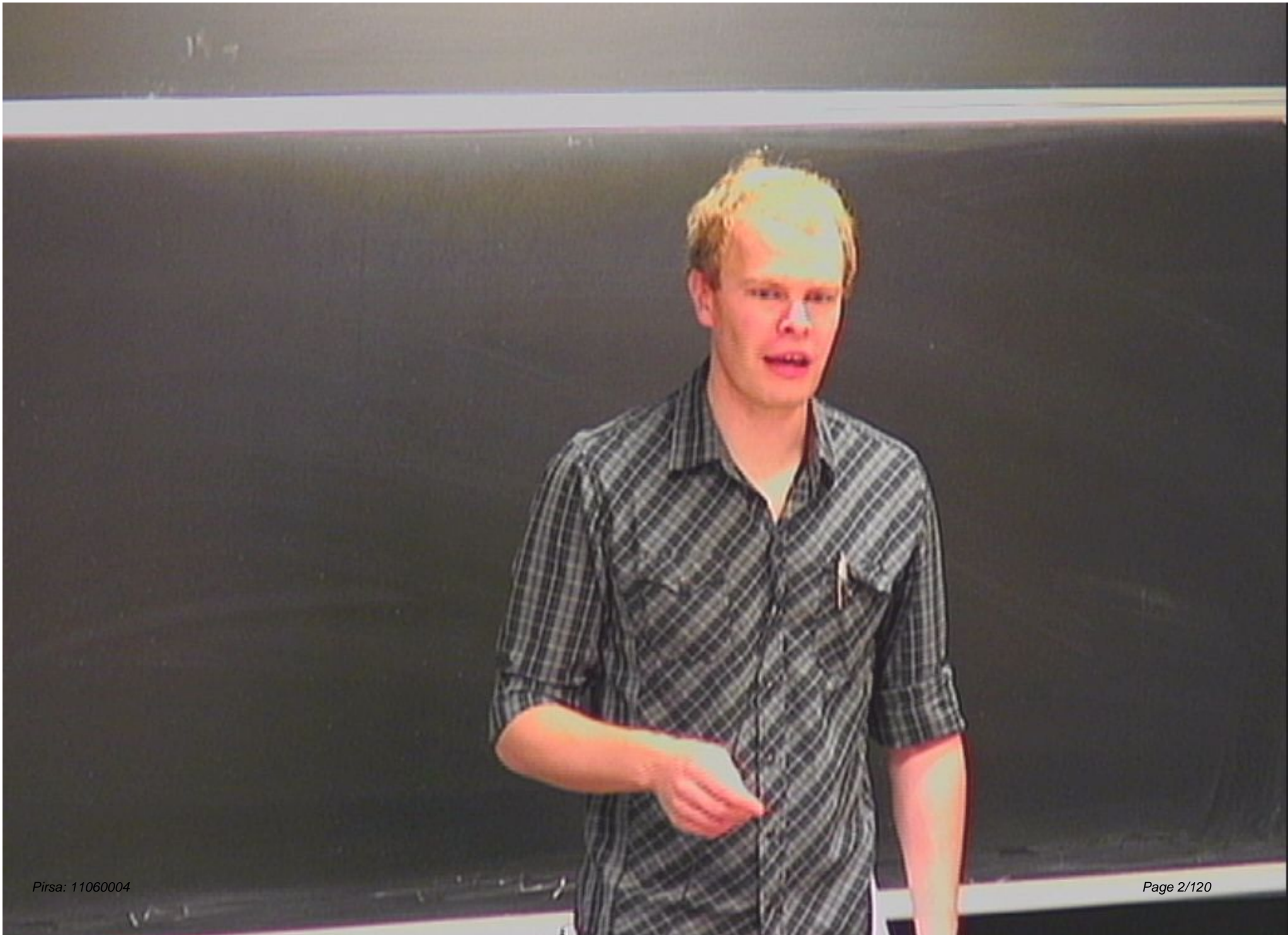


Title: Introduction to Tensor Network Algorithms - Lecture 1

Date: Jun 01, 2011 11:00 AM

URL: <http://pirsa.org/11060004>

Abstract: Tensor network algorithms are a powerful technique for the study of quantum systems in condensed matter physics. In this short series of lectures, I will present an applied perspective on tensor network algorithms. Topics to be covered will include motivation and methodology, graphical notation, Matrix Product States (MPS) and the Time-Evolving Block Decimation (TEBD) algorithm, identifying the capabilities and limitations of tensor network algorithms, the Multi-scale Entanglement Renormalisation Ansatz (MERA) and the study of systems at criticality, and the exploitation of global internal symmetries. The intent of this lecture series is to provide attendees with the necessary theoretical background to be able to understand and implement the more common tensor network algorithms.



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

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$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$|\uparrow\uparrow\rangle$$

$$|\uparrow\downarrow\rangle$$

$$|\downarrow\uparrow\rangle$$

⋮

Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{i_1} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right]_{i_1}^{(1) \kappa} \lambda_{\alpha}^{(1) \beta} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right]_{i_2}^{(2) \delta} \lambda_{\gamma}^{(2) \epsilon} \dots |i_1, \dots, i_n\rangle$$

Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

d^{\sim}

$$\left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right] \left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right] \dots \left[\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \right] |i_1, \dots, i_n\rangle$$

$$\begin{array}{c} \lambda_1 \\ \vdots \\ \lambda_n \end{array} \geq d^{\sim}$$

Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{\alpha} \left[\prod_{i=1}^n \langle i | \alpha \rangle \right] \alpha$$

α
 β
 γ
 \dots
 D



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$= \sum_{i_1, \dots, i_n} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \vdots \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]_{i_1, \dots, i_n}$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{\alpha} \lambda_{\alpha} \sqrt{\lambda_{\alpha}} \dots |i_1, \dots, i_n\rangle$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$= \sum_{i_1, \dots, i_n} \left[\prod_{\alpha=1}^n \delta_{i_\alpha}^{(\alpha)} \right] \dots$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

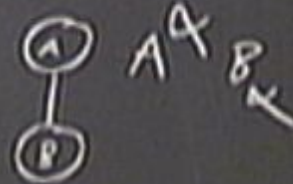
$$= \sum_{\alpha} \lambda_{\alpha} \sqrt{\lambda_{\alpha}} \left[\begin{array}{c} \alpha \\ i_1 \quad i_2 \quad \dots \quad i_n \end{array} \right] |i_1, \dots, i_n\rangle$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$= \sum_{i_1, \dots, i_n} \left[\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{\alpha} \left[\prod_{i=1}^n \lambda_{i, \alpha}^{(i)} \right] |\alpha\rangle$$

$$\sum_{\alpha} \lambda_{i, \alpha}^{(i)} \geq d^{\sum_{i=1}^n \lambda_{i, \alpha}^{(i)}}$$

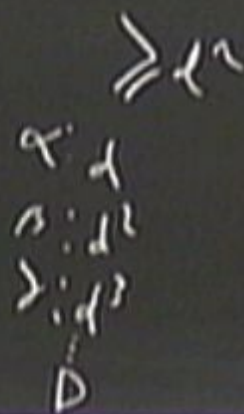


Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

d^n

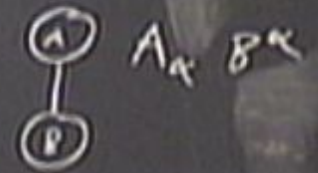
$$= \sum_{\alpha} \left[\prod_{i=1}^n \lambda_{\alpha}^{(i)} \right] \left[\prod_{i=1}^n \lambda_{\alpha}^{(i)} \right]$$



Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, i_2, \dots, i_n} c_{i_1, i_2, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{\alpha} \lambda_{\alpha} \sqrt{\frac{d_{\alpha}}{D}} \sum_{i_1, i_2, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$



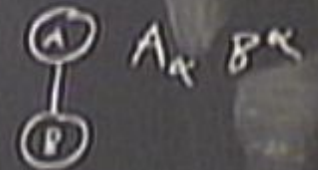
Introduction to Tensor Network Algorithms (I)

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

d^n

$$= \sum_{i_1, \dots, i_n} \left[\prod_{\alpha=1}^n \lambda_{i_1}^{(\alpha)} \lambda_{i_2}^{(\alpha)} \dots \lambda_{i_n}^{(\alpha)} \right]$$

$$\sum_{i_1, \dots, i_n} \lambda_{i_1}^{(\alpha)} \lambda_{i_2}^{(\alpha)} \dots \lambda_{i_n}^{(\alpha)} \geq d^n$$



Introduction to Tensor Network Algorithms (I)

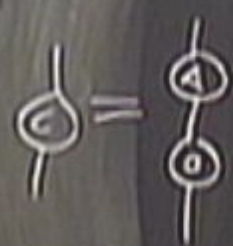
$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$= \sum_{i_1, \dots, i_n} \lambda_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$



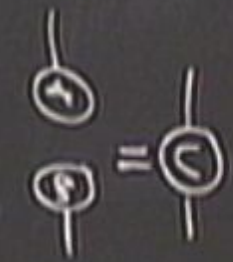
$$A_{\alpha}^{\beta} \quad B^{\alpha\beta}$$

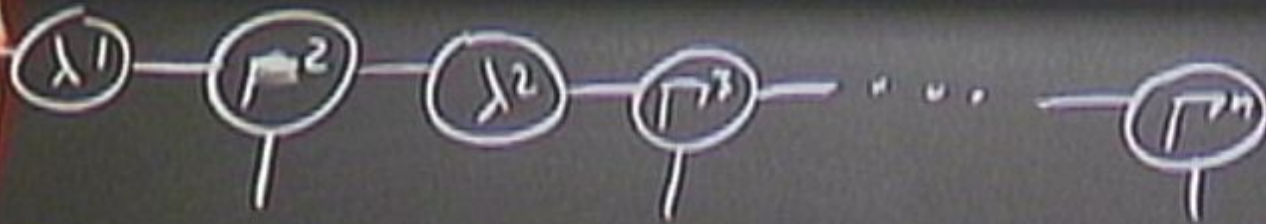
$$A_{\alpha}^{\alpha} = \text{Tr}(A)$$

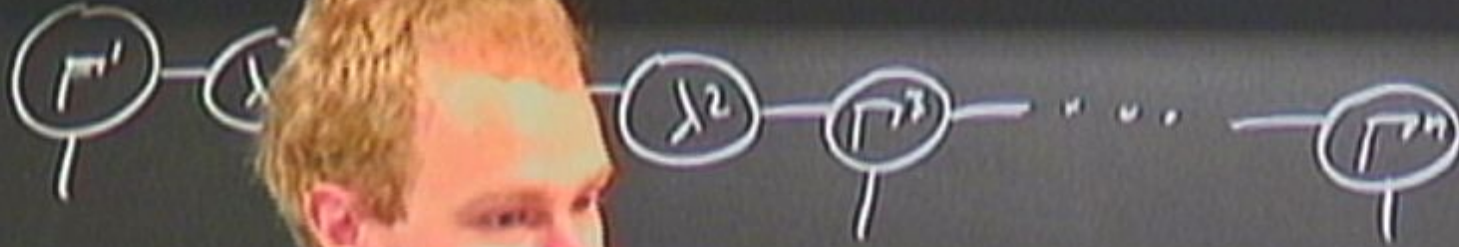


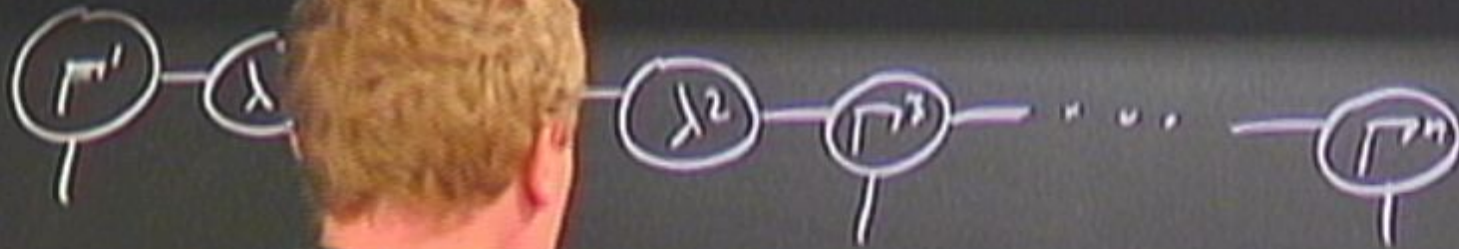
$$A_{\alpha}^{\beta} B^{\gamma\delta} = C_{\alpha}^{\gamma}$$

$$B^{\alpha\beta} = A^{\alpha\beta}$$

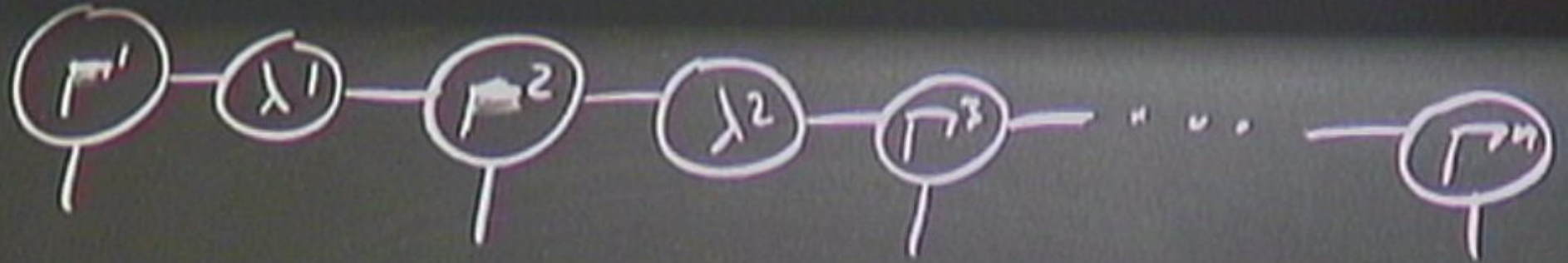








D



$\beta : d^2$
 $\lambda : d^3$
 D

$C = A^T B$



$\beta : d^2$
 $\lambda : d^3$
 D

$C = A^T B$



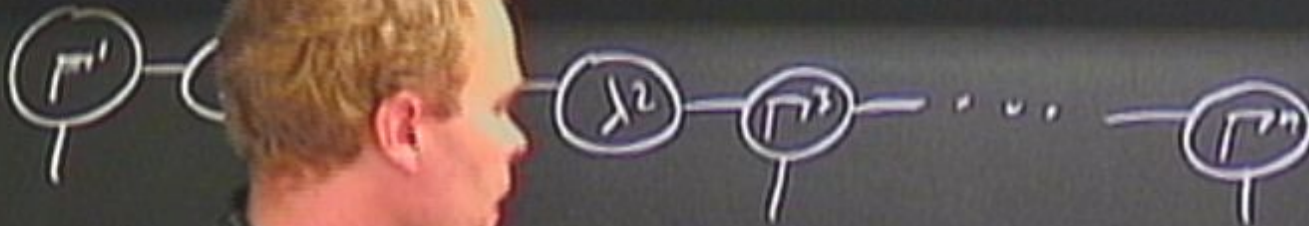
$\beta : d_1^2$
 $\gamma : d_1^3$
 $\delta : d_1^4$

$$C_{\beta} = A^T B$$



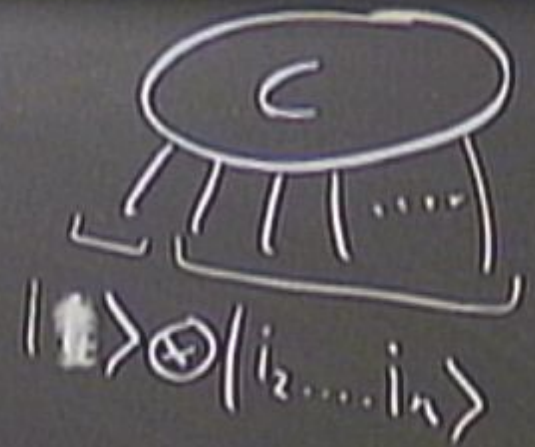
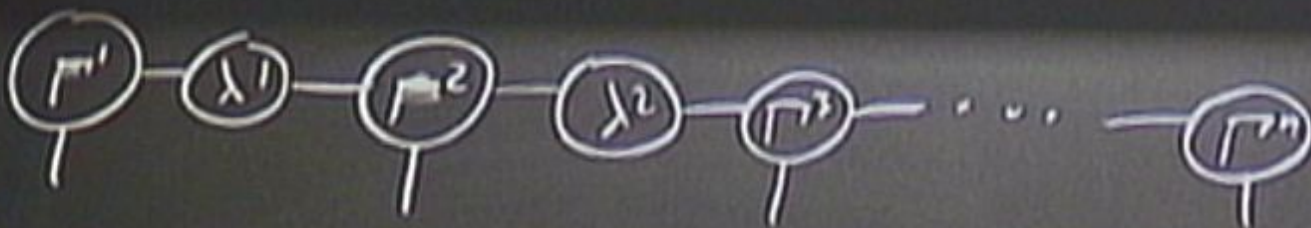
$\beta : d_1^2$
 $\gamma : d_1^3$
 $\delta : d_1^3$

$$C_{\beta} = A^T \beta$$



$$\begin{aligned} B &: d_1^2 \\ \lambda &: d_1^3 \\ D &: \dots \end{aligned}$$

$$C = A^T B$$



$\beta: d^2$
 $\lambda: d^3$
 $D: d^4$

$$C_B = A^T B$$

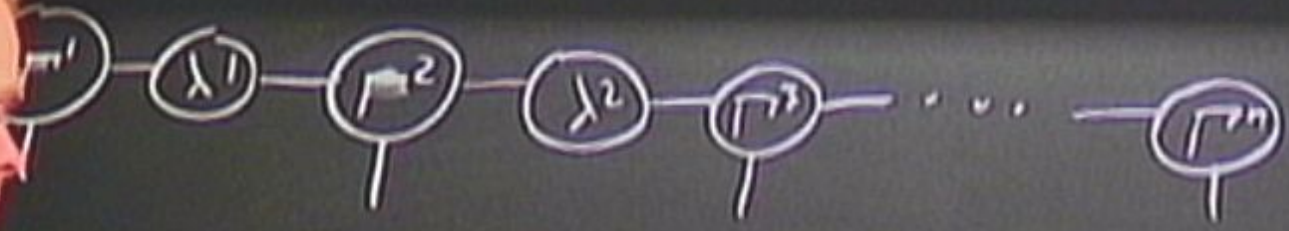


$$= \sum C_{i_1 \dots i_n} |i_1 \dots i_n\rangle$$

$$|i_1\rangle \otimes |i_2 \dots i_n\rangle$$

$B: d^2$
 $\lambda: d^3$
 $D: d^3$

$$C = A^T B$$



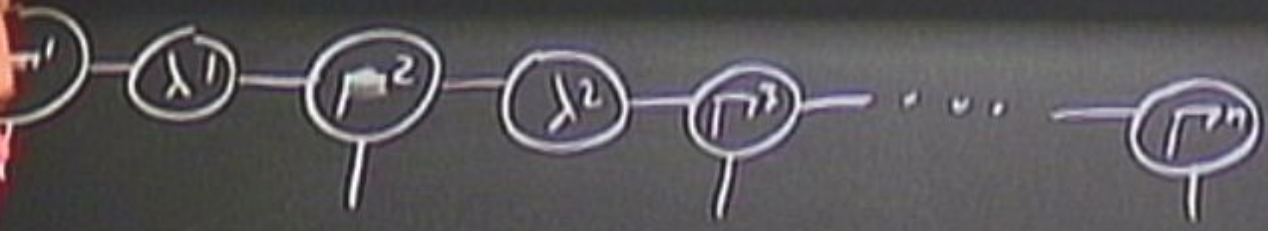
$$|\psi\rangle = \sum c_{i_1 \dots i_n} |i_1 \dots i_n\rangle$$

=



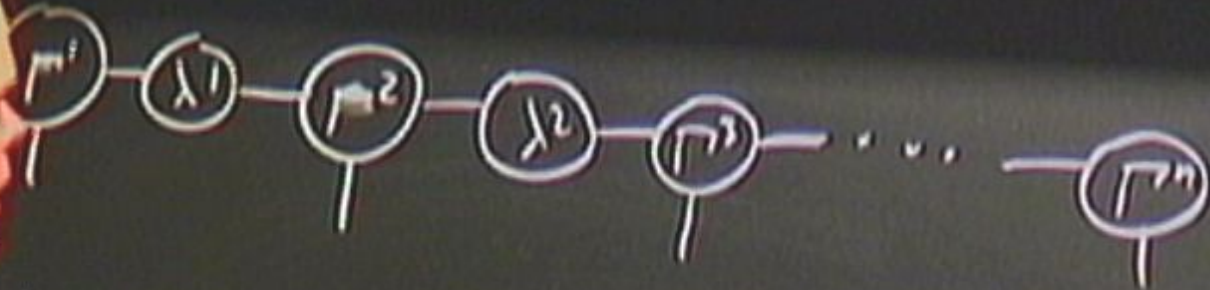
$\beta: d^2$
 $\lambda: d^3$
 $D: d^3$

$$C_B = A^T B$$



$$|\psi\rangle = \sum c_{i_1 \dots i_n} |i_1 \dots i_n\rangle$$

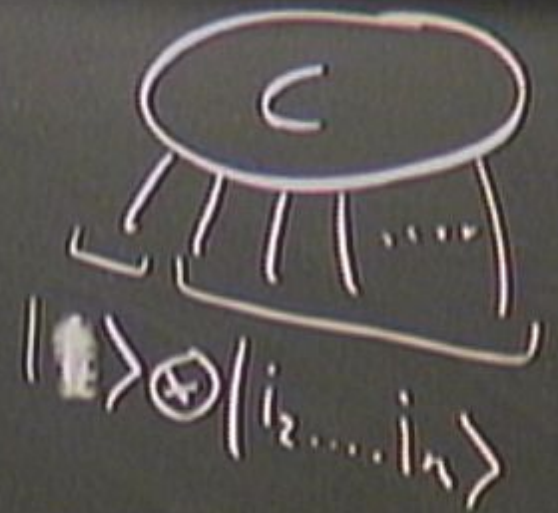
=

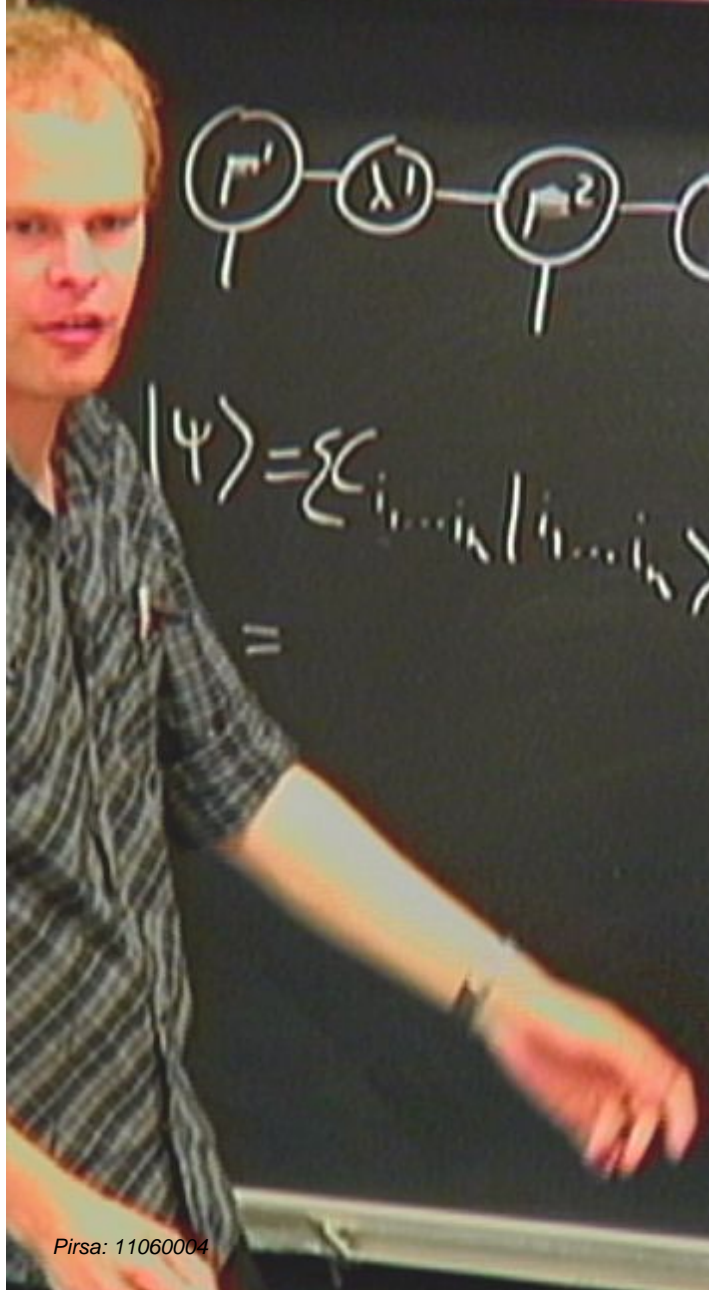


$$|\psi\rangle = \sum c_i |i_1 \dots i_n\rangle$$

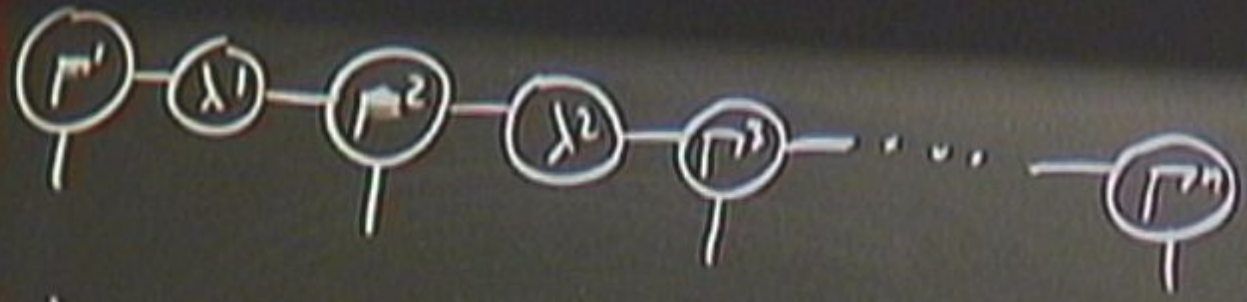
$$=$$

$$M = U S$$





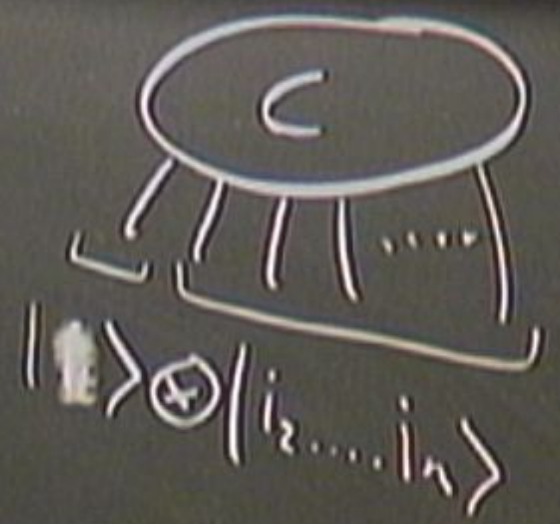
D

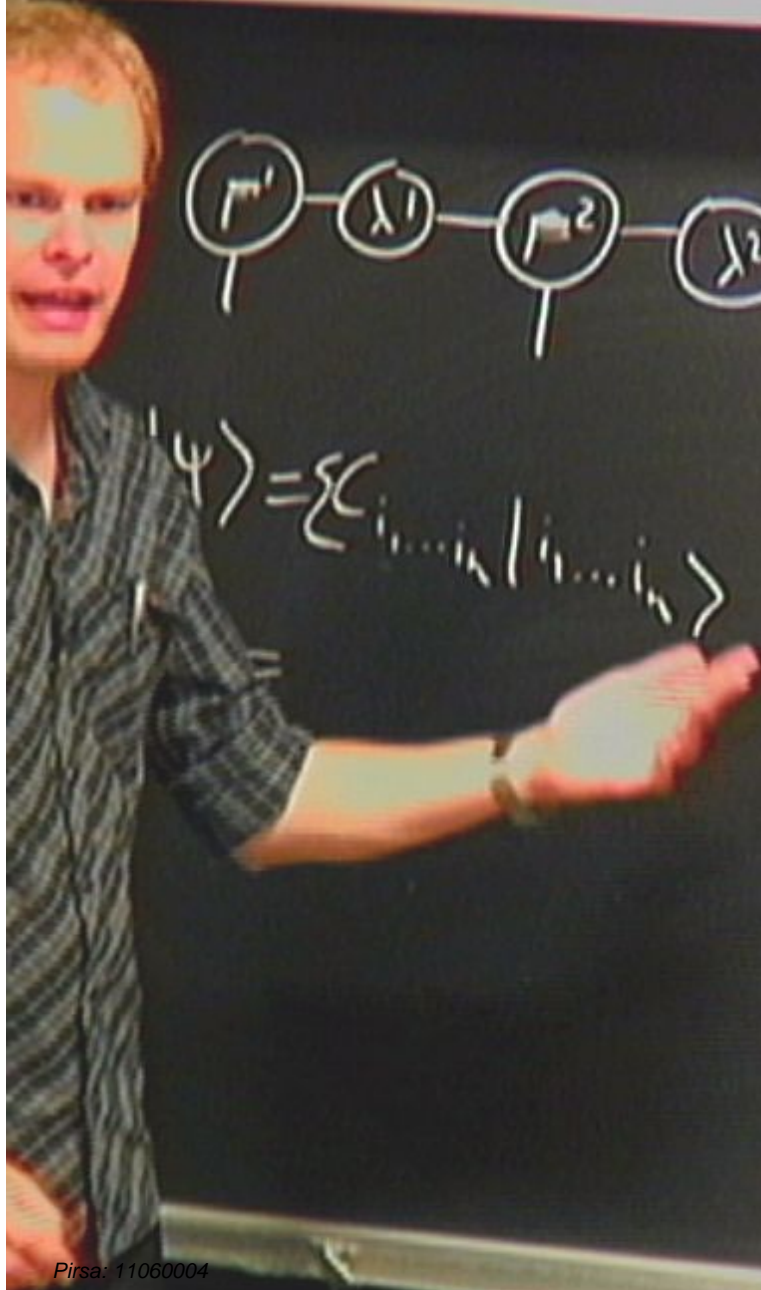


$$|\psi\rangle = \sum c_{i_1 \dots i_n} |i_1 \dots i_n\rangle$$

$$=$$

$$M = U S$$

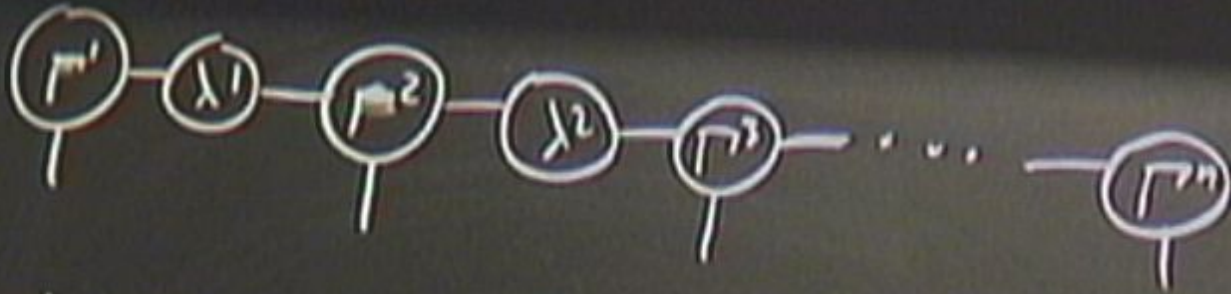




$$|\psi\rangle = \sum c_i |i_1 \dots i_n\rangle$$

$$M = U S V^\dagger$$

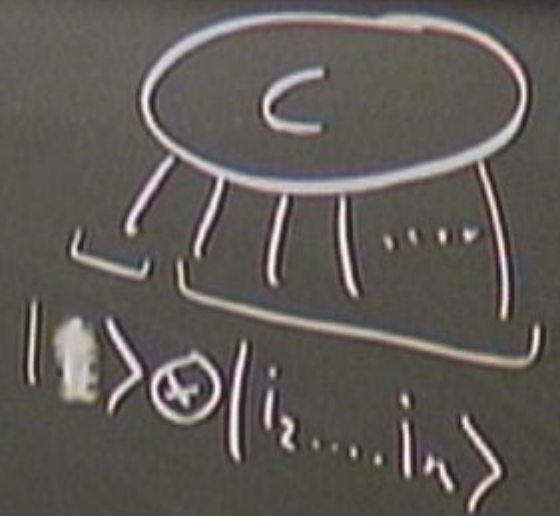


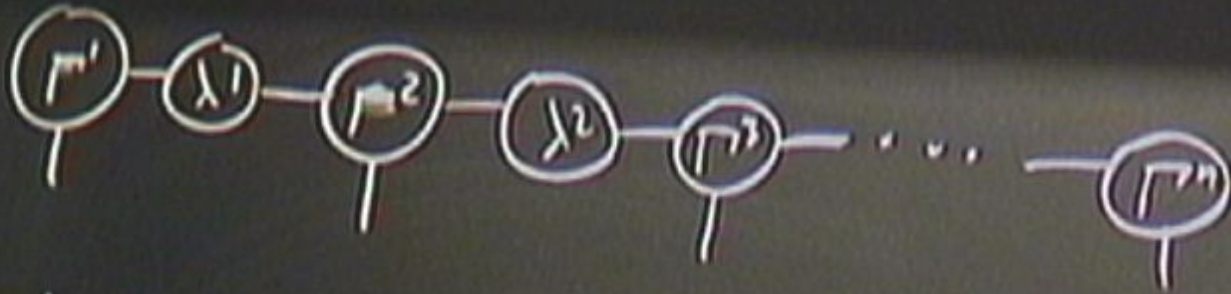


$$|\psi\rangle = \sum_{i=1}^n c_i |i_1 \dots i_n\rangle$$

$$=$$

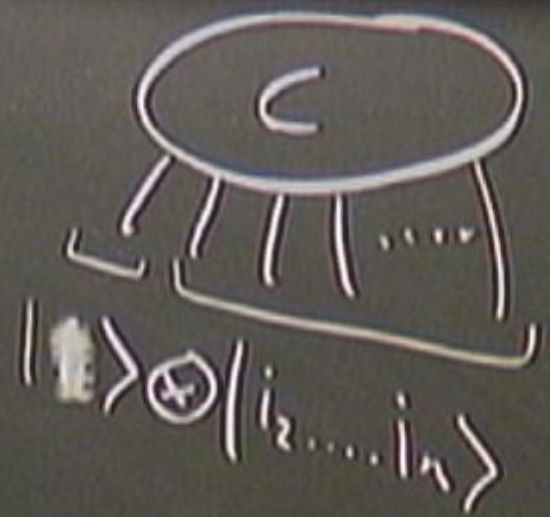
$$M = U S V^\dagger$$

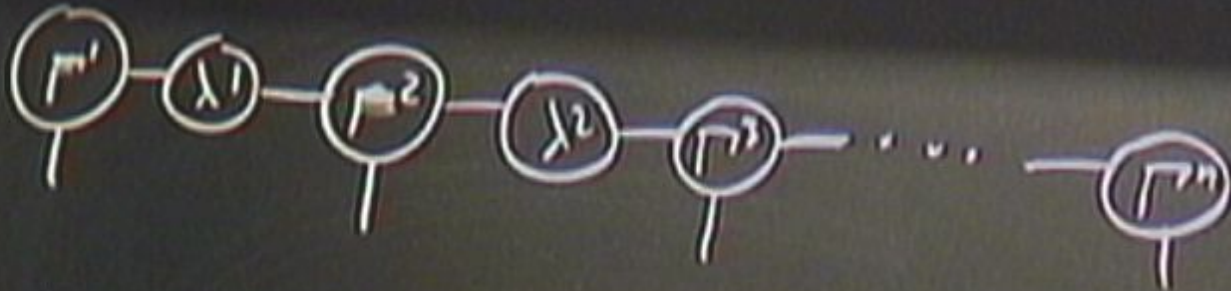




$$|\psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

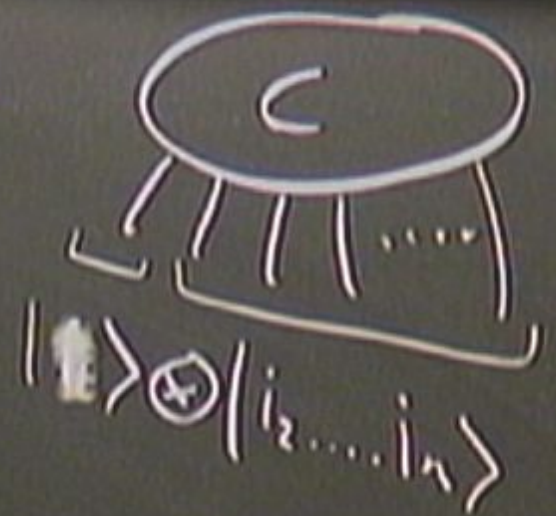
$$M = U S V^\dagger$$

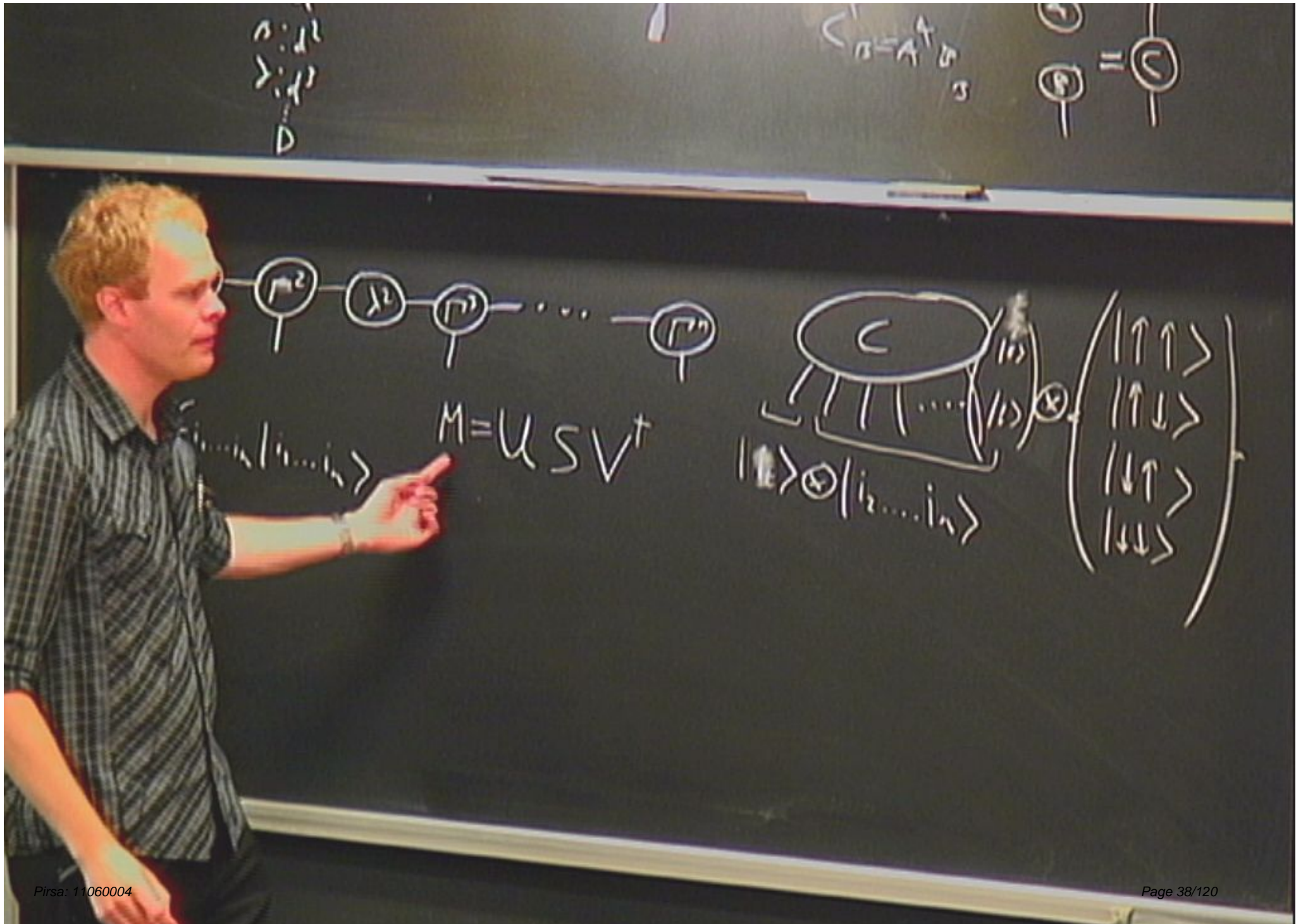




$$|\psi\rangle = \sum c_i |i_1 \dots i_n\rangle$$

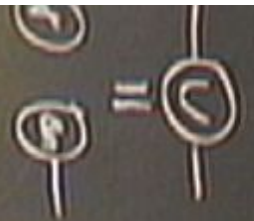
$$M = U S V^\dagger$$





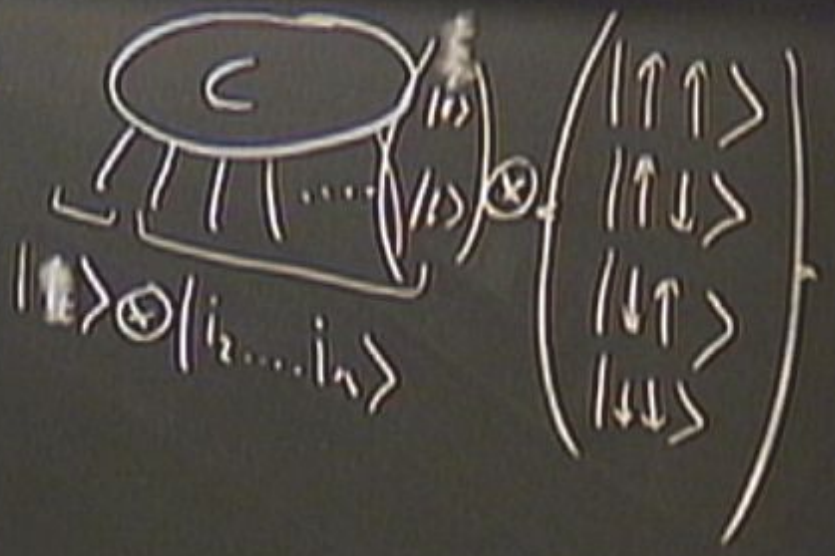
$\sigma: d_1$
 $\lambda: d_2$
D

$$C = A^T B$$



$\dots |i_1 \dots i_n\rangle$

$$M = U S V^T$$



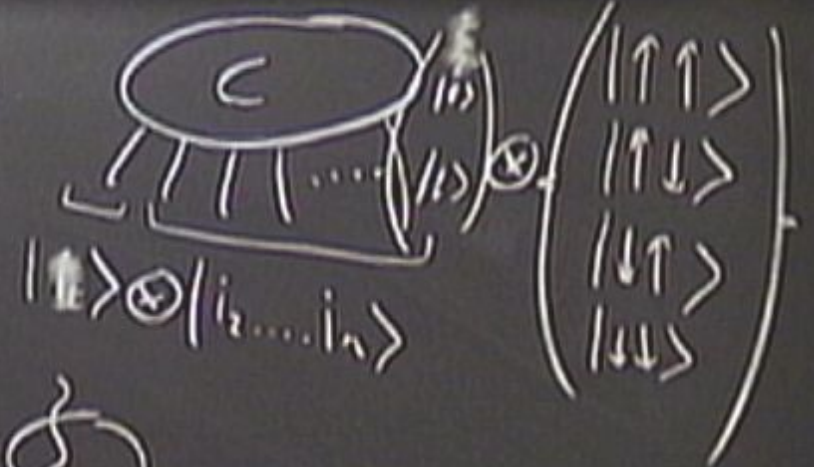
α_1
 β_1
 γ_1
 \dots
 α_n
 β_n
 γ_n
 \dots
 α

$B = A^T B$



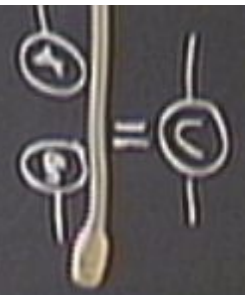
$|\psi\rangle = \sum c_i |i\rangle$
 $=$

$M = U S V^\dagger$



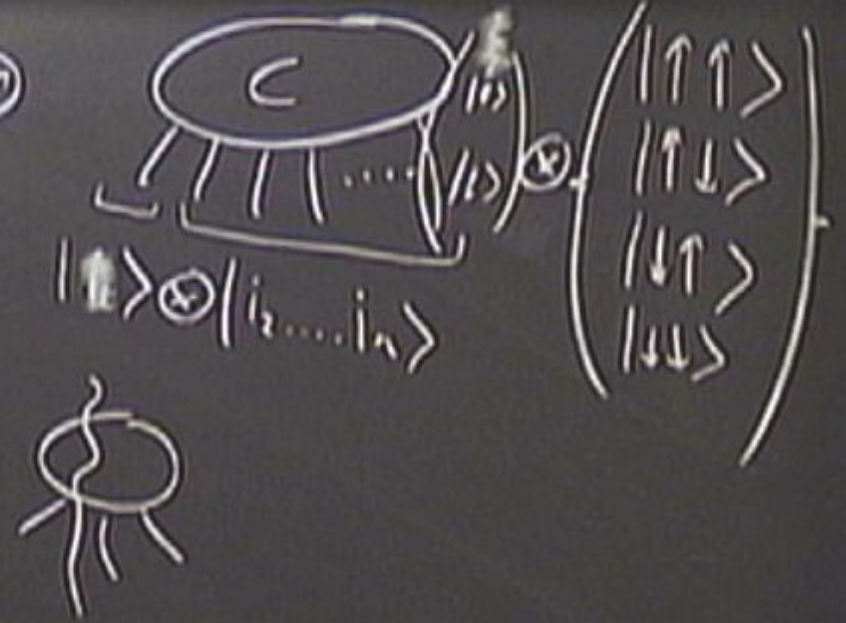
α
 β
 γ
 δ

$A = A^T B$



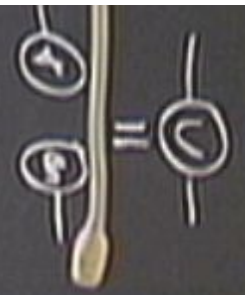
$|\psi\rangle = \xi$
 $=$

$M = U S V^T$



α
 β
 γ
 δ

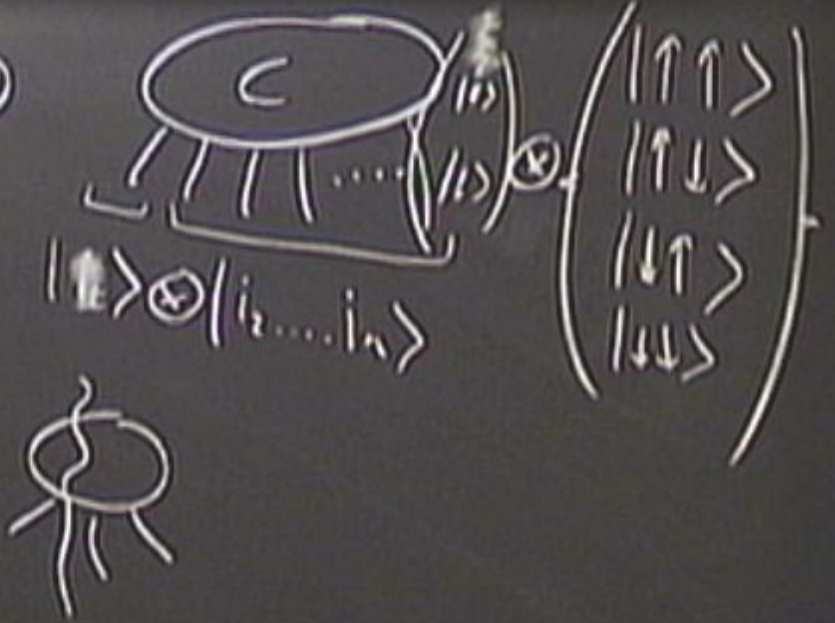
$$B = A^T A$$



$$|\psi\rangle = \sum c_i$$

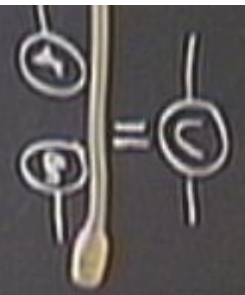
$$=$$

$$M = U S V^\dagger$$



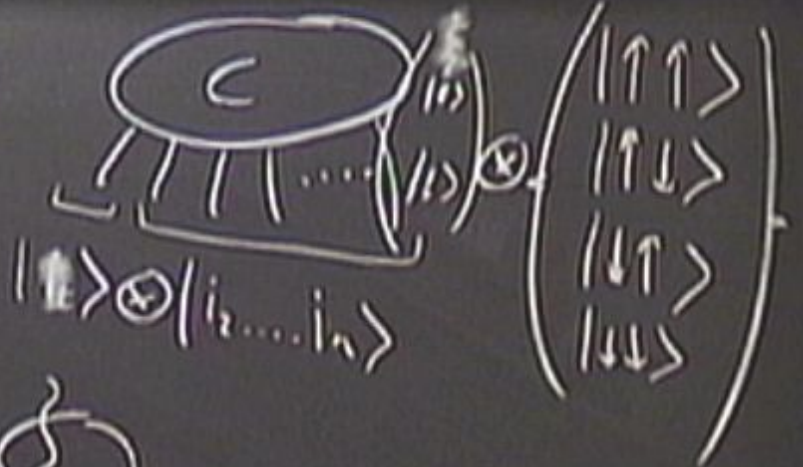
α
 β
 γ
 δ

$B = A^T B$



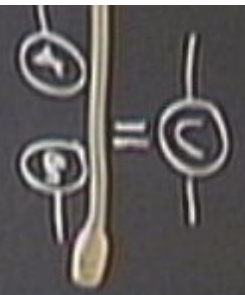
$|\psi\rangle = \sum c_i$
 $=$

$M = U S V^\dagger$



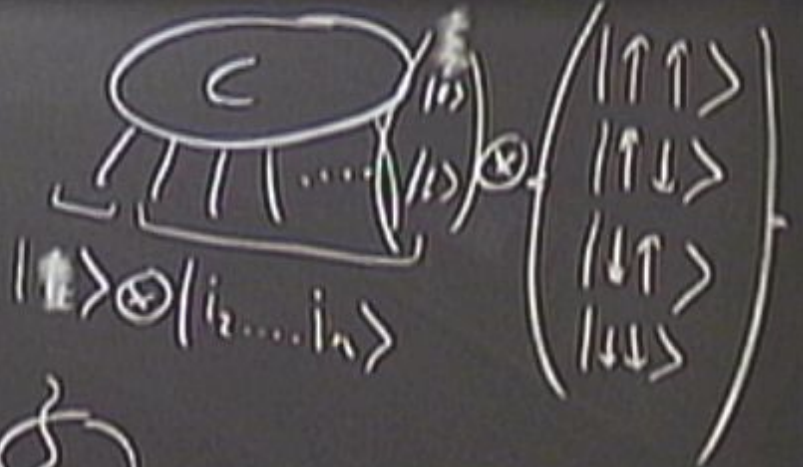
α
 β
 γ
 δ

$$A = A^T B$$



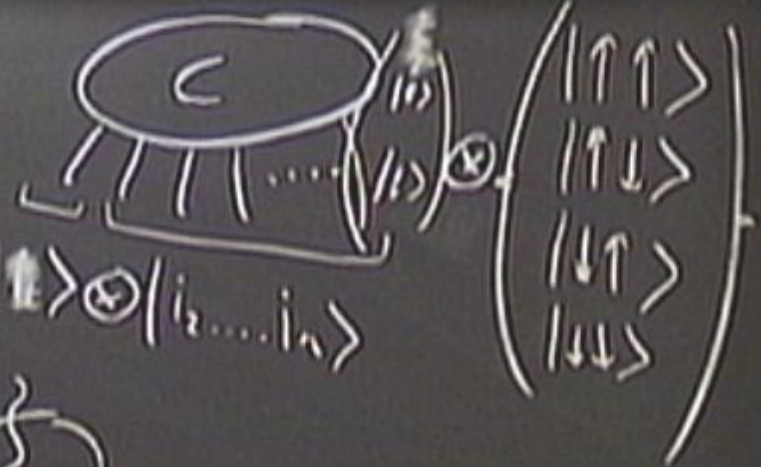
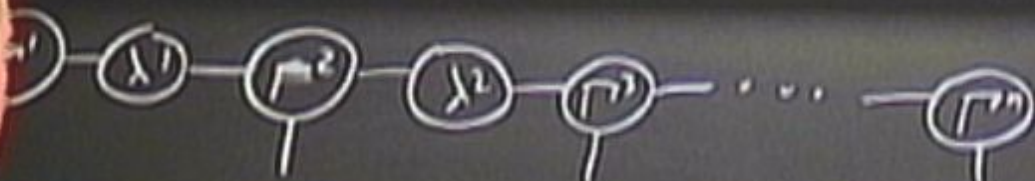
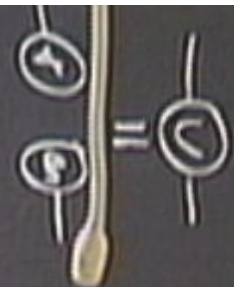
$$|\psi\rangle = \sum c_i |i\rangle$$

$$M = U S V^\dagger$$



λ_1
 λ_2
 λ_3
 \dots
 λ_n

$A = A^T$



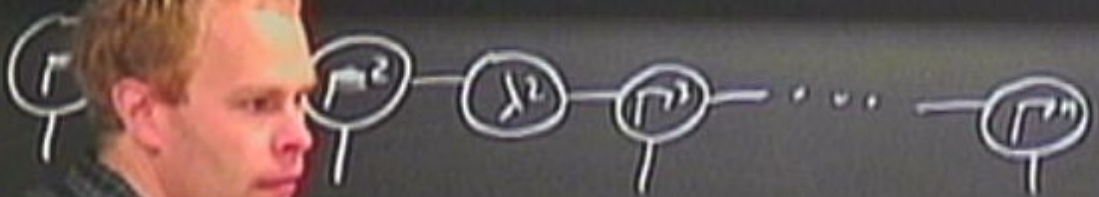
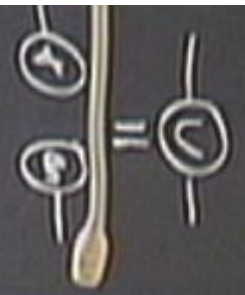
$|\psi\rangle = \sum c_i |i\rangle$

$M = U S V^T$



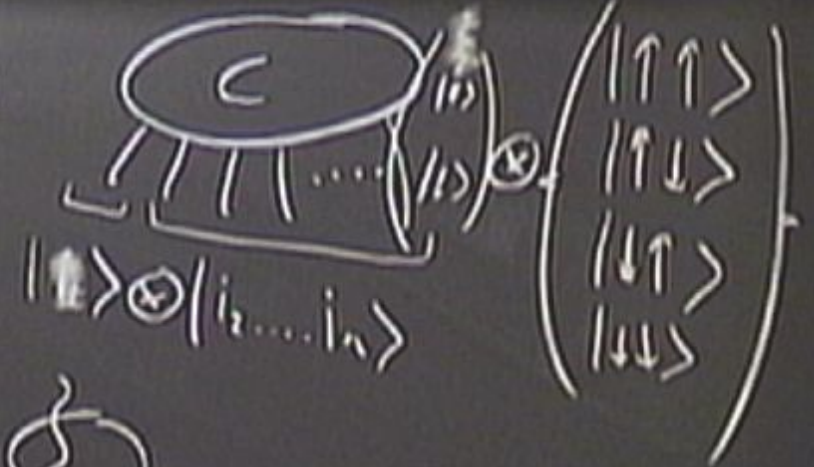
α
 β
 γ
 δ

$$A = A^\dagger$$



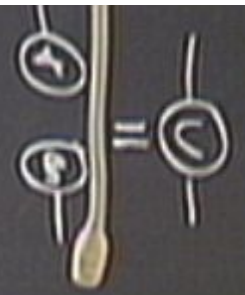
$$M = U S V^\dagger$$

$|i_1 \dots i_n\rangle$



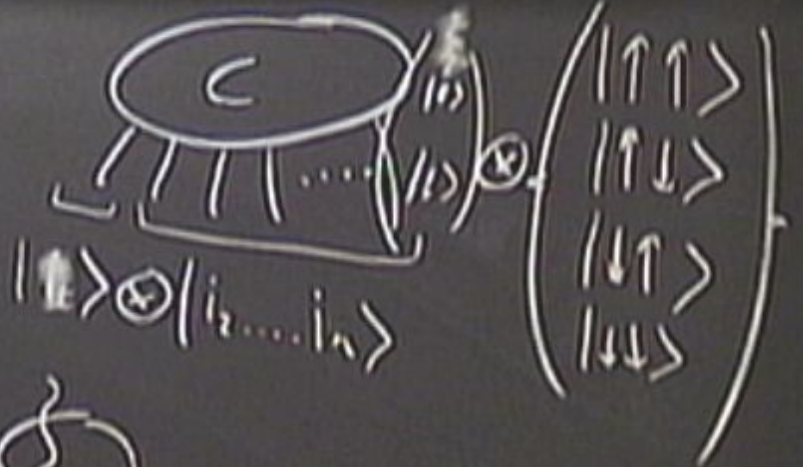
ρ
 $\rightarrow \rho$
 $\lambda_1 \rho_1$
 $\lambda_2 \rho_2$
 \dots
 $\lambda_n \rho_n$

$A^\dagger = A^T$



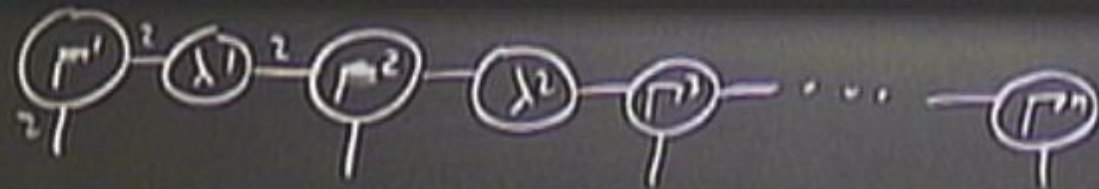
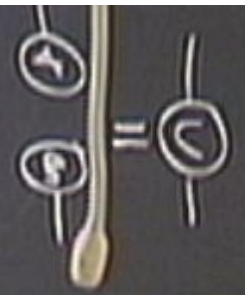
$|\psi\rangle = \sum c_{i_1 \dots i_n} |i_1 \dots i_n\rangle$
 $=$

$M = U S V^\dagger$



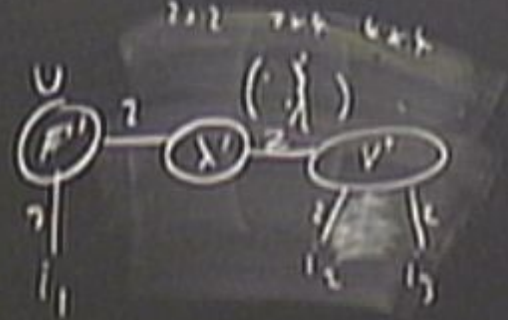
$\rho = \sum_i p_i |i\rangle\langle i|$
 $\rho = \sum_i p_i |i\rangle\langle i|$
 $\rho = \sum_i p_i |i\rangle\langle i|$

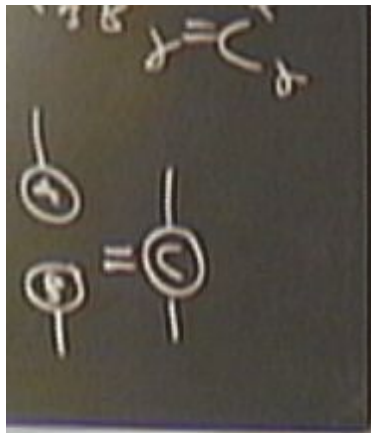
$A = A^\dagger$
 $B = B^\dagger$



$|\psi\rangle = \sum_i c_i |i\rangle$
 $=$

$M = U S V^\dagger$



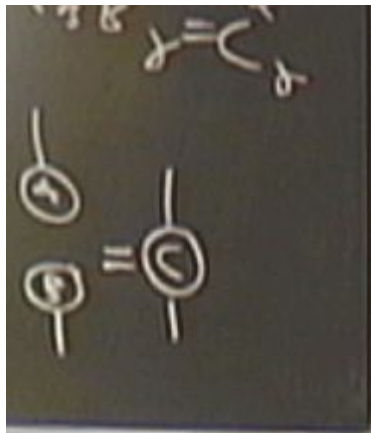


$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}$$

↓

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

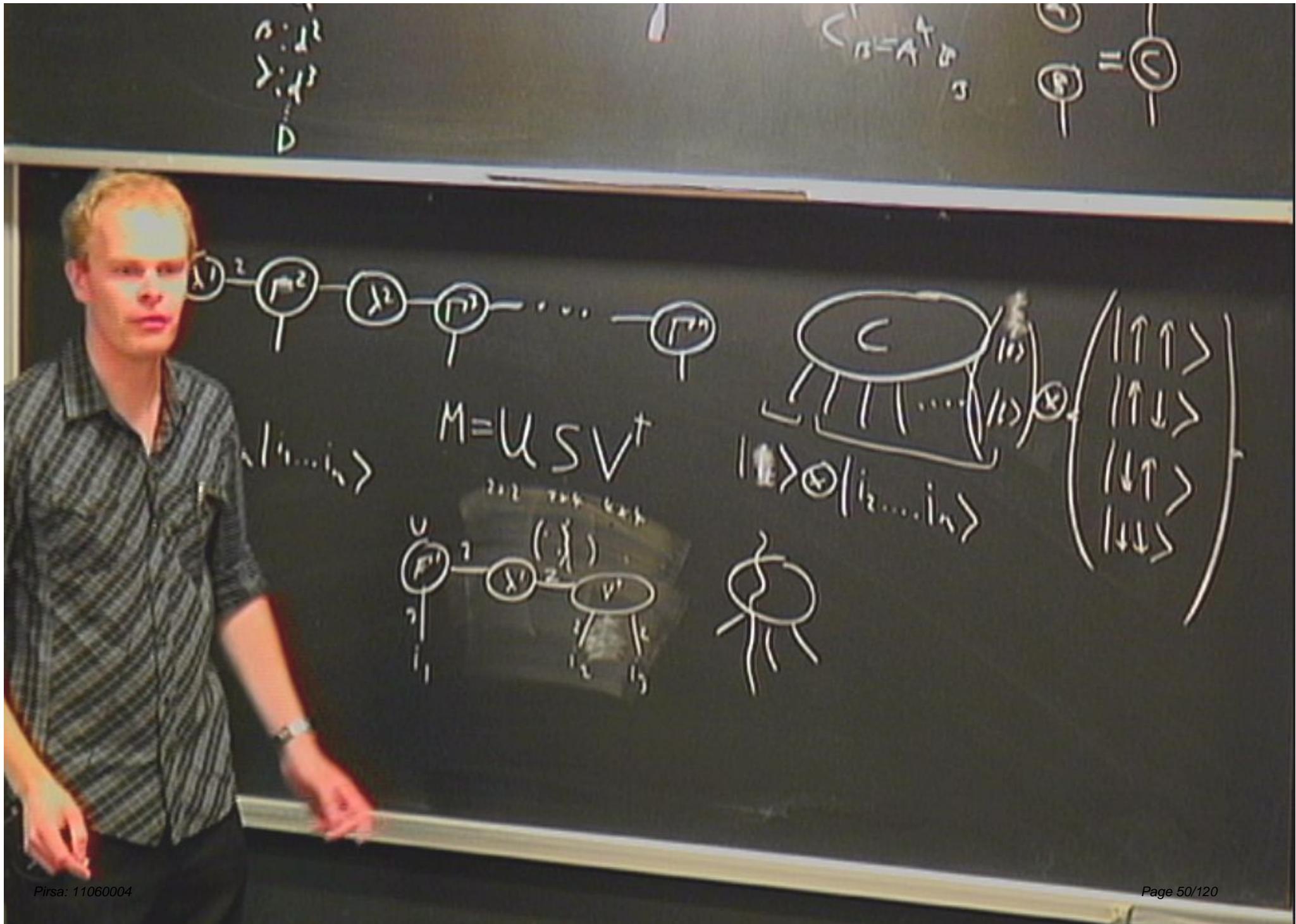
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$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}$$

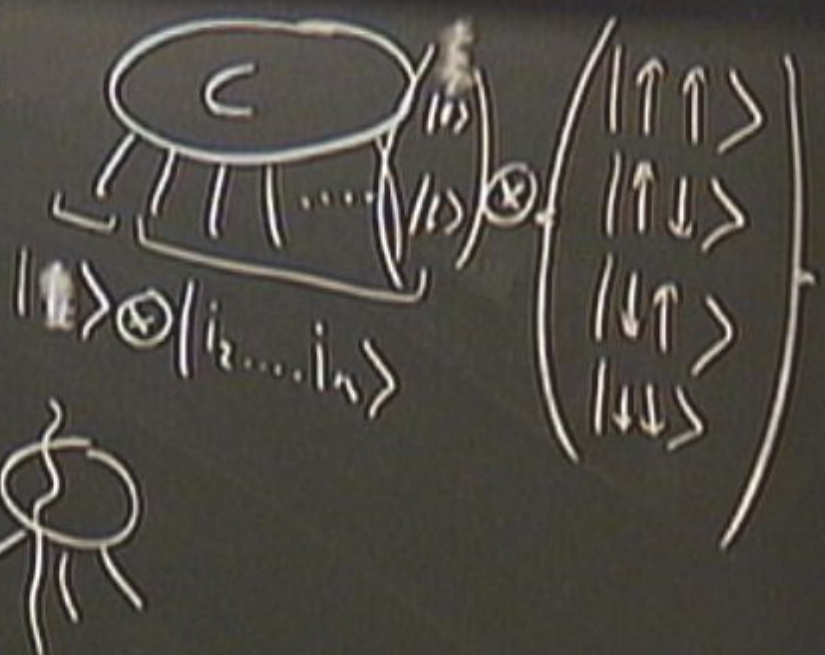
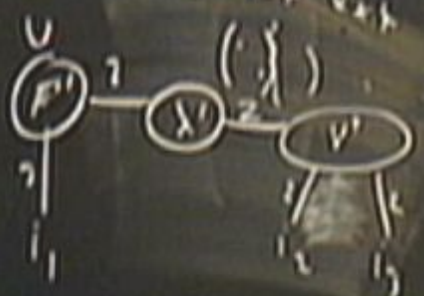
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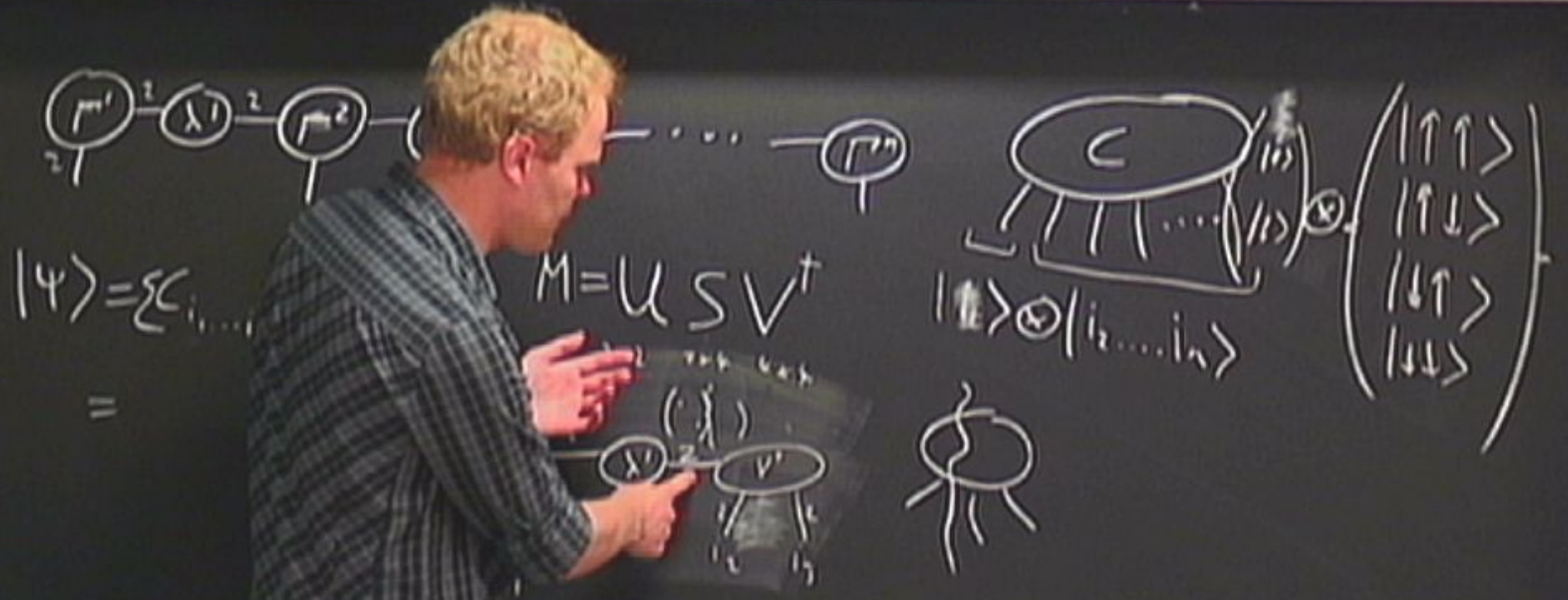
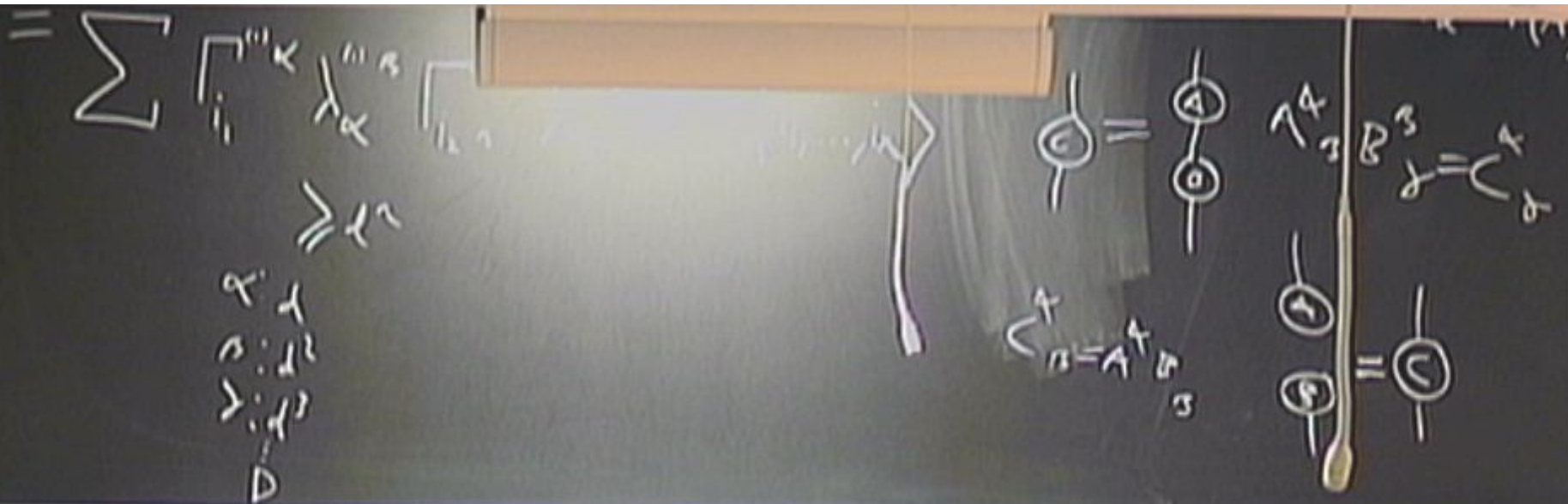
$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$



$$M = U S V^t$$

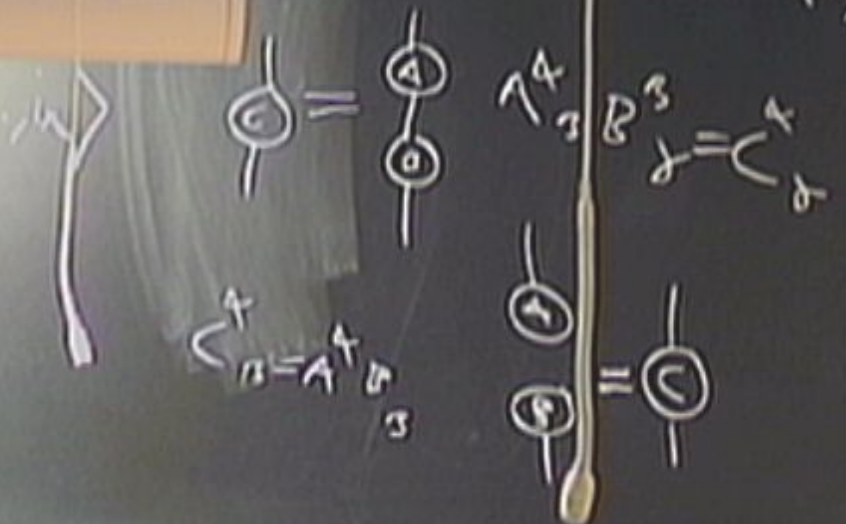
$|1 \dots i_n\rangle$





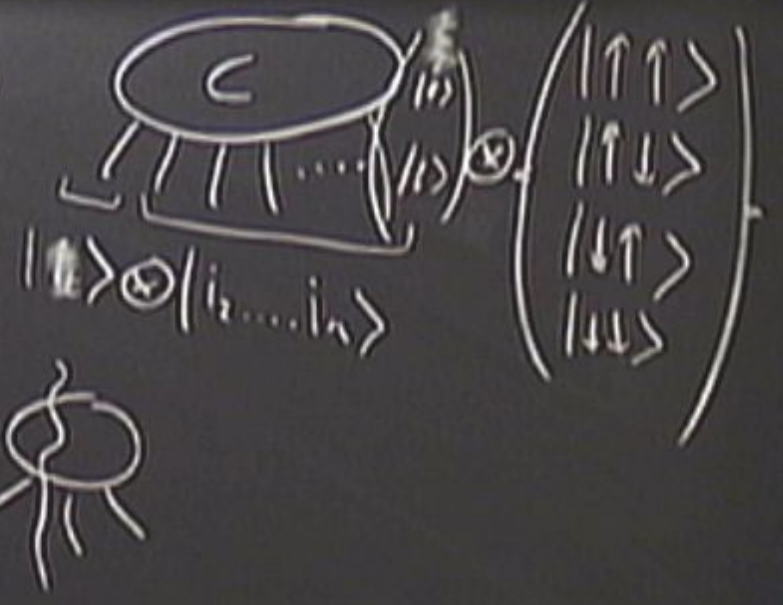
$$= \sum_k \left[\Gamma_k^{\dagger} \lambda_k \Gamma_k \right]$$

$$\sum_{\alpha} \Gamma_{\alpha}^{\dagger} \lambda_{\alpha} \Gamma_{\alpha}$$



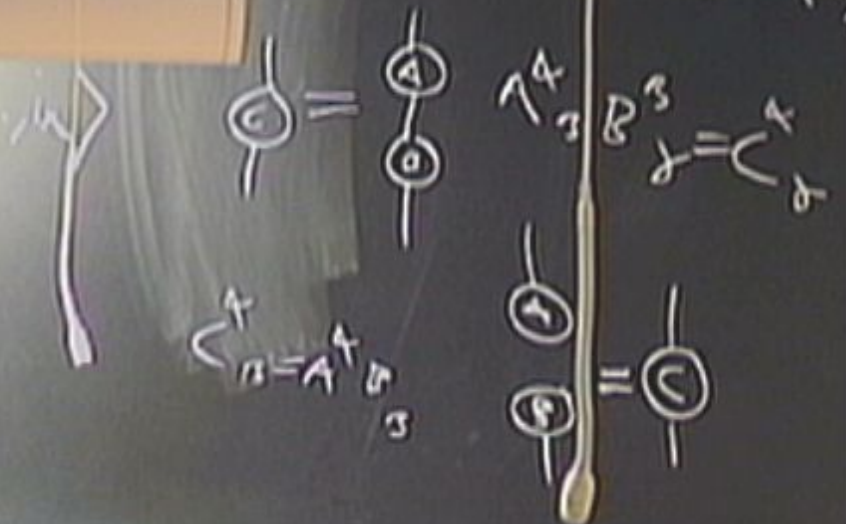
$$|\psi\rangle = \sum c_i \dots$$

$$M = U S V^{\dagger}$$



$$= \sum_{\alpha} \Gamma_{\alpha}^{\mu} \lambda_{\alpha}^{\nu} \Gamma_{\alpha}^{\nu}$$

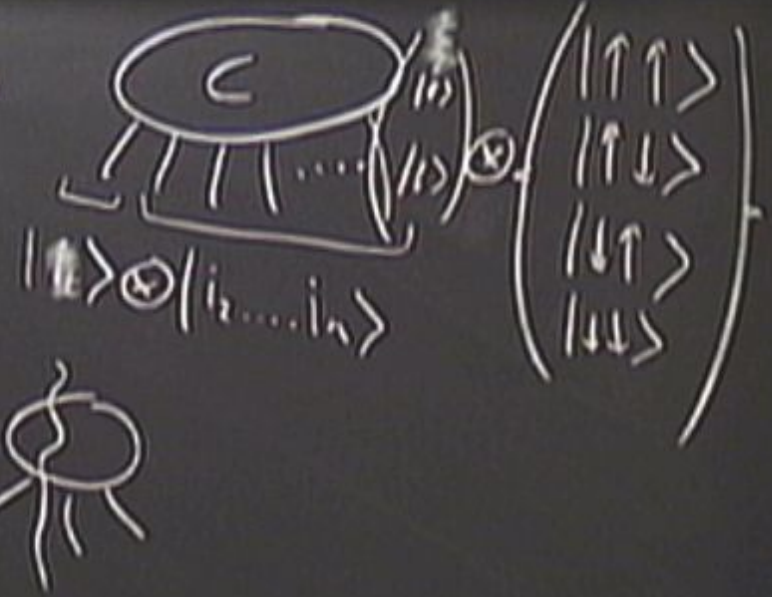
$$D_{\alpha}^{\mu} \lambda_{\alpha}^{\nu} \Gamma_{\alpha}^{\nu}$$



$$|\psi\rangle = \sum_{i_1, \dots, i_n} \dots$$

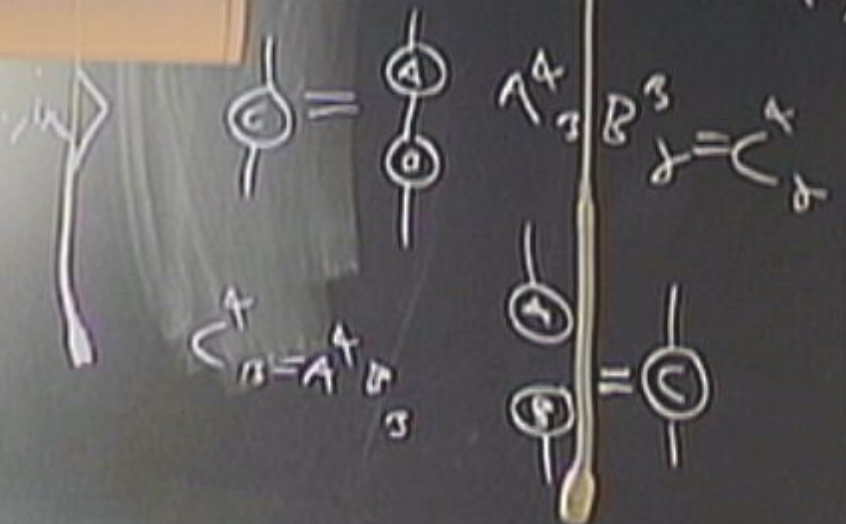
$$=$$

$$u S V^{\dagger}$$



$$= \sum_{i_1, i_2, \dots, i_n} \Gamma_{i_1}^{\mu_1} \lambda_{i_2}^{\mu_2} \dots \Gamma_{i_n}^{\mu_n}$$

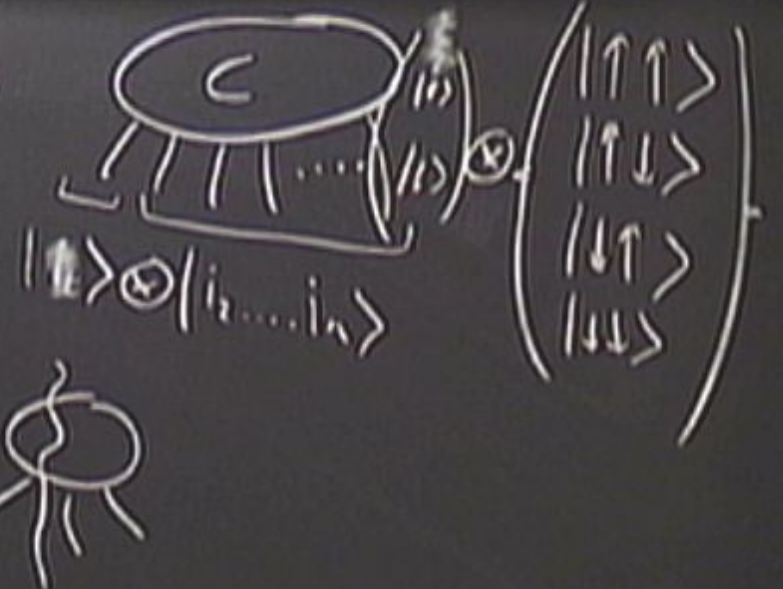
$$D_{i_1, i_2, \dots, i_n} \gg \lambda^2$$



$$|\psi\rangle = \sum c_i |i\rangle$$

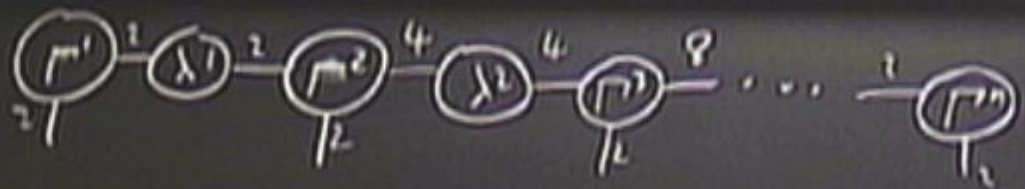
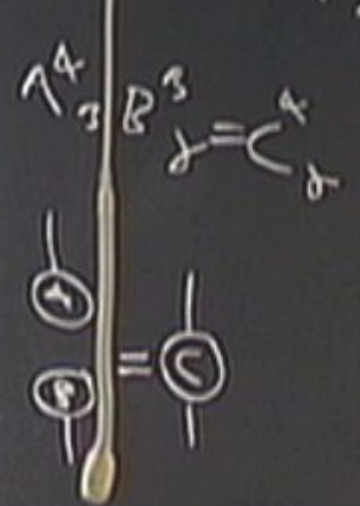
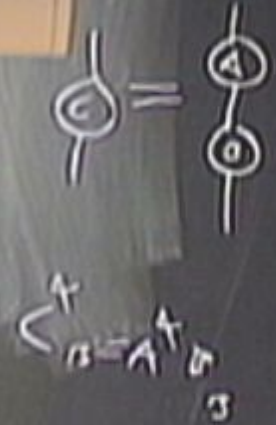
$$=$$

$$M = U S V^\dagger$$



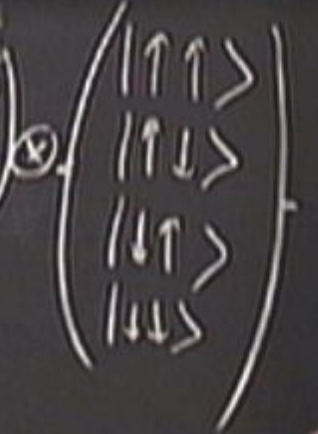
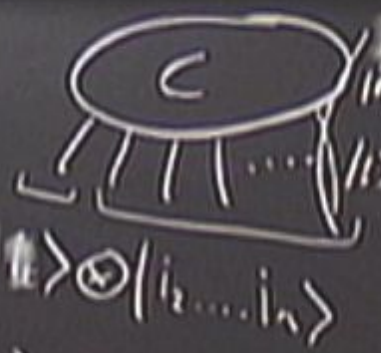
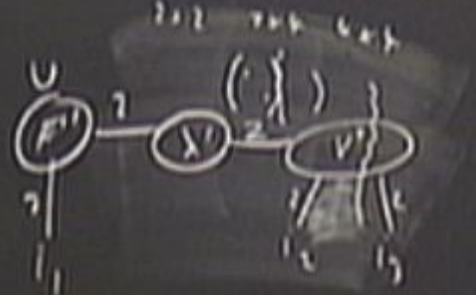
$$= \sum_{\lambda_1, \lambda_2, \dots, \lambda_n} \left[\prod_{i=1}^n \Gamma_i \right] \left[\prod_{i=1}^n \lambda_i \right]$$

$$\sum_{\lambda_1, \lambda_2, \dots, \lambda_n} \lambda_1 \lambda_2 \dots \lambda_n$$



$$|\psi\rangle = \sum c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$M = U S V^\dagger$$



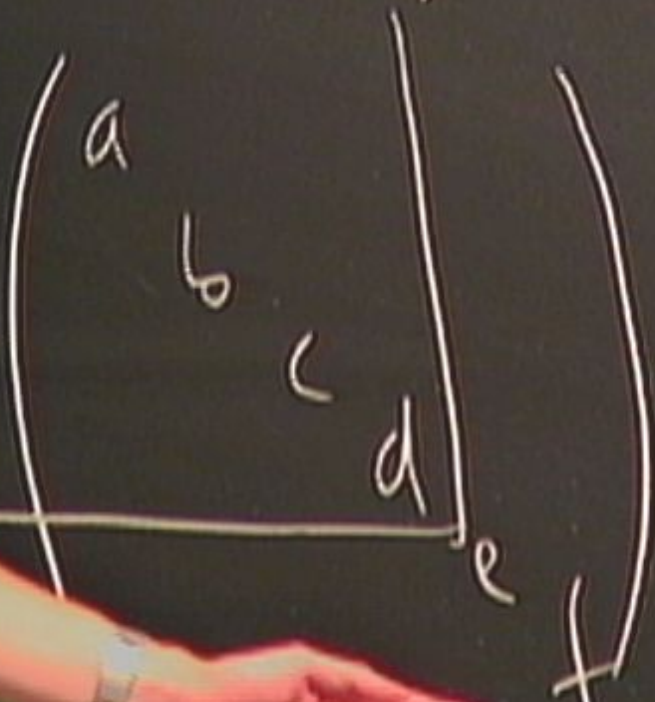
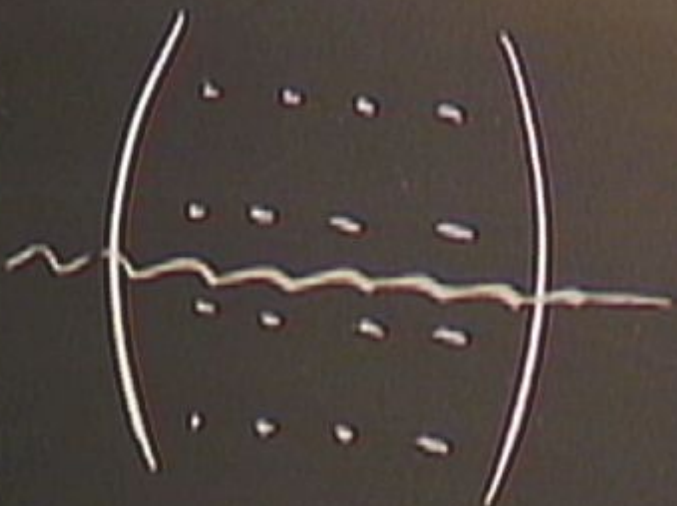
$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

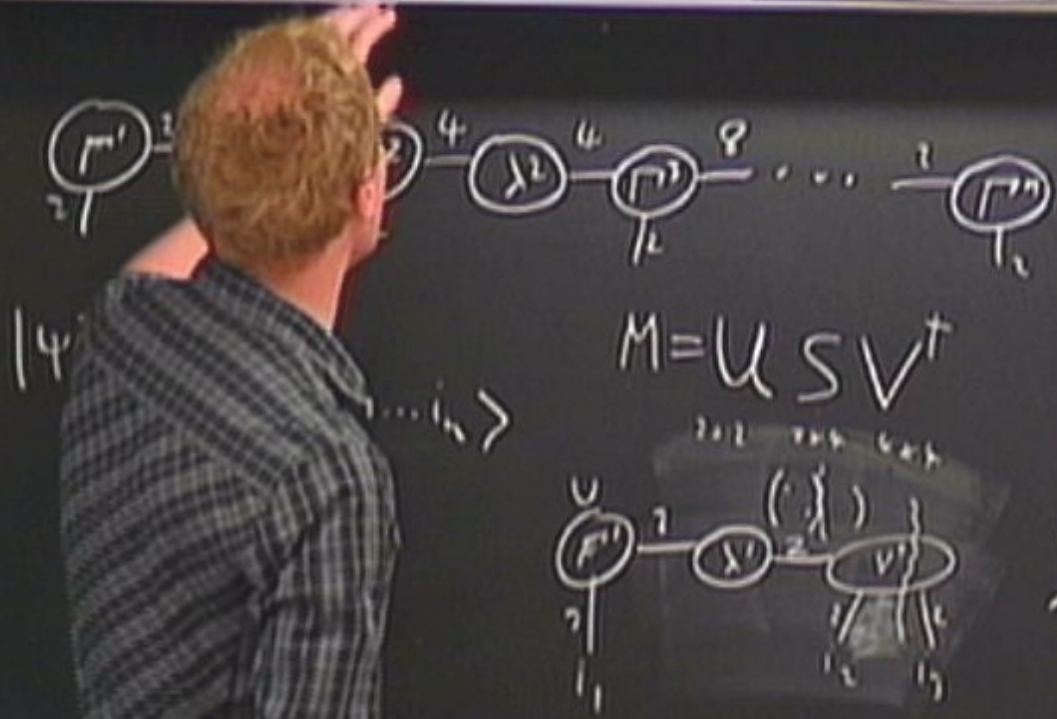
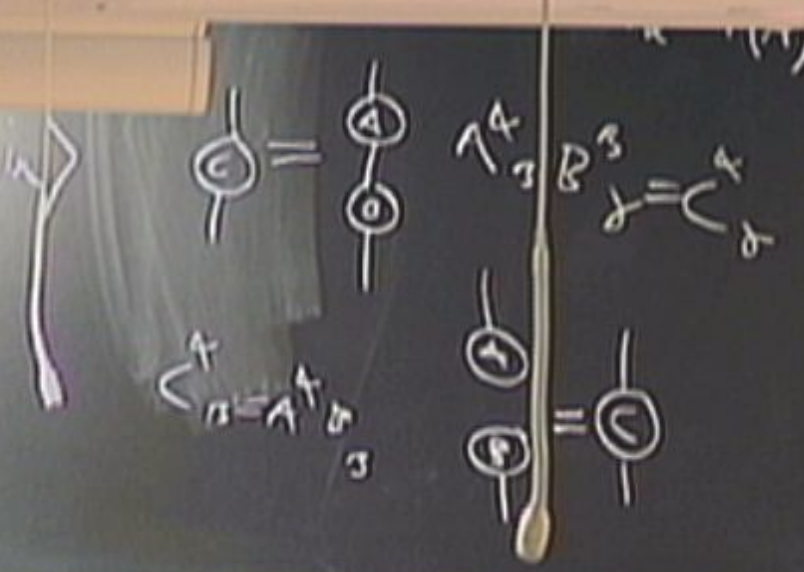
$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$

$$\begin{pmatrix} a^2 \\ c \end{pmatrix}$$

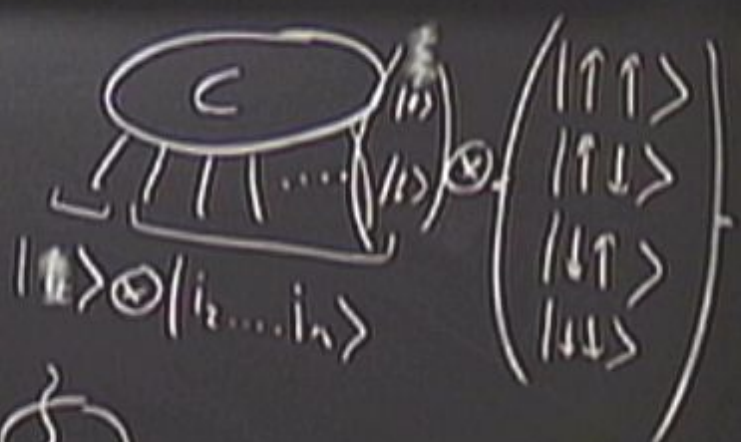


$$= \sum_{\lambda} \left[\prod_{i=1}^n \frac{1}{\lambda_i} \right] \left[\prod_{i=1}^n \lambda_i \right]$$

$$\sum_{\lambda} \frac{1}{\lambda_1 \lambda_2 \dots \lambda_n} \lambda_1 \lambda_2 \dots \lambda_n$$



$$M = U S V^T$$



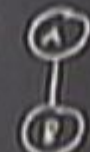
Introduction to Tensor Network Algorithms

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} C_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$$= \sum_{i_1, \dots, i_n} \lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \dots \lambda_{i_n}^{(n)} |i_1, \dots, i_n\rangle$$



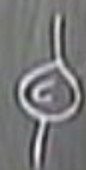
(I)



$A_{\alpha\beta}$

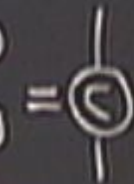
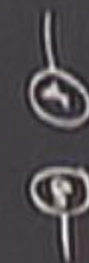


$$A_{\alpha\alpha} = \text{Tr}(A)$$



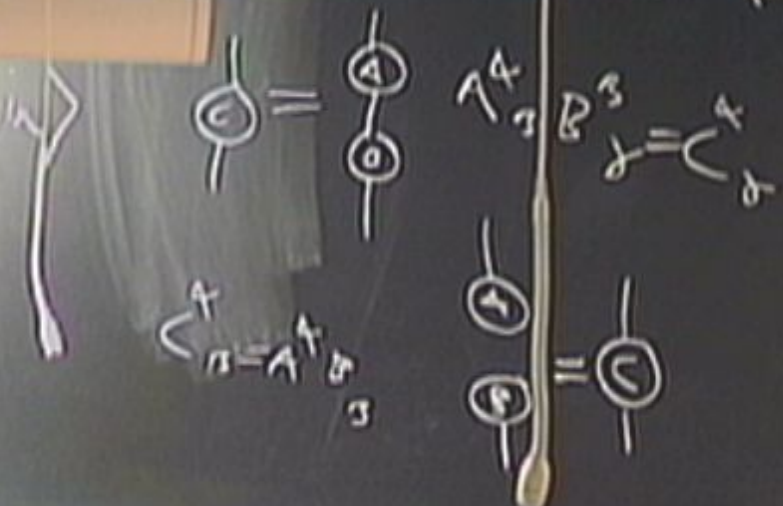
$$B_{\alpha\beta} = C_{\alpha\beta}$$

$$B = A^T$$



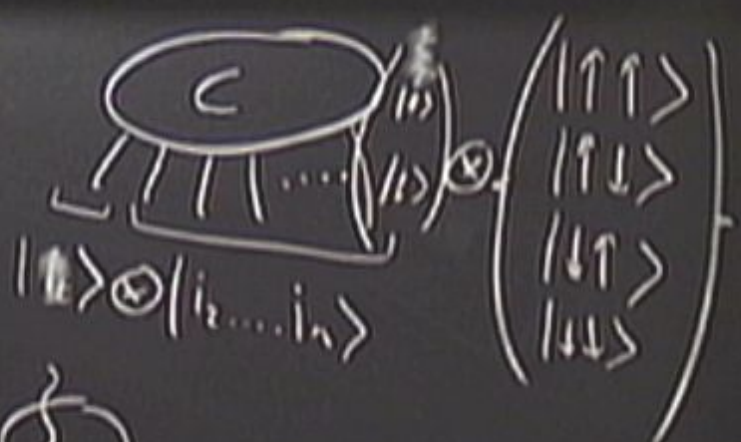
$$= \sum_{\lambda_1, \lambda_2, \dots, \lambda_n} \dots$$

$$D = 4$$



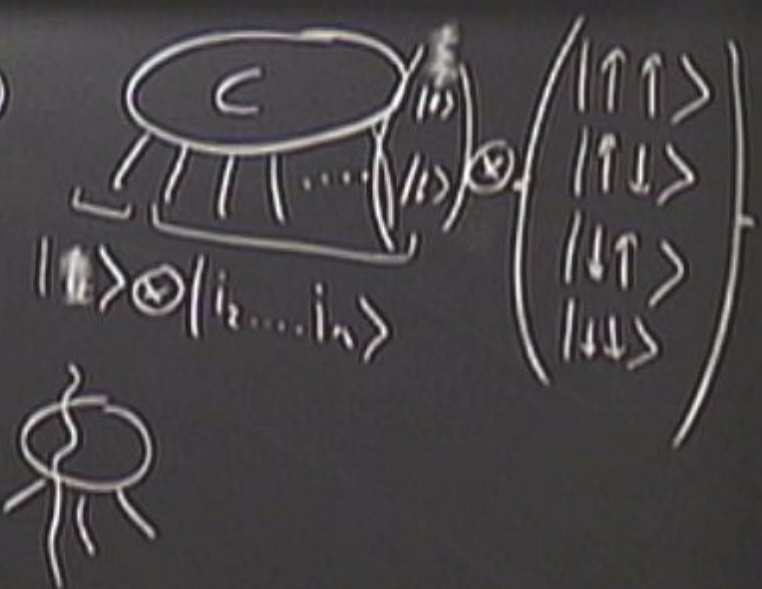
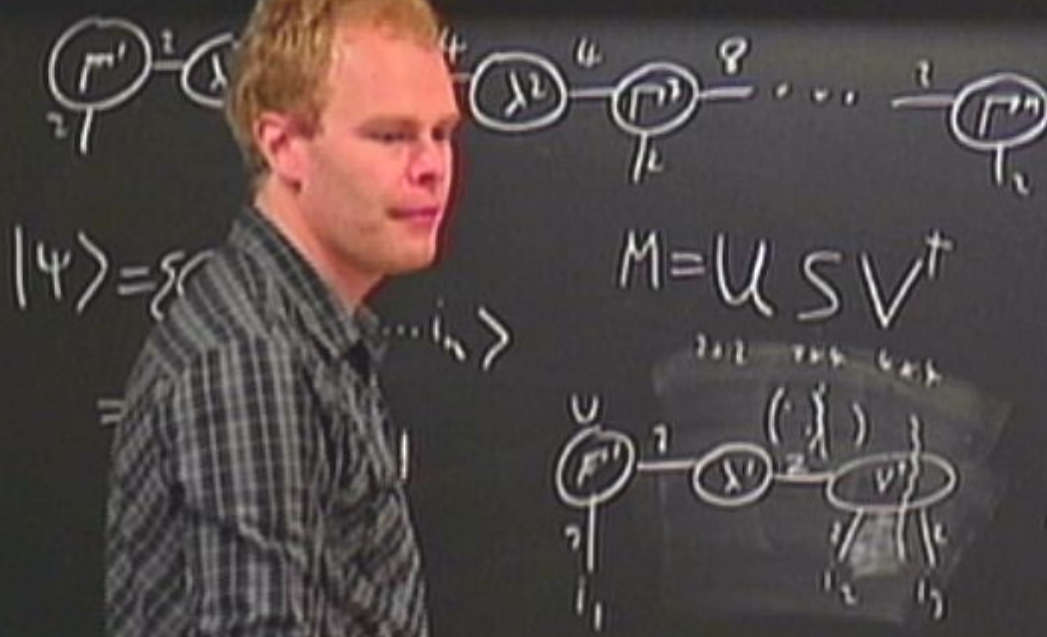
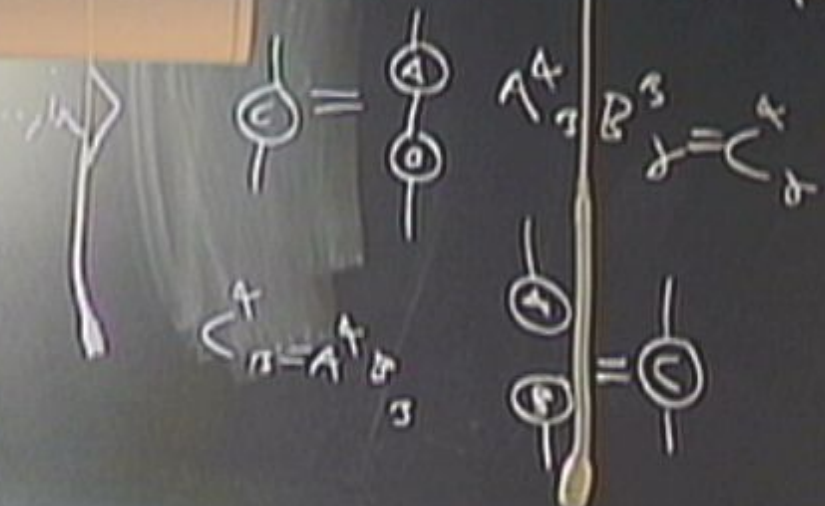
$$|\psi\rangle = \dots$$

$$M = U S V^\dagger$$



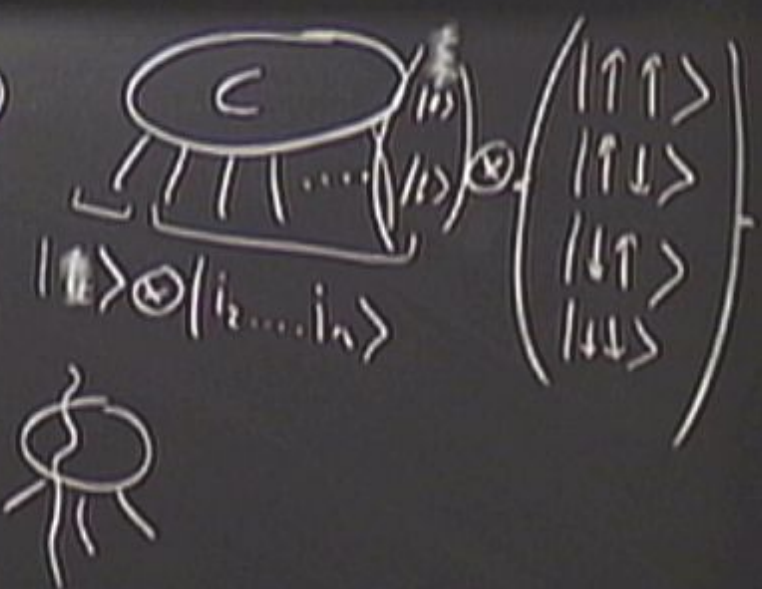
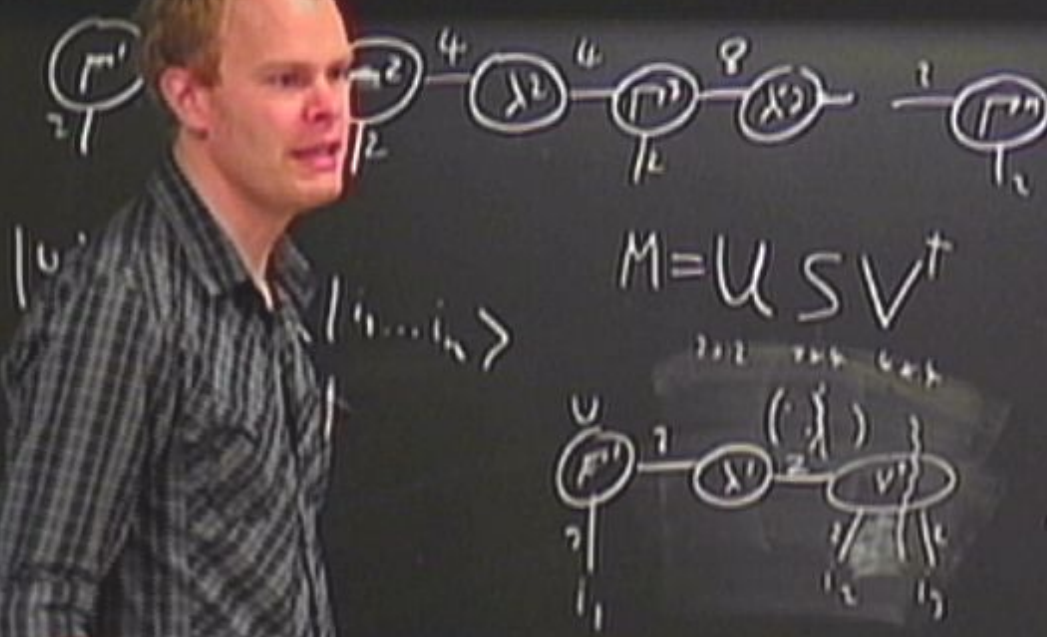
$$= \sum_{\lambda_1, \lambda_2, \dots, \lambda_n} \left[\prod_{i=1}^n \lambda_i \right] \left[\prod_{i=1}^n \lambda_i \right]$$

$$D = F$$



$$= \sum_{\lambda} \left[\dots \right]$$

$$D = 4$$



Introduction to Tensor Network Algorithms

$$|\Psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$d^{\sum n}$



$$\sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1, i_2, \dots, i_n\rangle$$

$\geq d^{\sum n}$

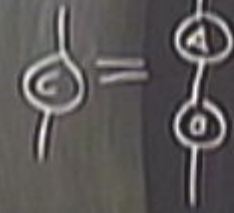
$$D = 4$$

(I)



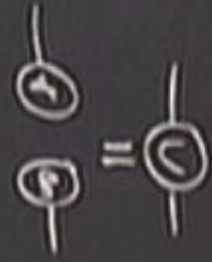
$$A^{\alpha} B^{\alpha}$$

$$A^{\alpha}_{\alpha} = \text{Tr}(A)$$



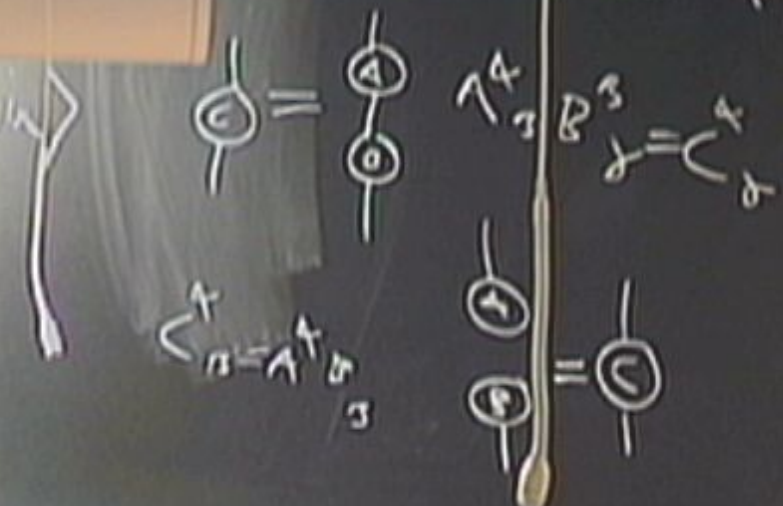
$$A^{\alpha} B^{\beta} C_{\alpha\beta}$$

$$C^{\alpha}_{\alpha} = A^{\alpha}_{\alpha}$$

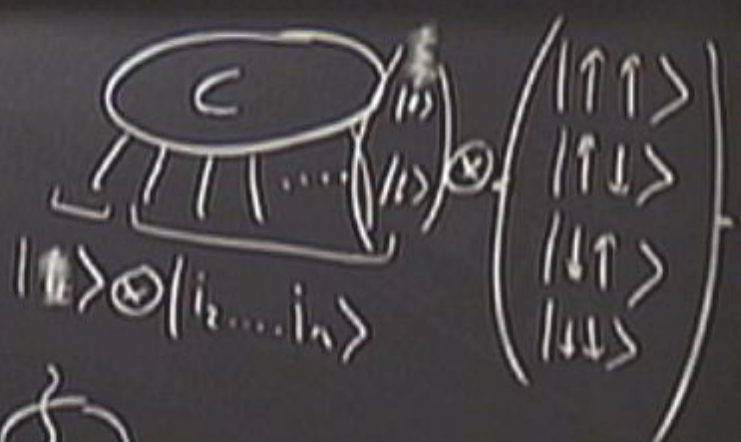


$$= \sum_{\lambda} \left[\dots \right] \left[\dots \right]$$

$$D = F$$



$$M = U S V^\dagger$$



$$\left(\begin{array}{cc|cc} a & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{array} \right)$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

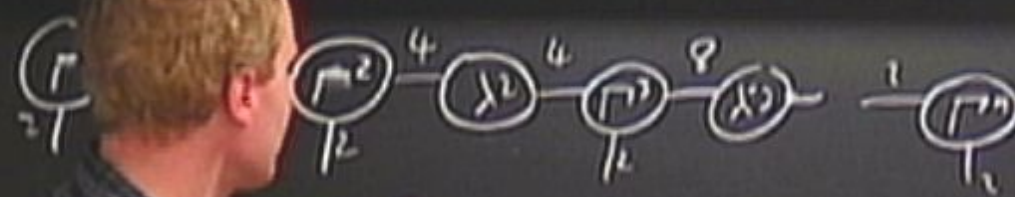
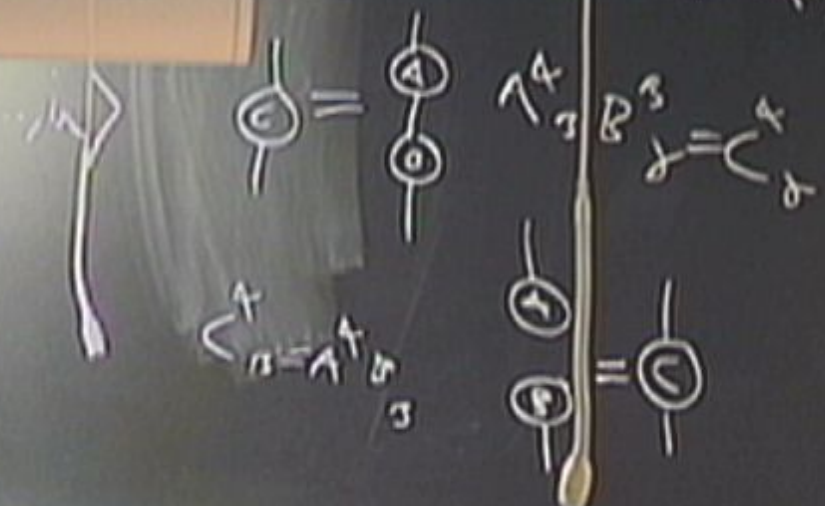
$$\sim \left(\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \right)$$

$$\left(\begin{array}{cccc|c} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ \hline & & & & e \\ & & & & f \end{array} \right)$$

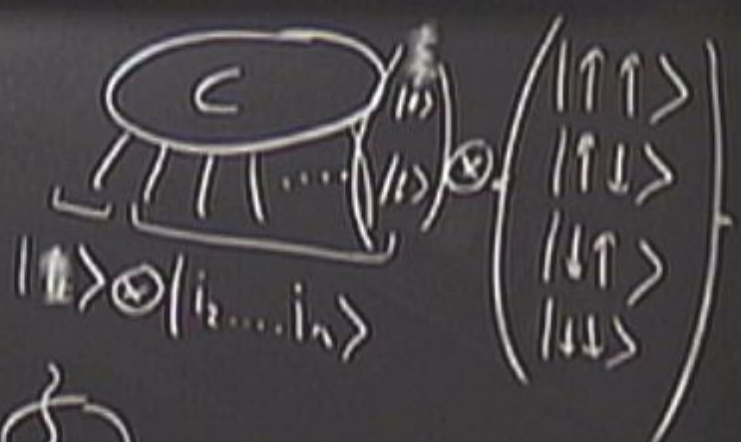


$$= \sum_{\lambda} \left[\frac{1}{\lambda} \right] \left[\frac{1}{\lambda} \right]$$

$$D = f$$



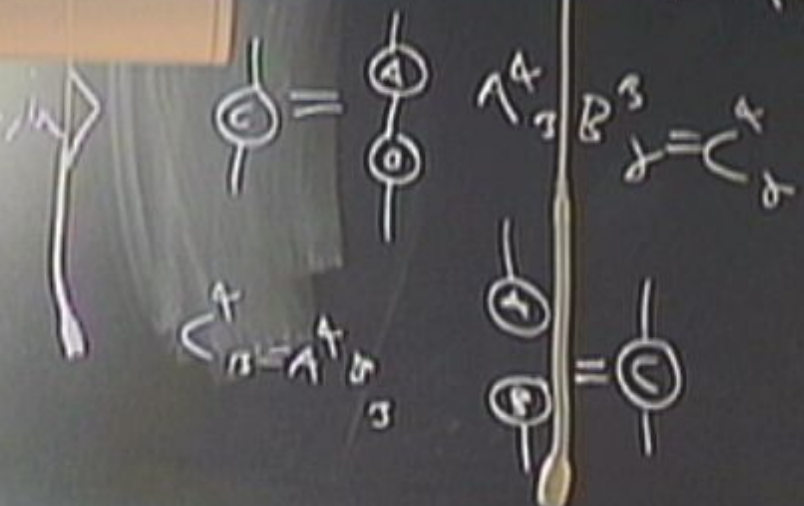
$$M = U S V^T$$



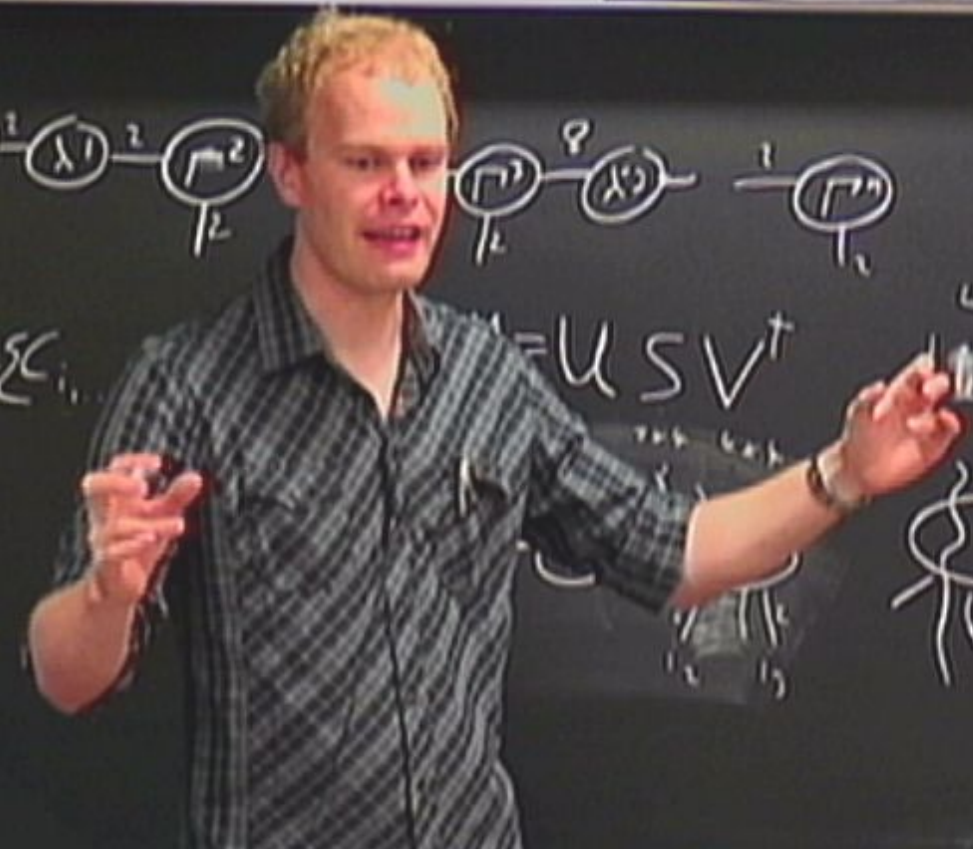
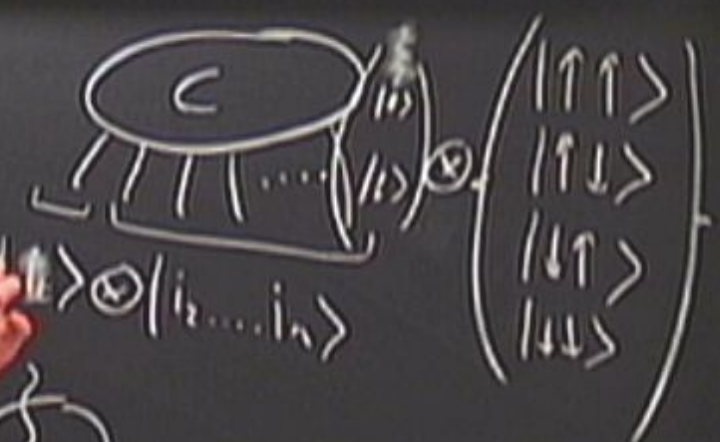
$$= \sum_{\lambda} \Gamma_{\lambda}^{\mu} \Gamma_{\lambda}^{\nu} \Gamma_{\lambda}^{\rho}$$

$$\sum_{\lambda} \Gamma_{\lambda}^{\mu} \Gamma_{\lambda}^{\nu} \Gamma_{\lambda}^{\rho} \approx \lambda^2$$

$$D = 4$$

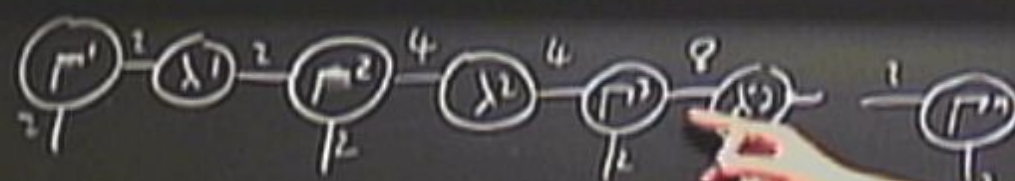
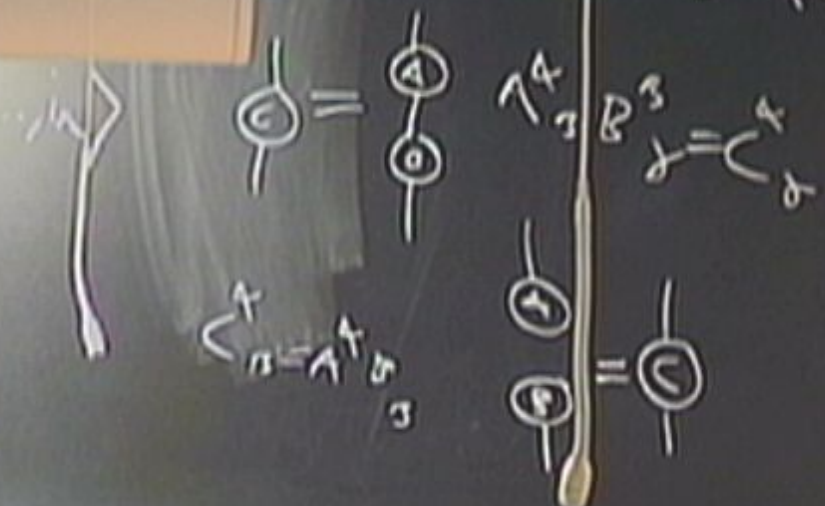


$$|\psi\rangle = \sum c_i |i\rangle = U S V^\dagger$$



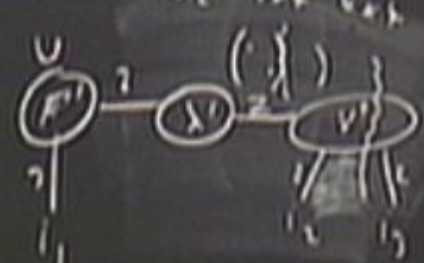
$$= \sum_{\lambda} \left[\prod_{i=1}^n \frac{1}{\lambda_i} \right] \left[\prod_{i=1}^n \lambda_i \right] \left[\prod_{i=1}^n \lambda_i \right]$$

$$D = \mathbb{F}$$



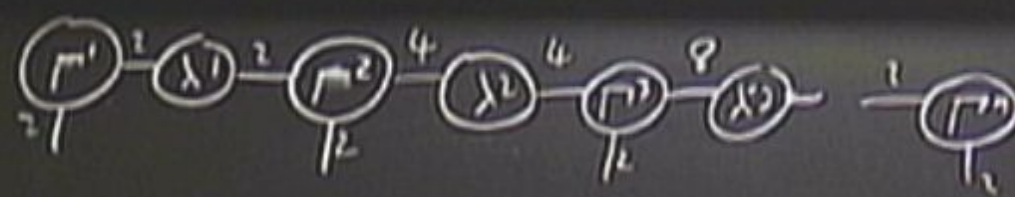
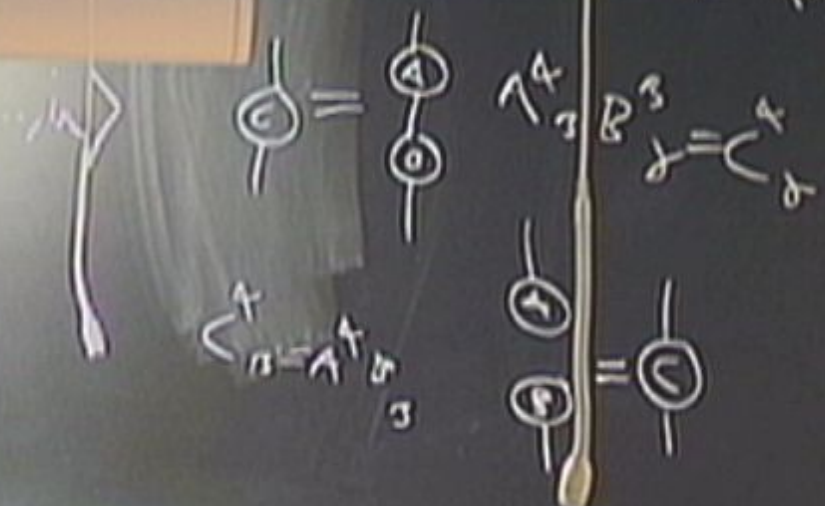
$$|\psi\rangle = \sum c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$M = U S V$$



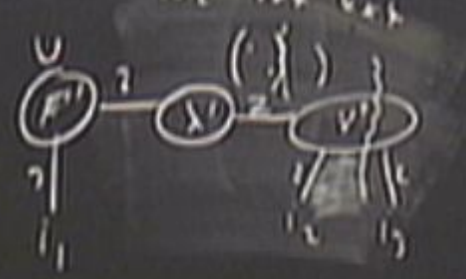
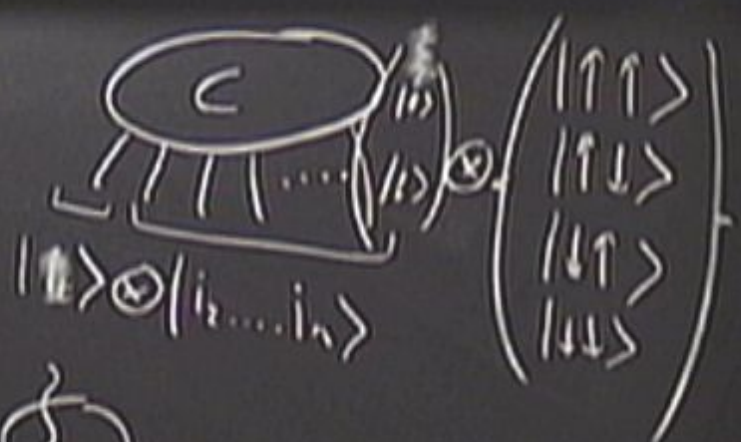
$$= \sum_{\lambda_1, \lambda_2, \dots, \lambda_n} \prod_{i=1}^n \Gamma_i^{\lambda_i} \lambda_i^{\lambda_i} \dots$$

$D = F$



$$|\psi\rangle = \sum c_{i_1, \dots, i_n} |i_1, \dots, i_n\rangle$$

$$M = U S V^\dagger$$



$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline a & & & & & & & \\ & b & & & & & & \\ & & c & & & & & \\ & & & d & & & & \\ & & & & e & & & \\ & & & & & f & & \end{array} \right)$$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \left(\begin{array}{cccc|cccc} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \right)$$

$$\left(\begin{array}{cccc|c} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \\ & & & & & f \end{array} \right)$$

$$\varphi \varphi \varphi \varphi \varphi \varphi \varphi \varphi$$

$$\begin{pmatrix} \{a\} & \{0\} & \{00\} \\ \{d\} & \{0\} & \{00\} \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \sim \begin{pmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \end{pmatrix}$$

$$\begin{matrix} \psi & \psi & \psi & \psi & \psi & \psi & \psi & \psi \\ | & | & | & | & | & | & | & | \\ \hline & & & & & & & \end{matrix}$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$

$$\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$



$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$



$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & \\ & \dots \\ & & \dots \\ & & & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$



$$S \sim \log L$$



$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & & & \\ \dots & & & \\ \dots & & & \\ \dots & & & \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$

$$\rho_1 \rho_2 \dots \rho_L$$

$$S \sim \log L$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & & \\ & \dots & \\ & & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$

$$\rho_1 \rho_2 \rho_3 \rho_4 \rho_5 \rho_6 \rho_7 \rho_8 \rho_9 \rho_{10}$$

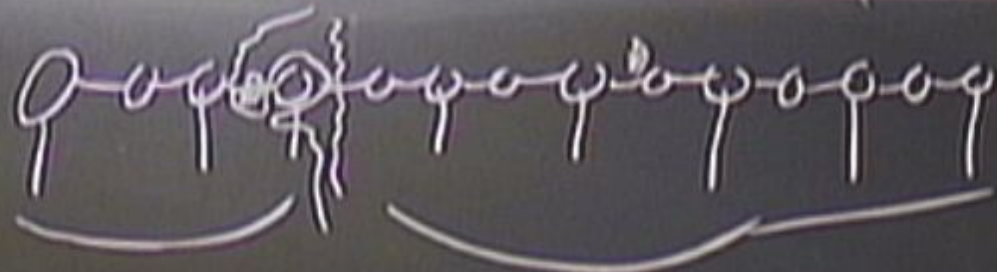
$$S \sim \log L$$

$$\begin{pmatrix} \{a\} & \{0\} & \{00\} \\ \{0\} & \{0\} & \{00\} \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & & \\ & \dots & \\ & & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & \\ & & & & & f \end{pmatrix}$$



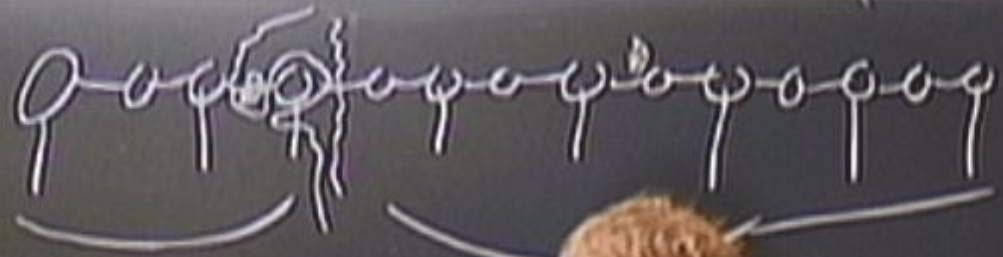
$$S \sim \log L \quad L D^a$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\sim \begin{pmatrix} \dots & & \\ & \dots & \\ & & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \end{pmatrix}$$



LD^2

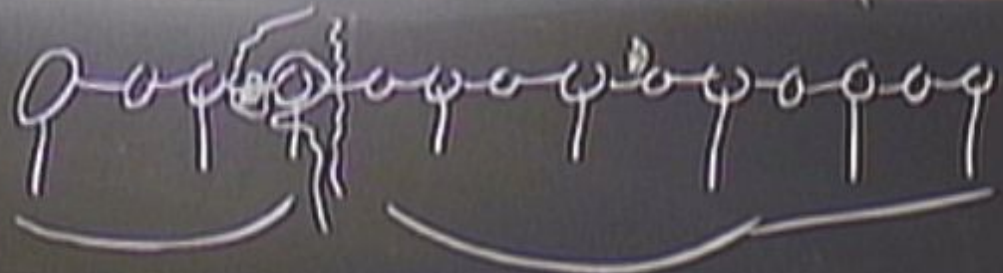


$$\begin{pmatrix} \{a\} & \{0\} & \{00\} \\ \{0\} & \{0\} & \{00\} \end{pmatrix}$$

$$\lambda = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

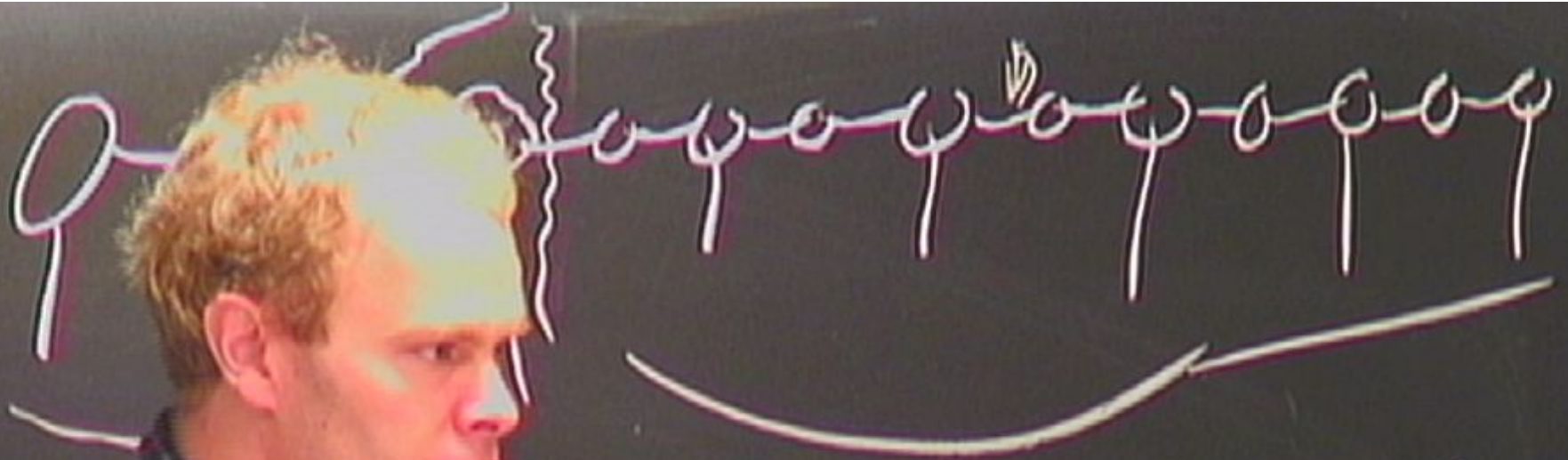
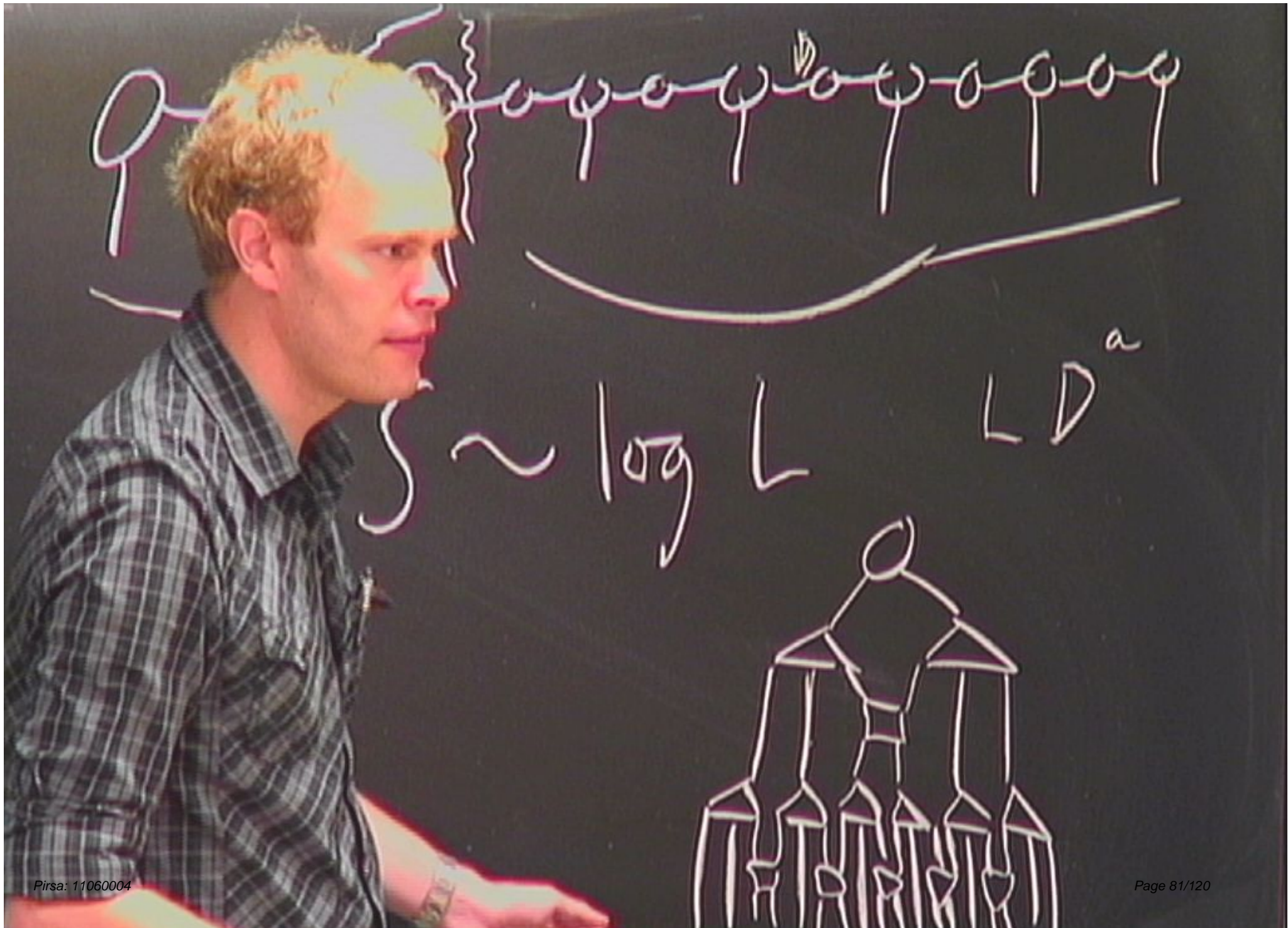
$$\sim \begin{pmatrix} \dots & \dots \\ \dots & \dots \\ \dots & \dots \\ \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} a & & & & \\ & b & & & \\ & & c & & \\ & & & d & \\ & & & & e \end{pmatrix}$$



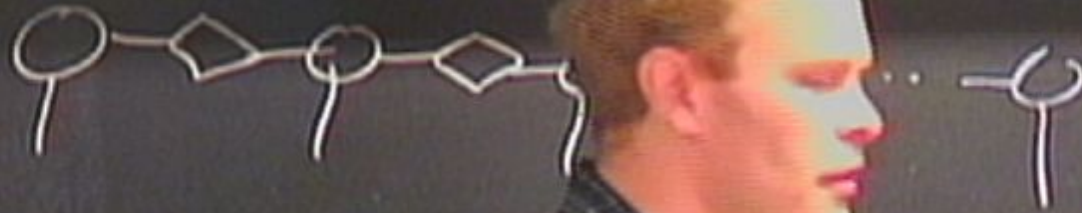
$\sim I$ LD^a





$$S \sim \log L \quad LD^a$$

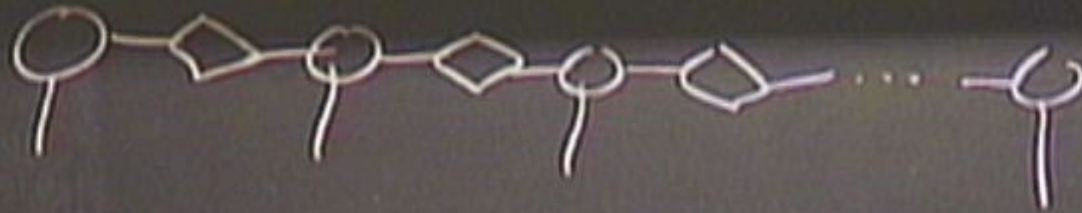




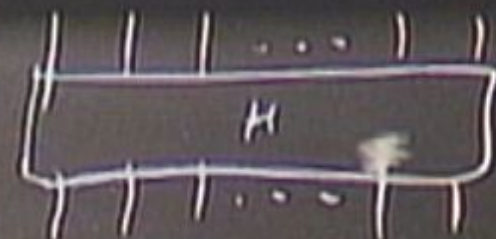
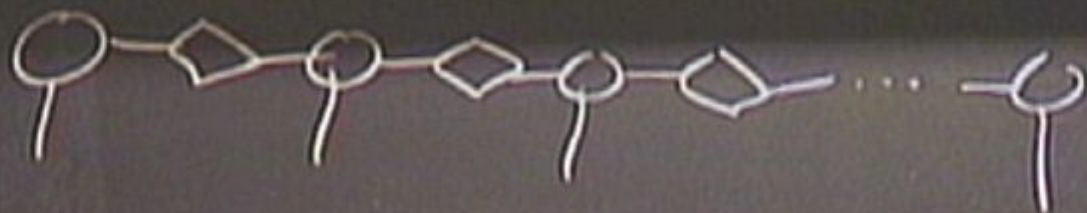


$$U = e^{-Ht}$$





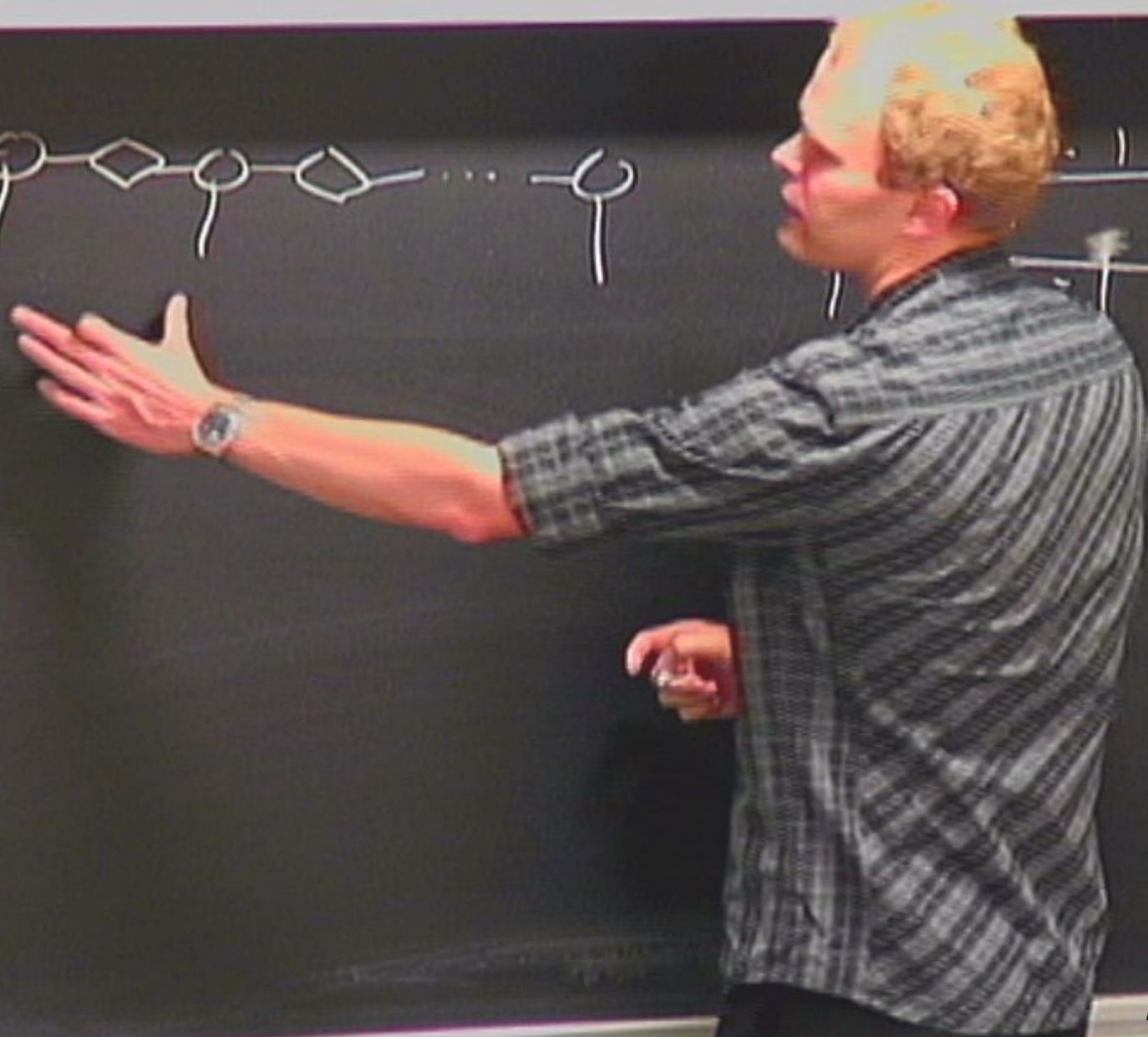
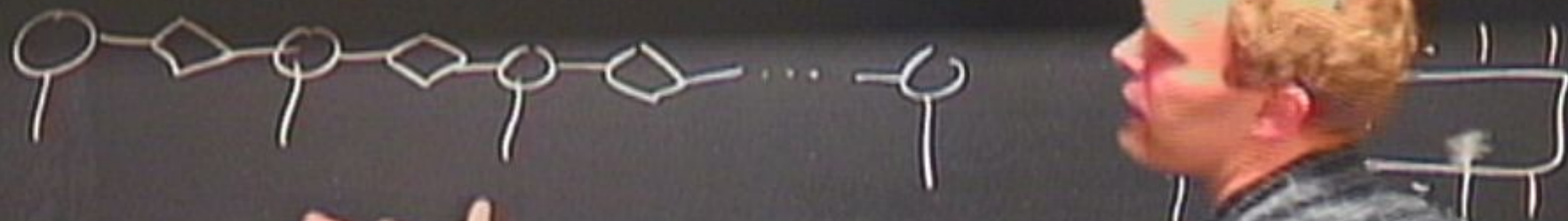
$$U = e^{-iHt}$$

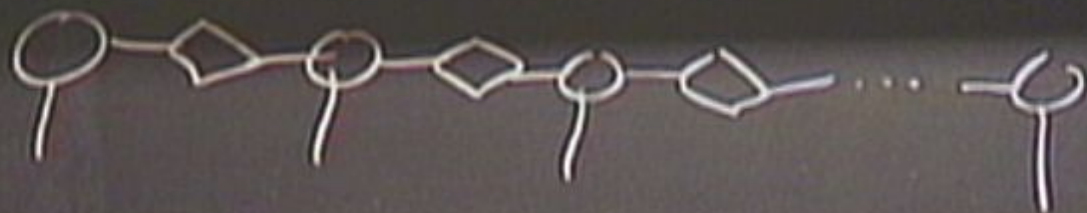


$$U = e^{-Ht}$$

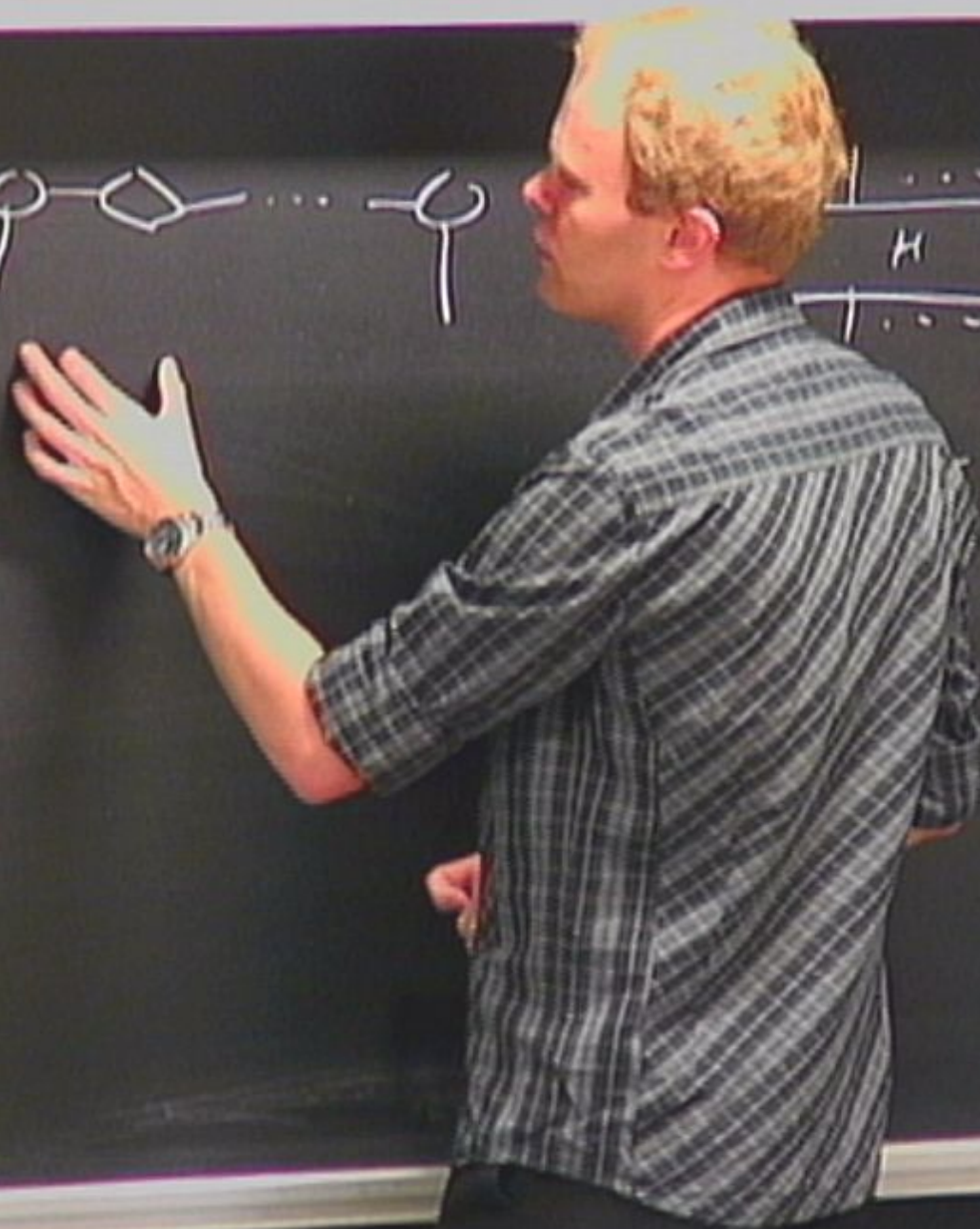


$$U = e^{-HSt}$$



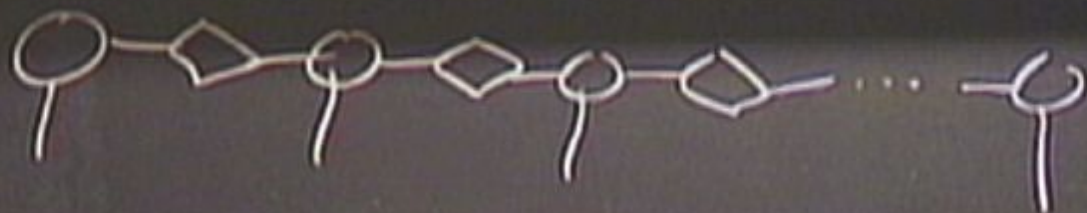


$$U = e^{-Ht}$$





$$U = e^{-Ht}$$

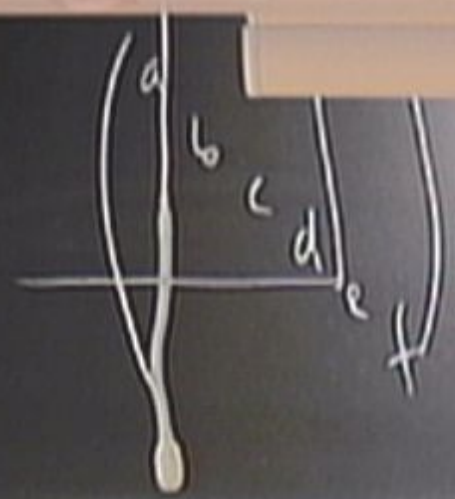


$$U = e^{-HSt}$$

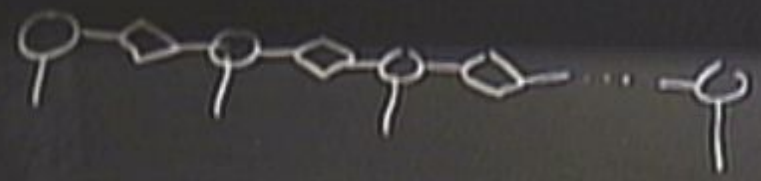


$$U = e^{-Ht}$$

(0 6)



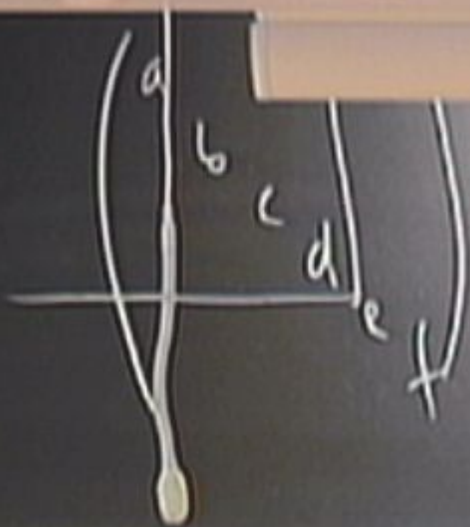
$\sim \log L$



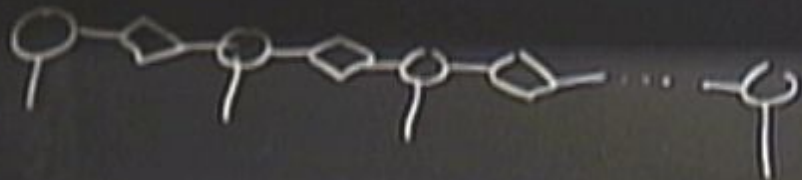
$$U = e^{-H\tau}$$

Suzuki-Trotter decomposition

(05)

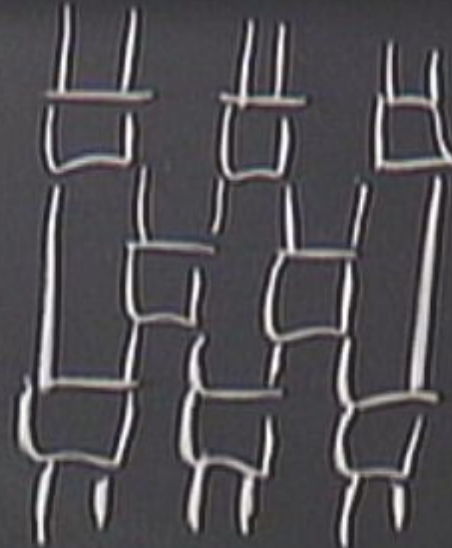
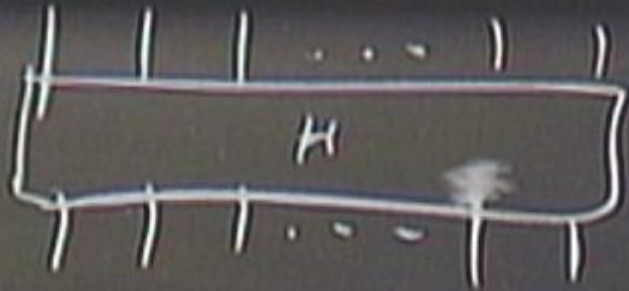


$\log L$



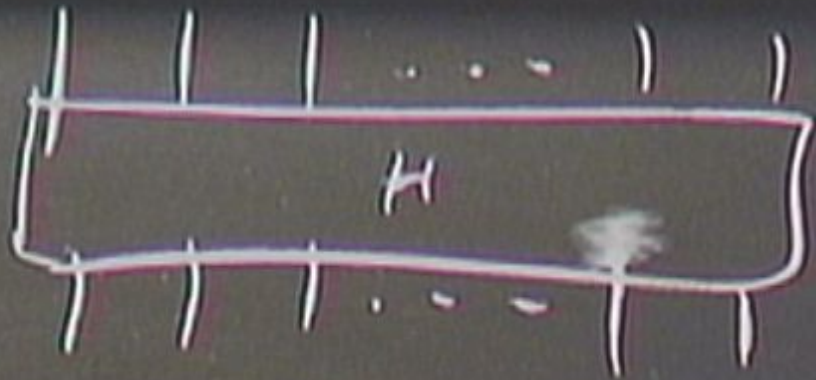
$$U = e^{-iHt}$$

Suzuki-Trotter decomposition

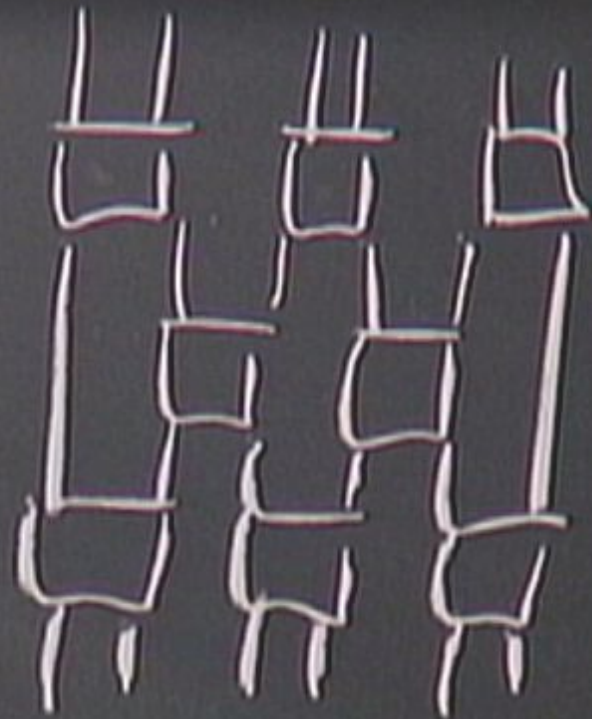


$$U = e^{-H\delta t}$$

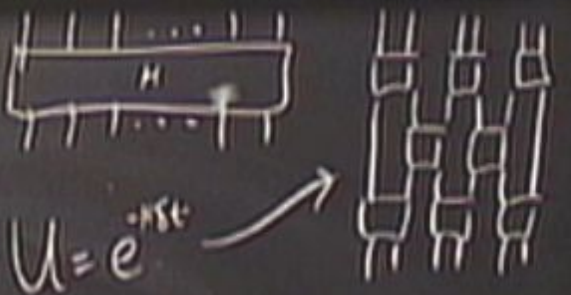
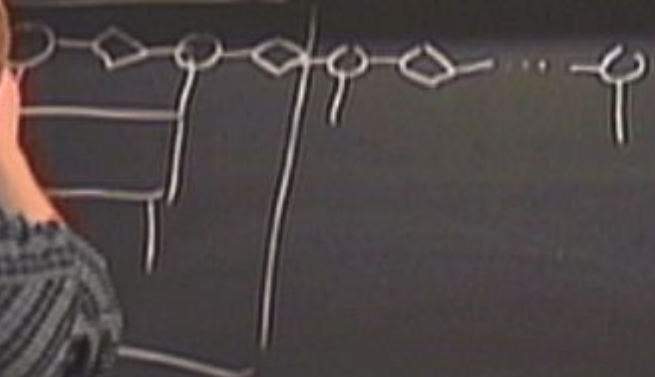
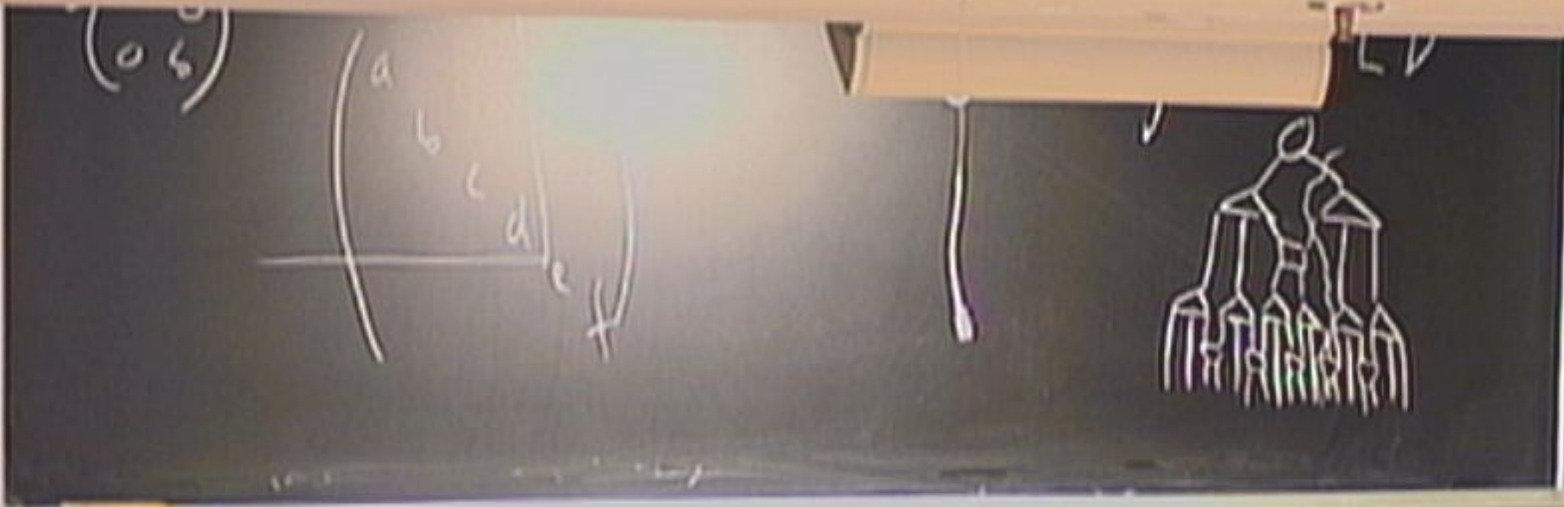
Suzuki-Trotter decomposition



$$U = e^{-H\delta t}$$

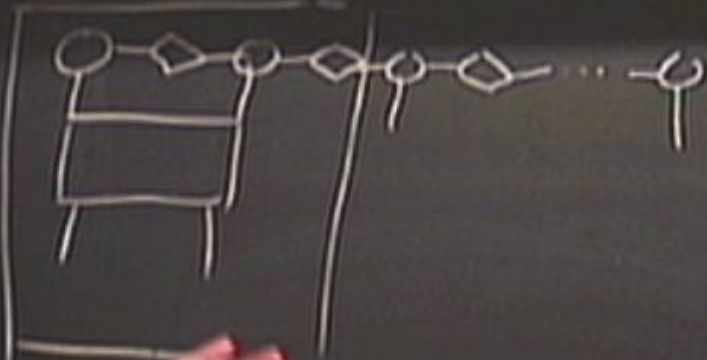


Suzuki-Trotter decomposition



$U = e^{-\tau H}$
 Suzuki-Trotter decomposition

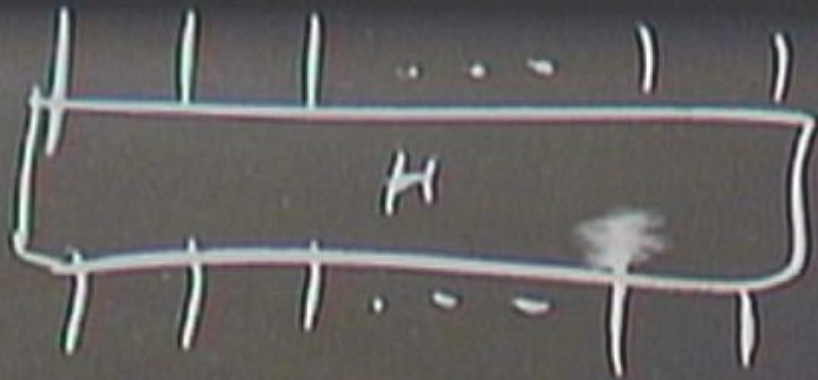
(0 6)



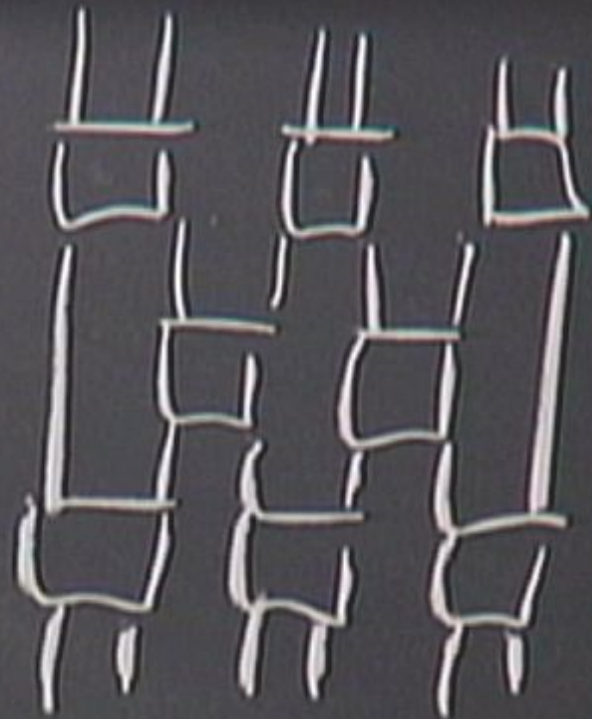
$$U = e^{-\kappa H t}$$



Suzuki-Trotter decomposition



$$U = e^{-iH\delta t}$$



Suzuki-Trotter decomposition

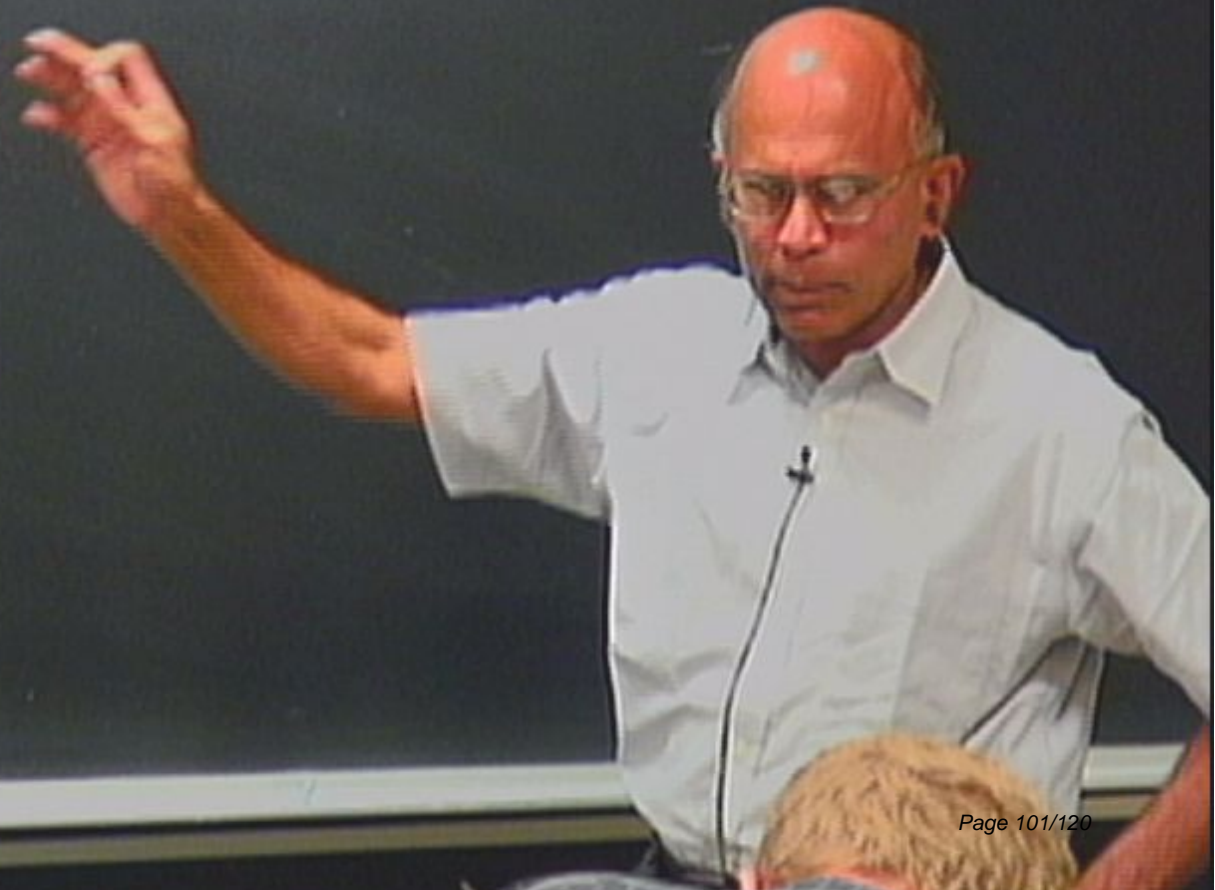
0 0 0 0 0
2

$S=1$ $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle$

0 0 0 0 0
i

$S=1$ $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

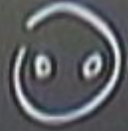
AKLT



0 0 0 0 0

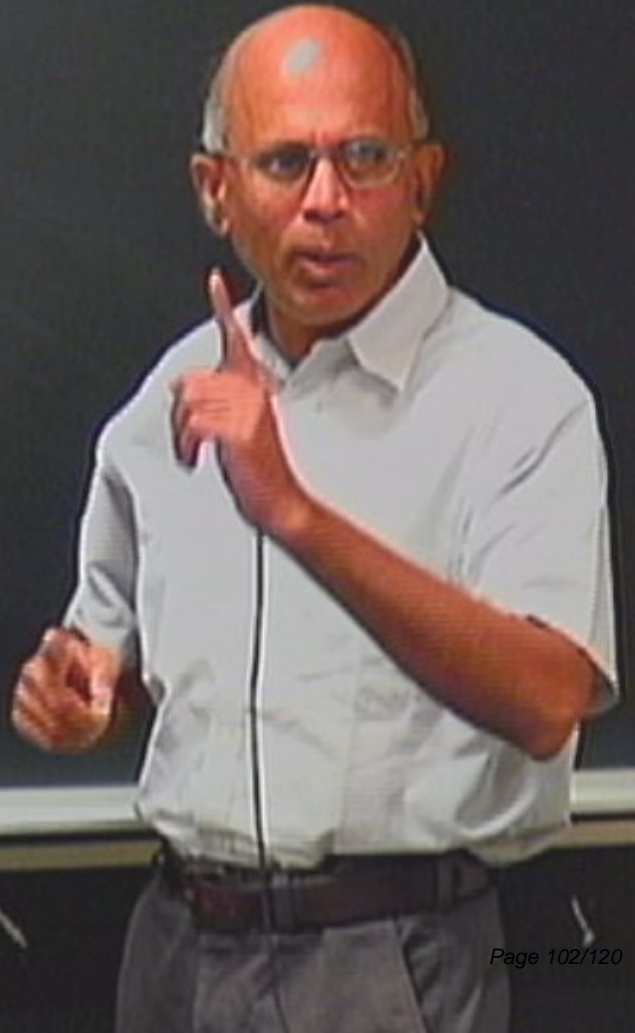
i

π



$S=1$ $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle$

AKLT



0 0 0 0 0

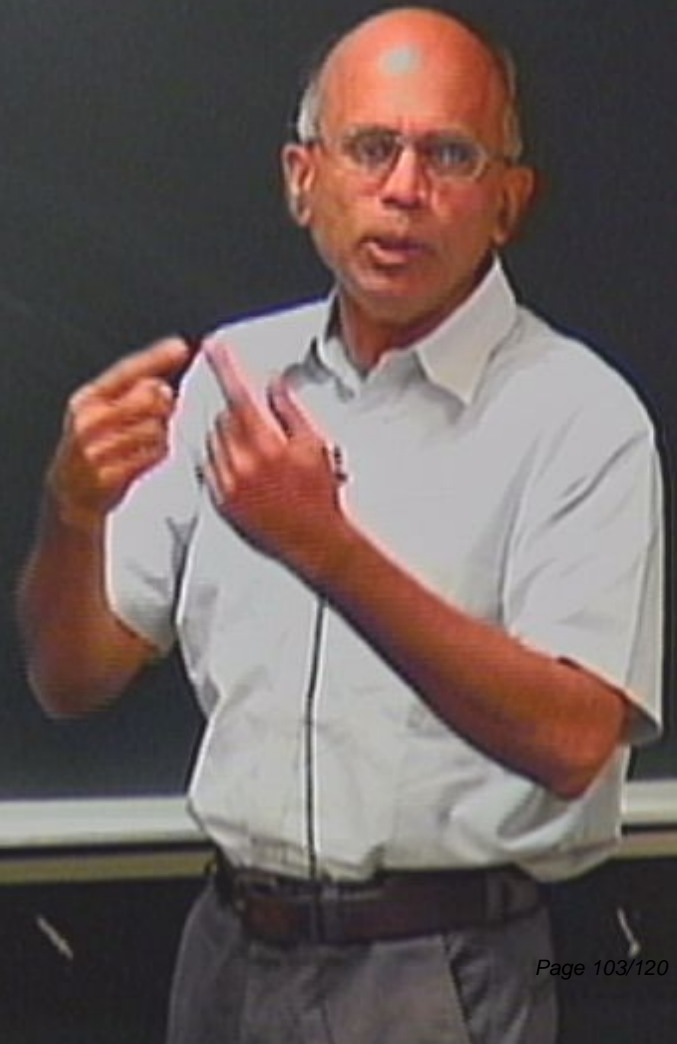
2

π



$S=1$ $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle$

AKLT



0 0 0 0 0

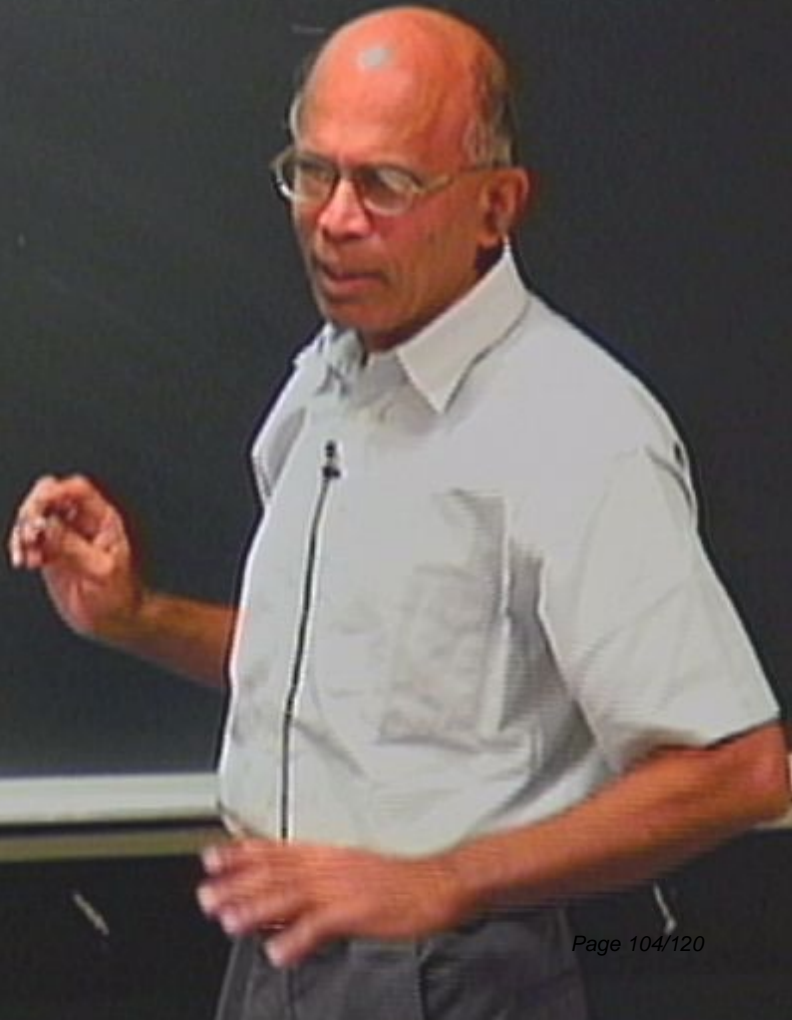
i

π



$S=1$ $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

AKLT



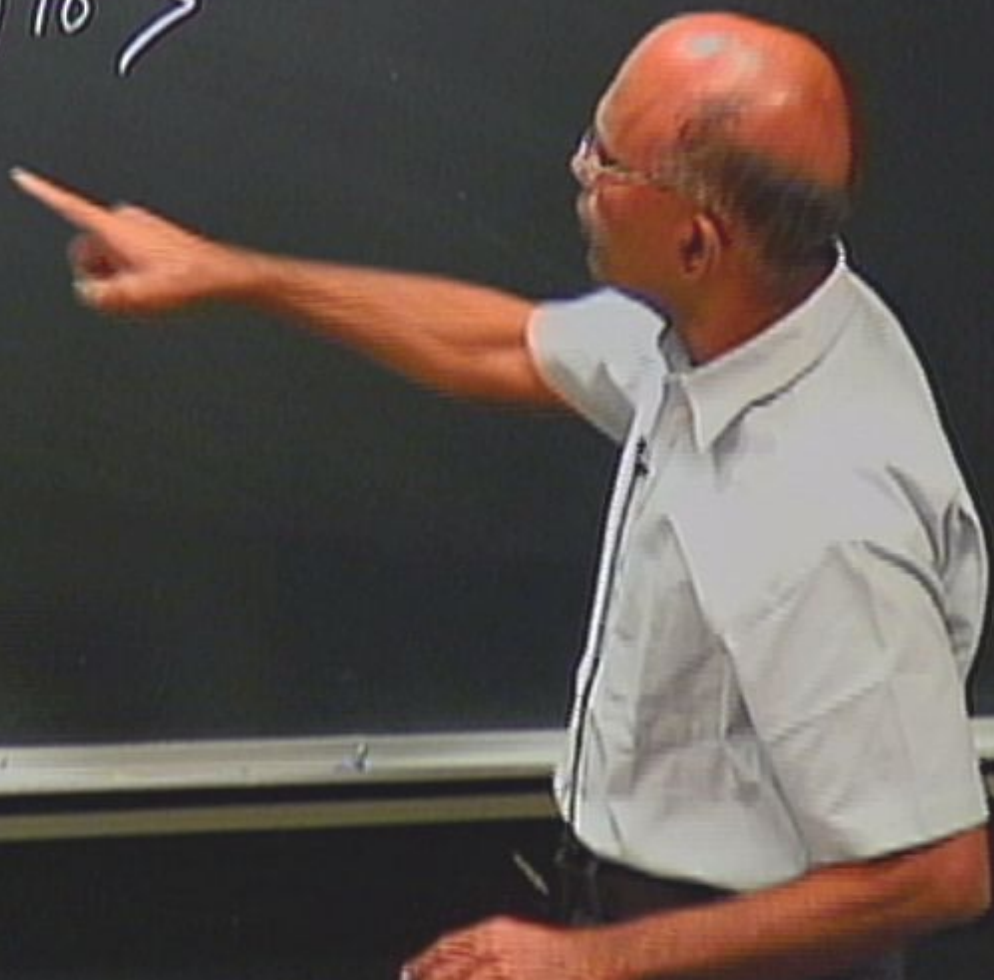
0 0 0 0 0

2

S=1 |↑↓, |0>, |↓>



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$



0 0 0 0 0

i

S=1 |↑↓, |0↓, |1↓⟩



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$



0 0 0 0 0

i

S=1 |↑↑, |0↑, |↓↓



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$



0 0 0 0 0
i

$S=1$ $|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

0 0 0 0 0
i

$S=1$ $|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i$$

$$\begin{pmatrix} 2 & 0 \\ - & - \end{pmatrix} \begin{pmatrix} 0 & 1 \\ - & - \end{pmatrix}$$

0 0 0 0 0
i

$S=1$ $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ - & - \end{pmatrix} \begin{pmatrix} 0 & 1 \\ - & - \end{pmatrix} \begin{pmatrix} 0 \\ - \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
i

$S=1$ $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ - & - \end{pmatrix} \begin{pmatrix} 0 \\ - \end{pmatrix} \begin{pmatrix} - \\ - \end{pmatrix}$$

$|\uparrow\rangle$ $|\downarrow\rangle$

$$S=1 \quad |\uparrow\rangle, |\downarrow\rangle, |0\rangle$$

0 0 0 0 0
i



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
 i

$S=1$ $|\uparrow\rangle, |\downarrow\rangle, |0\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

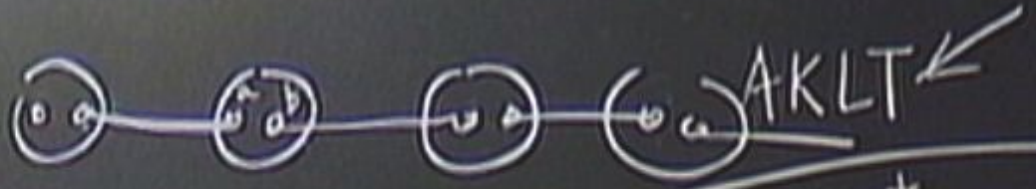
$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
i

S=1 $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

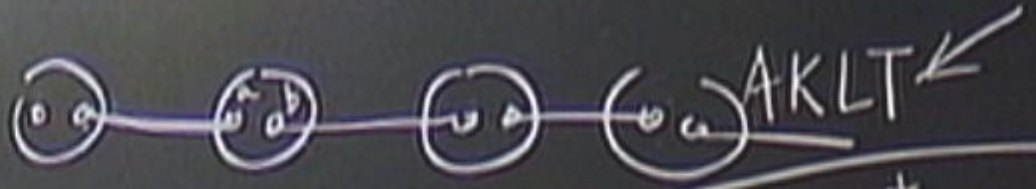
$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
i

S=1 $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

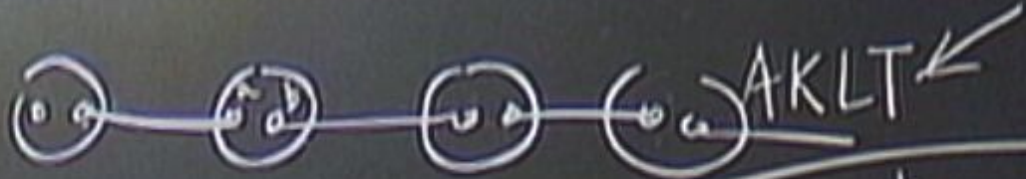
$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
 i

$S=1$ $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

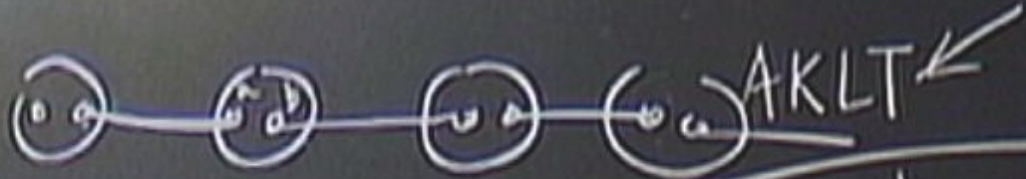
$$a_i^\dagger (b_i^\dagger a_{i+1}^\dagger) b_{i+1}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
 :

$S=1$ $|\uparrow\rangle, |\downarrow\rangle, |0\rangle$



$$\prod (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

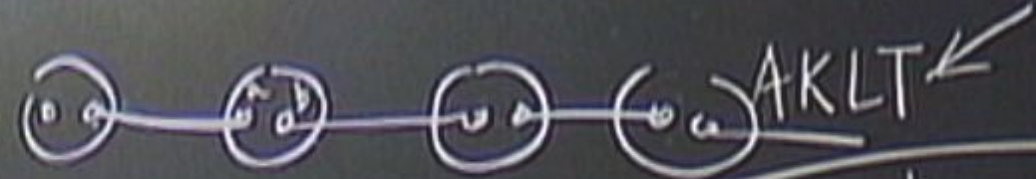
$$a_i^\dagger (b_i^\dagger a_{i+1}^\dagger) b_{i+1}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
i

$S=1$ $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

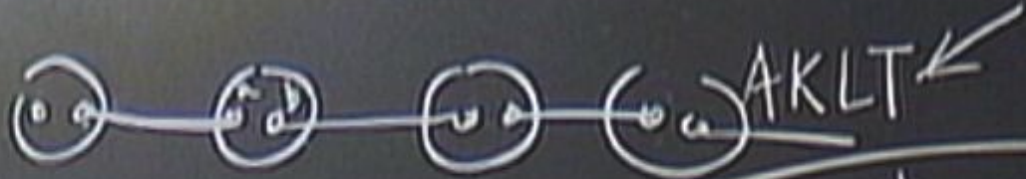
$$(b_i^\dagger a_{i+1}^\dagger) h_3$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad |0\rangle$

0 0 0 0 0
 i

$S=1$ $|\uparrow\rangle, |0\rangle, |\downarrow\rangle$



$$\prod (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |0\rangle$$

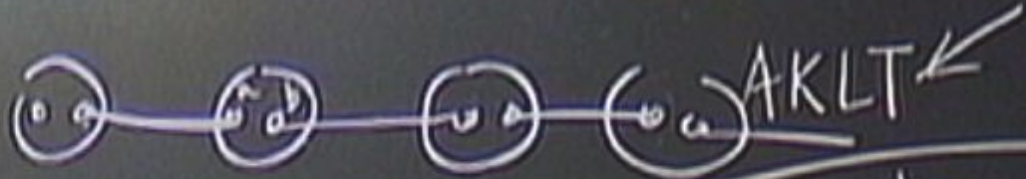
$$a_i^\dagger a_i + b_i^\dagger b_i = 2$$

$$a_1^\dagger (b_2^\dagger a_2^\dagger) b_3 - a_1^\dagger (b_2^\dagger)^2 a_3$$

$$\begin{pmatrix} 2 & 0 \\ - & - \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

0 0 0 0 0
i

S=1 $|\uparrow\rangle, |\downarrow\rangle, |\downarrow\rangle$



$$\prod_i (a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) |\downarrow\rangle$$

$$a_i^\dagger a_{i+1} + b_i^\dagger b_{i+1} = 2$$

$$a_1^\dagger (b_2^\dagger a_2^\dagger) b_3 - a_1^\dagger (b_2^\dagger)^2 a_3$$

$$\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$|\uparrow\rangle \quad |\downarrow\rangle \quad (a_i^\dagger)$

