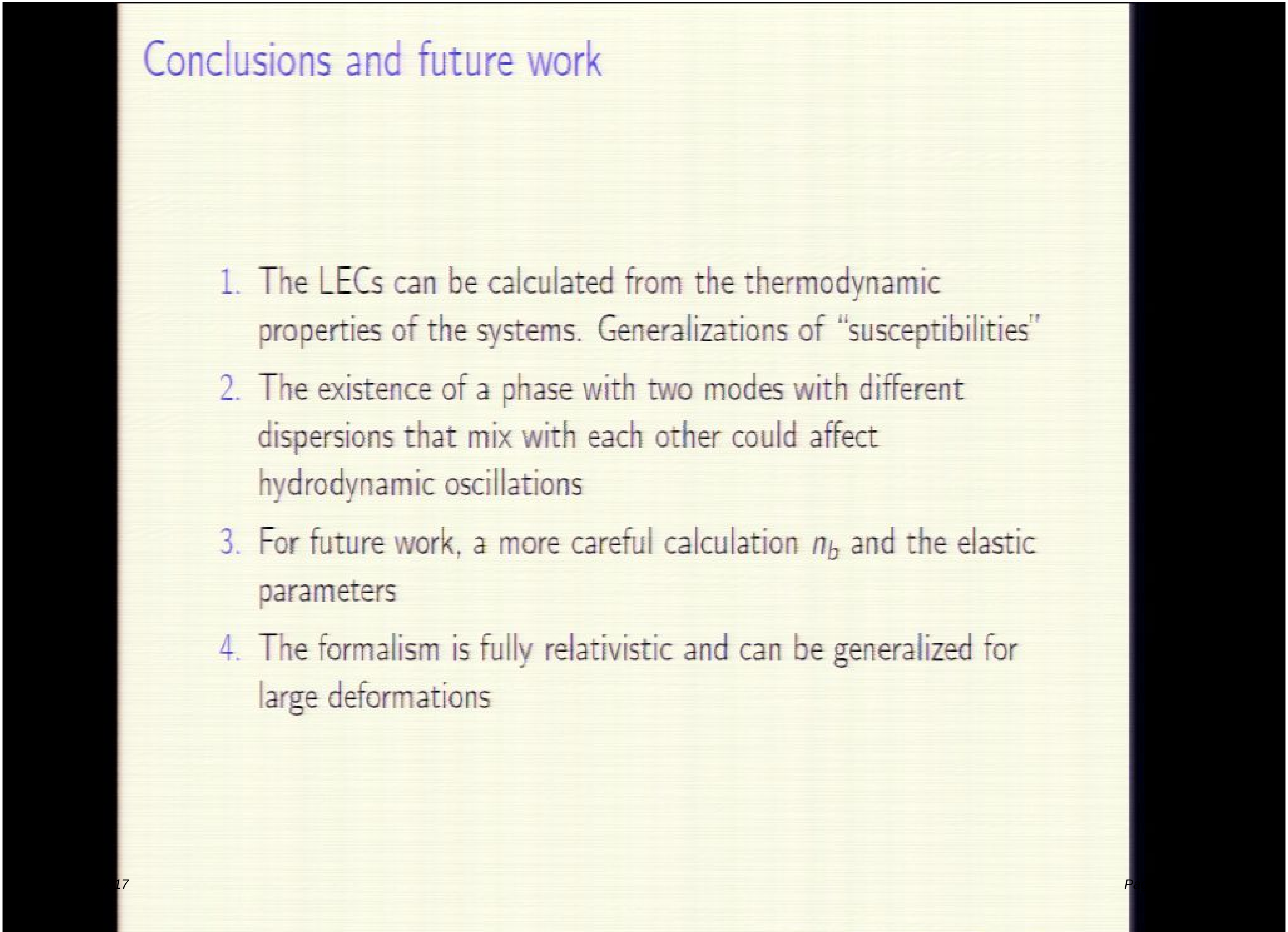

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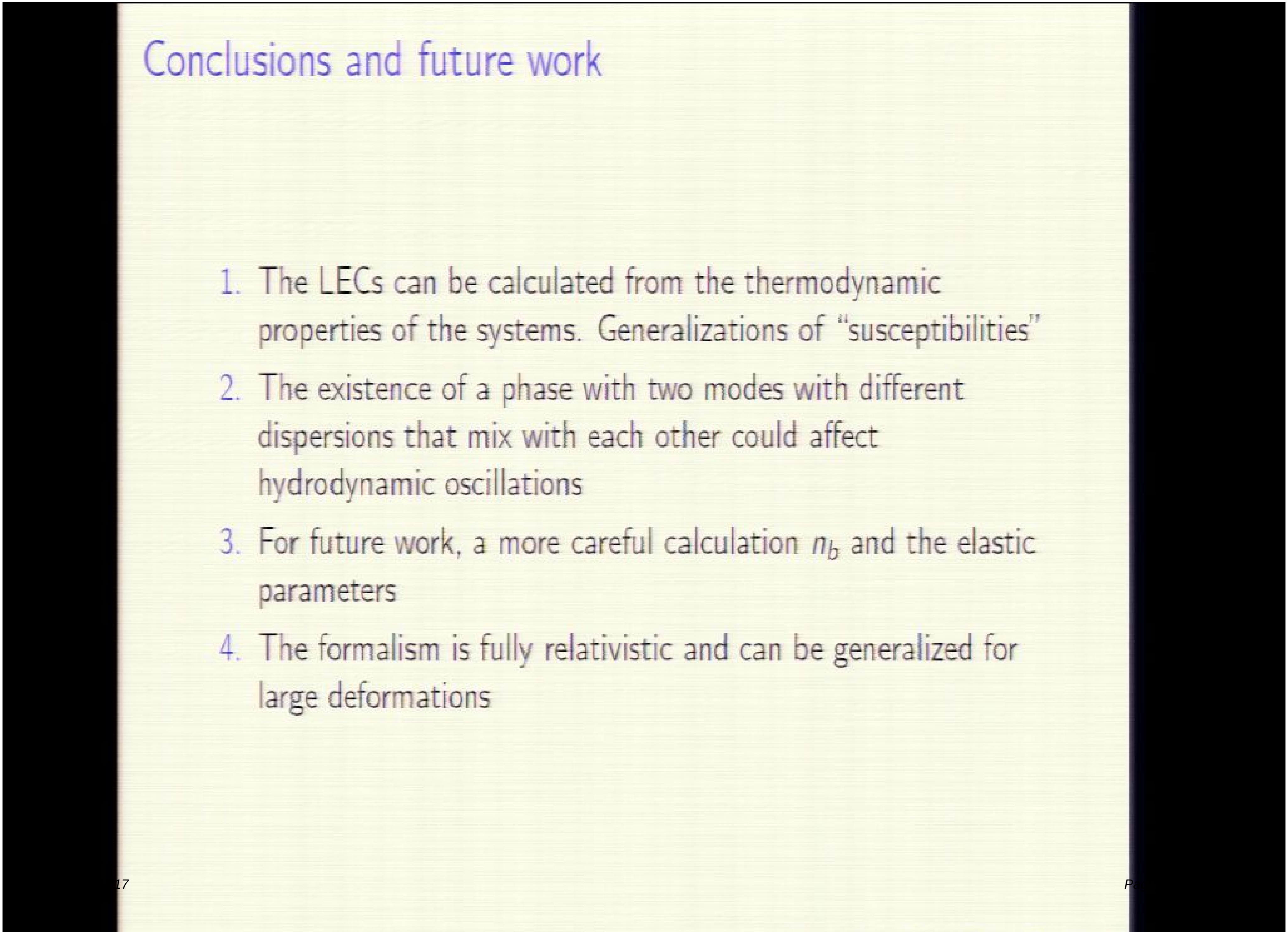

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