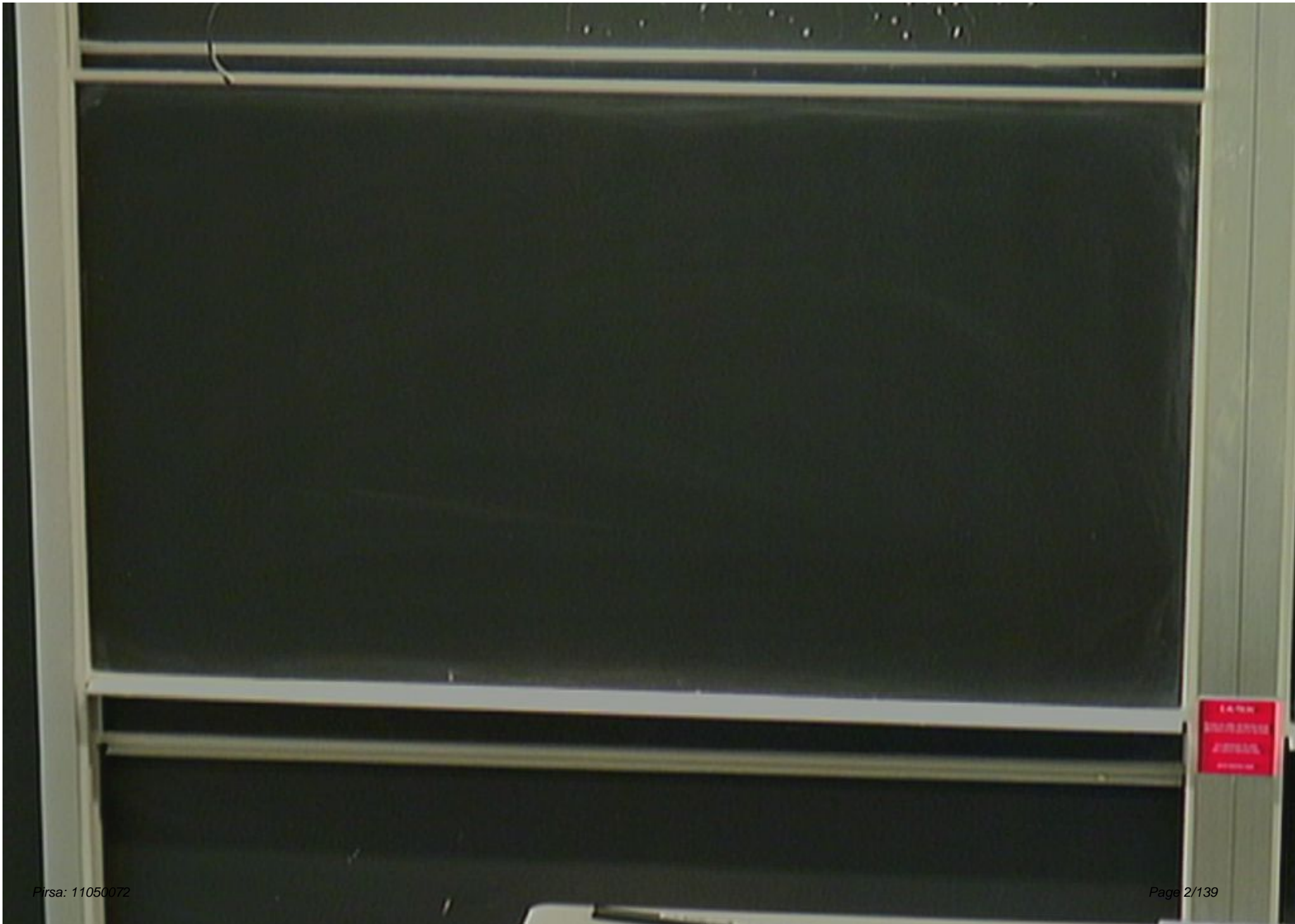


Title: Geometry & Topology for Physics - Lecture 3

Date: May 30, 2011 02:00 PM

URL: <http://pirsa.org/11050072>

Abstract:



Integration of

1

Integration of Forms.

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Integration of Forms.

1. Orientation.

Integration of Forms.

1. Orientation.



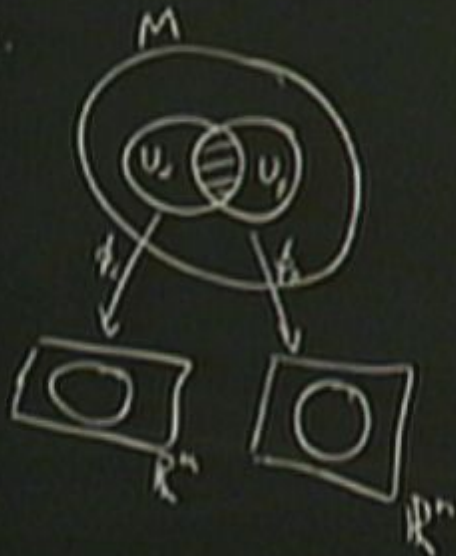
Integration of Forms.

1. Orientation.



Integration of Forms.

1. Orientation.



Integration of Forms.

1. Orientation.



A basis of $TM|_{U_i \cap U_j}$

Integration of Forms.

1. Orientation.



A basis of $TM|_{u_1 \cap u_2}$
is given either by $\left\{ \frac{\partial}{\partial x^i} \right\}$, $\left\{ \frac{\partial}{\partial y^j} \right\}$

Integration of Forms.

1. Orientation.



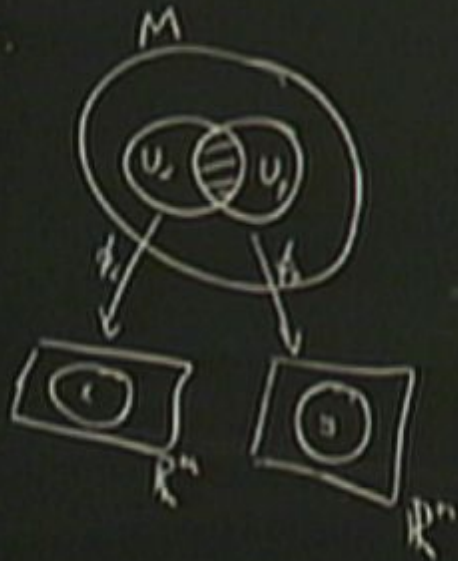
A basis of $TM|_{U_i \cap U_j}$ is given either by $\left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$ or $\left\{ \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \right\}$

M is orientable if

$$\left| \frac{\partial y^i}{\partial x^j} \right|$$

Integration of Forms.

1. Orientation.



A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{ \frac{\partial}{\partial x^i} \right\}$ or $\left\{ \frac{\partial}{\partial y^i} \right\}$

M is orientable if

$$\left| \frac{\partial y^i}{\partial x^i} \right| > 0 \text{ on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

Orientation.



A basis of TM|_{U_alpha U_beta}
is given either by $\left\{ \frac{\partial}{\partial x^i} \right\}$, $\left\{ \frac{\partial}{\partial y^i} \right\}$

M is orientable if

$$\left| \frac{\partial y^i}{\partial x^i} \right| > 0 \text{ on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

$$\frac{\partial}{\partial x^i} \cdot \frac{\partial}{\partial y^i}$$

Orientation



A basis of TM|_{U_alpha ∩ U_beta}
 is given either by $\left\{ \frac{\partial}{\partial x^r} \right\}$ or $\left\{ \frac{\partial}{\partial y^s} \right\}$

M is orientable if

$$\left| \frac{\partial y^v}{\partial x^r} \right| > 0 \text{ on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

$$\frac{\partial}{\partial x^r} = \frac{\partial y^v}{\partial x^r} \frac{\partial}{\partial y^v}$$



$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

M

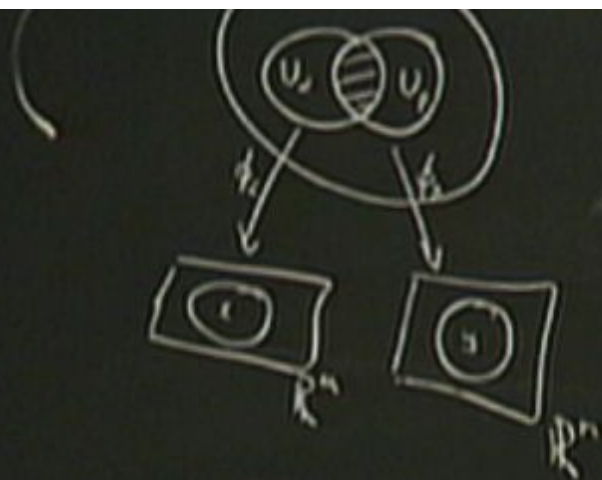
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$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

M is orientable \Leftrightarrow there exists a nowhere-vanishing

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

M is orientable \Leftrightarrow there exists a nowhere-vanishing, smooth top degree form ω on M



is given either by $\left\{ \frac{\partial}{\partial x^r} \right\}$ or $\left\{ \frac{\partial}{\partial y^s} \right\}$

M is orientable if

$$\left| \frac{\partial y^s}{\partial x^r} \right| > 0$$

on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial x^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$



1. Orientation.

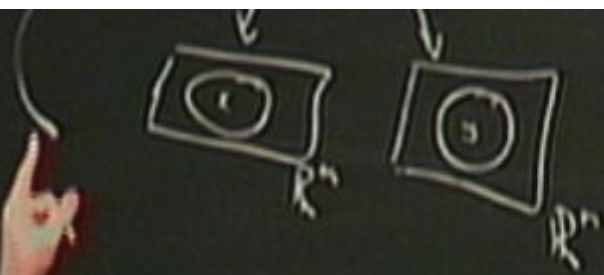


A basis of TM $|_{U_\alpha \cap U_\beta}$
 is given either by $\left\{ \frac{\partial}{\partial x^i} \right\}$, $\left\{ \frac{\partial}{\partial y^i} \right\}$

M is orientable if \exists atlas $\{U_\alpha, \phi_\alpha\}$ s.t.

$$\left| \frac{\partial y^i}{\partial x^i} \right| > 0 \quad \text{on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

$$= \frac{\partial y^i}{\partial x^i} \frac{\partial}{\partial y^i}$$



M is orientable if \exists an atlas $\{U_\alpha, \phi_\alpha\}$ s.t.
 $\left| \frac{\partial y^\nu}{\partial x^\mu} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$$

(1)

$$\phi_\alpha^*(dx^1 \wedge \dots \wedge dx^n)$$

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$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

M is orientable \Leftrightarrow there exists a nowhere-vanishing, smooth top degree form ω on M

Proof: (\Leftarrow)

$$\phi_* (dx^1 \wedge \dots \wedge dx^n) = f_x \omega \quad \text{where } f_x \text{ is nowhere-vanishing on } U_x.$$

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

M is orientable \Leftrightarrow there exists a nowhere-vanishing, smooth top degree form ω on M .

Proof: (\Leftarrow)

$$\phi_x^{-1}(dx^1 \wedge \dots \wedge dx^n) = f_x \omega \quad \text{where } f_x \text{ is nowhere-vanishing on } U_x.$$

$$\phi_y^{-1}(dy^1 \wedge \dots \wedge dy^n) = f_y \omega$$

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

M is orientable \Leftrightarrow there exists a nowhere-vanishing, smooth top degree form ω on M .

Proof. (\Rightarrow)

$$\begin{aligned} \phi_i^*(dx^1 \wedge \dots \wedge dx^n) &= f_i \omega & \text{where } f_i \text{ is nowhere-vanishing on } U_i. \\ \phi_j^*(dy^1 \wedge \dots \wedge dy^n) &= f_j \omega & \Rightarrow f_i > 0. \end{aligned}$$

\mathbb{R}^n \mathbb{R}^n $\left| \frac{\partial \phi_i}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps
 $\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$

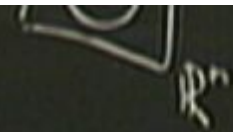
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Proof

$\phi_\alpha^*(dx^1 \wedge \dots \wedge dx^n) = f_\alpha \omega$ where f_α is nowhere-vanishing on U_α .
 $\phi_\beta^*(dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega \Rightarrow f_\alpha > 0$

$\phi_\beta^* \cdot \phi_\alpha^*(dx^1 \wedge \dots \wedge dx^n)$

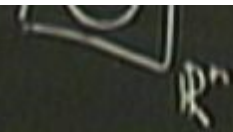
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\mathbb{R}^n  $\left| \frac{\partial f}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps
 $\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$

Proof (c)

$f_\alpha(dx^1 \wedge \dots \wedge dx^n) = f_\alpha \omega$ where f_α is nowhere-vanishing on U_α .
 $f_\beta(dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega \Rightarrow f_\alpha > 0$
 $f_\beta(dx^1 \wedge \dots \wedge dx^n) =$



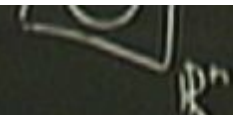
\mathbb{R}^n  $\left| \frac{\partial f_\alpha}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps
 $\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$

$f_\alpha (dx^1 \wedge \dots \wedge dx^n) = f_\alpha \omega$ where f_α is nowhere-vanishing on U_α
 $(dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega \rightarrow f_\beta > 0$

$(dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta} f_\beta^{-1} (dy^1 \wedge \dots \wedge dy^n)$



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\mathbb{R}^n  $\left| \frac{\partial \phi_i}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

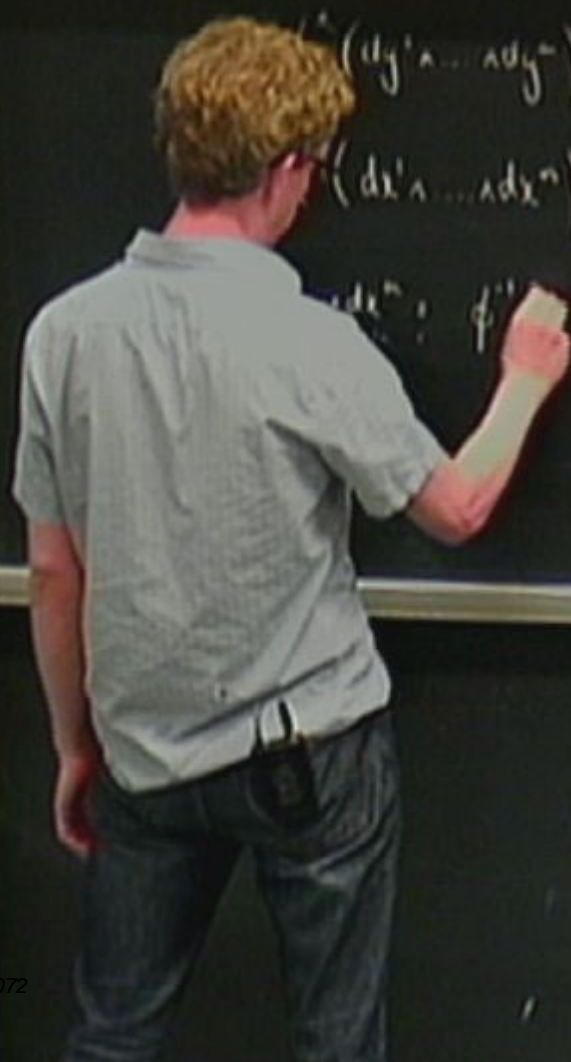
$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

$f_\alpha (dx^1 \wedge \dots \wedge dx^n) = f_\alpha \omega$ where f_α is nowhere vanishing on U_α

$$(dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega \Rightarrow f_\beta > 0$$

$$(dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta} \phi_\beta^* (dy^1 \wedge \dots \wedge dy^n)$$

$$dx^1 \wedge \dots \wedge dx^n = \phi_\beta^* (dy^1 \wedge \dots \wedge dy^n)$$



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$$\mathbb{R}^n \quad \mathbb{R}^n \quad \left| \frac{\partial f}{\partial x^i} \right| > 0 \quad \text{on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

$$f_\alpha^*(dy^1 \wedge \dots \wedge dy^n) = f_\alpha \omega \quad \rightarrow \quad f_\alpha > 0$$

$$(dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta} \phi_\beta^*(dy^1 \wedge \dots \wedge dy^n)$$

$$\dots \wedge dx^n = \left(\frac{f_\alpha}{f_\beta} \right)^n \left(\frac{f_\beta}{f_\alpha} \phi_\beta^*(dy^1 \wedge \dots \wedge dy^n) \right)$$



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$$\mathbb{R}^n \quad \mathbb{R}^n \quad \left| \frac{\partial f}{\partial x^k} \right| > 0 \quad \text{on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

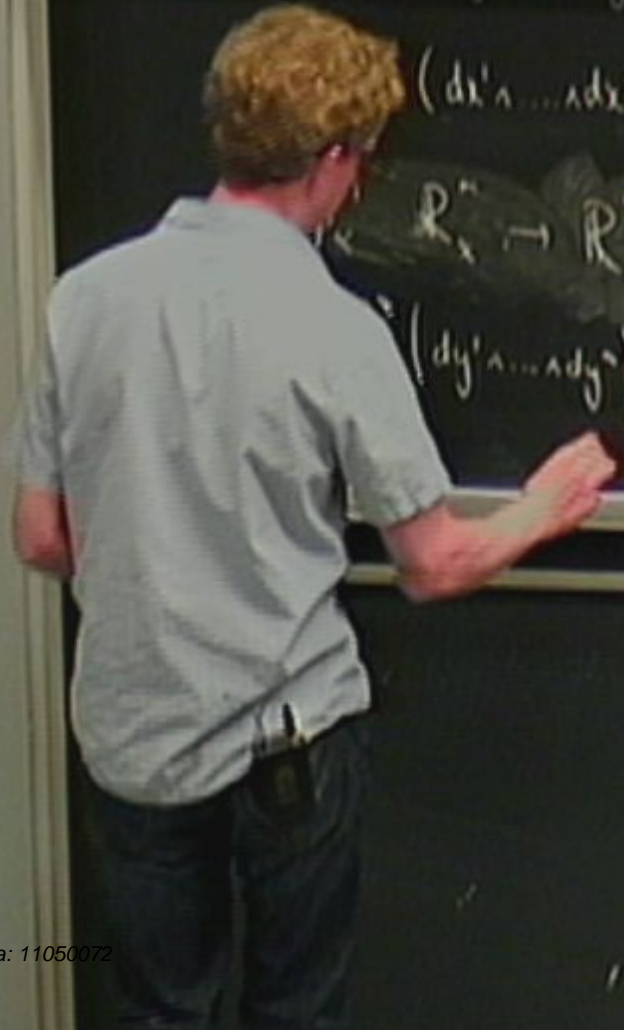
$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

$$f_i^*(dy^1 \wedge \dots \wedge dy^n) = f_i \omega \quad \rightarrow \quad f_i > 0$$

$$(dx^1 \wedge \dots \wedge dx^n) = \frac{f_n}{f_i} f_i^*(dy^1 \wedge \dots \wedge dy^n)$$

$$\mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$(dy^1 \wedge \dots \wedge dy^n)$$



2. 1. 1. 1.
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\mathbb{R}^n \mathbb{R}^n $\left| \frac{\partial f_i}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

$$f_i^*(dy^1 \wedge \dots \wedge dy^n) = f_i \omega \rightarrow f_i > 0$$

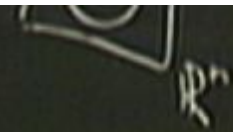
$$f_i^*(dx^1 \wedge \dots \wedge dx^n) = \frac{f_i}{f_j} f_j^*(dy^1 \wedge \dots \wedge dy^n)$$

$$f_i^*: \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$f_i^*(f_j^*)^*(dy^1 \wedge \dots \wedge dy^n)$$



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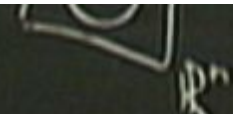
\mathbb{R}^n  $\left| \frac{\partial \phi_\alpha}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps
 $\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$

$\phi_\alpha^* (dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta} \phi_\beta^* (dy^1 \wedge \dots \wedge dy^n)$

$\phi_\alpha \cdot \phi_\beta^* (\omega)$
 $(\phi_\alpha \cdot \phi_\beta^*) (\omega) = (\phi_\alpha^{-1})^* \circ \phi_\beta^* (dy^1 \wedge \dots \wedge dy^n)$
 $= (\phi_\alpha^{-1})^* \cdot f_\beta \omega$



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\mathbb{R}^n  $\left| \frac{\partial \phi_\alpha}{\partial x^i} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^r} \frac{\partial}{\partial y^s}$$

$$f_\alpha (dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega \rightarrow f_\alpha > 0$$

$$\frac{f_\alpha}{f_\beta} \phi_\alpha^* (dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta} \phi_\beta^* (dy^1 \wedge \dots \wedge dy^n)$$

$\phi_\alpha \circ \phi_\beta^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\begin{aligned}
 (\phi_\alpha \circ \phi_\beta^{-1})^* (dy^1 \wedge \dots \wedge dy^n) &= (\phi_\beta^{-1})^* \circ \phi_\alpha^* (dy^1 \wedge \dots \wedge dy^n) \\
 &= (\phi_\beta^{-1})^* \circ f_\alpha \omega = \frac{f_\alpha(\phi_\beta^{-1}(y))}{f_\beta(y)} dx^1 \wedge \dots \wedge dx^n
 \end{aligned}$$

\mathbb{R}^n \mathbb{R}^n $\left| \frac{\partial \phi_i}{\partial x^k} \right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^r} = \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s}$$

$$f_2 (dy^1 \wedge \dots \wedge dy^n) = f_1 \omega \rightarrow f_1 > 0$$

$$\phi_2^* (dx^1 \wedge \dots \wedge dx^n) = \frac{f_2}{f_1} \phi_1^* (dy^1 \wedge \dots \wedge dy^n)$$

$\phi_1, \phi_2: \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\begin{aligned}
 (\phi_2 \circ \phi_1^{-1})^* (dy^1 \wedge \dots \wedge dy^n) &= (\phi_1^{-1})^* \circ \phi_2^* (dy^1 \wedge \dots \wedge dy^n) \\
 &= (\phi_1^{-1})^* \circ f_2 \omega = \frac{f_2(\phi_1(x))}{f_1(x)} dx^1 \wedge \dots \wedge dx^n
 \end{aligned}$$



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Partition of Unity -



Partition of Unity -

A \mathcal{U} is a collection of ^{smooth} functions $\rho_i : M \rightarrow \mathbb{R}$ s.t.

- $\text{supp}(\rho_i) \subset U_i$
- $\rho_i \geq 0$

Partition of Unity -

A p^o of 1 is a collection of ^{smooth} functions $\rho_i : M \rightarrow \mathbb{R}$ s.t

$$\text{Supp}(\rho_i) \subset U_i$$

$$\sum_i \rho_i = 1, \quad \rho_i \geq 0$$



Partition of Unity -

A p. of 1 is a collection of ^{smooth} functions $\rho_i : M \rightarrow \mathbb{R}$ s.t.

- $\text{Supp}(\rho_i) \subset U_i$
- $\sum_i \rho_i = 1, \rho_i \geq 0$

$$\begin{aligned}
 (\phi^{-1})^* (dy^1 \wedge \dots \wedge dy^n) &= (\phi^{-1})^* \phi^* (dy^1 \wedge \dots \wedge dy^n) \\
 &= (\phi^{-1})^* \phi^* \omega = \frac{f_2(f_1(x))}{f_1(x)} dx^1 \wedge \dots \wedge dx^n
 \end{aligned}$$

$$(dx^1 \wedge \dots \wedge dx^n)$$

$$\begin{aligned}
 (f_1' \dots f_n') (dy^1 \dots dy^n) &= (f_1')^n \cdot \phi_1 (dy^1 \dots dy^n) \\
 &= (f_1')^n \cdot f_1(x) = \frac{f_1'(x)}{f_1(x)} dx^1 \dots dx^n
 \end{aligned}$$

$$\begin{aligned}
 (f_1' \dots f_n')^n (dx^1 \dots dx^n) &= \lambda_+ (dy^1 \dots dy^n) \quad \text{with } \lambda_+ > 0 \\
 &= \phi_1^n (dx^1 \dots dx^n)
 \end{aligned}$$

$$\begin{aligned} (f_1, \dots, f_n) (dy^1, \dots, dy^n) &= (f_1, \dots, f_n) \circ \phi_1^* (dy^1, \dots, dy^n) \\ &= (f_1, \dots, f_n) \circ \phi_1^* = \frac{f_1(\lambda)}{f_1(\lambda)} dx^1, \dots, dx^n \end{aligned}$$

$$(f_1, \dots, f_n) (dx^1, \dots, dx^n) = \lambda_1 (dy^1, \dots, dy^n) \quad \text{with } \lambda_1 > 0$$

$$f_1^* (dx^1, \dots, dx^n) = f_1^*(\lambda) f_1^* (dy^1, \dots, dy^n)$$

$$\begin{aligned} (\phi_c^{-1})^* (dy^1 \wedge \dots \wedge dy^n) &= (\phi_c^{-1})^* \circ \phi_c^* (dy^1 \wedge \dots \wedge dy^n) \\ &= (\phi_c^{-1})^* \circ f_c^* = \frac{f_c \circ \phi_c^{-1}}{f_c} dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

$$(\phi_c \circ \phi_c^{-1})^* (dx^1 \wedge \dots \wedge dx^n) = \lambda_c (dy^1 \wedge \dots \wedge dy^n) \quad \text{with } \lambda_c > 0$$

$$\phi_c^* (dx^1 \wedge \dots \wedge dx^n) = \phi_c^*(\lambda_c) \phi_c^* (dy^1 \wedge \dots \wedge dy^n)$$

ϕ_c^{-1} is C^∞ smooth uniform on U_c .

$$\begin{aligned} (f_c^{-1})^* (dy^1 \wedge \dots \wedge dy^n) &= (f_c^{-1})^* \circ f_c^* (dy^1 \wedge \dots \wedge dy^n) \\ &= (f_c^{-1})^* \circ f_c^* \omega = \frac{f_c^*(\omega)}{f_c'(t)} dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

$$(f_c \circ f_c^{-1})^* (dx^1 \wedge \dots \wedge dx^n) = \lambda_c (dy^1 \wedge \dots \wedge dy^n) \quad \text{with } \lambda_c > 0$$

$$\omega_c \equiv f_c^* (dx^1 \wedge \dots \wedge dx^n) = f_c^*(\lambda) f_c^* (dy^1 \wedge \dots \wedge dy^n)$$

ω_c is a non-zero smooth n-form on U_c ; positive definite.

$$\omega = \sum_i p_i \omega_i$$

$$\begin{aligned} (\phi^* \omega) (dy^1 \wedge \dots \wedge dy^n) &= (\phi^* \omega)_i \phi_i^j (dy^1 \wedge \dots \wedge dy^n) \\ &= (\phi^* \omega)_i \phi_i^j = \frac{f_j(\phi(x))}{f_i(x)} dx^1 \wedge \dots \wedge dx^n \end{aligned}$$

$$(\phi^* \omega)_i (\phi_i^j)^2 (dx^1 \wedge \dots \wedge dx^n) = \lambda_\alpha (dy^1 \wedge \dots \wedge dy^n) \quad \text{with } \lambda_\alpha > 0$$

$$\omega_\alpha \equiv \phi_i^j (dx^1 \wedge \dots \wedge dx^n) = \phi_i^j(\lambda) \phi_i^k (dy^1 \wedge \dots \wedge dy^n)$$

ω_α is a non-zero smooth uniformly U_α , positive definite.

$$\omega = \sum_i \rho_i \omega_i$$

A ρ of I is a collection of n functions $\rho_i: M \rightarrow \mathbb{R}$ s.t.

- $\text{Supp}(\rho_i) \subset U_i$
- $\sum_i \rho_i = 1$, $\rho_i \geq 0$

To integrate: pick an orientation on M . Then for any $f: M \rightarrow \mathbb{R}$, define $\int_M f$ over M by

$$\int_M f \omega$$

A ρ of I is a collection of n functions $\rho_i: M \rightarrow \mathbb{R}$ s.t.

- $\text{Supp}(\rho_i) \subset U_i$
- $\sum_i \rho_i = 1$, $\rho_i \geq 0$

To integrate: pick an orientation on M . Then for any $f: M \rightarrow \mathbb{R}$, define
integral of f over M by

$$\int_M f \omega = \sum_i \int \rho_i f \omega_i$$

A ρ of I is a collection of n functions $\rho_i: M \rightarrow \mathbb{R}$ s.t.

- $\text{Supp}(\rho_i) \subset U_i$
- $\sum_i \rho_i = 1$, $\rho_i \geq 0$

To integrate: pick an orientation on M . Then for any $f: M \rightarrow \mathbb{R}$, define
integral of f over M by

$$\int_M f \omega = \sum_i \int_{U_i} \rho_i f \omega.$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_i, \phi_i)\}$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_i, \phi_i)\}$

- To see this, let V_j be another collection of open sets, and let ψ_j be a partition of unity subordinate to V_j .

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

- To see this, let V_β be another collection of open sets, and let χ_β be a p° of 1 subordinate to V_β

This is independent of choice of

- partition of unity
- of atlas $\{(U_i, \phi_i)\}$

- To see this, let V_j be another collection of open sets, and let χ_j be a p.o.f. of V_j subordinate to V_j

$$\sum_i \int_{U_i} p_i f_{U_i} = \sum_j \int_{V_j} p_j \chi_j$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_i, \phi_i)\}$

- To see this, let V_p be another collection of open sets and let χ_p be a p.s.f of 1 subordinate to V_p

$$\int_X f \omega = \sum_i \int_{U_i} \rho_i f \omega_i = \sum_{i,j} \int_{U_i} \rho_i \chi_j f \omega_i$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

- To see this, let V_β be another collection of sets, and let χ_β be a p.s.f of 1 subordinate to V_β
 $\text{Supp}(\rho_\beta \chi_\beta) \subset (U_\alpha \cap V_\beta)$

$$\int_M f \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\beta} \int_{U_\beta} \rho_\beta \chi_\beta f \omega_\beta$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

- To see this, let V_j be another collection of open sets, and let χ_j be a p^2 of 1

Subordinate to V_j

$$\text{Supp}(\rho_j \chi_j) \subset (U_\alpha \cap V_j)$$

$$\int_X f \rho_\alpha = \sum_{\alpha} \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha} \int_{V_j} \rho_\alpha \chi_j f \omega_\alpha = \sum_{\alpha} \int_{V_j} \rho_\alpha f \omega_\alpha$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

- this, let V_j be another collection of open sets, and let χ_j be a p.s.f. of 1 subordinate to V_j
 $\text{Supp}(\rho_j \chi_j) \subset (U_\alpha \cap V_j)$

$$\int_M f \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{j=1}^r \int_{U_\alpha} \rho_\alpha \chi_j f \omega_\alpha = \sum_{j=1}^r \int_{V_j} \rho_\alpha \chi_j f \omega_\alpha = \sum_{j=1}^r \int_{V_j} \chi_j f \omega_j$$

This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

- To see this, let V_j be another collection of open sets, and let χ_j be a p.o.f. subordinate to V_j

$$\text{Supp}(\rho_j \chi_j) \subset (U_\alpha \cap V_j)$$

$$\int_X f \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{j \neq \alpha} \int_{U_\alpha} \rho_\alpha \chi_j f \omega_\alpha = \sum_{j \neq \alpha} \int_{V_j} \rho_\alpha \chi_j f \omega_\alpha = \sum_j \int_{V_j} \chi_j f \omega_j$$

$$\omega_\alpha = \omega|_{U_\alpha} \quad \omega_j = \omega|_{V_j}$$

Stokes' Theorem



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Stokes' Theorem

Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d\mu =$$

$$d\mu = \frac{\partial \mu}{\partial x^i} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

Stokes' Theorem

Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \frac{\partial f(x)}{\partial x^i} dx^i \right) dx^1 \wedge \dots \wedge dx^n$$

$$d\mu = \frac{\partial f(x)}{\partial x^i} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$



Stokes' Theorem

Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$d\mu = \frac{\partial f(x)}{\partial x^i} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^n} f \left(\int_{\mathbb{R}^n} \frac{\partial f(x)}{\partial x^i} dx^i \right) dx^1 \wedge \dots \wedge dx^n = \int_{\mathbb{R}^n} [f(x) - f(-x)] dx^1 \wedge \dots \wedge dx^n$$

= 0



Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$d\mu = \frac{\partial f(x)}{\partial x^i} dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f(x)}{\partial x^i} dx^i \right) dx^1 \wedge \dots \wedge dx^{n-1} = \int_{\mathbb{R}^{n-1}} [f(+\infty) - f(-\infty)] dx^1 \wedge \dots \wedge dx^{n-1} = 0$$

Let $\mu \in \Omega_c^k(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d\mu = f\left(\int_{\mathbb{R}^n} \frac{\partial f}{\partial x^i} dx^i\right) dx^1 \wedge \dots \wedge dx^n = \int_{\mathbb{R}^n} [f(x) - f(-x)] dx^1 \wedge \dots \wedge dx^n = 0$$

Let $H^1 = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$

$$\mu \in \Omega^1(H^1) \rightarrow \mu = f(x,y) dx + g(x,y) dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx + dy$$

$$\int_{H'} d\mu = \int \left(\int \frac{\partial g}{\partial x} dx \right) dy - \int \left(\int \frac{\partial f}{\partial y} dy \right) dx$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx + dy$$

$$\int_{H'} d\mu = \int \left(\int \frac{\partial g}{\partial x} dx \right) dy - \int \int \frac{\partial f}{\partial y} dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx + dy$$

$$\int_C \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx + dy = \int_C \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx + dy$$

$$= \int_C [g(x, y) - g(x, 0)] dx - \int_C [f(x, 1) - f(x, 0)] dx$$

0

$$= \int_C f(x, 0) dx$$

$$\mu \in \mathcal{S}\mathcal{L}_c(\mathbb{H}^1) \Rightarrow \mu = f(x,y) dx + g(x,y) dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

$$= \int \left[\underbrace{g(-a,y) - g(-a,y)}_0 \right] dy - \int \left(\cancel{f(x,-a)} - f(x,0) \right) dx$$

$$= \int f(x,0) dx = \int_{\mathbb{H}^1} \mu$$

$$\mu \in \mathcal{S}\mathcal{L}_c(\mathbb{H}^1) \rightarrow \mu = f(x,y) dx + g(x,y) dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

$$= \int [g(\infty, y) - g(-\infty, y)] dy - \int [f(x, \infty) - f(x, 0)] dx$$

"
0

$$= \int f(x, 0) dx = \int_{\mathbb{H}^1} \mu$$

Proof (\Rightarrow)

$$M \in \Omega_c^k(M)$$

$$\int_M dM = \int_{\partial M} M$$

Proof (\Rightarrow)

$$M \subset \Omega_c^{-1}(M)$$

$$\int_M \omega_\mu = \int_{\Sigma M} \omega_\mu$$

$$\int_M \omega_\mu = \sum_i \int_{\alpha_i} \omega_\mu$$

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Proof (\Rightarrow)

$$\mu \in \Omega_c^{n-1}(M)$$

$$\int_M \mu = \int_{\partial M} \mu$$

$$\int_M \mu = \sum_i \int_{\partial M_i} \mu_i$$

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Proof (\Rightarrow)

$$\mu \in \Omega_c^{n-1}(M)$$

$$\int_M d\mu = \int_{\partial M} \mu$$

$$\int_M d\mu = \sum_i \int_{U_i} \alpha_i \lrcorner \mu, \text{ but } \text{Supp}(\alpha_i \lrcorner \mu) \subset \text{Supp}(\alpha_i) \cap \text{Supp}(\mu)$$

Proof (\Rightarrow)

$$M \in \Omega_c^{-1}(M)$$

$$\int_M d\mu = \int_{2M} \dots$$

$$\int_M d\mu = \sum_i \int_{U_i} \alpha_i d\mu|_{U_i}$$

but $\text{Supp}(\alpha_i d\mu) \subset \text{Supp}(\alpha_i) \cap \text{Supp}(d\mu)$

$$= \sum_i \int_{U_i} \alpha_i$$

is compact

Proof (\Rightarrow)

$$M \in \Omega_c^{-1}(M)$$

$$\int_M \mu_M = \int_{2M}$$

$$\sum \int d(\mu_M)_{\nu_i}, \text{ but } \text{Supp}(\mu_M) \subset \text{Supp}(\nu_i) \cap \text{Supp}(\mu_M)$$

$$= \sum \int \nu_i$$

is compact

This is independent of choice of \bullet
• partition of unity

de Rham Cohomology

$$d: \Omega^r(M) \rightarrow \Omega^{r+1}(M)$$

says

$$d^2 = 0$$

$$df^* = f^*d$$

where $f: N \rightarrow M$

This is independent of choice of
• partition of unity

de Rham Cohomology

$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ says $d^2=0$, $df^* = f^*d$
where $f: N \rightarrow M$ is smooth.

The p^{th} de-Rham cohomology of P , $H^p(M; \mathbb{R})$

This is a fundamental theorem

de Rham Cohomology:

$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ always $d^2=0$, $d f^* = f^* d$
when $f: N \rightarrow M$ is smooth.

The p -th cohomology group, $H^p(M; \mathbb{R})$, is defined to be



de Rham Cohomology

$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ says $d^2=0$, $df^* = f^*d$
where $f: N \rightarrow M$ is smooth.

The p^{th} de-Rham cohomology $H^p(M; \mathbb{R})$, is defined to be

$$H^p(M; \mathbb{R}) = \frac{\ker(d: \Omega^p \rightarrow \Omega^{p+1})}{\text{im}(d: \Omega^{p-1} \rightarrow \Omega^p)}$$

de Rham Cohomology

$$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$$

always $d^2 = 0$

$$d f^* = f^* d$$

when $f: N \rightarrow M$ is smooth.

The p -th de Rham cohomology group

$$H^p(M; \mathbb{R})$$

defined to be

$$H^p(M; \mathbb{R}) = \frac{\ker(d: \Omega^p \rightarrow \Omega^{p+1})}{\text{im}(d: \Omega^{p-1} \rightarrow \Omega^p)}$$

$$= H^p(M; \mathbb{R})$$

de Rham Cohomology.

$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ says $d^2=0$, $df^* = f^*d$
where $f: N \rightarrow M$ is smooth.

The p^{th} de-Rham cohomology $H^p(M; \mathbb{R})$, is defined to be

$$H^p(M, \mathbb{R}) = \frac{\ker(d: \Omega^p \rightarrow \Omega^{p+1})}{\text{im}(d: \Omega^{p-1} \rightarrow \Omega^p)} \quad \Leftrightarrow \quad \omega \in H^p(M, \mathbb{R})$$
$$\Leftrightarrow d\omega = 0$$

$\omega \neq d\eta$ if ω is non-trivial in $H^p(M; \mathbb{R})$

de Rham Cohomology

$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$ says $d^2 = 0$, $d f^* = f^* d$
where $f: N \rightarrow M$ is smooth.

The p^{th} de-Rham cohomology group, $H^p(M; \mathbb{R})$, is defined to be

$$H^p(M; \mathbb{R}) = \frac{\ker(d: \Omega^p \rightarrow \Omega^{p+1})}{\text{im}(d: \Omega^{p-1} \rightarrow \Omega^p)} \quad \Leftrightarrow \quad \omega \in H^p(M; \mathbb{R})$$

$$\Leftrightarrow d\omega = 0$$

$\omega \neq d\eta$ if ω is non-trivial in $H^p(M; \mathbb{R})$

$$d\omega = 0$$

de Rham Cohomology

$$d: \Omega^p(M) \rightarrow \Omega^{p+1}(M)$$

$$\text{always } d^2 = 0$$

$$df^* = f^*d$$

where $f: N \rightarrow M$ is smooth.

The p^{th} de-Rham cohomology group, $H^p(M; \mathbb{R})$, is defined to be

$$H^p(M; \mathbb{R}) = \frac{\ker(d: \Omega^p \rightarrow \Omega^{p+1})}{\text{im}(d: \Omega^{p-1} \rightarrow \Omega^p)}$$

$$\Leftrightarrow \omega \in H^p(M; \mathbb{R})$$

$$\Leftrightarrow d\omega = 0$$

$\omega \neq dq$ if ω is non-trivial in $H^p(M; \mathbb{R})$

$$d\omega = 0$$

$$\omega_1, \omega_2$$

$$a\omega_1 + b\omega_2$$

$$H^r(M) = \bigoplus_{p \geq 0} H^p(M)$$

$$H^r(M) = \bigoplus_{p \geq 0} H^p(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$H^*(M) = \bigoplus_{p \geq 0} H^p(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$[\omega] = [\omega + d\phi]$$

$$H^r(M) = \bigoplus_{p=0}^{\infty} H^p(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$([\omega] \in H^r(M)) \quad ([\omega] = [\omega + d\phi])$$

$$H^r(M) = \bigoplus_{p=0}^{\infty} H^p(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$[\eta] \in H^s(M) \quad ([\omega] = [\omega + d\phi])$$

then $[\omega] \wedge [\eta] \in H^{r+s}(M)$

$$H^r(M) = \bigoplus_{p \geq 0}^{\infty} H^p(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$[\eta] \in H^s(M) \quad ([\omega] = [\omega + d\phi])$$

$$[\omega] \wedge [\eta] \in H^{r+s}(M)$$

$$H^*(M) = \bigoplus_{r=0}^{\infty} H^r(M)$$

$[\omega] \in H^r(M)$ is the cohomology class of $\omega \in \Omega^r(M)$

$$[\eta] \in H^s(M) \quad ([\omega] = \dots)$$

then $[\omega] \wedge [\eta] \in H^{r+s}(M)$

$$[(\omega + d\phi) \wedge \eta] = [\omega \wedge \eta] + [d(\phi \wedge \eta)]$$

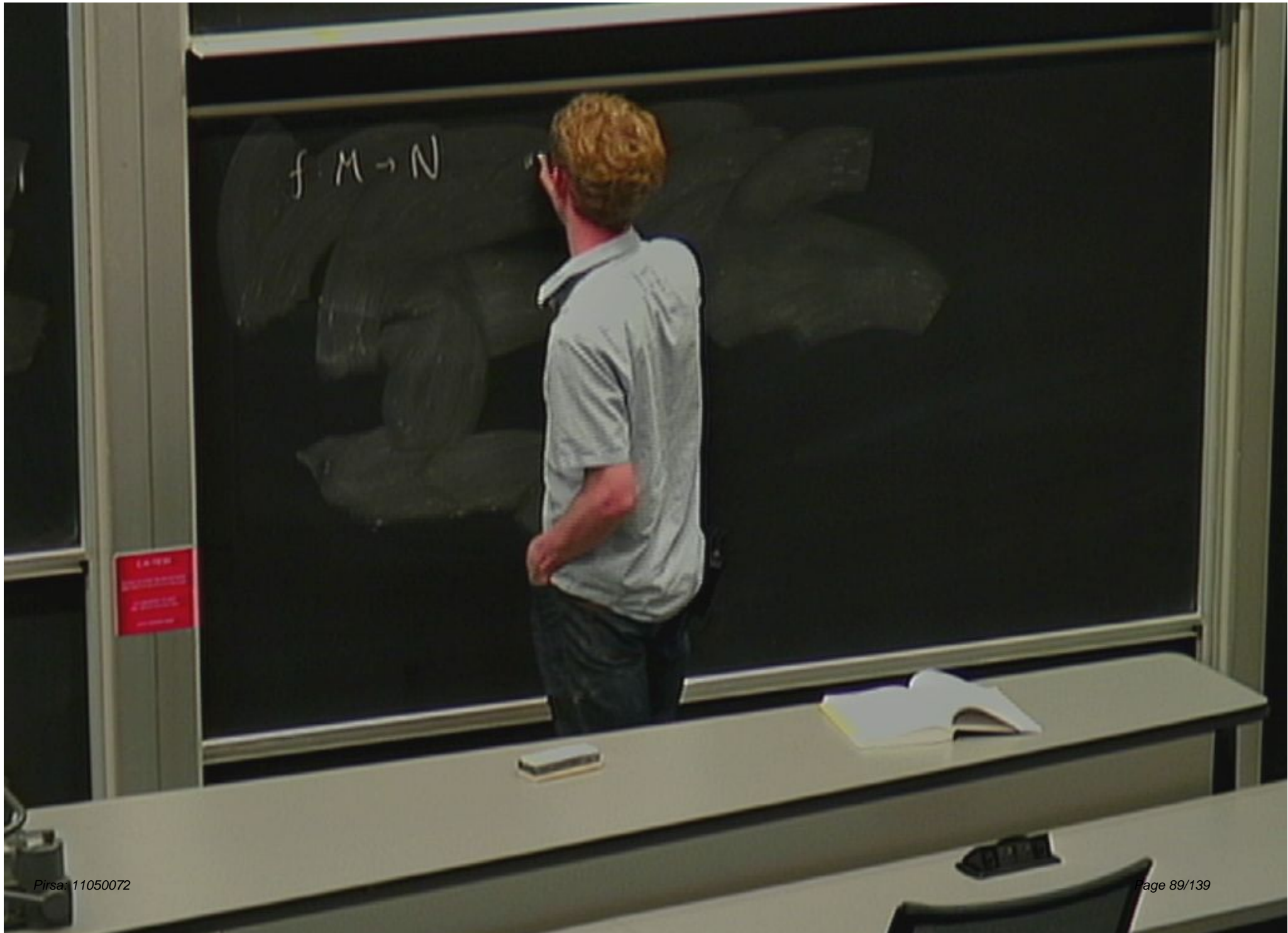
$$H^*(M) = \bigoplus_{p=0}^n H^p(M)$$

$[\omega] \in H^p(M)$ is the cohomology class of $\omega \in \Omega^p(M)$

$$[\eta] \in H^q(M) \quad ([\omega] = [\omega + d\phi])$$

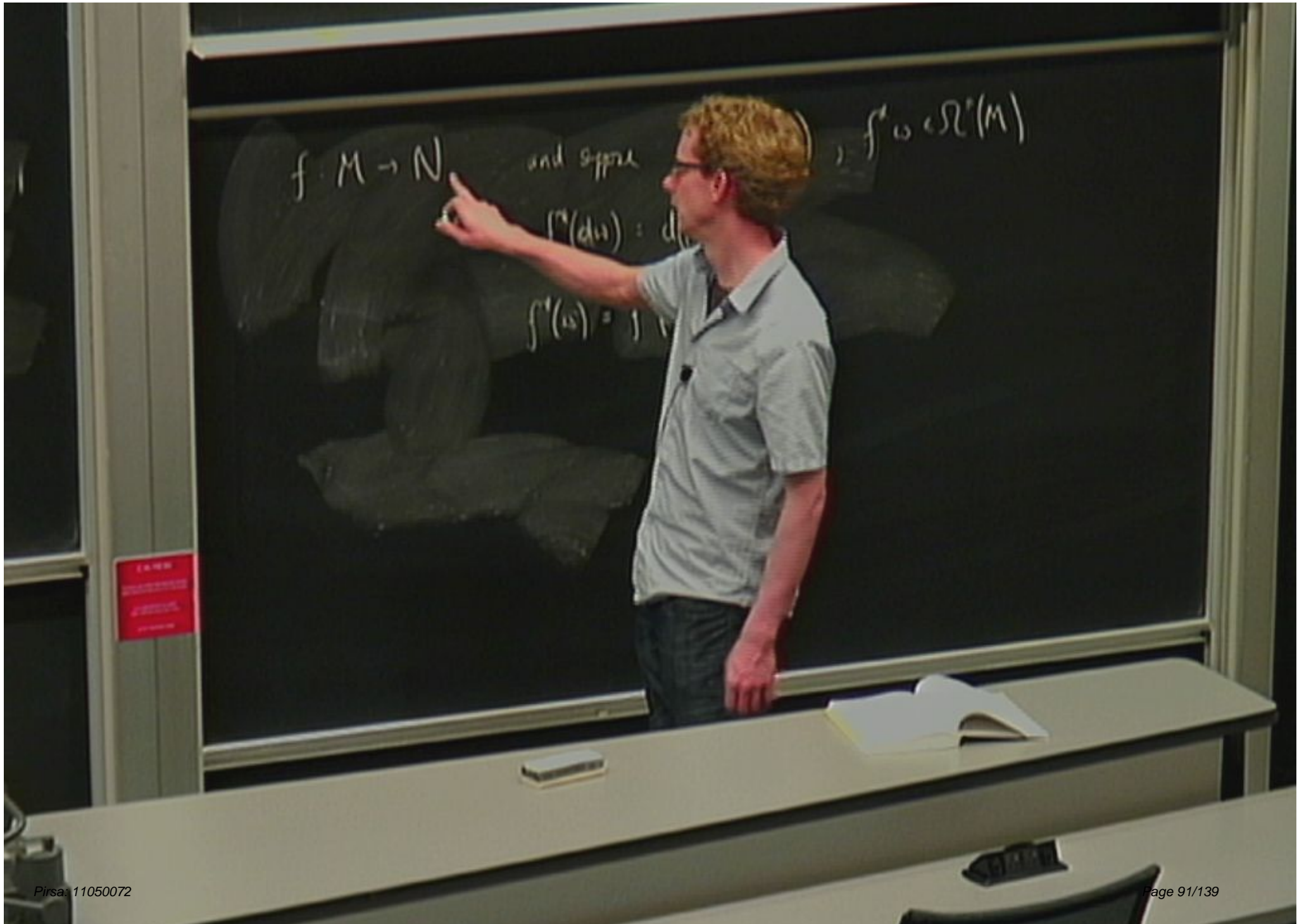
then $[\omega] \wedge [\eta] \in H^{p+q}(M)$ i.e. cohomology classes form a ring.

$$[(\omega + d\phi) \wedge \eta] = [\omega \wedge \eta] + \cancel{[d(\phi \wedge \eta)]}$$



$f: M \rightarrow N$ and suppose $\omega \in \Omega^1(N)$

$$f^*(d\omega) = d(f^*\omega)$$



$$f: M \rightarrow N$$

and suppose

$$f^*(du) = du$$

$$f^*(\omega) = \int \omega$$

$$f^*\omega \in \Omega^k(M)$$

$f: M \rightarrow N$ and suppose $\omega \in \Omega^r(N)$, $f^*\omega \in \Omega^r(M)$

$$f^*(du) = d(f^*u)$$

$$f^*(\omega) = f^*(d\eta) = d(f^*\eta)$$



Proof (\Rightarrow)

de Rham's Theorem

Suppose M compact (\Rightarrow)



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Proof (\Rightarrow)

de Rham's theorem.

Suppose M is compact ($\partial M = 0$) and $\omega \in \mathcal{R}^n(M)$, $d\omega = 0$

$$H_m(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int \omega$$

$n = \dim(M)$

Suppose M is compact ($\partial M = \emptyset$) nfd and $\omega \in \Omega^n(M)$, $d\omega = 0$.

$$H_n(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M \omega + d\int_M \omega = \int_M \omega + \int_{\partial M} \int_M \omega$$



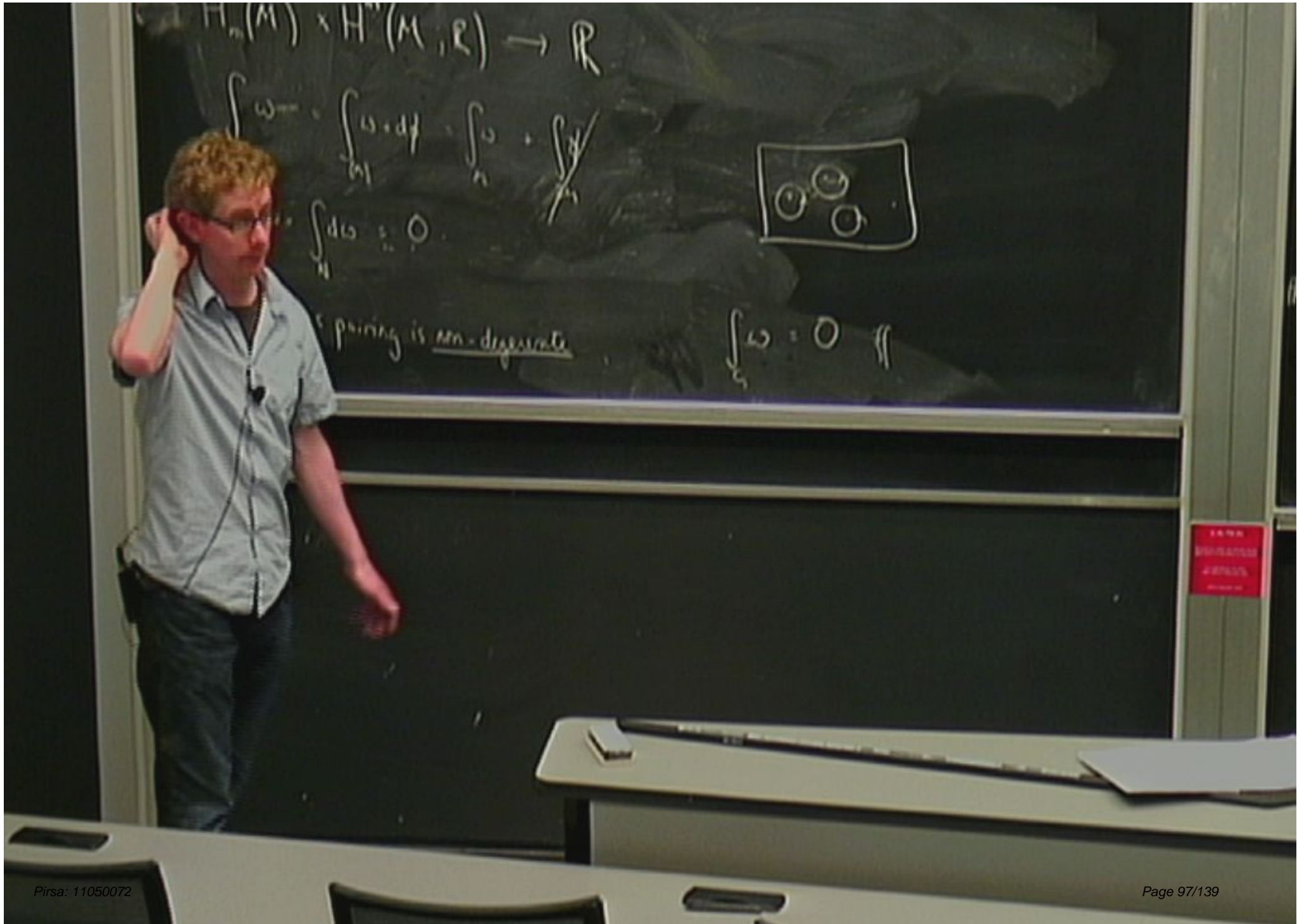
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Suppose M is compact ($\partial M = \emptyset$) nfdl and $\omega \in \Omega^n(M)$, $d\omega = 0$.

$$H_n(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M d\psi = \int_{\partial M} \psi = 0$$

$$\int_M \omega = 0$$



$$H_n(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M \omega + d\eta = \int_M \omega + \int_M d\eta$$

$$\int_M d\omega = 0$$



pairing is non-degenerate

$$\int_M \omega = 0 \iff$$

$$H_n(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M \omega + d\eta = \int_M \omega + 0$$

$$\int_M \omega = \int_M d\eta = 0$$

de Rham. This pairing is

$$\int_M \omega = 0 \iff$$



de Rham's Theorem

Suppose M is compact ($\partial M = \emptyset$) nfd and $\omega \in \mathcal{R}^n(M)$ $d\omega = 0$

$$\mathcal{H}_n(M) \times \mathcal{H}^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M \omega + d\tau = \int_M \omega + \int_M d\tau$$

$$\int_M \omega = \int_M d\omega = 0$$

de Rham: This pairing is non-degenerate

if $[\omega] = 0$

de Rham's Theorem

Suppose M is compact ($\partial M = \emptyset$) and $\omega \in D^k(M)$, $d\omega = 0$

$$H^k(M) \cong H^k(M, \mathbb{R})$$

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta$$

$$\int_M \omega = \int_M d\omega = 0$$



de Rham: This pairing is non-degenerate

$$\int_M \omega = 0 \quad \forall \omega \iff [\omega] = 0$$

Stokes' Theorem

Suppose M is compact ($\partial M = \emptyset$) nfd and $\omega \in \Omega^n(M)$, $d\omega = 0$

$$H_0(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M \omega + d\tau = \int_M \omega + \int_{\partial M} \tau$$

$$\int_M \omega = \int_M d\omega = 0$$



Then this pairing is non-degenerate

$$\int_M \omega = 0 \quad \forall \omega \iff [\omega] = 0$$

Stokes' Theorem

Suppose M is compact ($\partial M = \emptyset$) nfd and $\omega \in \Omega^n(M)$, $d\omega = 0$

$$H_0(M) \times H^n(M, \mathbb{R}) \rightarrow \mathbb{R}$$

$$\int_M \omega = \int_M d\psi = \int_{\partial M} \psi = 0$$



... This pairing is non-degenerate

$$\int_M \omega = 0 \quad \forall \omega \iff [\omega] = 0$$



Poincaré lemma

$$\begin{array}{c} \mathbb{R}^n \leftarrow \mathbb{R}^1 \\ \downarrow \quad \uparrow \\ \mathbb{R}^n \end{array}$$

$$\begin{array}{c} \Omega^1(\mathbb{R}^n \leftarrow \mathbb{R}^1) \\ \uparrow \quad \downarrow \\ \mathbb{R}^n \end{array}$$

$$\begin{array}{l} \pi: (x, t) \mapsto x \\ s: k \mapsto (k, 0) \end{array}$$

$$M \in \Omega^1(\mathbb{R}^n)$$



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Poincaré lemma

$$\begin{array}{c} \mathbb{R}^n \leftarrow \mathbb{R}^1 \\ \downarrow \uparrow \\ \mathbb{R}^n \end{array}$$

$$\begin{array}{c} \Omega^1(\mathbb{R}^n \leftarrow \mathbb{R}^1) \\ \uparrow \downarrow \\ \Omega^1(\mathbb{R}^n) \end{array}$$

$$\begin{array}{l} \pi: (x, t) \mapsto x \\ s: k \mapsto (k, 0) \end{array}$$

$$\begin{array}{l} \Gamma \\ \mathcal{M} \subset \Omega^1(\mathbb{R}^n) \\ \mathcal{M}(x) = \mathcal{M}(x) \end{array}$$

Poincaré lemma

$$\begin{array}{c} \mathbb{R}^n \leftarrow \mathbb{R}^k \\ \downarrow \quad \uparrow \\ \mathbb{R}^n \end{array}$$

$$\begin{array}{c} \Omega^1(\mathbb{R}^n \leftarrow \mathbb{R}^k) \\ \uparrow \quad \downarrow \\ \Omega^1(\mathbb{R}^n) \end{array}$$

$$\begin{array}{l} \pi: (t, x) \mapsto x \\ S: k \mapsto (t, 0) \end{array}$$

so if $M \in \Omega^1(\mathbb{R}^n)$
 $\pi^*(M) = M(x)$

$$\int_S M = \int_{\mathbb{R}^k} M(x(t)) dx^1 \wedge \dots \wedge dx^k$$

Poincaré lemma

$$\begin{array}{c}
 \mathbb{R}^n \leftarrow \mathbb{R}^k \\
 \downarrow \quad \uparrow \\
 \mathbb{R}^n
 \end{array}$$

$$\Omega^1(\mathbb{R}^n \leftarrow \mathbb{R}^k)$$

$$\pi \uparrow \quad \downarrow \sigma$$

$$\Omega^1(\mathbb{R}^n)$$

If $\omega \in \Omega^1(\mathbb{R}^n \leftarrow \mathbb{R}^k)$

$$\pi \cdot (x, t) \mapsto x$$

$$s \cdot k \mapsto (x, 0)$$

so if $\omega \in \Omega^1(\mathbb{R}^n)$
 $\pi^* \omega = \omega(x)$

$$\begin{aligned}
 \omega &= u_1(x, t) dx^1 + \dots + u_n(x, t) dx^n \\
 &+ \omega_{x^1 \dots x^k}(x, t) dx^{x^1} \dots dx^{x^k}
 \end{aligned}$$

Poincaré lemma

$$\begin{array}{c}
 \mathbb{R}^n \subset \mathbb{R}^k \\
 \downarrow \quad \uparrow \\
 \mathbb{R}^n
 \end{array}$$

$$\Omega^p(\mathbb{R}^n \subset \mathbb{R}^k)$$

$$\pi^* \uparrow \quad \downarrow \sigma^*$$

$$\Omega^p(\mathbb{R}^k)$$

If $\omega \in \Omega^p(\mathbb{R}^n \subset \mathbb{R}^k)$

$$\pi \cdot (x, t) \mapsto x$$

$$s \cdot k \mapsto (x, 0)$$

so if $\mu \in \Omega^p(\mathbb{R}^n)$

$$\pi^*(\mu) = \mu(x)$$

$$\omega = \omega_{i_1, \dots, i_p}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_p} dt$$

$$+ \omega_{i_1, \dots, i_p}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$s^* \omega = \omega_{i_1, \dots, i_p}(x, 0) dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$\pi \circ S = \text{id} \Leftrightarrow \text{id} \circ (\pi \circ S)^* = S^* \circ \pi^*$$

E.

$$\pi \circ S = \text{id} \Leftrightarrow \text{id} = (\pi \circ S)^* = S^* \circ \pi^*$$

But $\pi^* \circ S^* \neq \text{id}$ in general. But perhaps $\pi^* \circ S^*$ is equivalent to the identity on homology? If so,

$$1 - \pi^* \circ S^*$$

$$\pi \circ s = \text{id} \quad (\Leftrightarrow) \quad \text{id} = (\pi \circ s)^* = s^* \circ \pi^*$$

But $\pi^* \circ s^* \neq \text{id}$ in general. But perhaps $\pi^* \circ s^*$ is equivalent to the identity on cohomology? If so,

$$1 - \pi^* \circ s^* =: (dK + Kd) \quad \text{for some } K$$

$$\pi \circ S = \text{id} \Leftrightarrow \text{id} \circ (\pi \circ S)^* = S^* \circ \pi^*$$

But $\pi^* \circ S^* \neq \text{id}$ in general. But perhaps $\pi^* \circ S^*$ is equivalent to the identity on homology? If so,

$$I - \pi^* \circ S^* =: (IK + Kd) \quad \text{for some } K$$

define $K = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$

$$S^* = I \int_0^1 \omega(t, t)$$

$$\pi \circ S = \text{id} \Leftrightarrow \text{id} \circ (\pi \circ S)^* = S^* \circ \pi^*$$

But $\pi^* \circ S^* \neq \text{id}$ in general. But perhaps $\pi^* \circ S^*$ is equivalent to the identity on cohomology? If so,

$$I - \pi^* \circ S^* =: (dK + Kd) \quad \text{for some } K$$

Define $K : \hat{W} \rightarrow \hat{W}$

$$S^* \cdot I \int \omega(t, 1)$$

$$\pi \circ S = \text{id} \quad \Leftrightarrow \quad \text{id} = (\pi \circ S)^* = S^* \circ \pi^*$$

But $\pi^* \circ S^* \neq \text{id}$ in general. But perhaps $\pi^* \circ S^*$ is equivalent to the identity on cohomology? If so,

$$1 - \pi^* \circ S^* =: (dK + Kd) \quad \text{for some } K$$

Define $K: \tilde{\omega} \mapsto 0$

$$\tilde{\omega}' \mapsto \int \omega'(t, s)$$

$$\Omega^r(\mathbb{R}^n, \mathbb{R}^m) \rightarrow \Omega^r(\mathbb{R}^n, \mathbb{R}^m)$$

$$(1 - \pi^* \cdot \delta^*) \tilde{\omega} = \tilde{\omega}(x, t) \rightarrow \tilde{\omega}(x, 0)$$

$$\Omega^p(\mathbb{R}^n \times \mathbb{R}^m)$$

$$\pi^* \uparrow \quad \downarrow \sigma^*$$

$$\Omega^p(\mathbb{R}^n)$$

If $\omega \in \Omega^p(\mathbb{R}^n \times \mathbb{R}^m)$

$$\begin{aligned} \omega &= \underbrace{\omega_{a_1, \dots, a_p}(x, t)}_{\omega'} dx^{a_1} \wedge \dots \wedge dx^{a_p} \wedge dt \\ &+ \underbrace{\tilde{\omega}_{a_1, \dots, a_p}(x, t)}_{\tilde{\omega}} dx^{a_1} \wedge \dots \wedge dx^{a_p} \end{aligned}$$

$$\sigma^* \omega = \omega_{a_1, \dots, a_p}(x, t) dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

So if

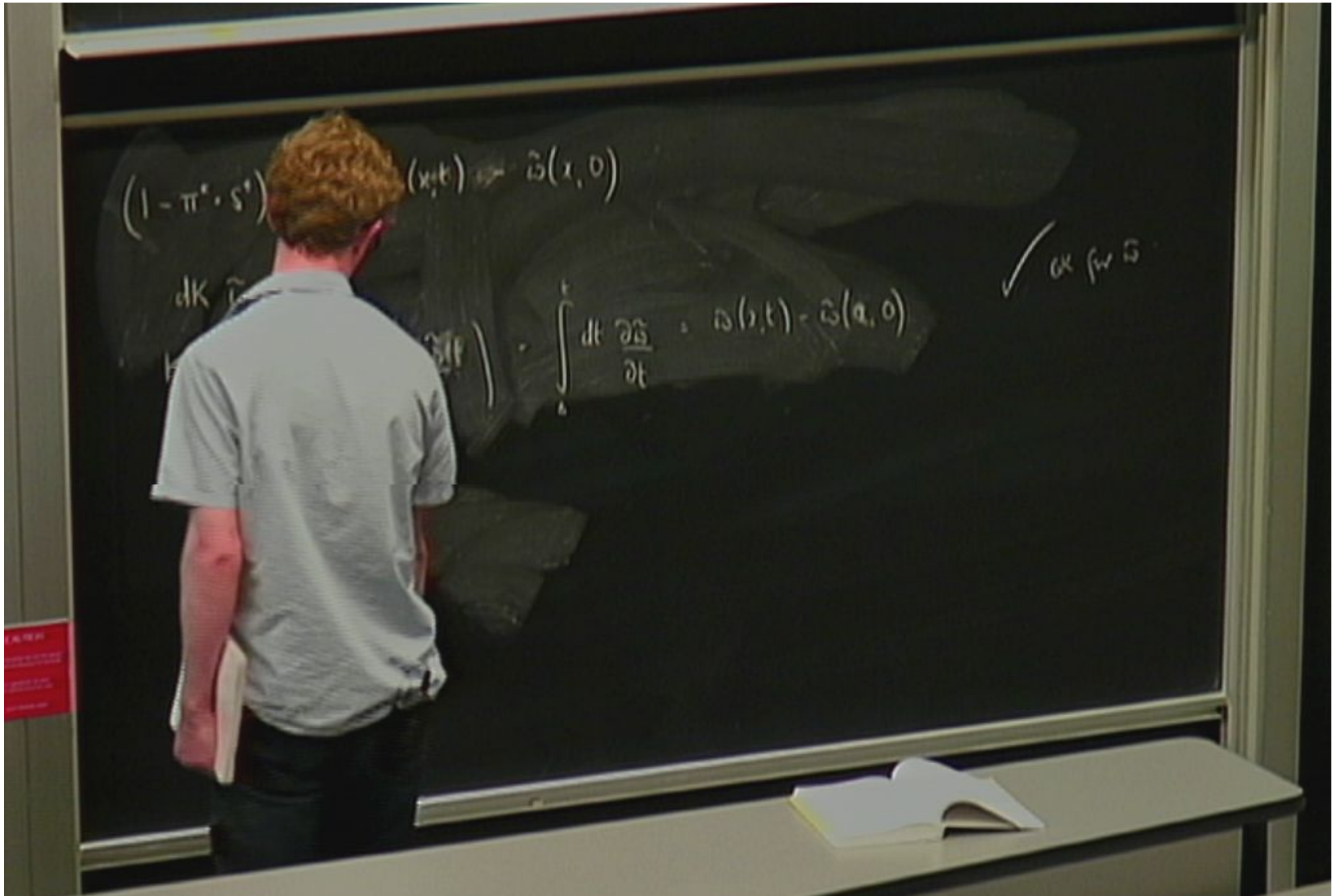
π^*

$$(1 - \pi \cdot s^*) \tilde{\omega} = \tilde{\omega}(x, t) \rightarrow \tilde{\omega}(x, 0)$$

$$dK \tilde{\omega} = 0$$

$$K(d\tilde{\omega}) = K\left(\frac{\partial \tilde{\omega}}{\partial t} dt\right)$$





$$(1 - \pi^2 \cdot s^2) \bar{\omega} = \bar{\omega}(x, t) = \bar{\omega}(x, 0)$$

$$dK \bar{\omega}$$

$$K(d\bar{\omega})$$

$$\int_a^t dt \frac{\partial \bar{\omega}}{\partial t} = \bar{\omega}(x, t) - \bar{\omega}(x, 0)$$

← $\omega(x, 0)$

$$(1 - \pi^* \cdot s^*) \omega'$$

Princal lemma

$$\begin{array}{c} \mathbb{R}^n \times \mathbb{R}^1 \\ \downarrow \quad \uparrow \\ \mathbb{R}^n \end{array}$$

$$\Omega^k(\mathbb{R}^n \times \mathbb{R}^1)$$

$$\begin{array}{c} \uparrow \quad \downarrow \\ \pi^* \quad \downarrow \sigma \end{array}$$

$$\Omega^k(\mathbb{R}^n)$$

If $\omega \in \Omega^k(\mathbb{R}^n \times \mathbb{R}^1)$

$$\pi \cdot (x, t) \mapsto x$$

$$s \cdot t \mapsto (x, 0)$$

so if $\mu \in \Omega^k(\mathbb{R}^n)$

$$\pi^*(\mu) = \mu(x)$$

$$\omega = \underbrace{\sum_{i_1, \dots, i_k} \omega_{i_1, \dots, i_k}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_k}}_{\omega'} + \underbrace{\sum_{i_1, \dots, i_{k-1}} \omega_{i_1, \dots, i_{k-1}}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}}_{\omega''}$$

$$s^*\omega = \omega_{i_1, \dots, i_k}(x, 0) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$(1 - \pi^* \cdot \xi^*) \omega' = \omega'(s, t)$$

$$d\omega' = dt' \frac{\partial \omega'}{\partial x^*}$$

$$(1 - \pi^* \cdot S^*) \omega' = \omega'(v, t)$$

$$d\omega' = dt' \frac{\partial \omega'}{\partial x^r}$$

$$K(d\omega') = \int_{\mathcal{M}} \frac{1}{\lambda^r} \frac{\partial \omega'}{\partial x^r} dt^r \wedge dt^s \wedge \dots \wedge dt^k \wedge dt^l$$



$$(1 - \pi^* \cdot S^*) \omega' = \omega'(b, t)$$

$$d\omega' = dt' \frac{\partial \omega'}{\partial x^{\mu}}$$

$$K(d\omega') = \int_{\mathcal{M}} \omega'_i \wedge dx^1 \wedge \dots \wedge dx^{n-1} \wedge dt'$$

$$dK(\omega') = d \int_{\mathcal{M}} \omega'_i \wedge dx^1 \wedge \dots \wedge dx^{n-1} \wedge dt'$$

$$= K(d\omega') \pm \omega'$$

$$(1 - \pi^* \cdot S^*) \omega' = \omega'(0, t)$$

$$d\omega' = dt' \frac{\partial \omega'}{\partial x^*}$$

$$K(d\omega') = \int_{x^*}^1 \omega'_i \cdot dt^* \wedge dx^* \wedge \dots \wedge dx^{n-1} \wedge dt$$

$$dK(\omega') = d \int_{x^*}^1 \omega'_i \cdot dt^* \wedge dx^* \wedge \dots \wedge dx^{n-1} \wedge dt$$

$$= K(d\omega') \pm \omega' \Rightarrow (dK - Kd)\omega' = \omega' \cdot (\dots)$$

Poincaré lemma

$$\begin{array}{ccc} \mathbb{R}^n & \subset & \mathbb{R}^n \\ \downarrow & & \uparrow \\ \mathbb{R}^n & & \mathbb{R}^n \end{array}$$

$$\begin{array}{ccc} \Omega^p(\mathbb{R}^n \subset \mathbb{R}^n) & & \\ \uparrow & & \downarrow \\ \Omega^p(\mathbb{R}^n) & & \end{array}$$

$$\begin{array}{l} \pi: (x, t) \mapsto x \\ S: t \mapsto (x, 0) \end{array}$$

so if $M \in \Omega^p(\mathbb{R}^n)$
 $\pi^*(M) = M(x)$

If $\omega \in \Omega^p(\mathbb{R}^n \subset \mathbb{R}^n)$

$$\omega = \underbrace{u_{i_1, \dots, i_p}(x, t)}_{\omega'} dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge dt + \underbrace{\tilde{\omega}_{i_1, \dots, i_p}(x, t)}_{\tilde{\omega}} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$S^* \omega = \omega_{i_1, \dots, i_p}(x, 0) dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$\pi \circ S = \text{id} \Leftrightarrow \text{id} = (\pi \circ S)^* = S^* \circ \pi^*$$

But $\pi^* \circ S^* \neq \text{id}$ in general. But perhaps $\pi^* \circ S^*$ is equivalent to the identity on

cozy? Yes!

$$\text{id} = S^* \circ \pi^* = \pi^* \circ S^* \text{ on cohomology}$$

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$$\text{id} = s^* \circ \pi^* = \pi^* \circ s^* \quad \text{on cohomology}$$

$$H^p(\mathbb{R}^n \times \mathbb{R}^1) = H^p(\mathbb{R}^n)$$

Poincaré lemma

$$M \subset \mathbb{R}^n$$

$$\pi \downarrow \uparrow \sigma$$

$$M$$

$$\Omega^0(M \subset \mathbb{R}^n)$$

$$\pi^* \uparrow \downarrow \sigma^*$$

$$\Omega^k(\mathbb{R}^n)$$

If $\omega \in \Omega^k(\mathbb{R}^n \subset \mathbb{R}^n)$

$$H^k(M \subset \mathbb{R}^n) = H^k(M)$$

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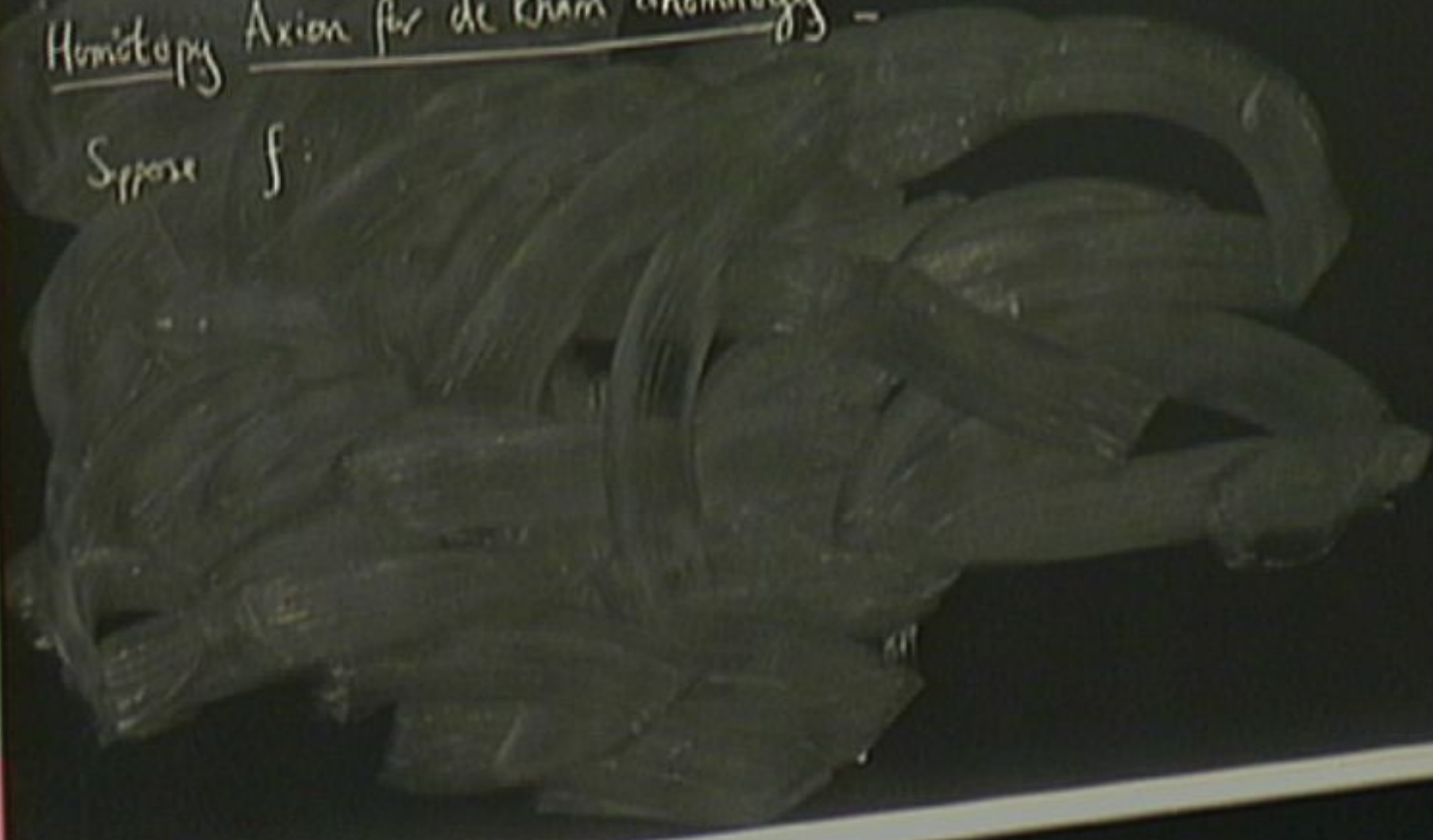
$$\pi^*(\mu) = \mu(x)$$

$$\omega = \underbrace{\sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_k}}_{\omega'} + \underbrace{\sum_{i_1 < \dots < i_{k-1}} \omega_{i_1, \dots, i_{k-1}}(x, t) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}}_{\omega''}$$

$$\sigma^* \omega = \sum_{i_1 < \dots < i_k} \omega_{i_1, \dots, i_k}(x, 0) dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

Homotopy Axiom for de Rham cohomology -

Suppose f :



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Suppose $f: M \rightarrow N$ and $g: M \rightarrow N$ are homotopic

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$M \subset \mathbb{R}^n$

$\begin{array}{c} \uparrow \\ \uparrow \\ M \end{array}$

$f = F \circ s_1 \quad g = F \circ s_0$

$h: N \rightarrow M$

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and suppose $h \circ f$ is ~~isomorphic~~ to identity



$$h: N \rightarrow M$$

and suppose $h \circ f$ is ~~isomorphic~~ to identity



$h: N \rightarrow M$
and suppose $h \circ f$ is homotopic to identity on M , and $f \circ h$ is homotopic to id
on N . then

$$H^p(N, \mathbb{R}) = H^p(M, \mathbb{R})$$