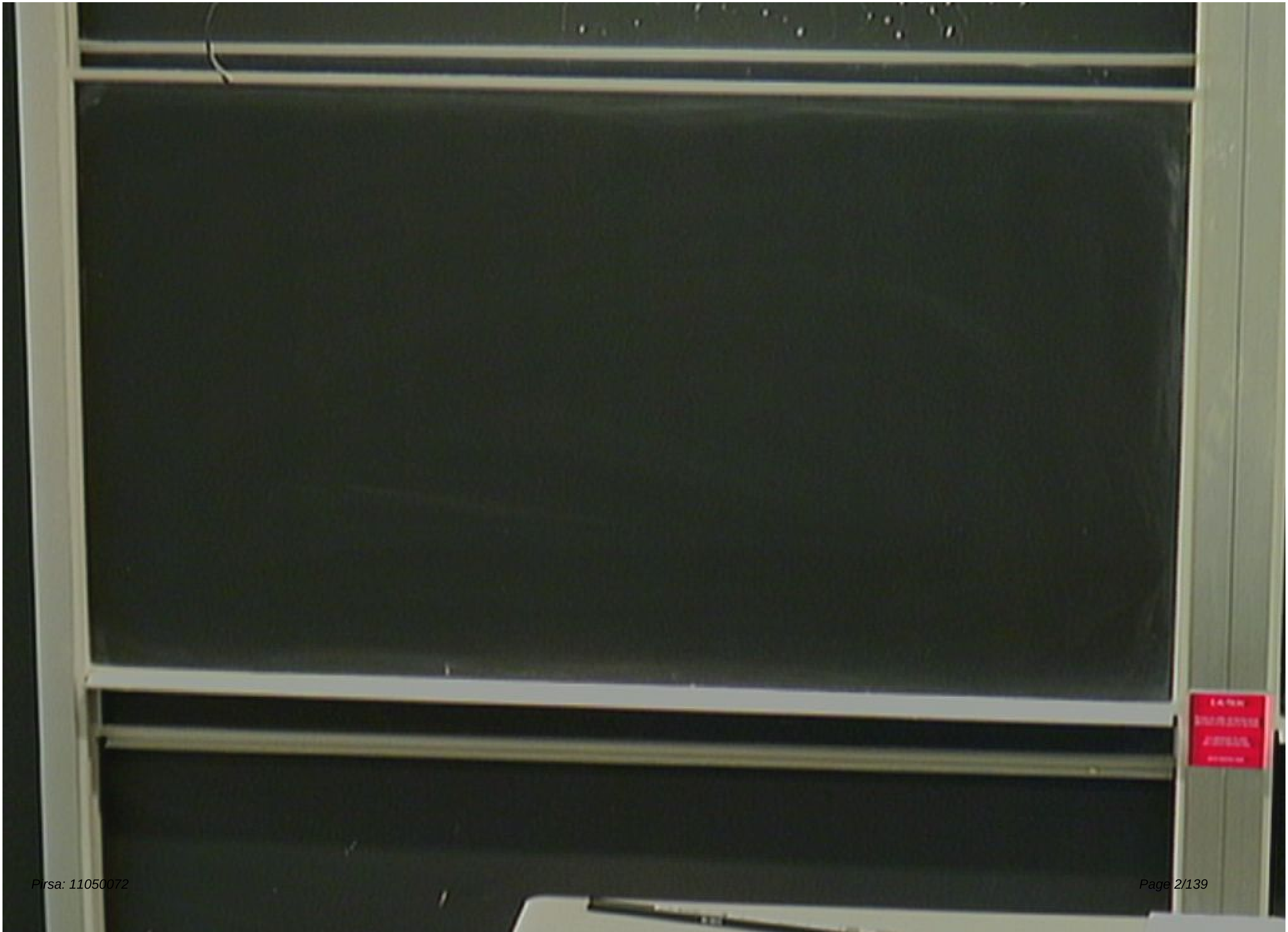
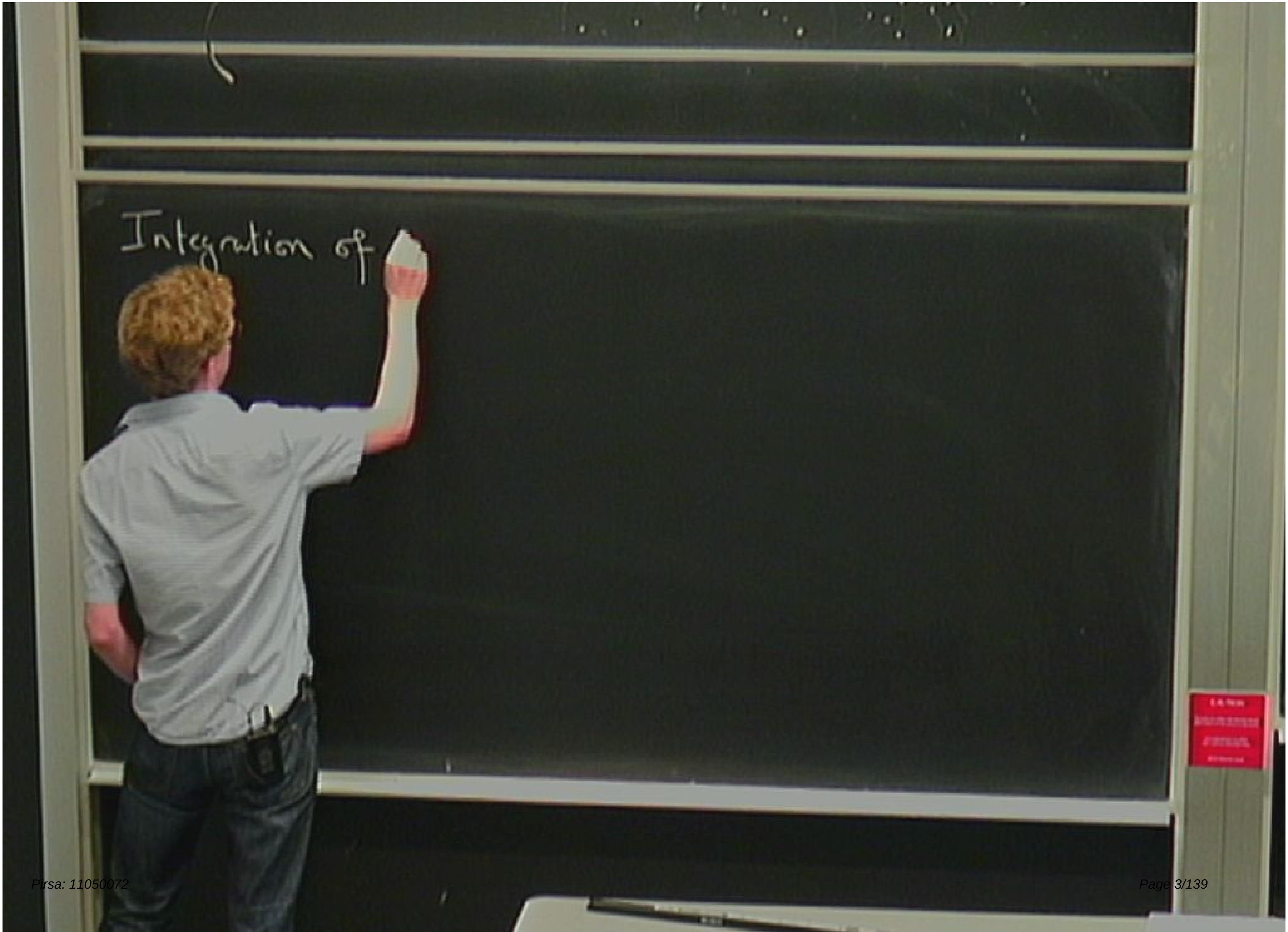
Title: Geometry & Topology for Physics - Lecture 3

Date: May 30, 2011 02:00 PM

URL: http://pirsa.org/11050072

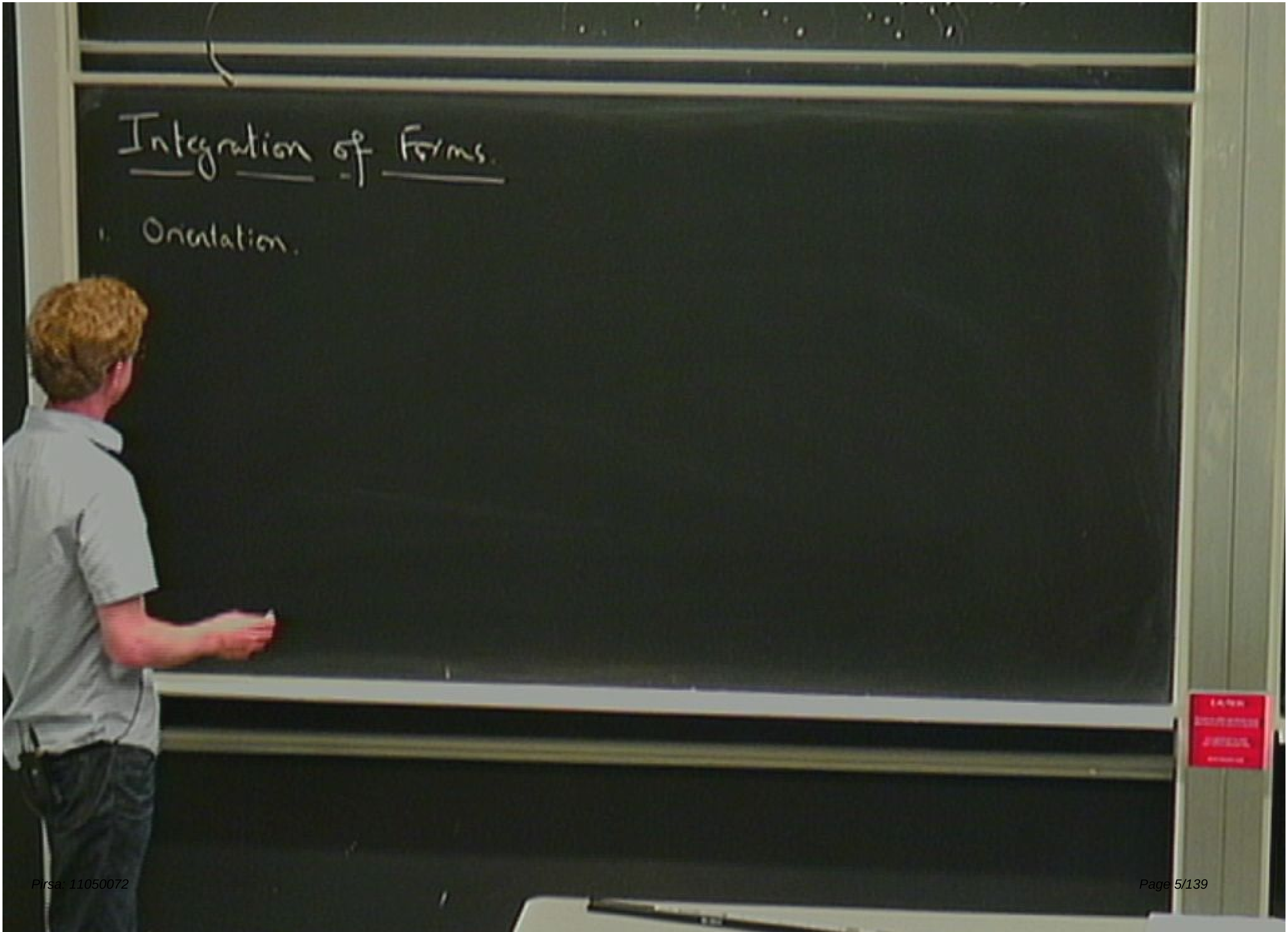
Abstract:





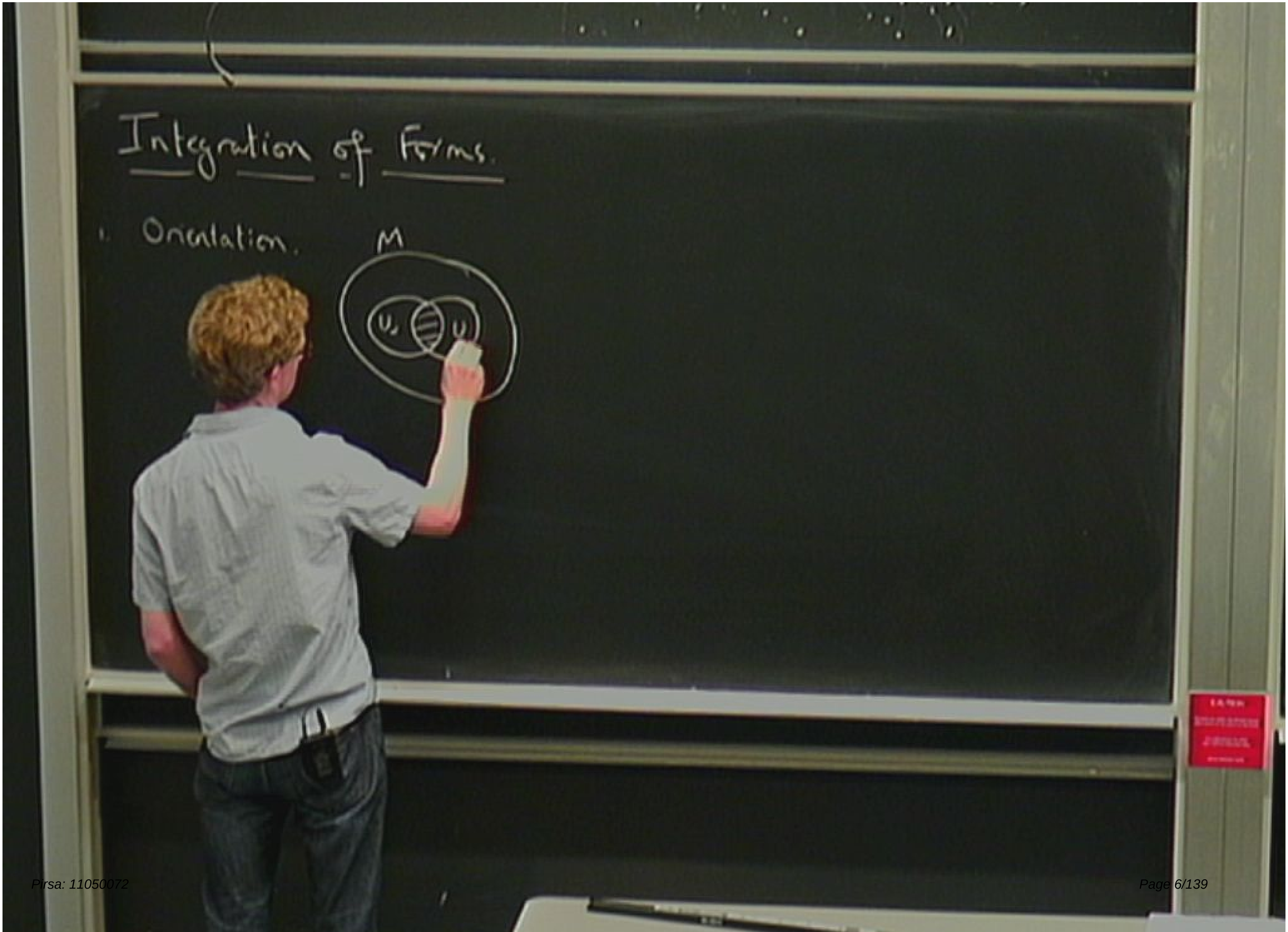
Integration of

# Integration of Forms



# Integration of Forms.

1. Orientation.



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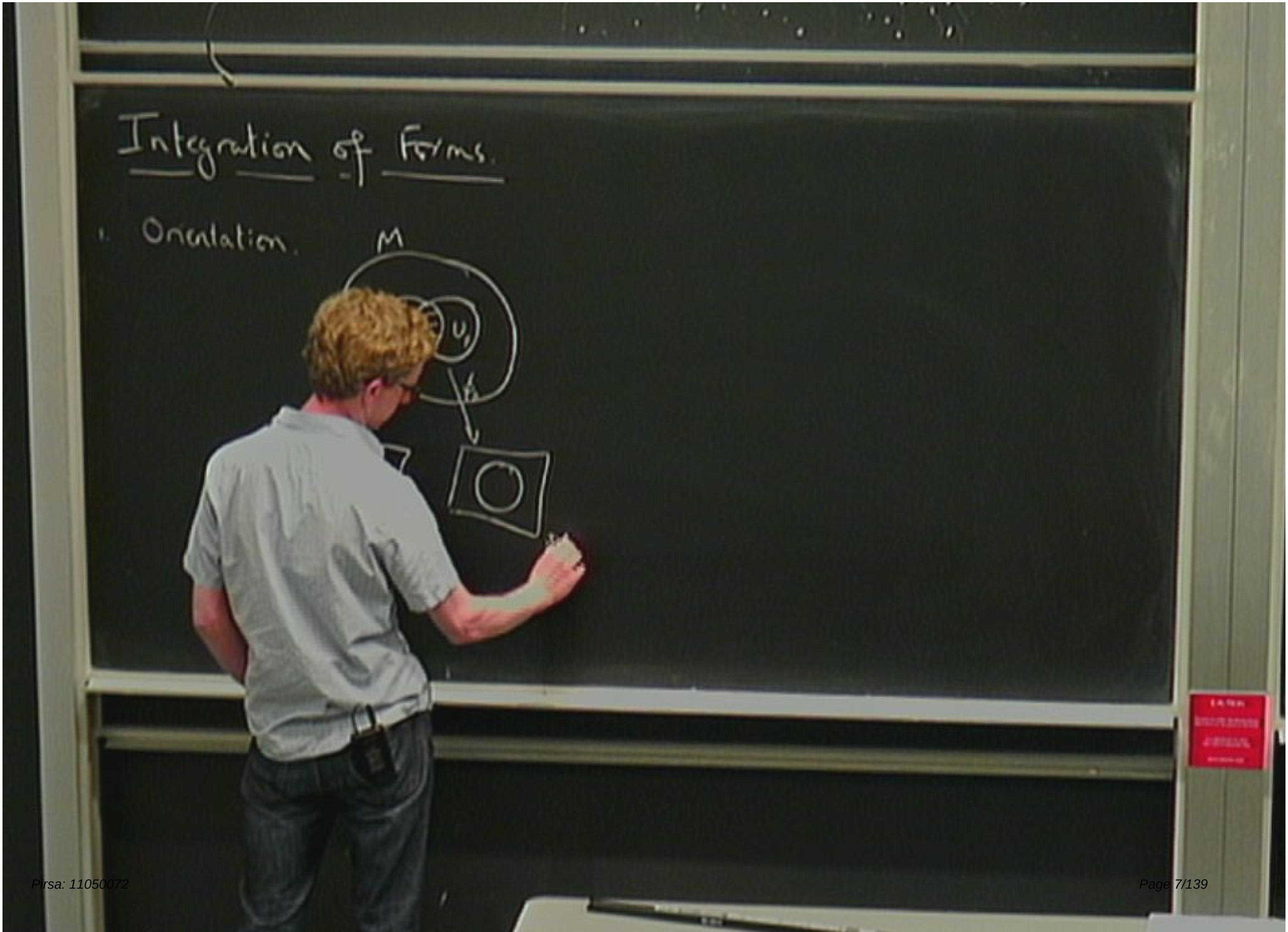

Integration of Forms.

1. Orientation.

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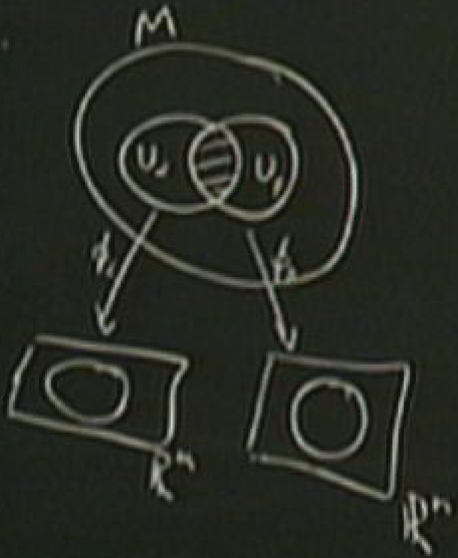
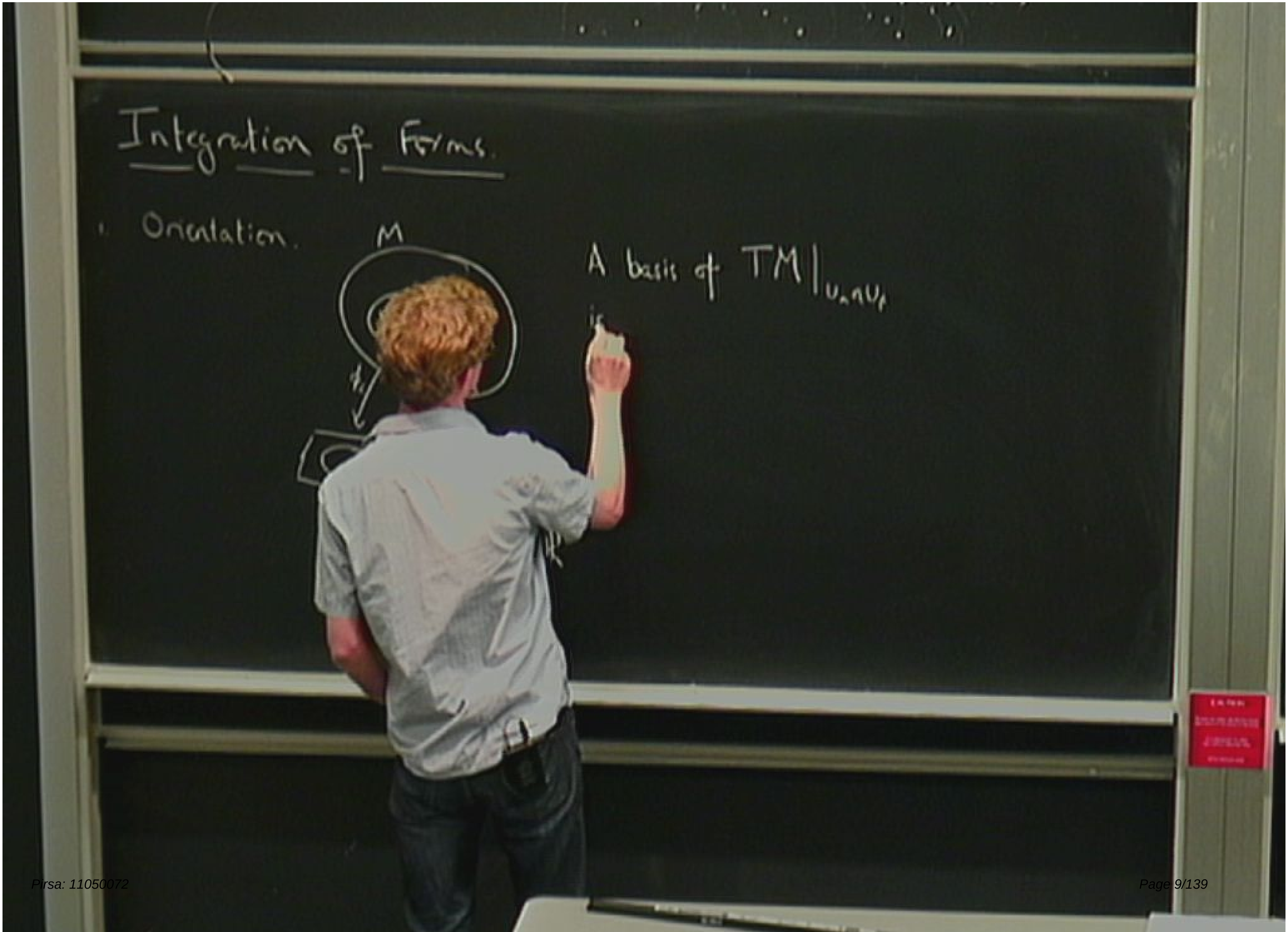


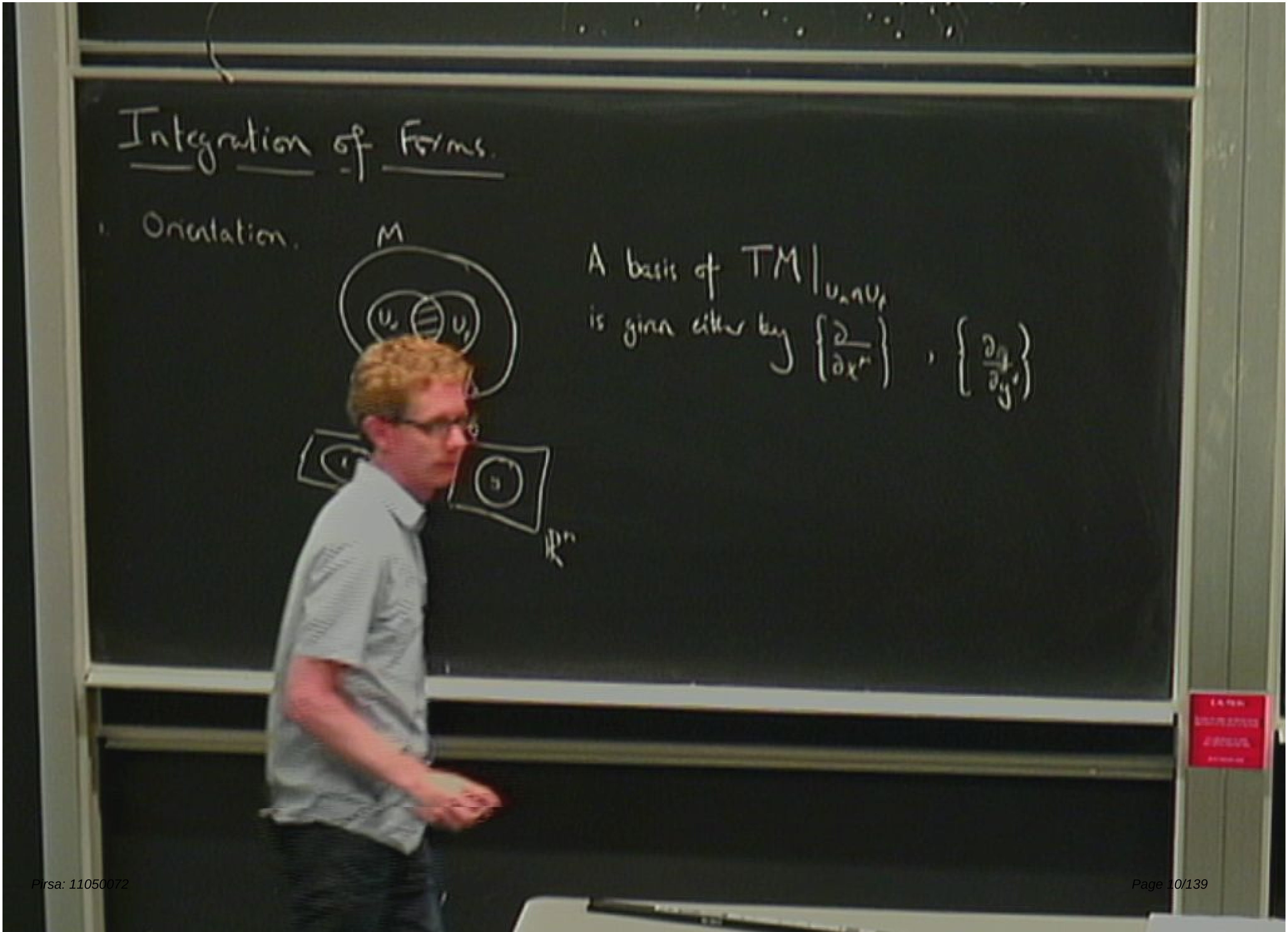


# Integration of Forms.

1. Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$

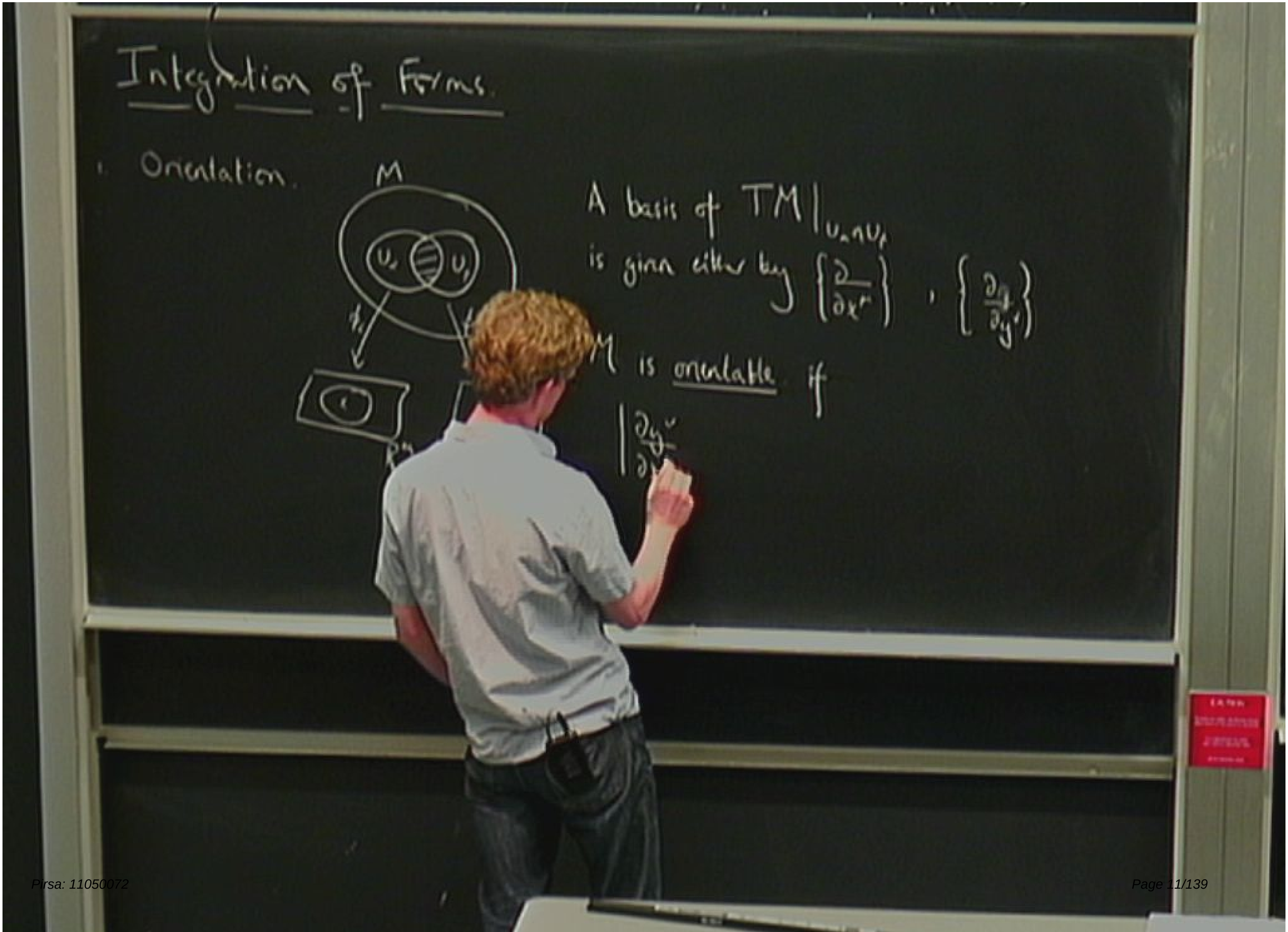
is



# Integration of Forms.

1. Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$, $\left\{\frac{\partial}{\partial y^\nu}\right\}$



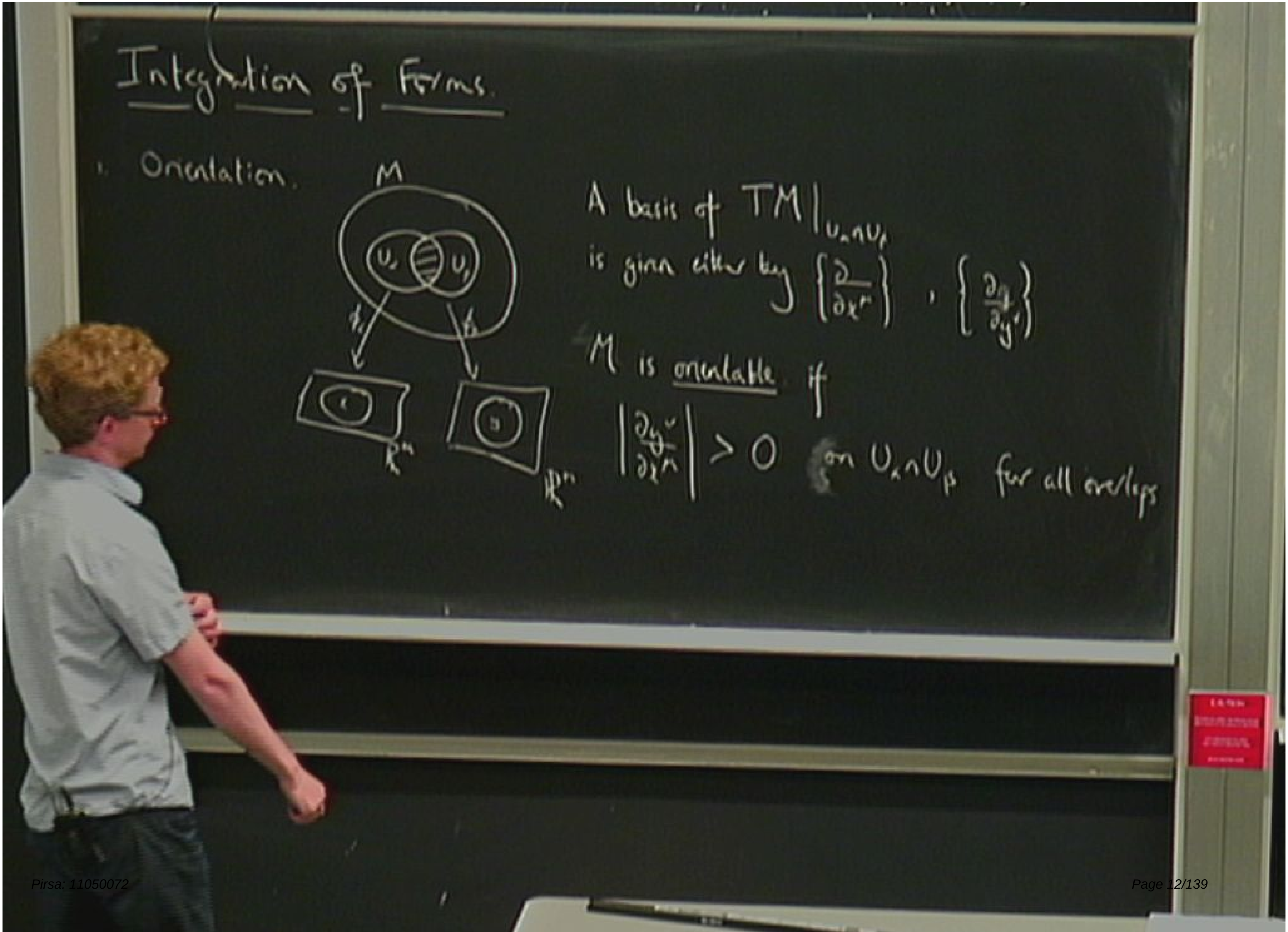
# Integration of Forms.

1. Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$, $\left\{\frac{\partial}{\partial y^\nu}\right\}$

M is orientable if

$\left|\frac{\partial y^\nu}{\partial x}\right.$



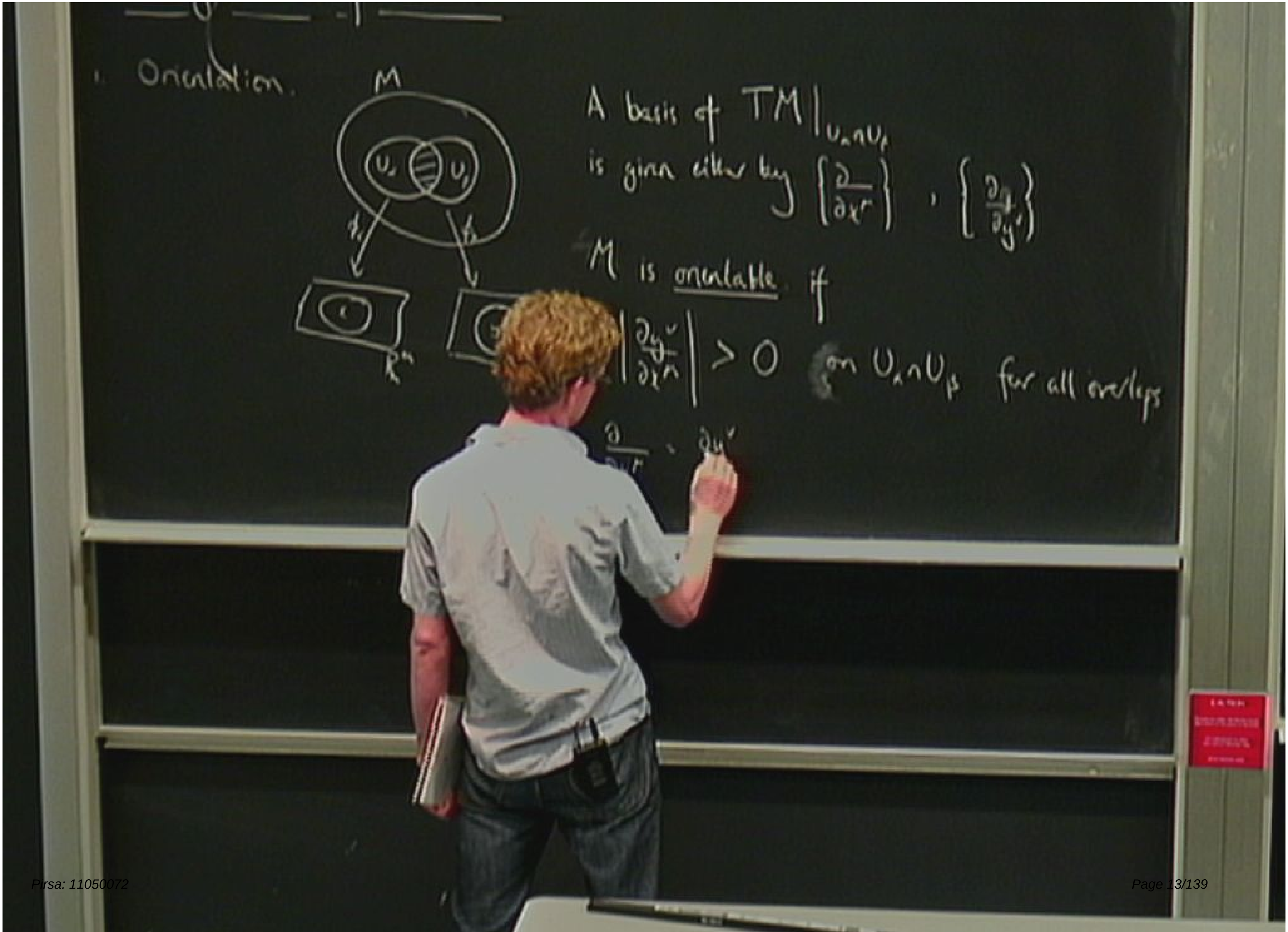
# Integration of Forms.

1. Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$, $\left\{\frac{\partial}{\partial y^\nu}\right\}$

M is <u>orientable</u> if

$$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0$$ on $U_\alpha \cap U_\beta$ for all overlaps



1. Orientation.

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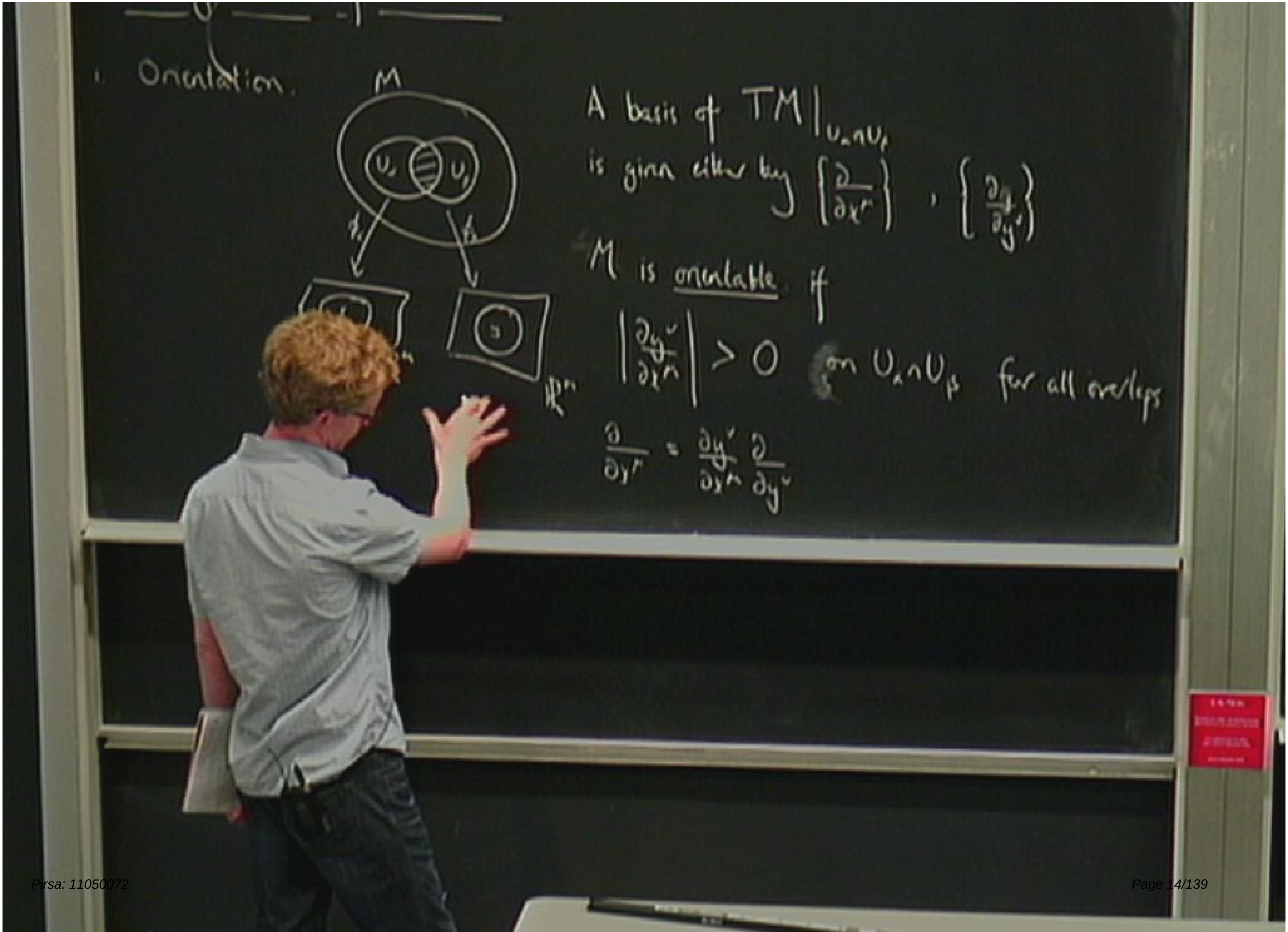
M is <u>orientable</u> if

$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$\frac{\partial}{\partial x^\mu} = \partial y^\nu$



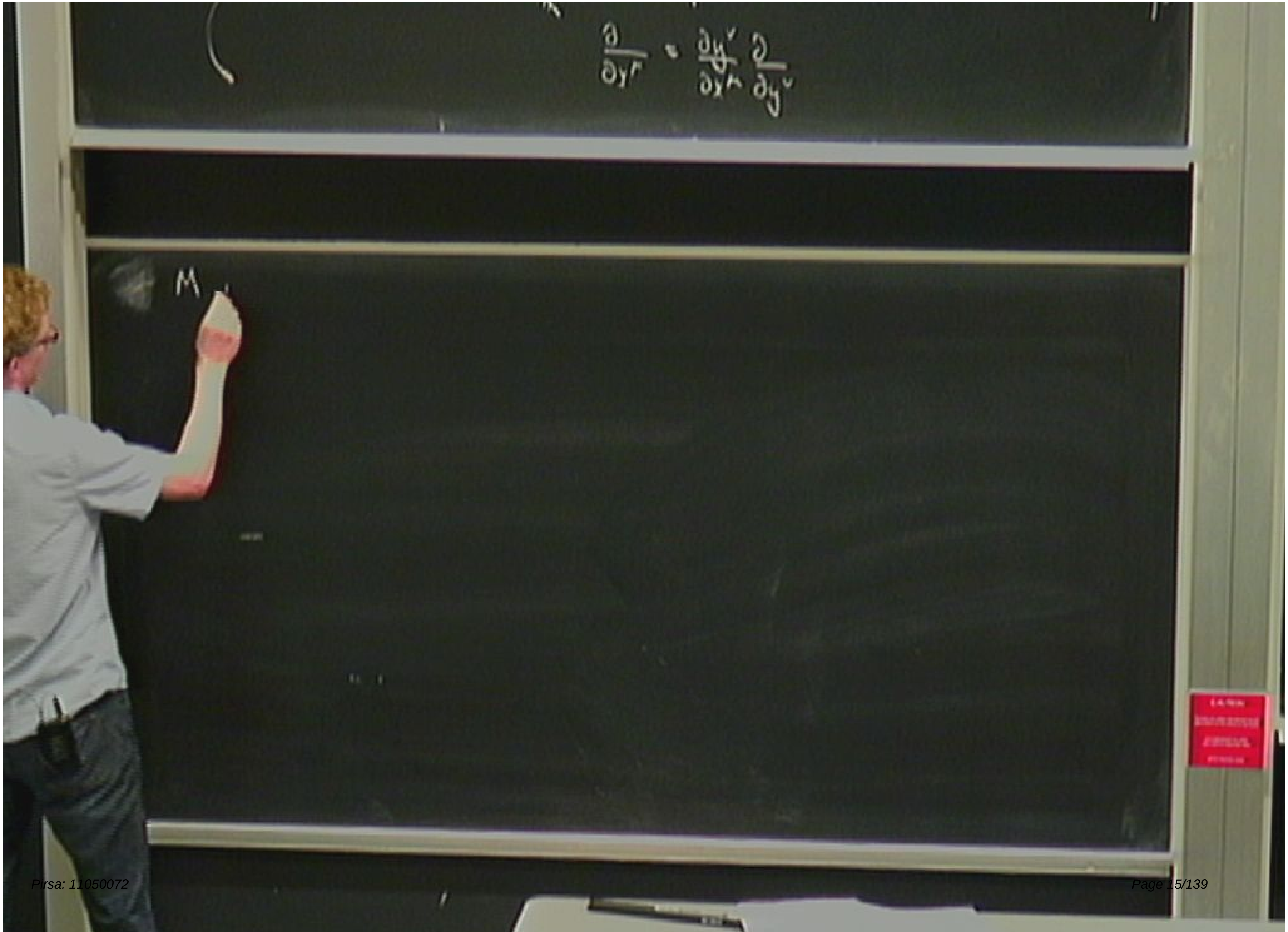
Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$, $\left\{\frac{\partial}{\partial y^\nu}\right\}$

M is <u>orientable</u> if

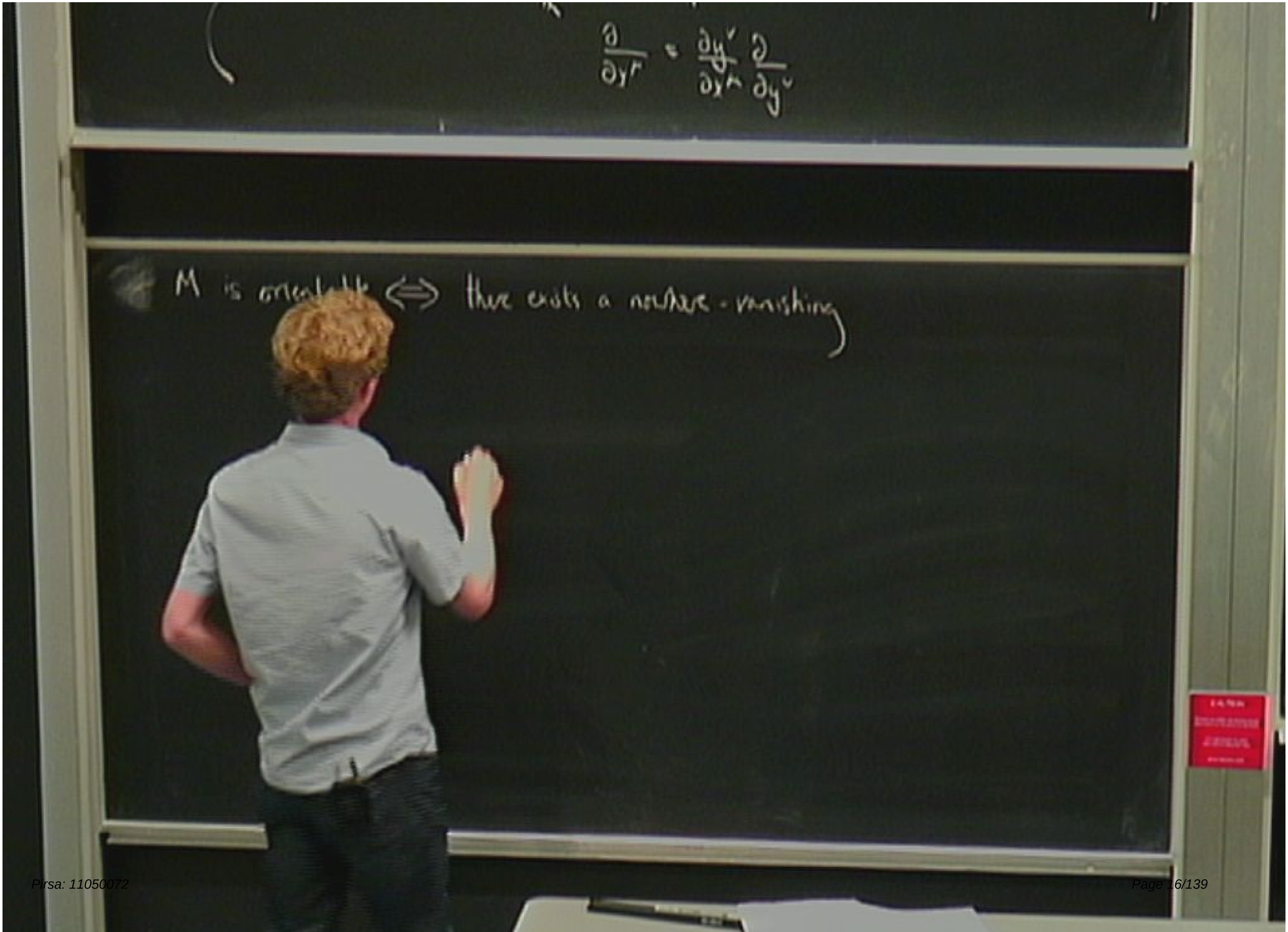
$$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0 \quad \text{on } U_\alpha \cap U_\beta \quad \text{for all overlaps}$$

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$



$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$$

M

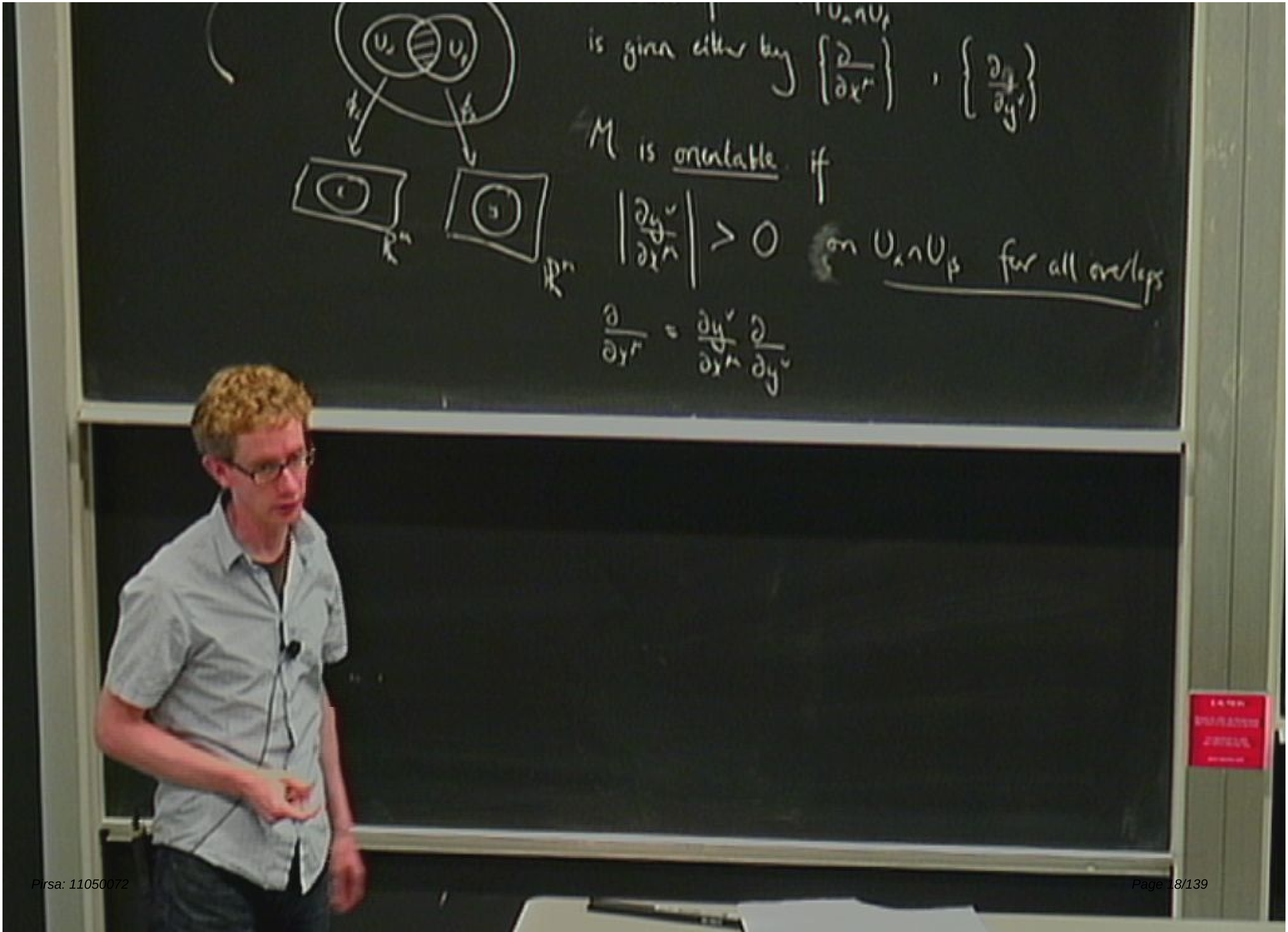


$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

M is orientable $\Longleftrightarrow$ there exists a nowhere-vanishing

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

M is orientable $\Longleftrightarrow$ there exists a nowhere-vanishing, smooth top degree form $\omega$ on M )



$U_\alpha \cap U_\beta$

is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$ , $\left\{\frac{\partial}{\partial y^\nu}\right\}$

$M$ is <u>orientable</u> if

$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

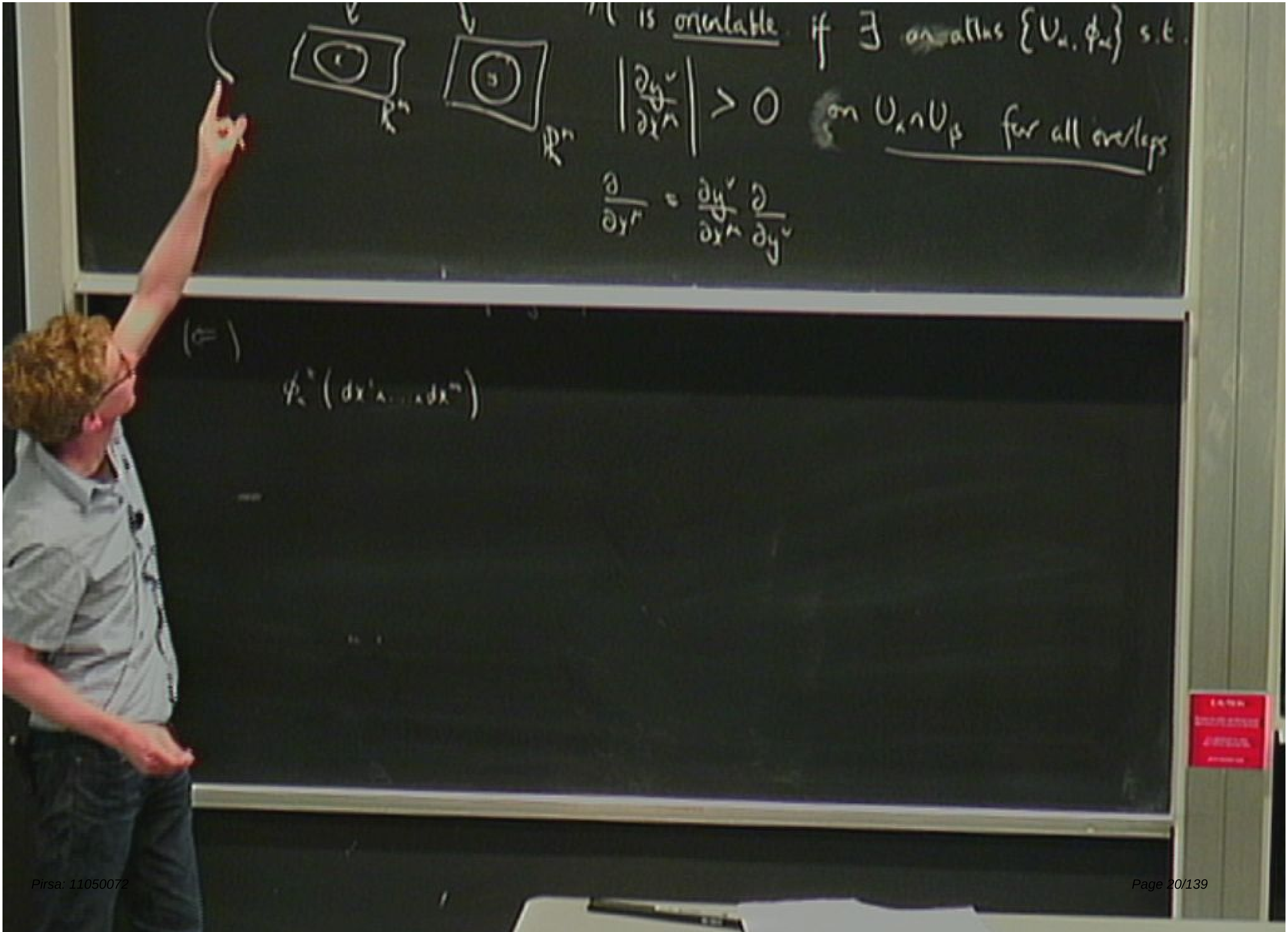
Orientation.

A basis of $TM|_{U_\alpha \cap U_\beta}$ is given either by $\left\{\frac{\partial}{\partial x^\mu}\right\}$, $\left\{\frac{\partial}{\partial y^\nu}\right\}$

$M$ is <u>orientable</u> if $\exists$ an atlas $\{U_\alpha, \phi_\alpha\}$ s.t.

$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$= \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$



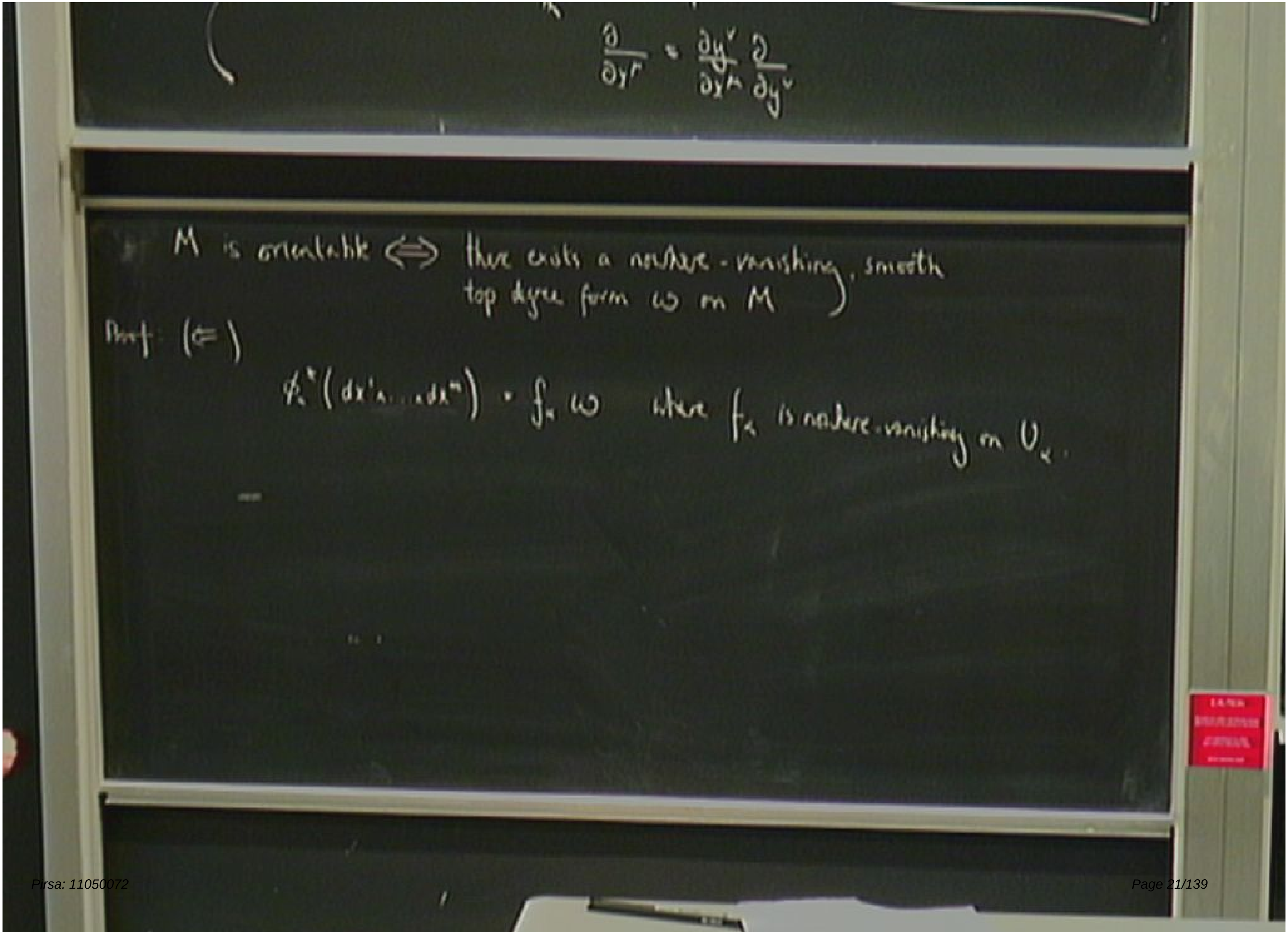
$M$ is orientable if $\exists$ an atlas $\{U_\alpha, \phi_\alpha\}$ s.t.

$$\left|\frac{\partial y^\nu}{\partial x^\mu}\right| > 0 \text{ on } U_\alpha \cap U_\beta \text{ for all overlaps}$$

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

$(\Leftarrow)$

$$\phi_\alpha^*(dx^1 \wedge \ldots \wedge dx^m)$$

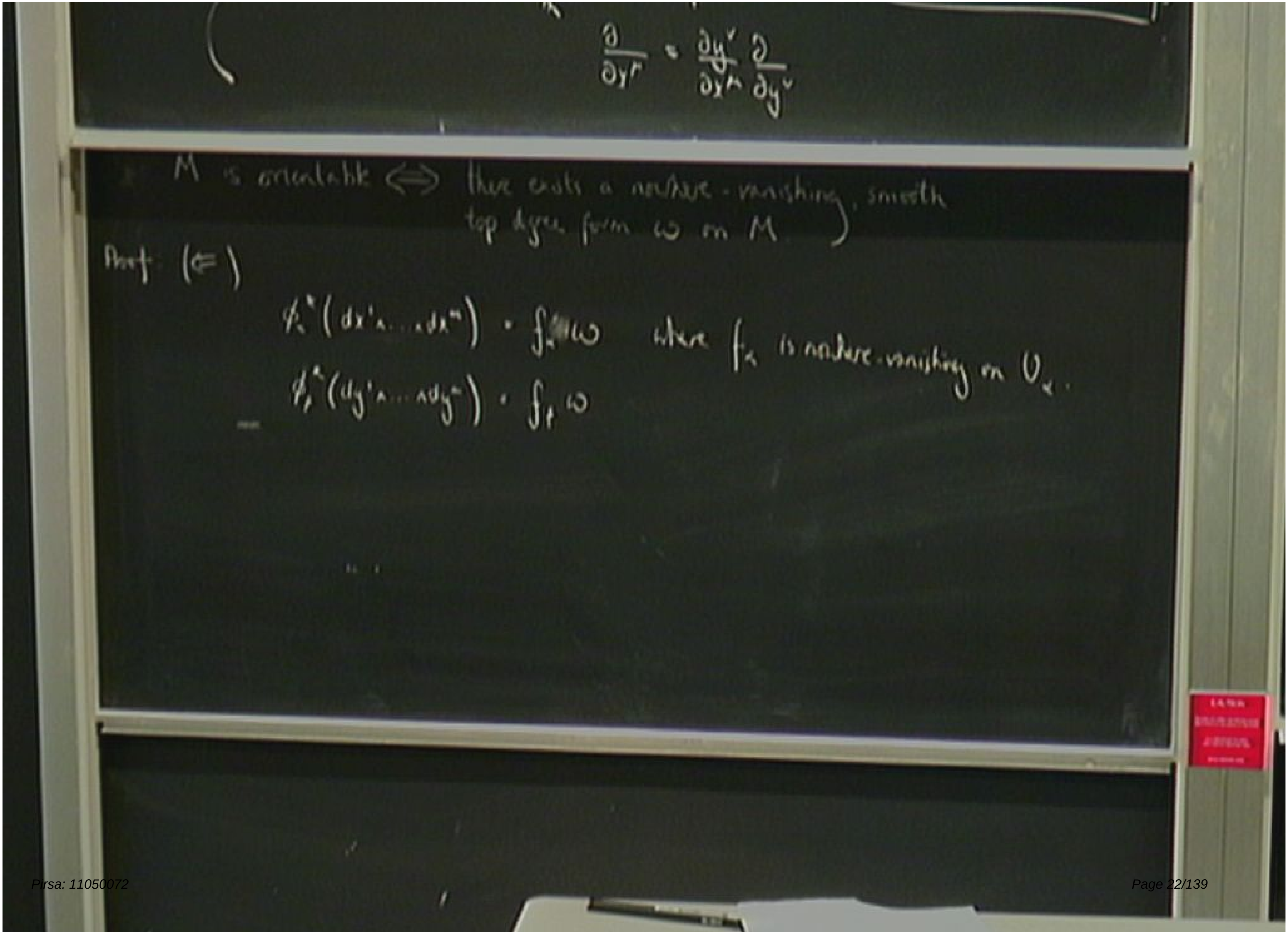


$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

M is orientable $\Longleftrightarrow$ there exists a nowhere-vanishing, smooth top degree form $\omega$ on M

Proof: $(\Leftarrow)$

$\phi_\alpha^*(dx^1\wedge\ldots\wedge dx^m) = f_\alpha\,\omega$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$.



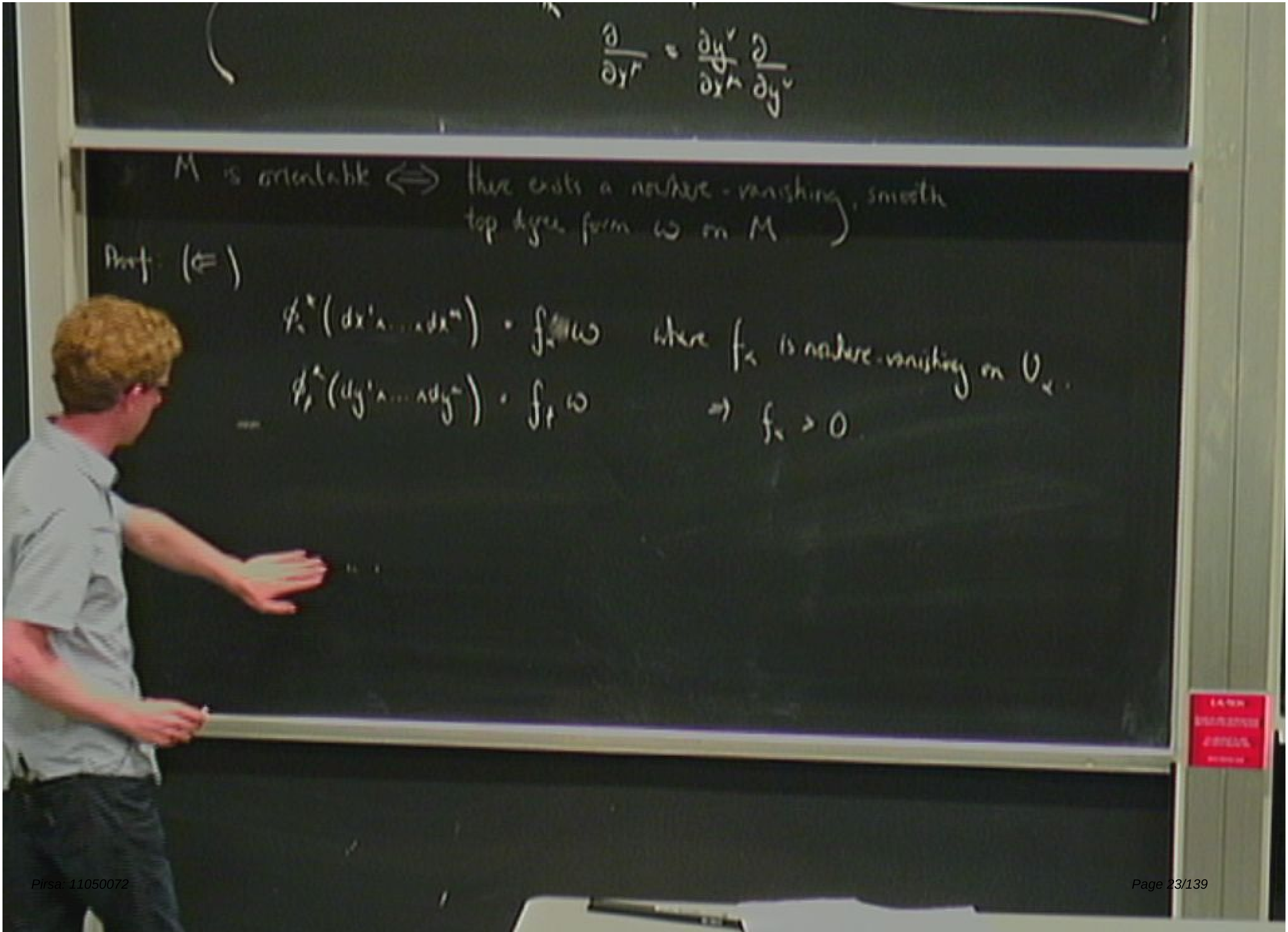
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

M is orientable $\Longleftrightarrow$ there exists a nowhere-vanishing, smooth top degree form $\omega$ on M.

Proof: ($\Leftarrow$)

$$\phi_\alpha^*(dx^1\wedge\ldots\wedge dx^m) = f_\alpha\,\omega$$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$.

$$\phi_\beta^*(dy^1\wedge\ldots\wedge dy^m) = f_\beta\,\omega$$



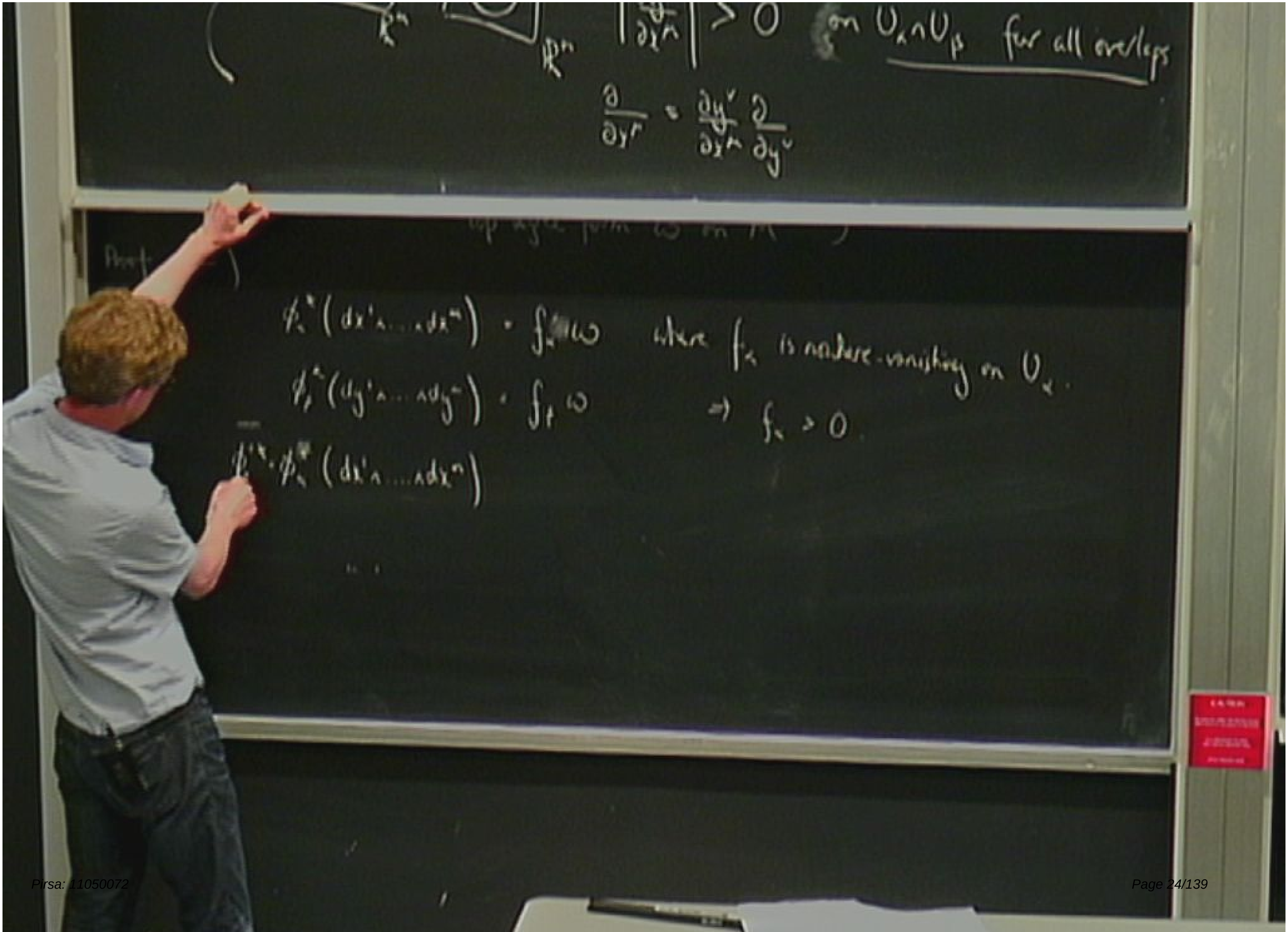
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$$

$M$ is orientable $\Longleftrightarrow$ there exists a nowhere-vanishing, smooth top degree form $\omega$ on $M$

Proof: $(\Leftarrow)$

$$\phi_\alpha^*(dx^1 \wedge \dots \wedge dx^n) = f_\alpha \omega$$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$.

$$\phi_\beta^*(dy^1 \wedge \dots \wedge dy^n) = f_\beta \omega$$ $\Rightarrow$ $f_\alpha > 0$.



$\mathbb{R}^n$ $\mathbb{R}^n$ $\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

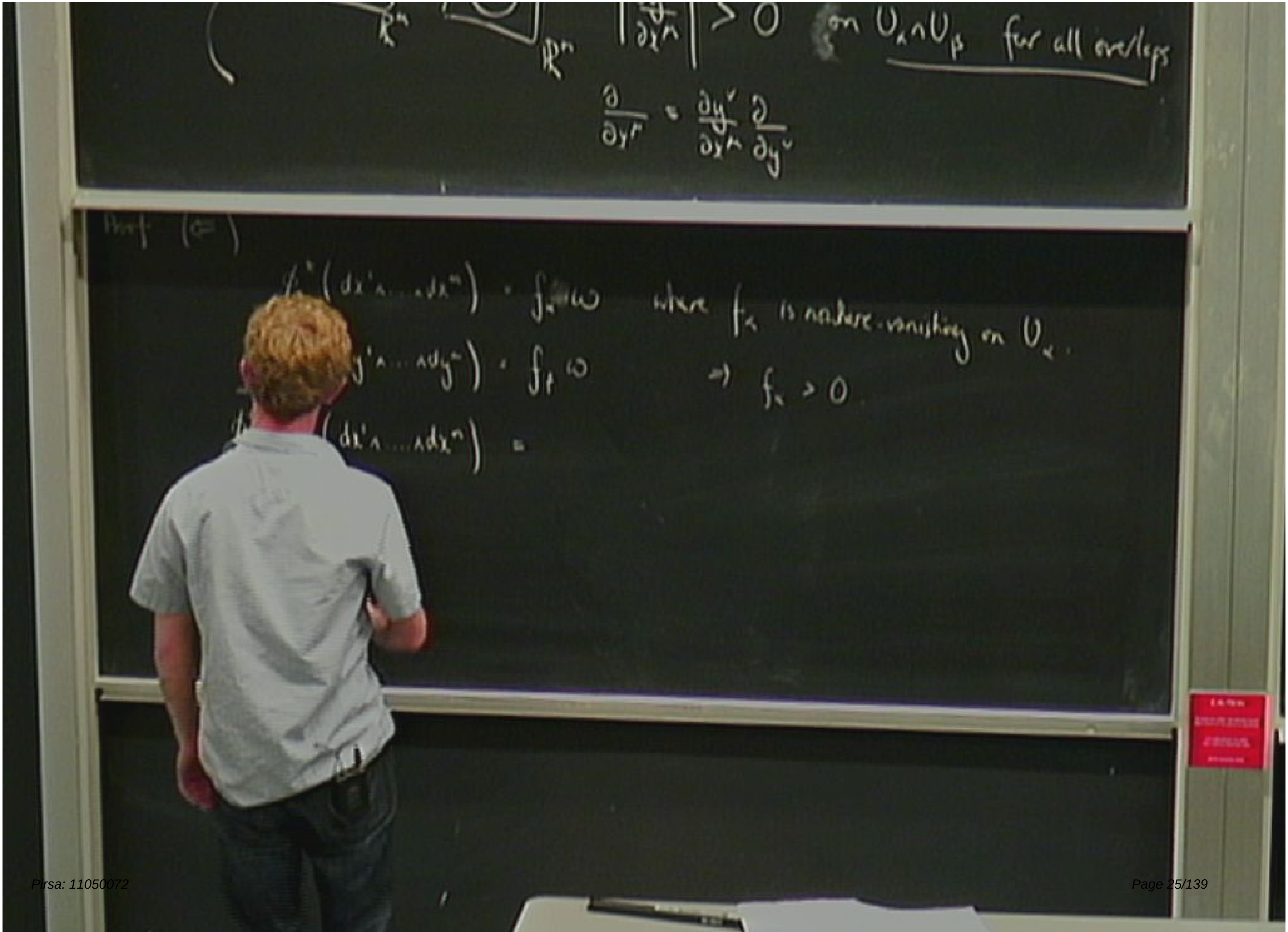
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

Proof

$\phi_\alpha^{*}(dx^1\wedge\ldots\wedge dx^n) = f_\alpha\,\omega$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$.

$\phi_\beta^{*}(dy^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega$ $\Rightarrow$ $f_\alpha > 0$.

$\phi_\beta^{-1*}\cdot\phi_\alpha^{*}(dx^1\wedge\ldots\wedge dx^n)$



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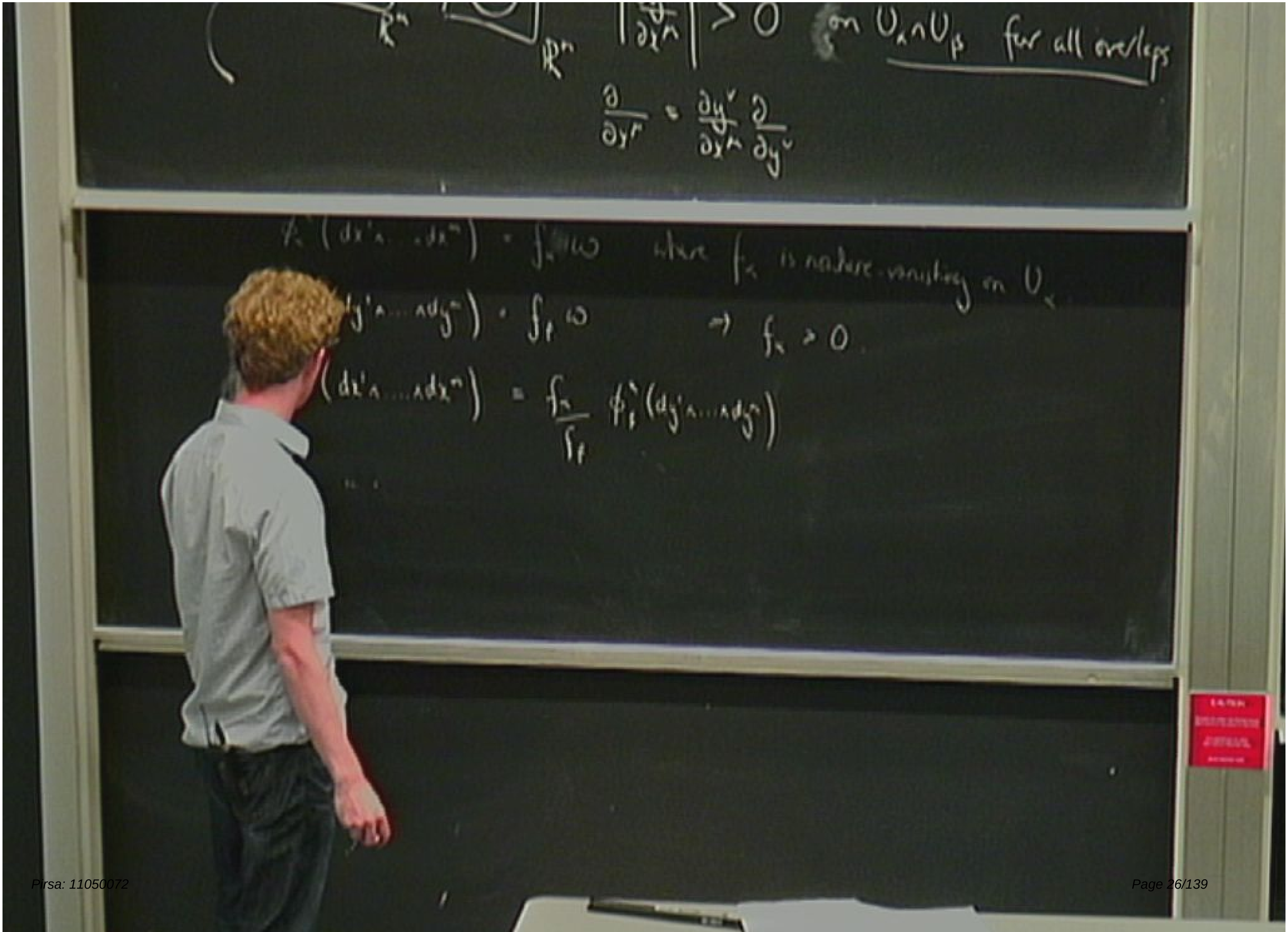
Proof ($\Leftarrow$)

$\phi^*(dx^1\wedge\ldots\wedge dx^n) = f_\alpha\,\omega$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$.

$y^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega$ $\Rightarrow$ $f_\alpha > 0$.

$(dx^1\wedge\ldots\wedge dx^n) =$



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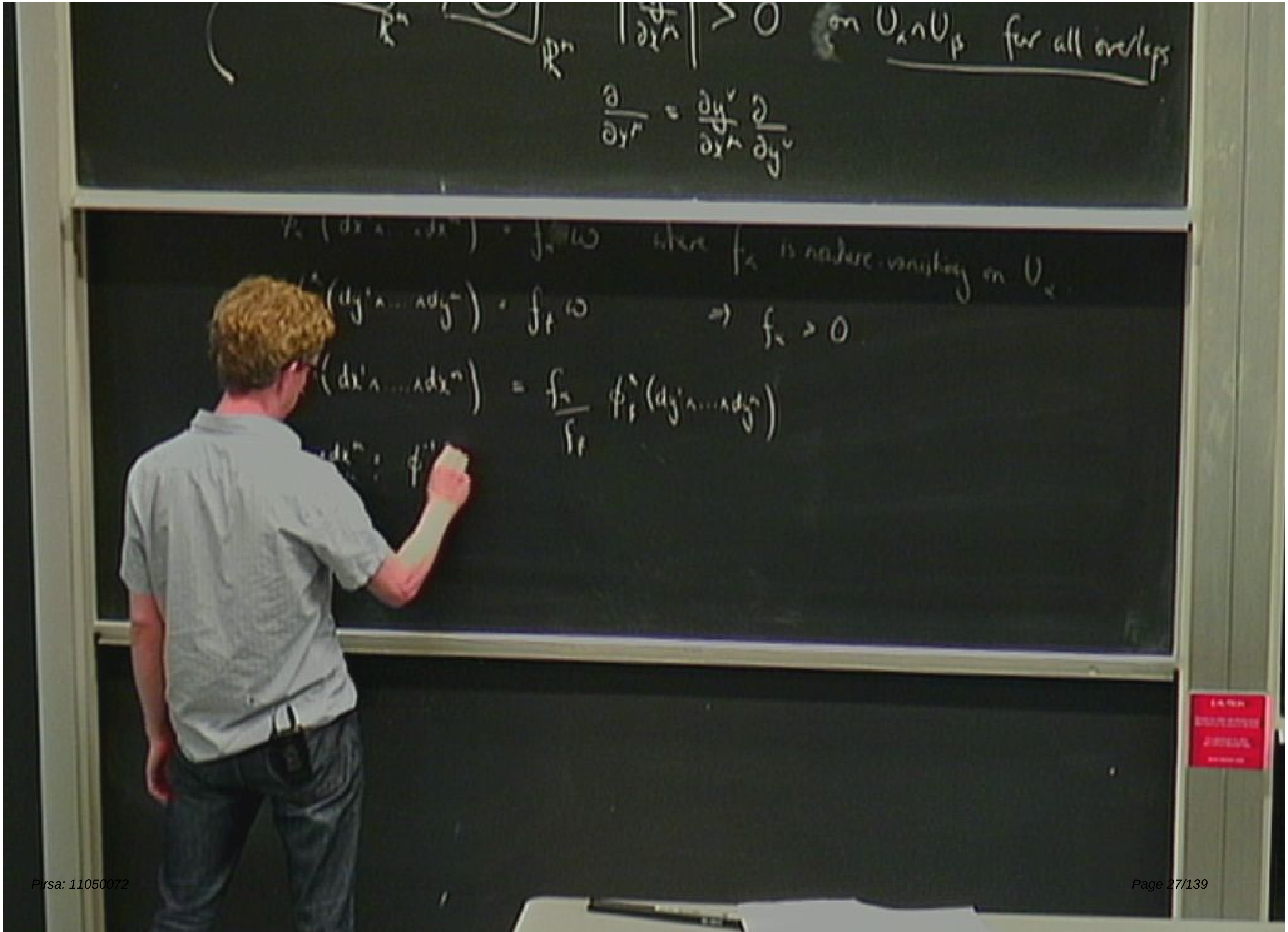
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}$$

$\phi_\alpha^*(dx^1 \wedge \ldots \wedge dx^n) = f_\alpha \omega$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$

$dy^1 \wedge \ldots \wedge dy^n) = f_\beta \omega$ $\Rightarrow$ $f_\alpha > 0$.

$(dx^1 \wedge \ldots \wedge dx^n) = \frac{f_\alpha}{f_\beta} \phi_\beta^*(dy^1 \wedge \ldots \wedge dy^n)$



$\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

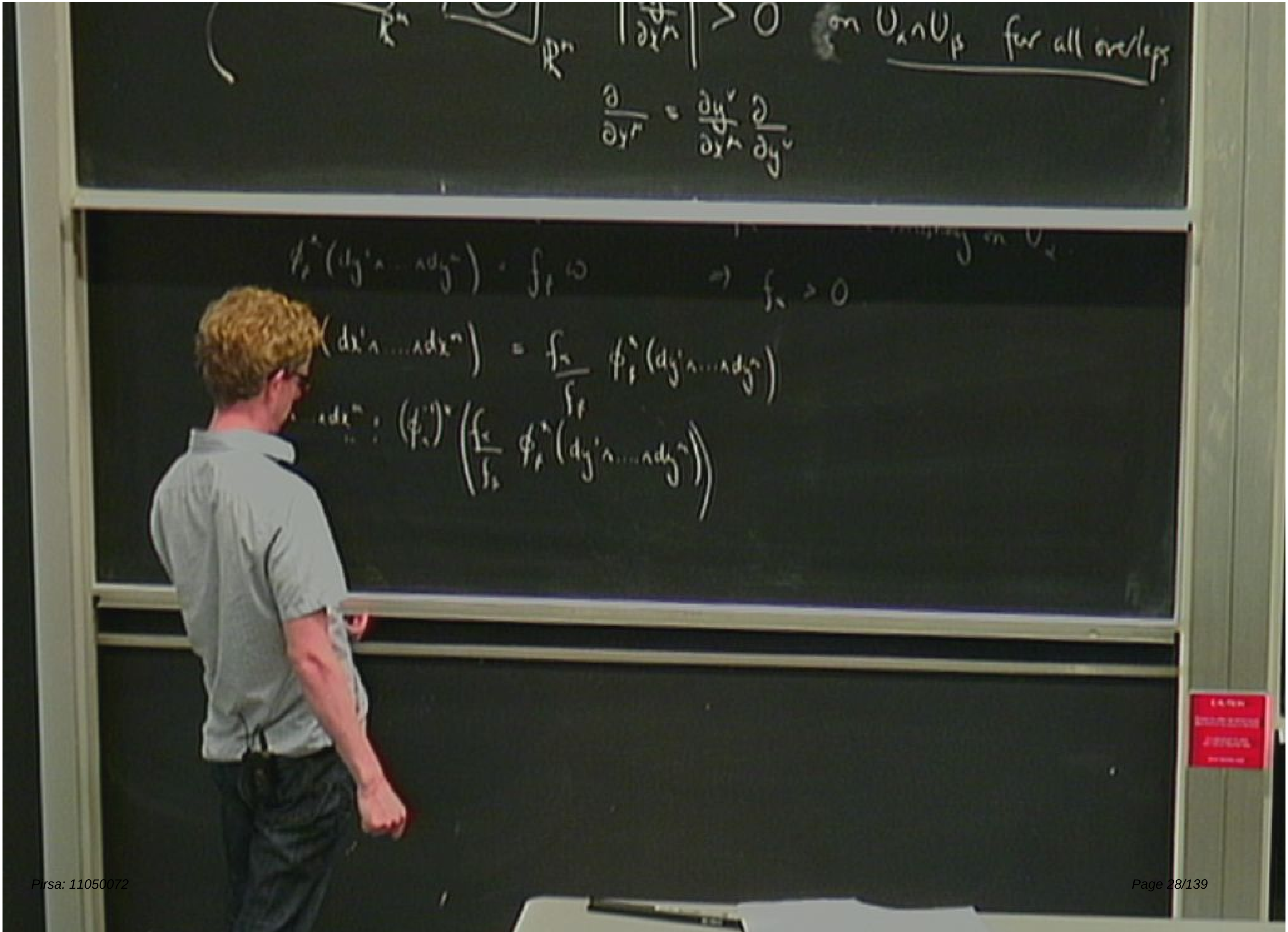
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

$\phi_\alpha^*(dx^1\wedge\dots\wedge dx^n) = f_\alpha\,\omega$ where $f_\alpha$ is nowhere-vanishing on $U_\alpha$

$\phi_\beta^*(dy^1\wedge\dots\wedge dy^n) = f_\beta\,\omega$ $\Rightarrow$ $f_\alpha > 0$

$(dx^1\wedge\dots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\,\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$

$dx^n$ : $\phi^{*}$



$\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

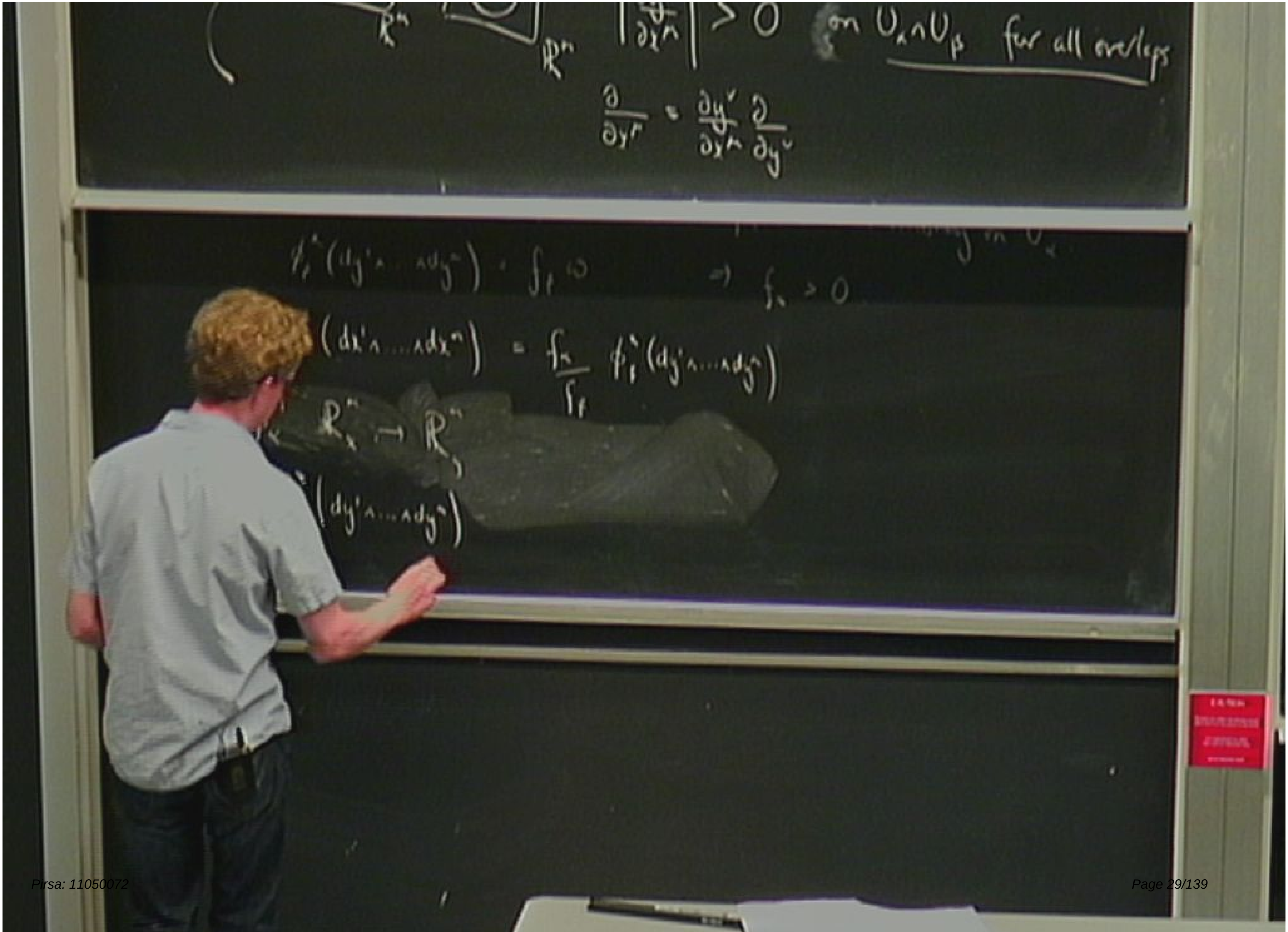
$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

vanishing on $U_\alpha$

$$\phi_\beta^{*}(dy^1\wedge\dots\wedge dy^n) = f_\beta\,\omega \quad \Rightarrow \quad f_\alpha > 0$$

$$(dx^1\wedge\dots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\,\phi_\beta^{*}(dy^1\wedge\dots\wedge dy^n)$$

$$\dots\wedge dx^n = (\phi_\alpha^{-1})^{*}\left(\frac{f_\alpha}{f_\beta}\,\phi_\beta^{*}(dy^1\wedge\dots\wedge dy^n)\right)$$



$\mathbb{R}^n$ $\mathbb{R}^n$ $\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

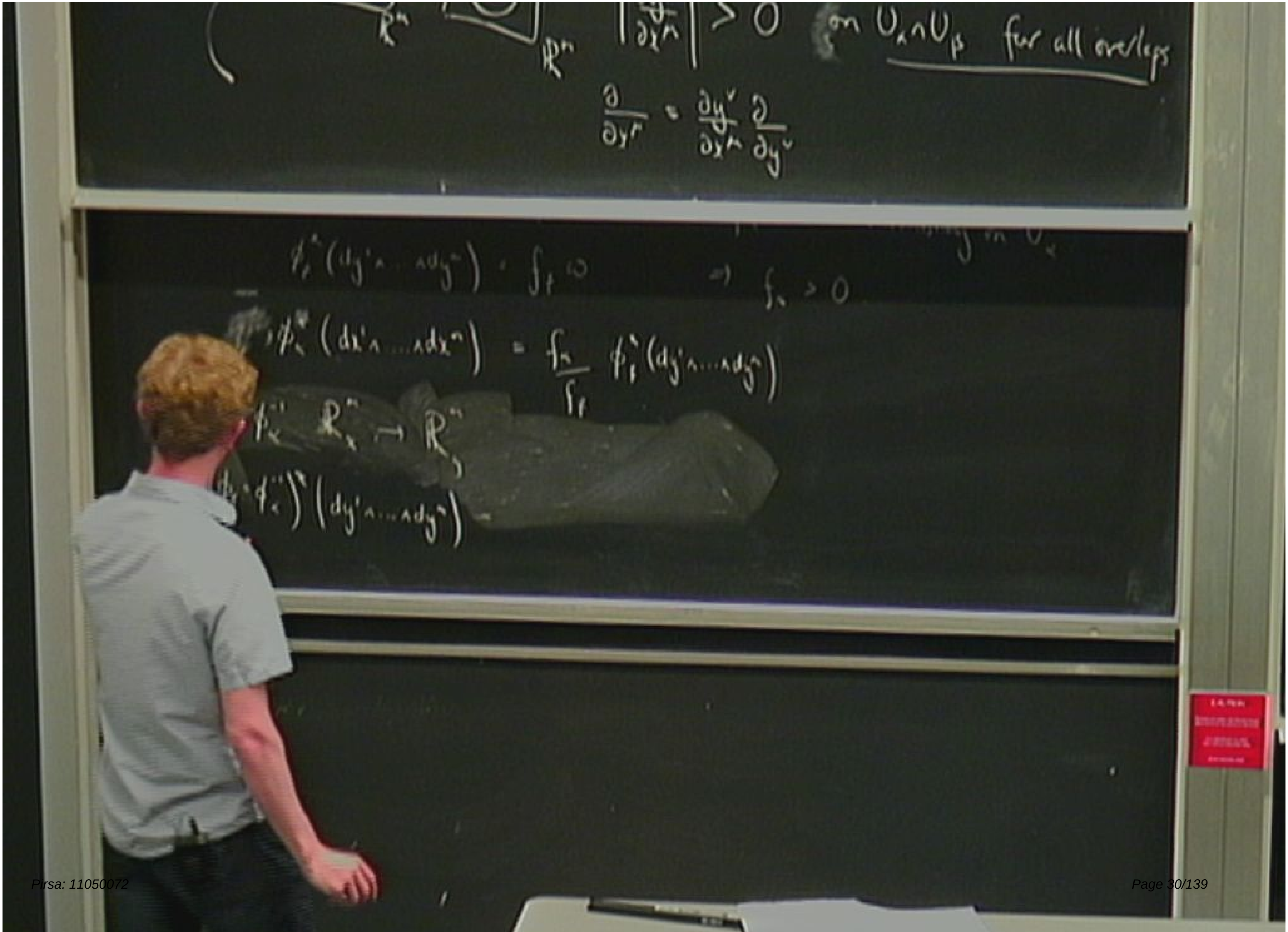
on $U_\alpha$

$\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega$ $\Rightarrow$ $f_\alpha > 0$

$(dx^1\wedge\ldots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$

$\mathbb{R}^n_x \to \mathbb{R}^n_y$

$(dy^1\wedge\ldots\wedge dy^n)$



$\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

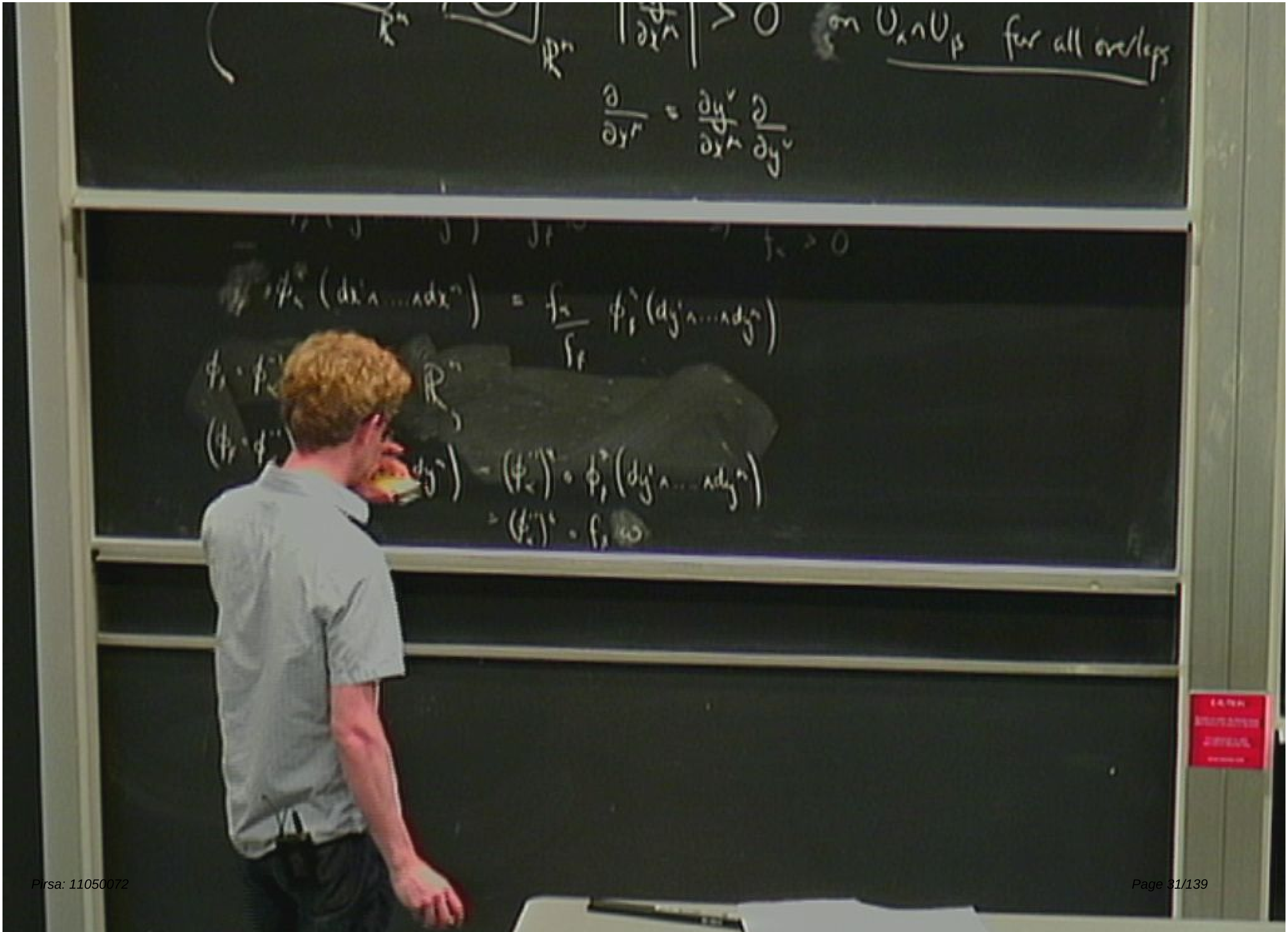
on $U_\alpha$

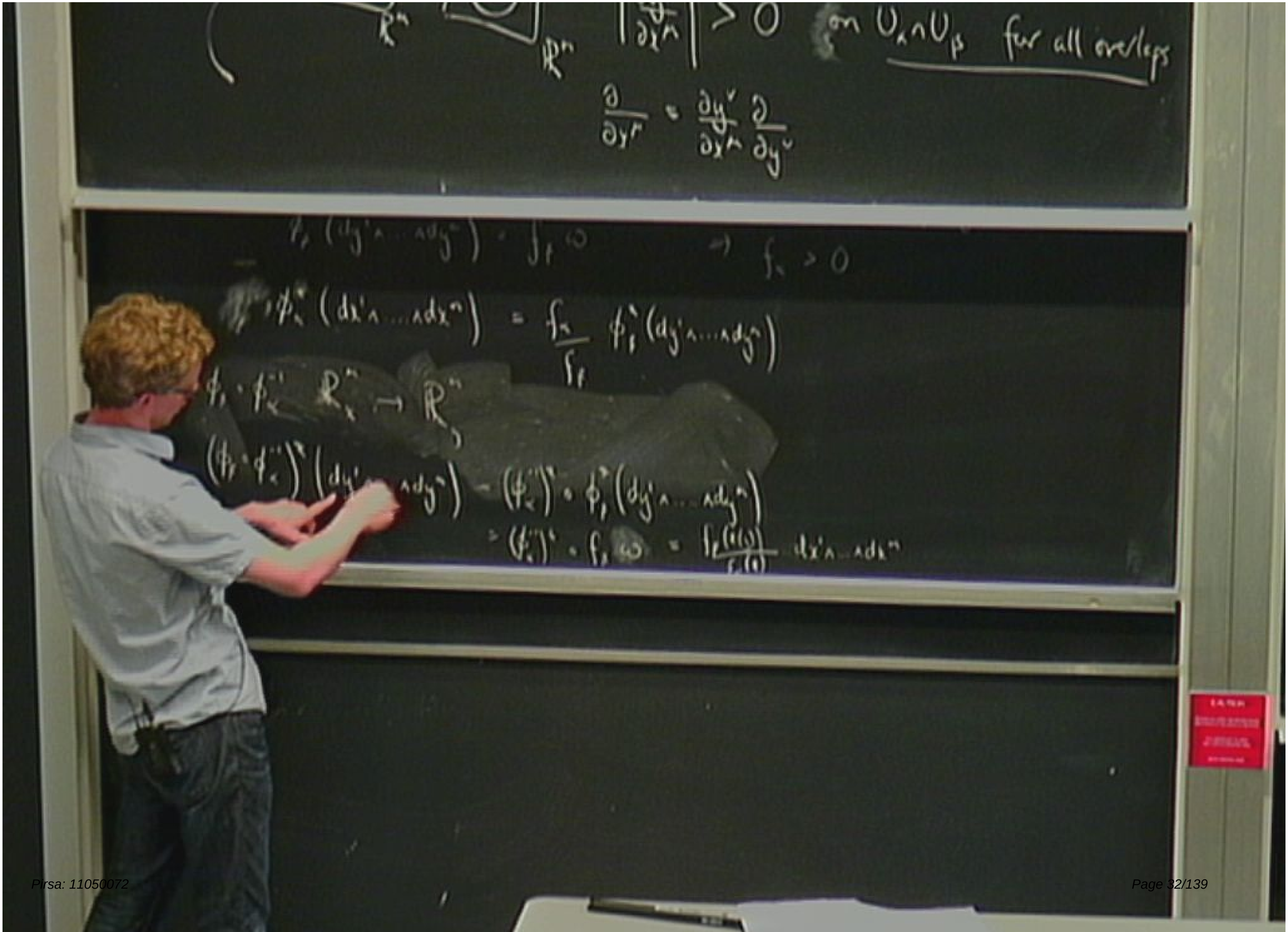
$$\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega \quad\Rightarrow\quad f_\alpha > 0$$

$$\phi_\alpha^*(dx^1\wedge\ldots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\,\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$$

$$\phi_\alpha^{-1}\quad \mathbb{R}^n_x \to \mathbb{R}^n_y$$

$$(\phi_\beta\circ\phi_\alpha^{-1})^*(dy^1\wedge\ldots\wedge dy^n)$$


$\mathbb{R}^n$
$\mathbb{R}^n$
$\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps
$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$
$f_\alpha > 0$
$\phi_\alpha^*(dx^1 \wedge \dots \wedge dx^n) = \frac{f_\alpha}{f_\beta}\phi_\beta^*(dy^1 \wedge \dots \wedge dy^n)$
$\phi_\beta \circ \phi_\alpha^{-1}$
$\mathbb{R}^n_y$
$= (\phi_\alpha^{-1})^* \circ \phi_\beta^*(dy^1 \wedge \dots \wedge dy^n)$
$= (\phi_\alpha^{-1})^* \circ f_\beta \omega$



$\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

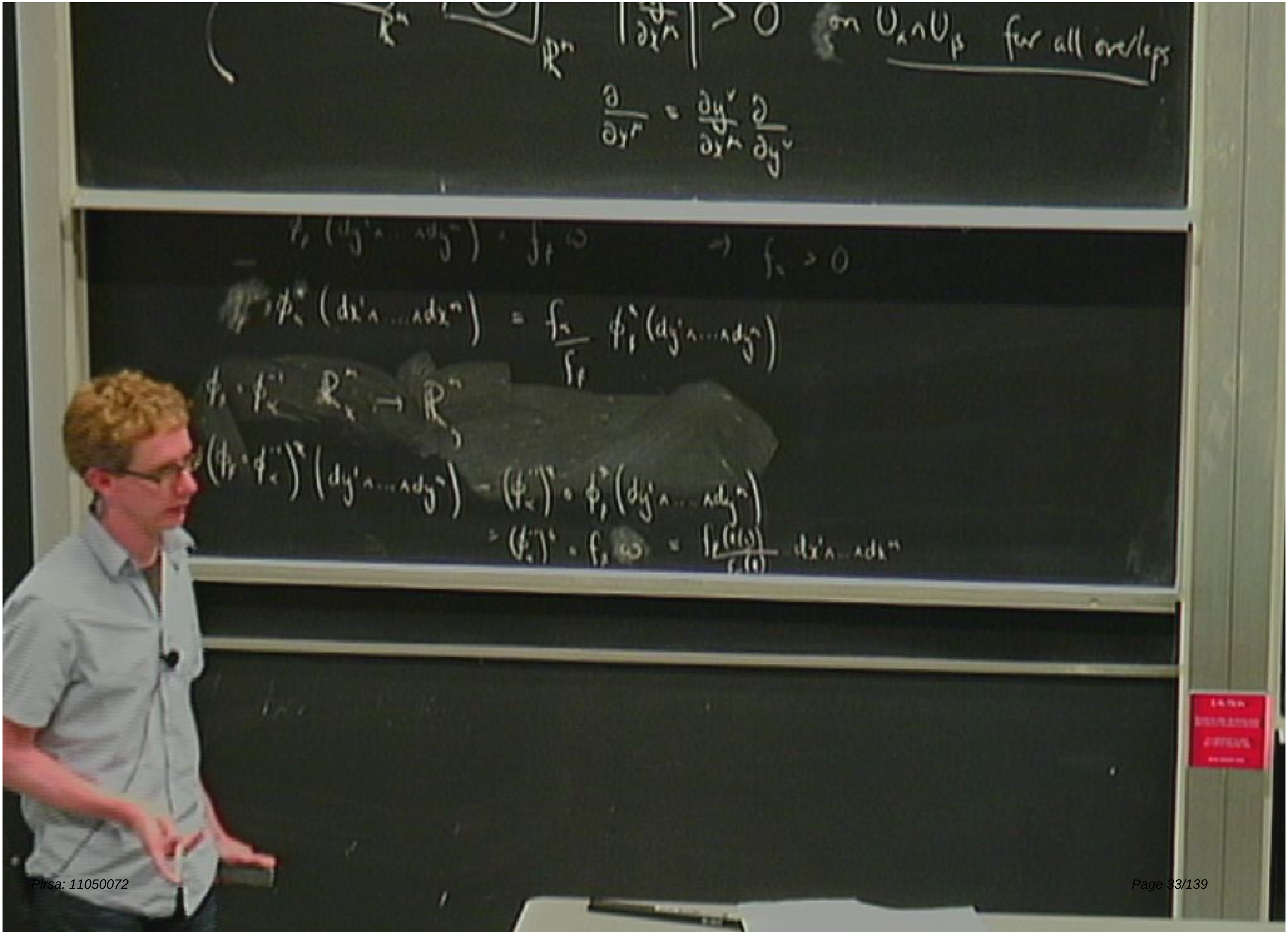
$$\phi_\beta(dy^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega \quad \Rightarrow \quad f_\alpha > 0$$

$$\phi_\alpha^*(dx^1\wedge\ldots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\,\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$$

$$\phi_\beta\circ\phi_\alpha^{-1}\ \mathbb{R}^n_x \to \mathbb{R}^n_y$$

$$(\phi_\beta\circ\phi_\alpha^{-1})^*(dy^1\wedge\ldots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$$

$$= (\phi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f(\phi)}\,dx^1\wedge\ldots\wedge dx^n$$



$\mathbb{R}^n$ $\mathbb{R}^n$ $\left|\frac{\partial y}{\partial x^\mu}\right| > 0$ on $U_\alpha \cap U_\beta$ for all overlaps

$$\frac{\partial}{\partial y^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}$$

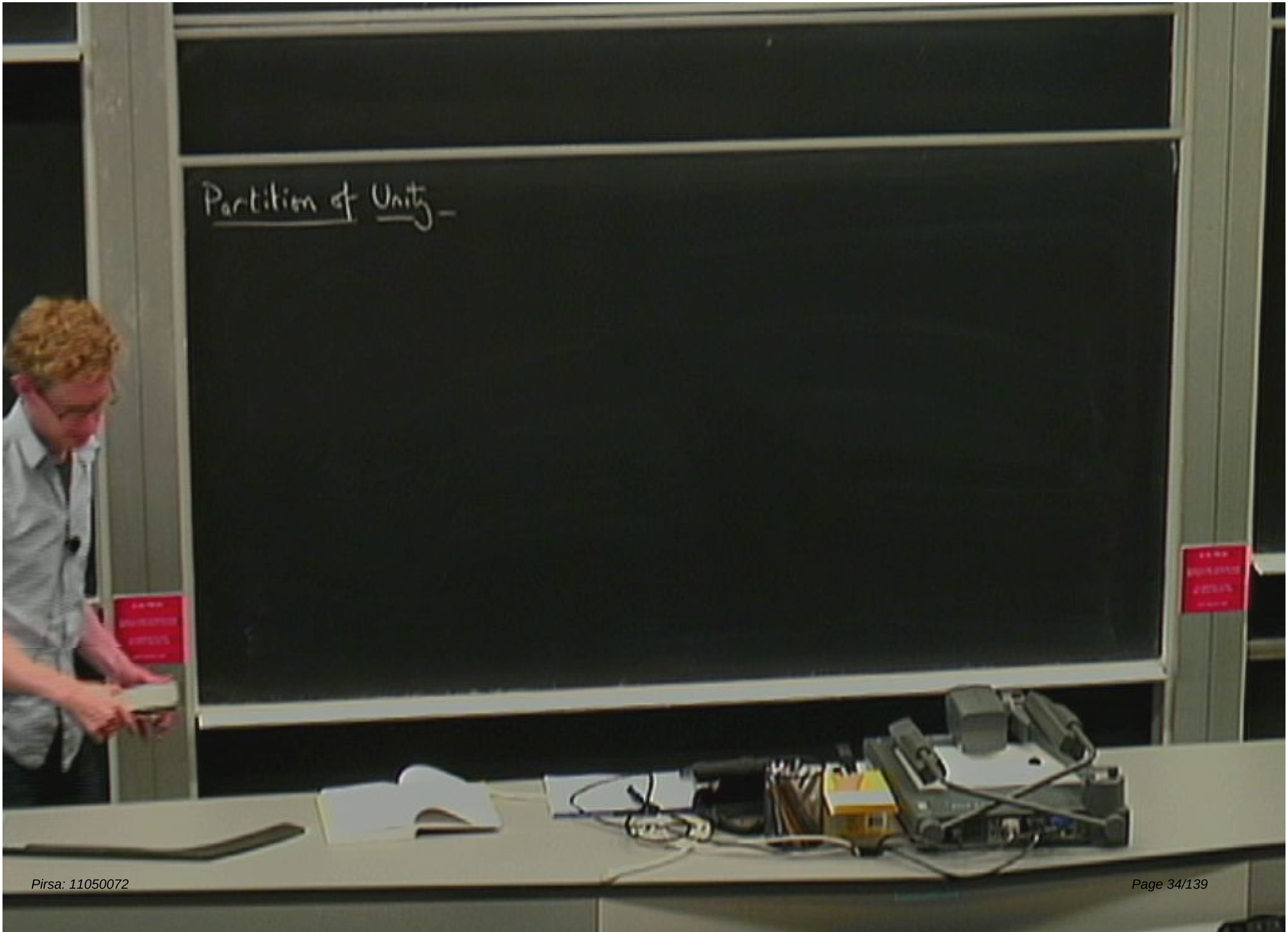
$\phi_\beta(dy^1\wedge\ldots\wedge dy^n) = f_\beta\,\omega$ $\rightarrow$ $f_\alpha > 0$

$$\phi_\alpha^*(dx^1\wedge\ldots\wedge dx^n) = \frac{f_\alpha}{f_\beta}\,\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$$

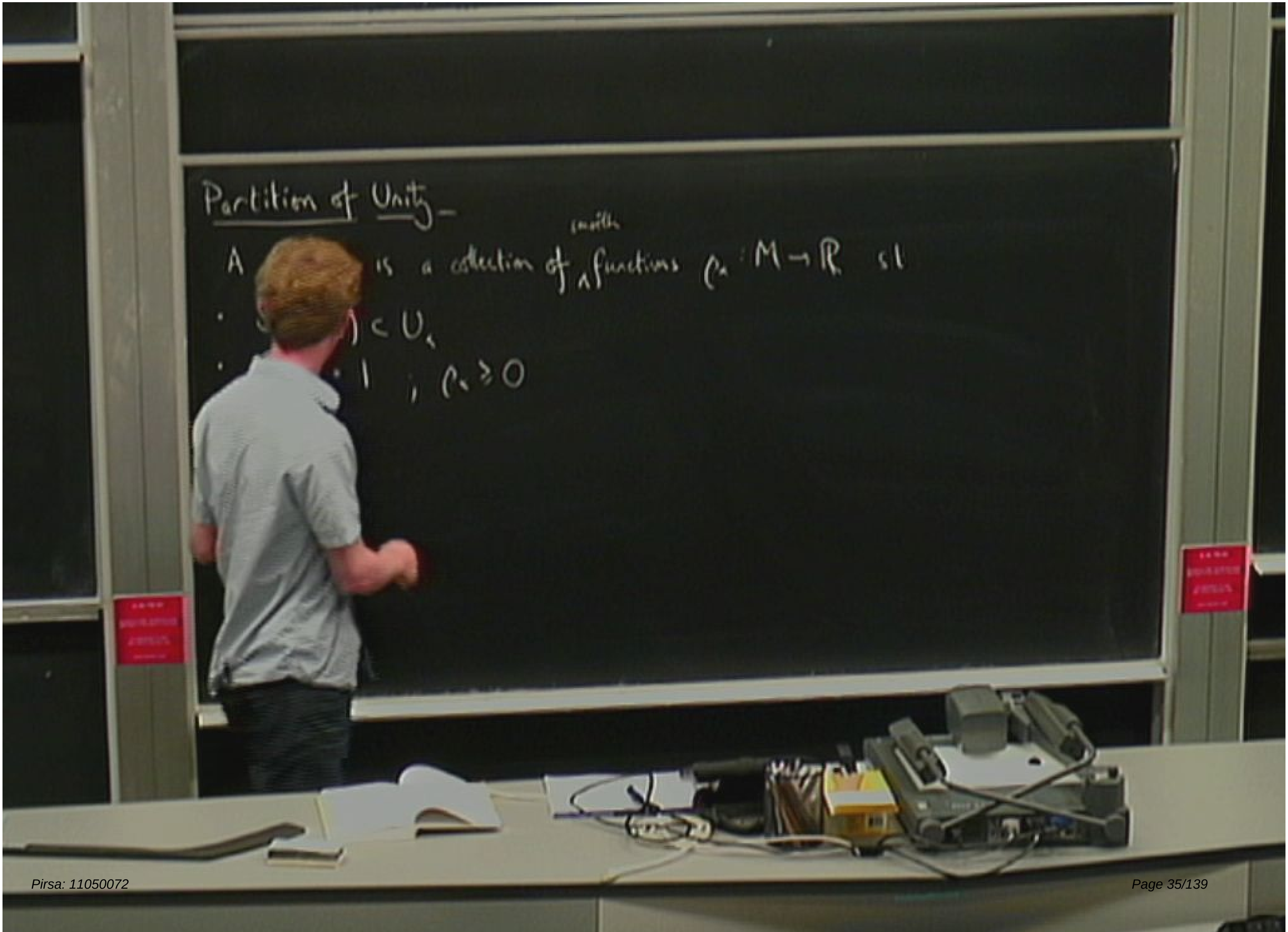
$\phi_\beta\circ\phi_\alpha^{-1}\ \mathbb{R}^n_x \rightarrow \mathbb{R}^n_y$

$$(\phi_\beta\circ\phi_\alpha^{-1})^*(dy^1\wedge\ldots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\ldots\wedge dy^n)$$

$$= (\phi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f_\alpha(\phi)}\,dx^1\wedge\ldots\wedge dx^n$$



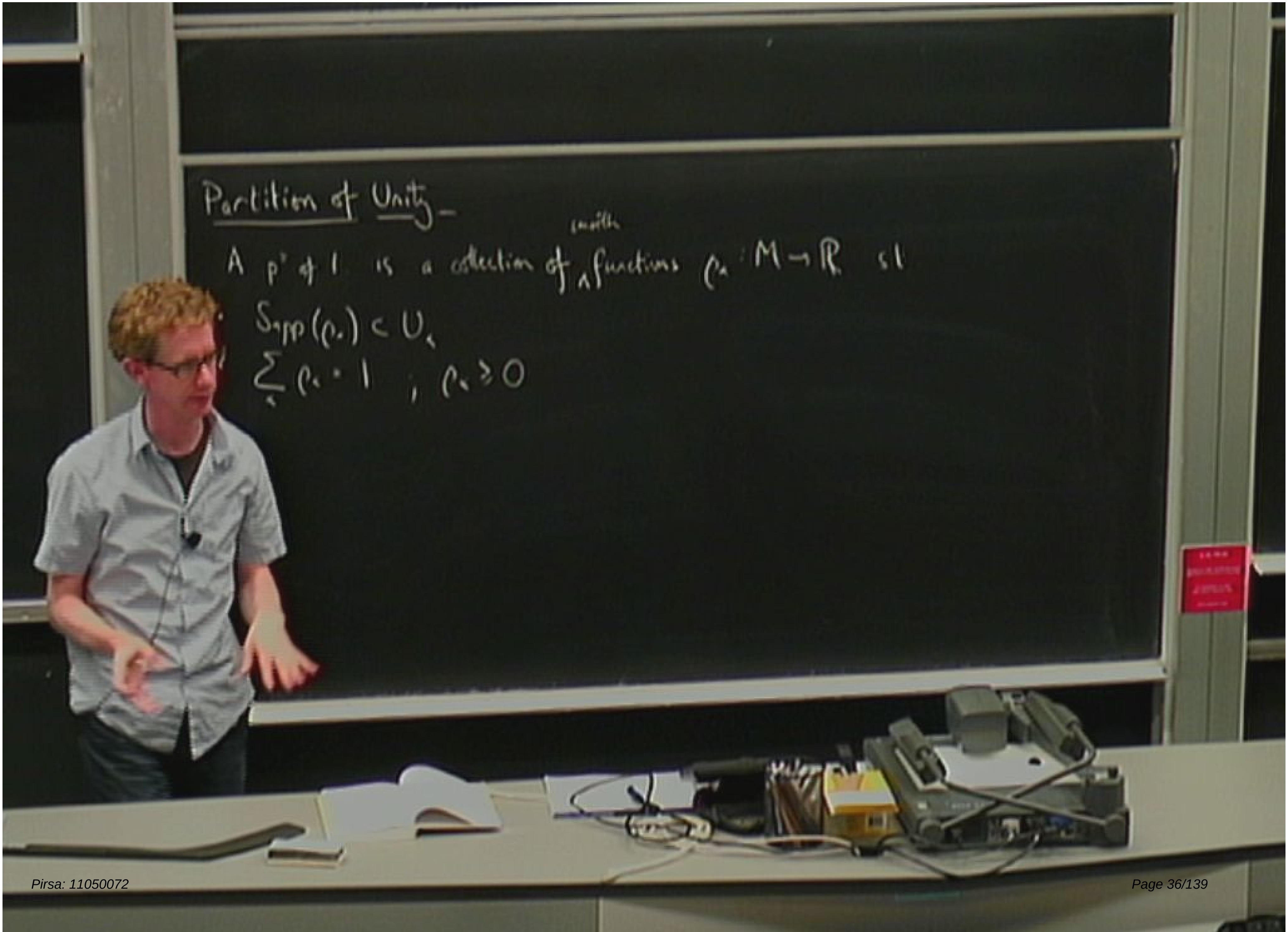
Partition of Unity –



Partition of Unity –

A is a collection of smooth functions $\rho_\alpha : M \to \mathbb{R}$ st

- $) \subset U_\alpha$
- $1 \;;\; \rho_\alpha \geq 0$

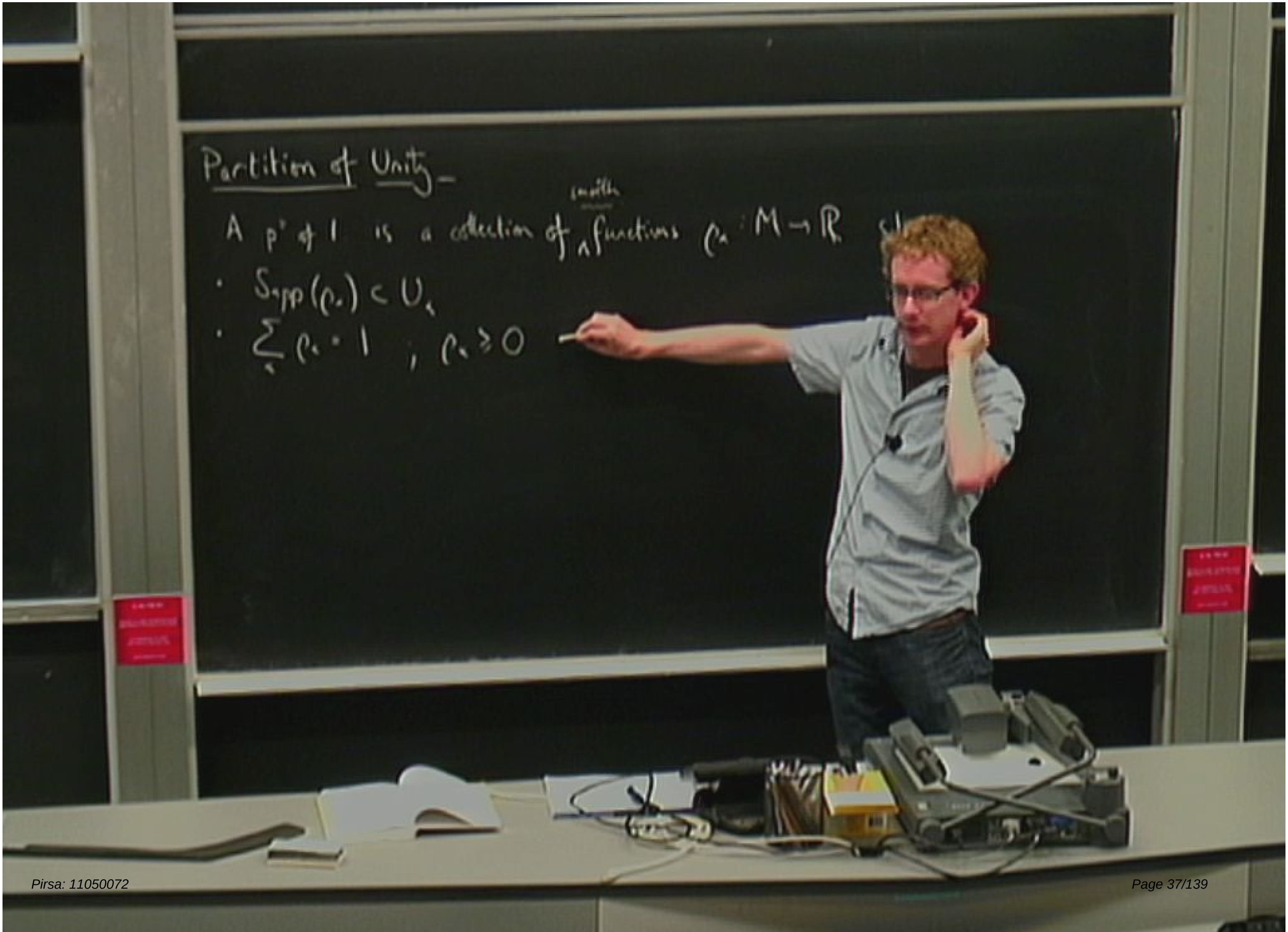


## Partition of Unity –

A p' of 1 is a collection of smooth functions $\rho_\alpha : M \to \mathbb{R}$ st

$\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$

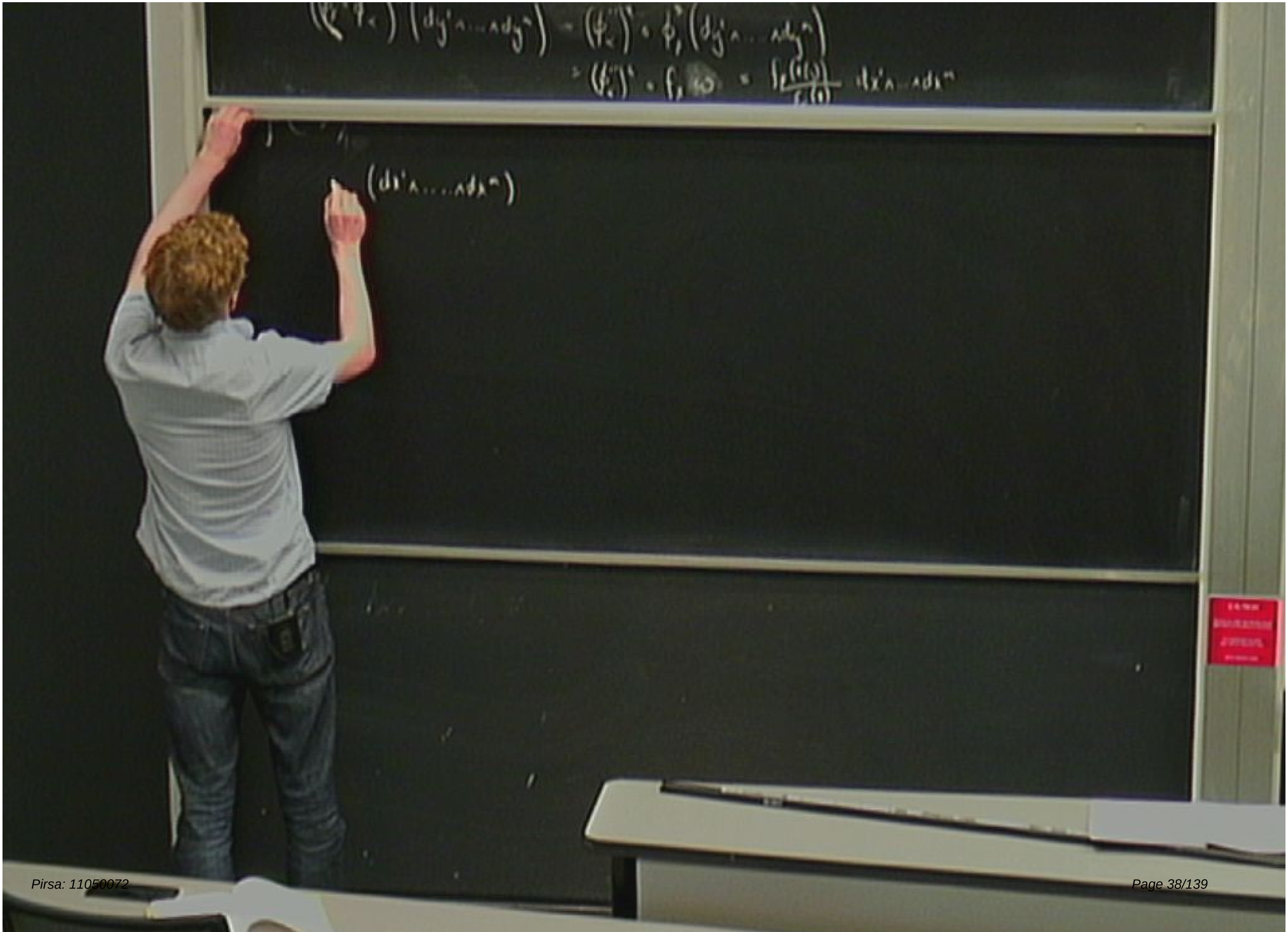
$\sum_\alpha \rho_\alpha = 1$ ; $\rho_\alpha \geq 0$

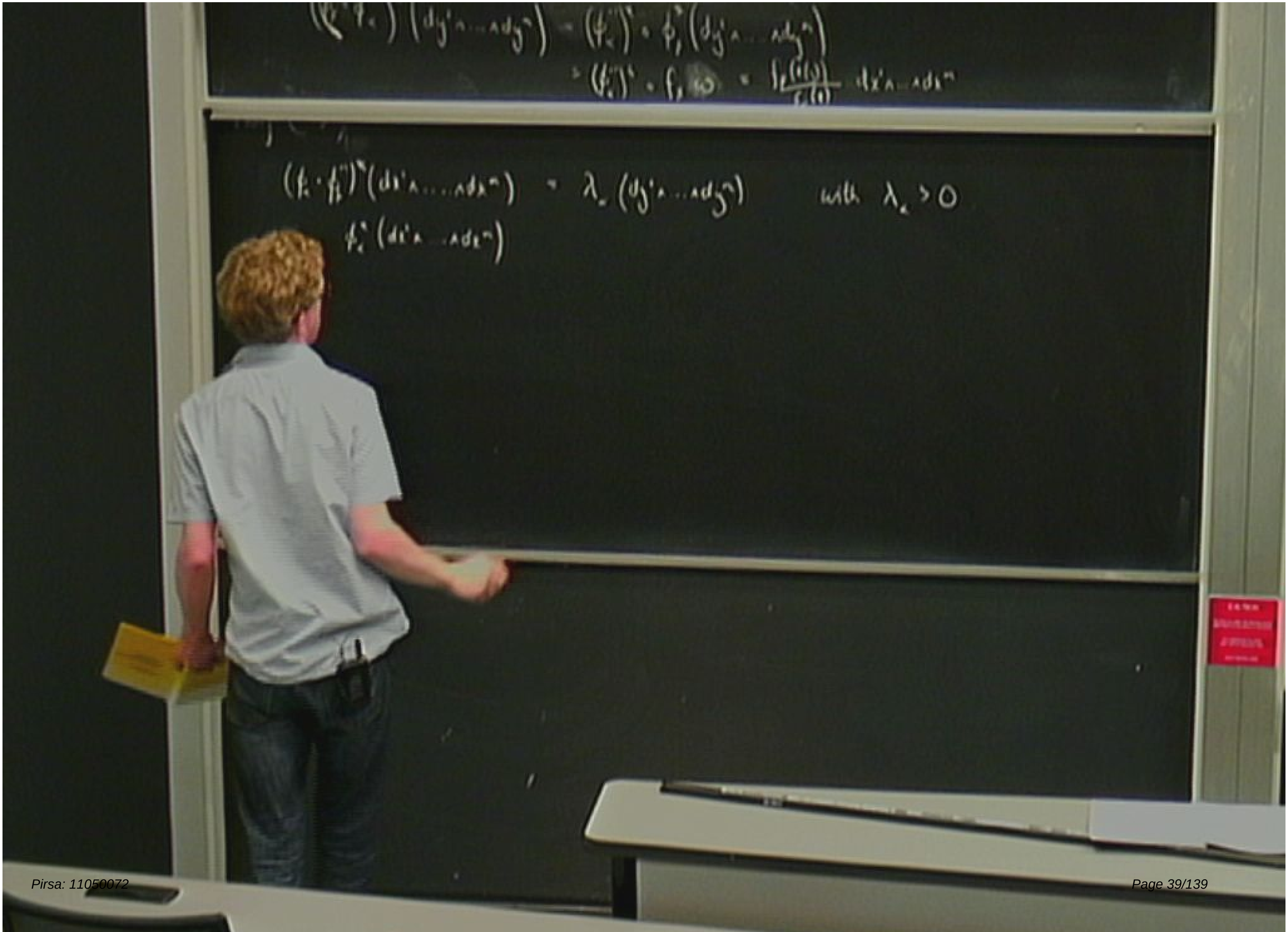


Partition of Unity –

A p' of 1 is a collection of smooth functions $\rho_\alpha : M \to \mathbb{R}$ s

- $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$
- $\sum_\alpha \rho_\alpha = 1$ ; $\rho_\alpha \geq 0$


$(\varphi_\beta^{-1}\varphi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\varphi_\alpha^{-1})^*\circ\varphi_\beta^*(dy^1\wedge\dots\wedge dy^n)$
$= (\varphi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\varphi(x))}{f(0)}\,dx^1\wedge\dots\wedge dx^n$
$(dx^1\wedge\dots\wedge dx^n)$

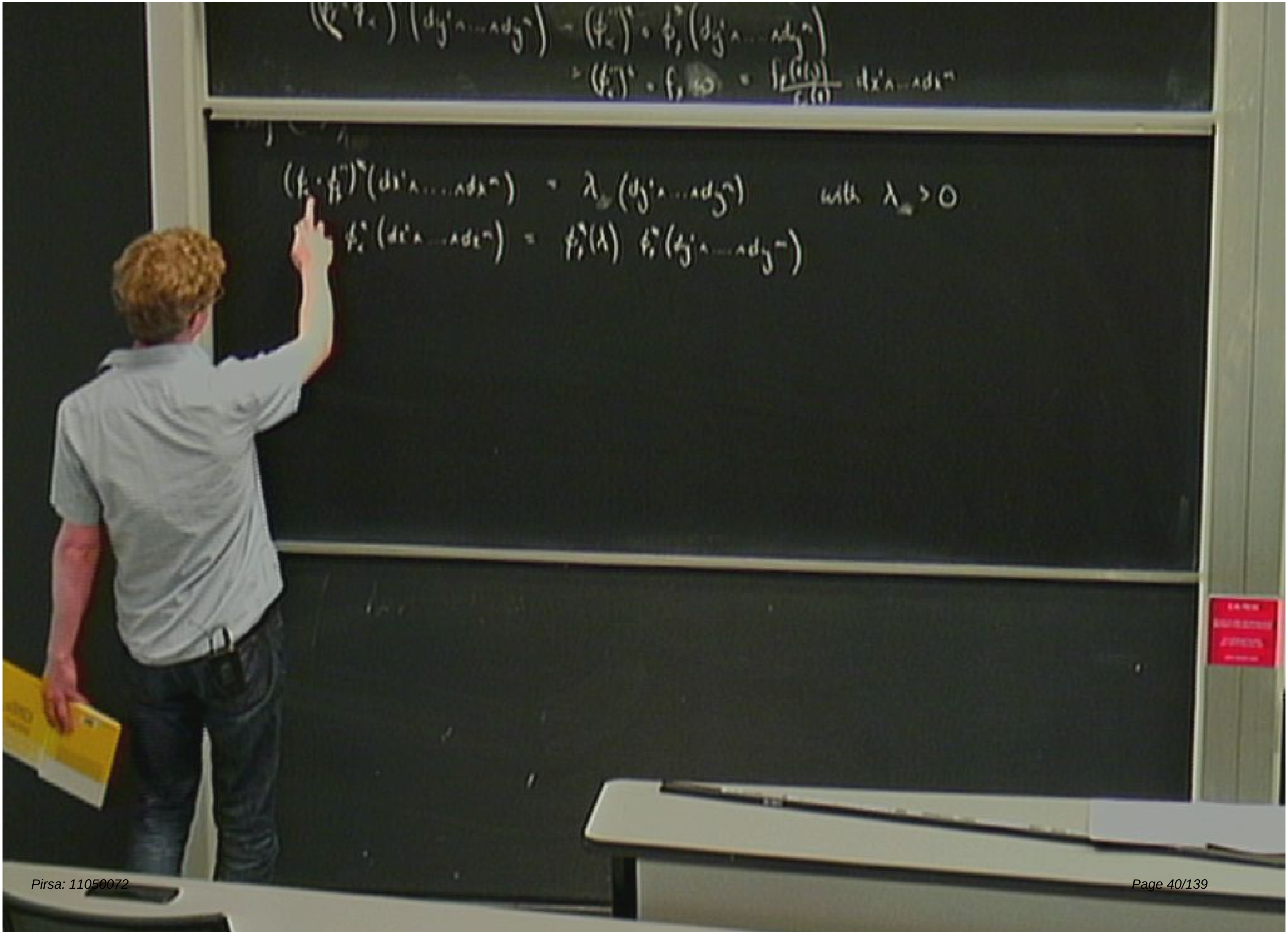


$$(\phi_\beta^{-1}\circ\phi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$$

$$= (\phi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f(x)}\,dx^1\wedge\dots\wedge dx^n$$

$$(\phi_\alpha\circ\phi_\beta^{-1})^*(dx^1\wedge\dots\wedge dx^n) = \lambda_\alpha\,(dy^1\wedge\dots\wedge dy^n) \quad \text{with } \lambda_\alpha > 0$$

$$\phi_\alpha^*(dx^1\wedge\dots\wedge dx^n)$$

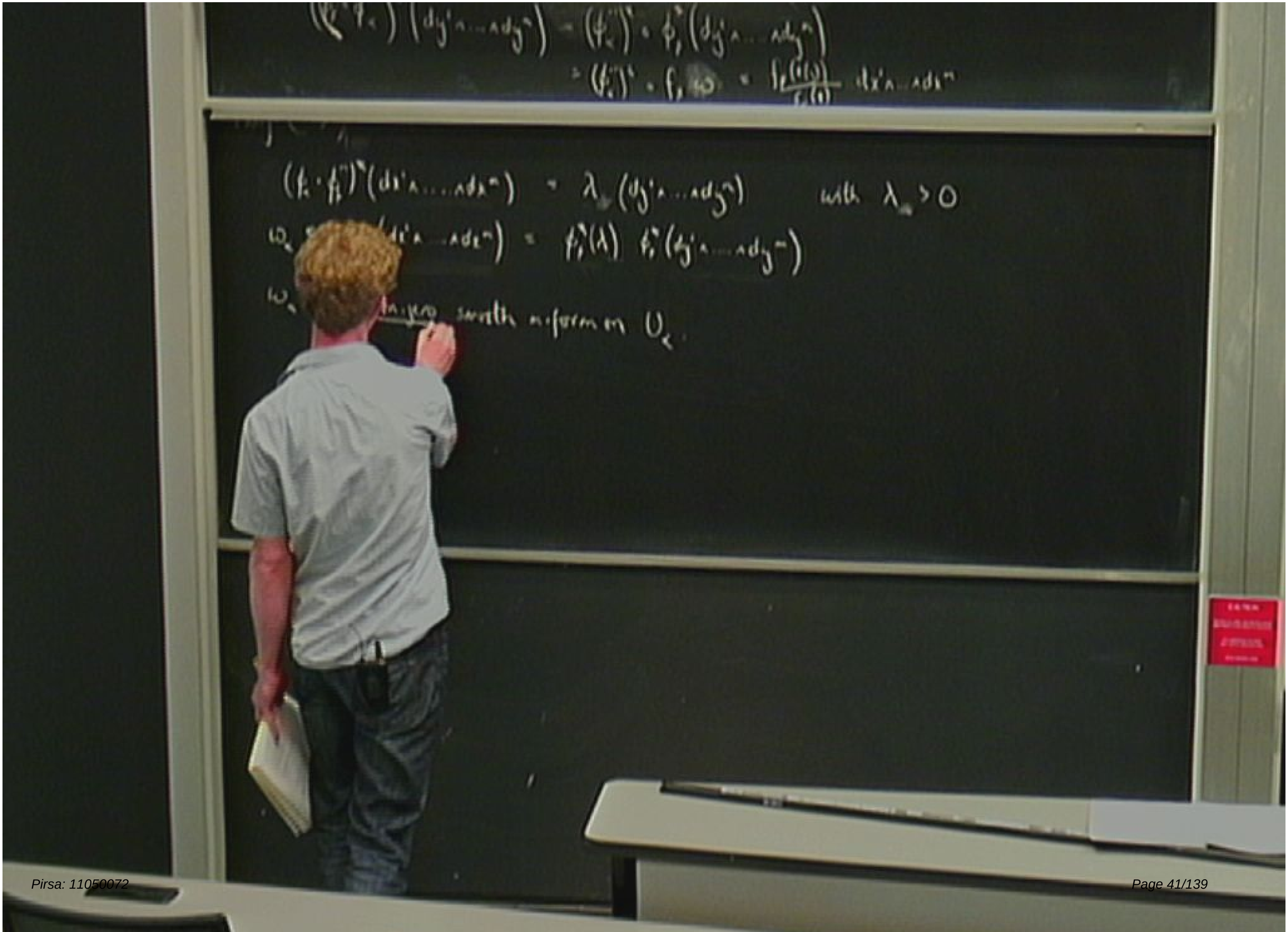


$$(\phi_\beta^{-1}\phi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$$

$$= (\phi_\alpha^{-1})^*\circ f_\beta(\omega) = \frac{f_\beta(\phi(x))}{f(0)}\,dx^1\wedge\dots\wedge dx^n$$

$$((\phi_\beta\circ\phi_\alpha^{-1})^*(dx^1\wedge\dots\wedge dx^n) = \lambda_{\alpha\beta}(dy^1\wedge\dots\wedge dy^n) \quad \text{with } \lambda_{\alpha\beta} > 0$$

$$\phi_\alpha^*(dx^1\wedge\dots\wedge dx^n) = \phi_\beta^*(\lambda)\ \phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$$



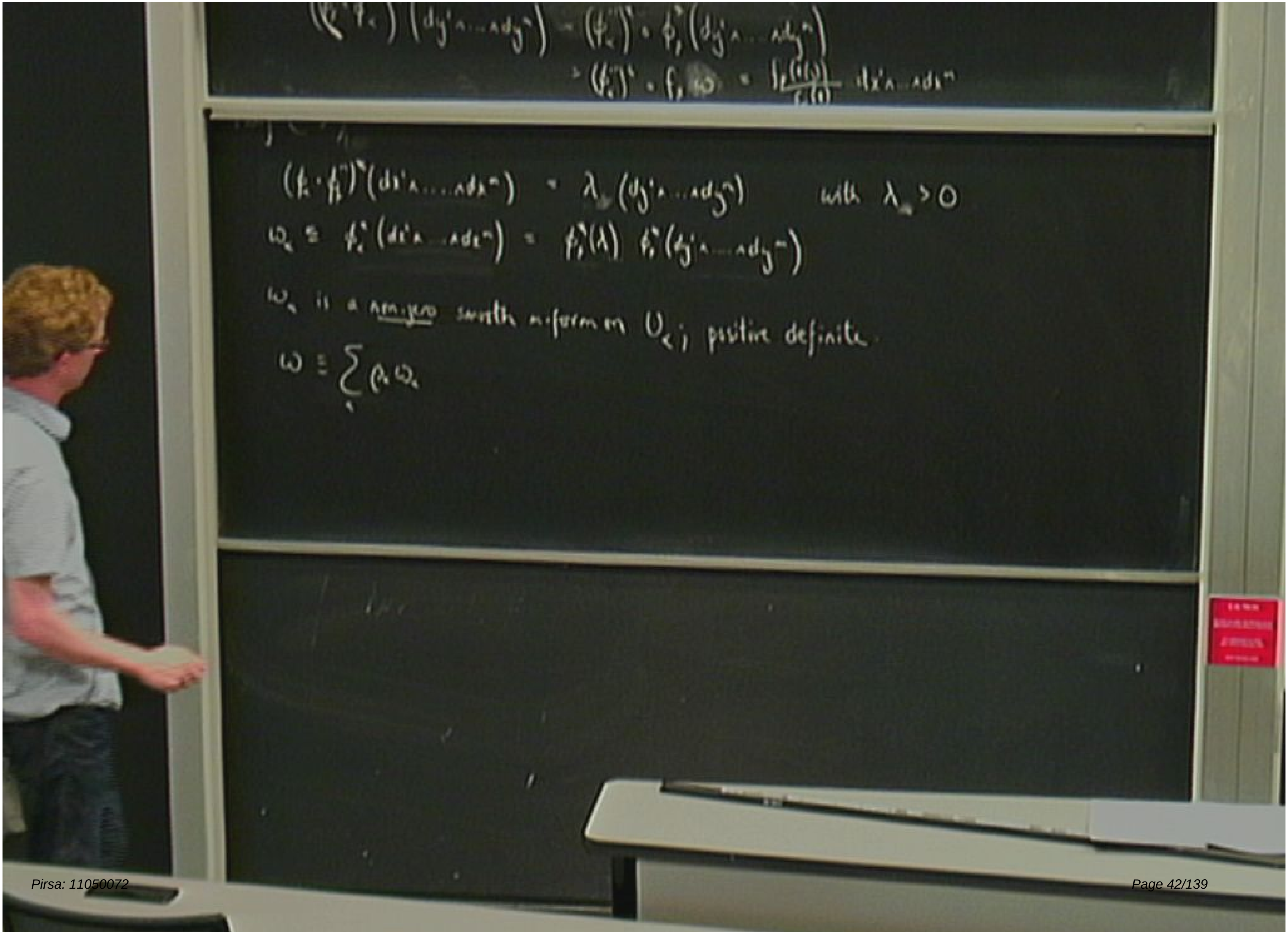
$(\phi_\beta^{-1}\circ\phi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$

$= (\phi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f_\alpha(x)}\,dx^1\wedge\dots\wedge dx^n$

$((\phi_\alpha\circ\phi_\beta^{-1})^*(dx^1\wedge\dots\wedge dx^n) = \lambda_{\alpha\beta}(dy^1\wedge\dots\wedge dy^n)$ with $\lambda_{\alpha\beta} > 0$

$\omega_\alpha$ [illegible] $(dx^1\wedge\dots\wedge dx^n) = \phi_\beta^*(\lambda)\ \phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$

$\omega_\alpha$ [illegible] nonzero smooth n-form on $U_\alpha$.



$$(\phi_\beta^{-1}\circ\phi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\phi_\alpha^{-1})^*\circ\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$$

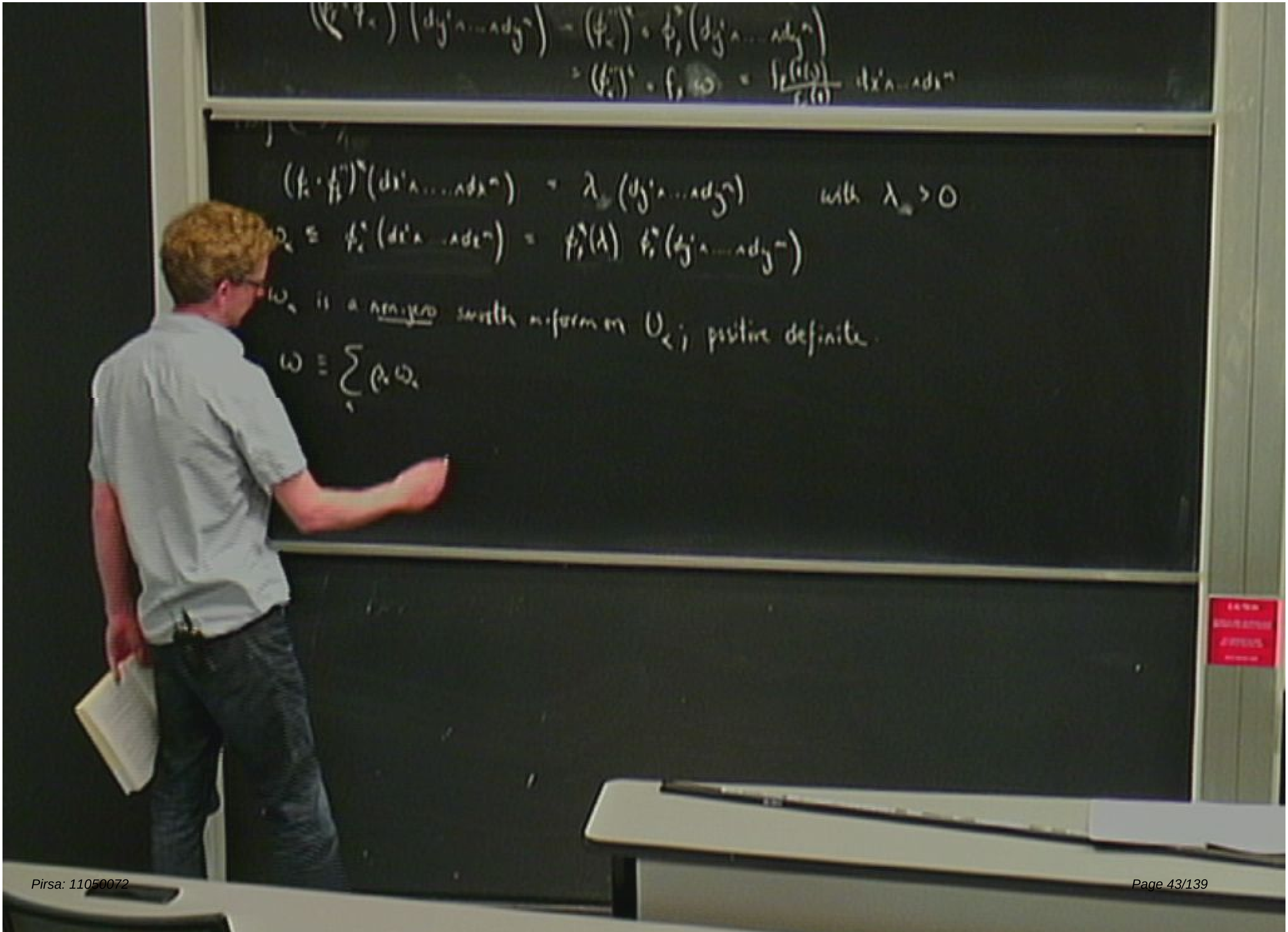
$$= (\phi_\alpha^{-1})^*\circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f(0)}\,dx^1\wedge\dots\wedge dx^n$$

$$((\phi_\alpha\circ\phi_\beta^{-1})^*(dx^1\wedge\dots\wedge dx^n) = \lambda_{\alpha\beta}(dy^1\wedge\dots\wedge dy^n) \quad \text{with } \lambda_{\alpha\beta} > 0$$

$$\omega_\alpha \equiv \phi_\alpha^*(dx^1\wedge\dots\wedge dx^n) = \phi_\beta^*(\lambda)\,\phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$$

$\omega_\alpha$ is a <u>nonzero</u> smooth n-form on $U_\alpha$; positive definite.

$$\omega \equiv \sum_\alpha \rho_\alpha\omega_\alpha$$



$(\phi_\beta^{-1}\phi_\alpha)^*(dy^1\wedge\dots\wedge dy^n) = (\phi_\alpha^{-1})^* \circ \phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$

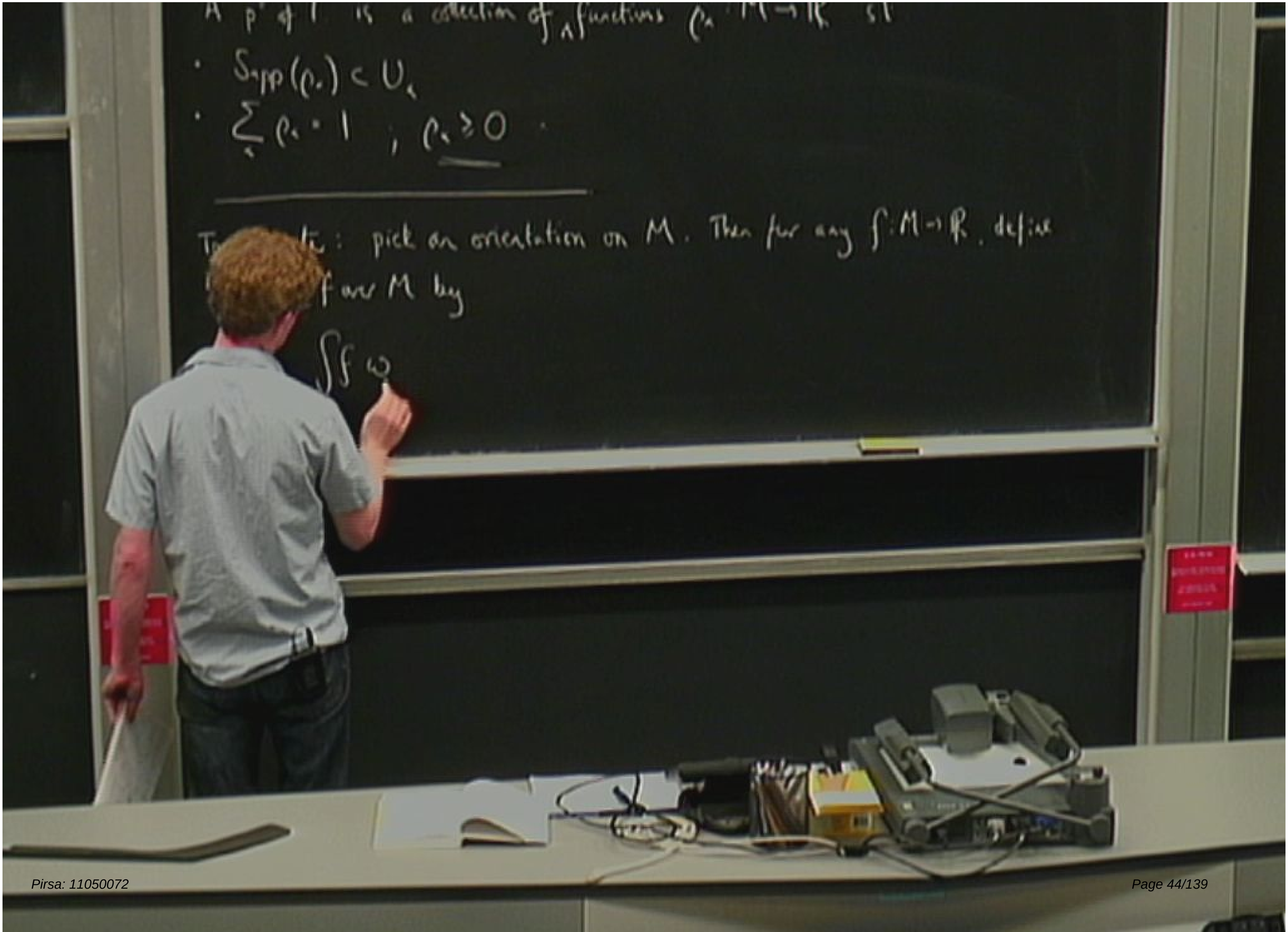
$= (\phi_\alpha^{-1})^* \circ f_\beta\,\omega = \frac{f_\beta(\phi(x))}{f(0)}\,dx^1\wedge\dots\wedge dx^n$

$((\phi_\alpha\cdot\phi_\beta^{-1})^*(dx^1\wedge\dots\wedge dx^n) = \lambda_{\alpha\beta}(dy^1\wedge\dots\wedge dy^n)$ with $\lambda_{\alpha\beta} > 0$

$\omega_\alpha \equiv \phi_\alpha^*(dx^1\wedge\dots\wedge dx^n) = \phi_\beta^*(\lambda)\ \phi_\beta^*(dy^1\wedge\dots\wedge dy^n)$

$\omega_\alpha$ is a nonzero smooth n-form on $U_\alpha$; positive definite.

$\omega \equiv \sum_\alpha \rho_\alpha \omega_\alpha$



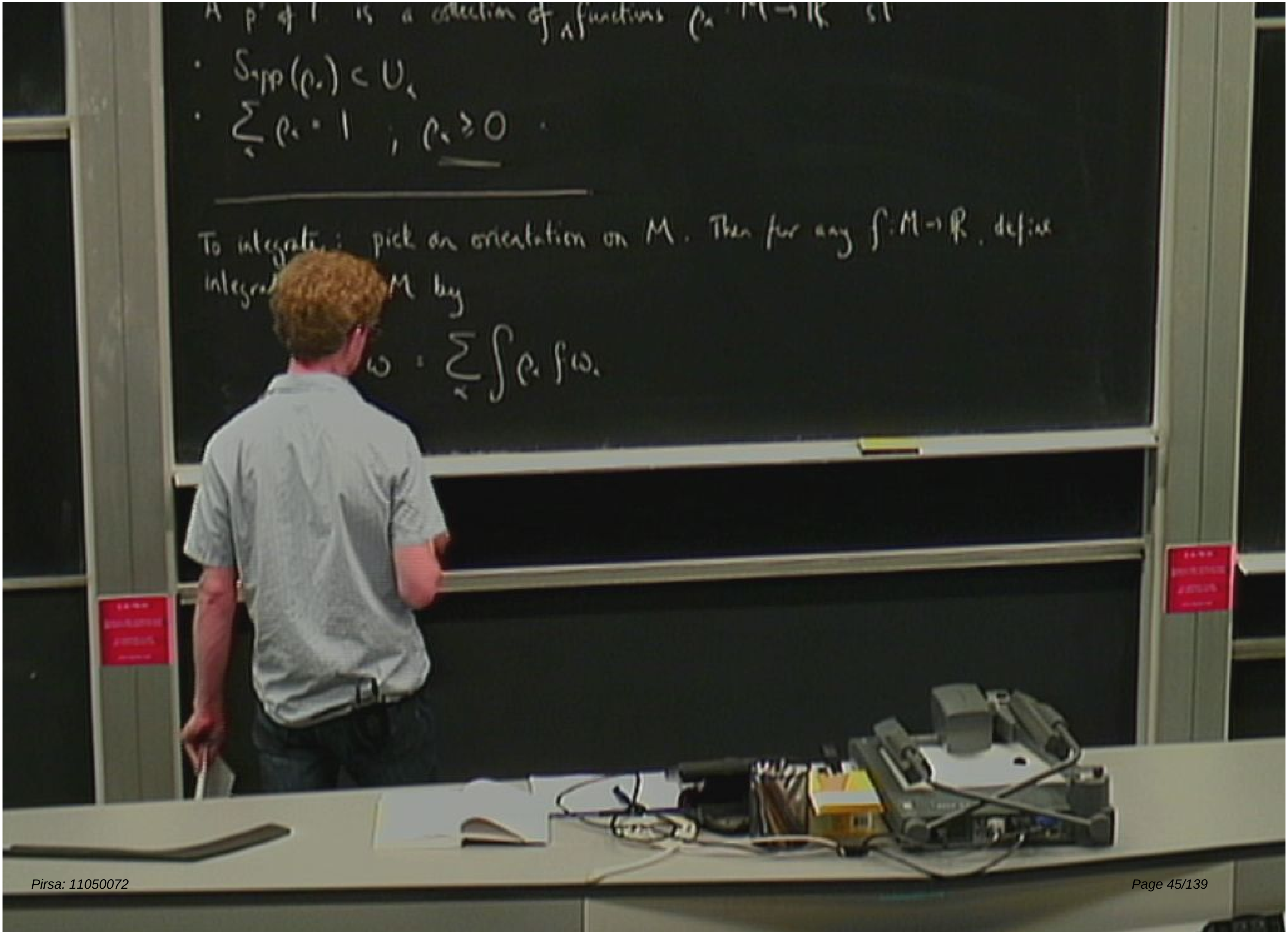
A p° of 1 is a collection of functions $\rho_\alpha : M \to \mathbb{R}$ s.t.

- $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$
- $\sum_\alpha \rho_\alpha = 1$ ; $\underline{\rho_\alpha \geq 0}$ .

---

T[illegible]te: pick an orientation on $M$. Then for any $f : M \to \mathbb{R}$, define [illegible] $f$ over $M$ by

$$\int f \omega$$



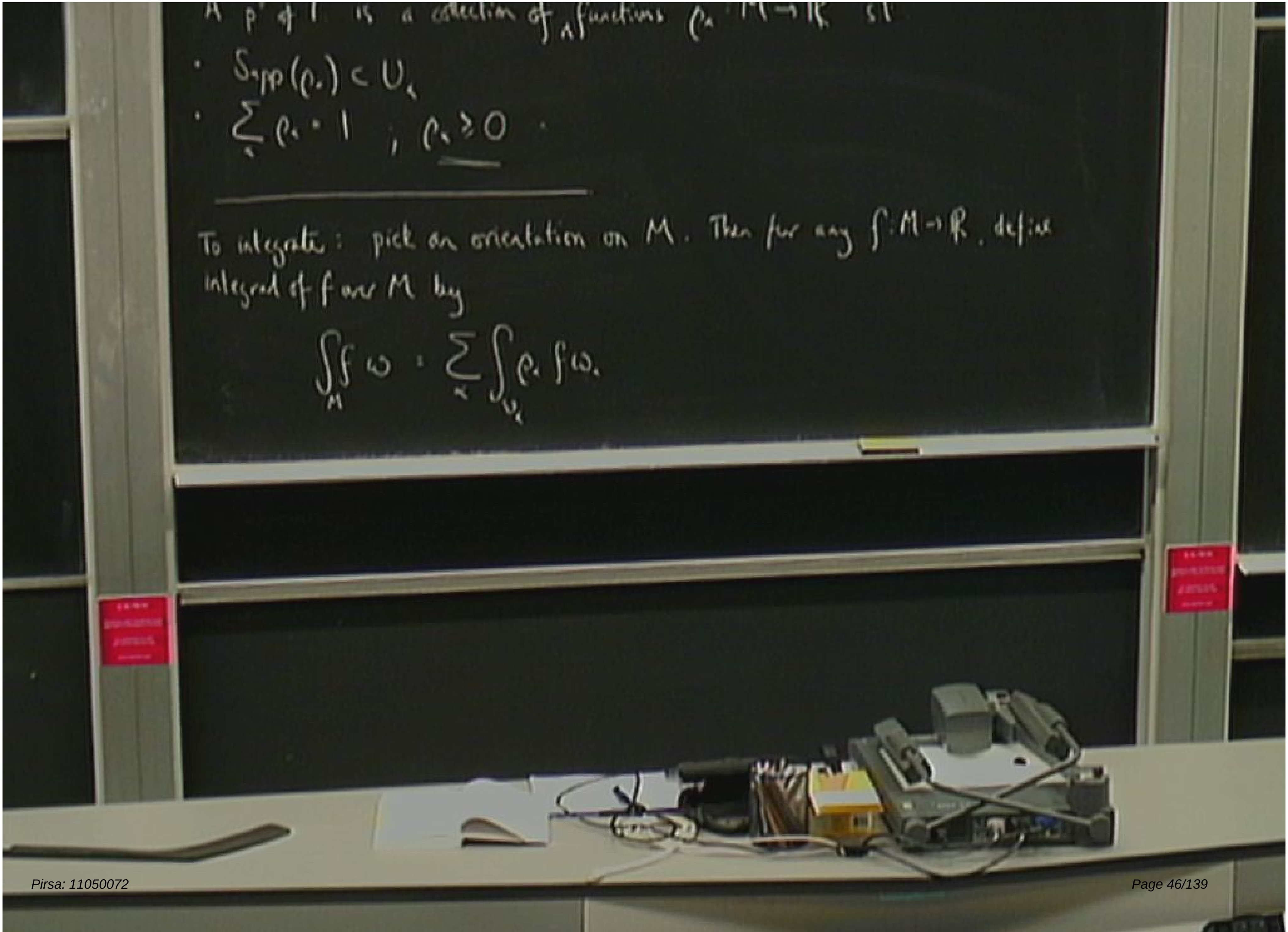
A p of U is a collection of $_\wedge$functions $\rho_\alpha : M \to \mathbb{R}$ s.t.

- $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$
- $\sum_\alpha \rho_\alpha = 1$ ; $\underline{\rho_\alpha \geq 0}$ .

---

To integrate : pick an orientation on M. Then for any $f : M \to \mathbb{R}$, define integral M by

$$\omega = \sum_\alpha \int \rho_\alpha f \omega .$$



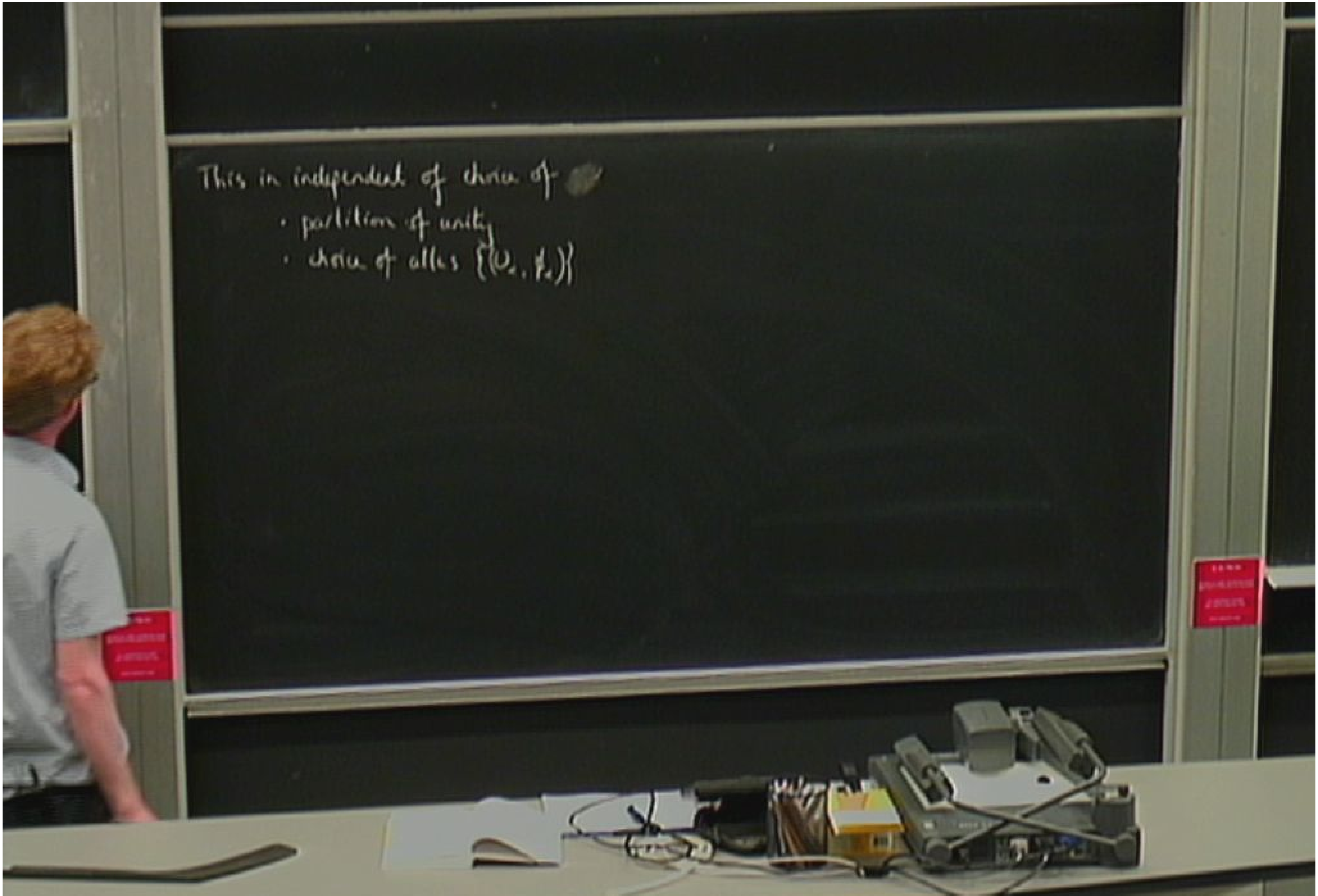
A p' of Γ is a collection of functions $\rho_\alpha : M \to \mathbb{R}$ s.t.

- $\mathrm{Supp}(\rho_\alpha) \subset U_\alpha$
- $\sum_\alpha \rho_\alpha = 1$ ; $\underline{\rho_\alpha \geq 0}$ .

---

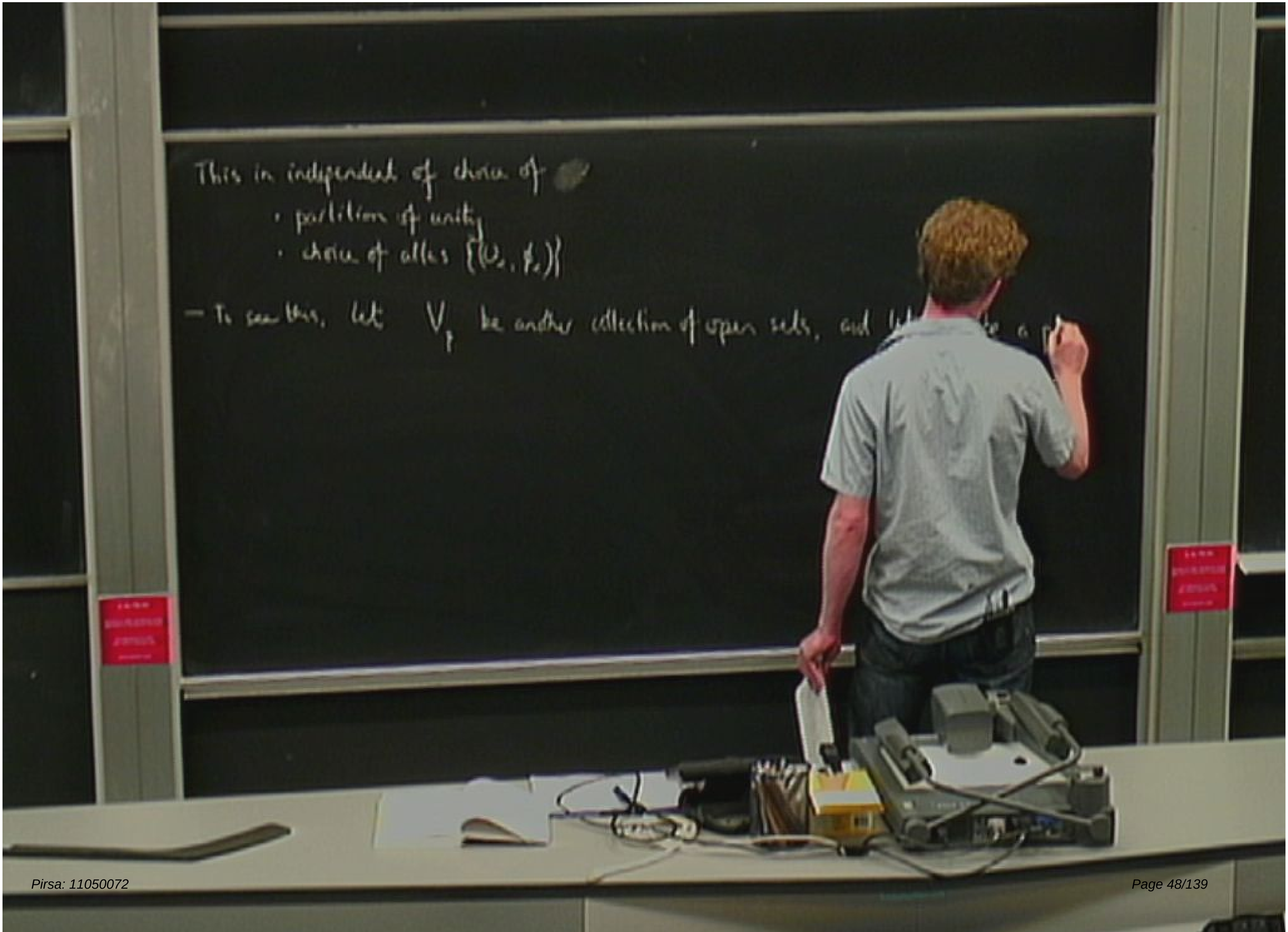
To integrate: pick an orientation on M. Then for any $f : M \to \mathbb{R}$, define integral of f over M by

$$\int_M f\,\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha$$



This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$



This is independent of choice of

- partition of unity
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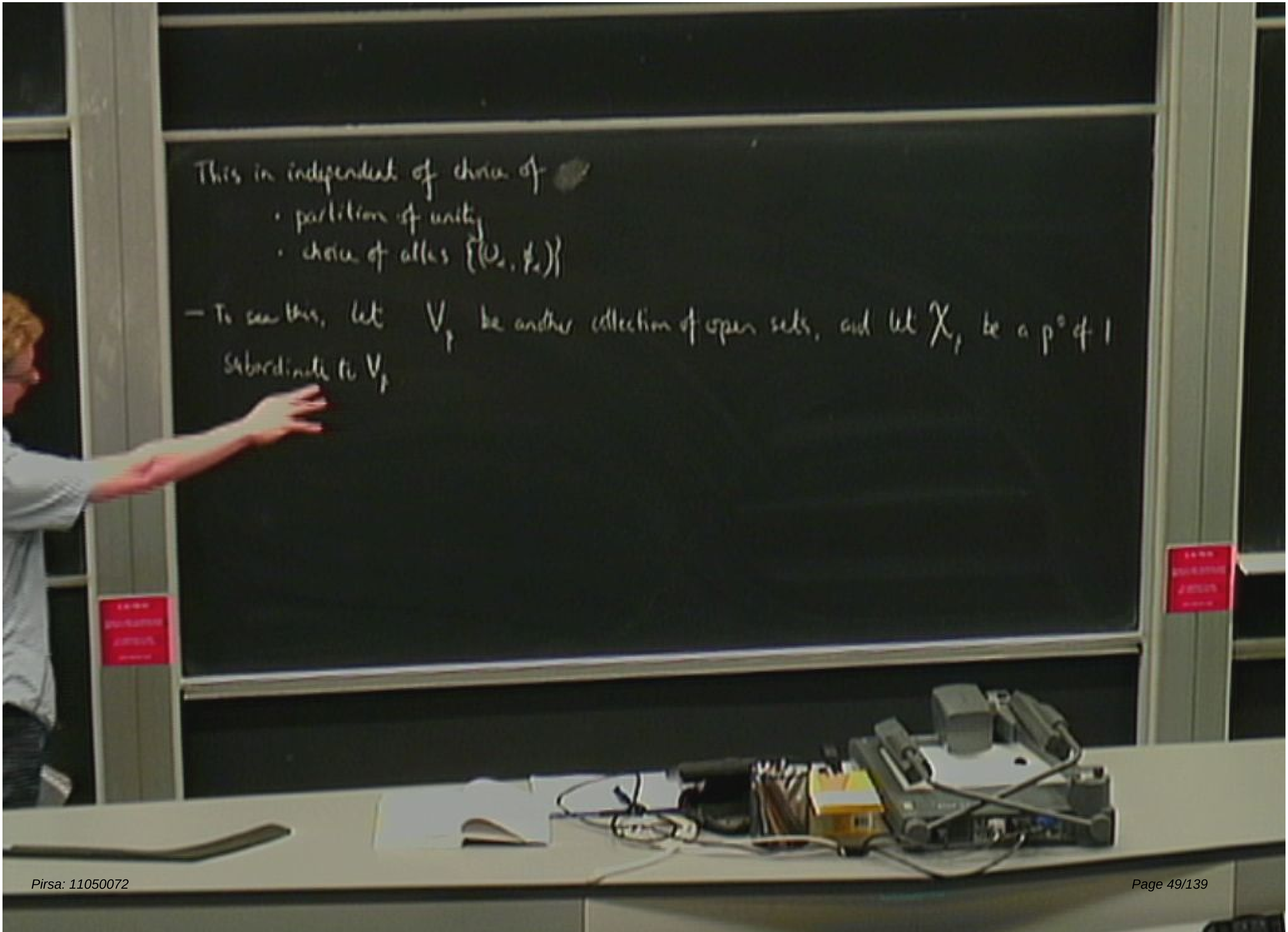
— To see this, let $V_\beta$ be another collection of open sets, and



This is independent of choice of

- partition of unity
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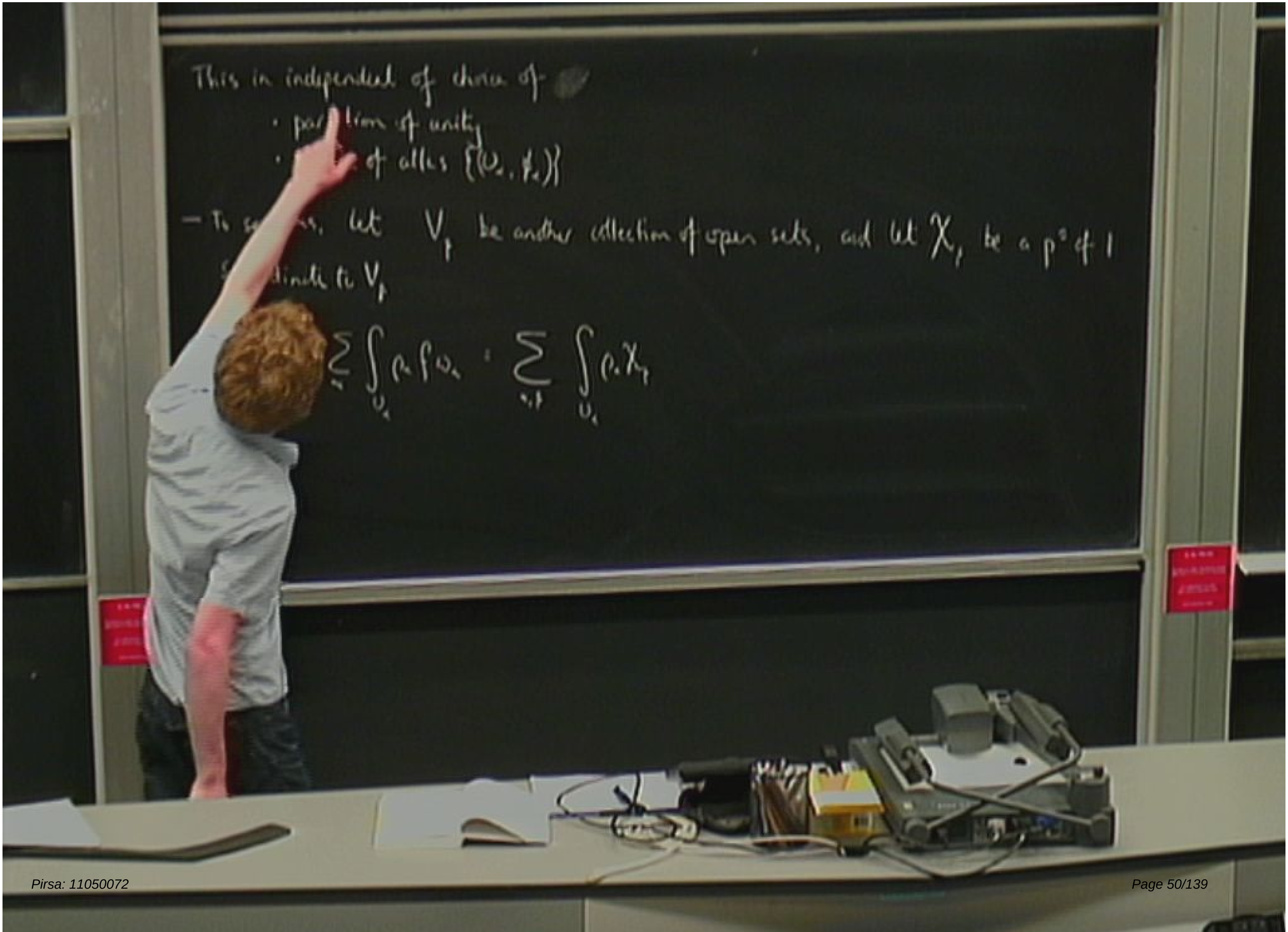
— To see this, let $V_\beta$ be another collection of open sets, and let $\chi_\beta$ be a p° of 1 subordinate to $V_\beta$

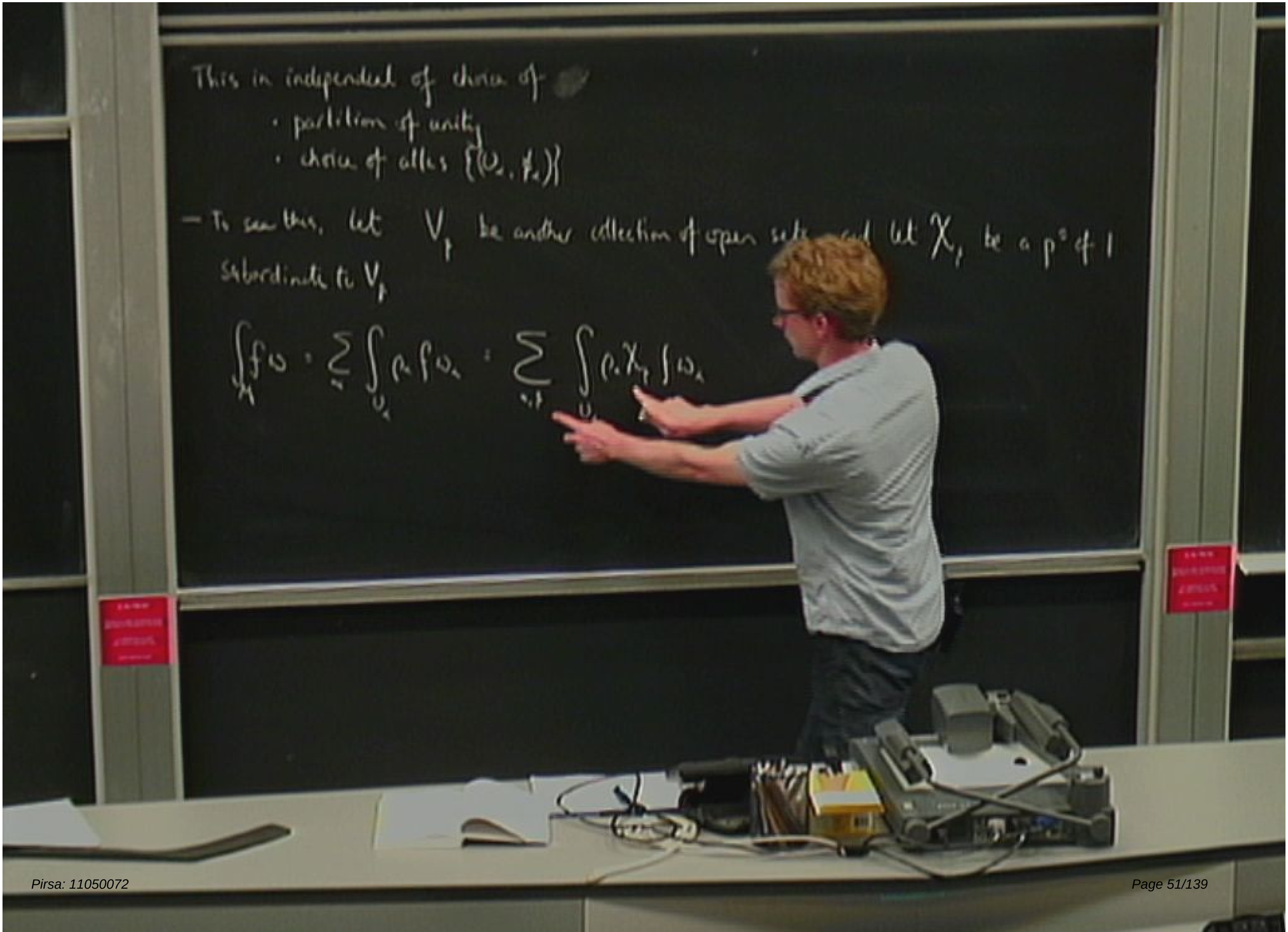


This is independent of choice of

- par[illegible]tion of unity
- [illegible] of atlas $\{(U_\alpha, \phi_\alpha)\}$

— To se[illegible]s, let $V_\beta$ be another collection of open sets, and let $\chi_\beta$ be a p° of 1 [illegible]dinate to $V_\beta$

$$\sum_{\alpha} \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta$$



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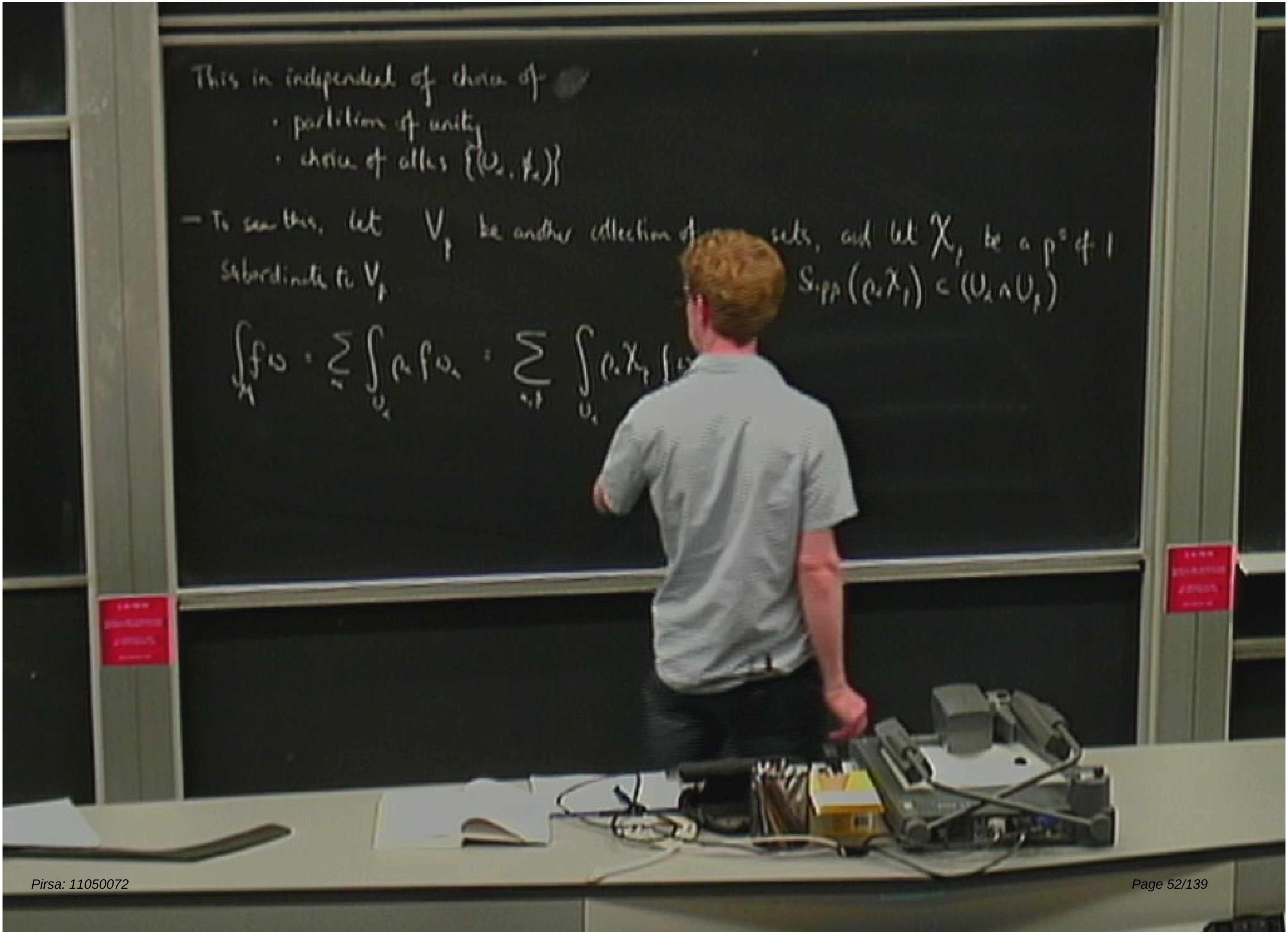
– To see this, let $V_\beta$ be another collection of open sets and let $\chi_\beta$ be a p° of 1 subordinate to $V_\beta$

$$\int_M f\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_U \rho_\alpha \chi_\beta f \omega_\alpha$$



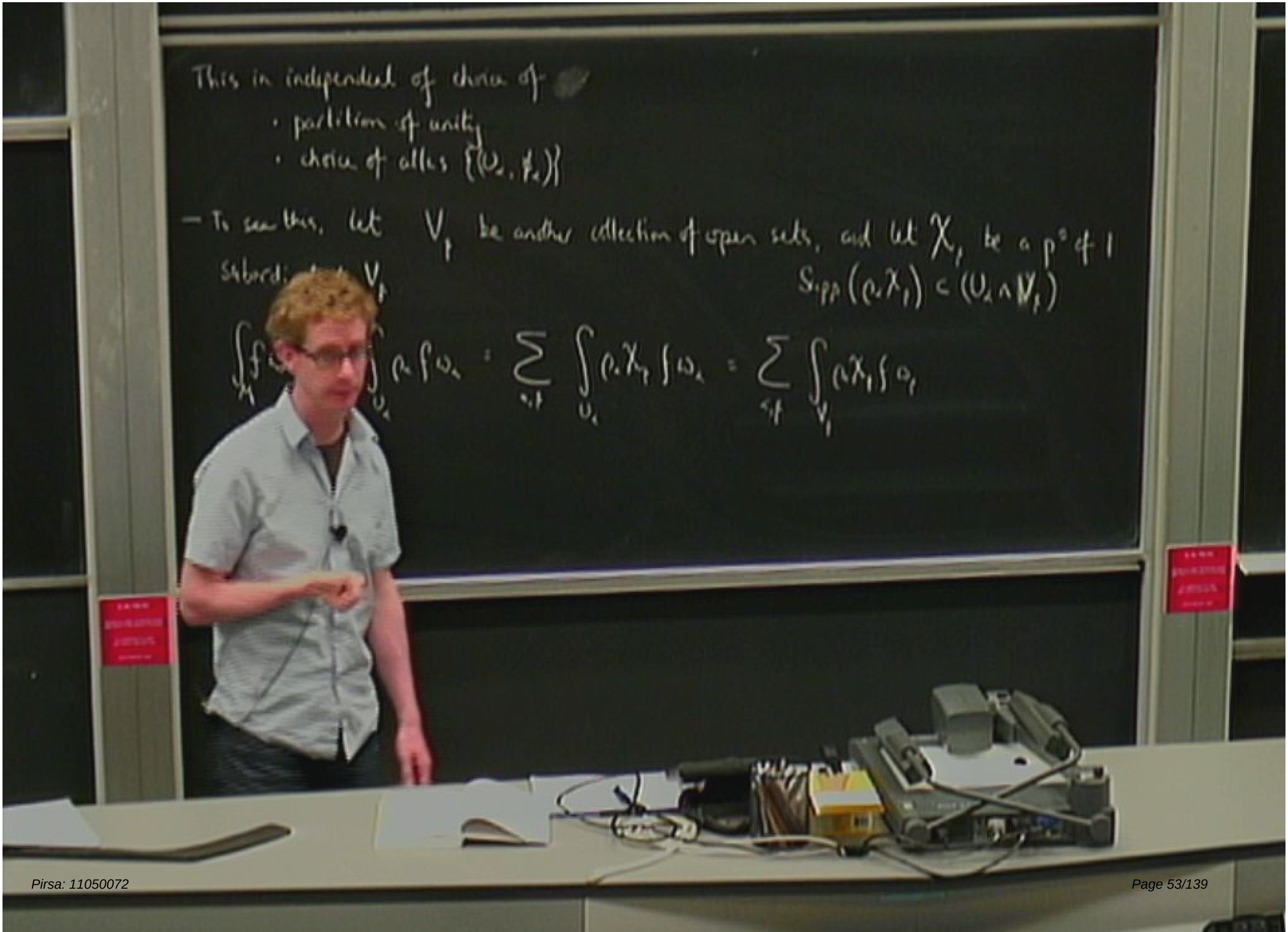
This is independent of choice of

- partition of unity
- choice of atlas $\{(U_\alpha, \phi_\alpha)\}$

— To see this, let $V_\beta$ be another collection of sets, and let $\chi_\beta$ be a p° of 1 subordinate to $V_\beta$

$\mathrm{Supp}(\rho_\alpha \chi_\beta) \subset (U_\alpha \cap U_\beta)$

$$\int_M f\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta f \omega$$



This in independent of choice of

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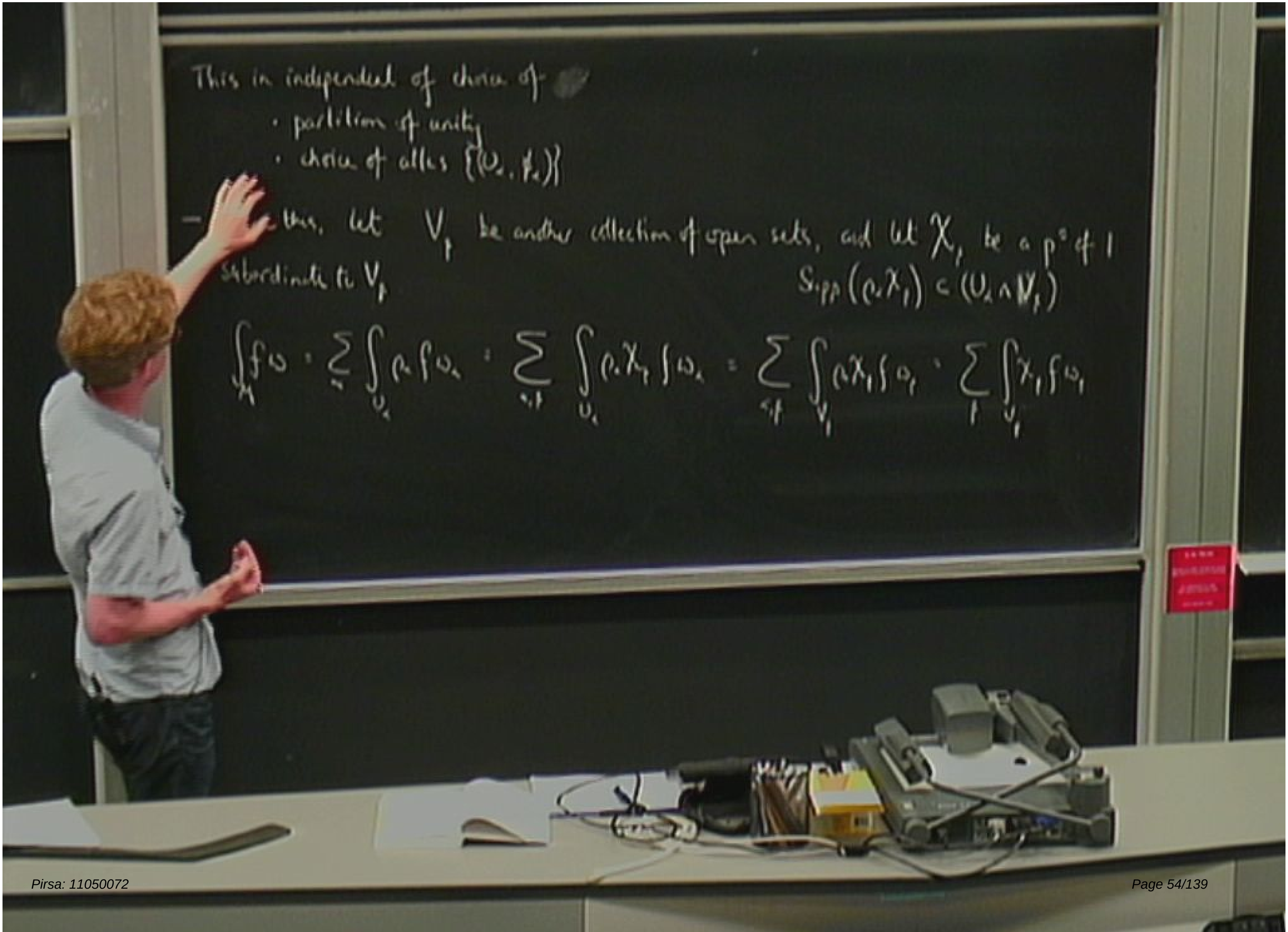
— To see this, let $V_\beta$ be another collection of open sets, and let $\chi_\beta$ be a p° of 1 subordi[illegible] $V_\beta$

$$\mathrm{Supp}(\rho_\alpha \chi_\beta) \subset (U_\alpha \cap V_\beta)$$

$$\int_M f [illegible] \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta f \omega_\alpha = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta f \omega_\beta$$



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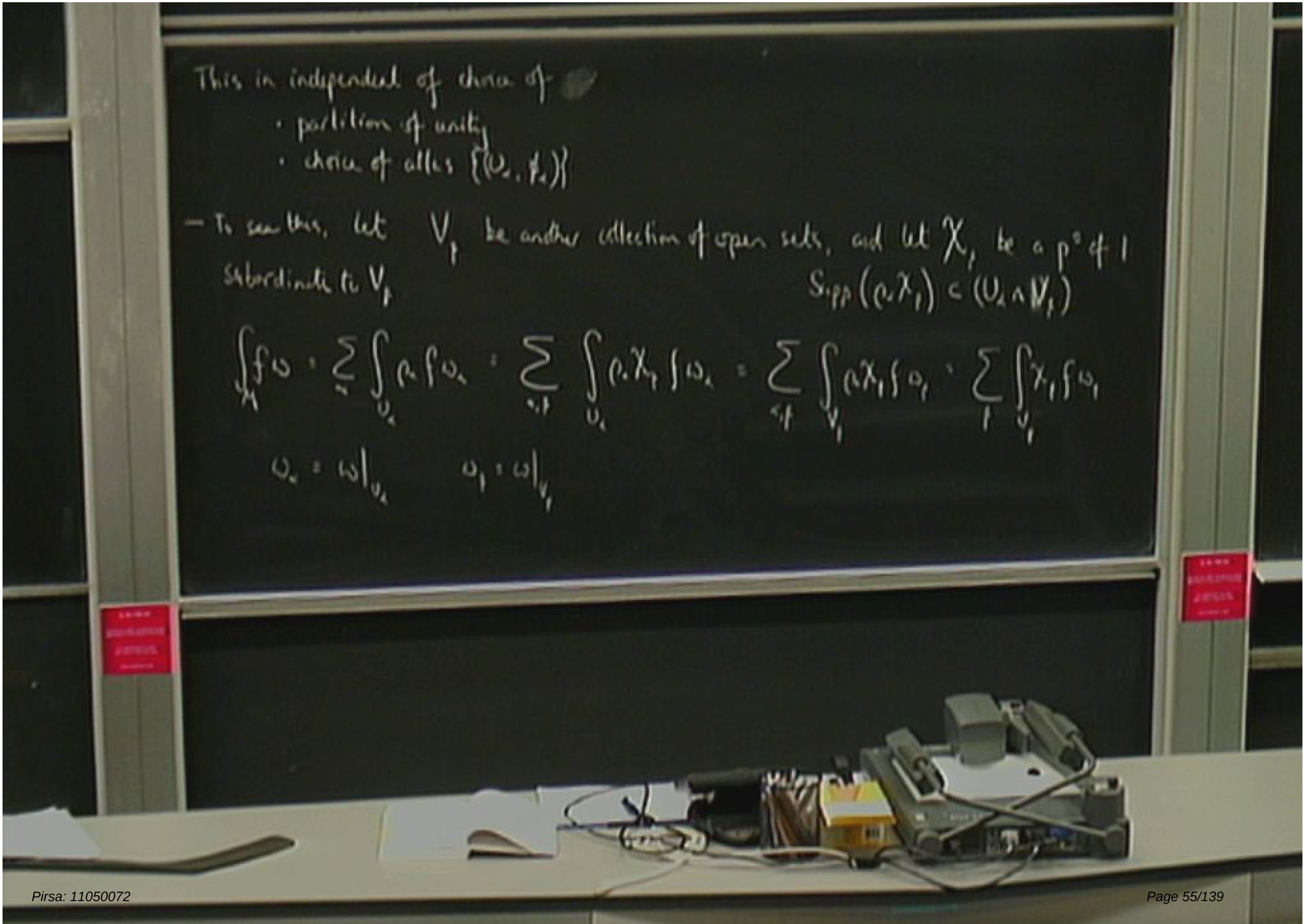
– [illegible] this, let $V_\beta$ be another collection of open sets, and let $\chi_\beta$ be a p° of 1 subordinate to $V_\beta$

$\mathrm{Supp}(\rho_\alpha \chi_\beta) \subset (U_\alpha \cap V_\beta)$

$$\int_M f\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta f \omega_\alpha = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta f \omega_\beta = \sum_\beta \int_{V_\beta} \chi_\beta f \omega_\beta$$



This is independent of choice of:

- partition of unity
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— To see this, let $V_\beta$ be another collection of open sets, and let $\chi_\beta$ be a p° of 1 subordinate to $V_\beta$

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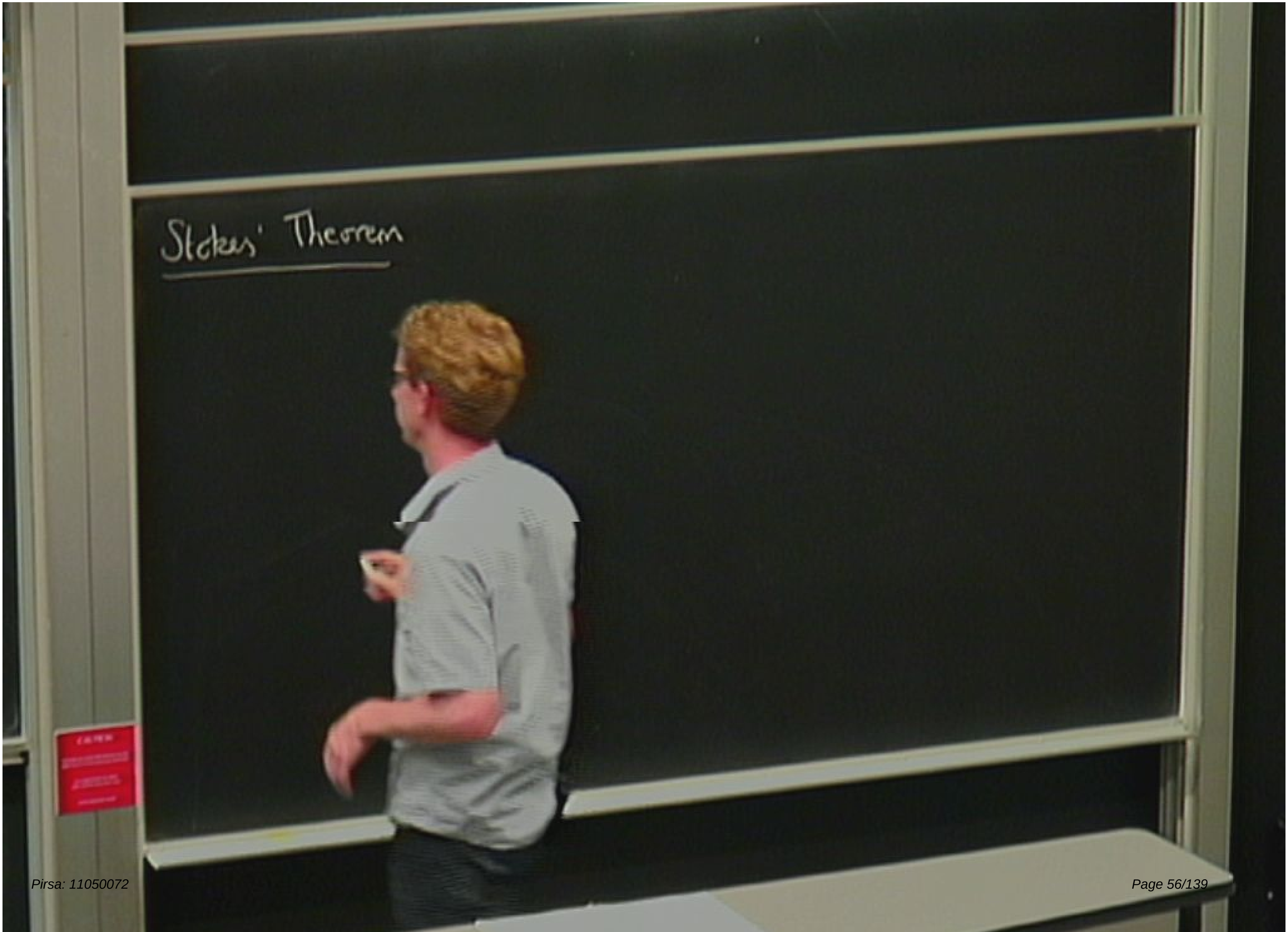
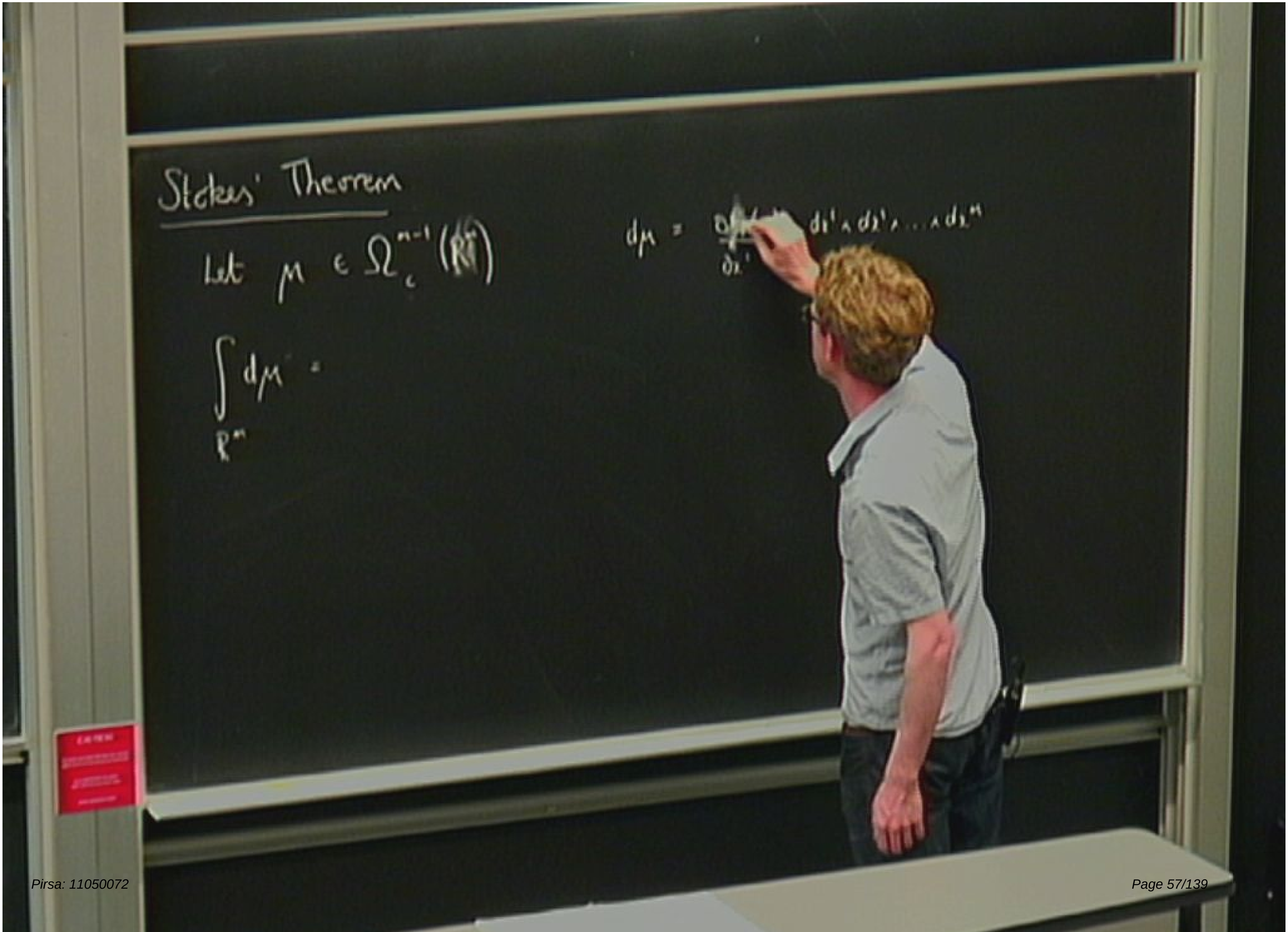
$$\int_M f\omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f \omega_\alpha = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta f \omega_\alpha = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta f \omega_\beta = \sum_\beta \int_{V_\beta} \chi_\beta f \omega_\beta$$

$\omega_\alpha = \omega|_{U_\alpha}$ $\qquad$ $\omega_\beta = \omega|_{V_\beta}$



## Stokes' Theorem

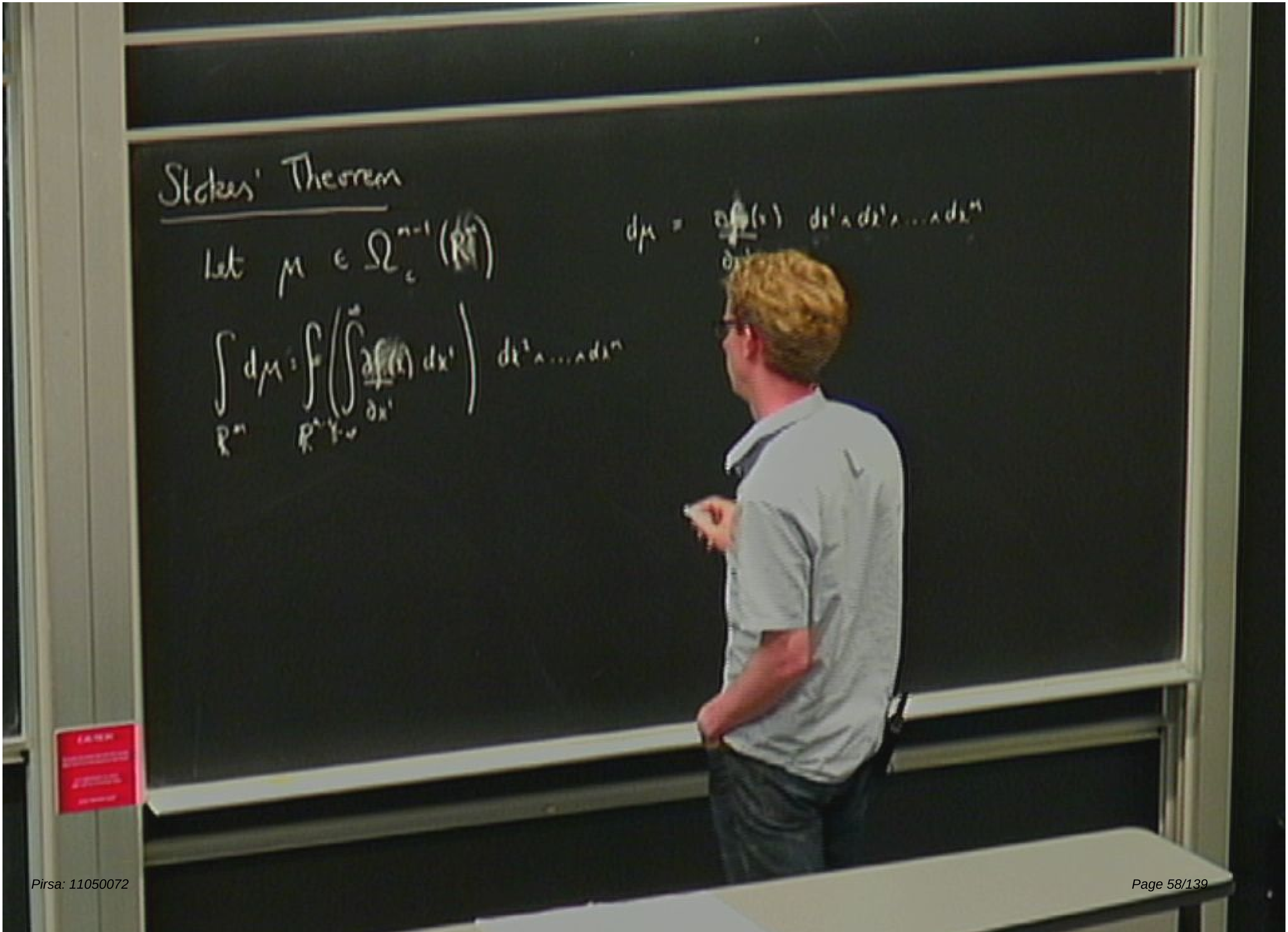


## Stokes' Theorem

Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d\mu =$$

$$d\mu = \frac{\partial f\,\text{[illegible]}}{\partial x^1}\, dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

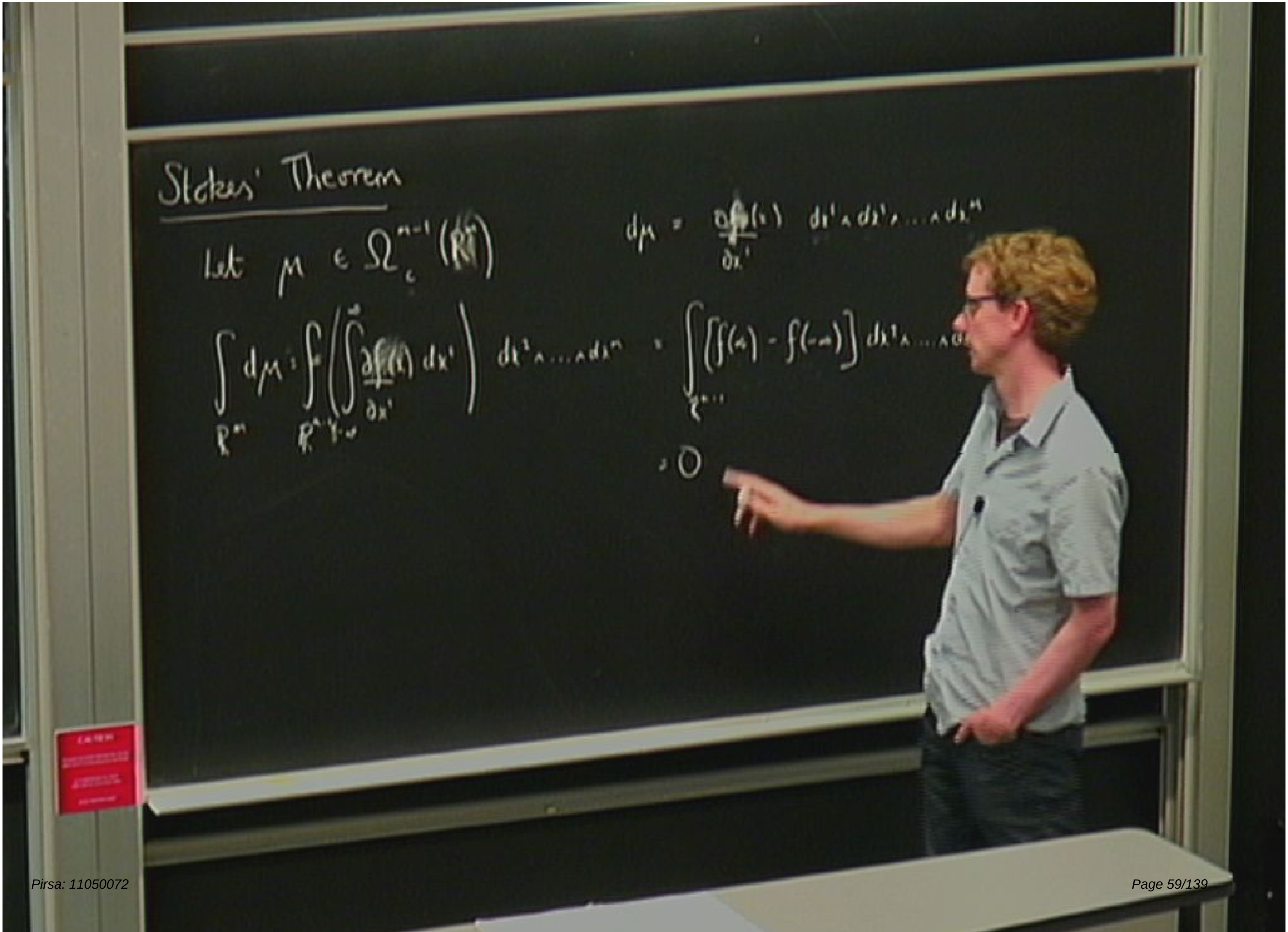


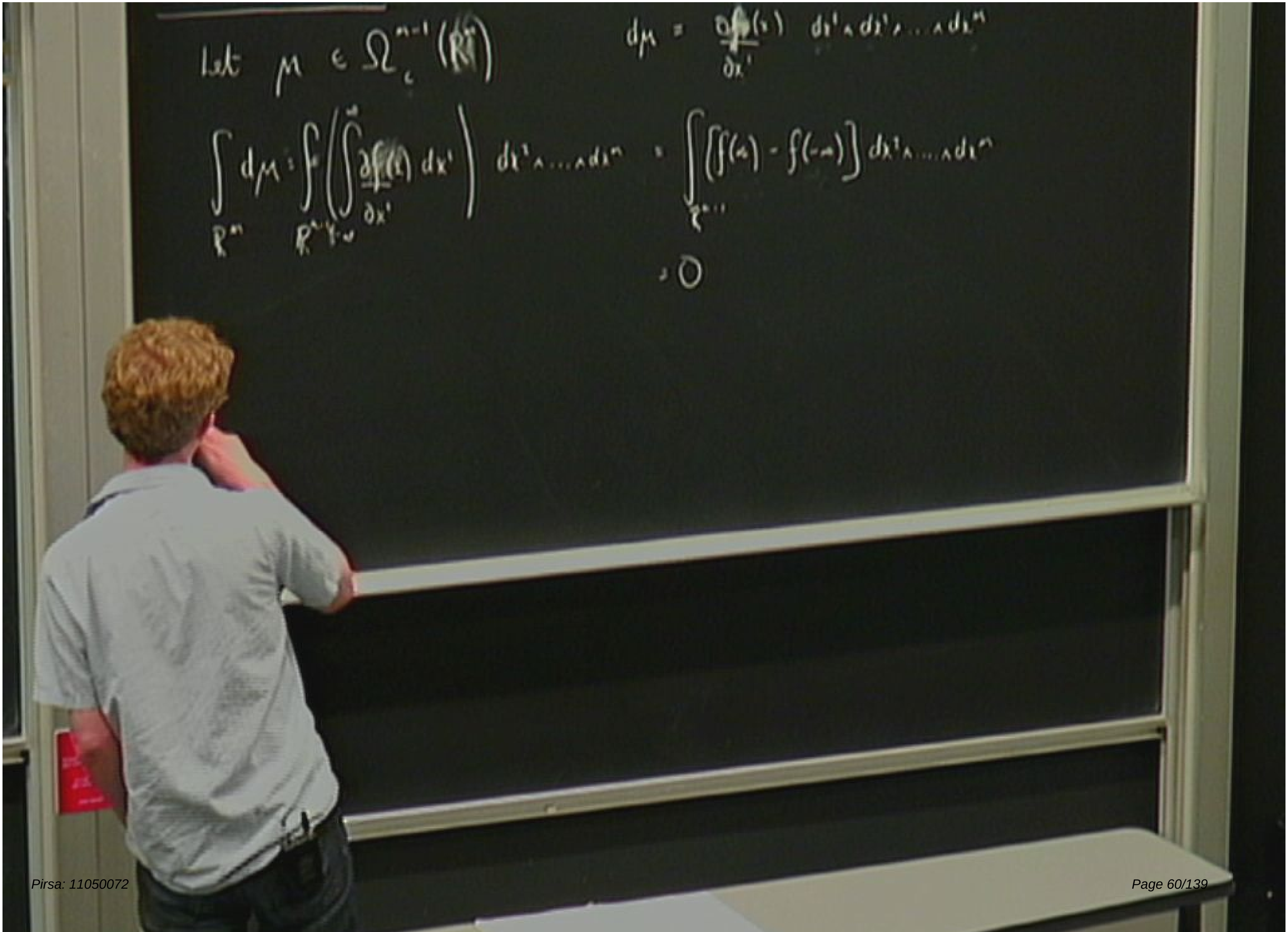
## Stokes' Theorem

Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial f_1}{\partial x^1}(x)\, dx^1 \right) dx^2 \wedge \dots \wedge dx^n$$

$$d\mu = \frac{\partial f_1}{\partial x^1}(x)\; dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$



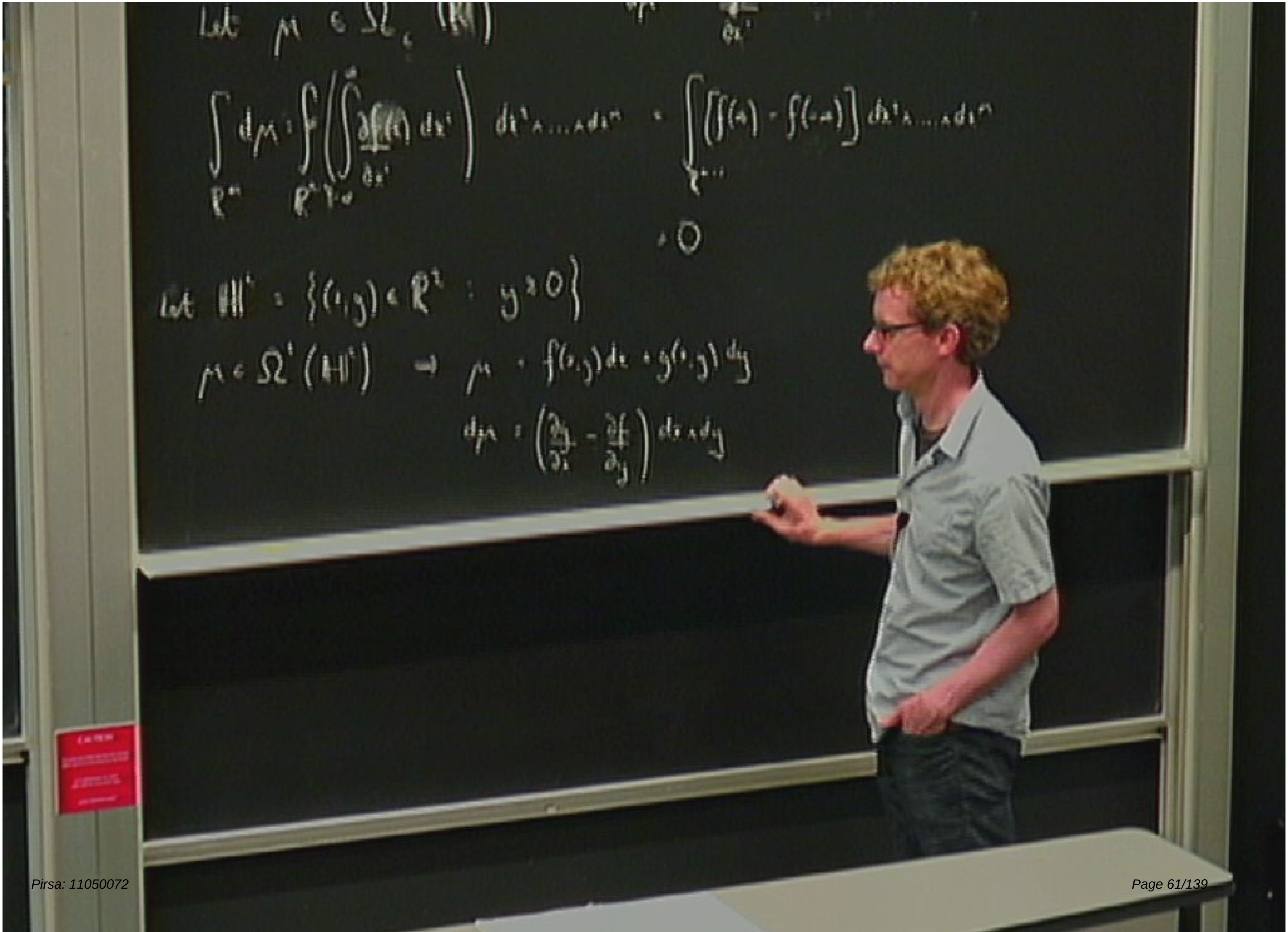


Let $\mu \in \Omega_c^{n-1}(\mathbb{R}^n)$

$$d\mu = \frac{\partial f}{\partial x^1}(x)\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$$

$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^1}(x)\, dx^1 \right) dx^2 \wedge \ldots \wedge dx^n = \int_{\mathbb{R}^{n-1}} \left[ f(\infty) - f(-\infty) \right] dx^2 \wedge \ldots \wedge dx^n$$

$$= 0$$



Let $\mu \in \Omega_c(\mathbb{R}^n)$

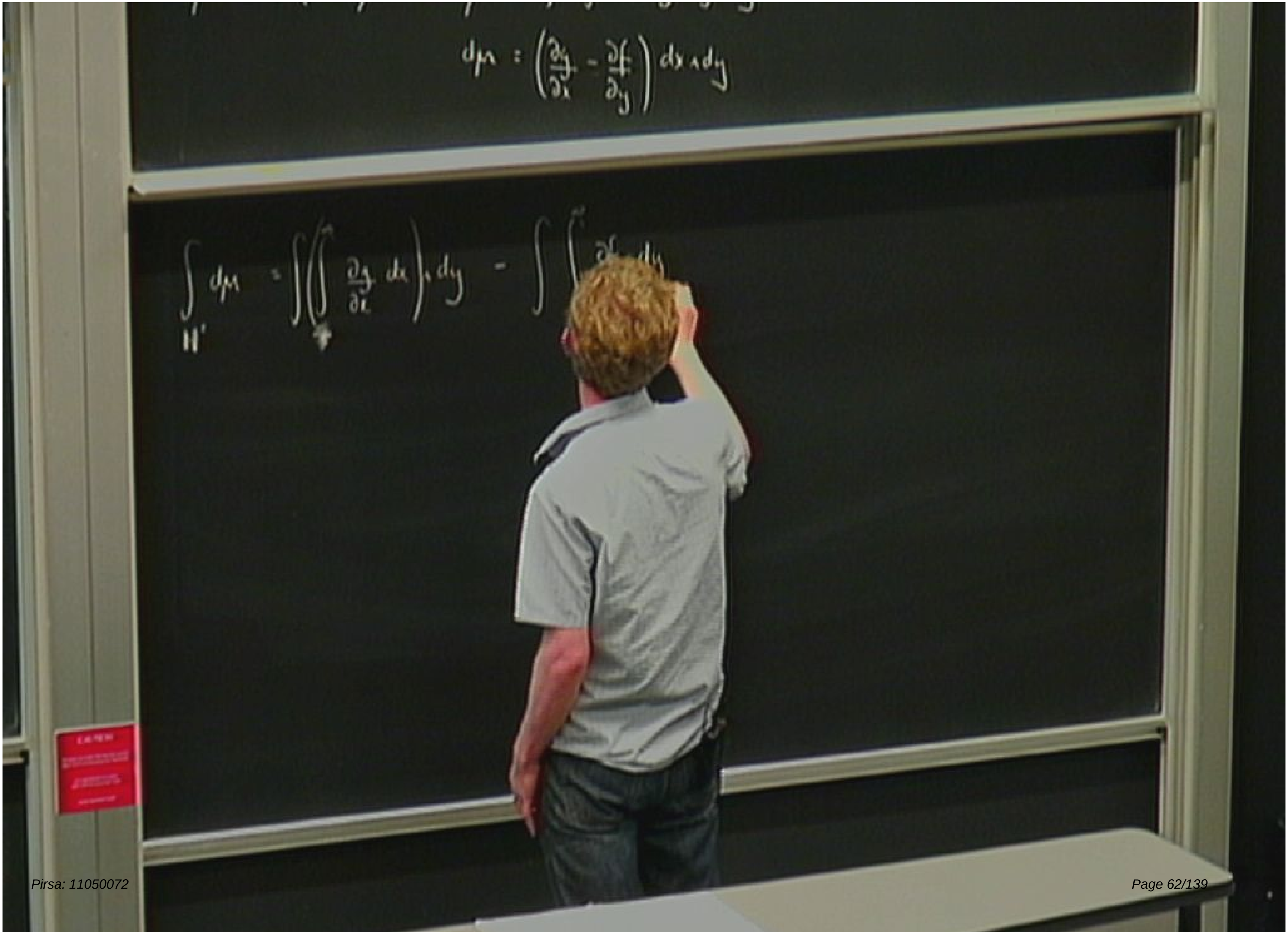
$$\int_{\mathbb{R}^n} d\mu = \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^1}(x)\, dx^1 \right) dx^2 \wedge \dots \wedge dx^n = \int_{\mathbb{R}^{n-1}} \left[ f(\infty) - f(-\infty) \right] dx^2 \wedge \dots \wedge dx^n$$

$$= 0$$

Let $\mathbb{H}^2 = \{ (x,y) \in \mathbb{R}^2 : y \geq 0 \}$

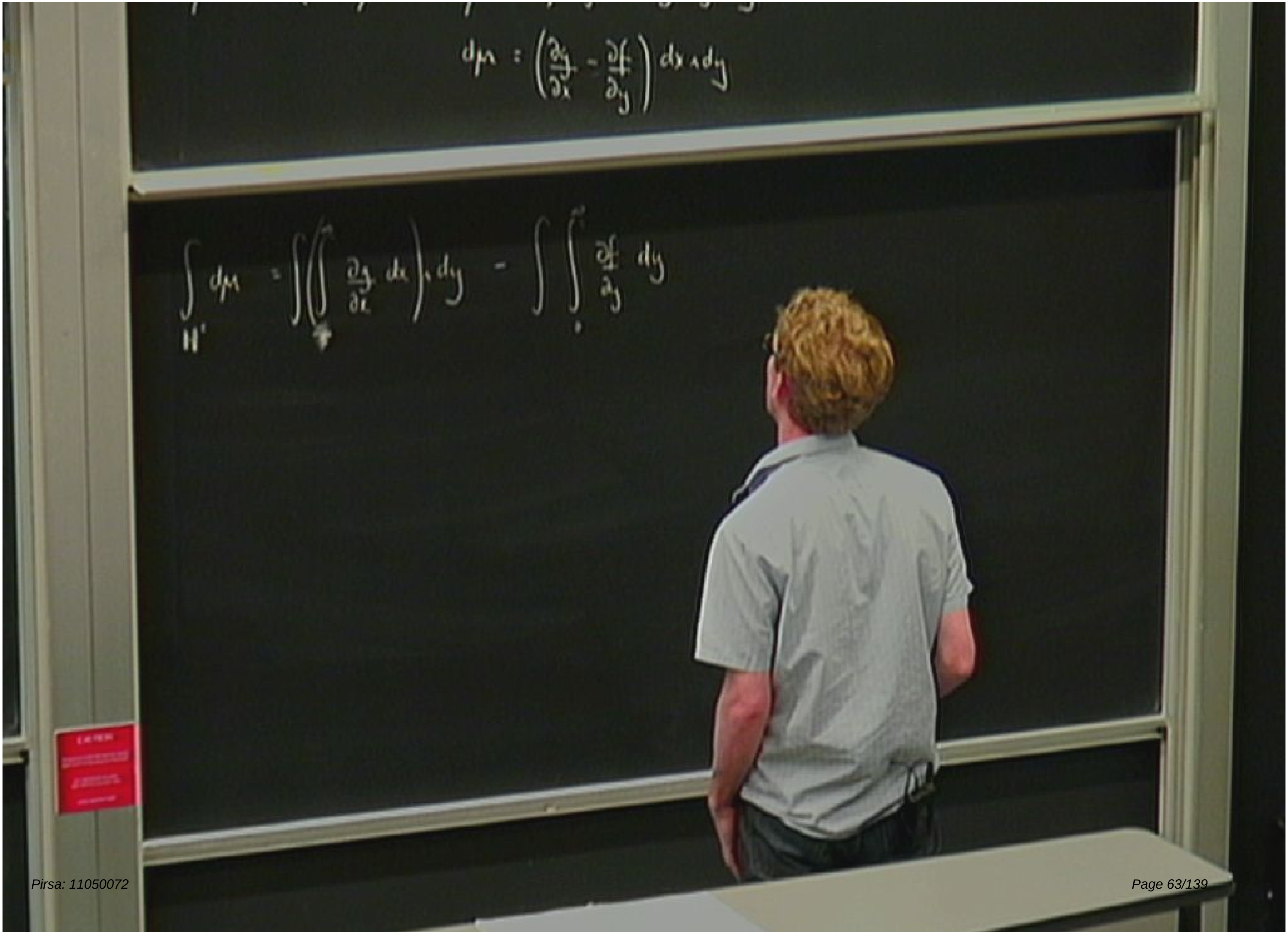
$$\mu \in \Omega^1(\mathbb{H}^2) \Rightarrow \mu = f(x,y)\,dx + g(x,y)\,dy$$

$$d\mu = \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$



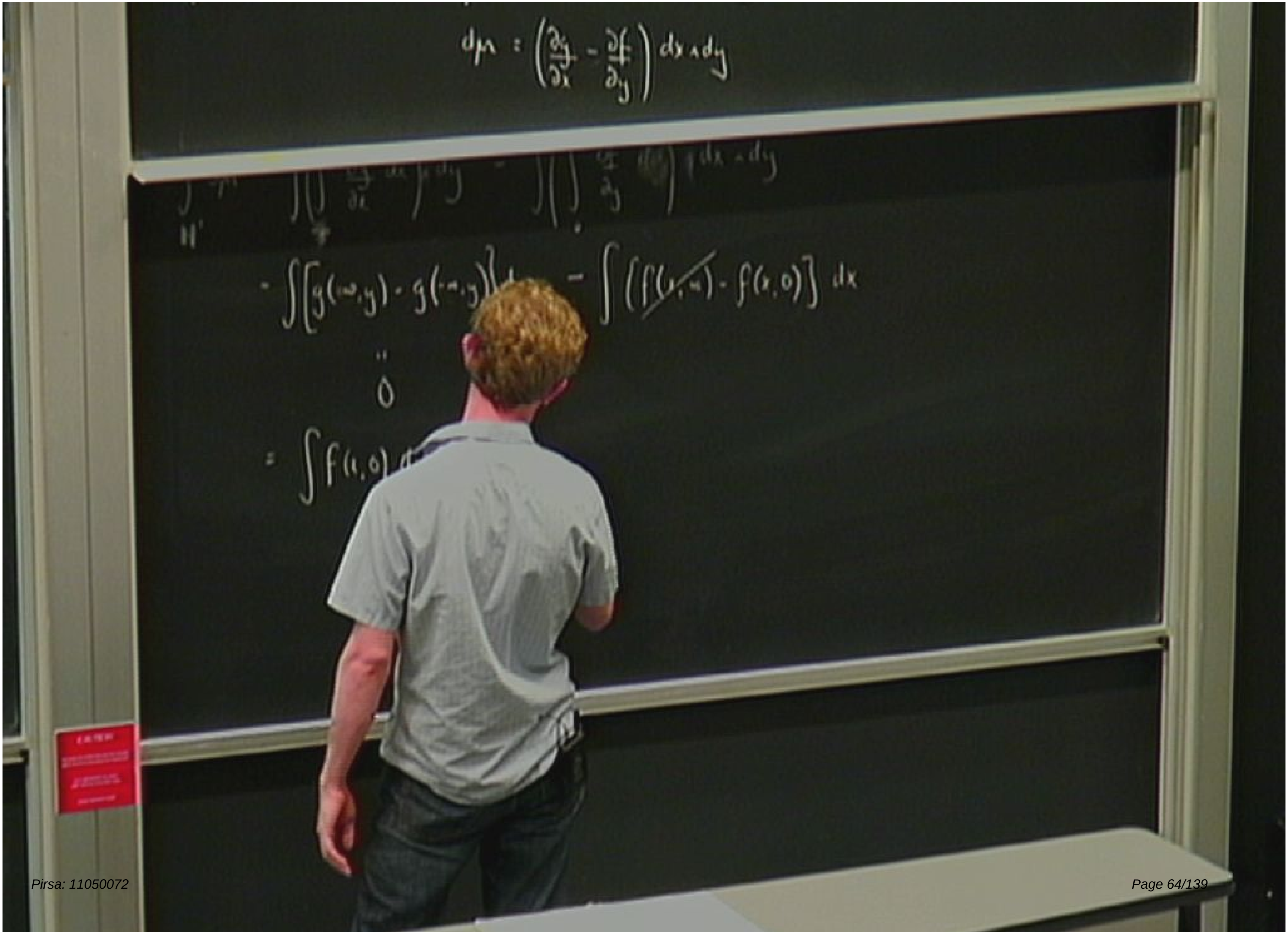
$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\int_{H'} d\mu = \int\left(\int \frac{\partial g}{\partial x}\, dx\right) \wedge dy - \int \int \partial f \; dy$$



$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\int_{H^+} d\mu = \int\left(\int \frac{\partial g}{\partial x}\, dx\right) dy - \int \int_0 \frac{\partial f}{\partial y}\, dy$$

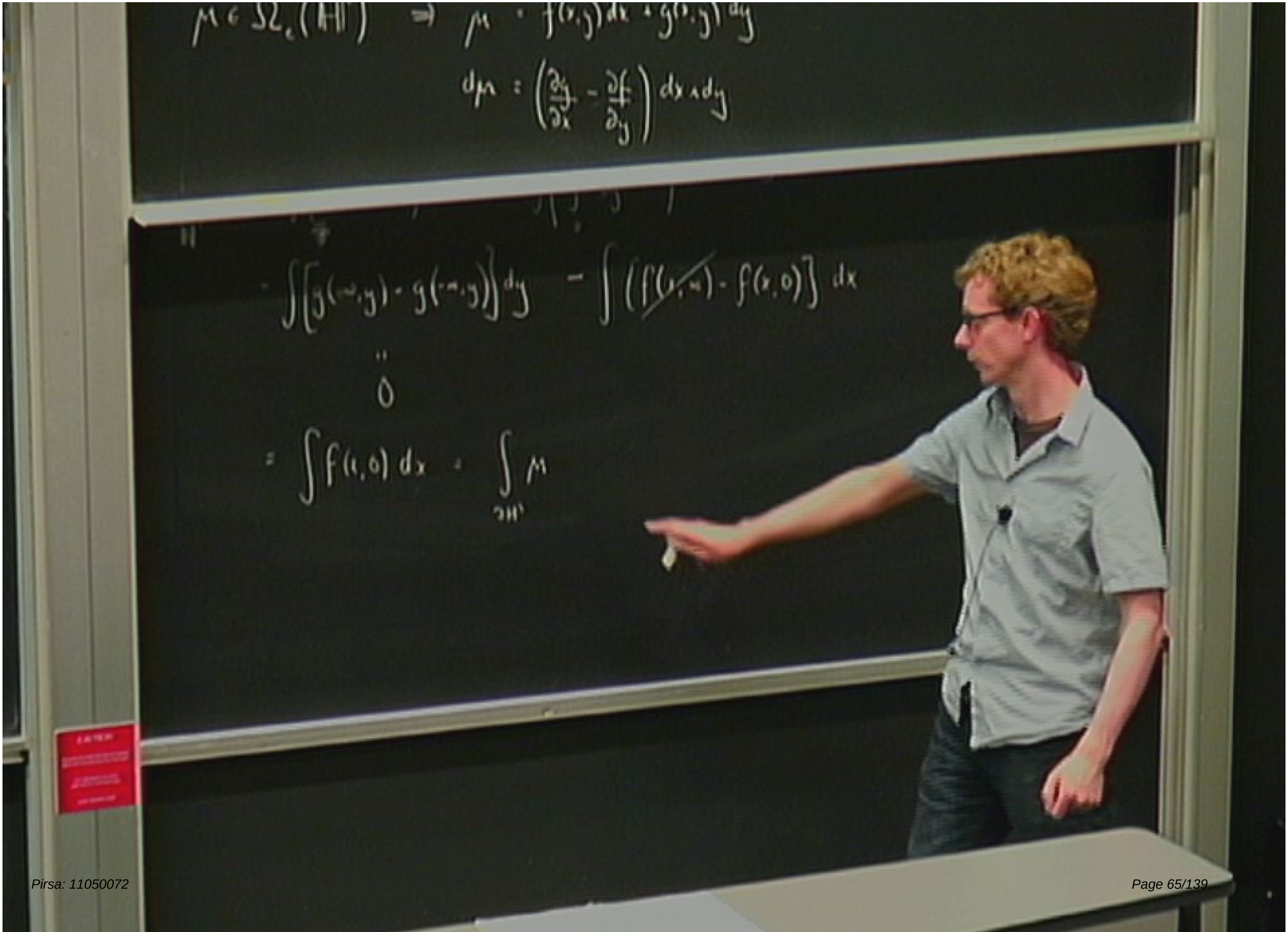


$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$\int_{H'} d\mu = \iint \frac{\partial g}{\partial x}\, dx \, dy - \iint \frac{\partial f}{\partial y} \, dx \wedge dy$$

$$= \int \left[\underset{\overset{\shortparallel}{0}}{g(\infty,y) - g(-\infty,y)}\right] dy - \int \left[\cancel{f(x,\infty)} - f(x,0)\right] dx$$

$$= \int f(x,0)$$



$$\mu \in \Omega_c(\mathbb{H}') \Rightarrow \mu = f(x,y)\,dx + g(x,y)\,dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$= \int \underset{\overset{\shortparallel}{0}}{\left[g(\infty,y) - g(-\infty,y)\right]} dy - \int \left(\left[\cancel{f(x,\infty)} - f(x,0)\right] dx\right.$$

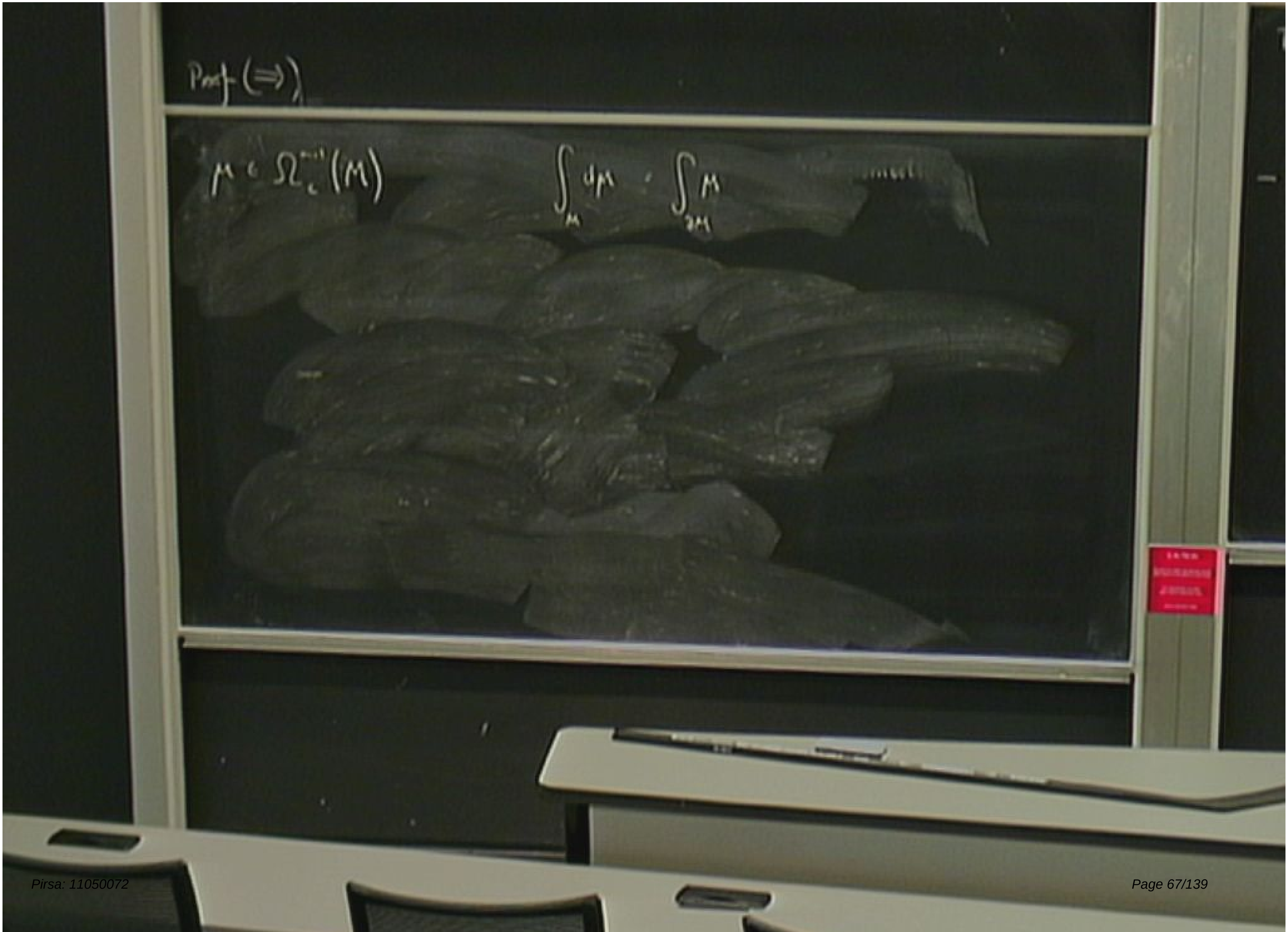
$$= \int f(x,0)\,dx = \int_{\partial \mathbb{H}'} \mu$$

$$\mu \in \Omega_c(\mathbb{H}') \Rightarrow \mu = f(x,y)\,dx + g(x,y)\,dy$$

$$d\mu = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$$

$$= \int \underbrace{\left[g(+\infty,y) - g(-\infty,y)\right]}_{0} dy \;-\; \int \left[\cancel{f(x,\infty)} - f(x,0)\right] dx$$

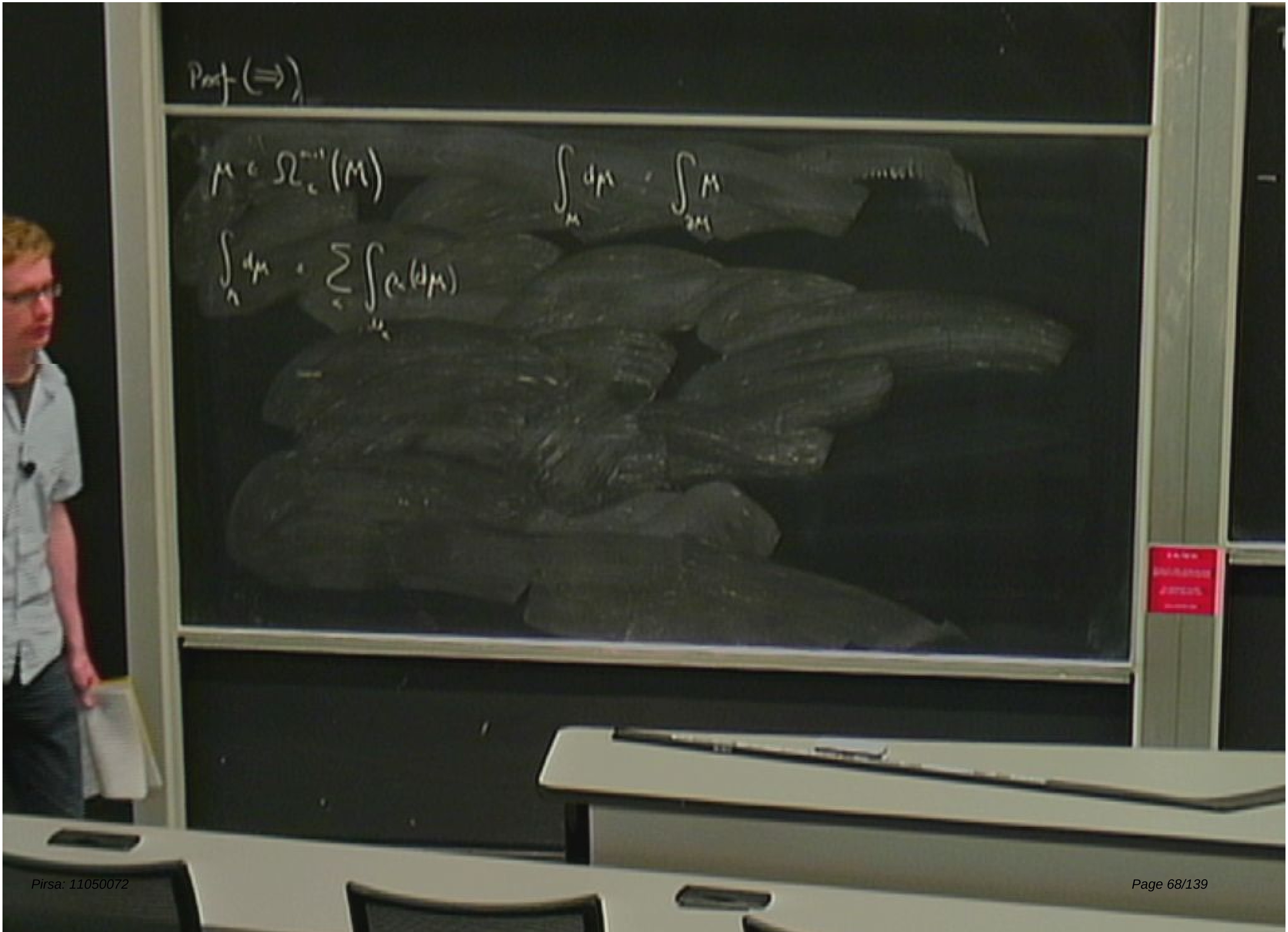
$$= \int f(x,0)\,dx = \int_{\partial \mathbb{H}'} \mu$$



Proof ($\Rightarrow$)

$\mu \in \Omega_c^{n-1}(M)$

$$\int_M d\mu = \int_{\partial M} \mu$$

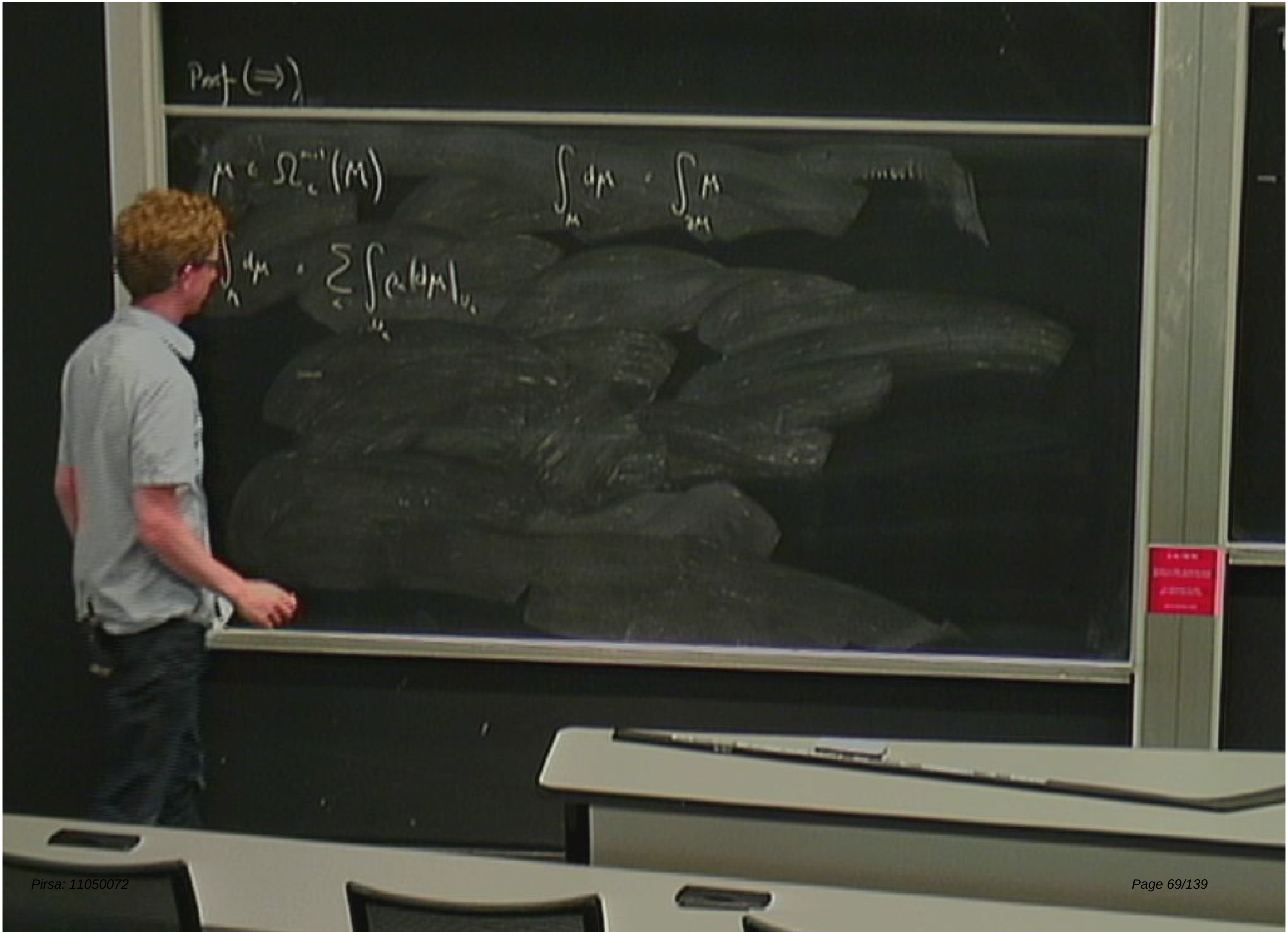


Proof ($\Longrightarrow$)

$\mu \in \Omega_c^{n-1}(M)$

$$\int_M d\mu = \int_{\partial M} \mu$$

$$\int_M d\mu = \sum_\alpha \int_{U_\alpha} \rho_\alpha (d\mu)$$

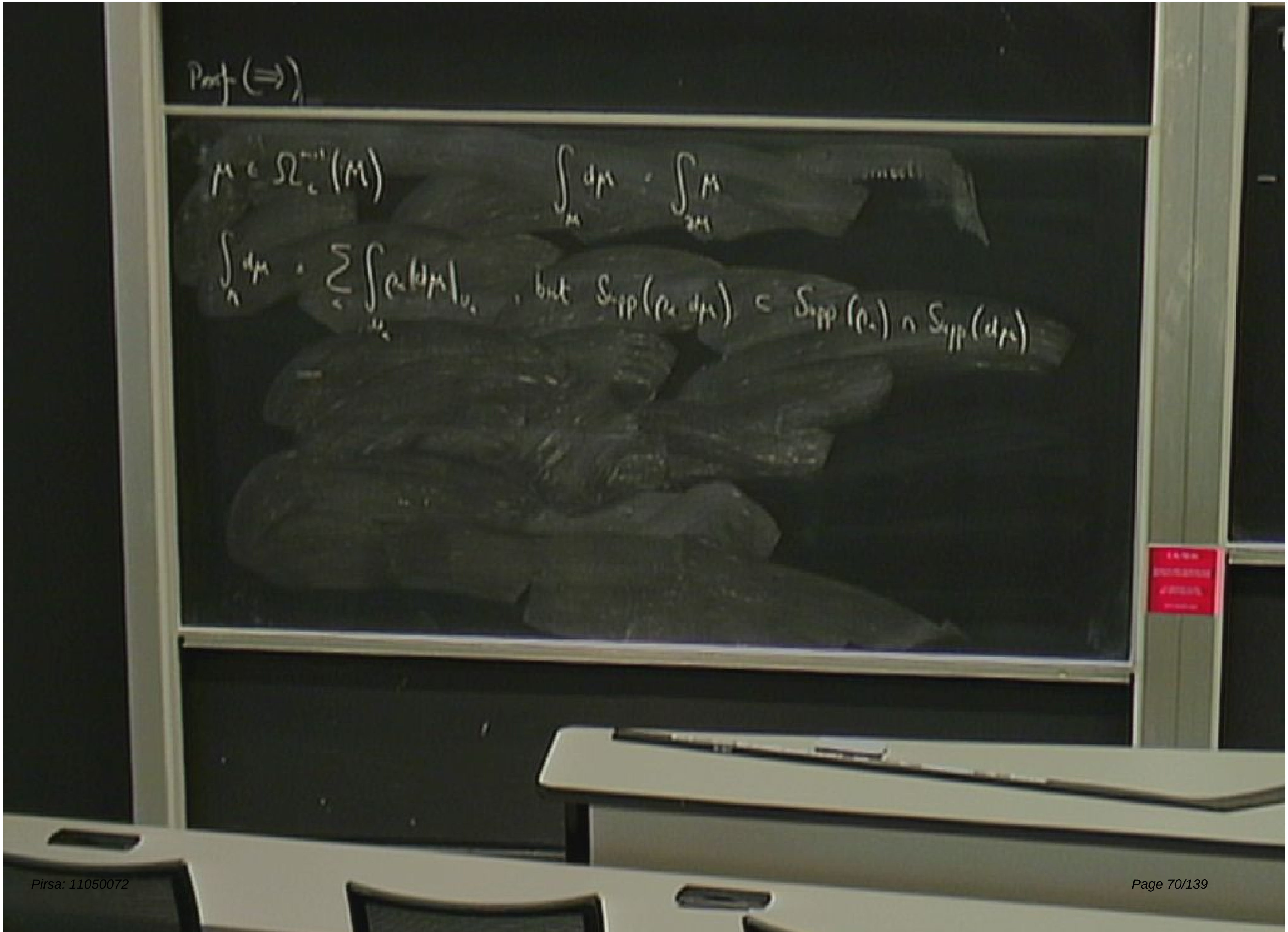


Proof (⟹)

$\mu \in \Omega^{n-1}_c(M)$

$$\int_M d\mu = \int_{\partial M} \mu$$

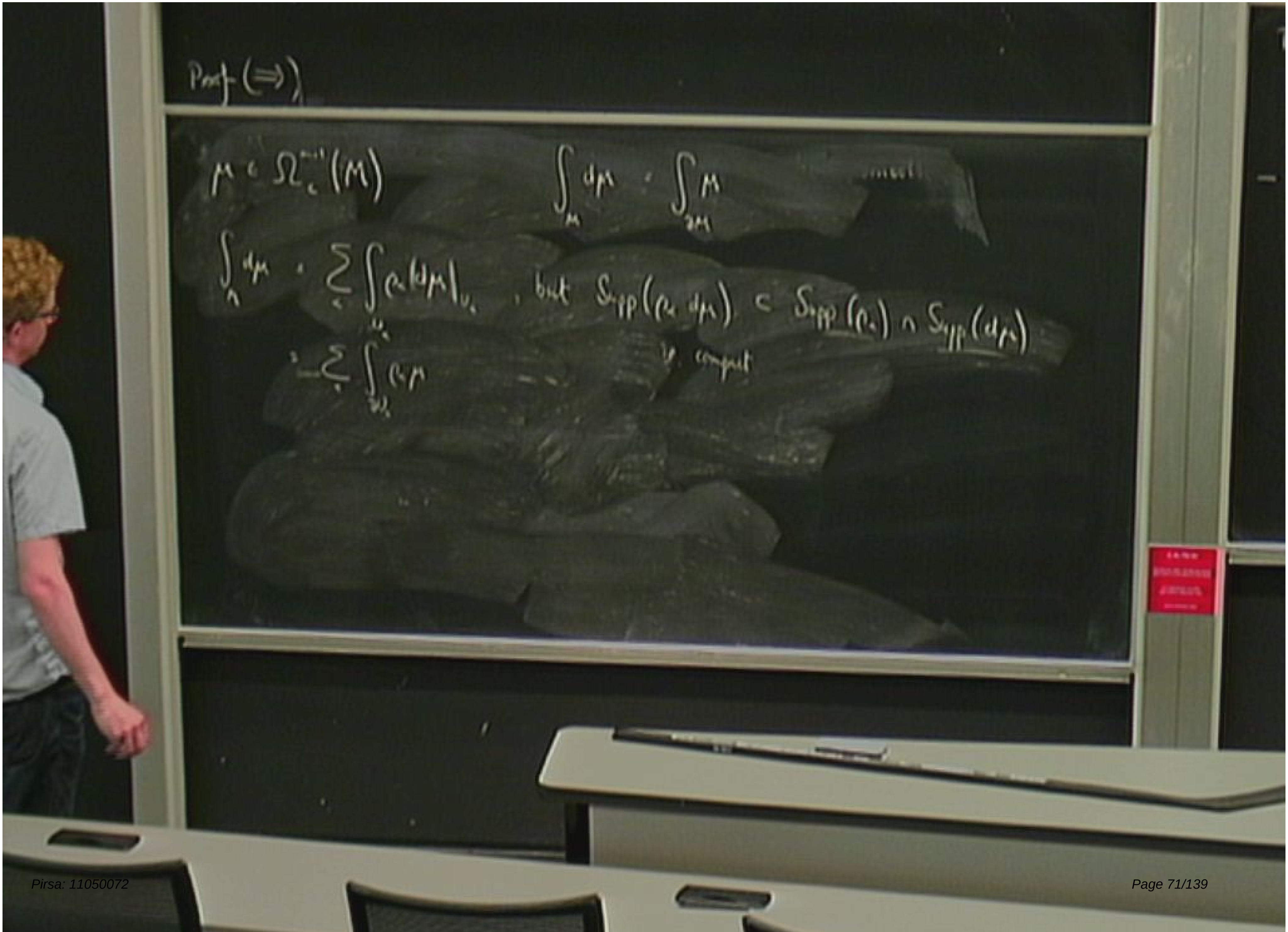
$$\int_M d\mu = \sum_i \int_{U_i} (\rho_i d\mu)|_{U_i}$$



Proof (⟹)

$\mu \in \Omega_c^{n-1}(M)$ $\qquad$ $\int_M d\mu = \int_{\partial M} \mu$

$\int_M d\mu = \sum_\alpha \int_{U_\alpha} \rho_\alpha d\mu|_{U_\alpha}$ , but $\mathrm{Supp}(\rho_\alpha d\mu) \subset \mathrm{Supp}(\rho_\alpha) \cap \mathrm{Supp}(d\mu)$



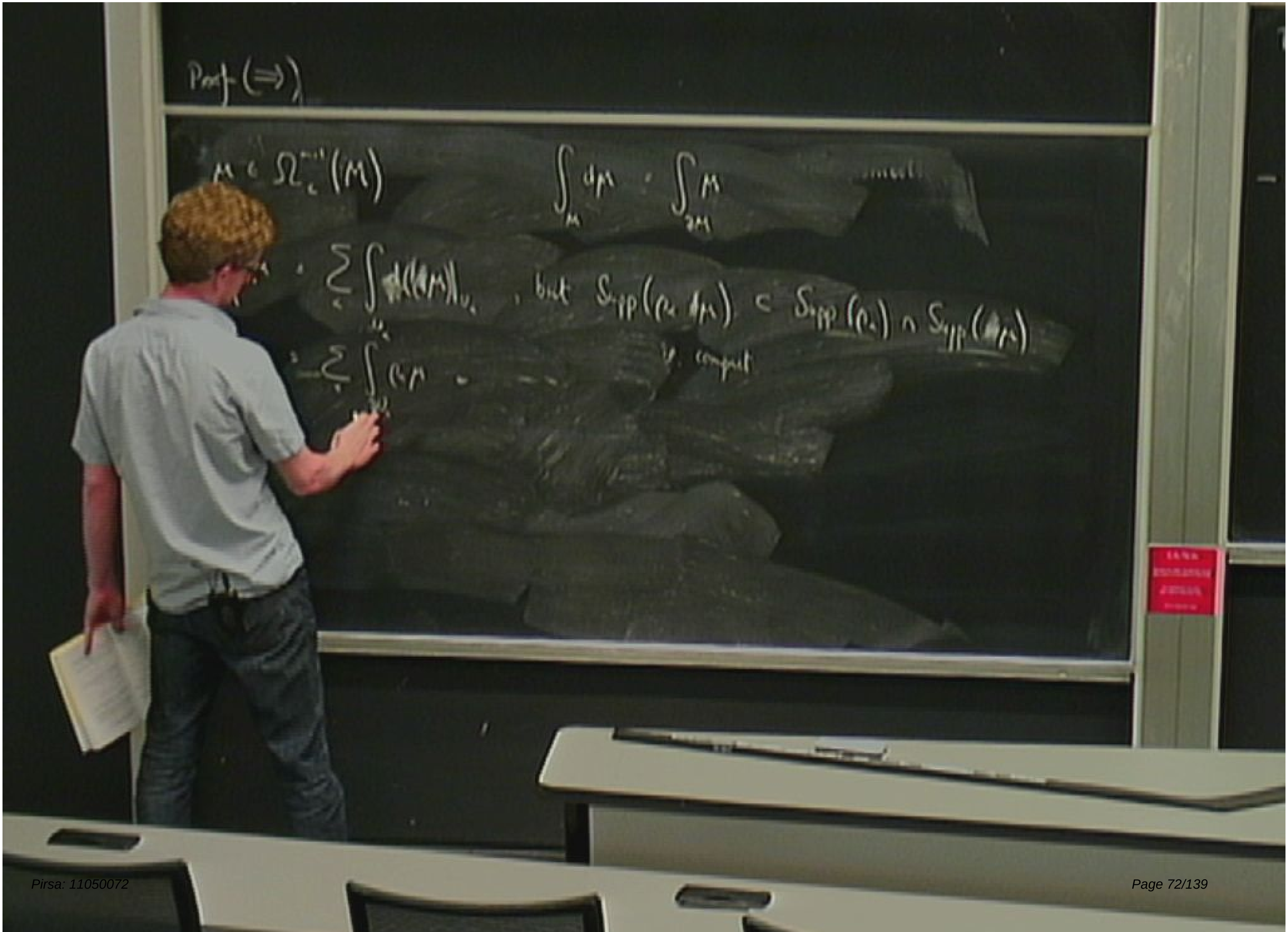
Proof ($\Rightarrow$)

$\mu \in \Omega_c^{n-1}(M)$ $\int_M d\mu = \int_{\partial M} \mu$ want

$\int_M d\mu = \sum_\alpha \int_{U_\alpha} (\rho_\alpha d\mu)|_{U_\alpha}$ , but $\operatorname{Supp}(\rho_\alpha d\mu) \subset \underline{\operatorname{Supp}(\rho_\alpha)} \cap \underline{\operatorname{Supp}(d\mu)}$

compact

$= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \mu$



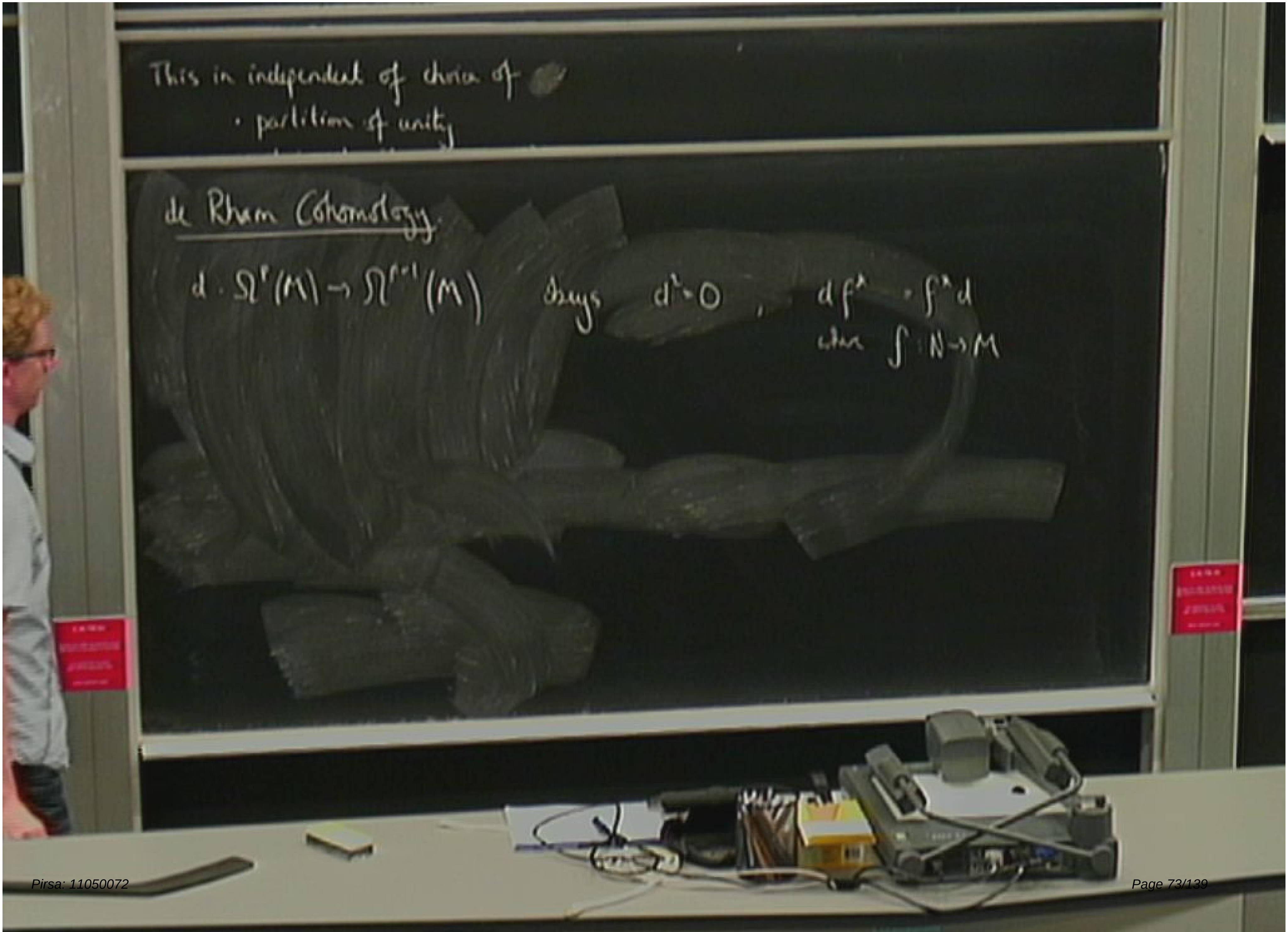
Proof ($\Rightarrow$)

$\mu \in \Omega_c^{n-1}(M)$   $\int_M d\mu = \int_{\partial M} \mu$   [illegible]

$= \sum_i \int_{U_i} d(\rho_i \mu)|_{U_i}$ , but $\mathrm{Supp}(\rho_i d\mu) \subset \mathrm{Supp}(\rho_i) \cap \mathrm{Supp}(d\mu)$

compact

$= \sum_i \int_{U_i} \rho_i \mu = \ldots$



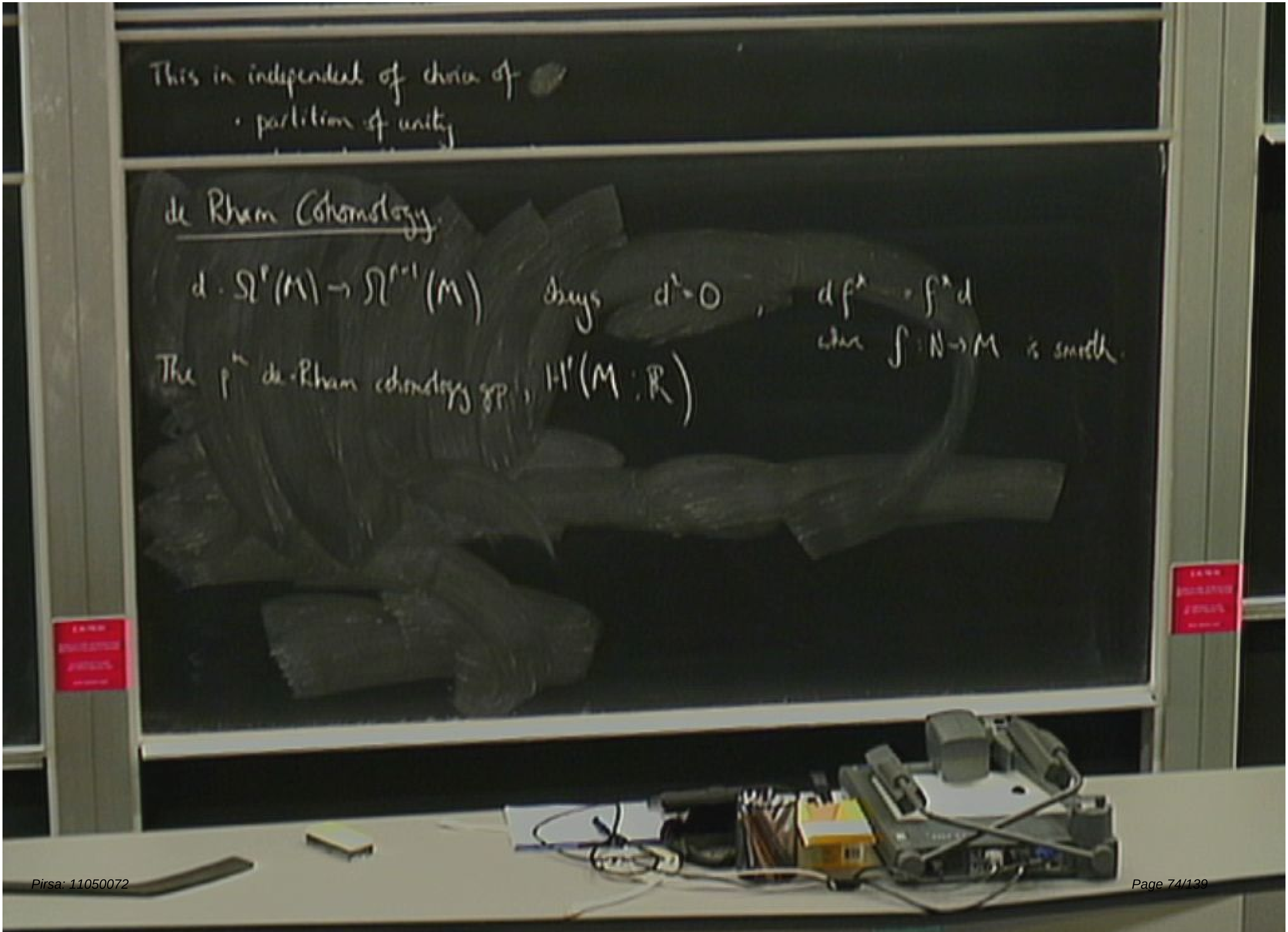
This is independent of choice of

- partition of unity

## de Rham Cohomology.

$d : \Omega^p(M) \to \Omega^{p+1}(M)$ obeys $d^2 = 0$ , $d f^* = f^* d$

where $f : N \to M$



This in independent of choice of
- partition of unity

## de Rham Cohomology.

$d : \Omega^p(M) \to \Omega^{p+1}(M)$ obeys $d^2 = 0$, $d f^* = f^* d$

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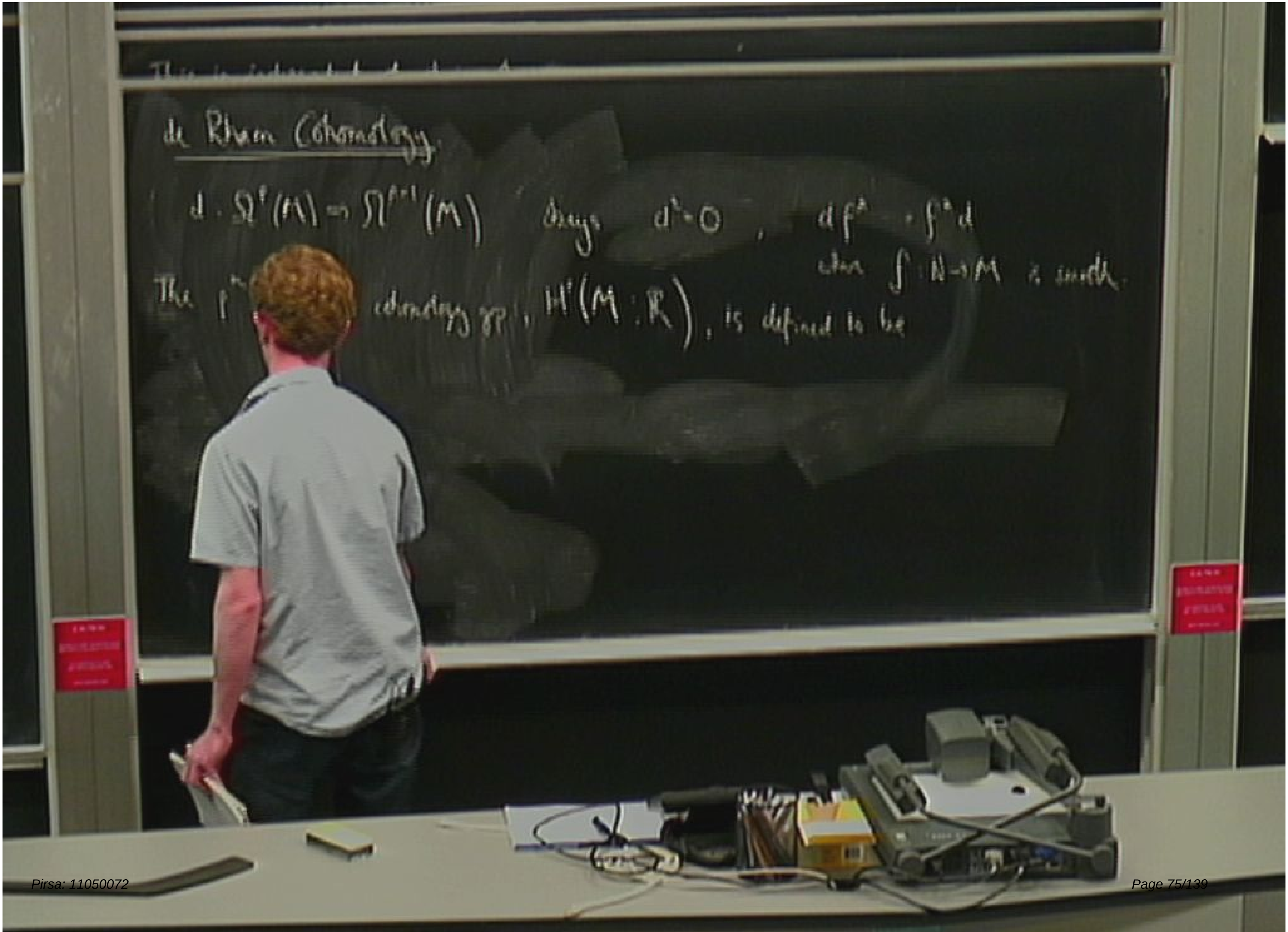
The $p^{th}$ de Rham cohomology gp, $H^p(M ; \mathbb{R})$



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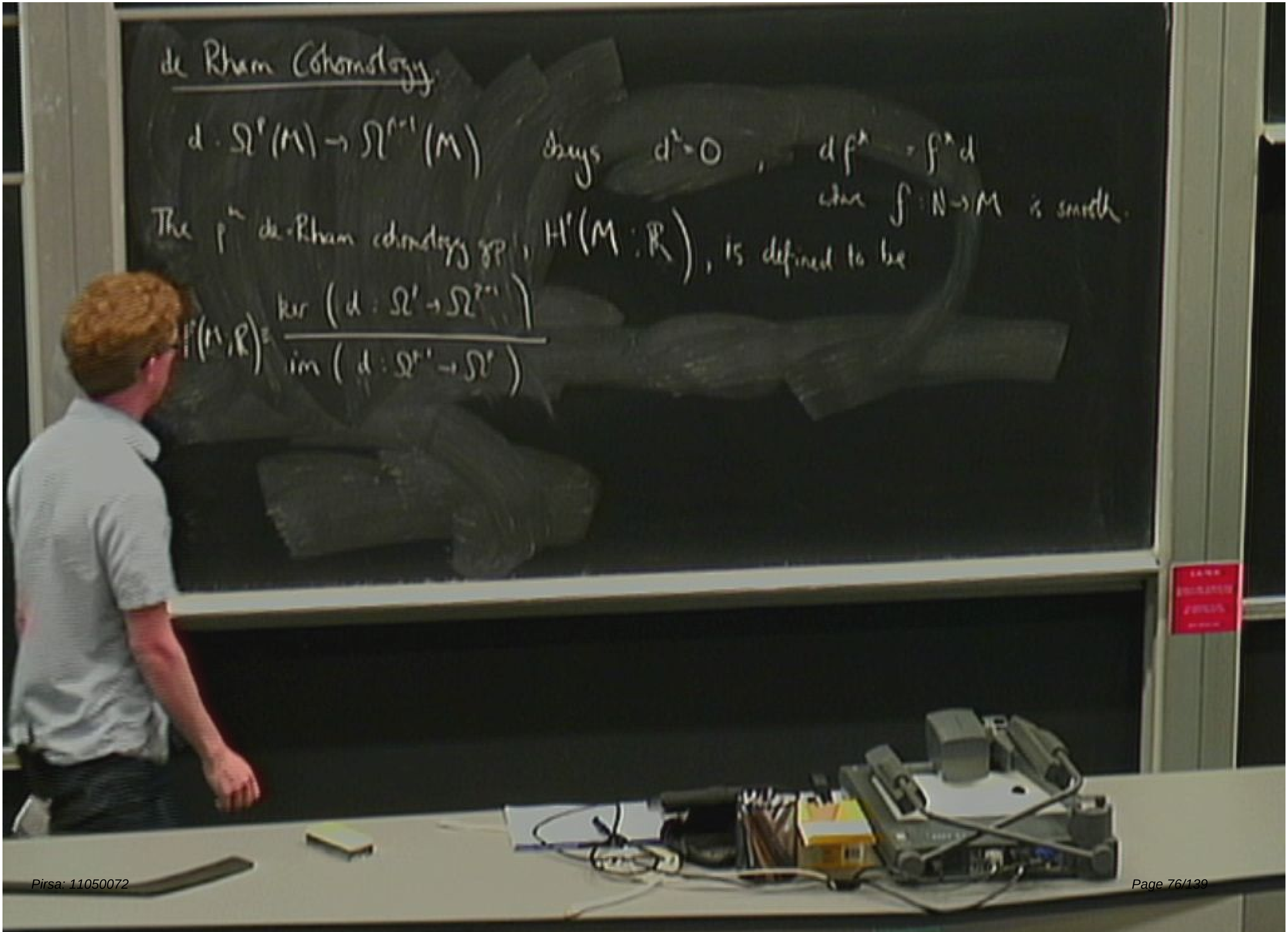

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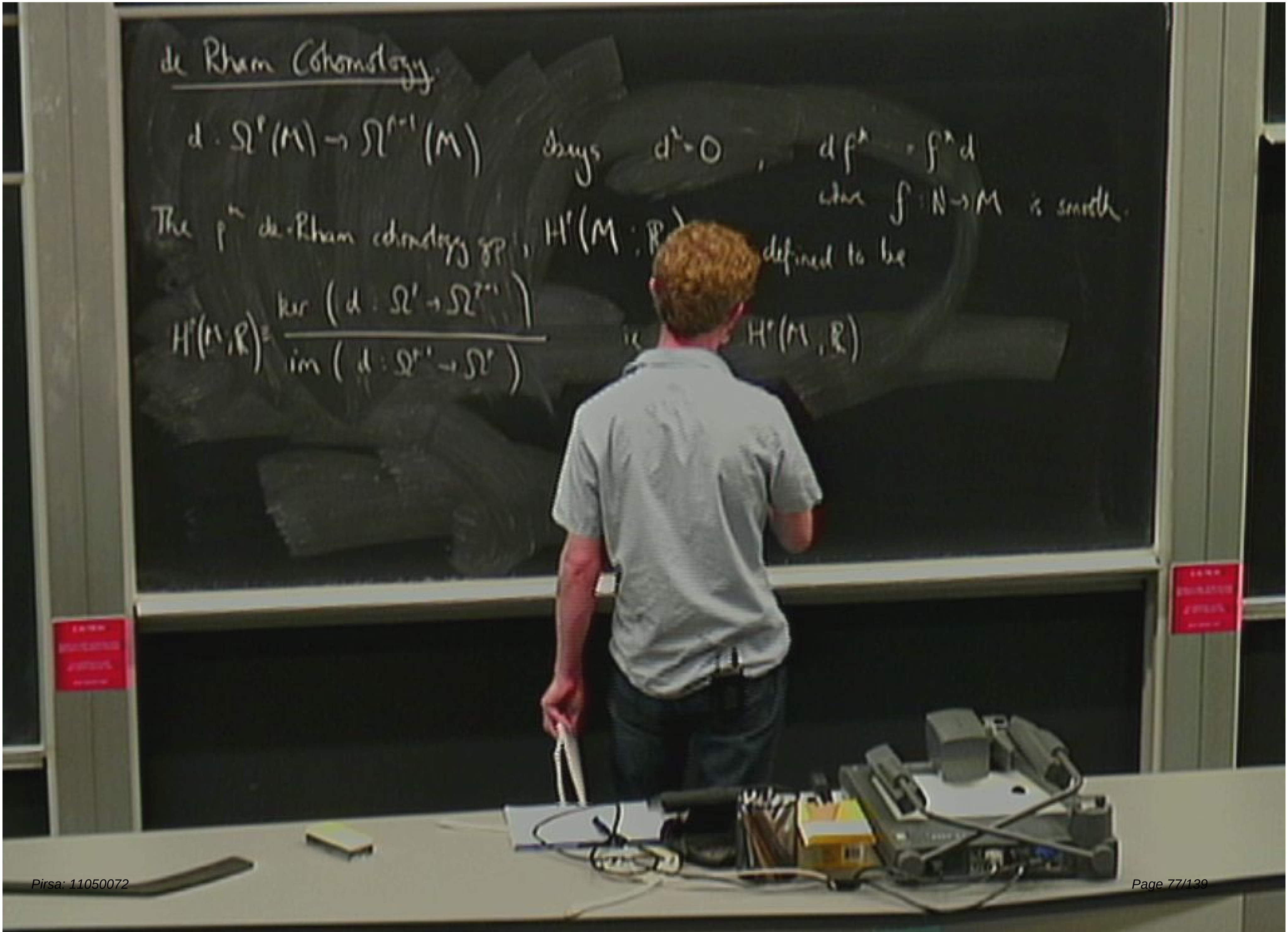

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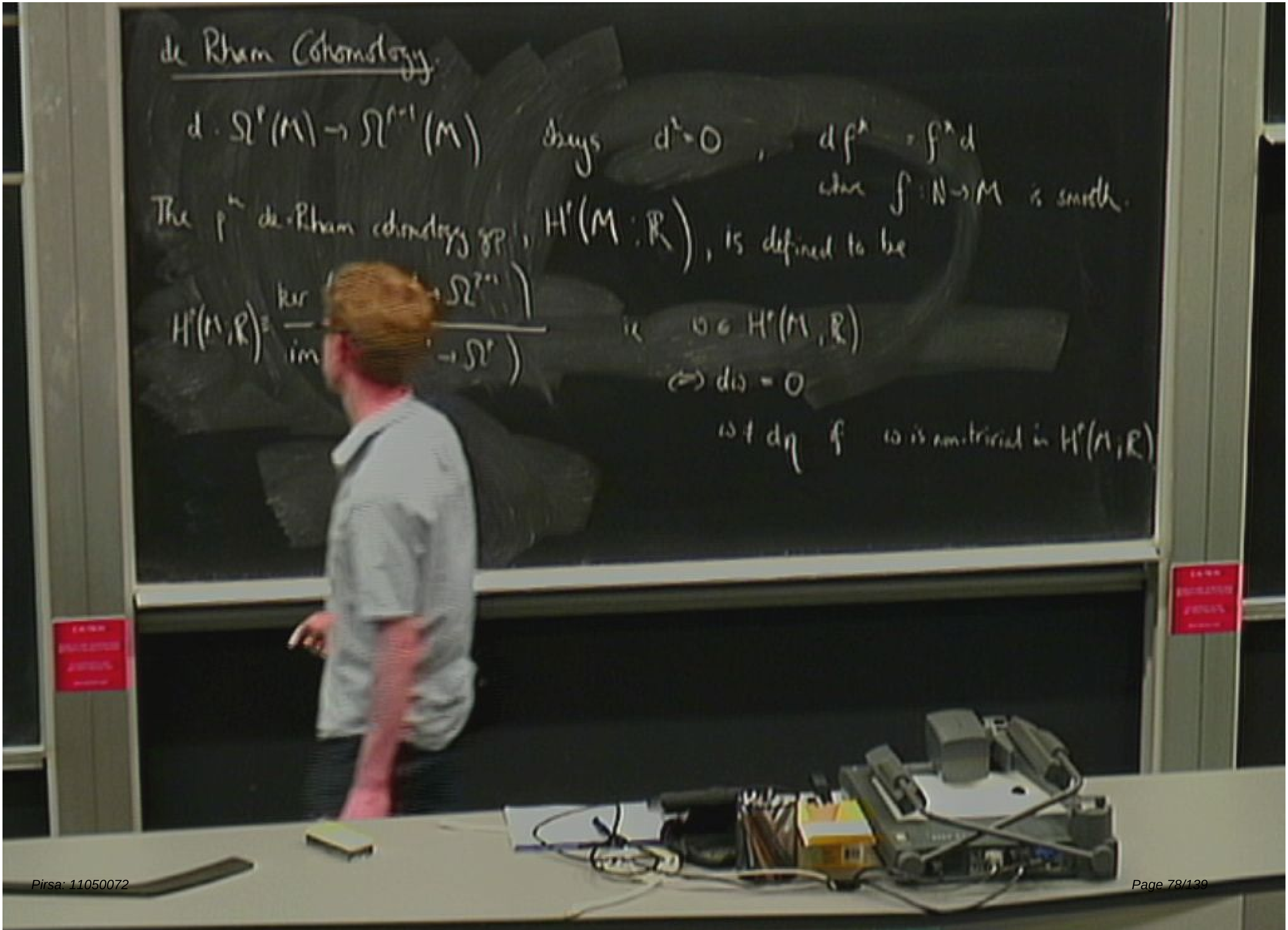

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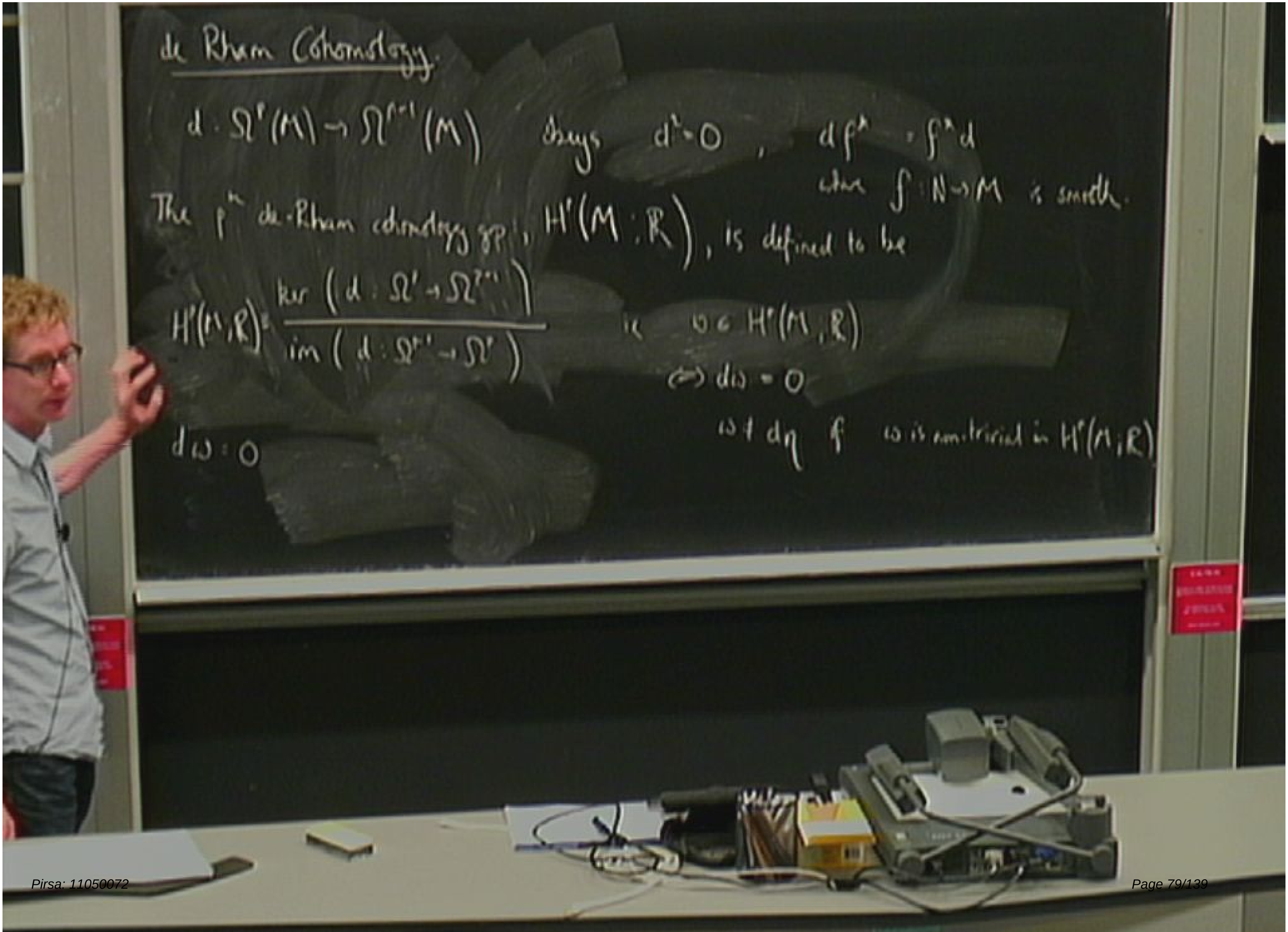

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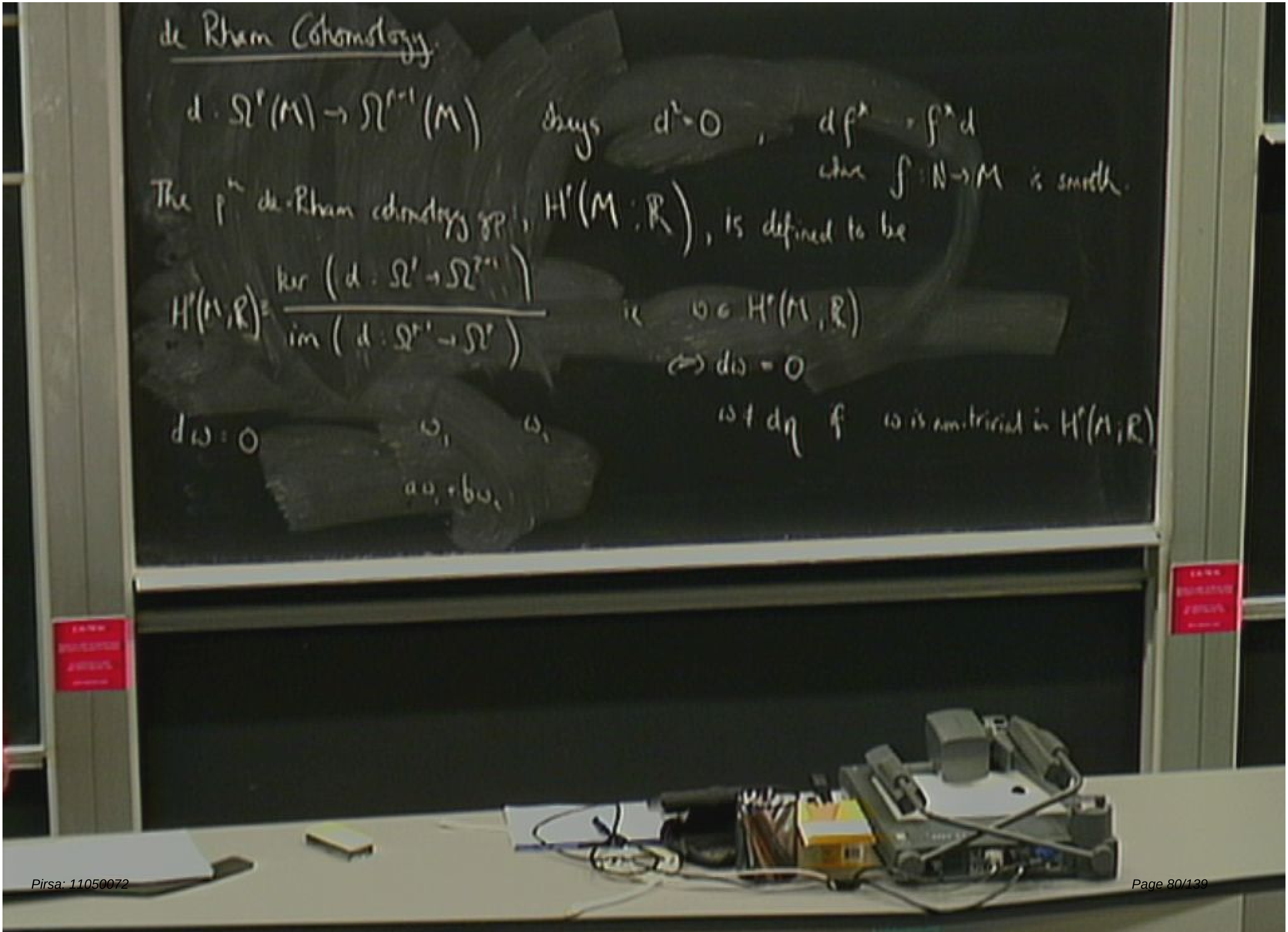
$\omega \neq d\eta$ if $\omega$ is non-trivial in $H^p(M;\mathbb{R})$

$d\omega = 0$



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ie $\omega \in H^p(M;\mathbb{R})$

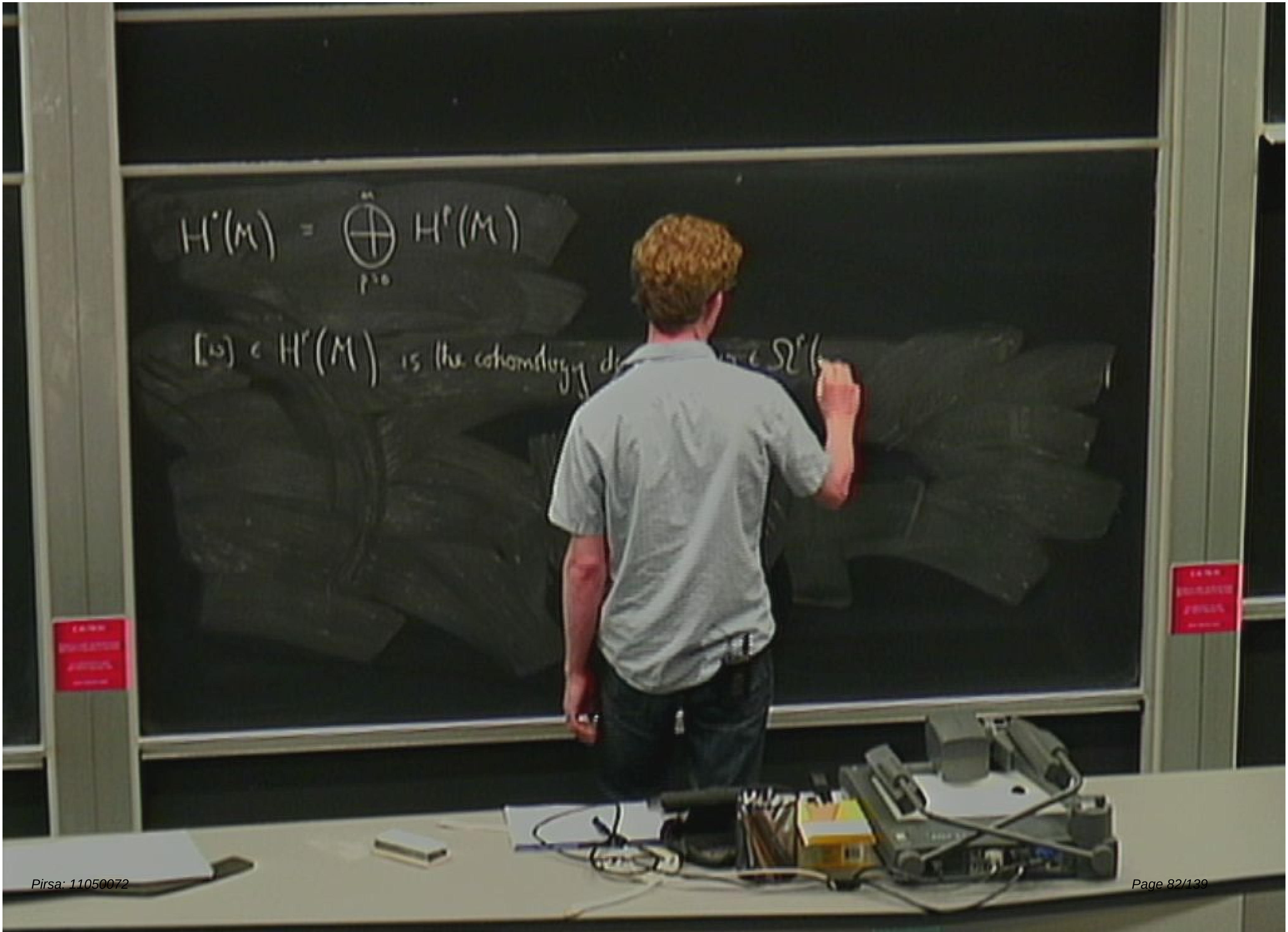
$\Leftrightarrow d\omega = 0$

$\omega \neq d\eta$ if $\omega$ is nontrivial in $H^p(M;\mathbb{R})$

$d\omega = 0$ $\omega_1$ $\omega_2$

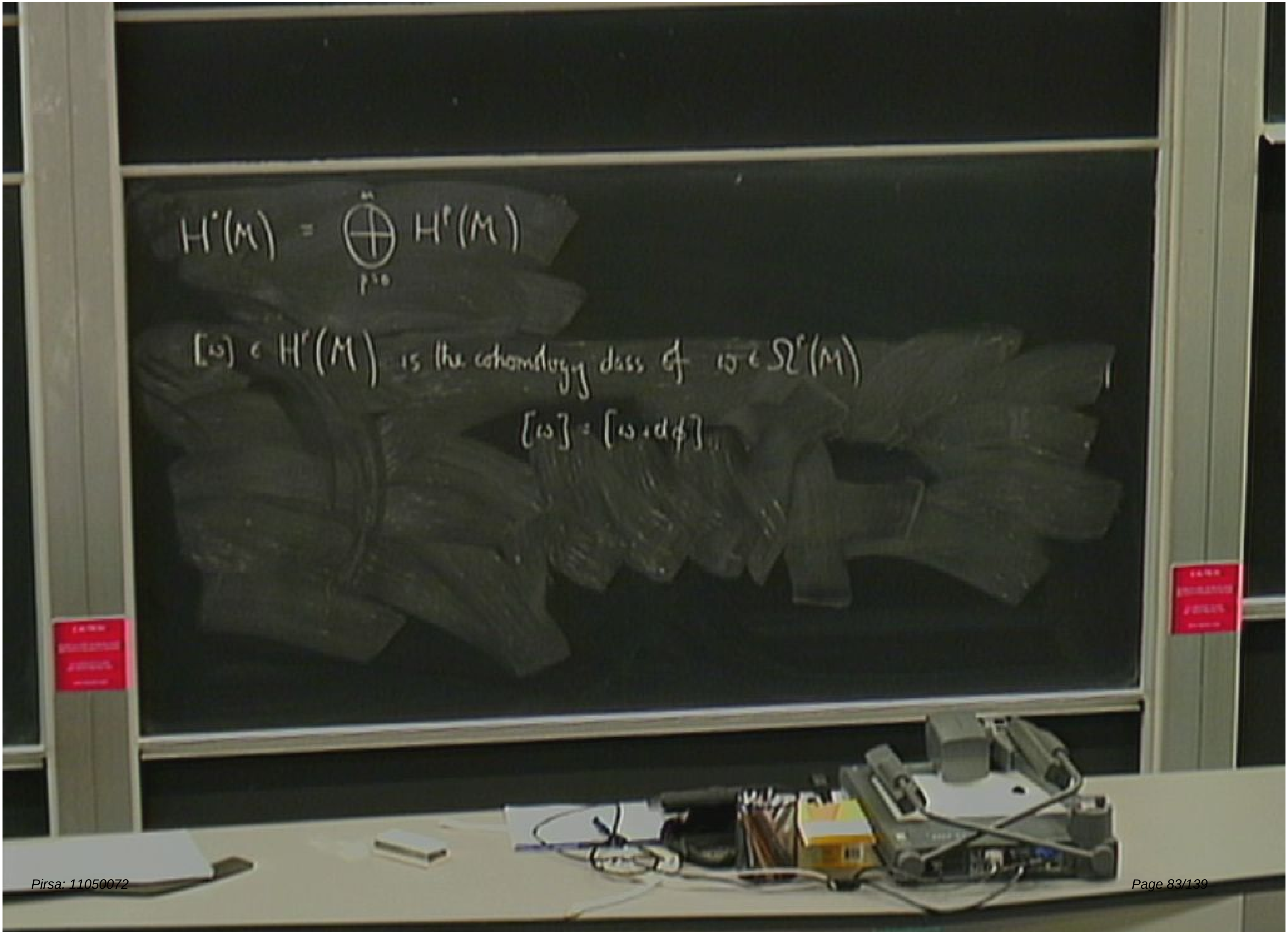
$a\omega_1 + b\omega_2$

$$H^{\bullet}(M) = \bigoplus_{p=0}^{m} H^{p}(M)$$



$$H^\bullet(M) = \bigoplus_{p\geq 0}^{m} H^p(M)$$

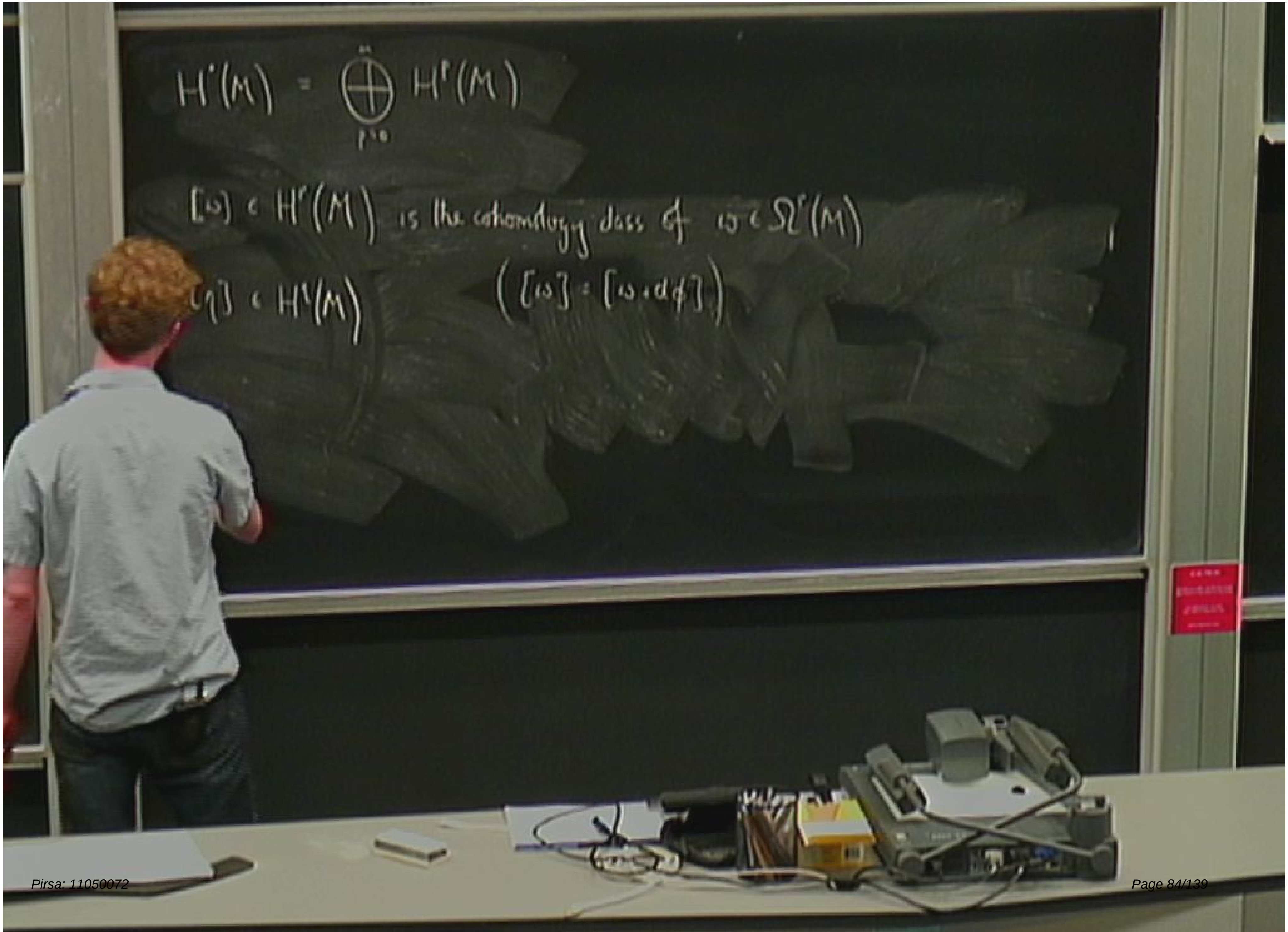
$[\omega] \in H^p(M)$ is the cohomology cl… $\in \Omega^p($



$$H^{\bullet}(M) = \bigoplus_{p=0}^{m} H^p(M)$$

$[\omega] \in H^p(M)$ is the cohomology class of $\omega \in \Omega^p(M)$

$$[\omega] = [\omega + d\phi]$$

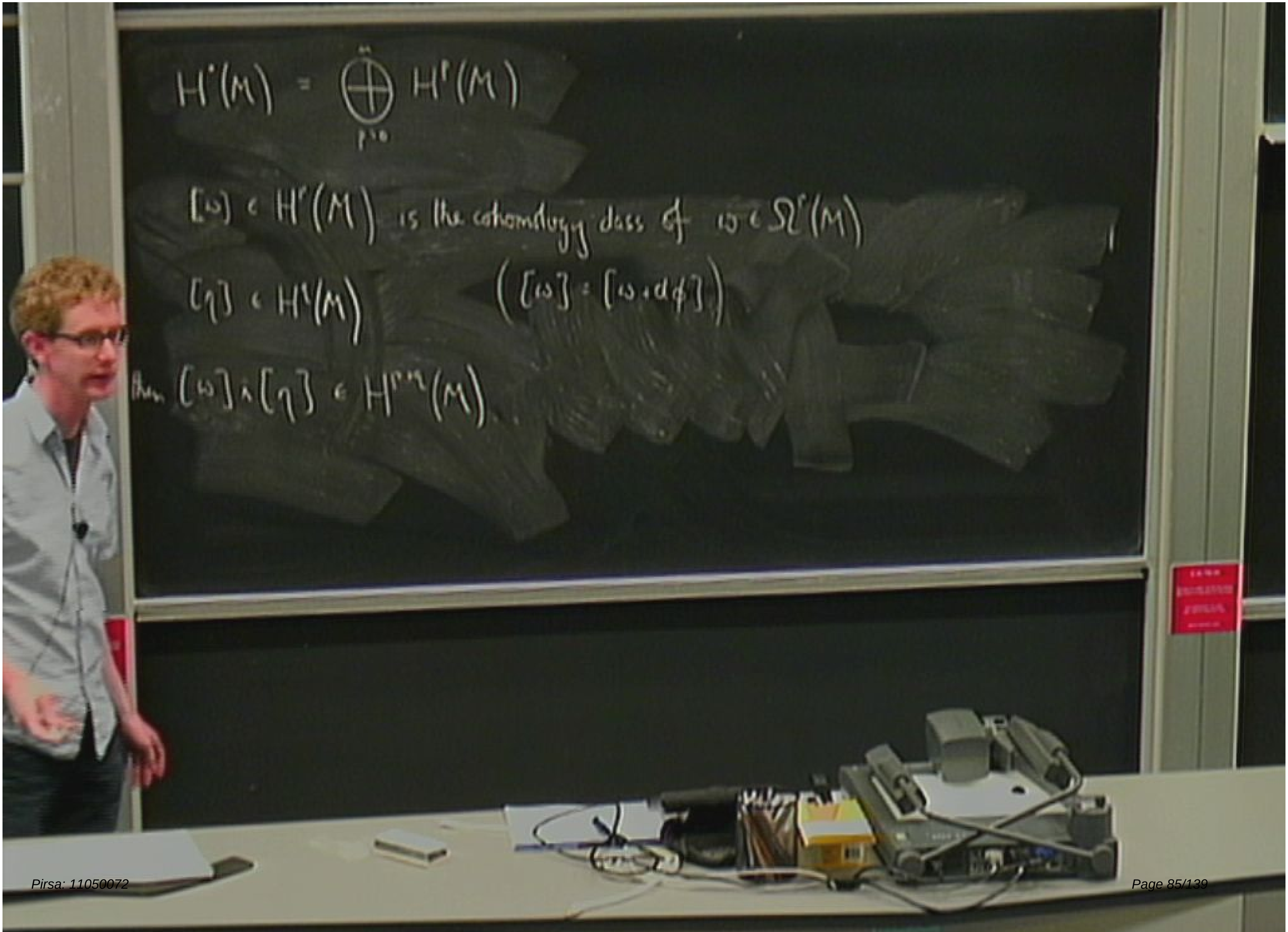


$$H^{*}(M) = \bigoplus_{p=0}^{n} H^{p}(M)$$

$[\omega] \in H^{p}(M)$ is the cohomology class of $\omega \in \Omega^{p}(M)$

$$\left( [\omega] = [\omega + d\phi] \right)$$

$[\eta] \in H^{q}(M)$



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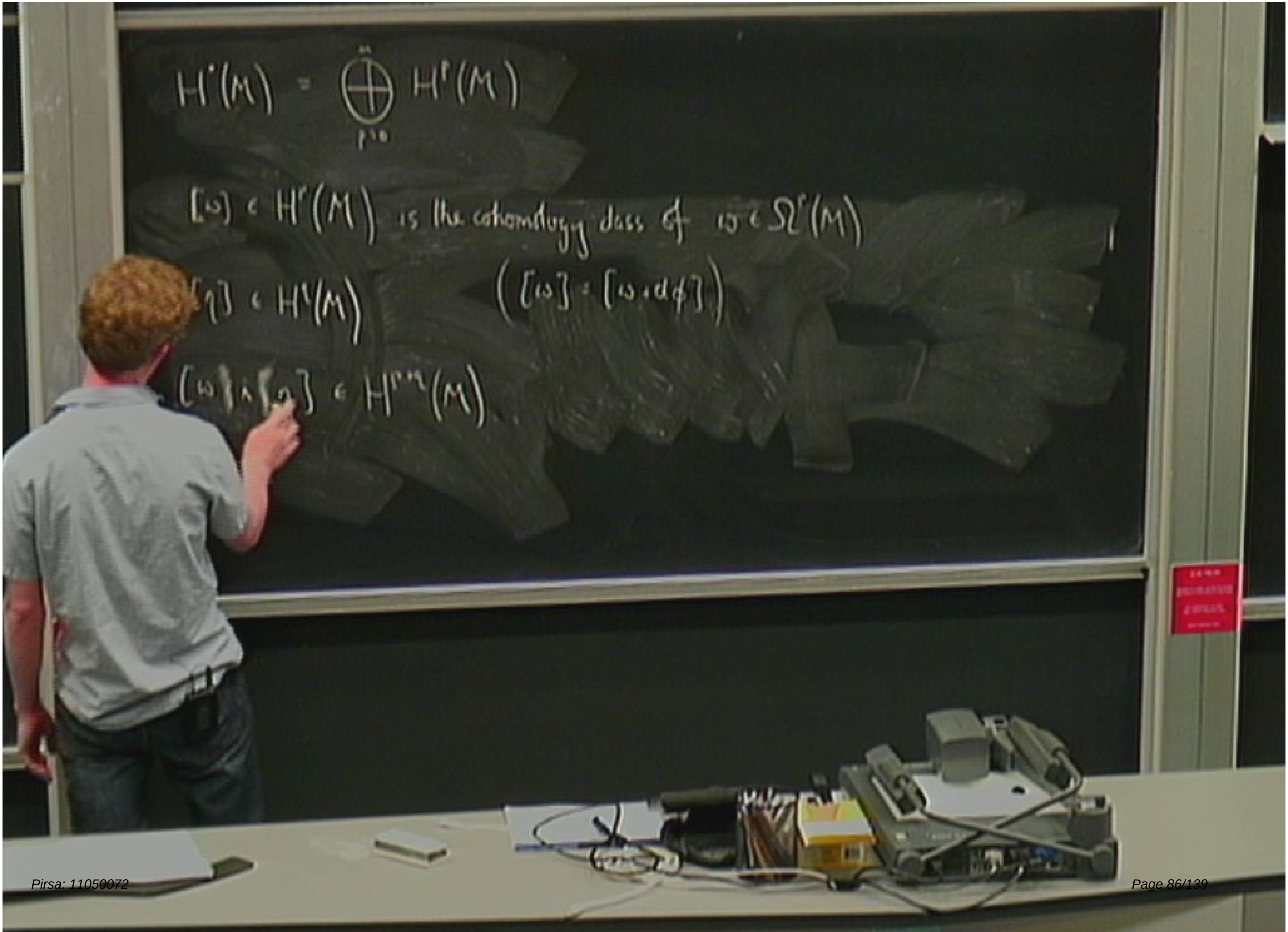

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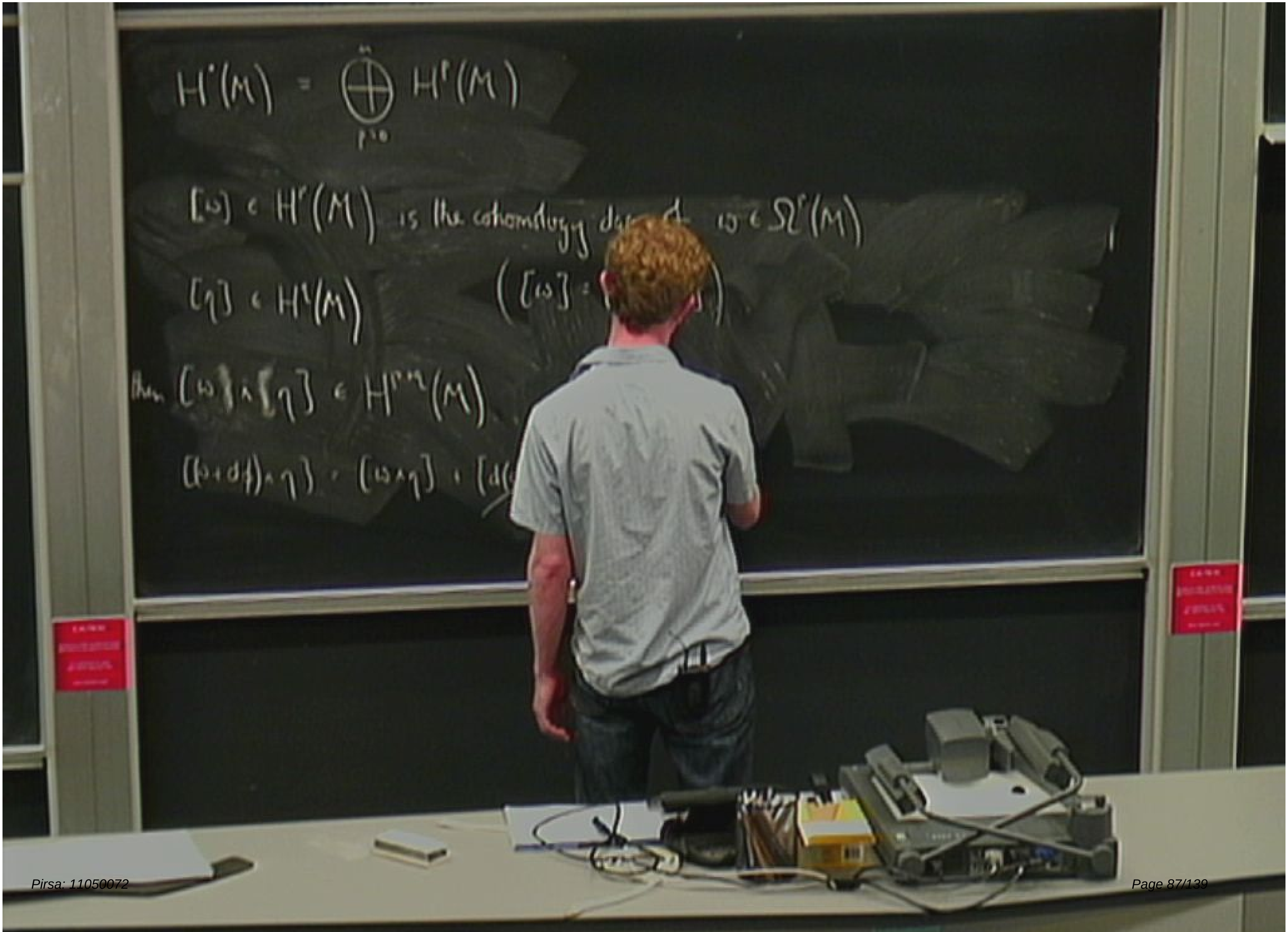


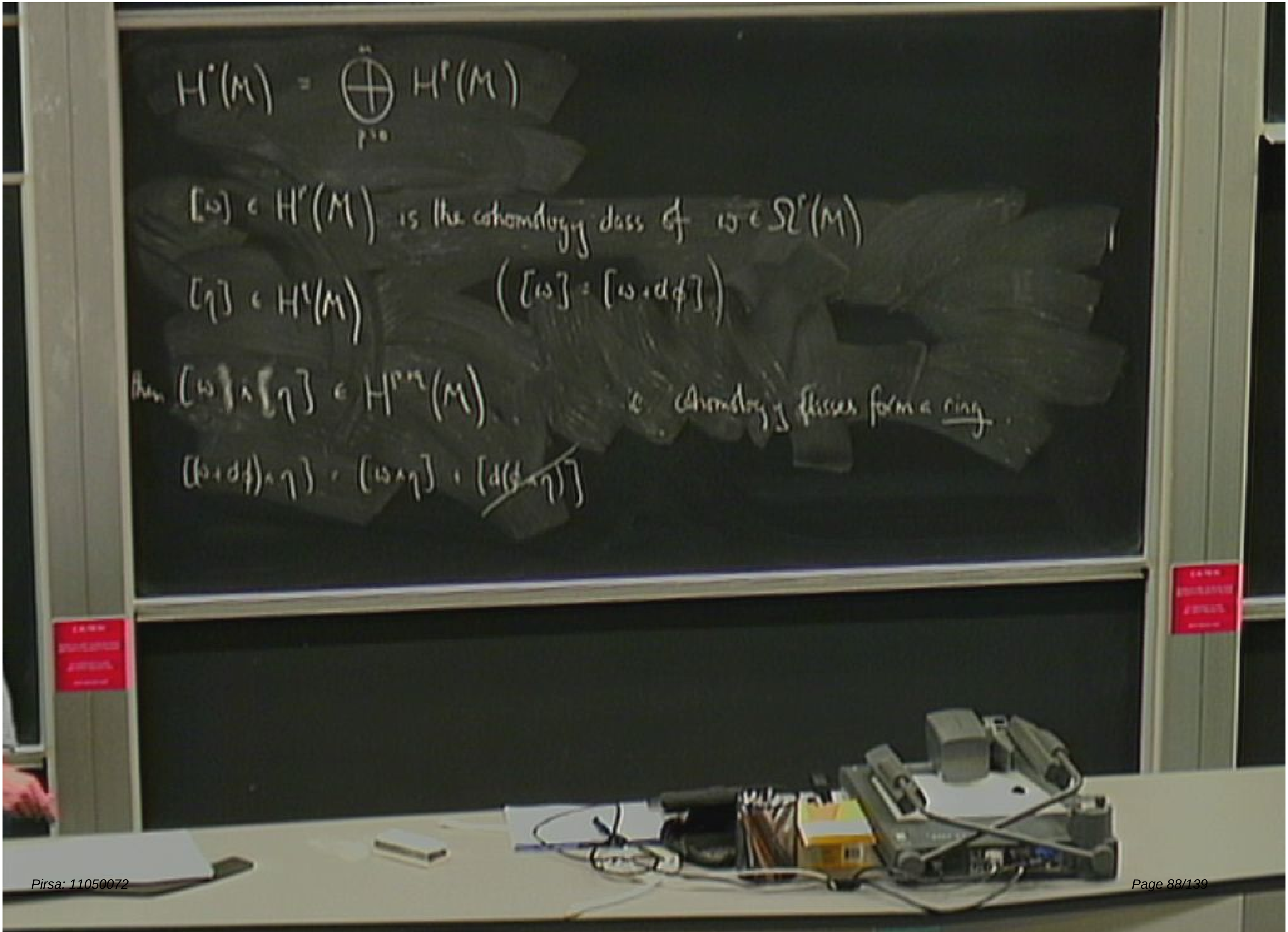
$$H^*(M) = \bigoplus_{p=0}^{n} H^p(M)$$

$[\omega] \in H^p(M)$ is the cohomology class of $\omega \in \Omega^p(M)$

$[\eta] \in H^q(M)$ $\quad ([\omega] =$ [illegible] $)$

then $[\omega] \wedge [\eta] \in H^{p+q}(M)$

$[(\omega + d\phi) \wedge \eta] = [\omega \wedge \eta] + [d($ [illegible]



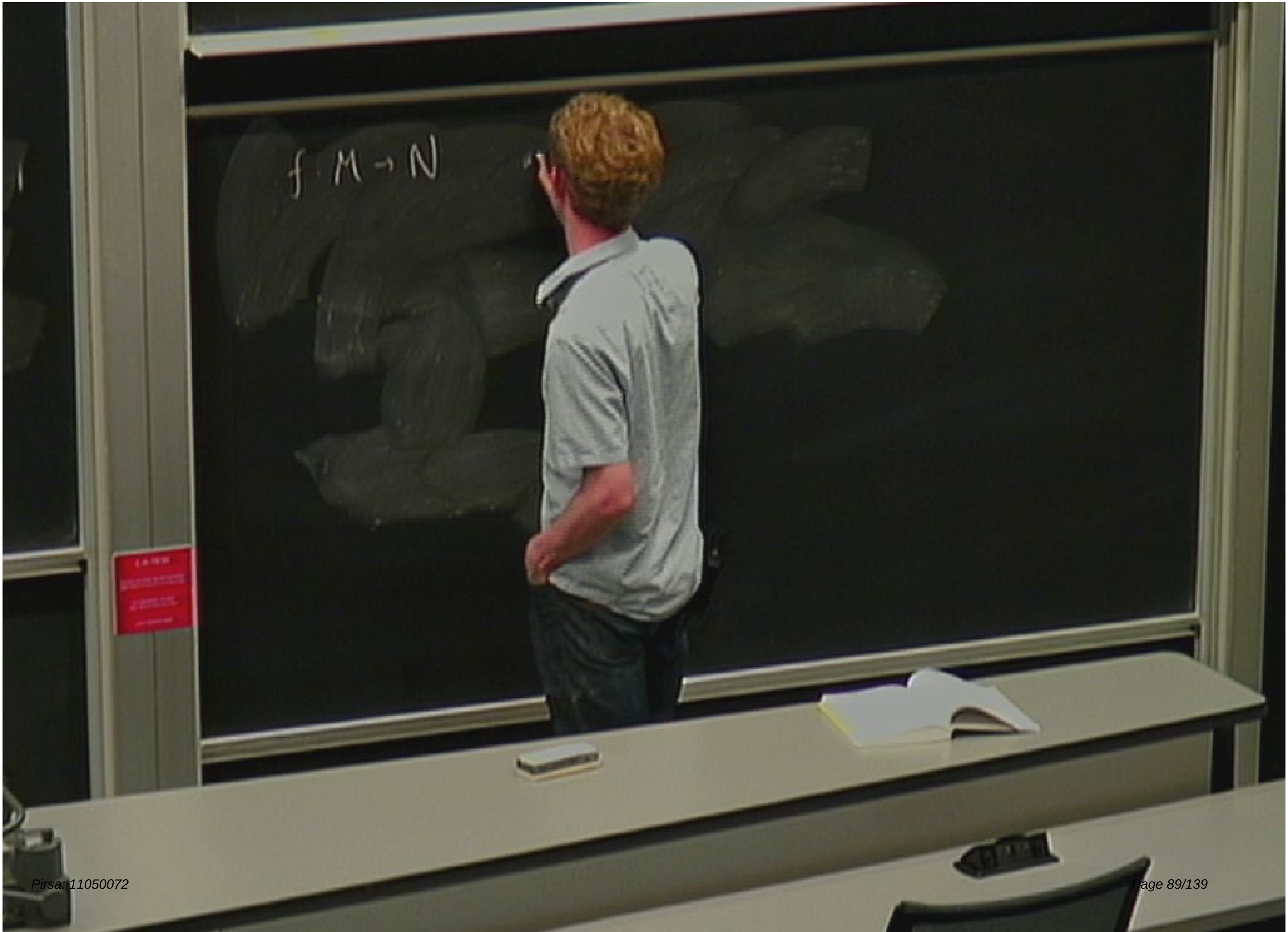
$$H^*(M) = \bigoplus_{p=0}^{n} H^p(M)$$

$[\omega] \in H^p(M)$ is the cohomology class of $\omega \in \Omega^p(M)$

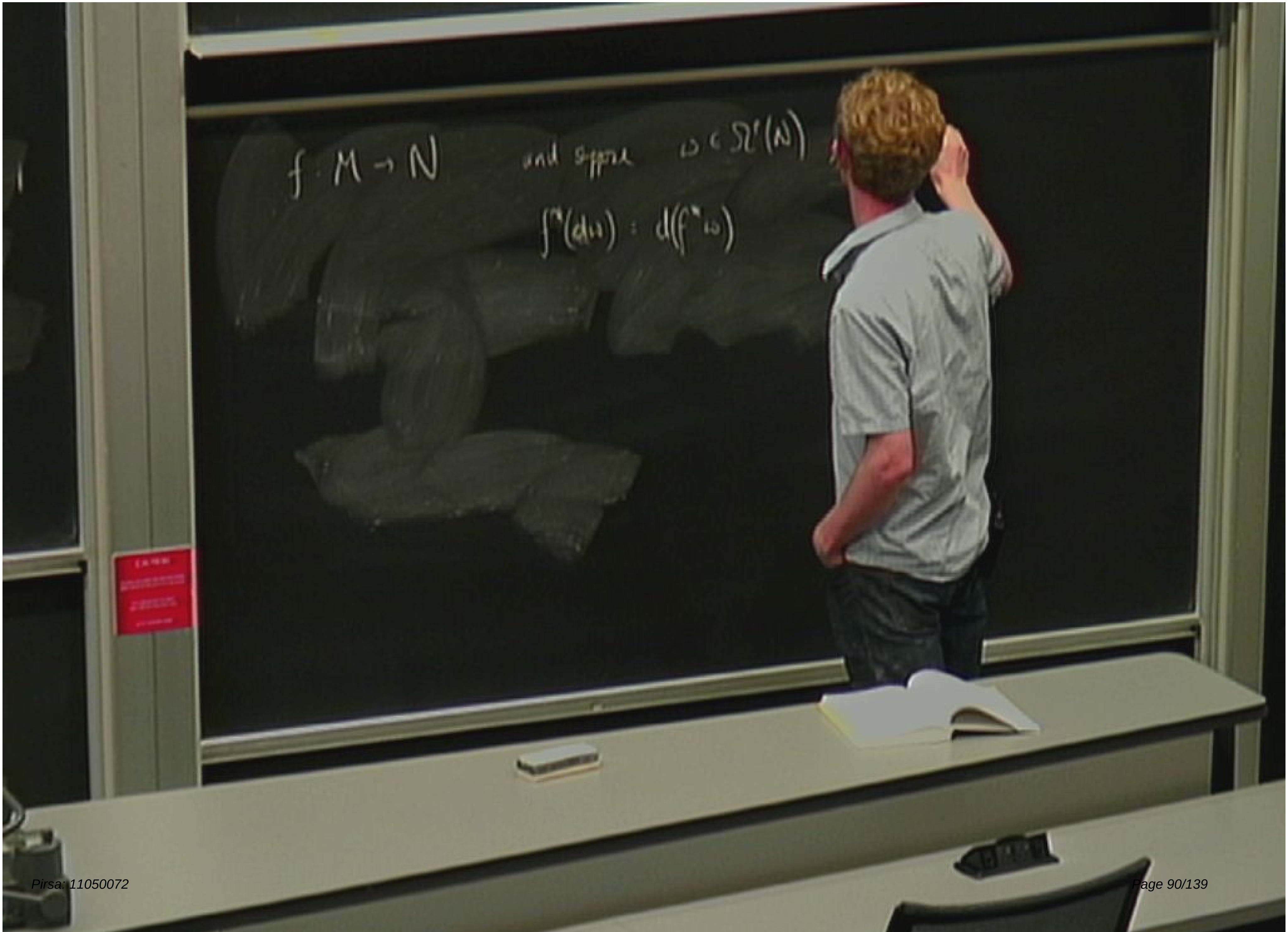
$[\eta] \in H^q(M)$ $\quad \left( [\omega] = [\omega + d\phi] \right)$

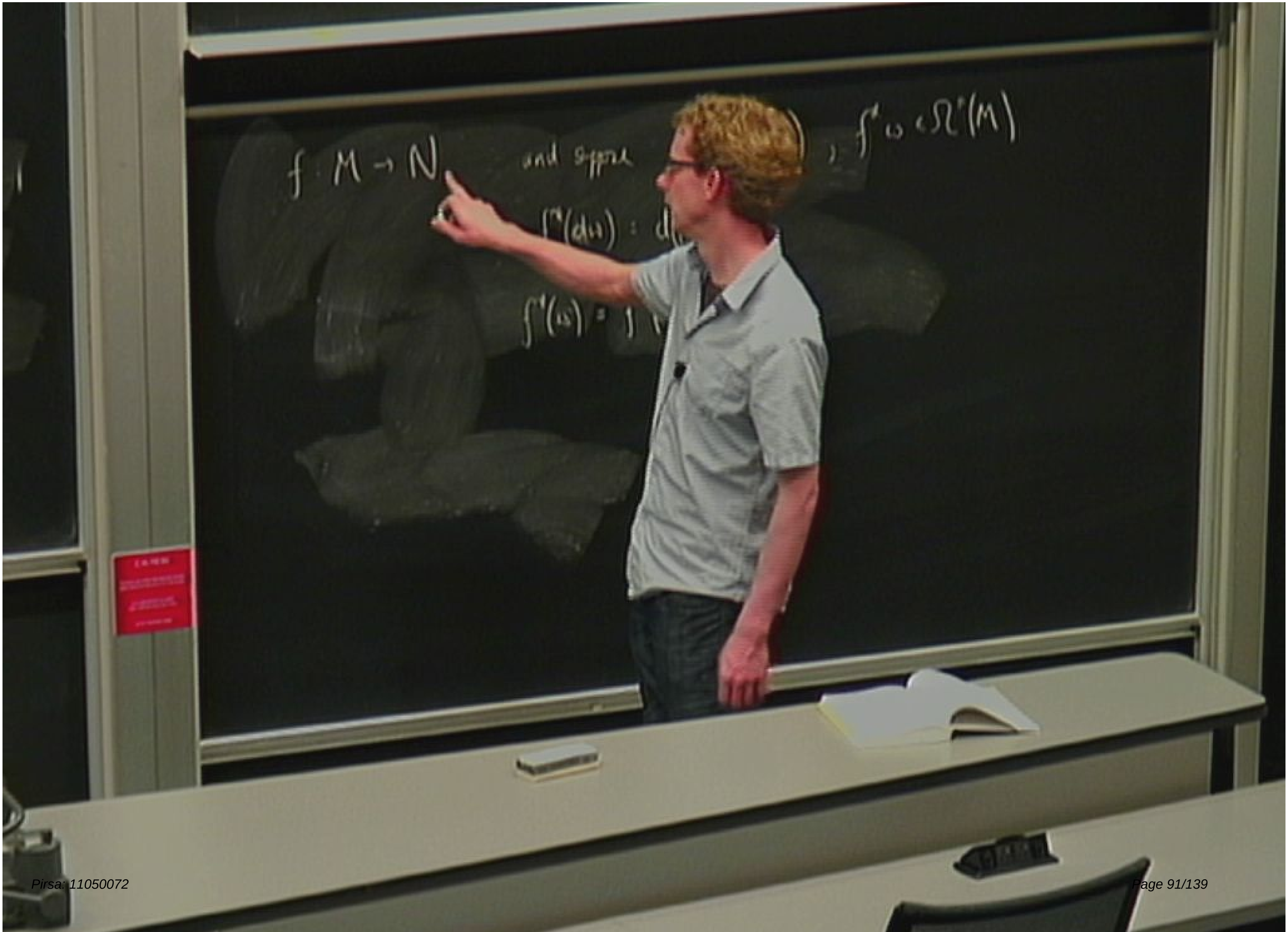
then $[\omega] \wedge [\eta] \in H^{p+q}(M)$ ie. cohomology classes form a <u>ring</u>.

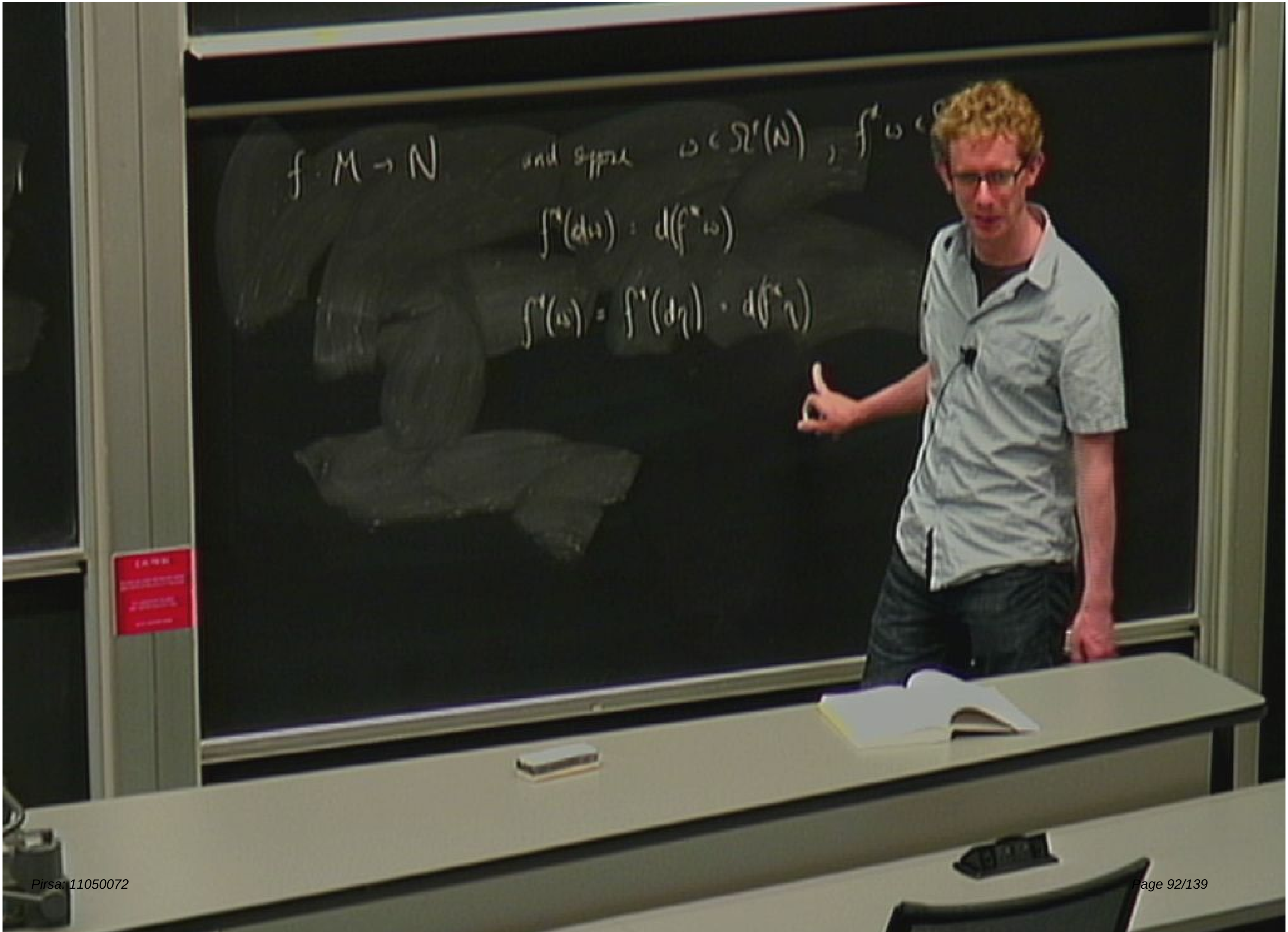
$$[(\omega + d\phi) \wedge \eta] = [\omega \wedge \eta] + [d(\phi \wedge \eta)]$$



$f: M \to N$


$f : M \to N$ and suppose $\omega \in \Omega^l(N)$
$f^*(d\omega) = d(f^*\omega)$

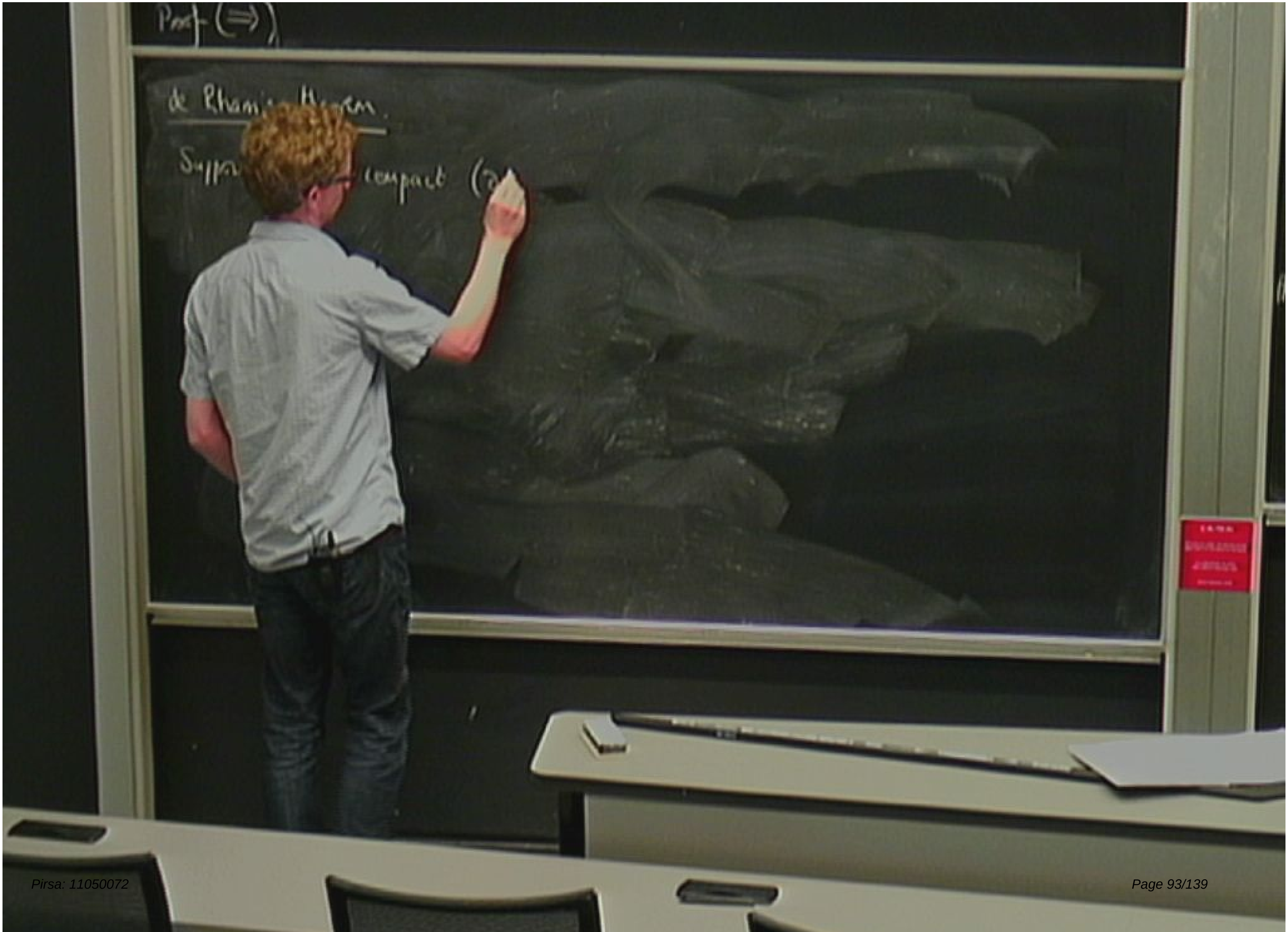

f : M → N
and suppose
, f*ω ∈ Ω^p(M)
f*(dω) = d(
f*(ω) = ∫ (

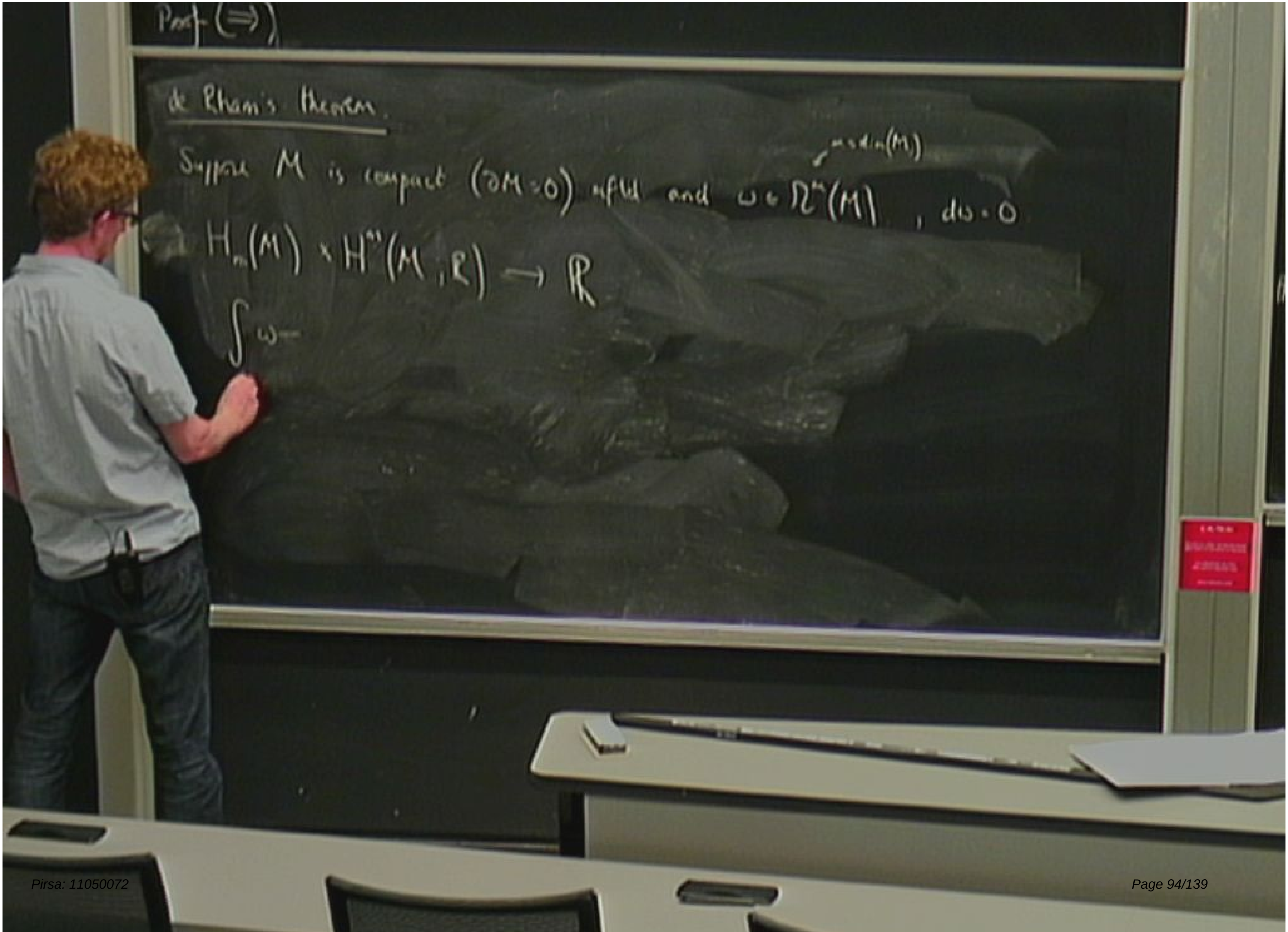


$f : M \to N$ and suppose $\omega \in \Omega^1(N)$ , $f^*\omega \in$

$$f^*(d\omega) = d(f^*\omega)$$

$$f^*(\omega) = f^*(d\eta) = d(f^*\eta)$$


Proof (⇒)
de Rham
Supp
compact (



Proof ($\Longrightarrow$)

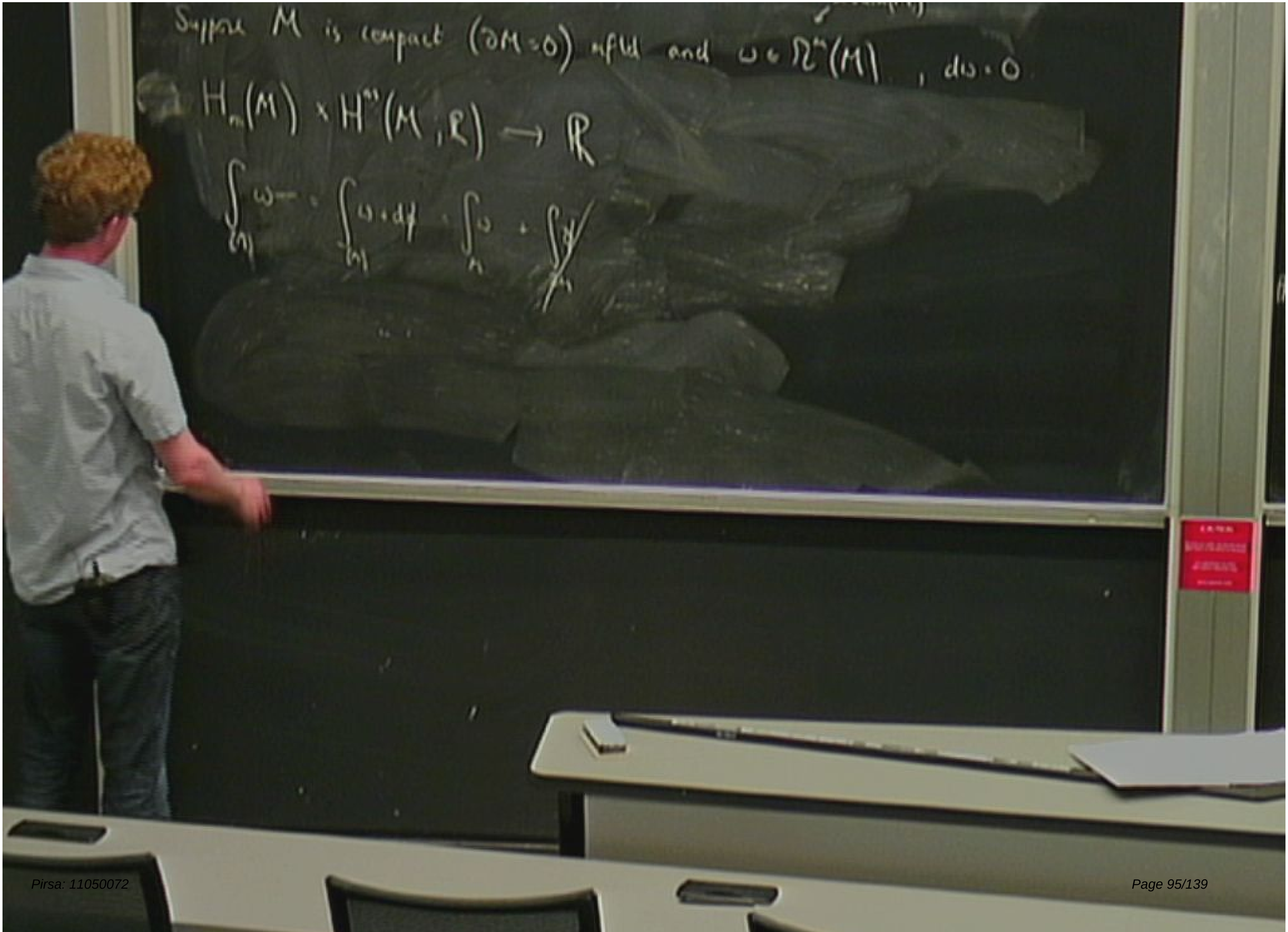
## de Rham's theorem

$$m = \dim(M)$$

Suppose $M$ is compact ($\partial M = 0$) mfld and $\omega \in \Omega^m(M)$ , $d\omega = 0$

$$H_m(M) \times H^m(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

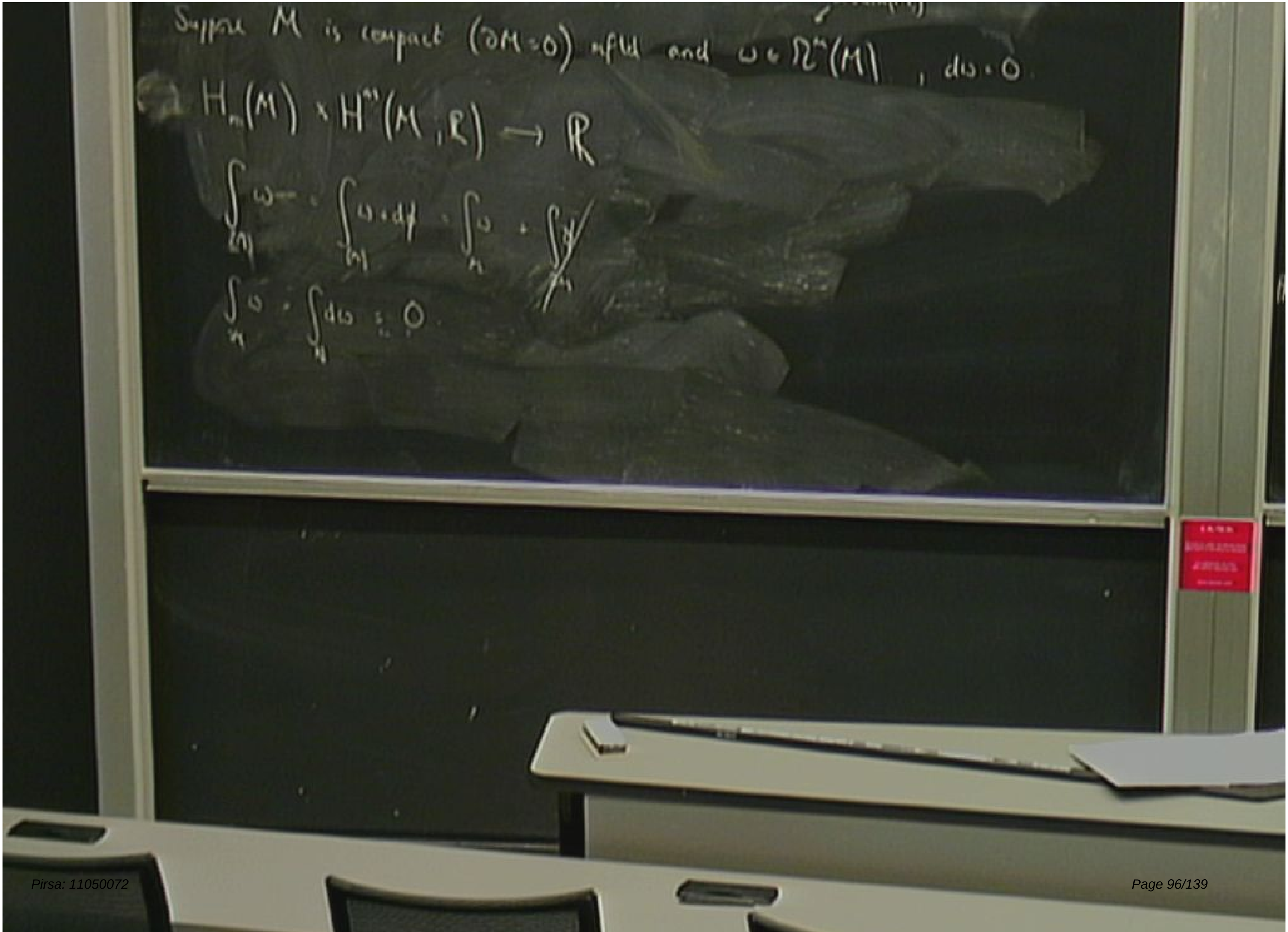
$$\int \omega -$$



Suppose $M$ is compact $(\partial M = 0)$ mfld and $\omega \in \Omega^m(M)$, $d\omega = 0$

$$H_m(M) \times H^m(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int_{[M]} \omega = \int_{[M]} \omega + d\psi = \int_M \omega + \cancel{\int_M \psi}$$

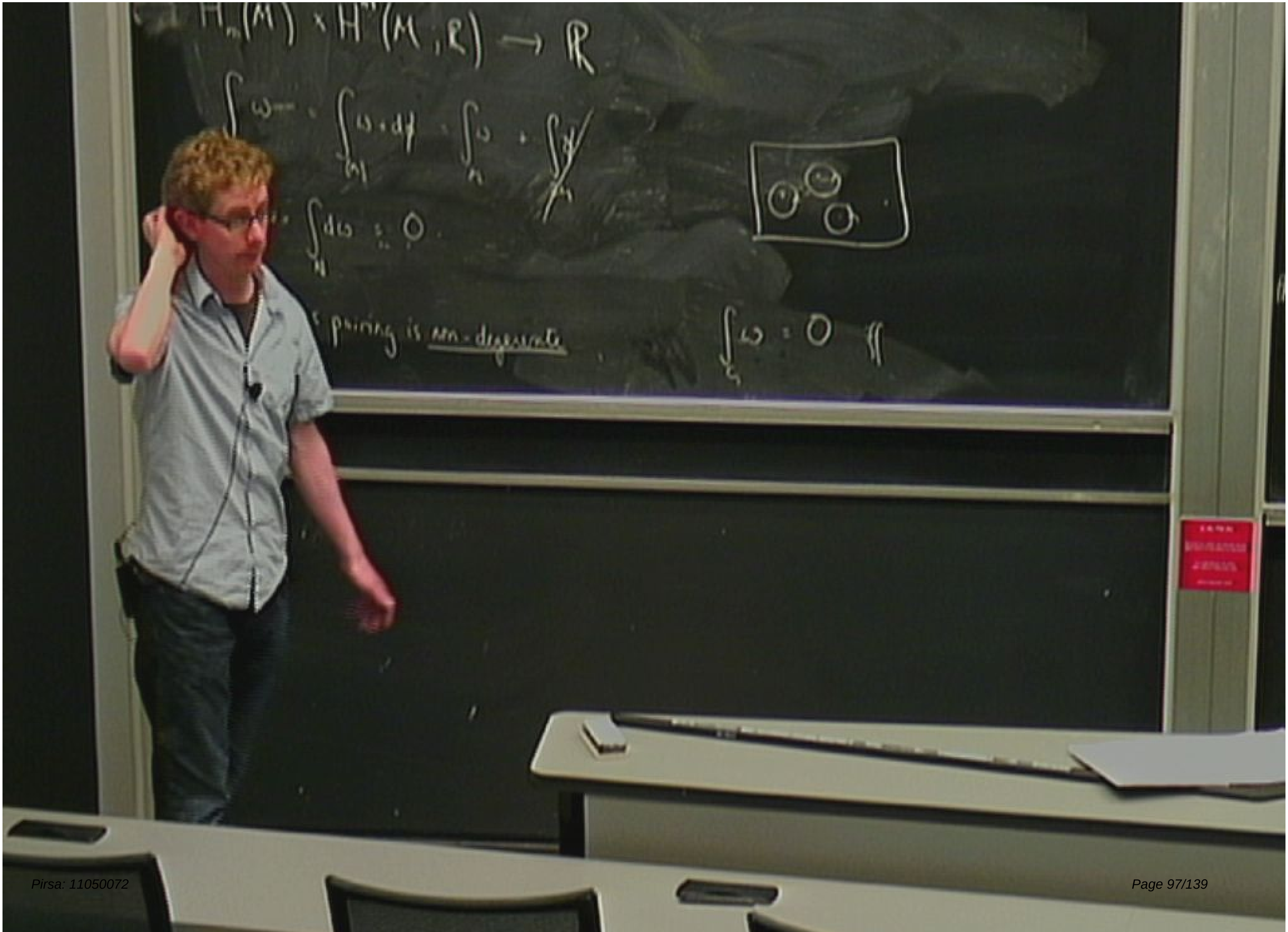


Suppose $M$ is compact ($\partial M = 0$) mfld and $\omega \in \Omega^m(M)$, $d\omega = 0$

$$H_m(M) \times H^m(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int_{[M]} \omega = \int_{[M]} \omega + d\phi = \int_M \omega + \cancel{\int_M d\phi}$$

$$\int_M \omega = \int_M d\omega = 0$$



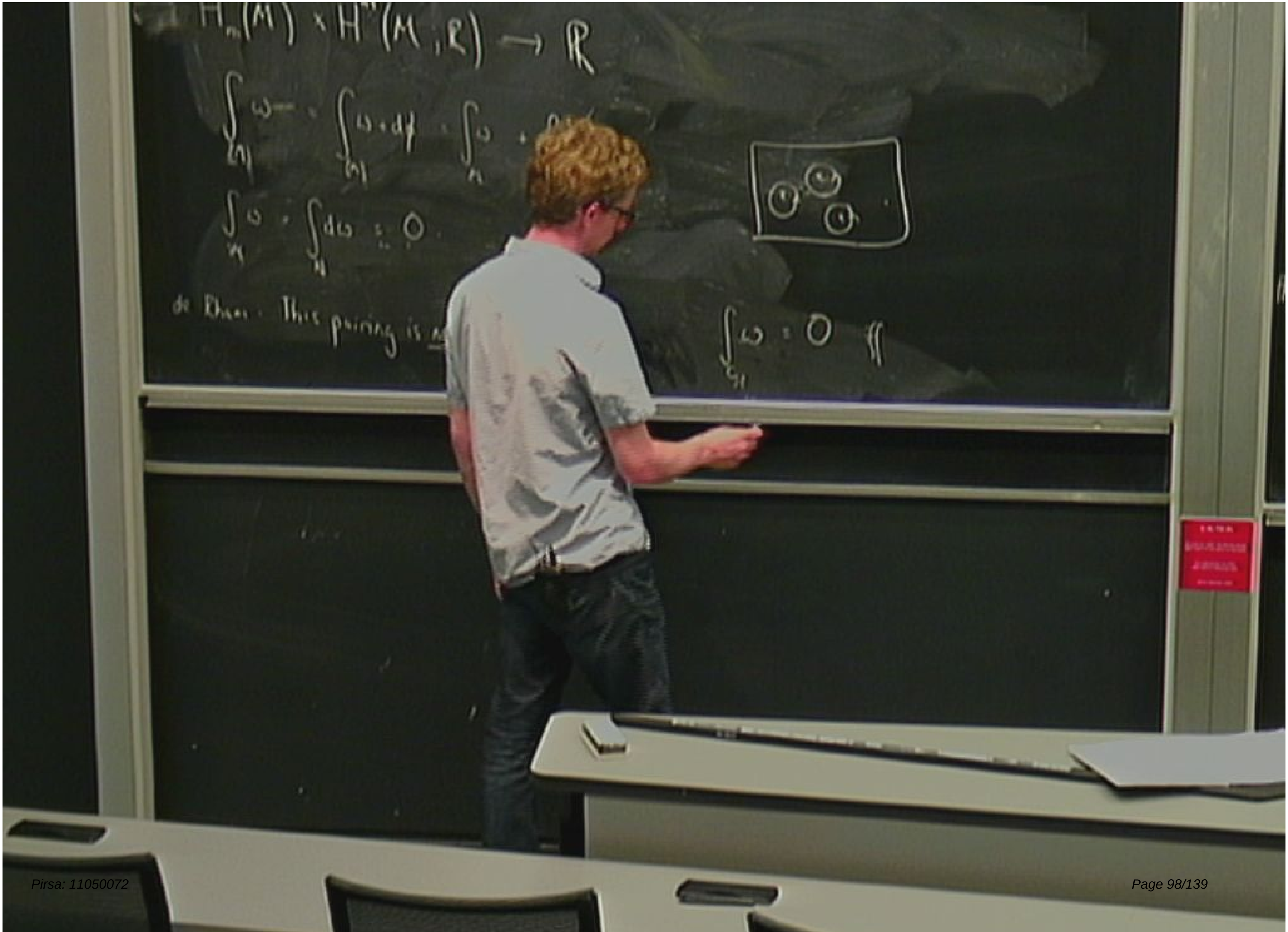
$H_m(M) \times H^m(M;\mathbb{R}) \longrightarrow \mathbb{R}$

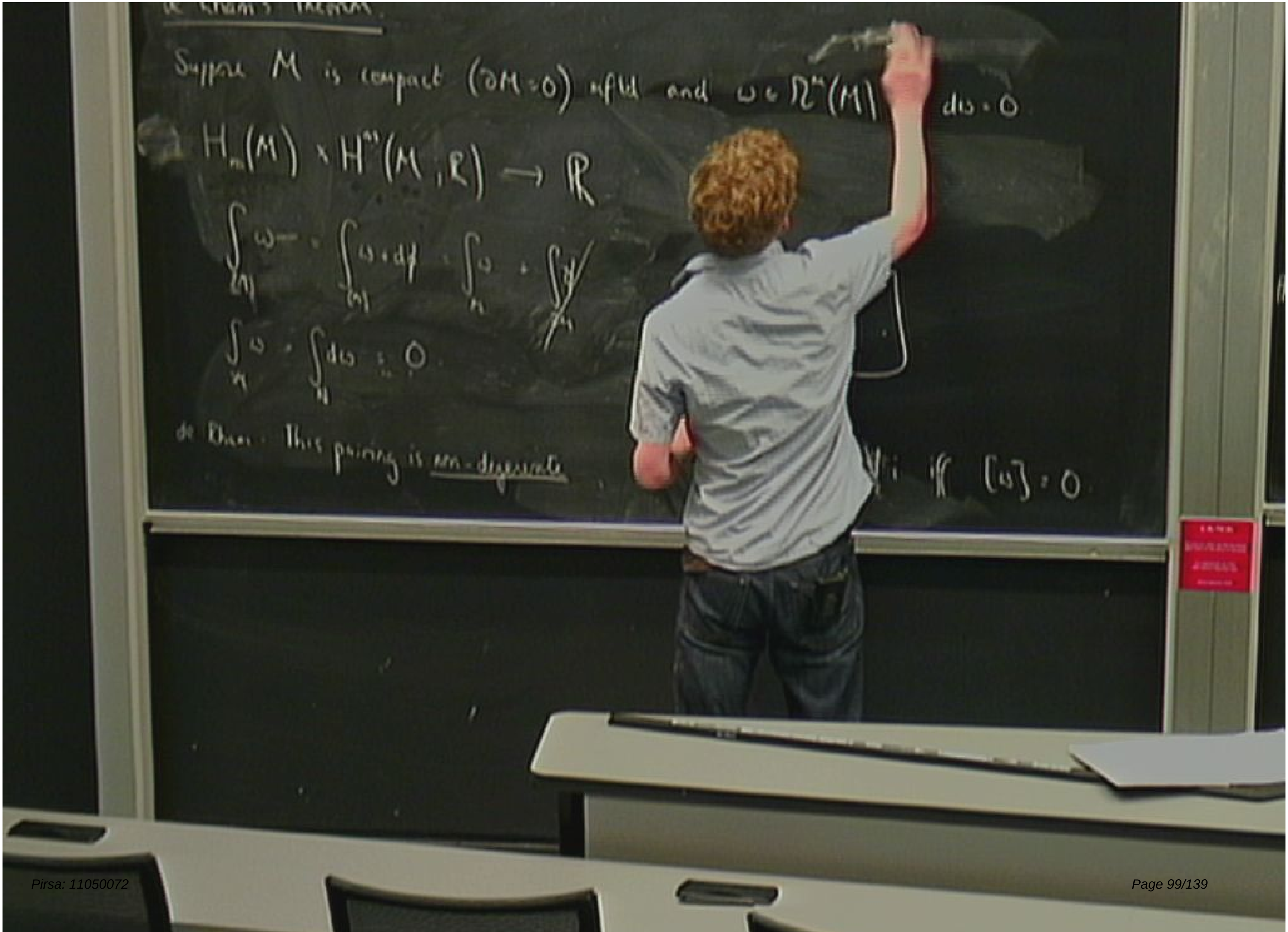
$\int \omega = \int_M \omega + d\phi = \int_M \omega + \cancel{\int_M d\phi}$

$\int_M d\omega = 0$.

pairing is <u>non-degenerate</u>

$\int_C \omega = 0$ iff


$H_m(M) \times H^m(M;\mathbb{R}) \longrightarrow \mathbb{R}$
$\int_{[M]} \omega = \int_{[M]} \omega + d\eta = \int_M \omega + 0$
$\int_M \omega = \int_N d\omega = 0$
de Rham: This pairing is n
$\int_{C_1} \omega = 0$ ∀



de Rham's theorem.

Suppose $M$ is compact $(\partial M = 0)$ mfld and $\omega \in \Omega^m(M)$ $d\omega = 0$

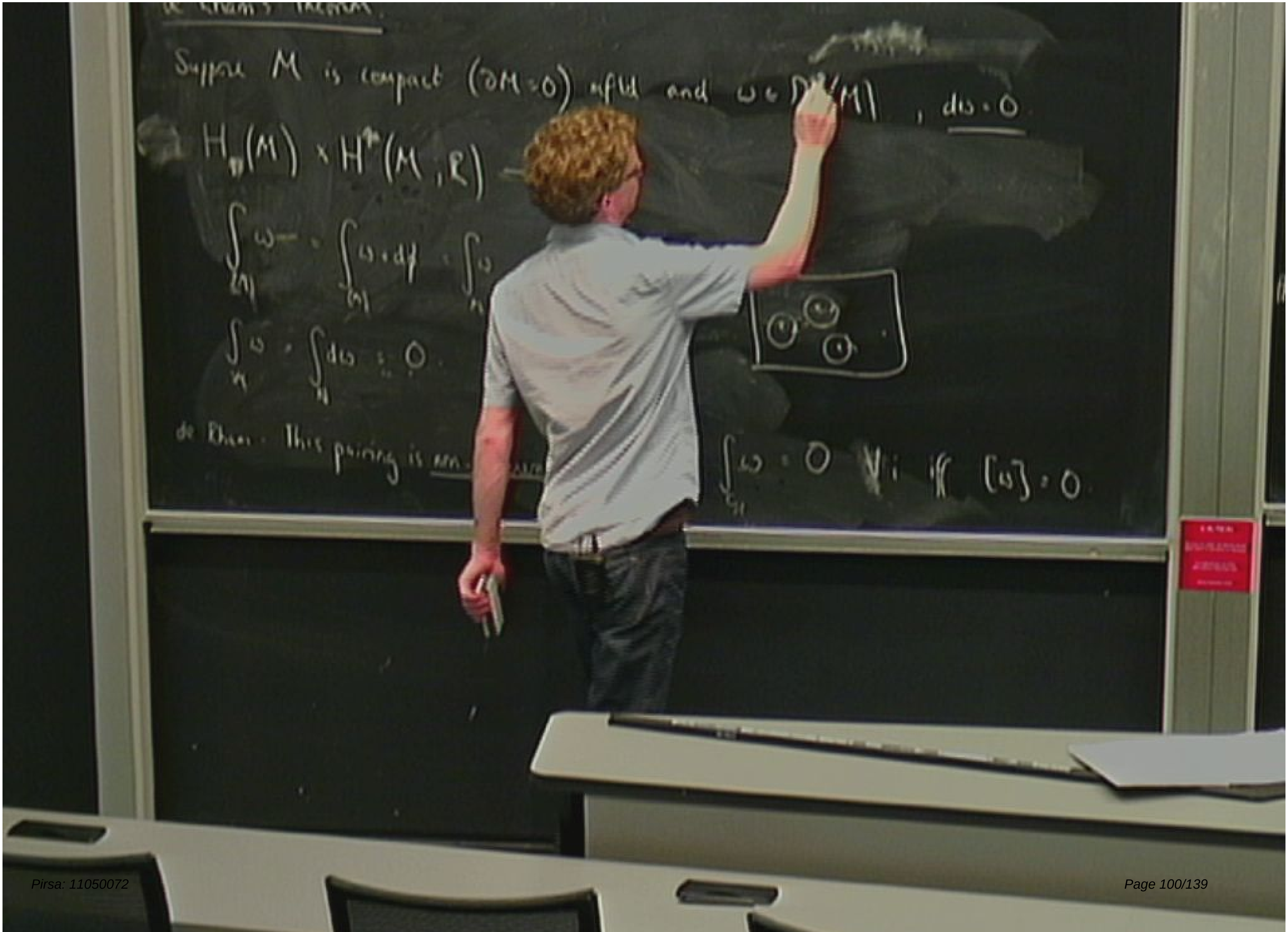
$$H_m(M) \times H^m(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int_{[M]} \omega = \int_{[M]} \omega + df = \int_M \omega + \cancel{\int_M df}$$

$$\int_M \omega = \int_N d\omega = 0.$$

de Rham: This pairing is non-degenerate

iff $[\omega] = 0$.



de Rham's theorem.

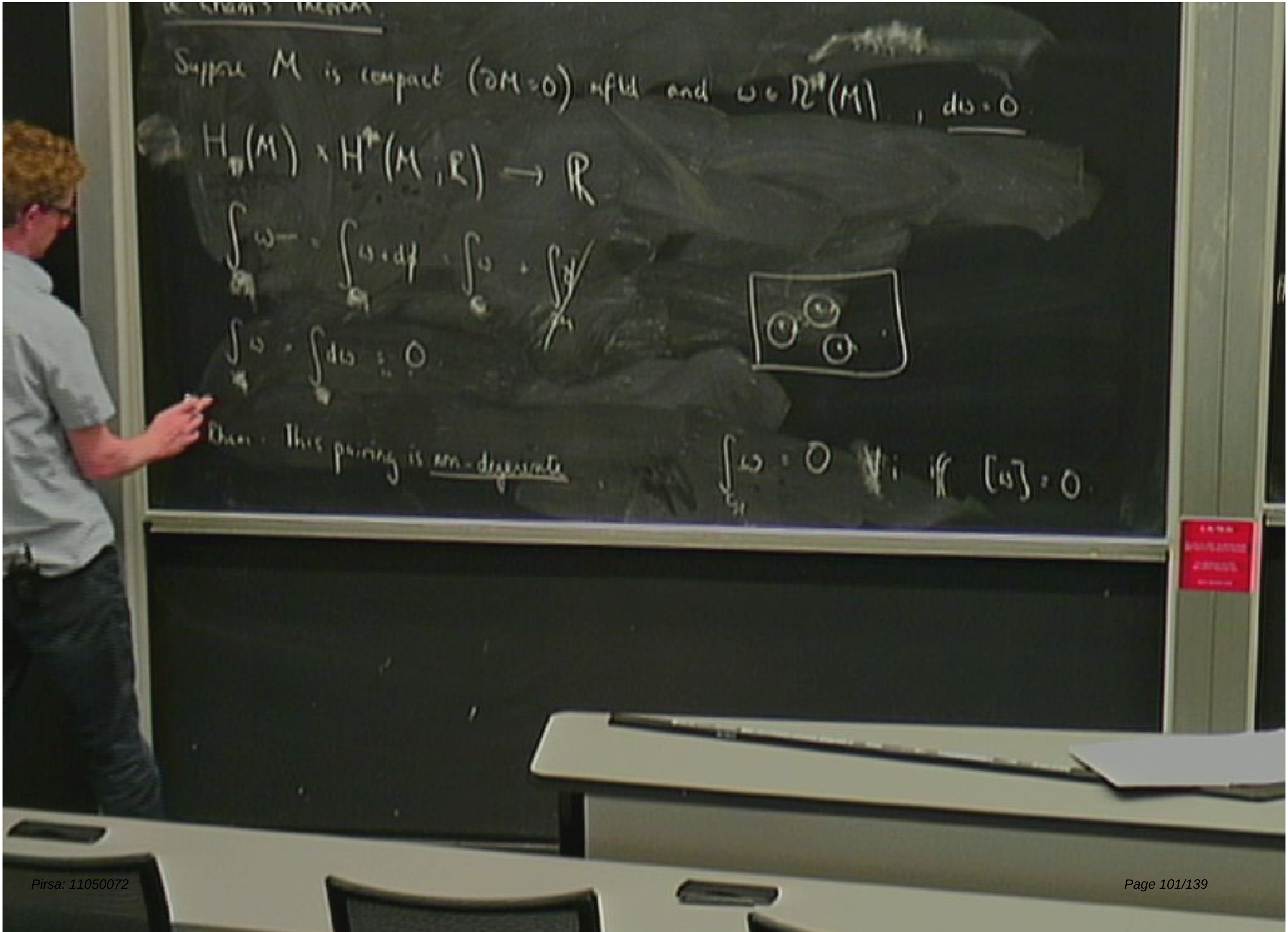
Suppose $M$ is compact $(\partial M = 0)$ mfld and $\omega \in$ [illegible] $(M)$, $\underline{d\omega = 0}$

$H_p(M) \times H^p(M, \mathbb{R}) \to$

$\int_{\partial N} \omega = \int_{\partial N} \omega + d\eta = \int_{N} \omega$

$\int_{\partial N} \omega = \int_{N} d\omega = 0.$

de Rham: This pairing is non-degenerate ... $\int_{\sigma_i} \omega = 0 \ \forall i$ iff $[\omega] = 0.$



de Rham's theorem

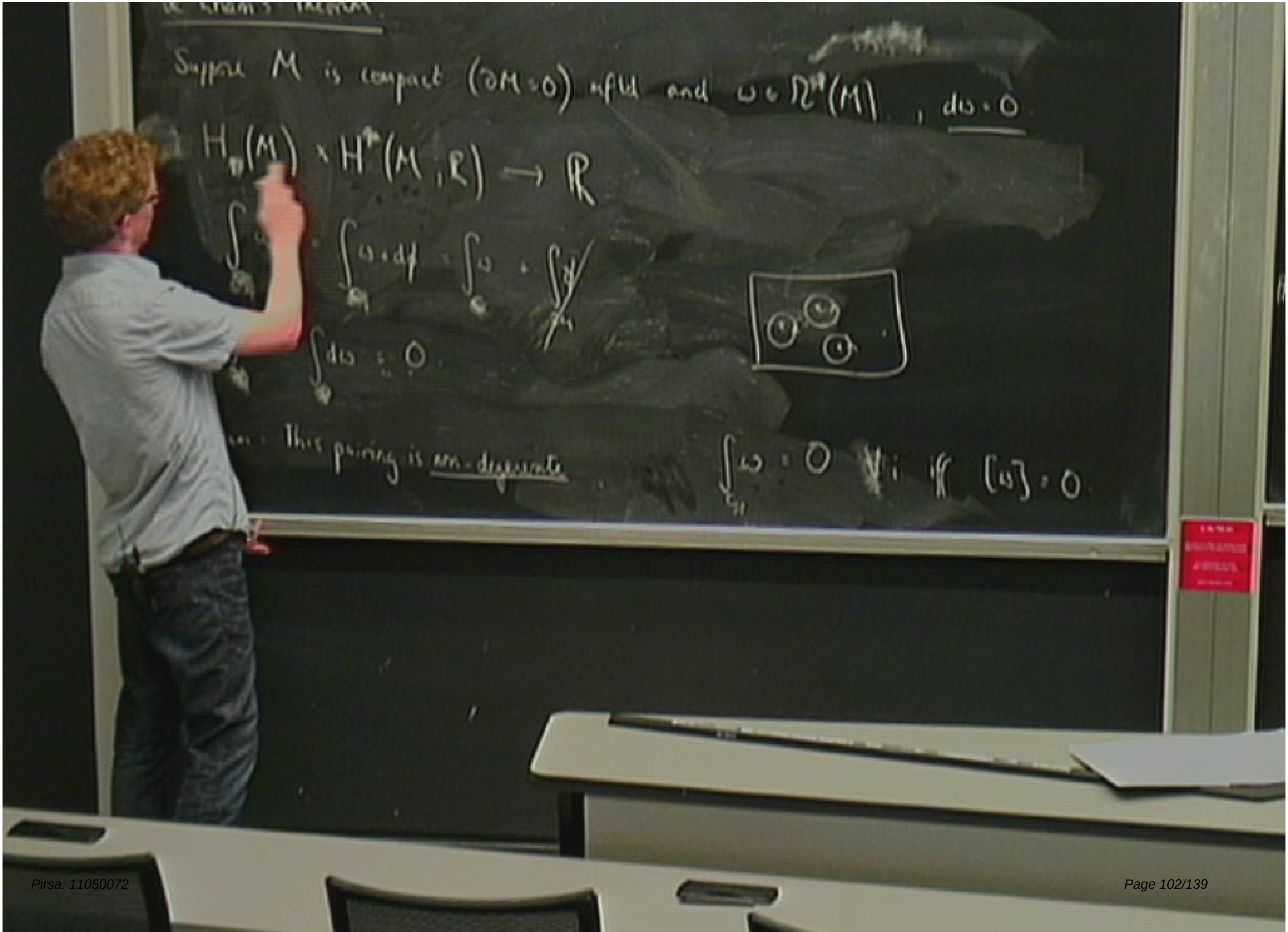
Suppose $M$ is compact $(\partial M = 0)$ mfld and $\omega \in \Omega^p(M)$, $\underline{d\omega = 0}$.

$$H_p(M) \times H^p(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int \omega = \int \omega + d\phi = \int \omega + \cancel{\int \phi}$$

$$\int \omega = \int d\omega = 0.$$

Thm: This pairing is non-degenerate. $\int_{c_i} \omega = 0 \ \forall i$ iff $[\omega] = 0$.



de Rham's Theorem

Suppose $M$ is compact $(\partial M = 0)$ mfld and $\omega \in \Omega^p(M)$, $\underline{d\omega = 0}$

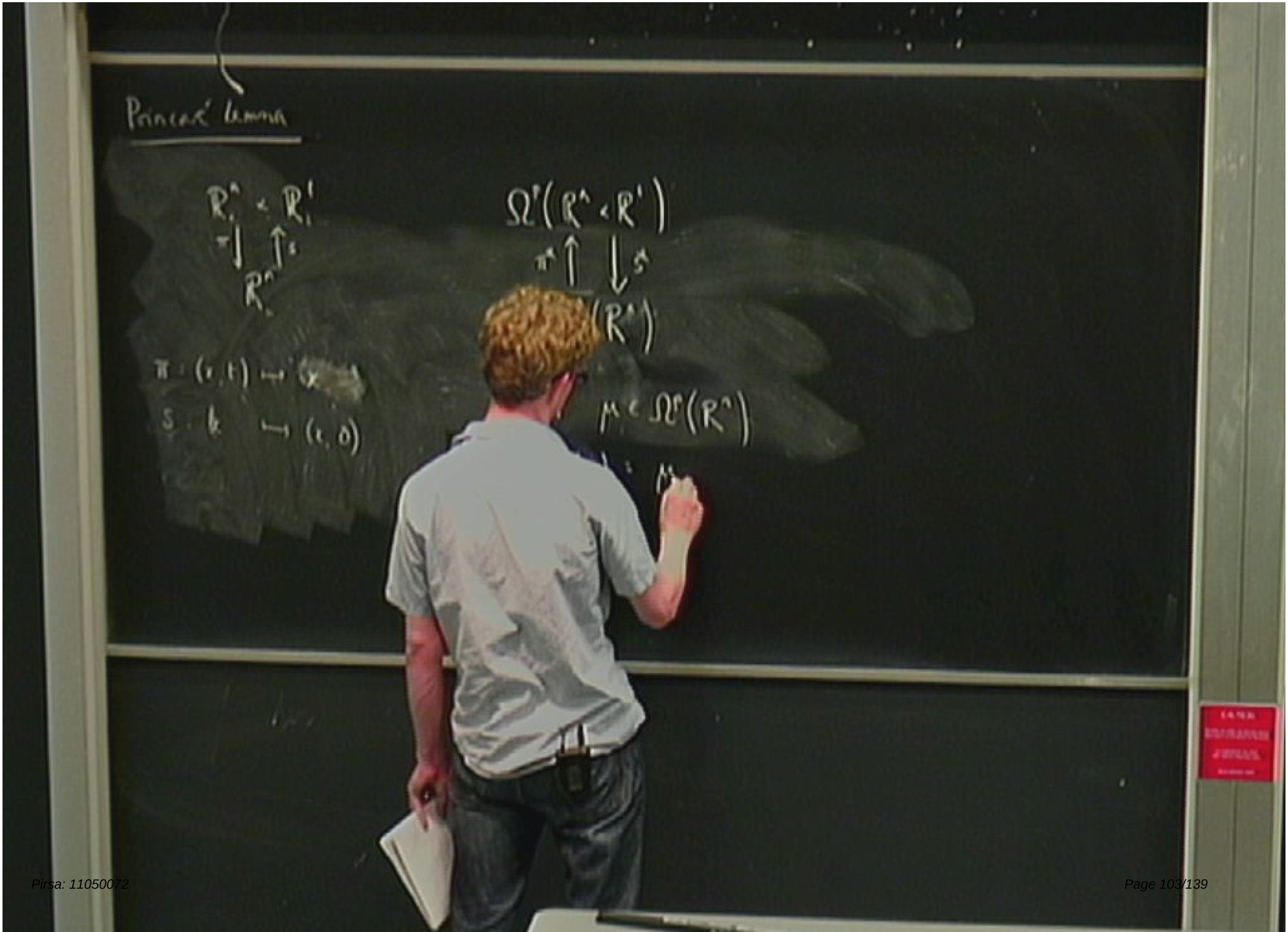
$$H_p(M) \times H^p(M;\mathbb{R}) \longrightarrow \mathbb{R}$$

$$\int_{\sigma} \quad = \int_{\sigma} \omega + d\phi = \int_{\sigma} \omega + \cancel{\int_{\partial\sigma} \phi}$$

$$\int_{\sigma} d\omega = 0.$$

This pairing is <u>non-degenerate</u>.

$$\int_{c_i} \omega = 0 \ \forall i \ \text{iff} \ [\omega] = 0.$$



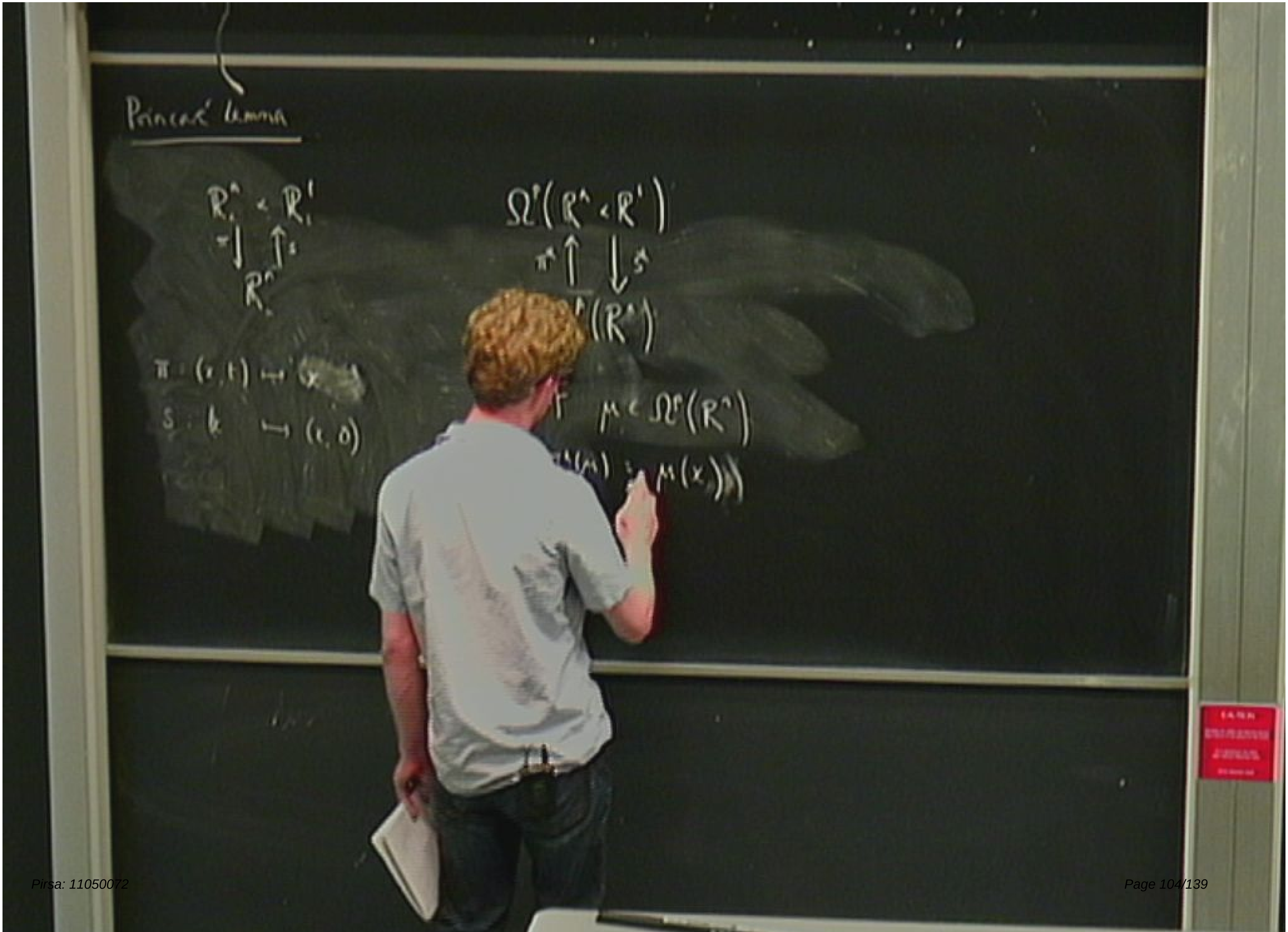
Poincaré Lemma

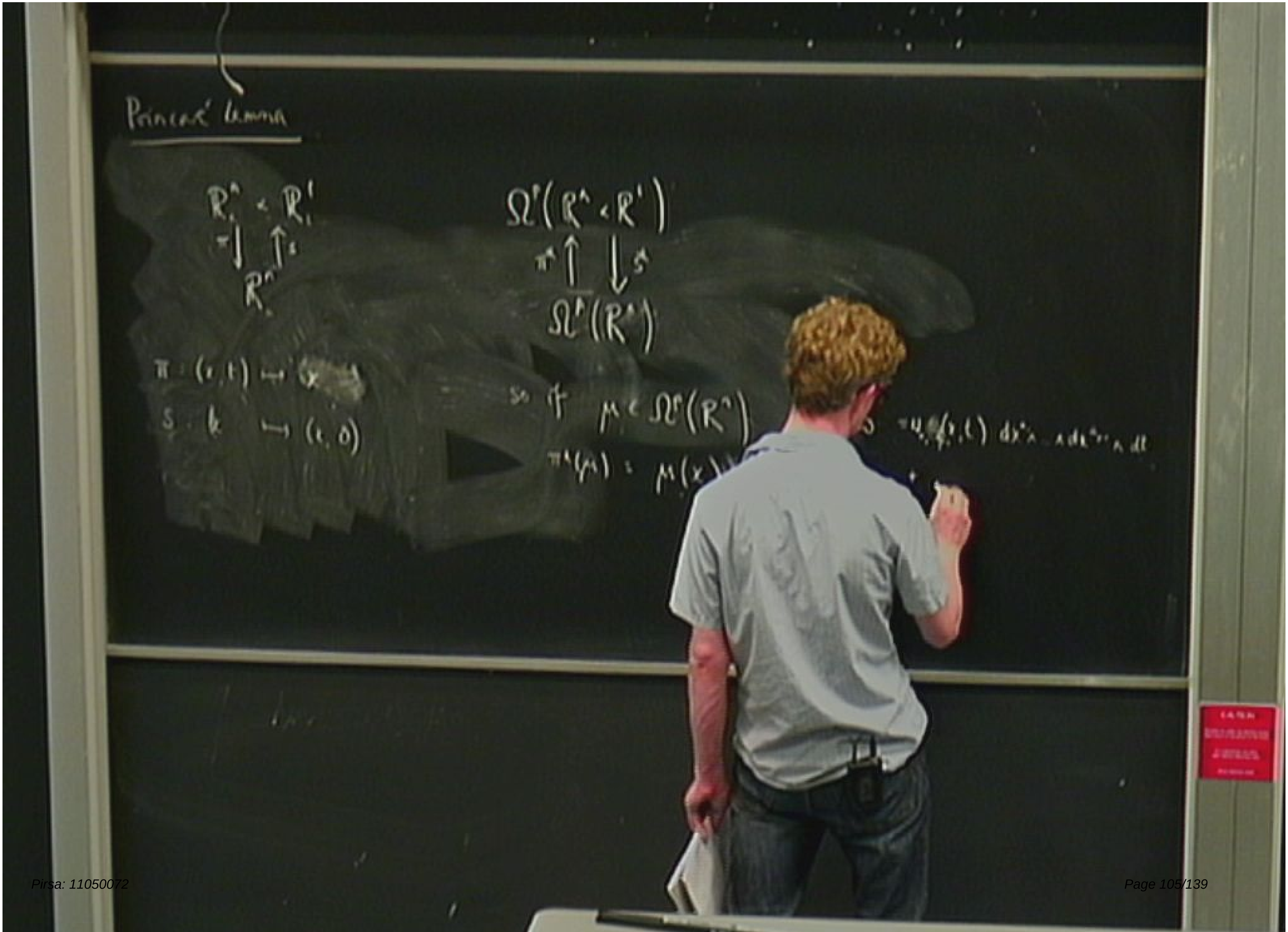
$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\ \mathbb{R}^n \arrow[u, shift right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \Omega^p(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] \\ \Omega^p(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

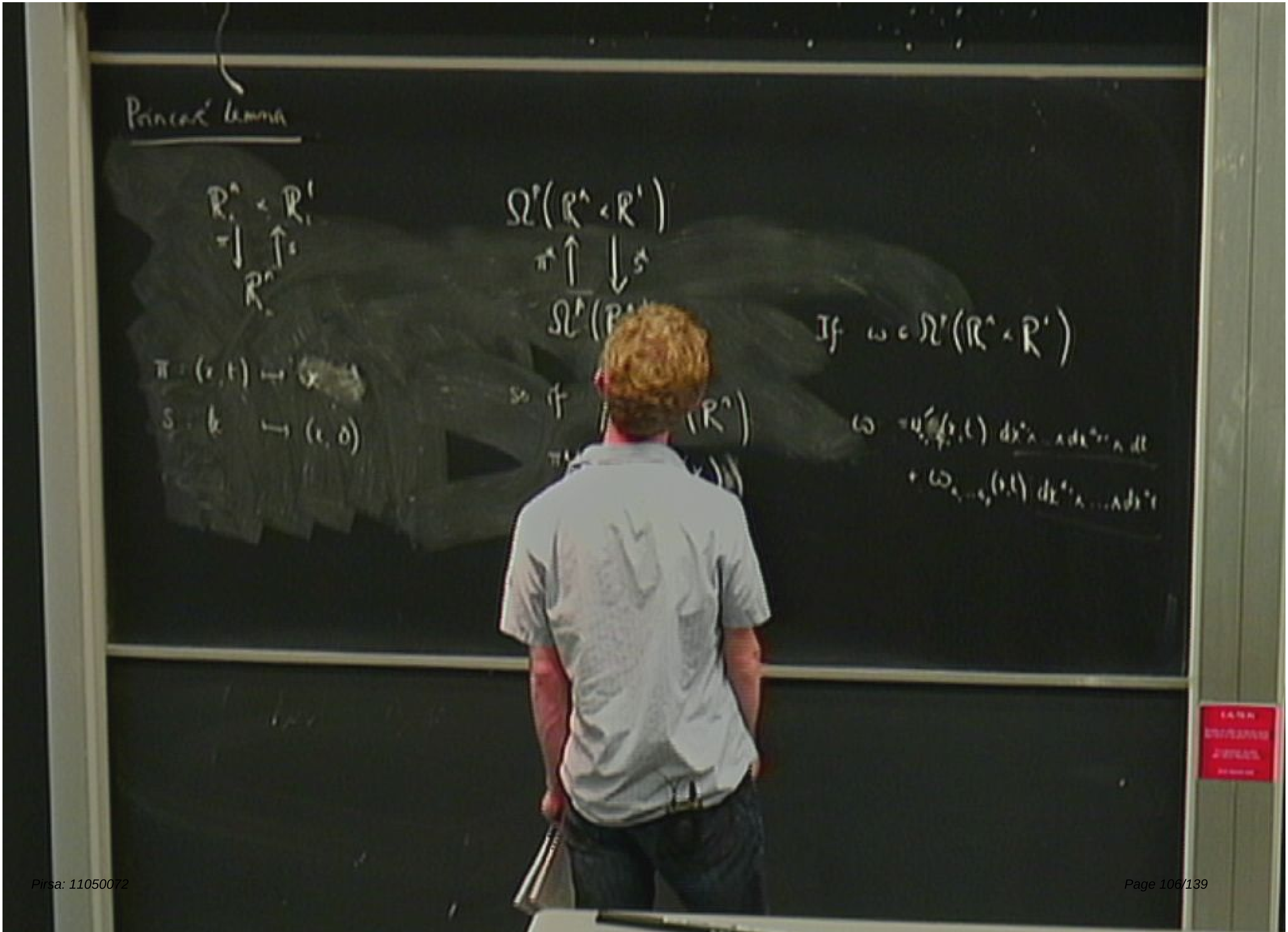
$\pi : (x, t) \mapsto x$

$s : x \mapsto (x, 0)$

$\mu \in \Omega^p(\mathbb{R}^n)$


Poincaré Lemma
π : (x, t) ↦ x
s : x ↦ (x, 0)
π*
s*
μ ∈ Ω*(ℝⁿ)


Poincaré Lemma
R^n × R^1
π ↓ ↑ s
R^n
Ω^p(R^n × R^1)
π* ↑ ↓ s*
Ω^p(R^n)
π : (x, t) ↦ x
s : x ↦ (x, 0)
so if μ ∈ Ω^p(R^n)
π*(μ) = μ(x)
dx ∧ … ∧ dx ∧ dt



Poincaré Lemma

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\
\mathbb{R}^n \arrow[u, shift right, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^p(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] \\
\Omega^p(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"]
\end{tikzcd}$$

$\pi : (x,t) \mapsto x$

$s : x \mapsto (x,0)$

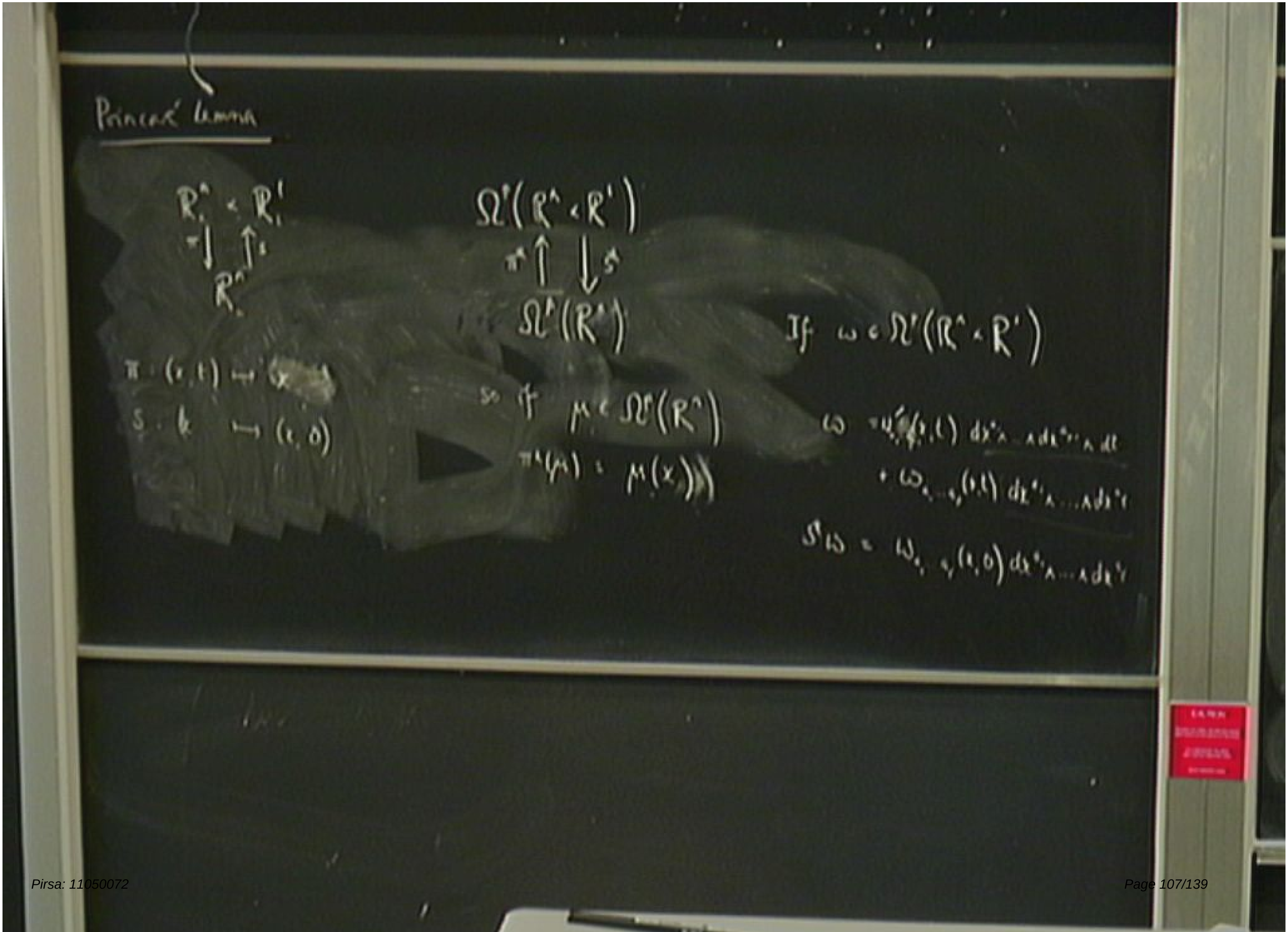
So if ... $(\mathbb{R}^n)$

$\pi^*$ ...

If $\omega \in \Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$

$$\omega = \omega'_{i_1 \ldots i_{p-1}}(x,t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_{p-1}} \wedge dt$$

$$+ \omega_{i_1 \ldots i_p}(x,t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$



## Poincaré Lemma

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\ \mathbb{R}^n \arrow[u, shift right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \Omega^r(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] \\ \Omega^r(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

$\pi : (x, t) \mapsto (x)$

$s : x \mapsto (x, 0)$

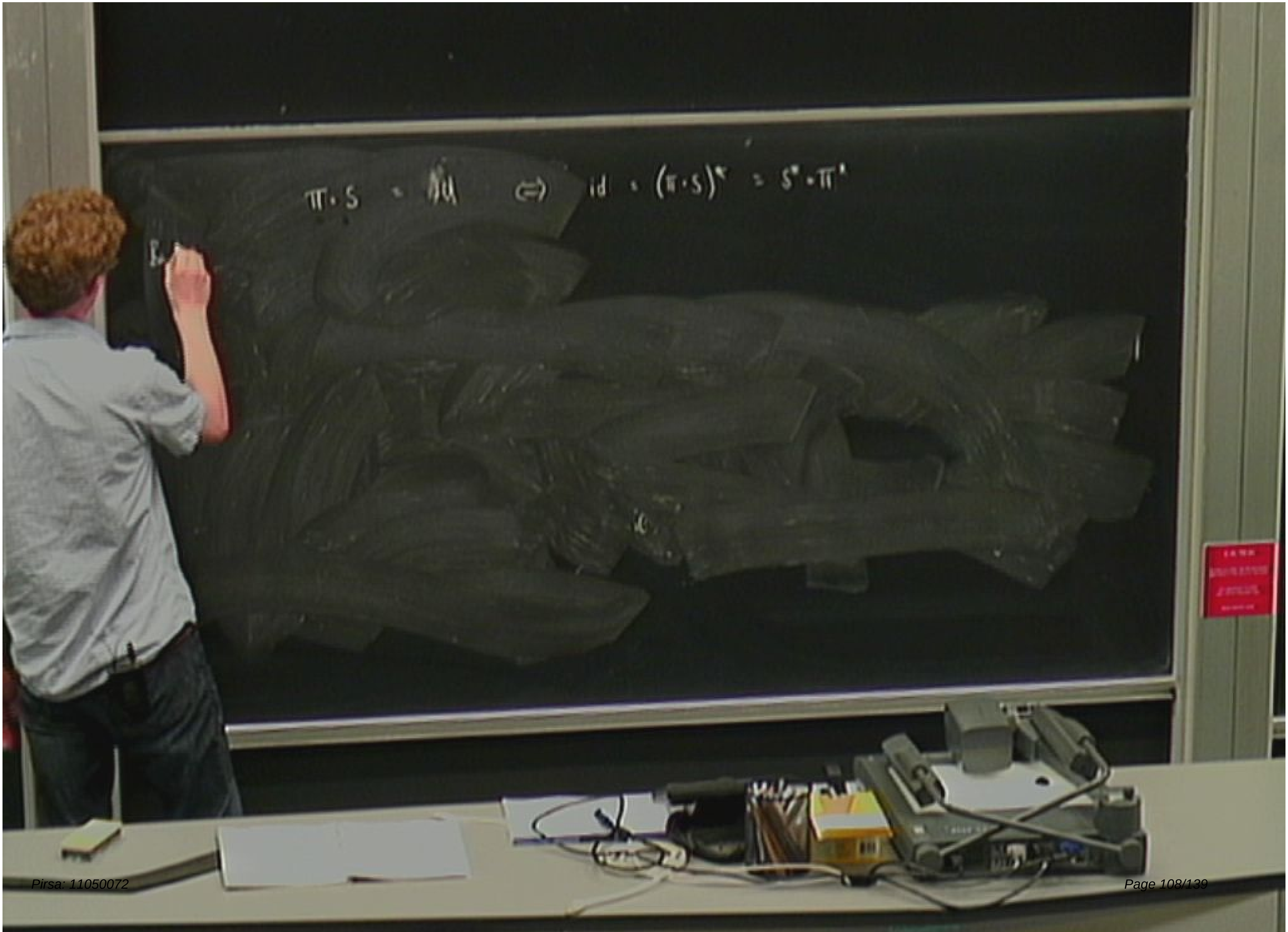
so if $\mu \in \Omega^r(\mathbb{R}^n)$

$\pi^*(\mu) = \mu(x)$

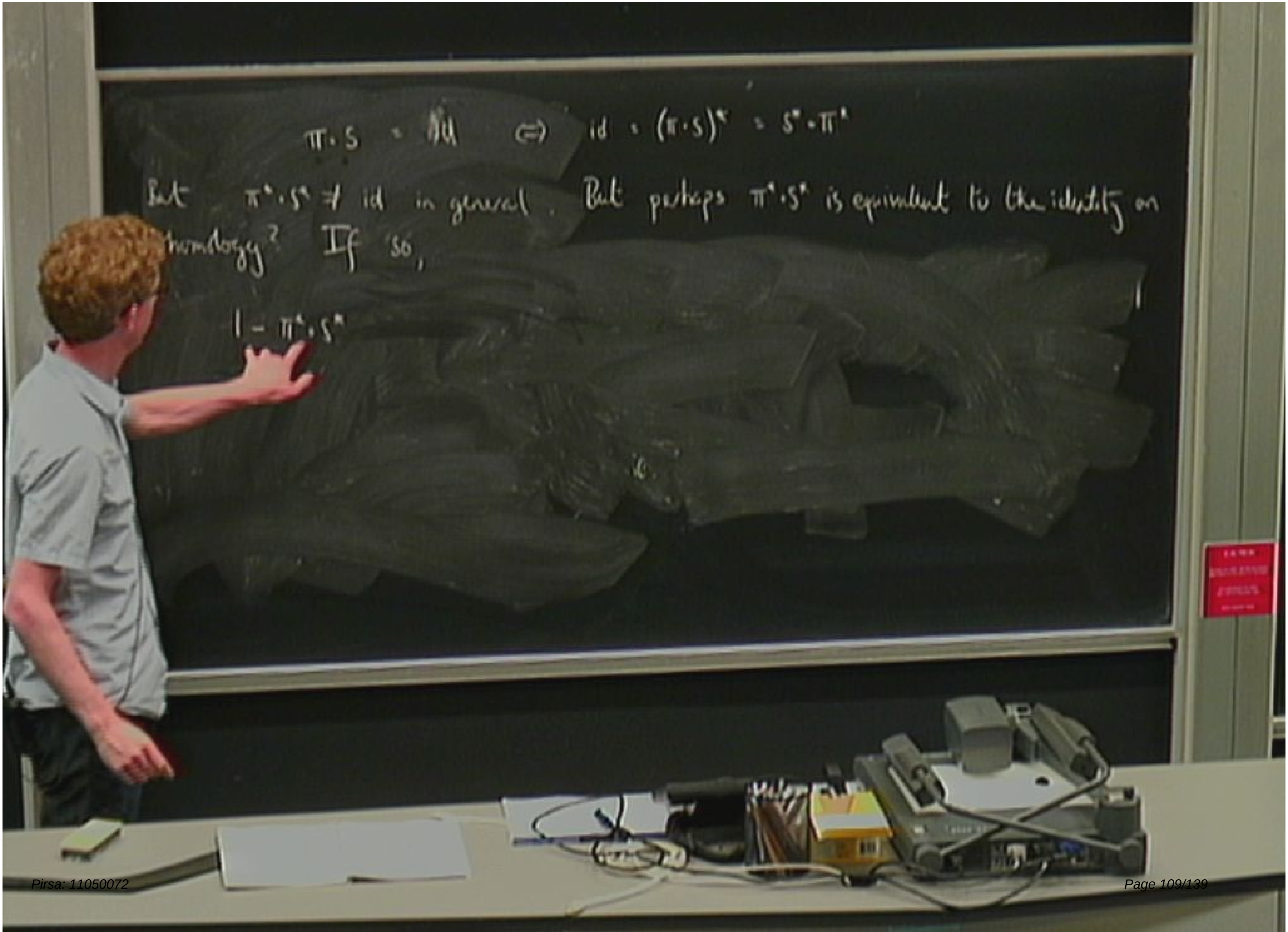
If $\omega \in \Omega^r(\mathbb{R}^n \times \mathbb{R}^1)$

$$\omega = \omega_{i_1 \ldots i_{r-1}}(x,t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_{r-1}} \wedge dt + \omega_{i_1 \ldots i_r}(x,t)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

$$s^*\omega = \omega_{i_1 \ldots i_r}(x,0)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$



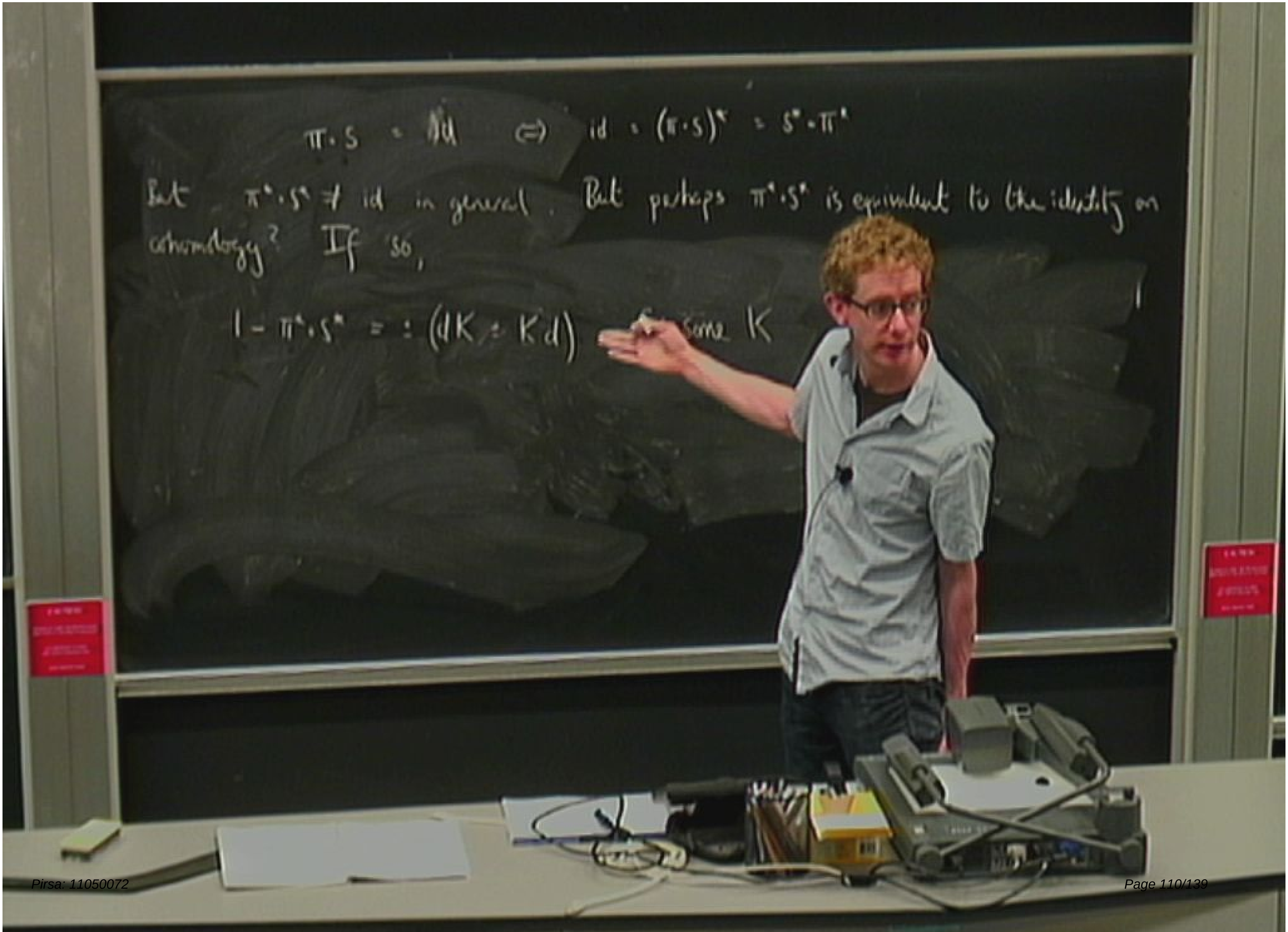
$$\pi \circ s = \mathrm{id} \iff \mathrm{id} = (\pi \circ s)^* = s^* \circ \pi^*$$



$$\pi \circ s = \mathrm{id} \iff \mathrm{id} = (\pi \circ s)^* = s^* \circ \pi^*$$

But $\pi^* \circ s^* \neq \mathrm{id}$ in general. But perhaps $\pi^* \circ s^*$ is equivalent to the identity on homology? If so,

$$1 - \pi^* \circ s^*$$



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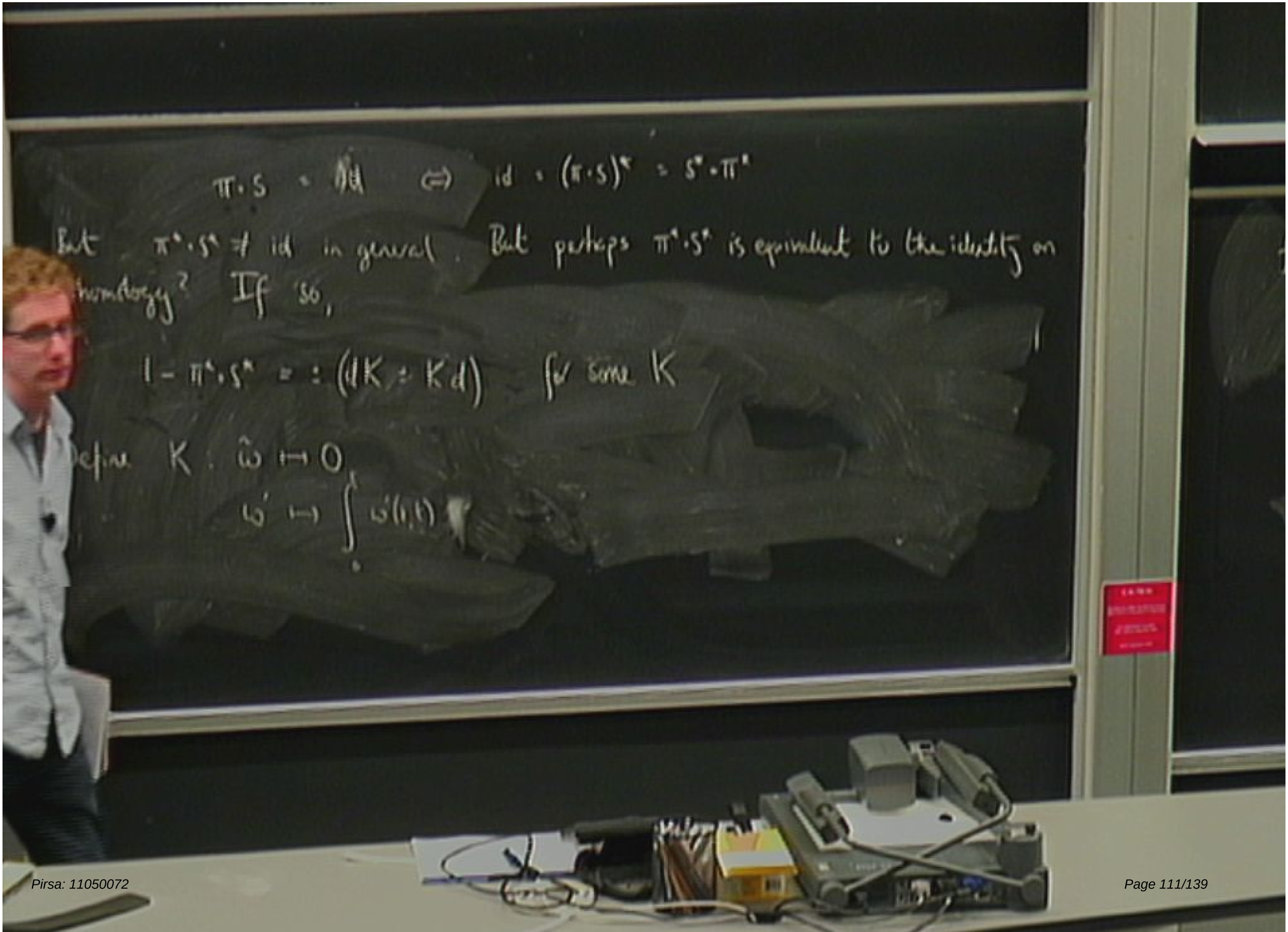

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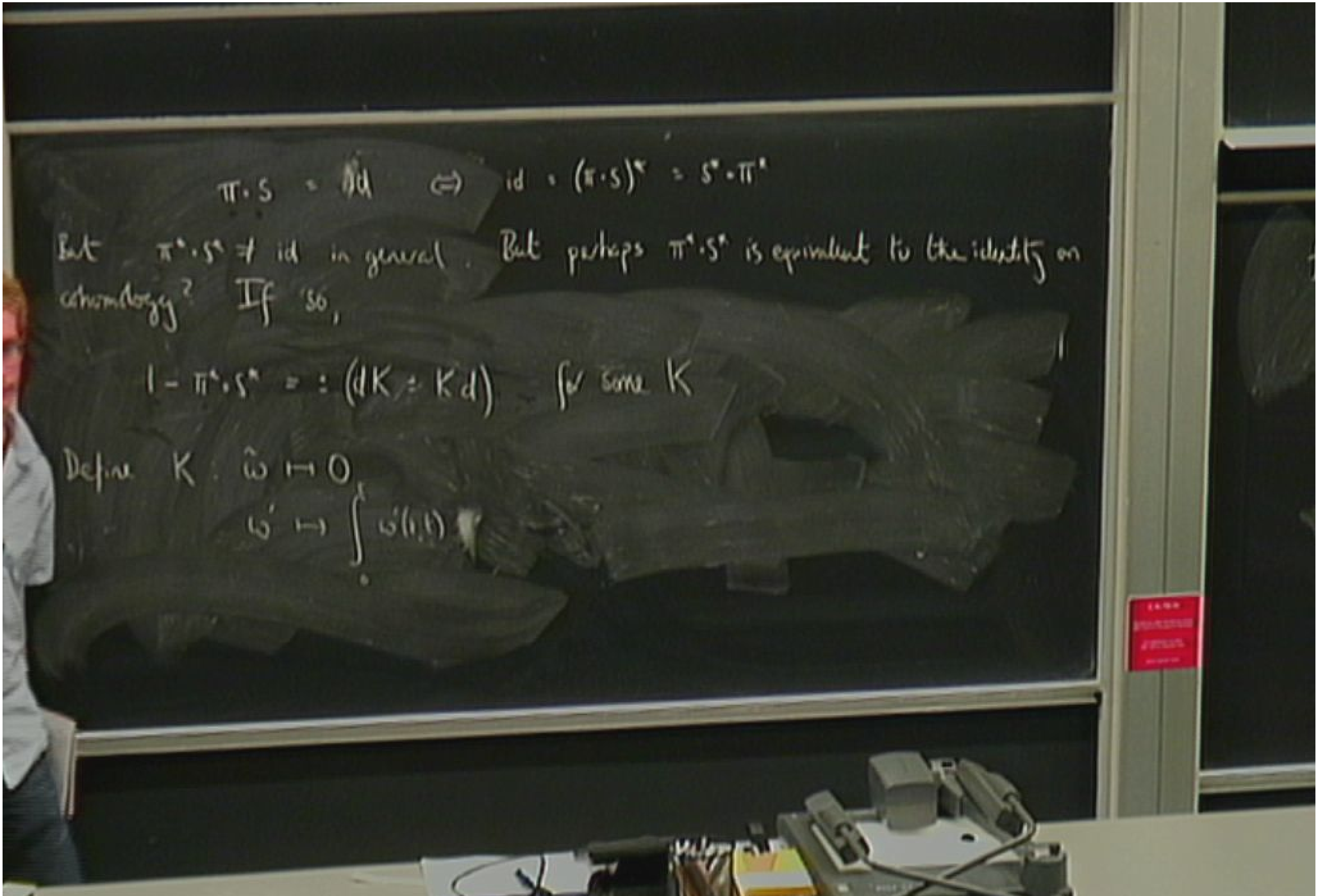
Define $K : \hat{\omega} \mapsto 0$

$$\omega' \mapsto \int_0^1 \omega'(\cdot,t)$$



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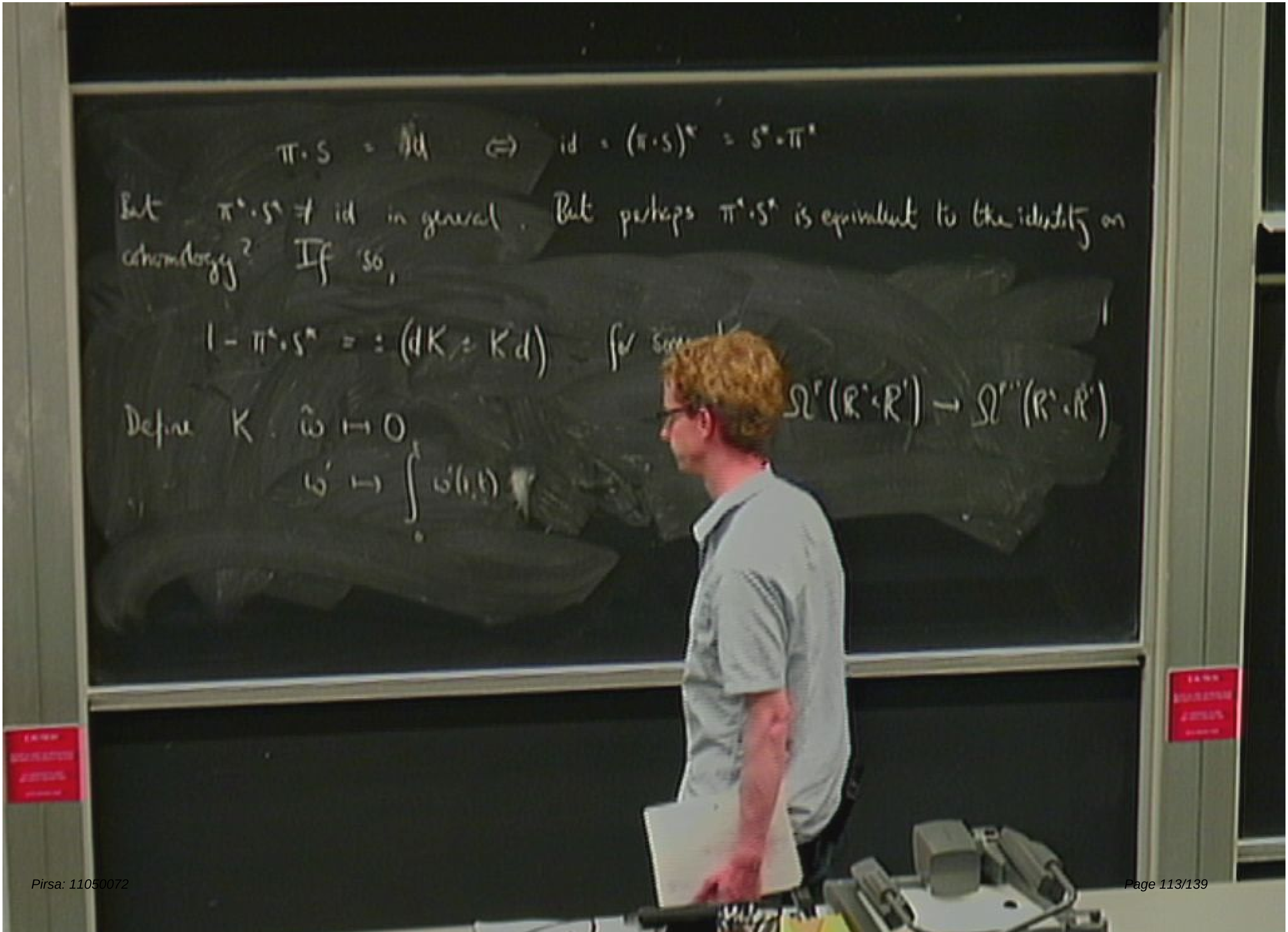

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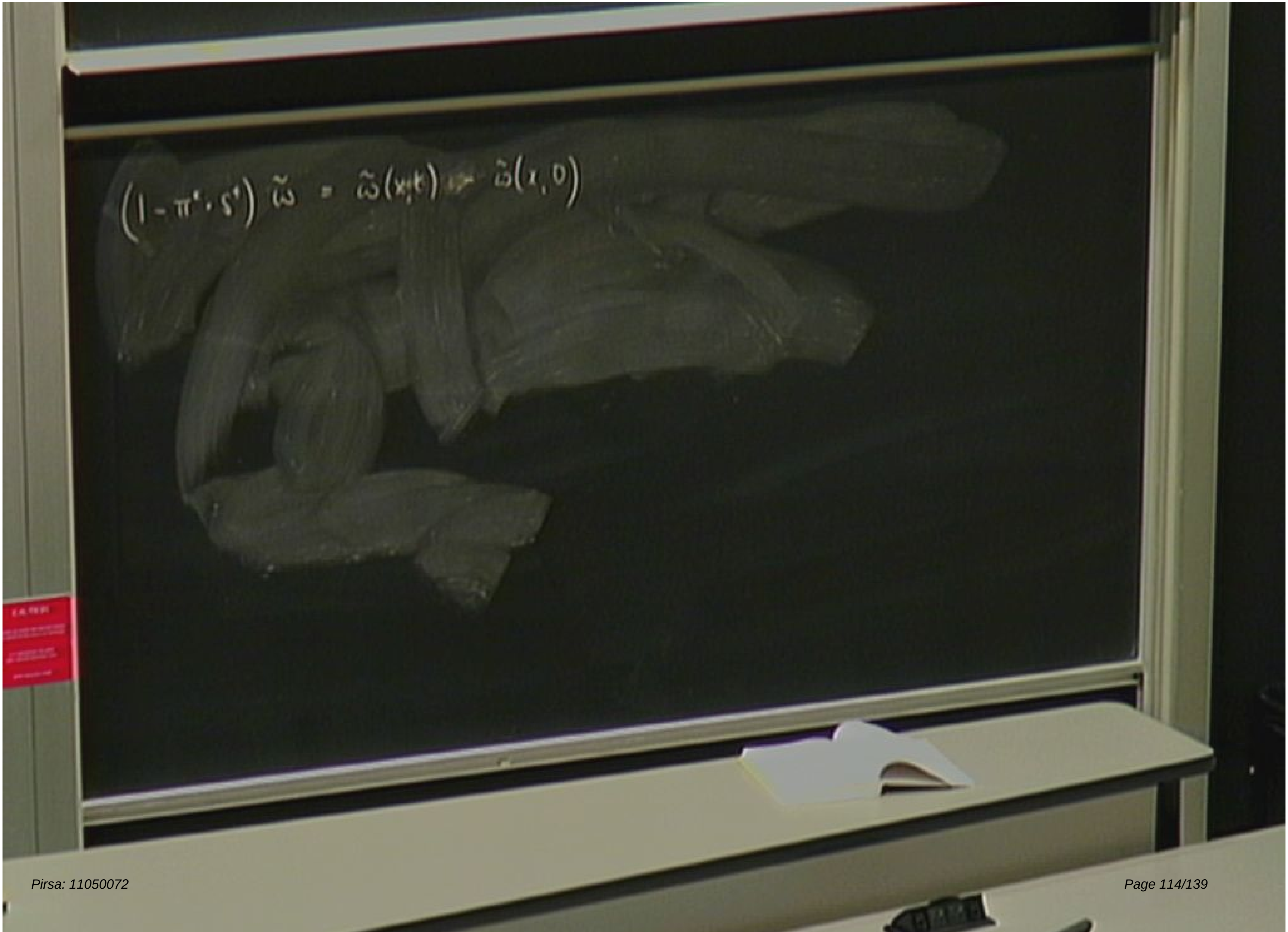
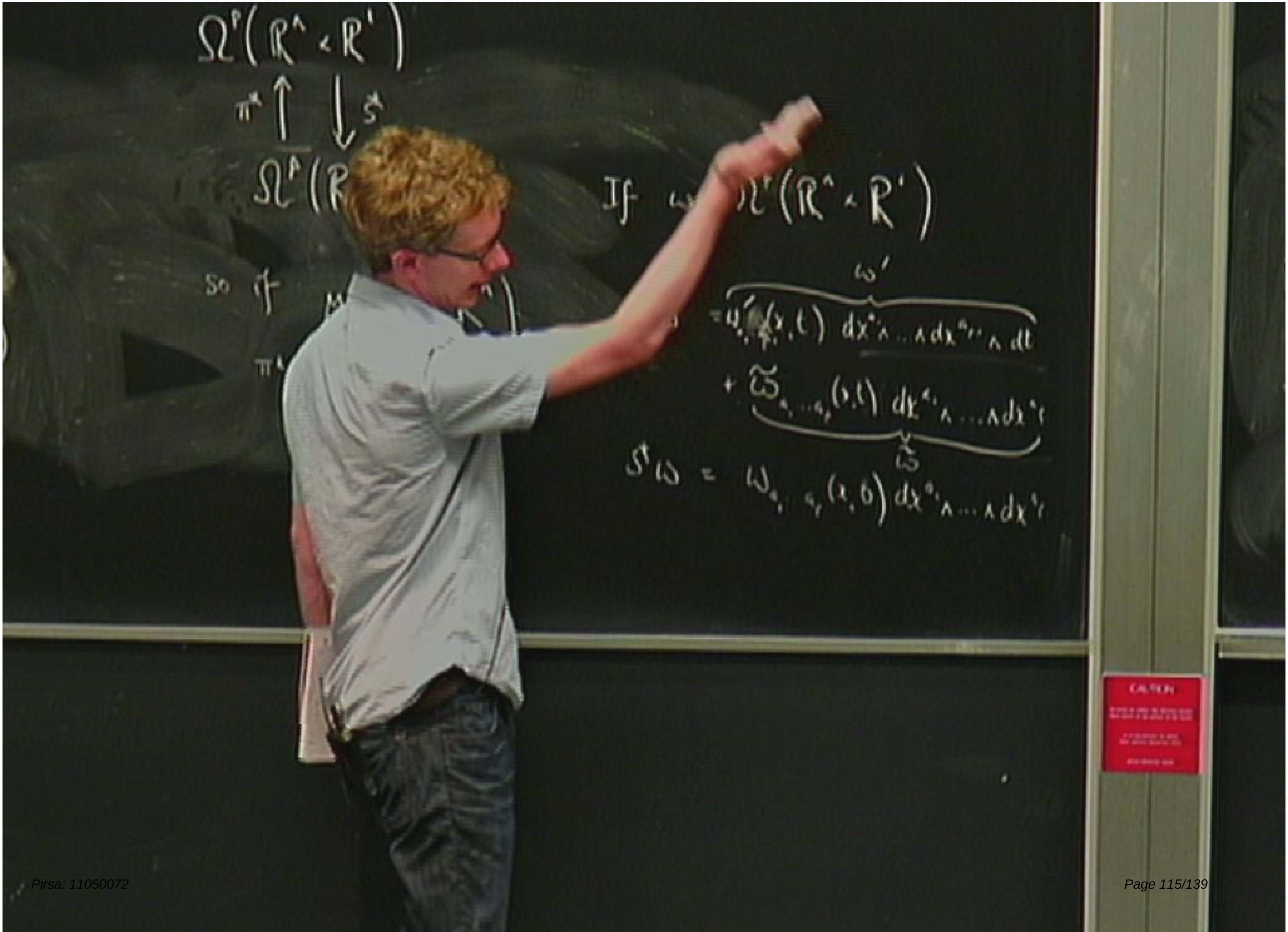
$$\Omega^r(\mathbb{R}^n \times \mathbb{R}^1) \to \Omega^{r-1}(\mathbb{R}^n \times \mathbb{R}^1)$$

Define $K$: $\hat{\omega} \mapsto 0$

$$\omega' \mapsto \int_0^1 \omega'(x,t)$$



$$\left(1 - \pi' \cdot S'\right) \tilde{\omega} = \tilde{\omega}(x;t) - \tilde{\omega}(x,0)$$



$$\Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$$

$$\pi^* \uparrow \quad \downarrow s^*$$

$$\Omega^p(\mathbb{R}$$

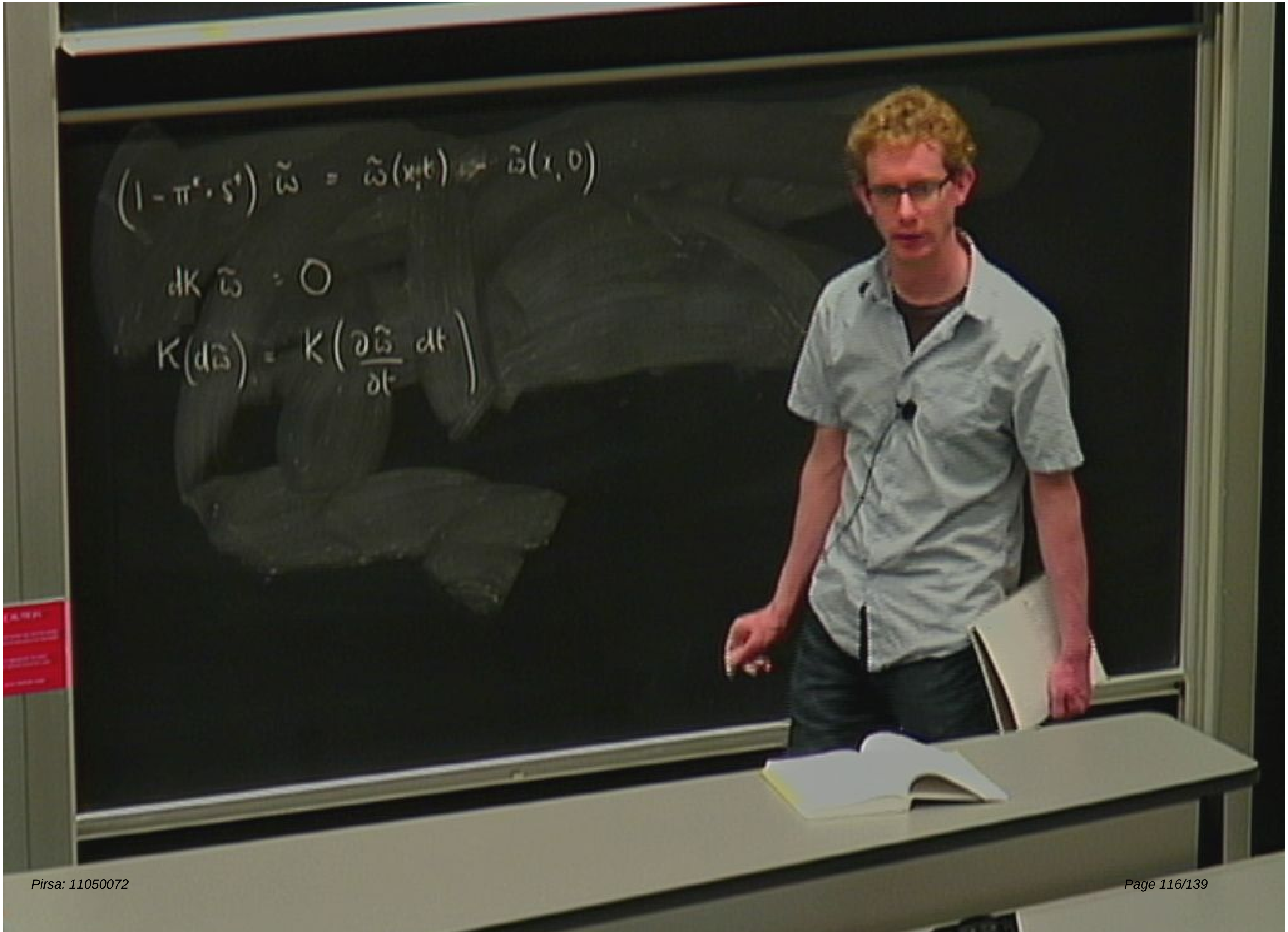
so if

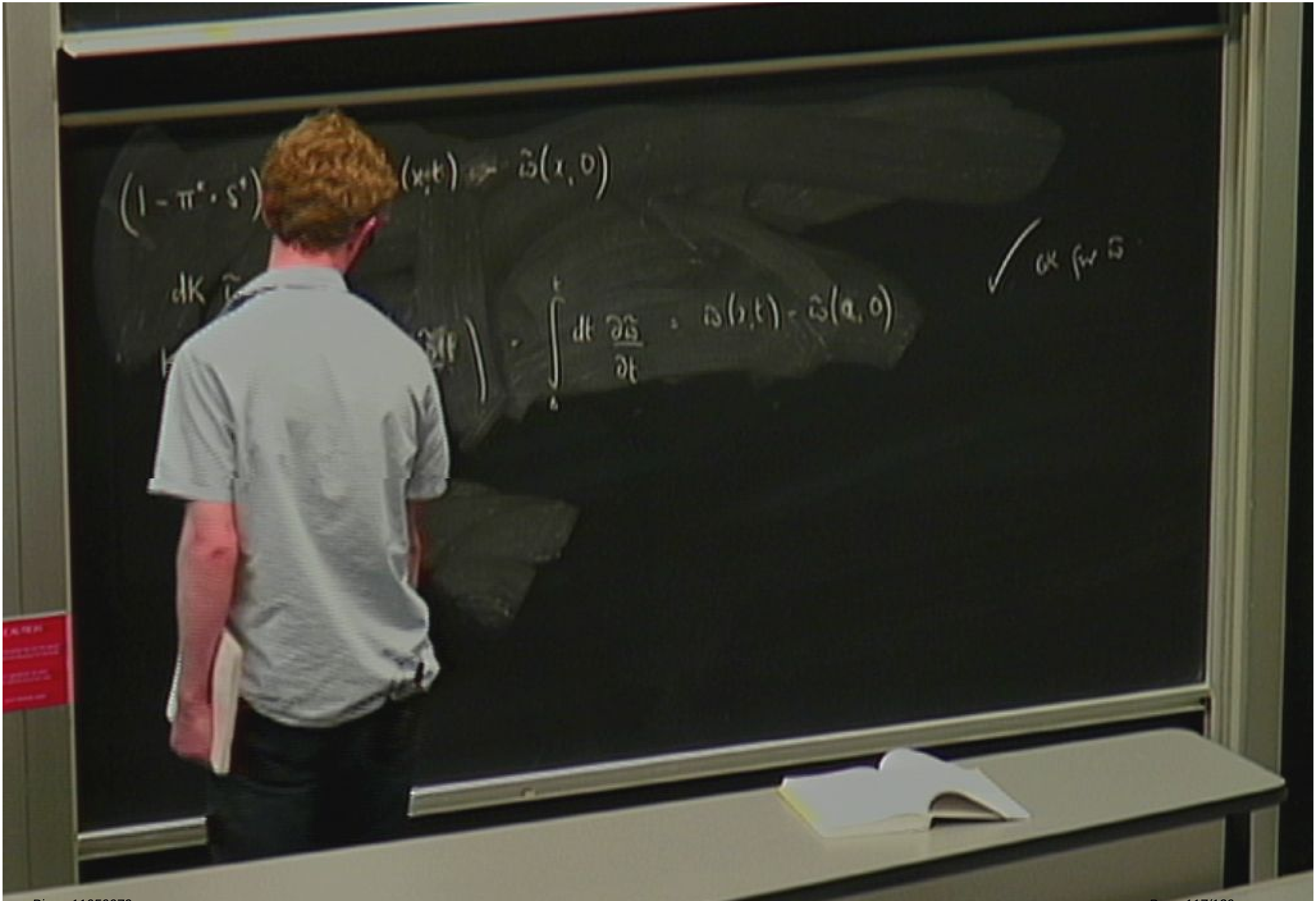
$\pi^*$

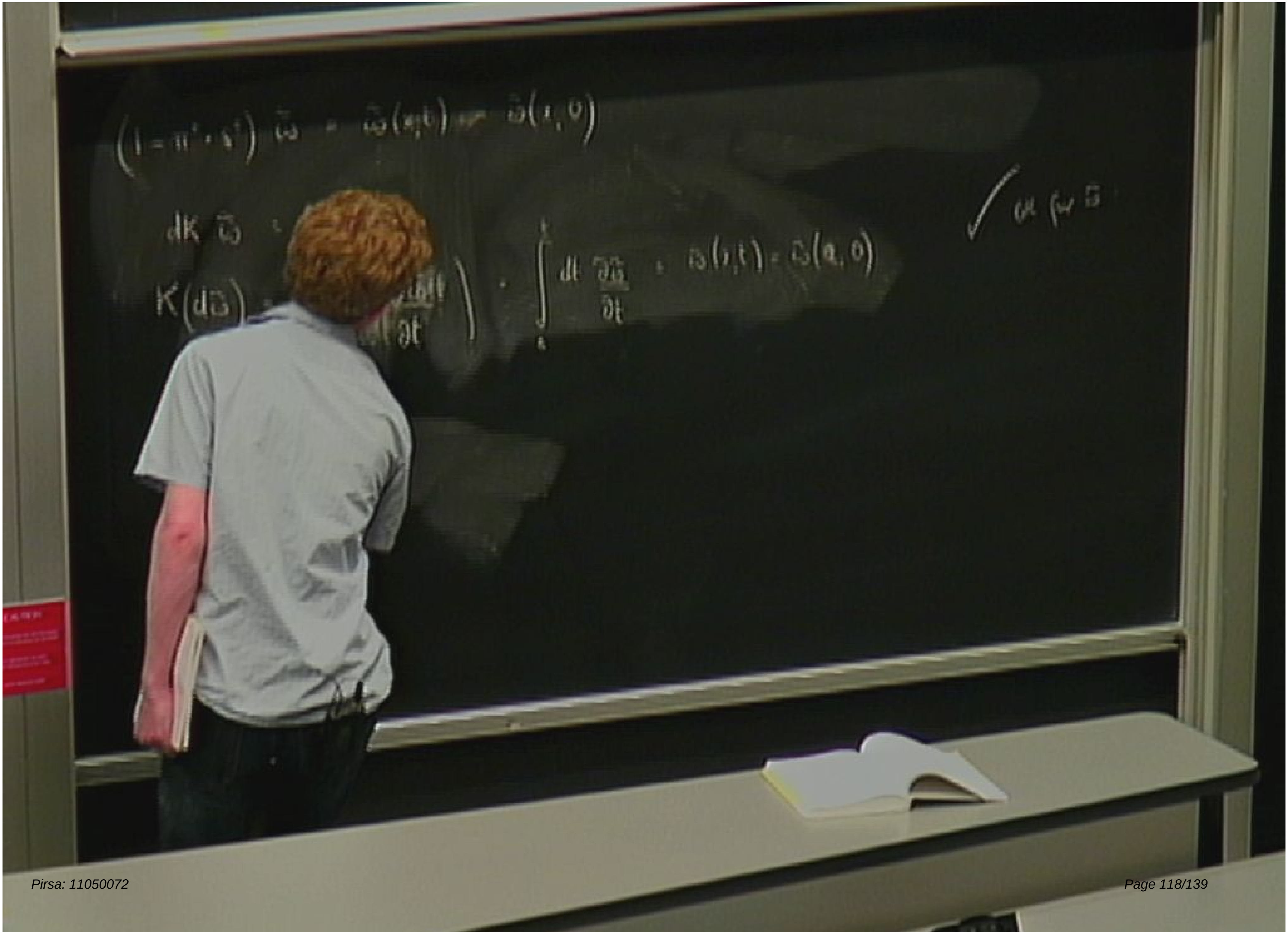
If $\omega$ $\Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$

$$= \overbrace{\omega'(x,t)\, dx^{a_1} \wedge \ldots \wedge dx^{a_{p-1}} \wedge dt}^{\omega'} + \underbrace{\tilde{\omega}_{a_1 \ldots a_p}(x,t)\, dx^{a_1} \wedge \ldots \wedge dx^{a_p}}_{\tilde{\omega}}$$

$$s^*\omega = \omega_{a_1 \ldots a_p}(x,0)\, dx^{a_1} \wedge \ldots \wedge dx^{a_p}$$


$(1 - \pi^* \circ s^*)\, \breve{\omega} = \hat{\omega}(x,t) - \hat{\omega}(x,0)$
$dK\, \tilde{\omega} = 0$
$K(d\hat{\omega}) = K\left(\frac{\partial \hat{\omega}}{\partial t}\, dt\right)$


(1 - π* · s*)
(x, t)
ω̃(x, 0)
✓ ok for ω̃
dK
∫ dt ∂ω̃/∂t = ω(b, t) - ω̃(a, 0)

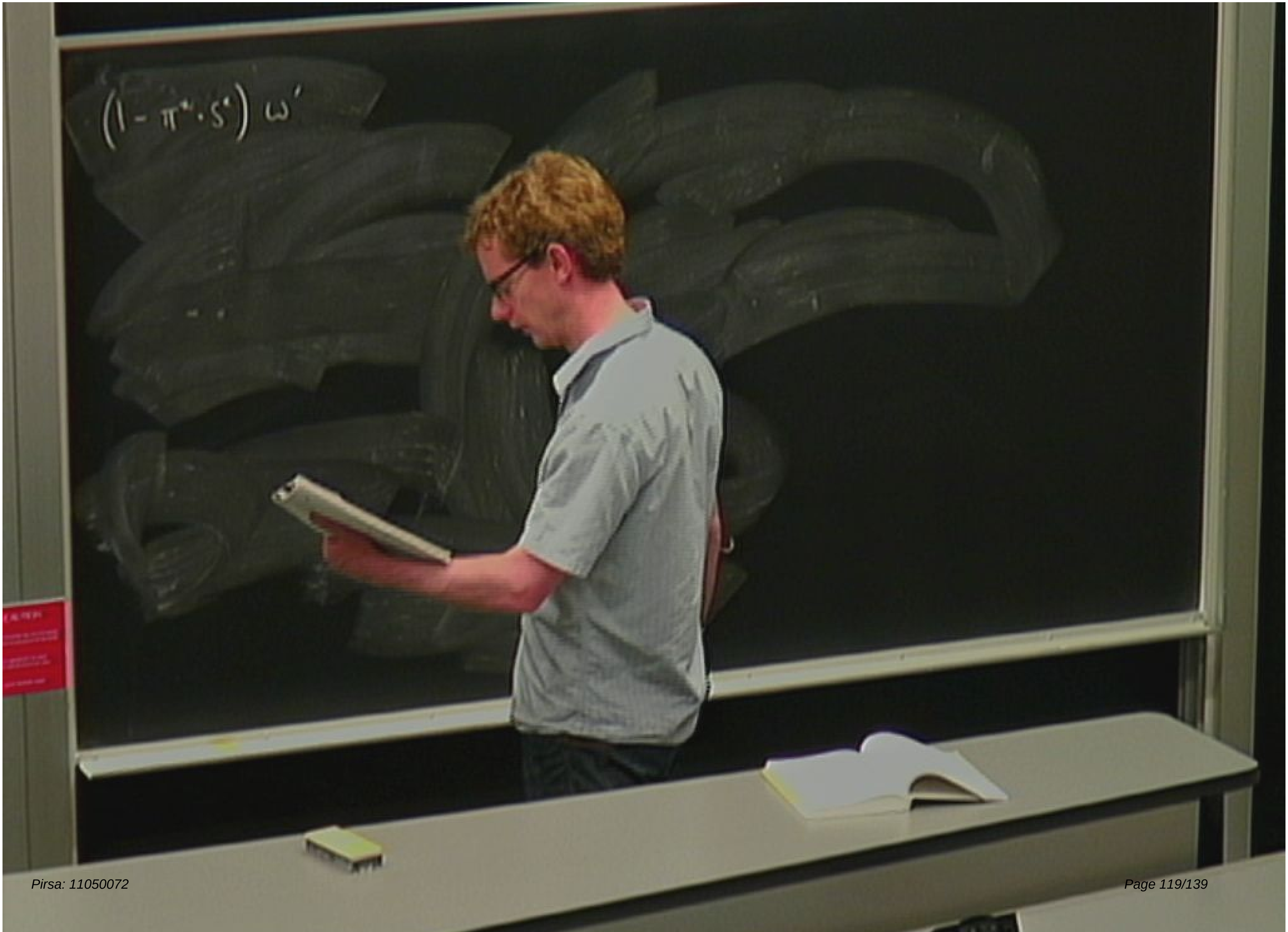


$$\left(1 - \pi^* s^*\right)\tilde{\omega} = \tilde{\omega}(x,t) - \tilde{\omega}(x,0)$$

$$dK\,\tilde{\omega} =$$

$$K(d\tilde{\omega}) = \left( \frac{\partial\tilde{\omega}}{\partial t} \right) = \int_0^t dt\,\frac{\partial\tilde{\omega}}{\partial t} = \tilde{\omega}(x,t) - \tilde{\omega}(x,0)$$

✓ ok for $\tilde{\omega}$


(1 - π* · s*) ω'

## Poincaré Lemma

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R}^1 \arrow[d, "\pi"', shift right] \\ \mathbb{R}^n \arrow[u, "s"', shift right] \end{tikzcd}$$

$$\begin{tikzcd} \Omega^p(\mathbb{R}^n \times \mathbb{R}^1) \arrow[d, "s^*", shift left] \\ \Omega^p(\mathbb{R}^n) \arrow[u, "\pi^*", shift left] \end{tikzcd}$$

$\pi : (x,t) \mapsto (x)$

$s : x \mapsto (x,0)$

so if $\mu \in \Omega^p(\mathbb{R}^n)$

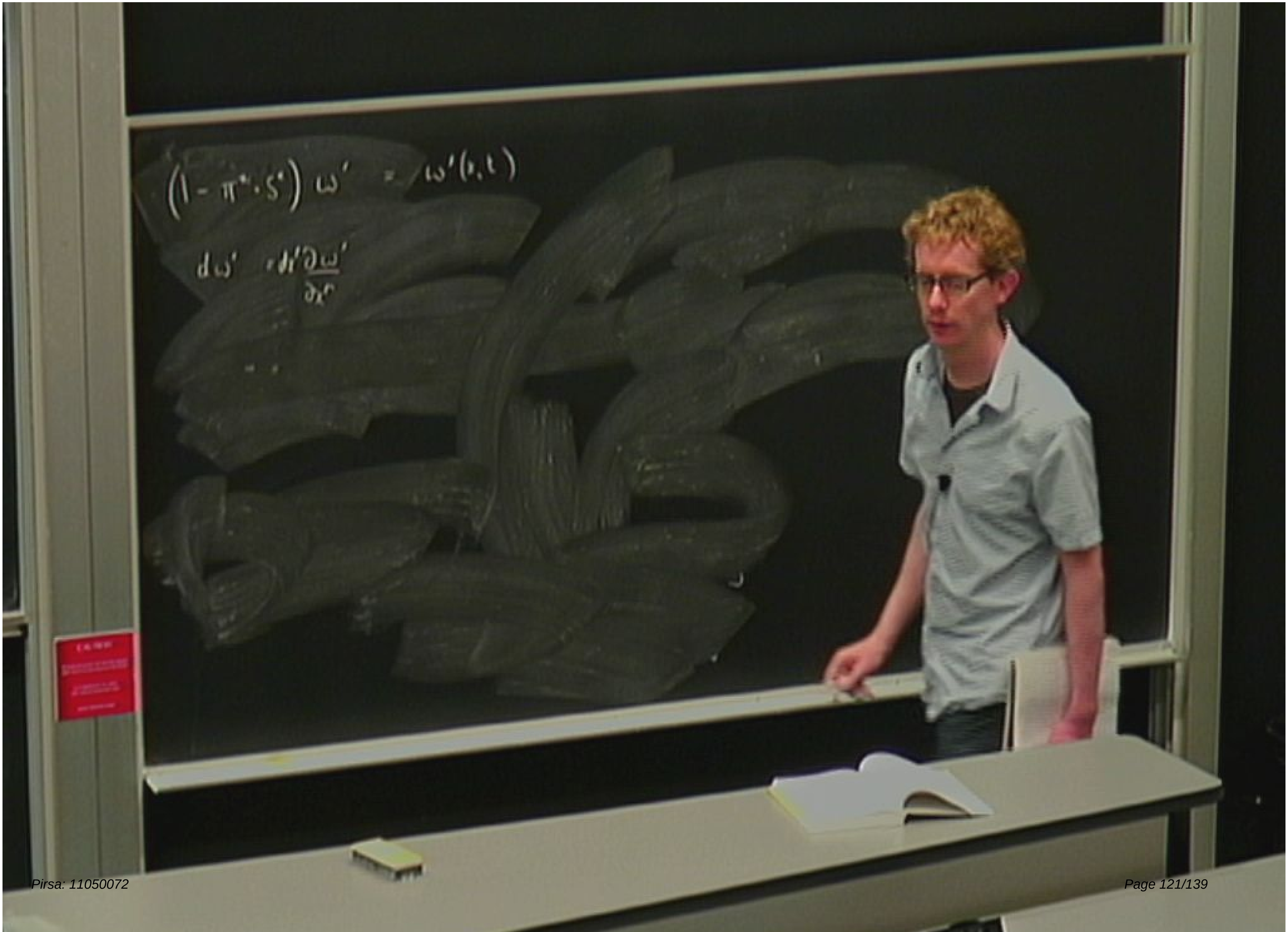
$\pi^*(\mu) = \mu(x))$

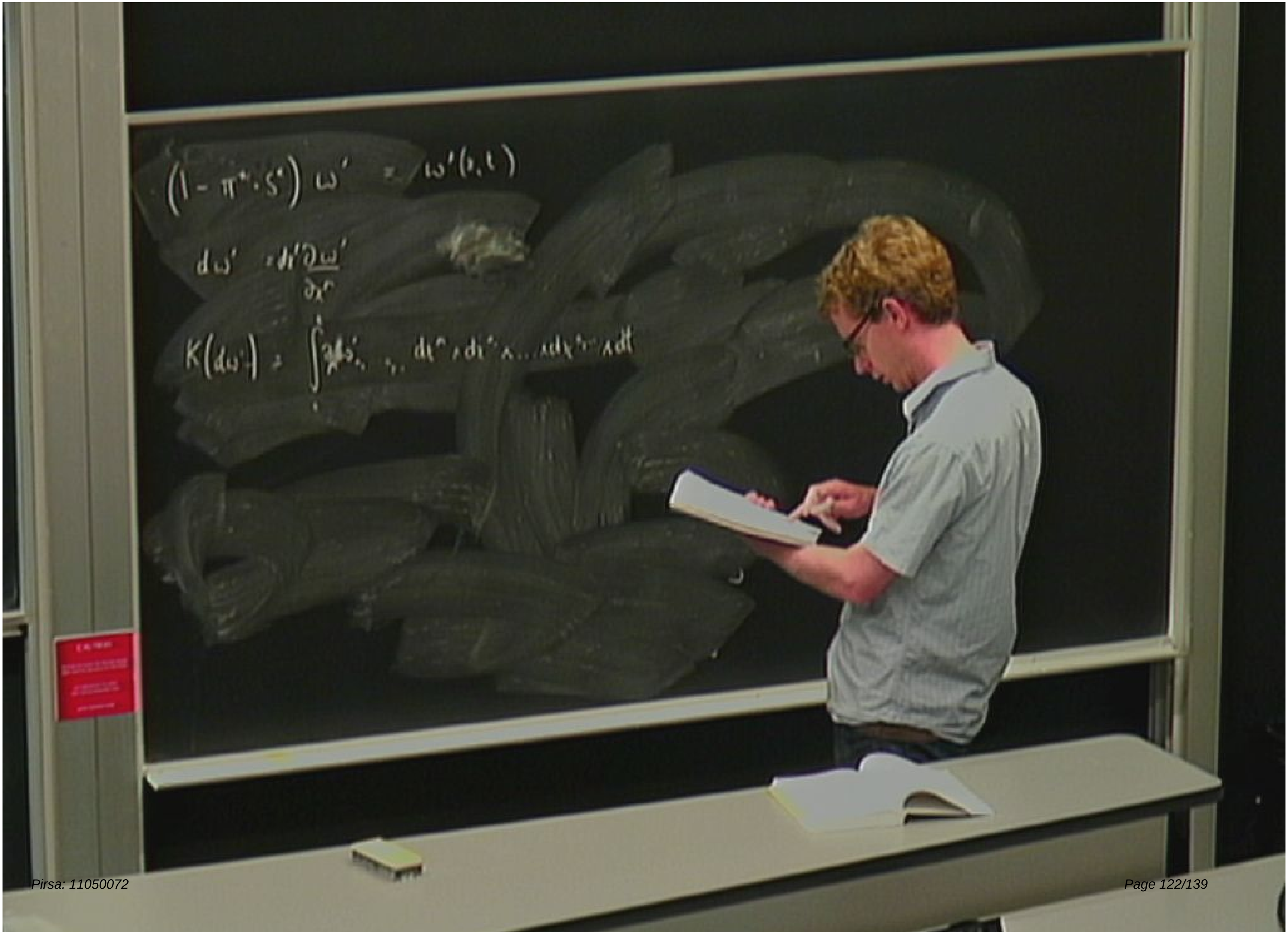
If $\omega \in \Omega^p(\mathbb{R}^n \times \mathbb{R}^1)$

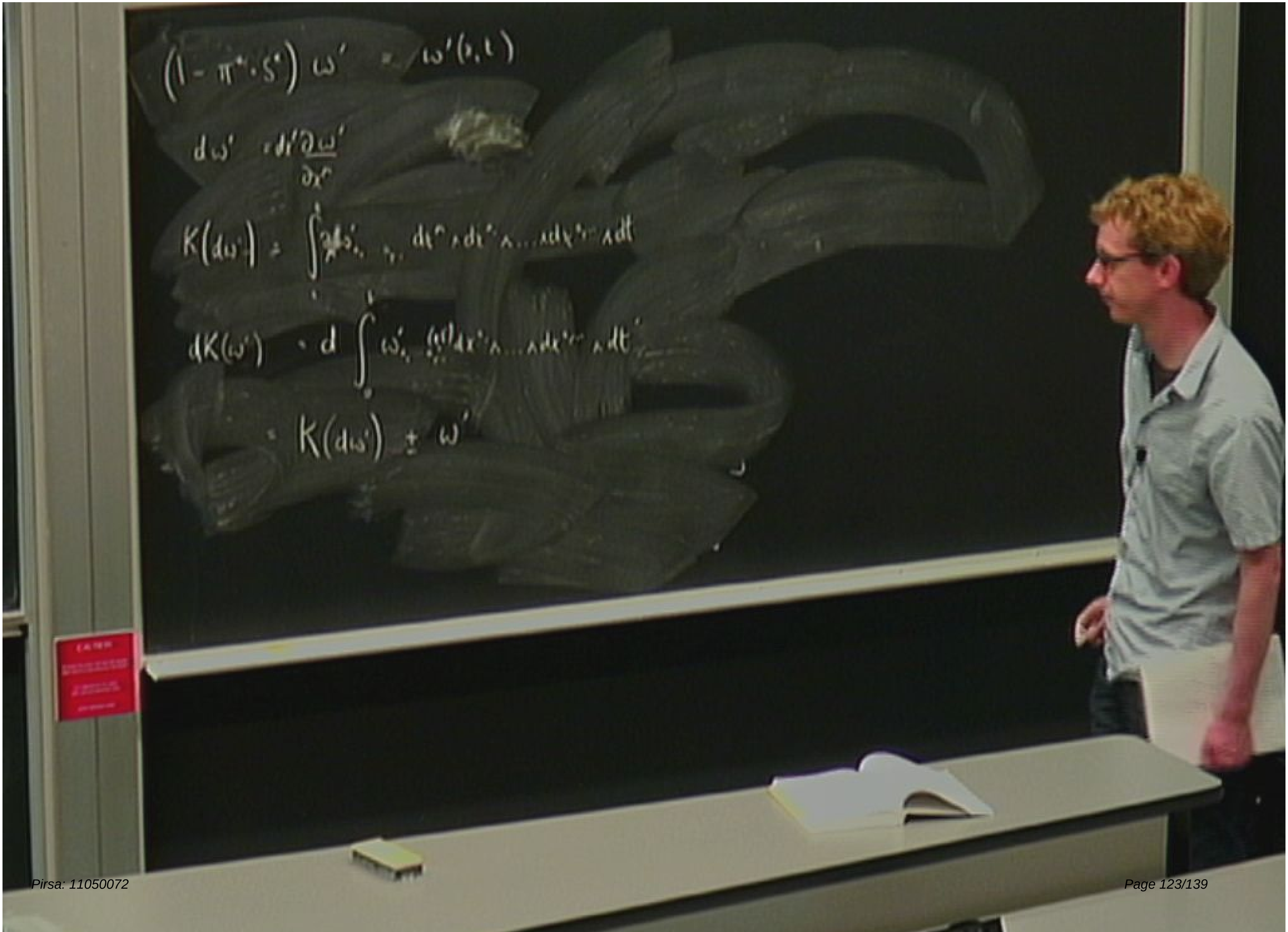
$$\omega = \overbrace{\omega'_{a_1 \ldots a_{p-1}}(x,t)\, dx^{a_1} \wedge \ldots \wedge dx^{a_{p-1}} \wedge dt}^{\omega'}$$

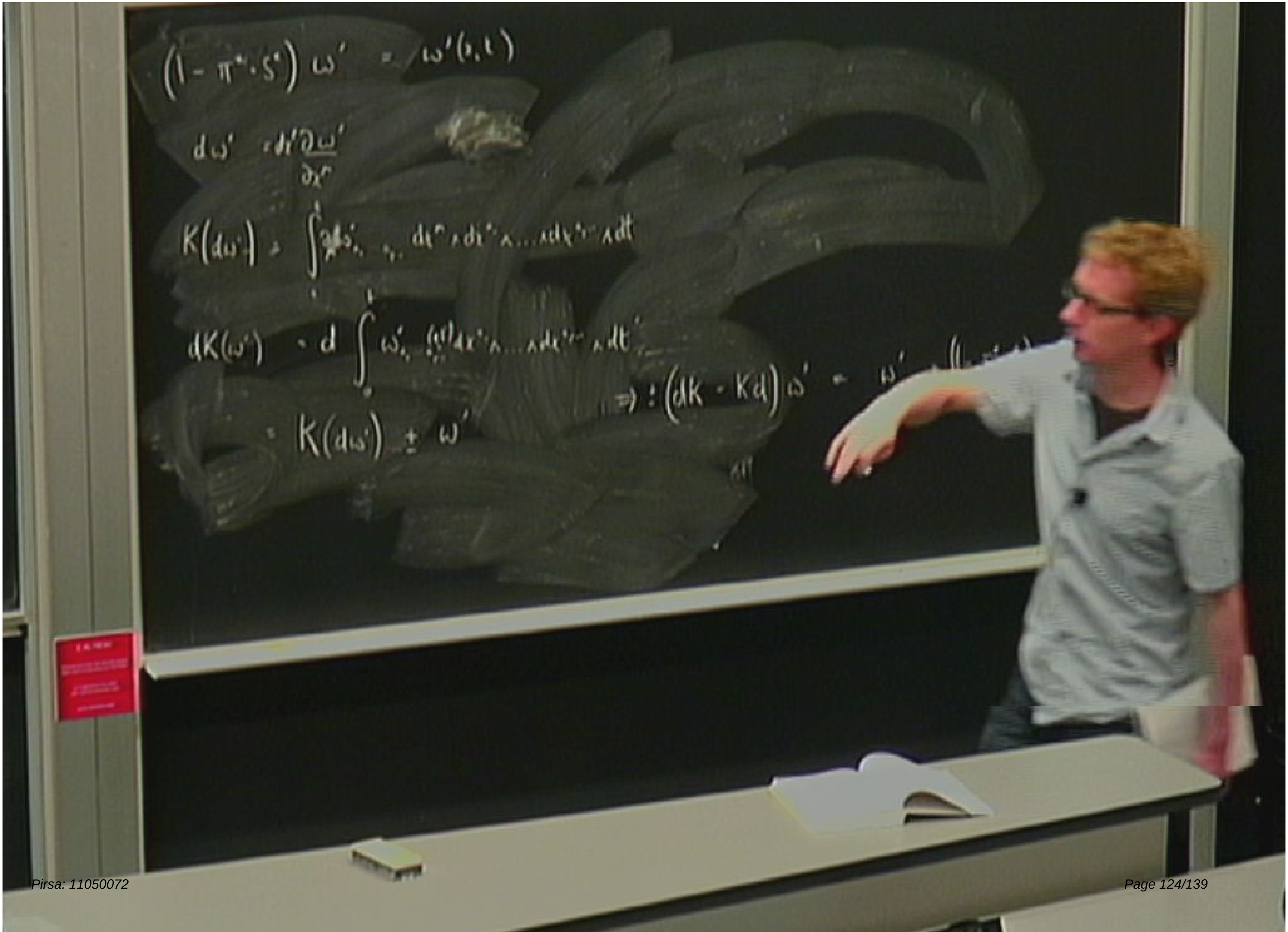
$$+ \underbrace{\tilde{\omega}_{a_1 \ldots a_p}(x,t)\, dx^{a_1} \wedge \ldots \wedge dx^{a_p}}_{\tilde{\omega}}$$

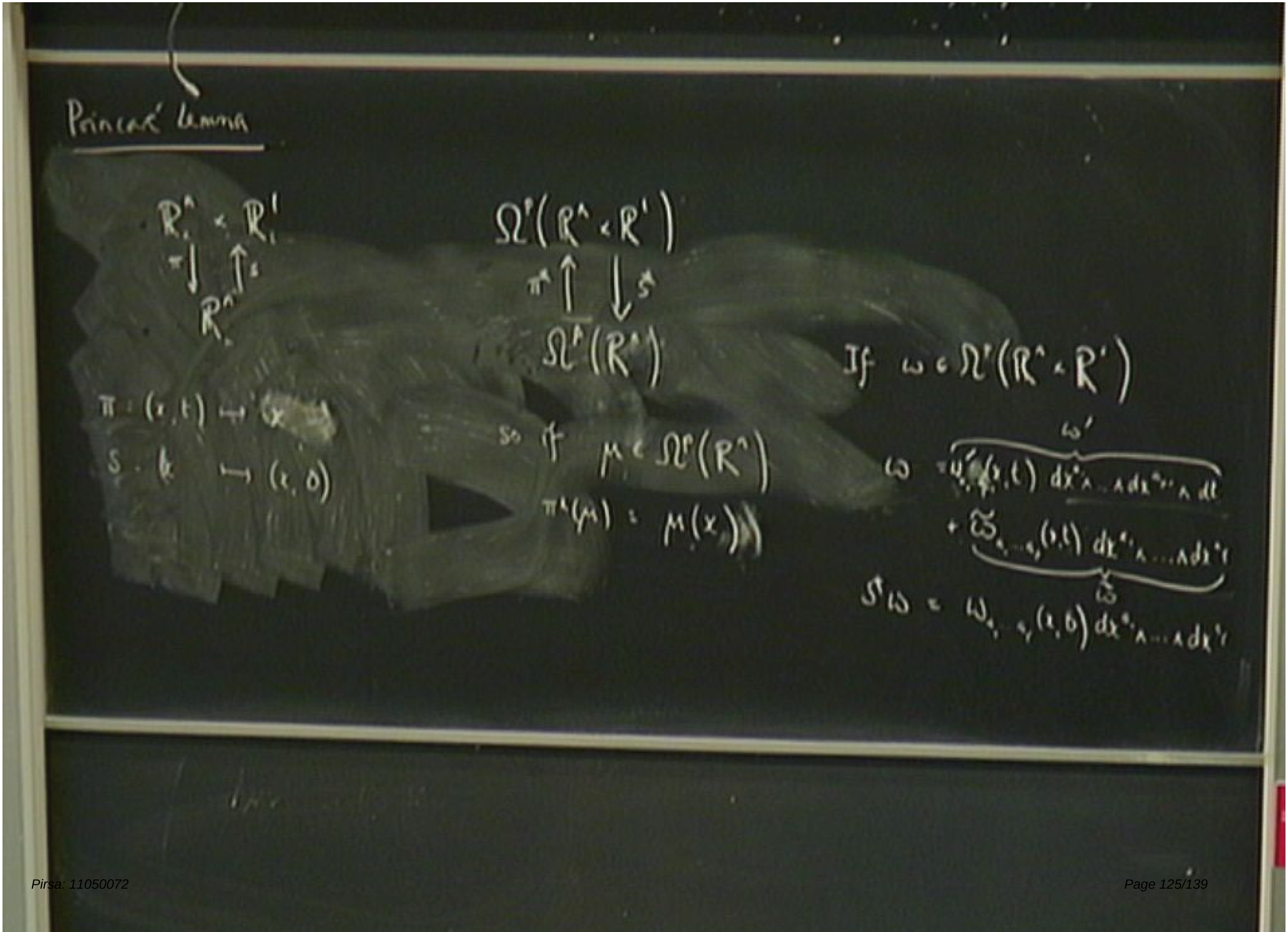
$$s^*\omega = \omega_{a_1 \ldots a_p}(x,0)\, dx^{a_1} \wedge \ldots \wedge dx^{a_p}$$


(1 - π*·S*) ω' = ω'(x,t)
dω' = dx' ∂ω'/∂x^μ


$(1 - \pi^{*} \cdot S^{*})\ \omega' = \omega'(x, t)$
$d\omega' = dx^{\mu}\frac{\partial \omega'}{\partial x^{\mu}}$
$K(d\omega') = \int_{0}^{t} \ldots dx^{\mu} \wedge dx^{\nu} \wedge \ldots \wedge dx^{n-1} \wedge dt$


$(1 - \pi^* \circ s^*)\,\omega' = \omega'(x,t)$
$d\omega' = dt\,\frac{\partial \omega'}{\partial x^\mu}$
$K(d\omega') = \int_0^1 \ldots dt \wedge dx^{\mu} \wedge \ldots \wedge dx^{\mu_n} \wedge dt$
$dK(\omega') = d \int_0^1 \omega'_{\mu \ldots}\, dx^{\mu} \wedge \ldots \wedge dx^{\mu_n} \wedge dt$
$= K(d\omega') \pm \omega'$


(1 − π* · s*) ω′ = ω′(x, t)
dω′ = dt ∂ω′/∂x
K(dω′) = ∫ … dx ∧ dx ∧ … ∧ dx ∧ dt
dK(ω′) = d ∫ ω′ … dx ∧ … ∧ dx ∧ dt
= K(dω′) ± ω′
⇒ (dK ± Kd) ω′ = ω′ …



## Poincaré Lemma

$$\begin{tikzcd} \mathbb{R}^n_x \times \mathbb{R}^1_t \arrow[d, shift right, "\pi"'] \\ \mathbb{R}^n_x \arrow[u, shift right, "s"'] \end{tikzcd}$$

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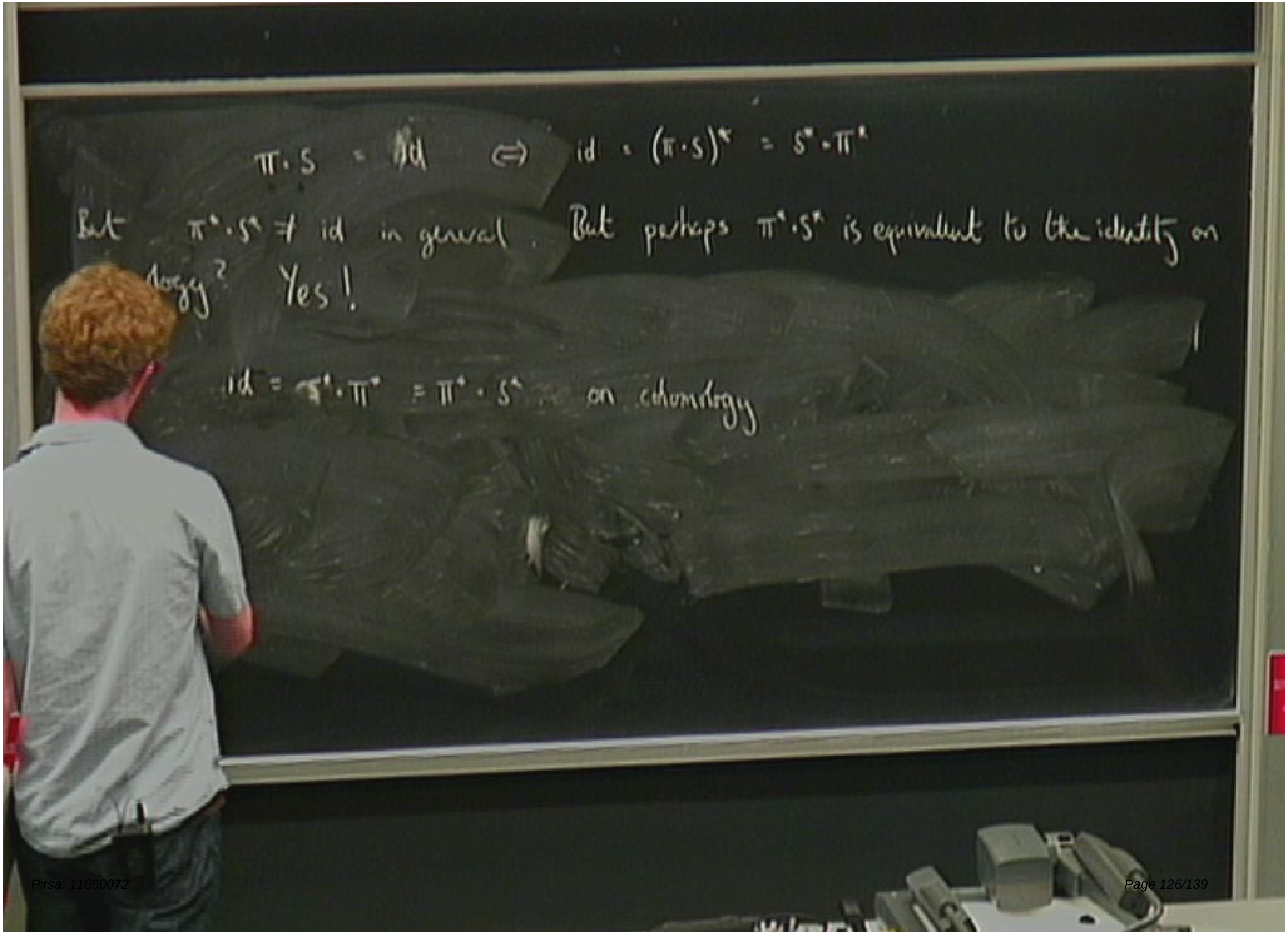

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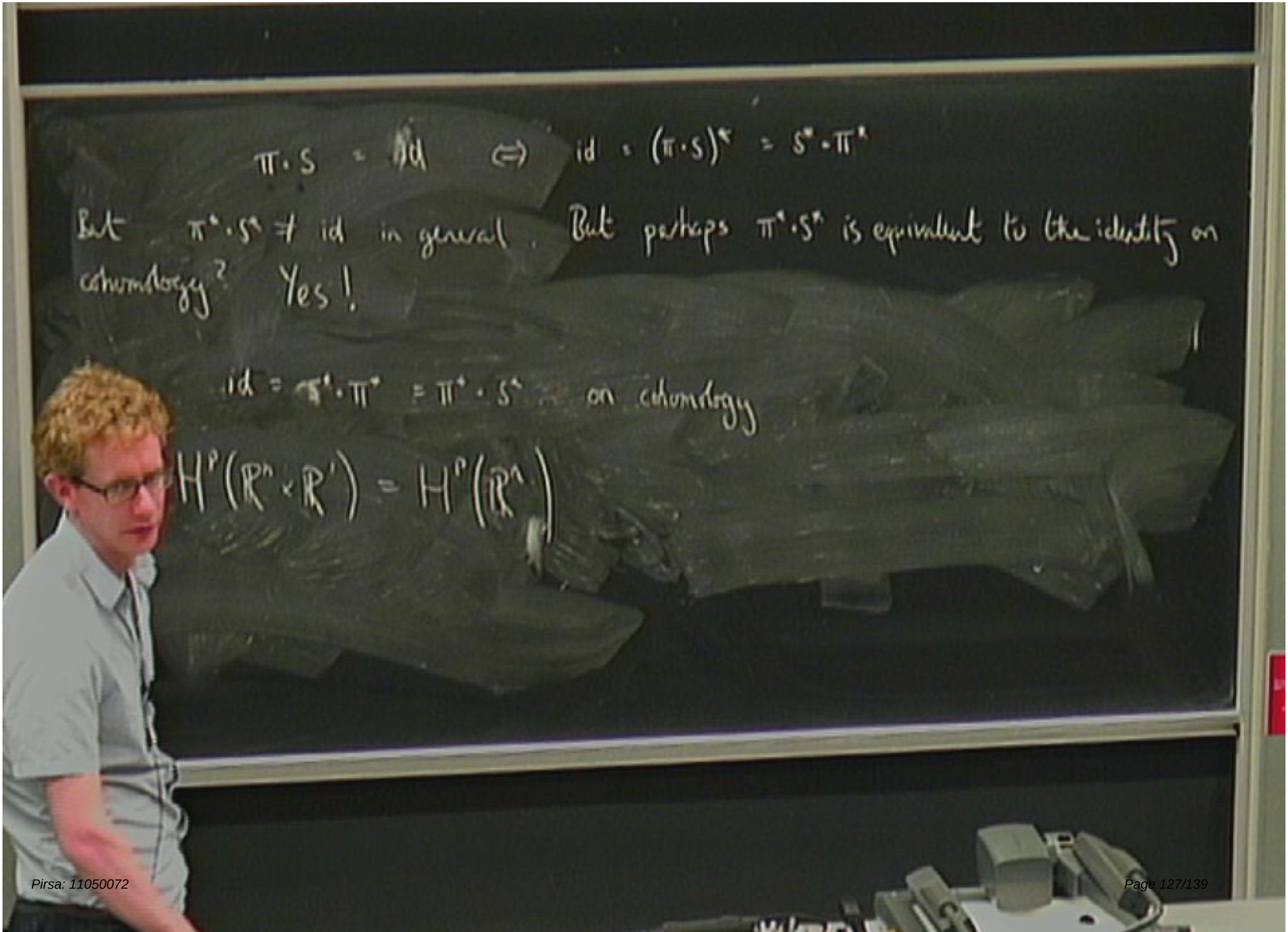

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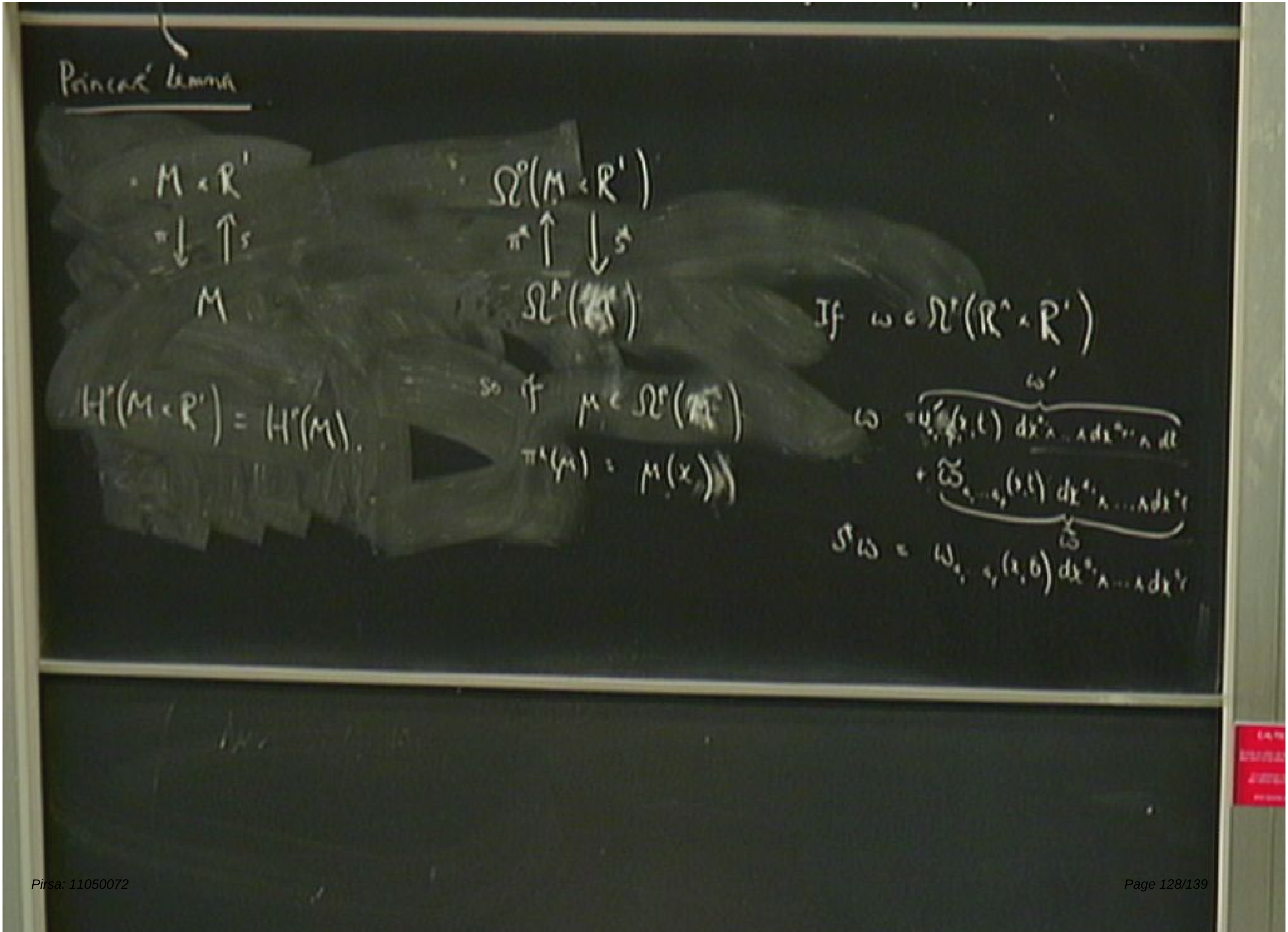

## Poincaré Lemma

$$\begin{tikzcd} M \times \mathbb{R}^1 \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} \Omega^p(M \times \mathbb{R}^1) \arrow[d, shift left, "s^*"] \\ \Omega^p(M) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

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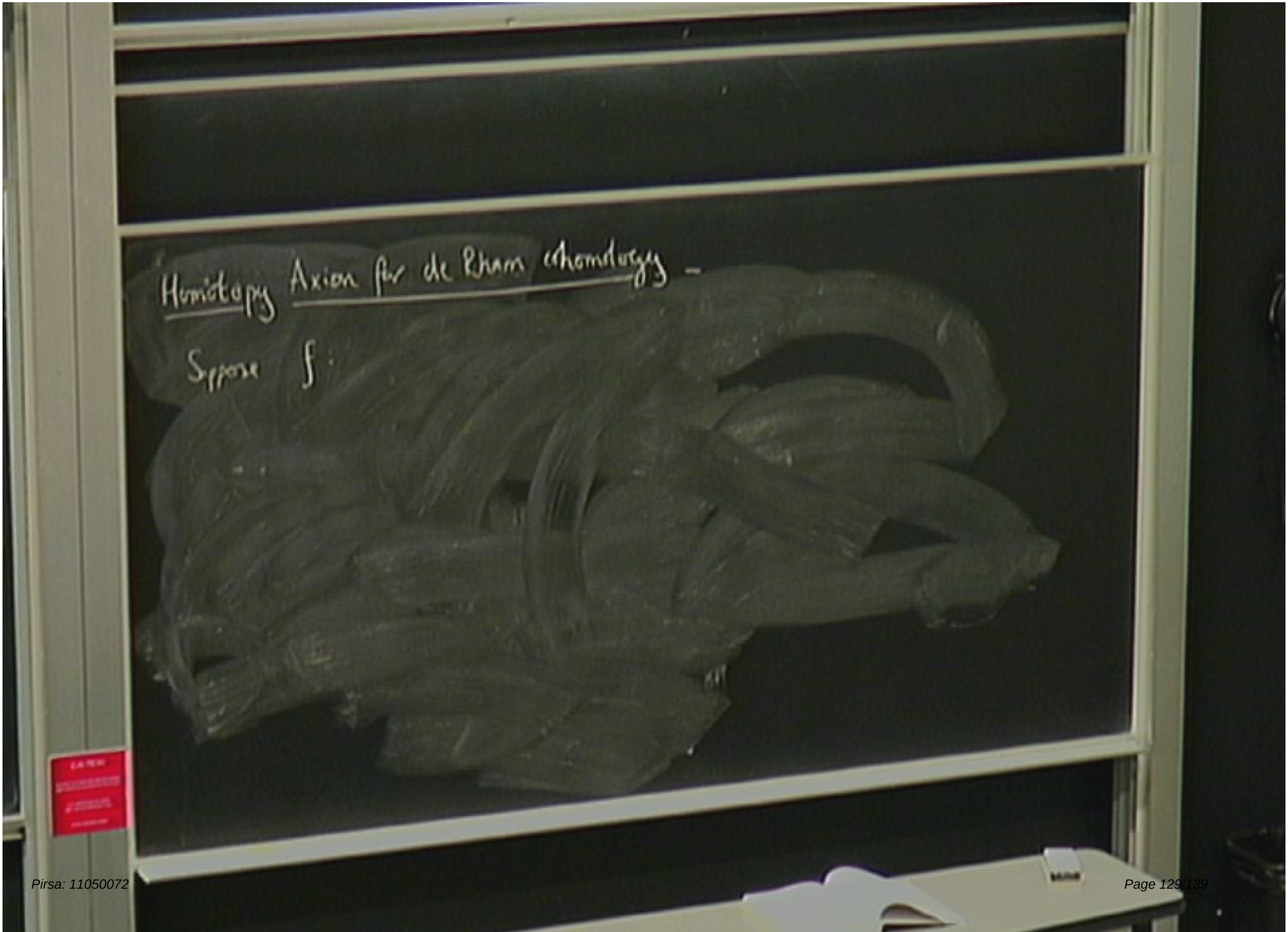
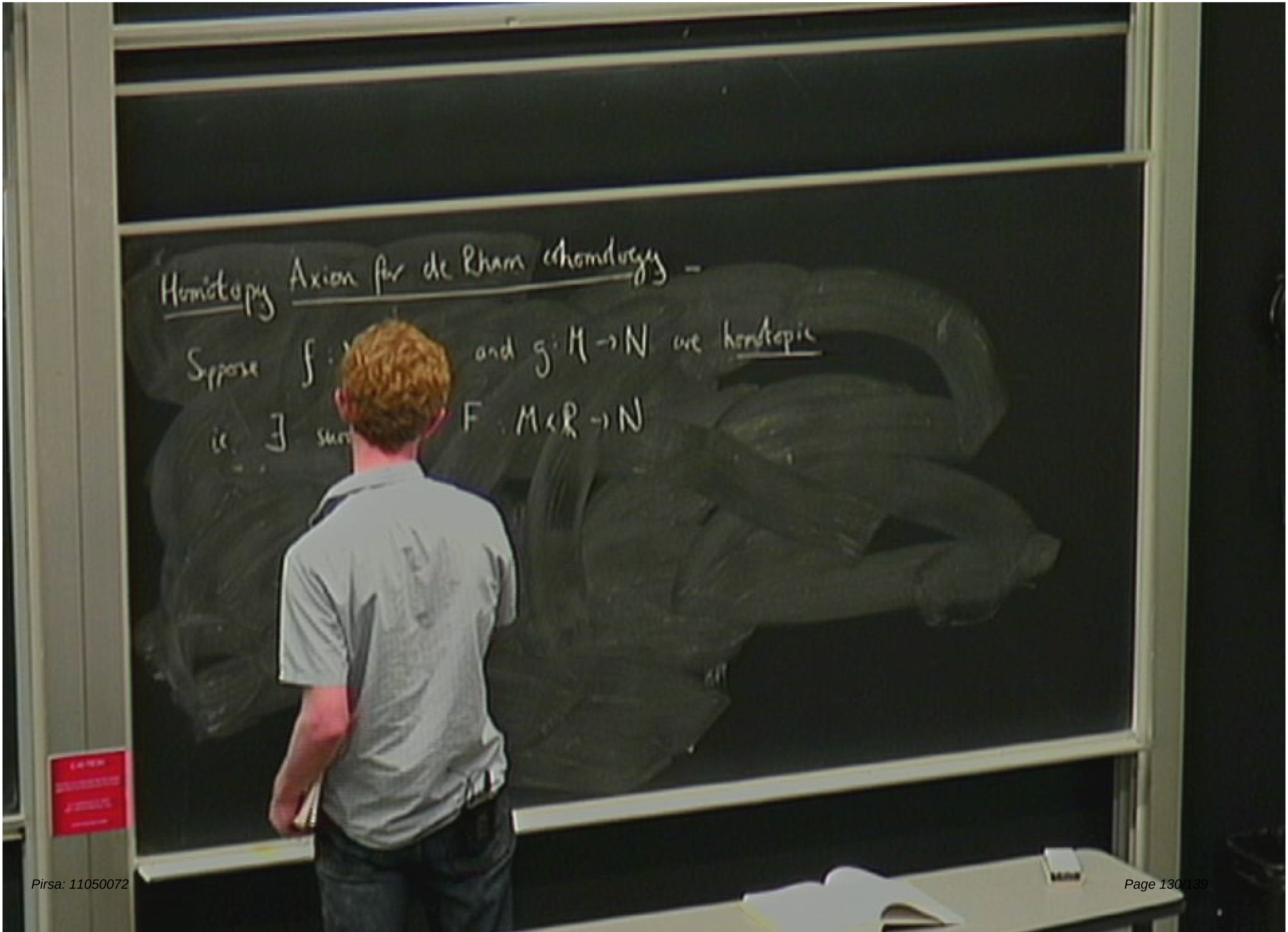


## Homotopy Axiom for de Rham cohomology –

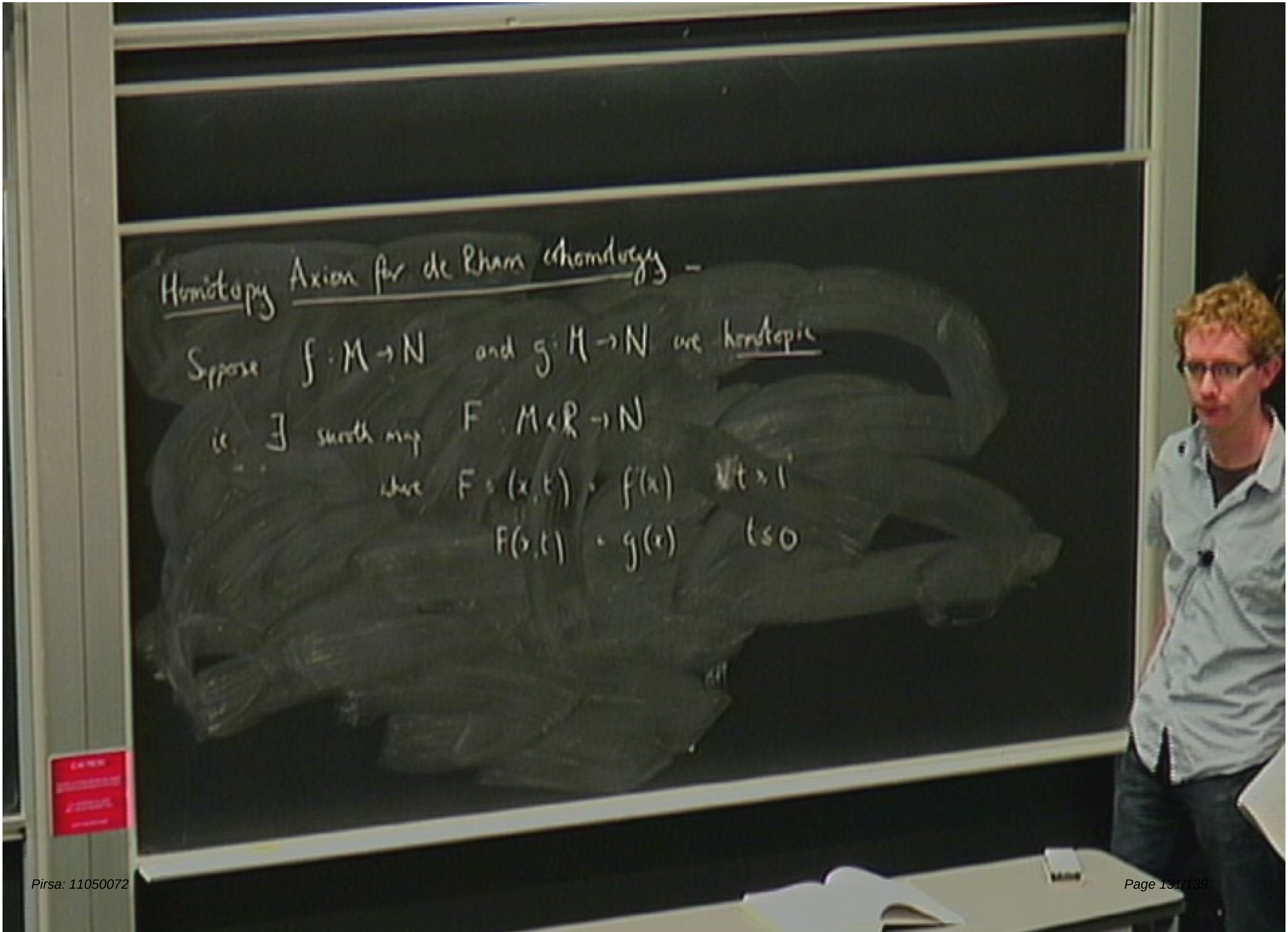
Suppose f :



Homotopy Axiom for de Rham cohomology –

Suppose $f :$ and $g : M \to N$ are homotopic

i.e. $\exists$ sm $F : M \times \mathbb{R} \to N$



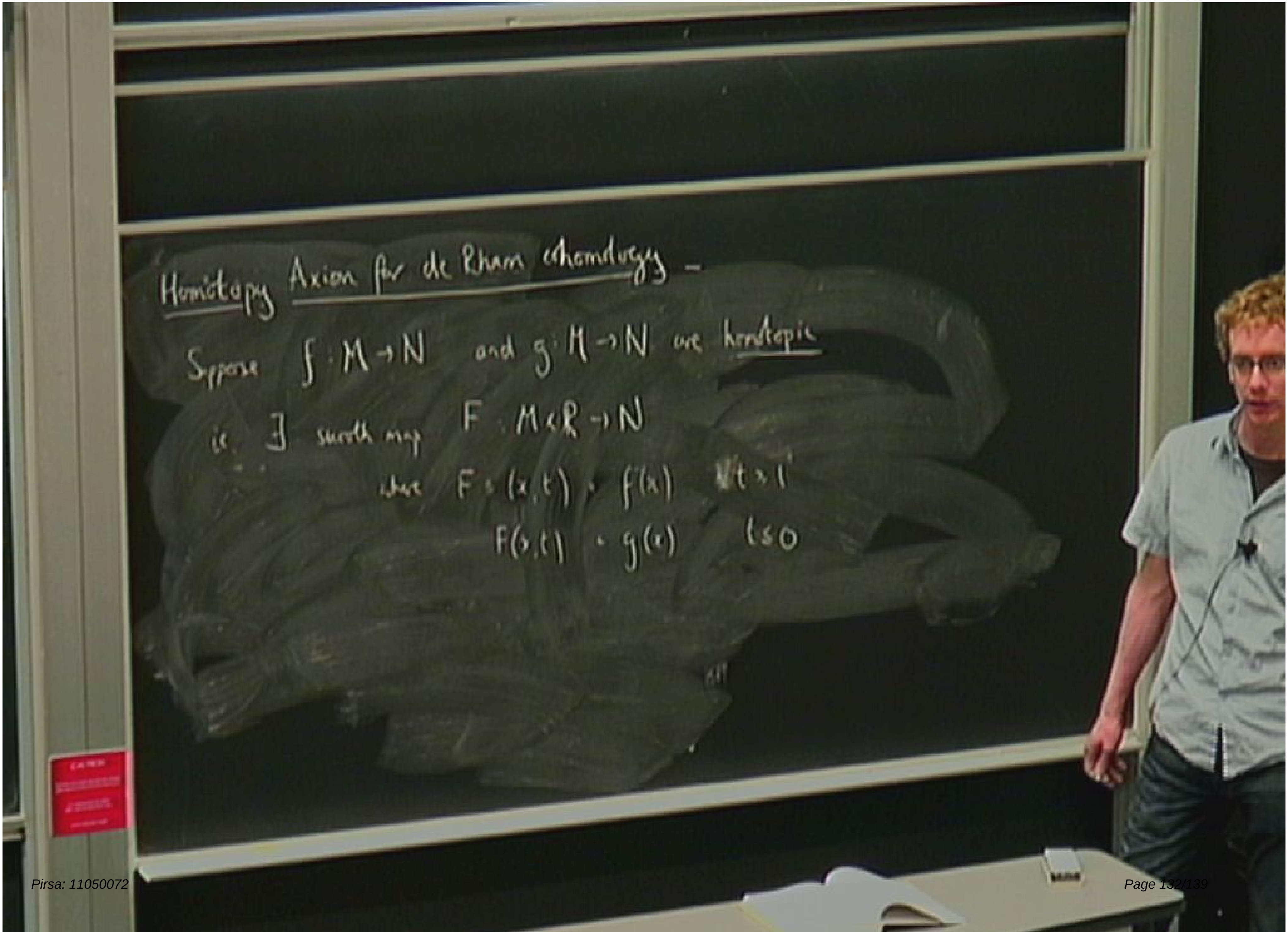
## Homotopy Axiom for de Rham cohomology –

Suppose $f : M \to N$ and $g : M \to N$ are homotopic

ie. $\exists$ smooth map $F : M \times \mathbb{R} \to N$

where $F(x,t) = f(x)$ $\forall t > 1$

$F(x,t) = g(x)$ $t \leq 0$



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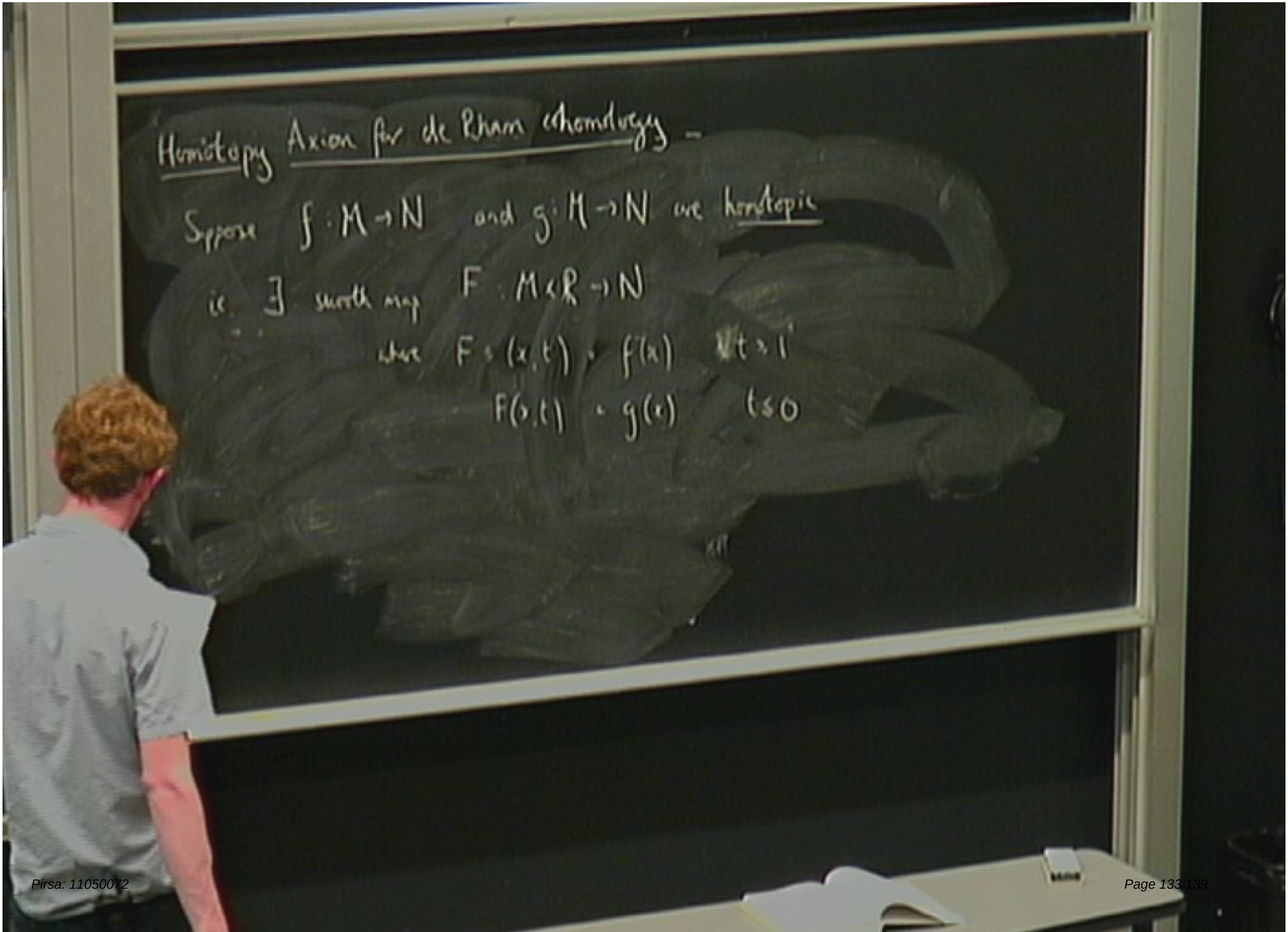

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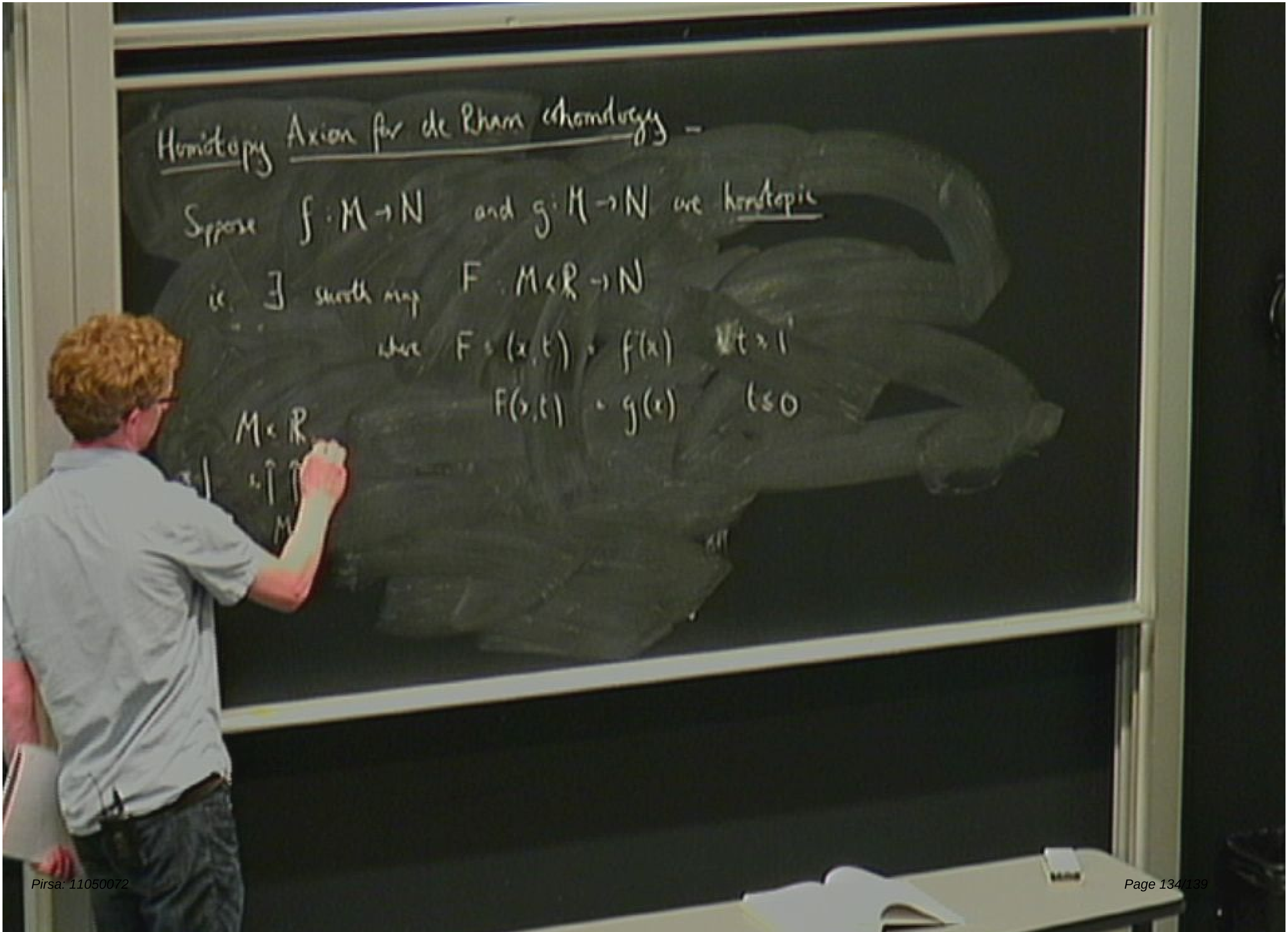

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$$\begin{tikzcd} M \times \mathbb{R} \arrow[d] \\ M \arrow[u, bend left] \arrow[u, bend right] \end{tikzcd}$$

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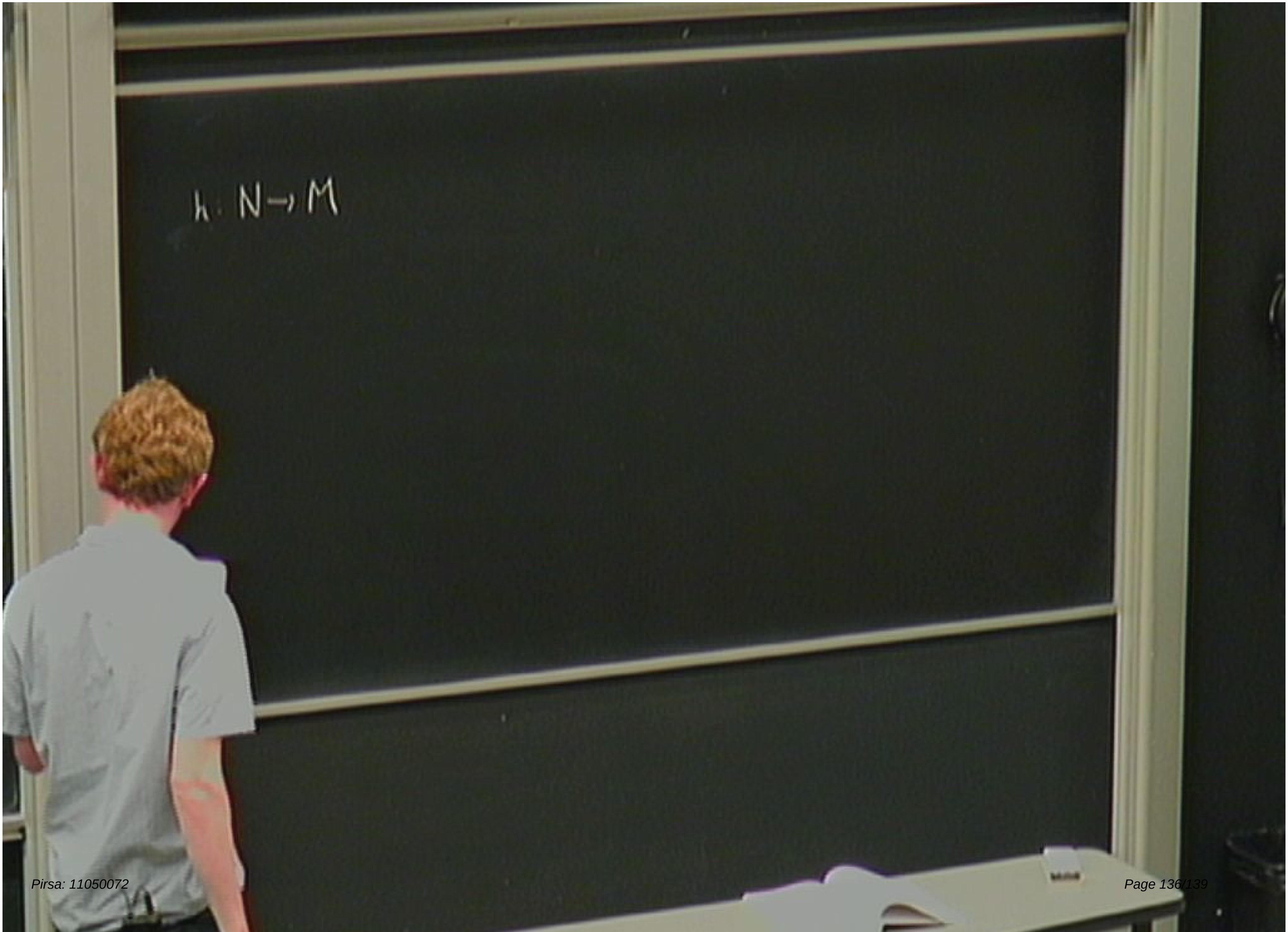
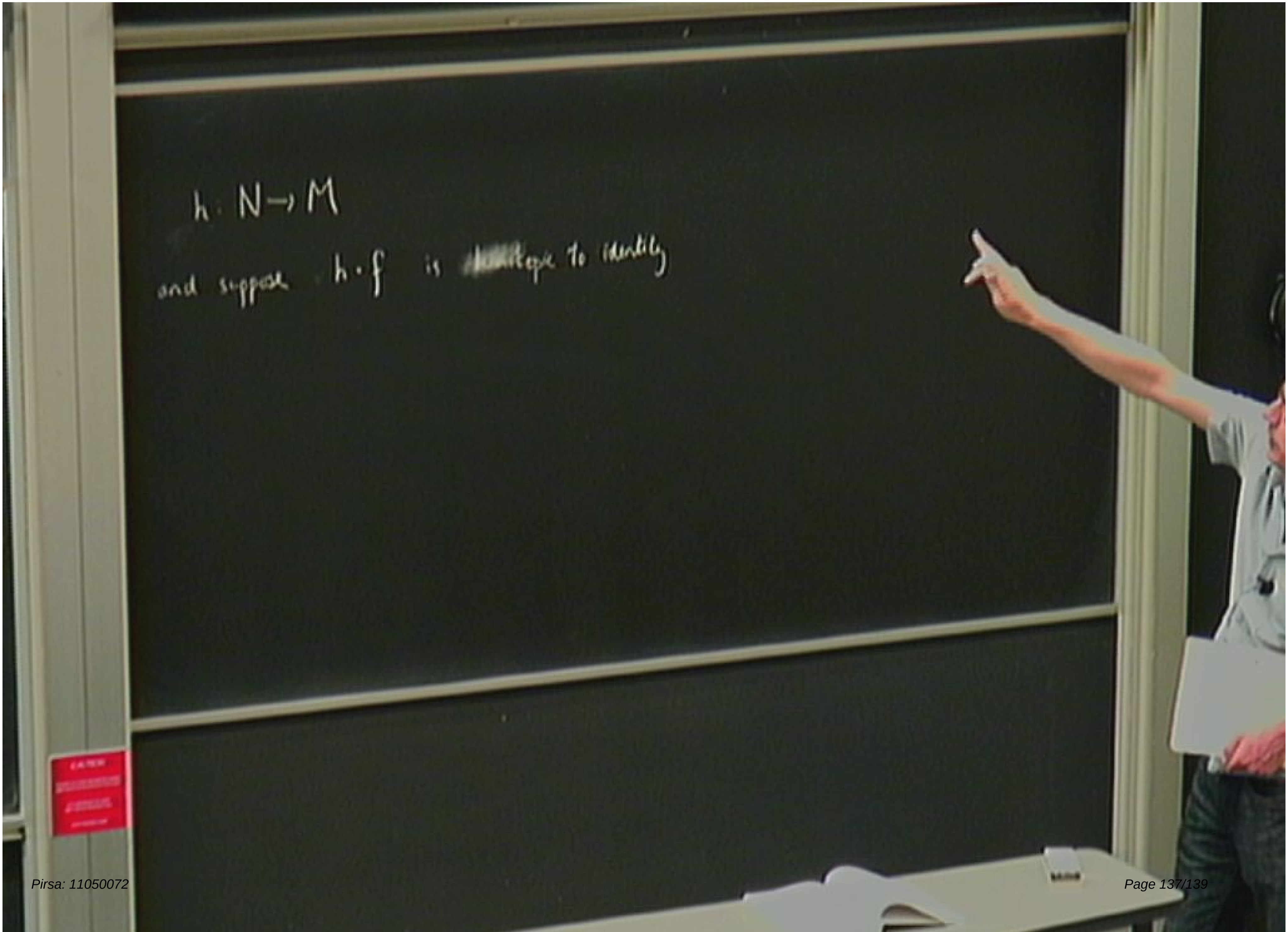
$$\begin{tikzcd} M \times \mathbb{R} \arrow[d, "\pi"'] \\ M \arrow[u, bend left, "s_0"] \arrow[u, bend right, "s_1"'] \end{tikzcd}$$

$f = F \circ s_1$ $g = F \circ s_0$



$h: N \to M$



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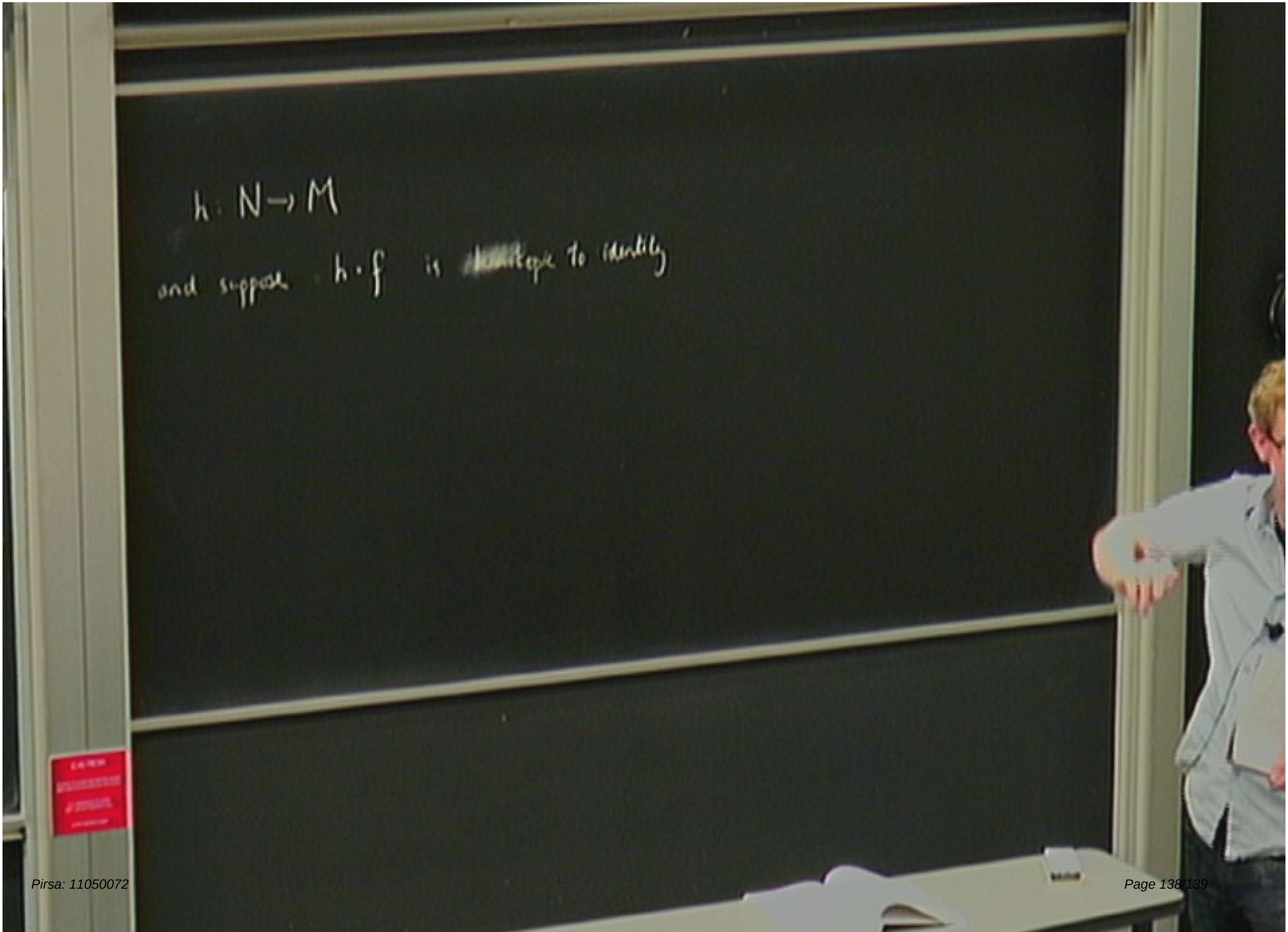
and suppose $h \circ f$ is homotopic to identity



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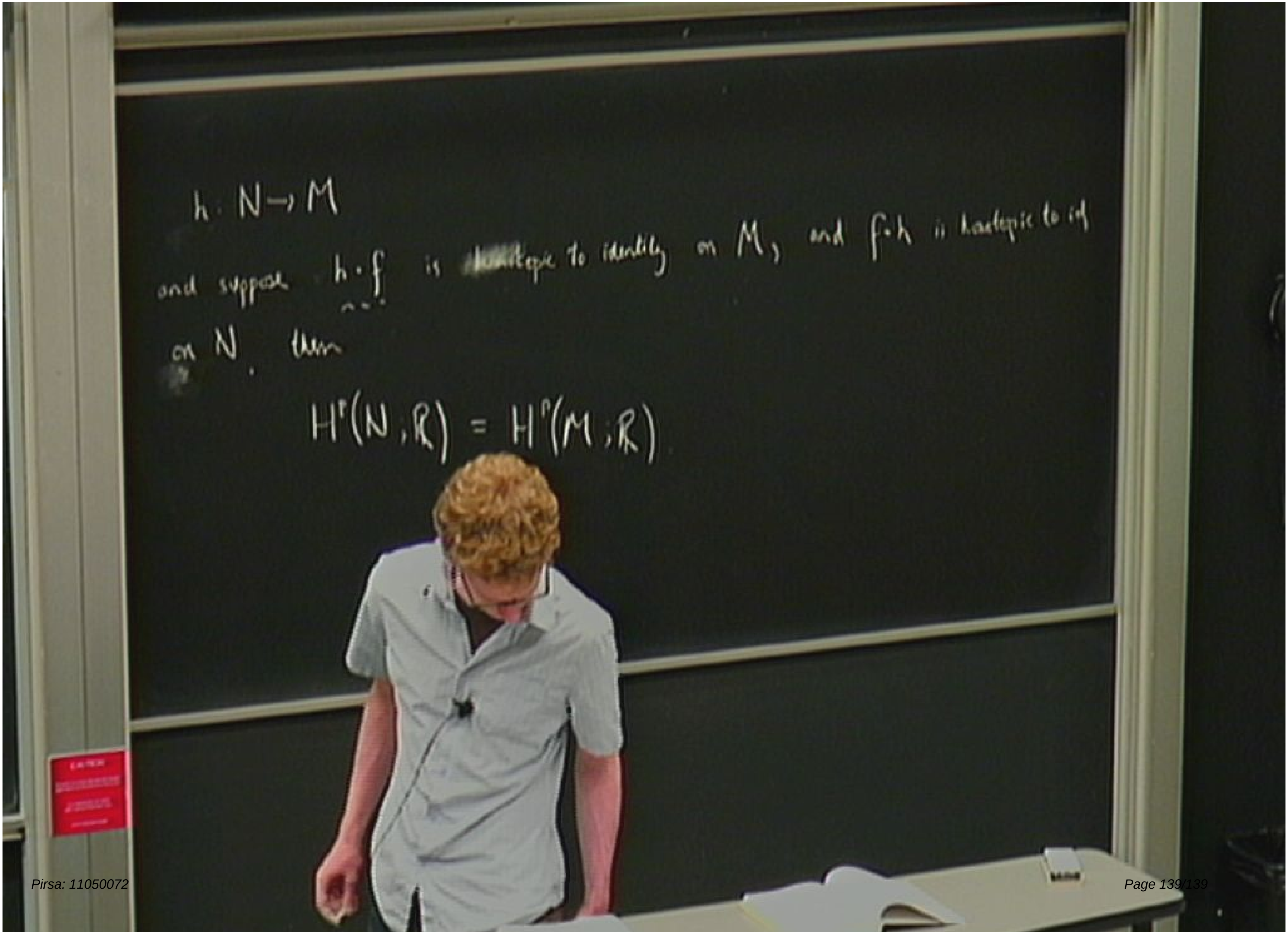

$h : N \to M$

and suppose $h \circ f$ is homotopic to identity on $M$, and $f \circ h$ is homotopic to id on $N$. then

$$H^p(N;\mathbb{R}) = H^p(M;\mathbb{R})$$