Title: Part 1: Monte-Carlo approach to the gauge/gravity duality

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Abstract: The gauge/gravity duality may give a nonperturbative formulation of superstring/M theory, and hence, if one can study the nonperturbative dynamics of the gauge theory, it would be useful to understand the nonperturbative aspects of superstring theory. Although researches in this direction were not successful for long time because of the notorious difficulties in lattice SUSY, however, recent progress made it possible; nonperturbative formulations free from the parameter fine-tuning were proposed, some of them are confirmed to work numerically, and nontrivial evidence for the validity of the gauge/gravity duality has been obtained. In these talks I review the state of the art in this field. I start with reviewing basics of the Monte-Carlo. Then I explain how to put supersymmetric theories on computer and show actual numerical results. 1st talk : basics of Monte-Carlo simulation. 2nd talk : 1d SYM (matrix quantum mechanics). 3rd talk : how to put 2d, 3d and 4d SYM on computer. In the talks I concentrate on basic ideas and omit technical details (e.g. algorithms to accelerate simulations). They will be explained after the talks if people are interested in. References: 1st talk : standard textbooks e.g. Heinz J. Rothe, "Lattice Gauge Theories: An Introduction", Third Edition, World Scientific. 2nd talk : 0706.1647 [hep-lat], 0707.4454 [hep-th], 0811.2081 [hep-th], 0811.3102 [hep-th], 0911.1623 [hep-th], 1012.2913 [hep-th]. 3rd talk : hep-lat/0302017, hep-lat/0311021, 1010.2948 [hep-lat] (2d SYM); hep-th/0211139 (3d SYM); 1004.5513 [hep-lat], 1009.0901 [hep-lat] (4d SYM)

Monte Carlo approach to the gauge/gravity duality

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May 2011 @ Perimeter Institute

Plan

- What can be done? (or what I want to do)
- What is "Monte Carlo"?

↑ today

- Difficulty of lattice SUSY and its cure
- Some numerical results

What can be done?

(My personal motivation)

Q.
What is quantum gravity?

Q.

What is quantum gravity?

A.

Superstring theory
may give the answer...

Q.

What is ‘superstring’?
Did anybody give
the nonperturbative formulation?

Q.

What is 'superstring'?

Did anybody give

the nonperturbative formulation?

A.

Not yet, but several proposals.

the most concrete :

AdS/CFT or

Gauge/Gravity correspondence

Maldacena, hep-th/9711200

"In principle, we can use this duality to give a definition of M/string theory on flat spacetime as (a region of) the large N limit of the field theories. Notice that this is a non-perturbative proposal for defining such theories, since the corresponding field theories can, in principle, be defined non-perturbatively."

Q.
Is it only of conceptual importance?

Q.
Is it only of conceptual importance?

A.
It is a practical tool for explicit calculations.

One of the goals of this talk is
to make you convince this point!

SYM	STRING
large-N, strong coupling	SUGRA
large-N, finite coupling	Classical string ($SUGRA+\alpha'$)
finite-N, finite coupling	Quantum string ($g_{string}>0$)

SYM$_{\text{difficult}}$	STRING
large-N, strong coupling	SUGRA easier
large-N, finite coupling	Classical string (SUGRA+α') more difficult
finite-N, finite coupling	Quantum string (g_{string}>0) very difficult

SYM$_{\text{difficult}}$ STRING

SYM		STRING
large-N, strong coupling	⟵	SUGRA easier
large-N, finite coupling	⟶	Classical string ($SUGRA+\alpha'$) more difficult
finite-N, finite coupling	⟶	Quantum string ($g_{string}>0$) very difficult

The opposite direction of the dictionary can be useful !

Q.

How can we study
the strongly coupled theory?

Q.

How can we study
the strongly coupled theory?

A.

Monte-Carlo simulation!

Example: 1d SYM(D0 brane)

$$S = N \int_0^\beta dt Tr\{\frac{1}{2}(D_t X_i)^2 - \frac{1}{4}[X_i, X_j]^2 + \frac{1}{2}\bar{\psi}\Gamma^0 D_t \psi - \frac{i}{2}\bar{\psi}\Gamma^i [X_i, \psi]\}$$

SYM thermodynamics

=

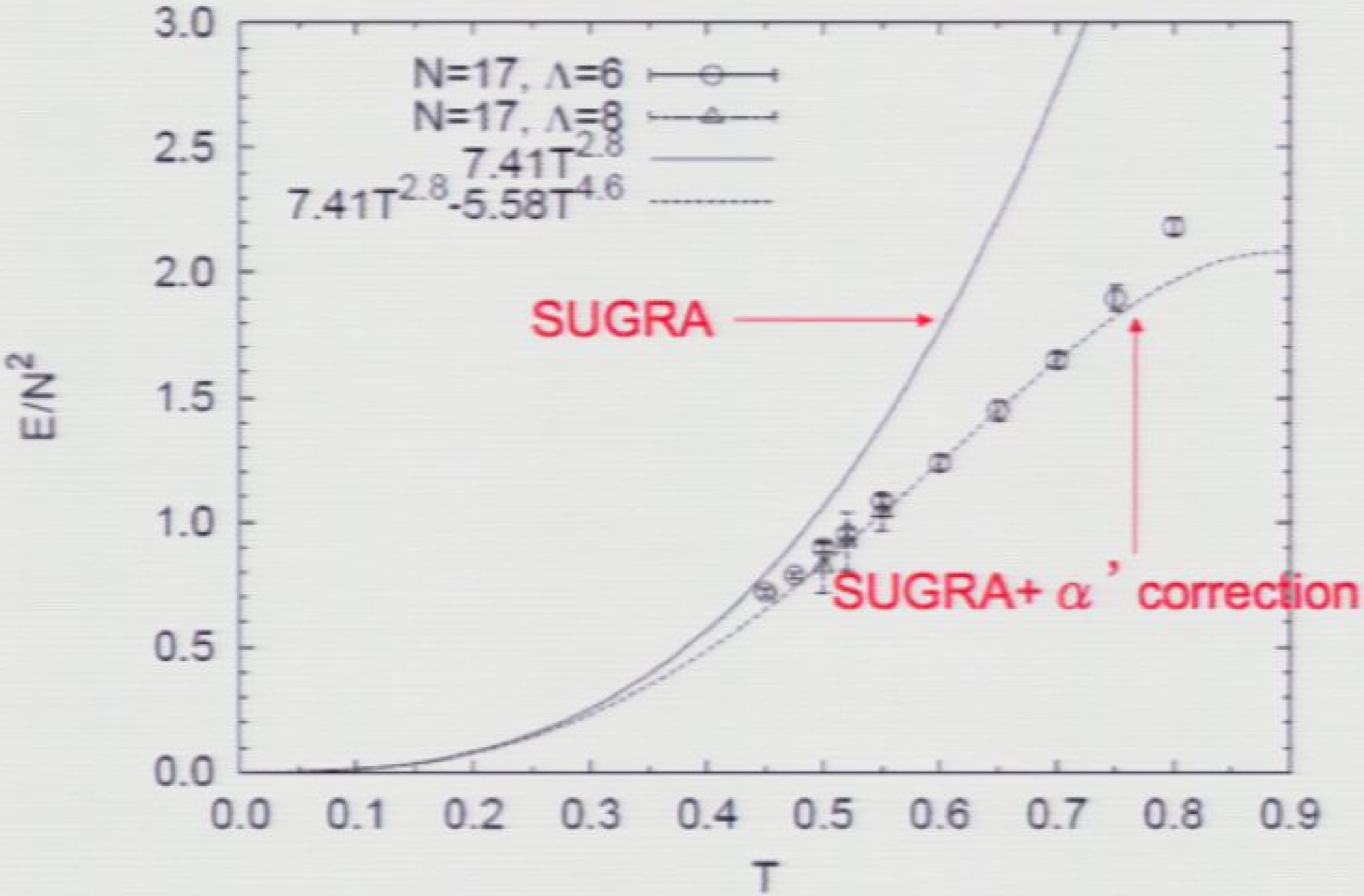
Black hole (black 0-brane)

thermodynamics

The dictionary

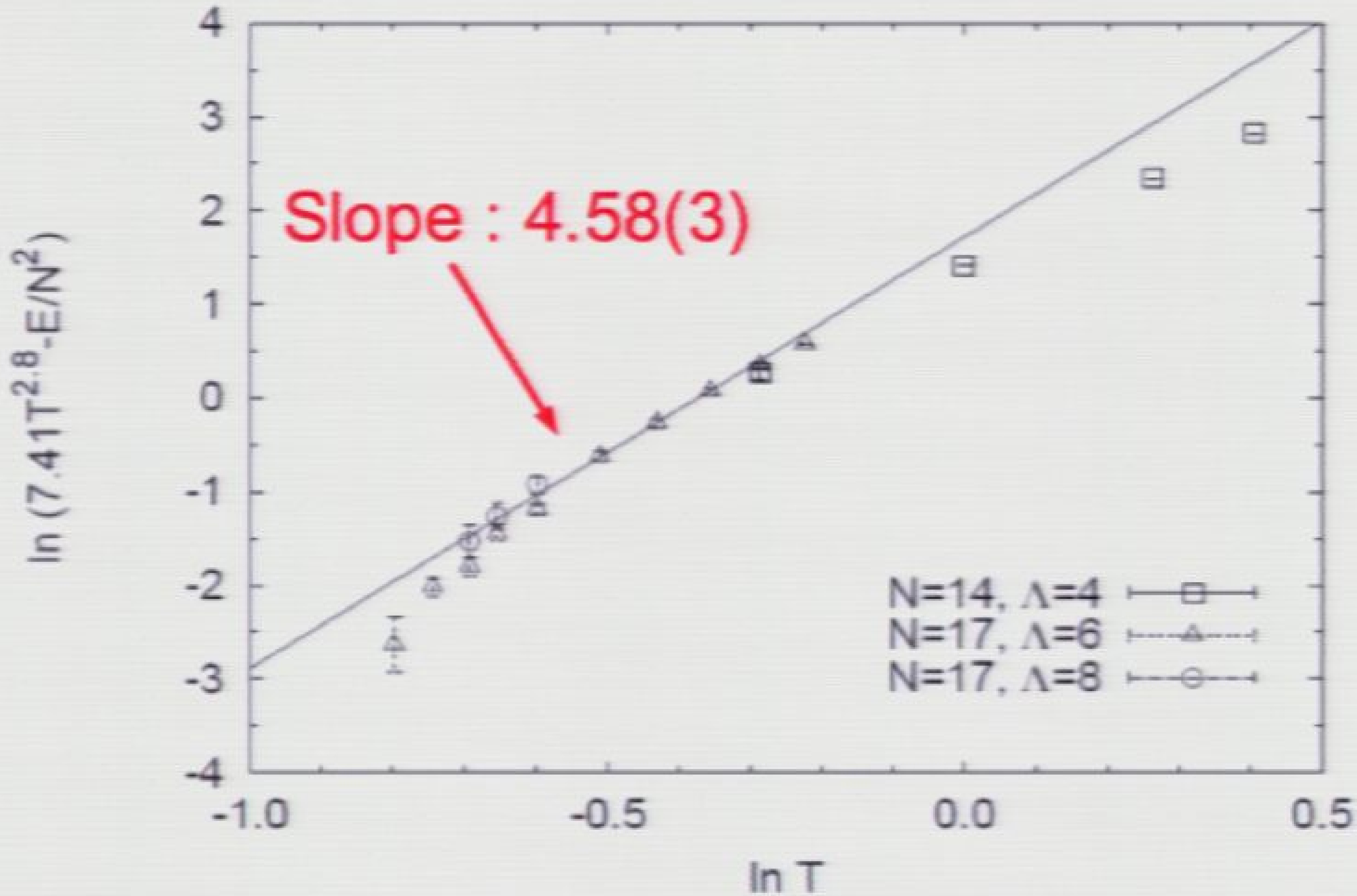
Gravity	SYM
ADM mass	Energy density
minimal surface	Wilson/Polyakov loop
mass of field excitation	'generalized' conformal dimension

simulation result (1d)



Anagnostopoulos-M.H.-Nishimura-Takeuchi, 2007

M.H.-Hyakutake-Nishimura -Takeuchi, 2008



M.H.-Hyakutake-Nishimura -Takeuchi, 2008

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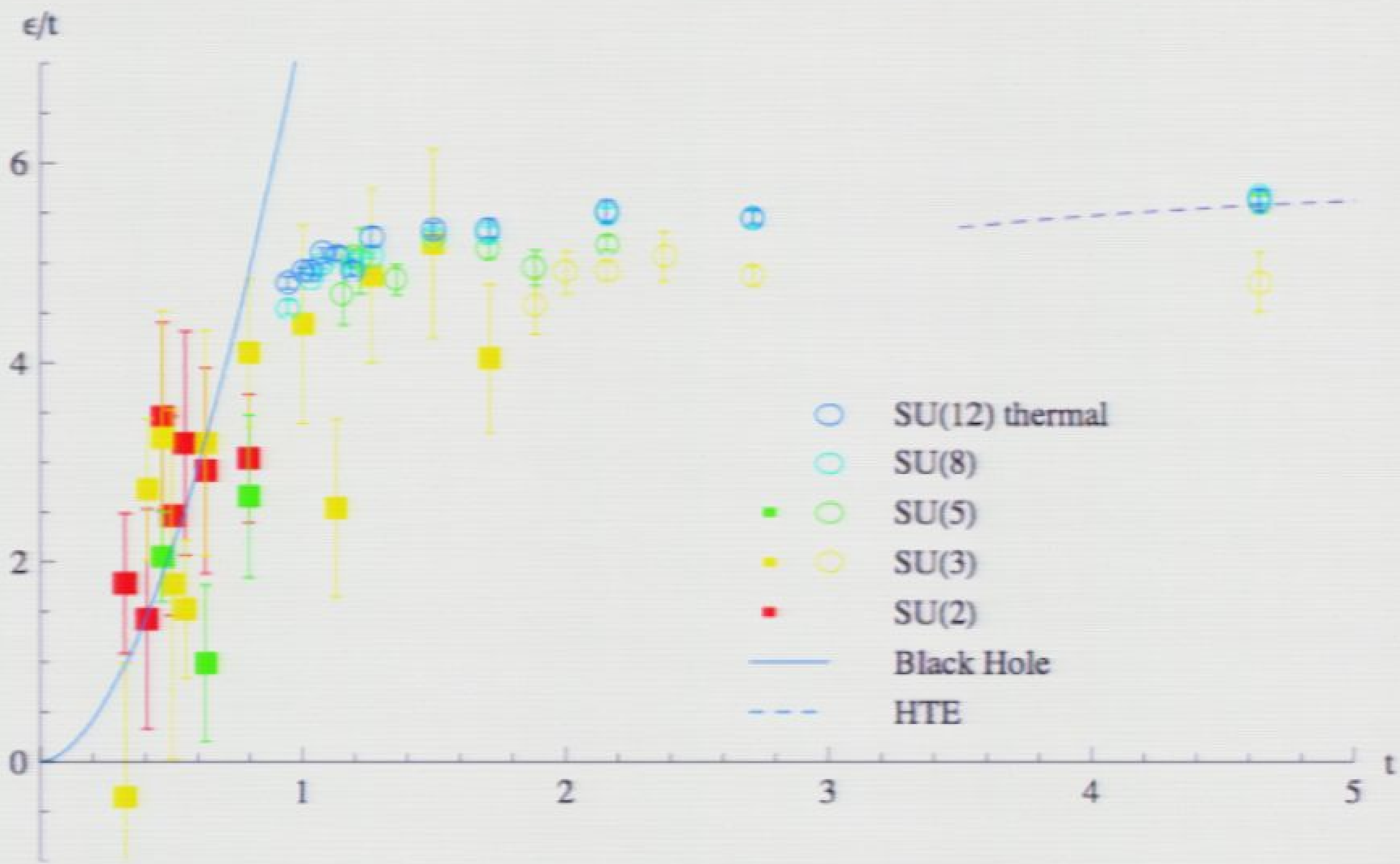
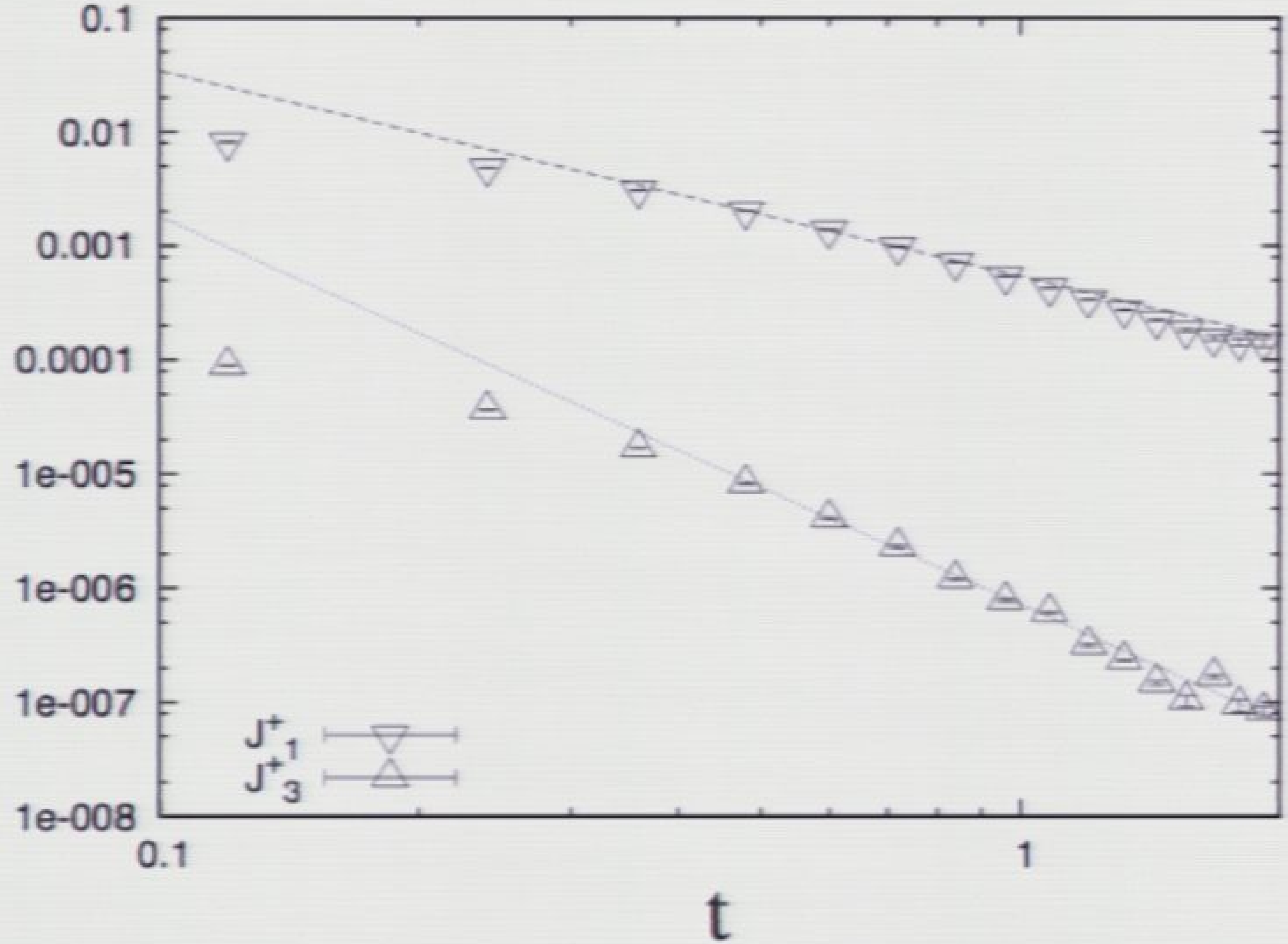


Catterall-Wiseman, 2008



two-point functions, SU(2)

M.H.-Nishimuea-Sekino-Yoneya, to appear soon


Basics of Monte Carlo

Keywords

- Markov Chain
- detailed balance condition
- importance sampling
- Metropolis algorithm
- 'Dynamical fermion' vs 'Quench'
- Sign problem

John von Neumann
1903-1957

The principle of Monte-Carlo

- Consider field theory on Euclidean spacetime with the action $S[\phi]$.
- Generate field configurations with probability $e^{-S[\phi]}$. Then,

$$\langle \mathcal{O} \rangle = \frac{\int [d\phi] \mathcal{O}[\phi] e^{-S[\phi]}}{\int [d\phi] e^{-S[\phi]}} \simeq \frac{1}{n} \sum_{i=1}^{n} \mathcal{O}[\phi_i]$$

- Such a set of configurations can be generated as long as $e^{-S[\phi]} > 0$ (not 'probability' otherwise...)

Algorithm

- generate a chain of field configurations with the transition probability $P[C \to C']$

$$C_0 \to C_1 \to C_2 \to \cdots$$

- 'Markov chain' : transition probability from C_k to C_{k+1} does not depend on $C_0, \ldots, C_{k-1}$

$w_k[C]$: probability of obtaining C at k-th step

$$w_{k+1}[C] = \sum_{C'} w_k[C'] P[C' \to C]$$

Choose $P[C \to C']$ so that

$$\lim_{k \to \infty} w_k[C] \propto e^{-S[C]}$$

Algorithm (cont'd)

$$w_{k+1}[C] = \sum_{C'} w_k[C'] P[C' \to C]$$

↓ $k \to \infty$

$$e^{-S[C]} = \sum_{C'} e^{-S[C']} P[C' \to C]$$ necessary

↑ sum over C'

a stronger condition

$$e^{-S[C]} P[C \to C'] = e^{-S[C']} P[C' \to C]$$

<u>*the detailed balance condition*</u>

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Algorithm (cont'd)

- *Ergodicity* : for any C and C', there is a finite transition probability with finite steps. (As long as C' exists with nonzero probability, of course.)

Theorem

If a Markov chain satisfies the detailed balance condition and the ergodicity,

$$\lim_{k\to\infty} w_k[C] \propto e^{-S[C]}$$

Proof? After the talk, on a black board :)

Algorithm (cont'd)

'algorithm' = choice of $P[C \to C']$

- Metropolis
  simplest & the basis of all others
- Hybrid Monte Carlo (HMC) ← useful for fermions
- Rational Hybrid Monte Carlo (RHMC) ← useful for fermions

.....etc etc...

Nicholas Constantine Metropolis

Metropolis algorithm

(Metropolis-Rosenbluth-et al, 1953)

- Consider the Gaussian integral,

$$S[x] = \frac{x^2}{2}, \qquad Z = \int_{-\infty}^{\infty} dx e^{-S[x]}.$$

(1) vary the 'field' x randomely:

$$x \to x + \Delta x, \qquad -0.5 < \Delta x < 0.5$$

(2) accept the new 'configuration' with a probability

$\min\{1, e^{-\Delta S}\}$ where $\Delta S = S[x + \Delta x] - S[x]$

'Metropolis test'

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'Metropolis test'

Metropolis (cont'd)

- Ergodicity is satisfied.
- Detailed balance is also OK:

$$P[x \to x + \Delta x] = \begin{cases} 0 & (|\Delta x| \geq 0.5) \\ 1 & (|\Delta x| < 0.5 \text{ and } \Delta S < 0) \\ e^{-\Delta S} & (|\Delta x| < 0.5 \text{ and } \Delta S > 0) \end{cases}$$

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Numerical example (Gaussian integral)

Metropolis (cont'd)

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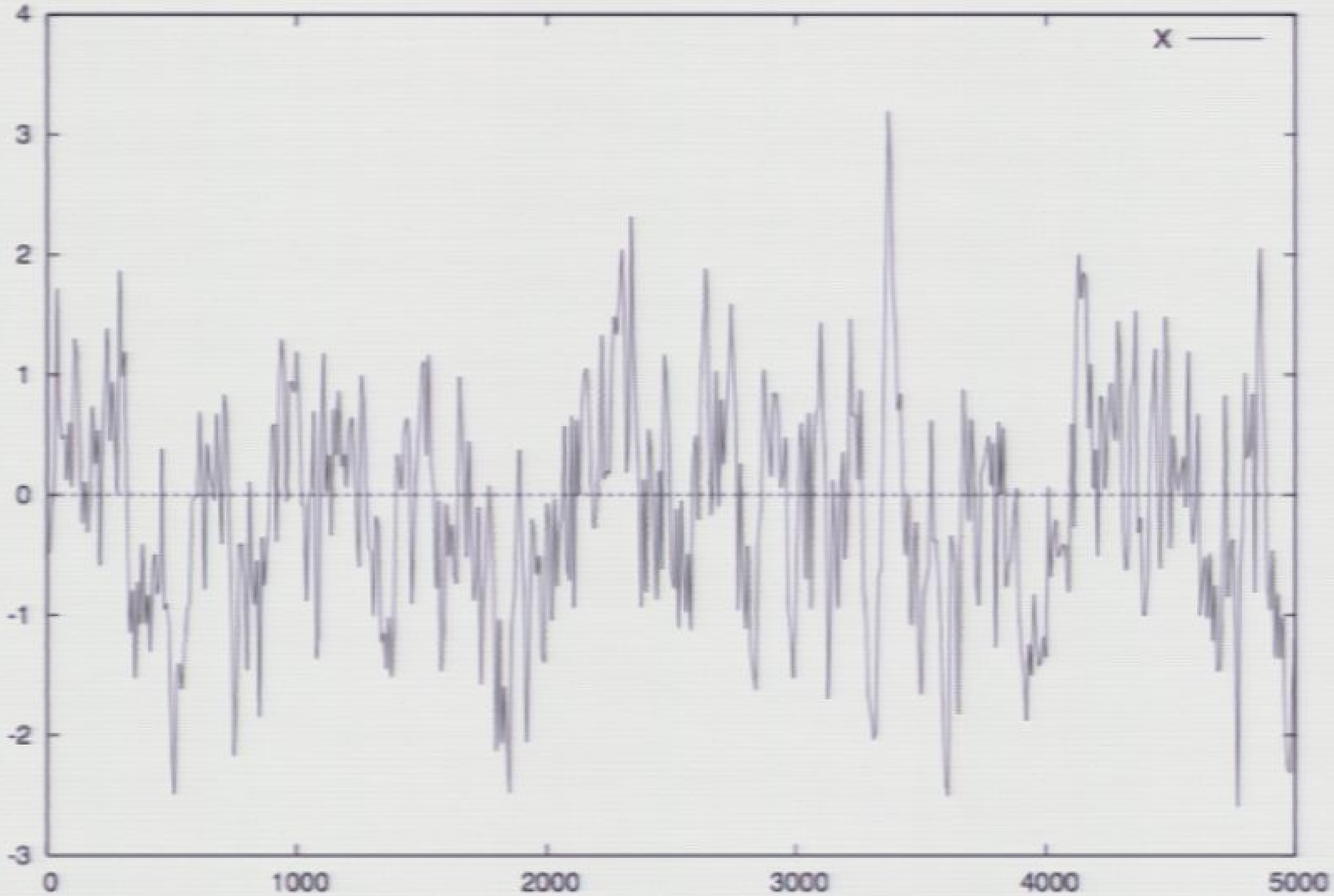
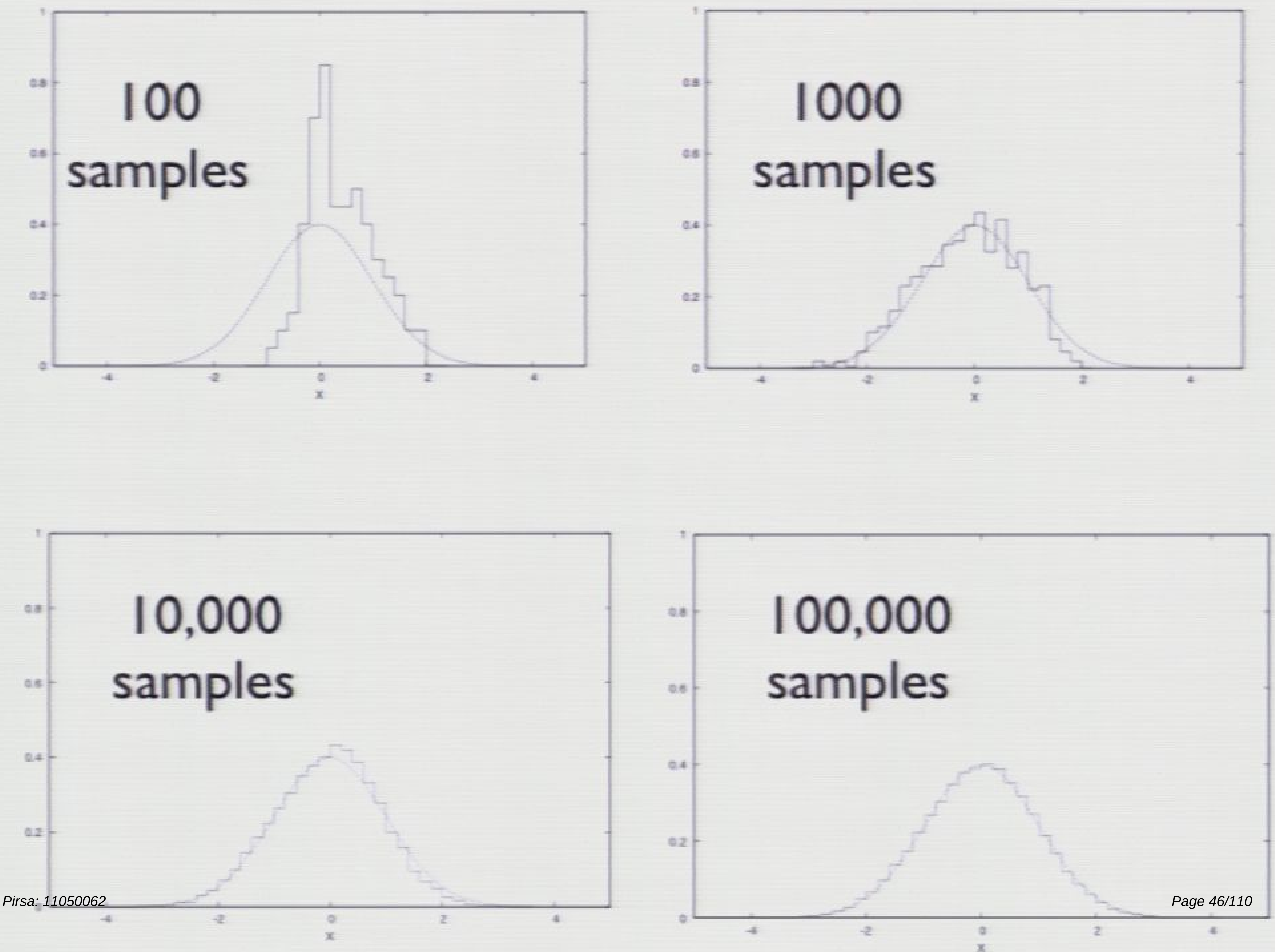
Numerical example (Gaussian integral)

Initial condition : x=0

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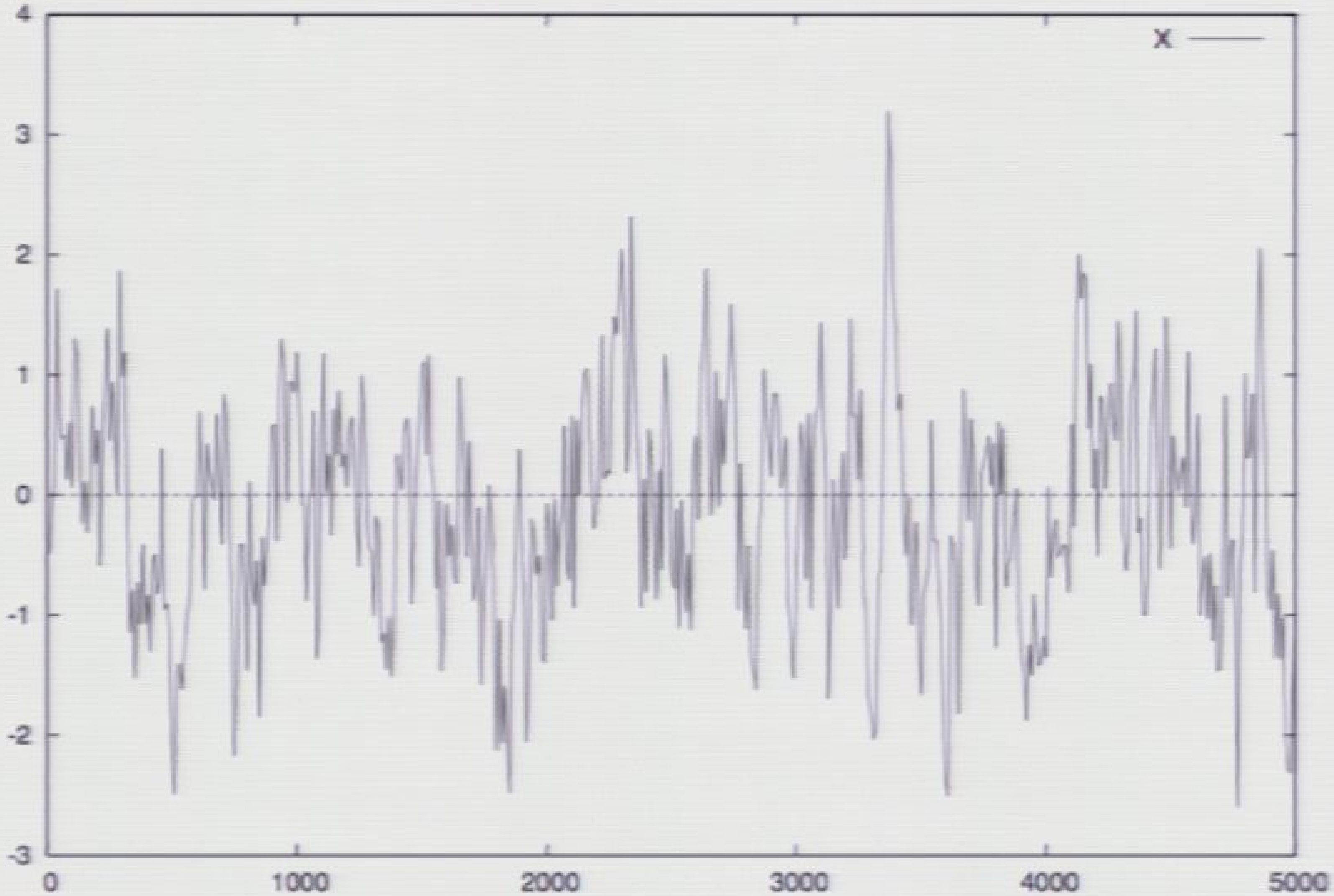
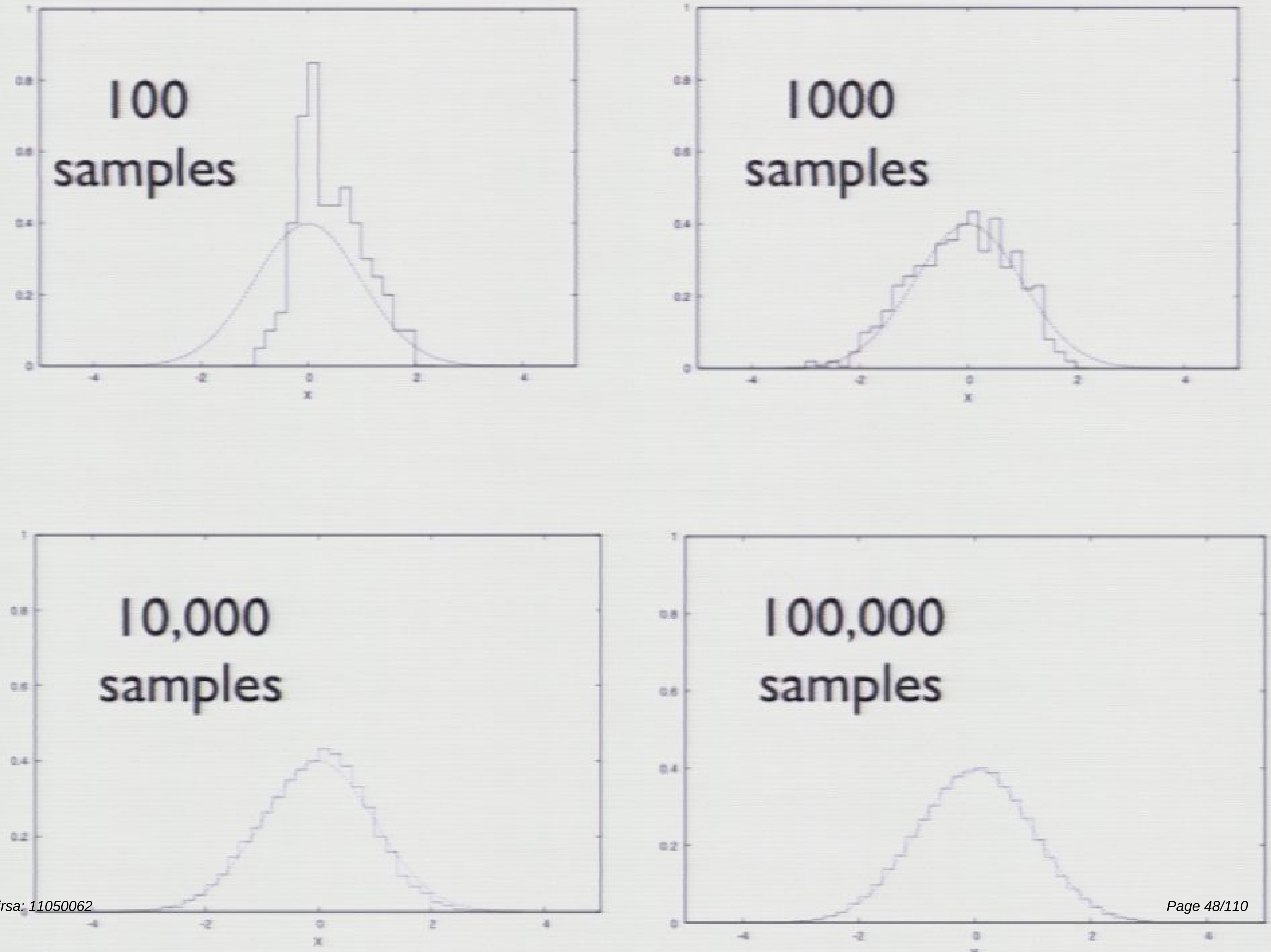
Initial condition : x=0

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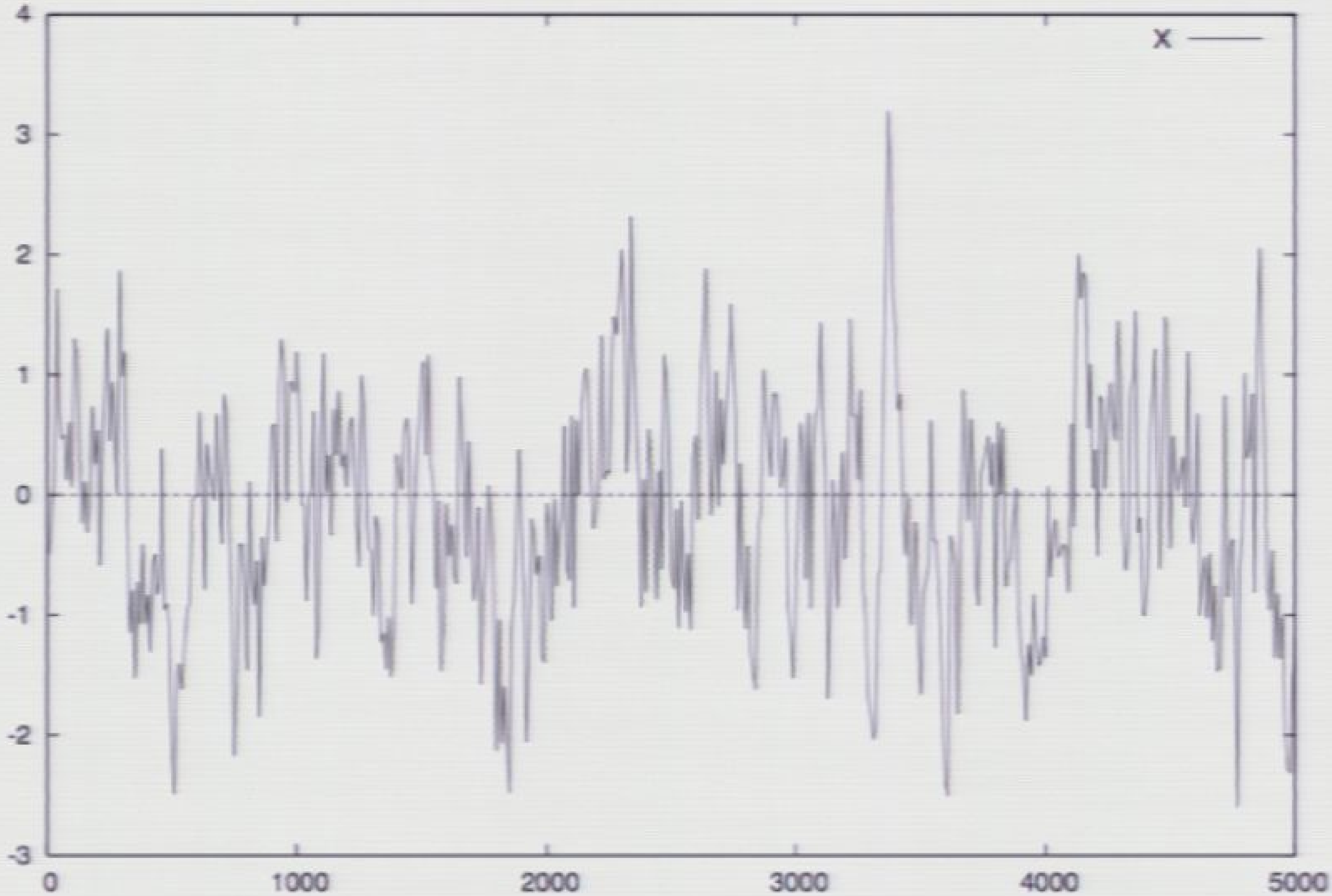

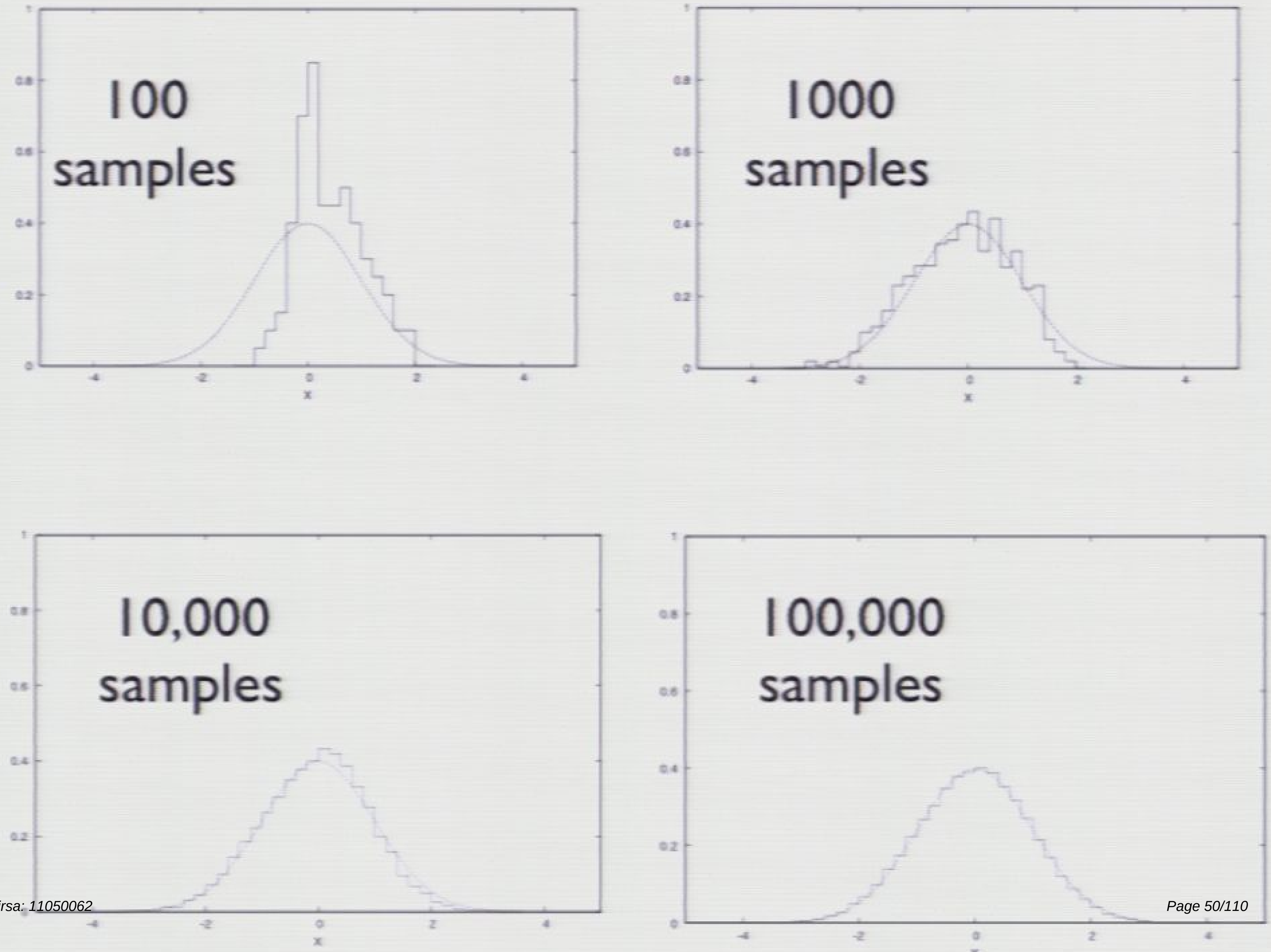
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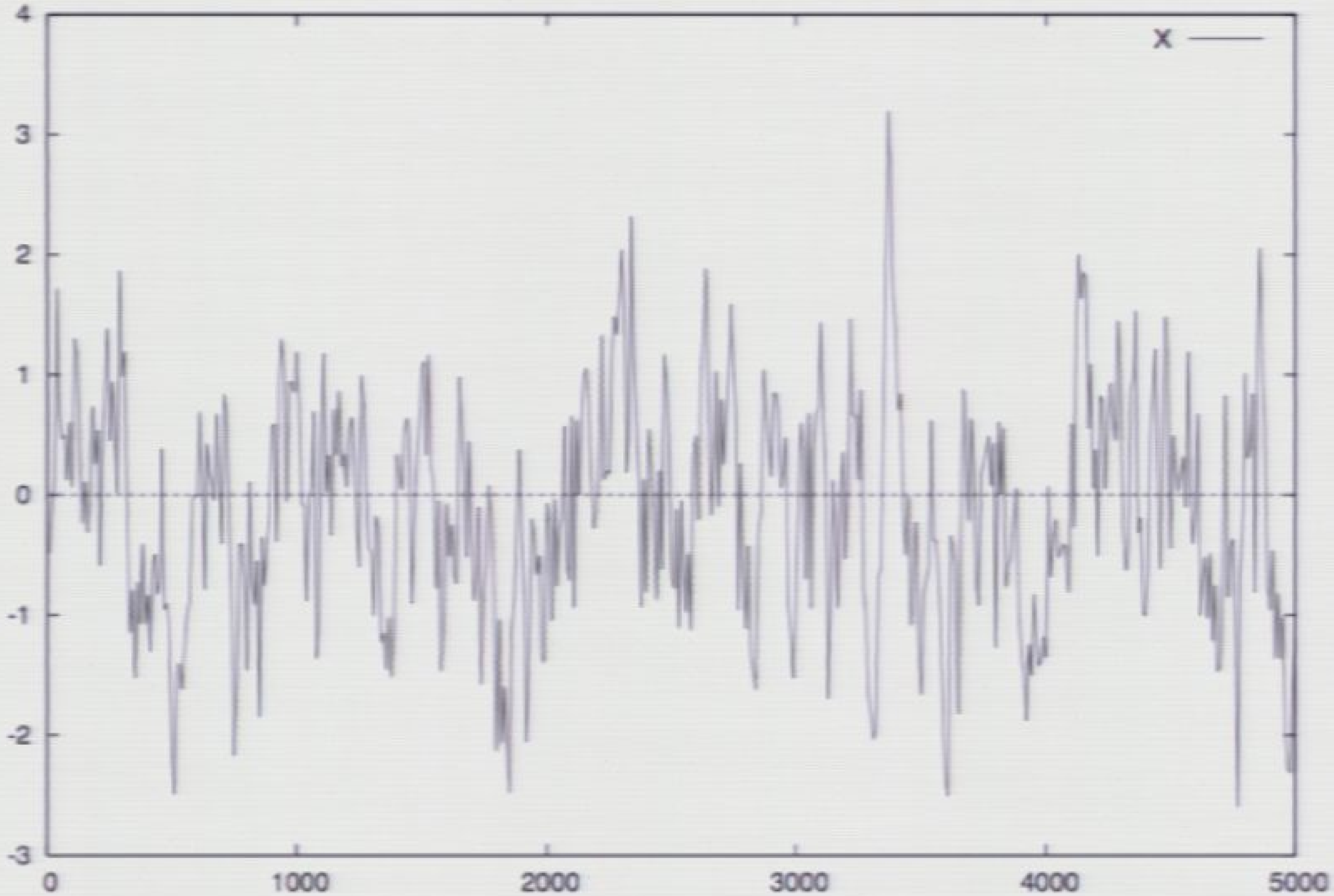

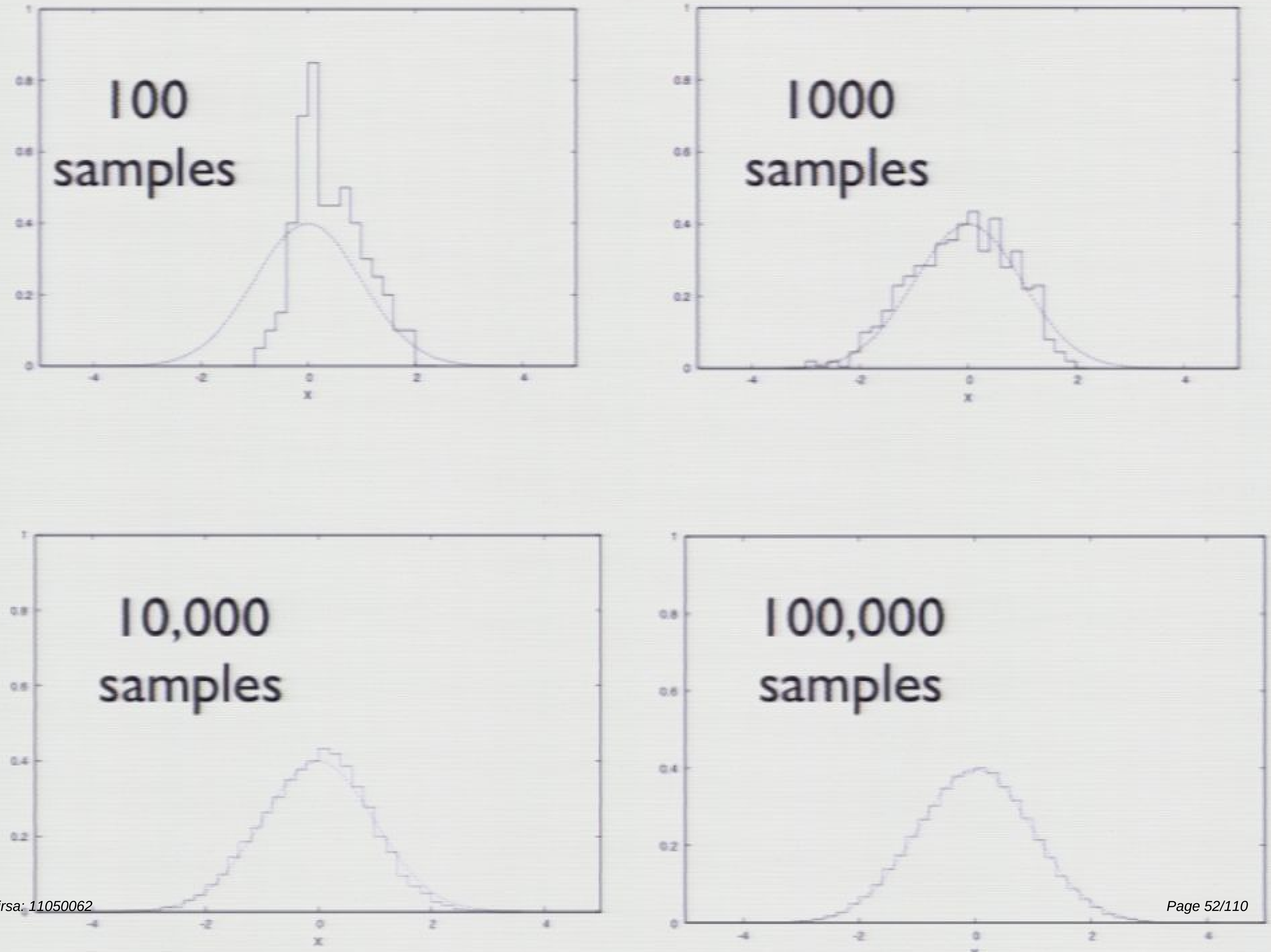
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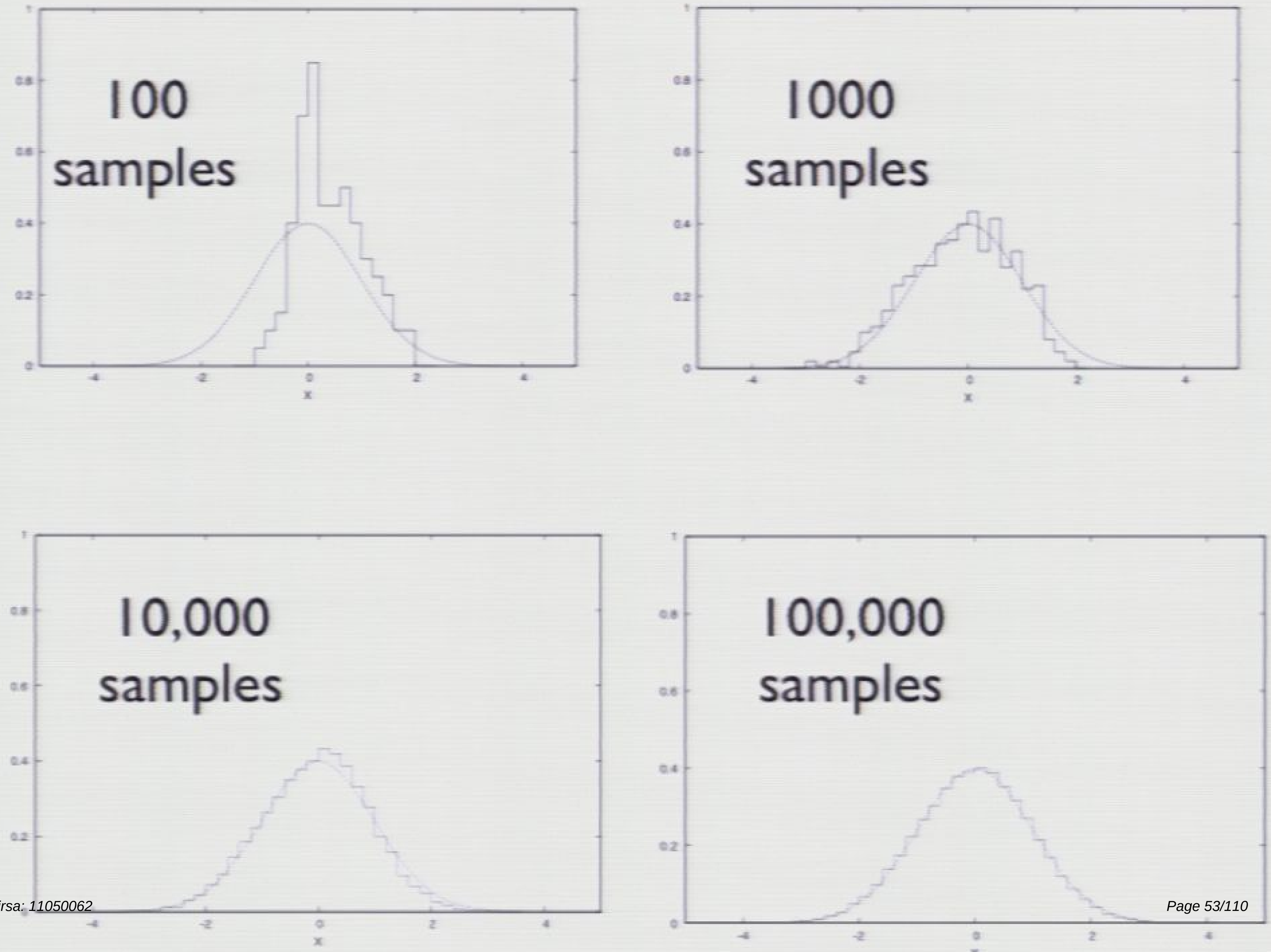
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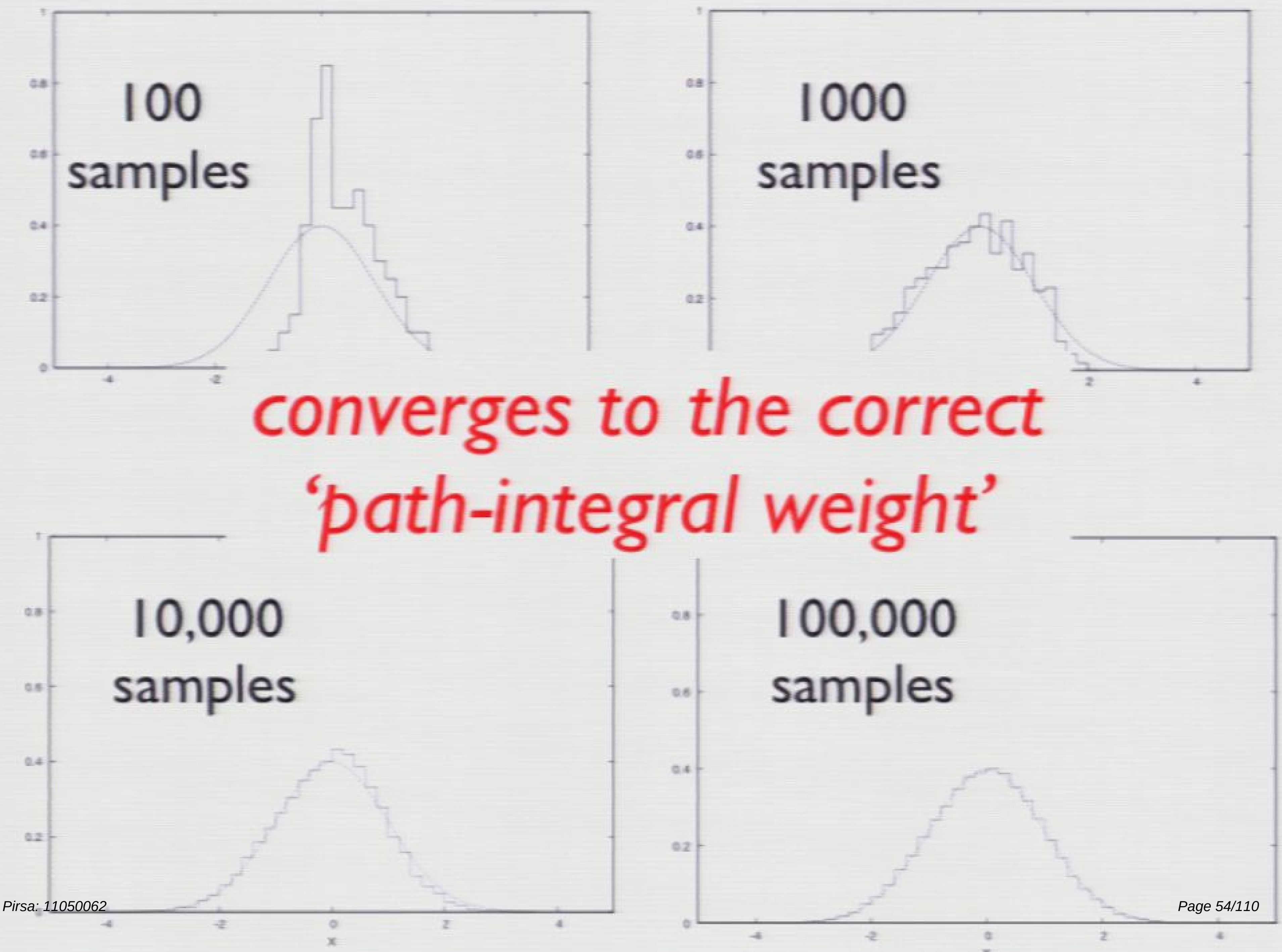
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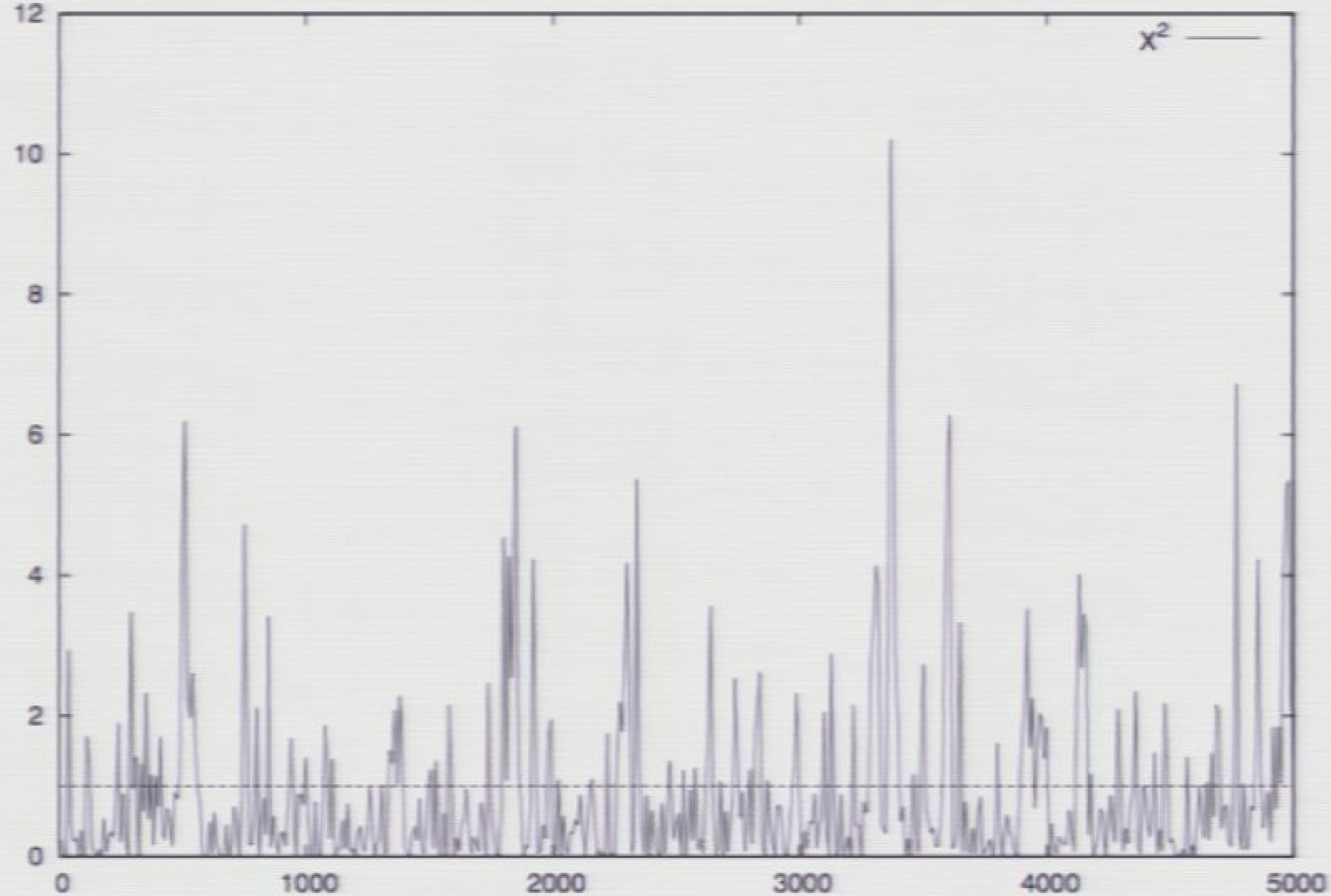
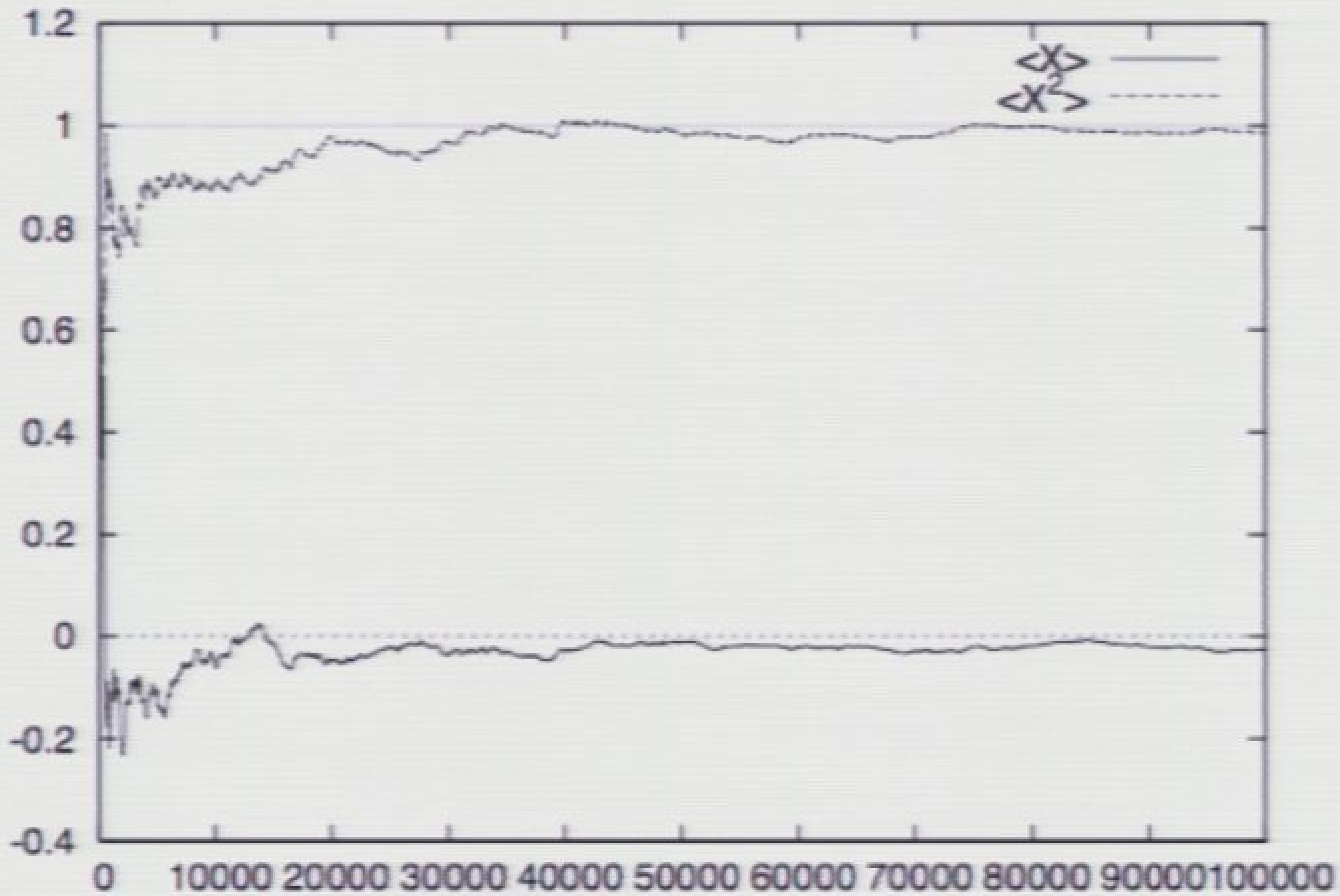
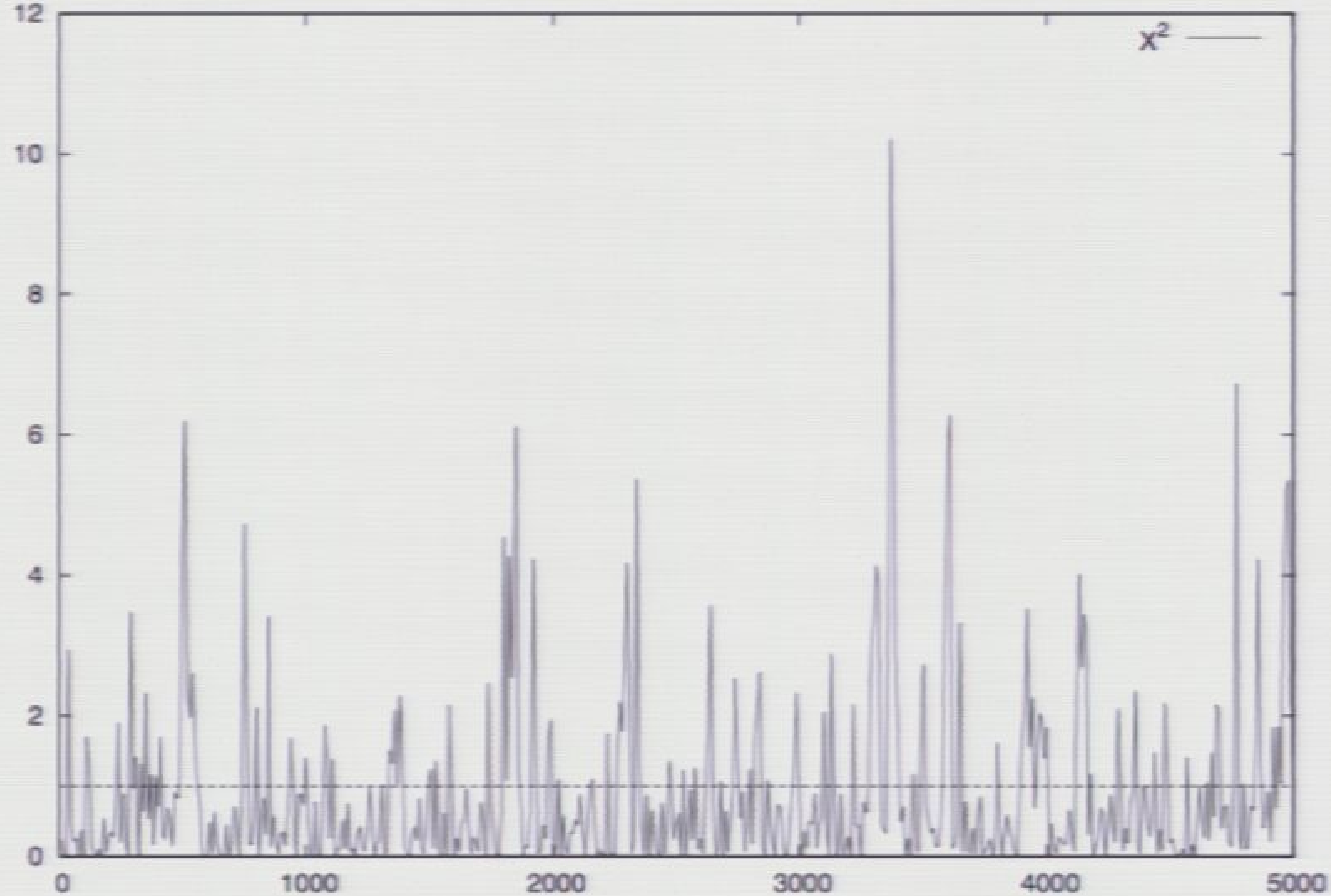


history of x^2



<u>Note</u> In typical YM simulations, with better algorithm, reasonable results can be obtained from 100 - 1000 configurations, if the theory does not suffer from the 'sign problem'.



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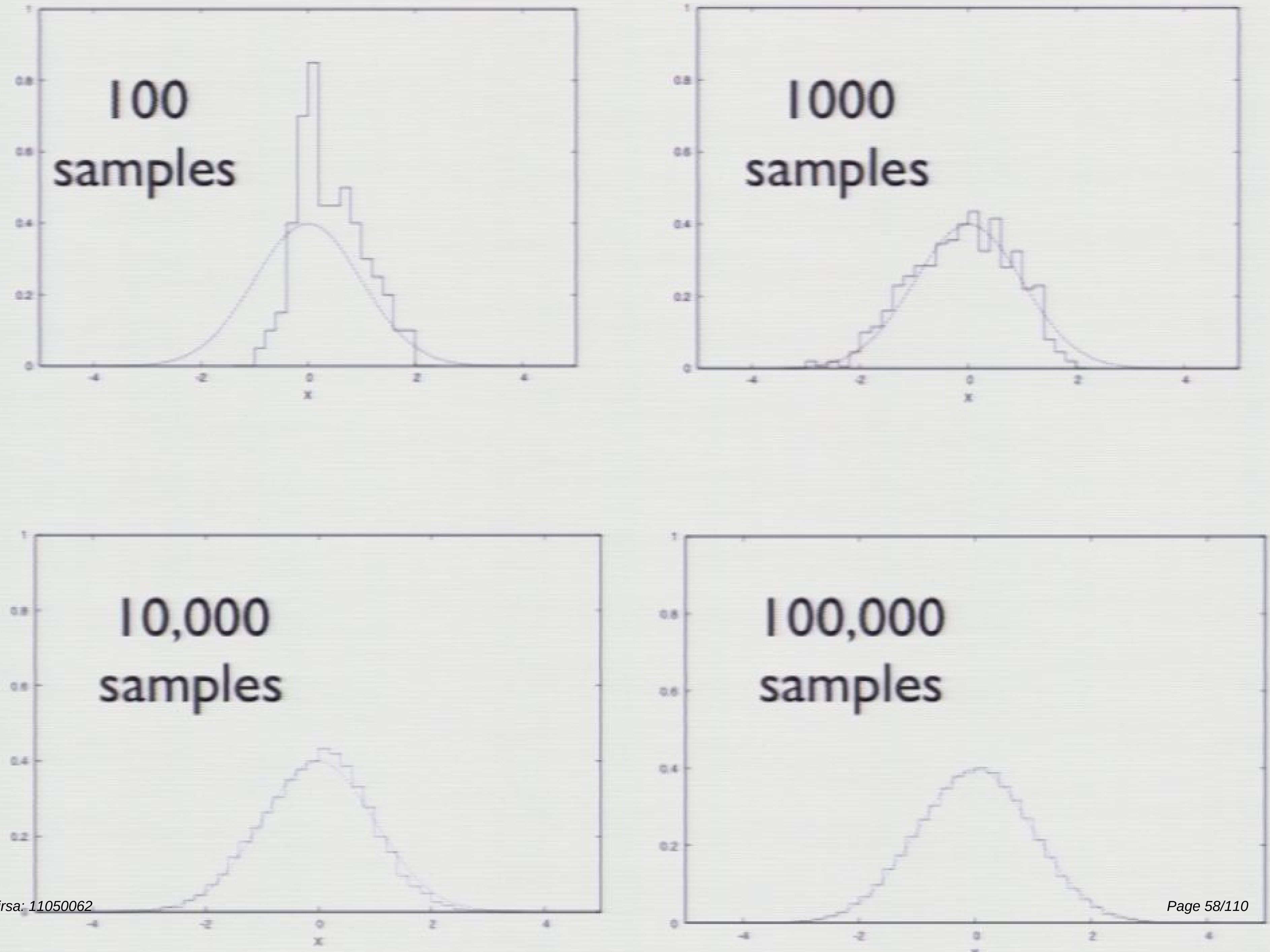
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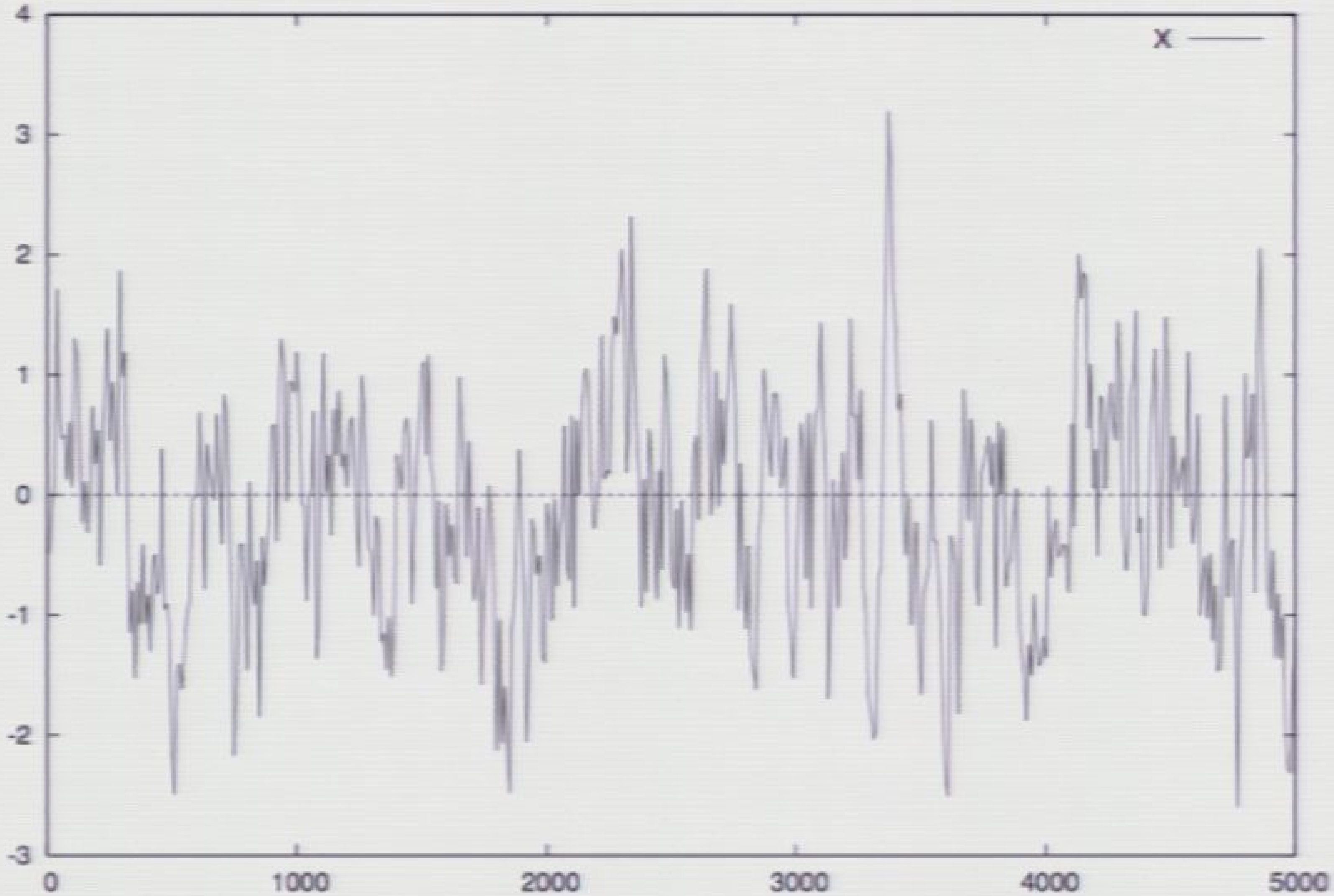

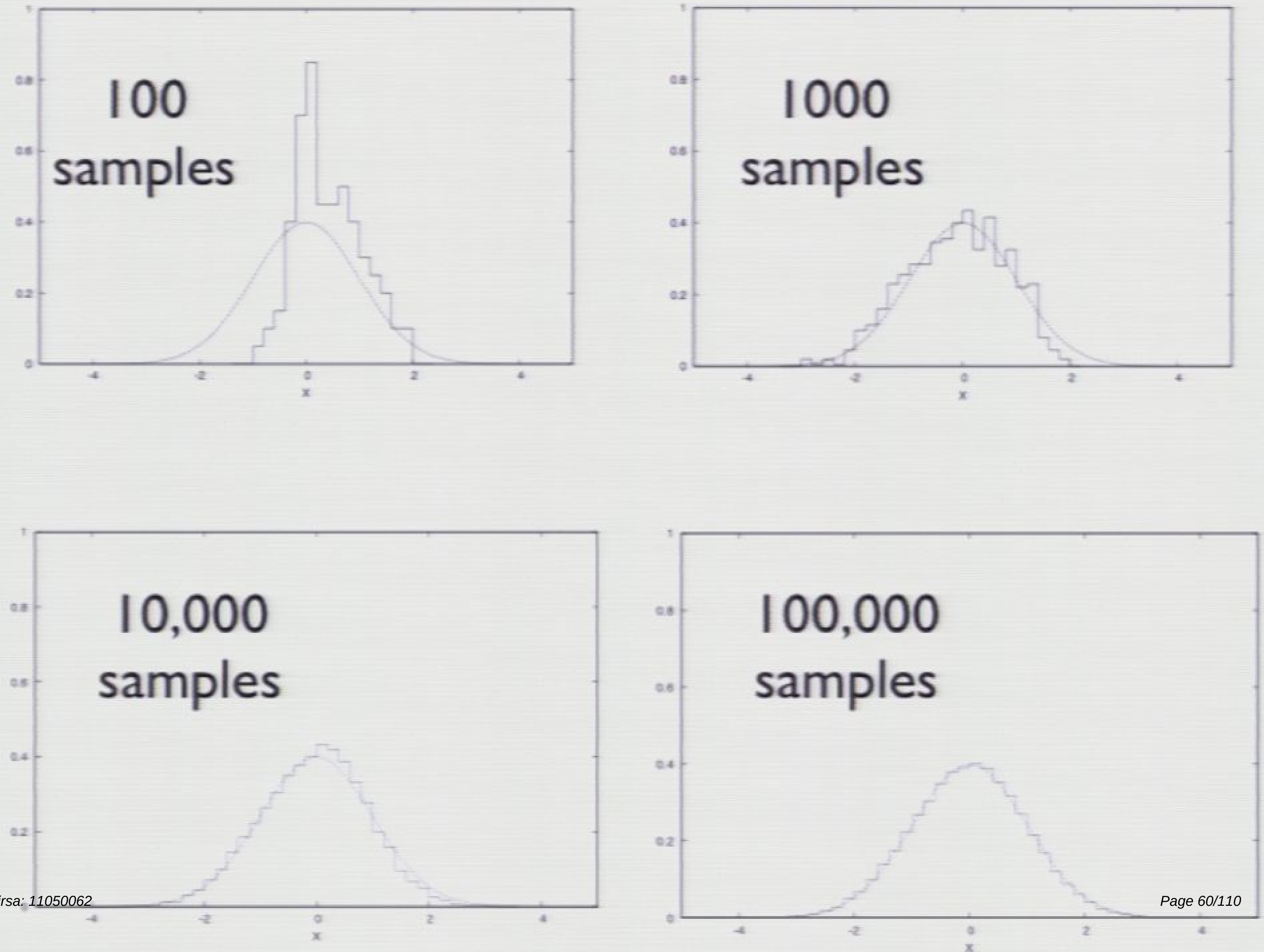
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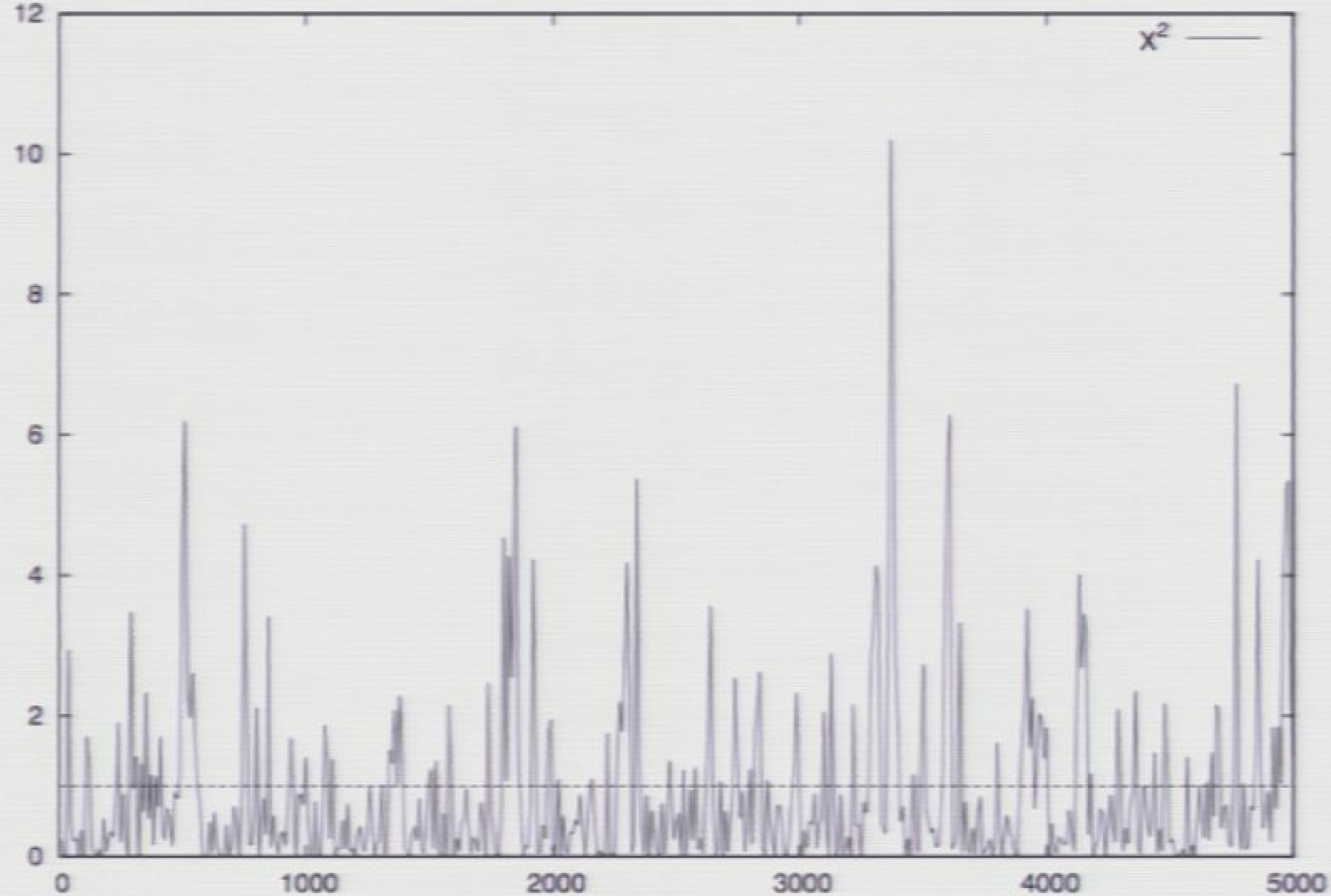

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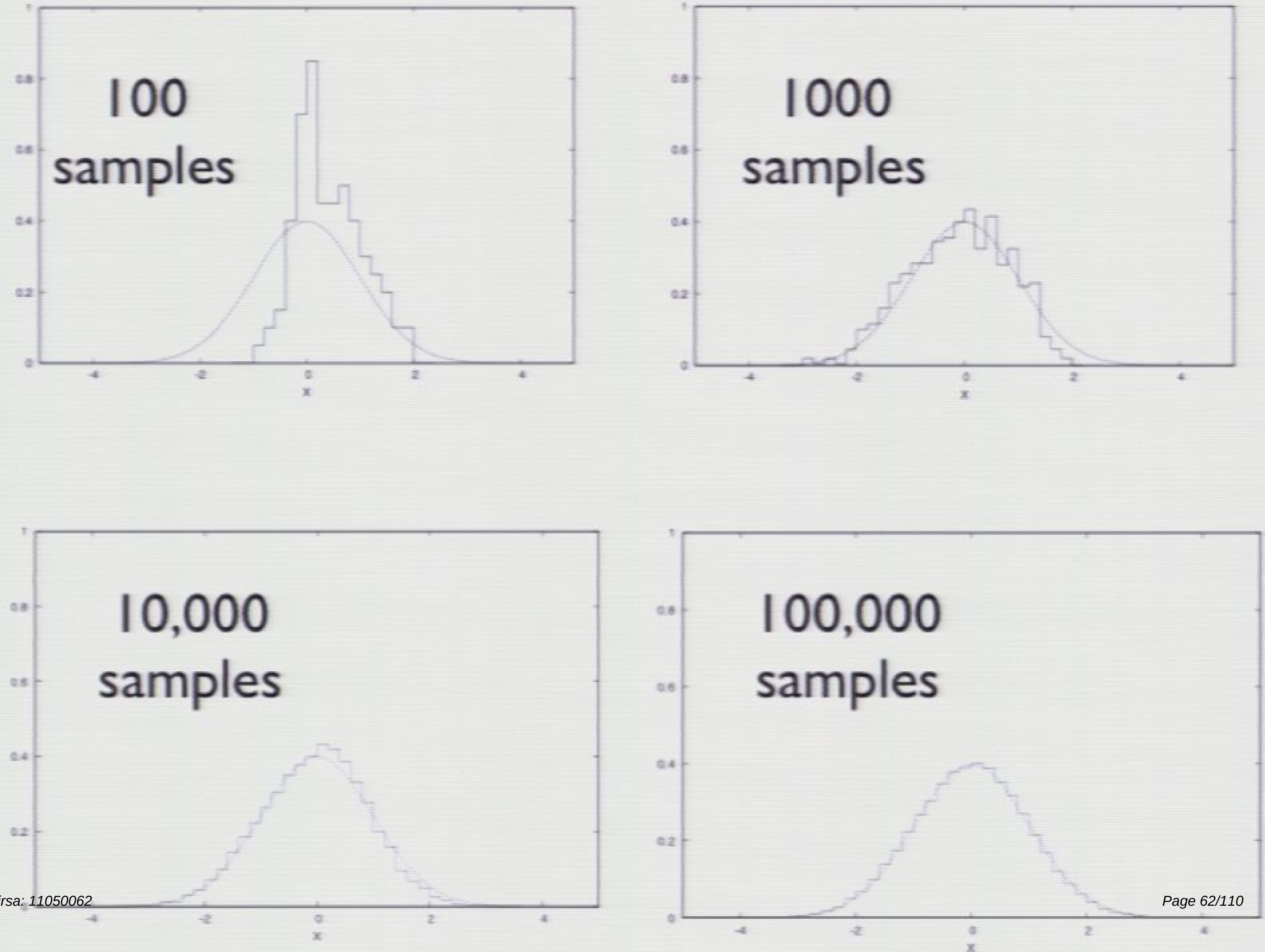
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Algorithm (cont'd)

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Theorem

If a Markov chain satisfies the detailed balance condition and the ergodicity,

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Proof? After the talk, on a black board :)

The principle of Monte-Carlo

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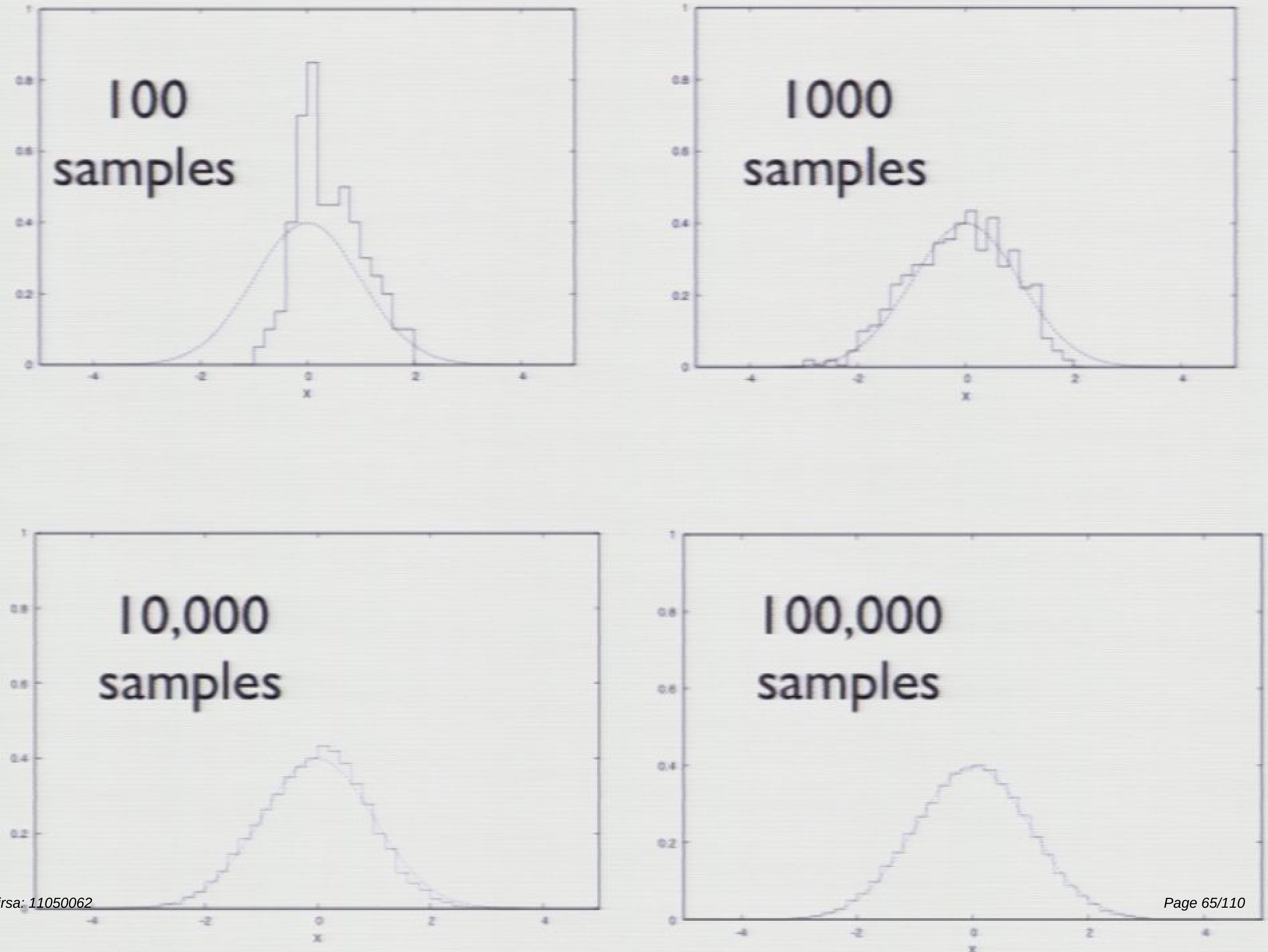
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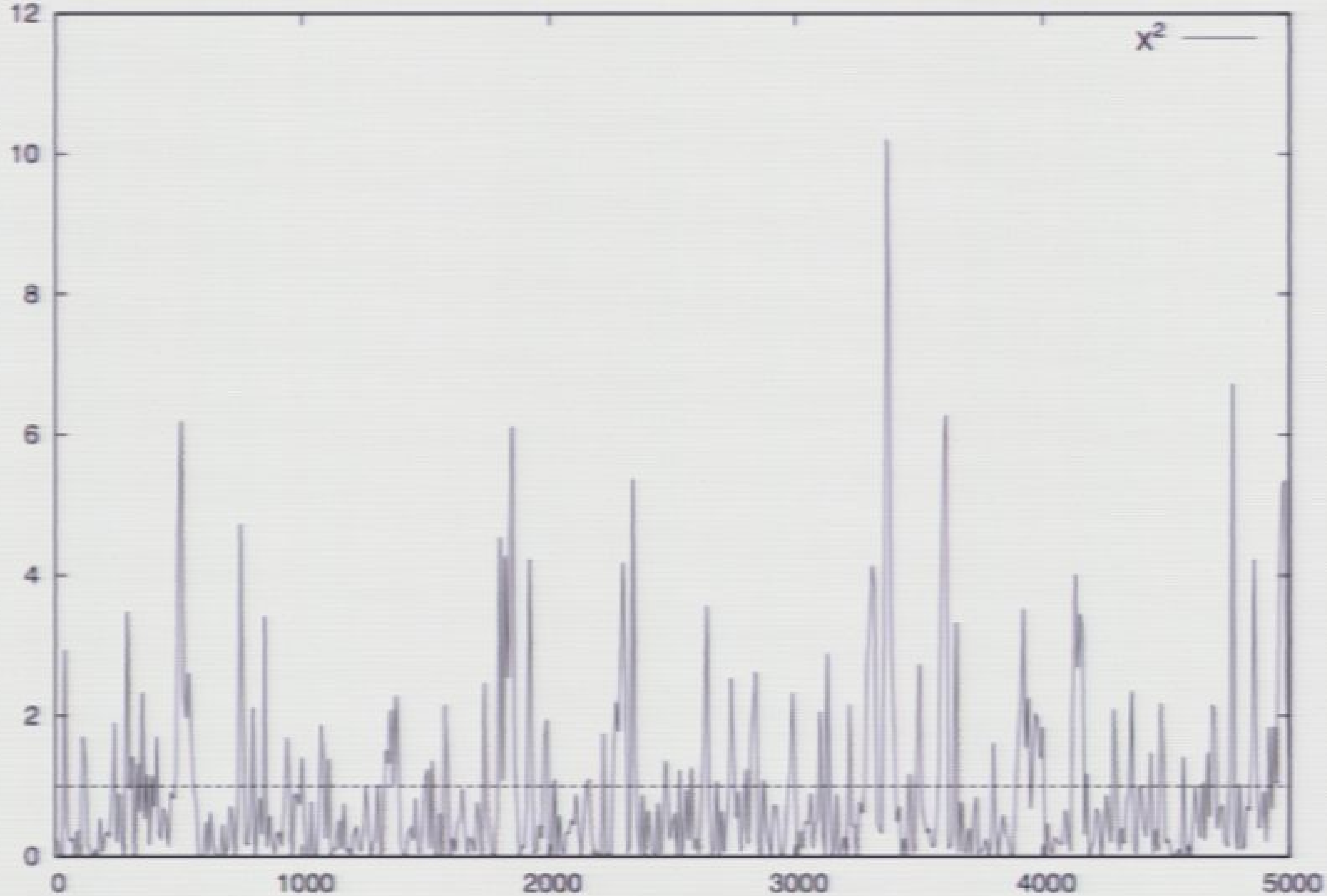

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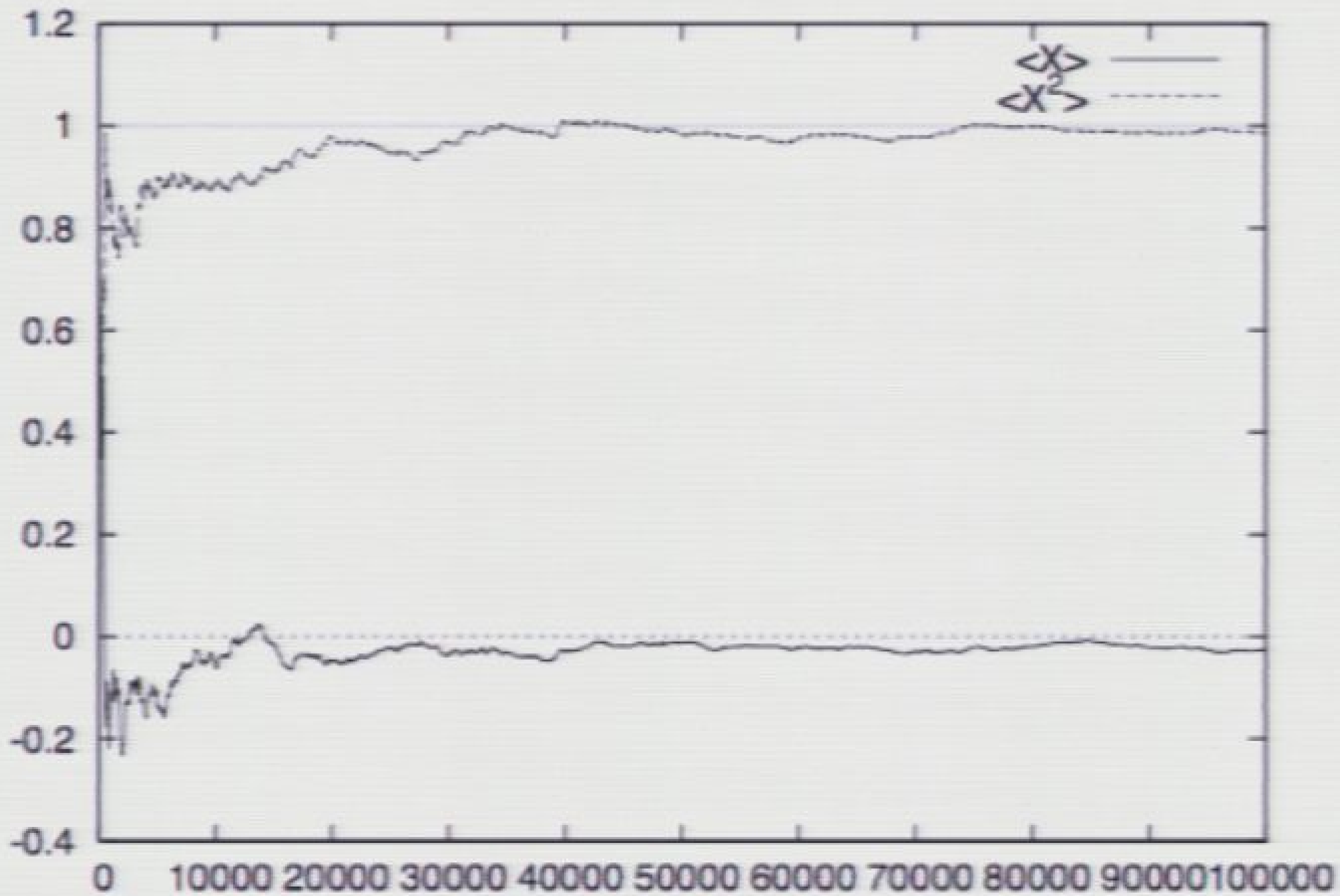
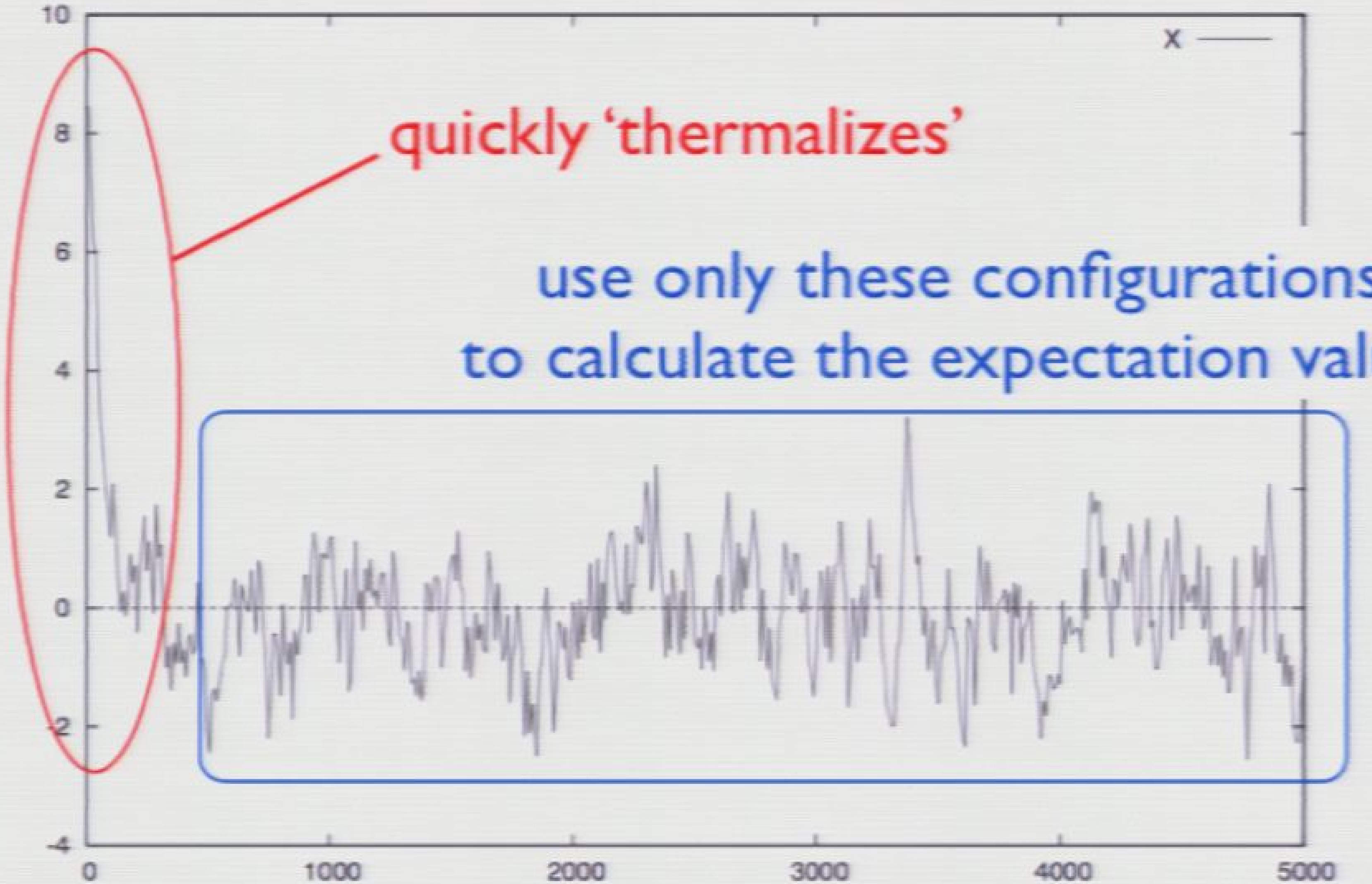


Note In typical YM simulations, with better algorithm, reasonable results can be obtained from 100 - 1000 configurations, if the theory does not suffer from the 'sign problem'.

Initial condition : x=10



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FERMIONS

$$S = S_B + S_F, \qquad S_F = \int d^4x \bar{\psi} D \psi$$

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Fermions appear in a bilinear form.

(if not.. make them bilinear by introducing auxiliary fields!)

→ can be integrated out *by hand*.

$$\int [dA][d\psi] e^{-S_B[A] - S_F[A,\psi]} = \int [dA] \det D[A] \cdot e^{-S_B[A]}$$

So, simply use the 'effective action',

$$S_{eff}[A] = S_B[A] - \log \det D[A]$$

(crucial assumption : det D > 0)

Fermion is <u>expensive</u>

cost for calculating S_B : N^3V

cost for det D :

N^3V^3(fundamental fermion)

N^6V^3(adjoint fermion)

(size of D is $NV\times NV$ (fundamental), $N^2V\times N^2V$(adjoint))

For this reason, until recently '*quench approximation*' was popular.

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- Quench is not appropriate for SYM.
- In order to perform the dynamical fermion simulation, we should use better algorithms developed in lattice QCD. The most powerful is the rational Hybrid Monte-Carlo (RHMC) algorithm.

(I can explain later, if somebody is interested in.)

Sign problem

- 'Probability' must be real positive.
- Life is sometimes hard... path integral weight e^{-S} can be *complex*! (after the Wick rotation)
    - Chern-Simons term (pure imaginary!)
    - Finite baryon chemical potential
    - Yukawa coupling
    - Super Yang-Mills

det D is complex

Such path integral measures cannot be generated by the Markov-chain Monte-Carlo method :'-(

‘reweighting method’

- Use the ‘phase-quenched’ effective action

$$S_{eff}[A] = S_B[A] - \log|\det D[A]|$$

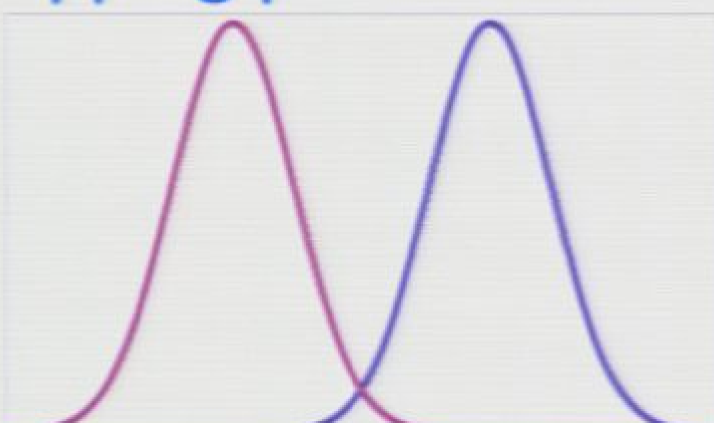
- Phase can be taken into account by the ‘phase reweighting’ :

$$\begin{aligned}\langle \mathcal{O} \rangle &= \frac{\int [dA] \det D \cdot e^{-S_B} \cdot \mathcal{O}}{\int [dA] \det D \cdot e^{-S_B}} \\ &= \frac{\int [dA](phase) \cdot |\det D| \cdot e^{-S_B} \cdot \mathcal{O} / \int [dA] |\det D| \cdot e^{-S_B}}{\int [dA](phase) \cdot |\det D| \cdot e^{-S_B} / \int [dA] |\det D| \cdot e^{-S_B}} \\ &= \frac{\langle (phase) \cdot \mathcal{O} \rangle_{phase\ quench}}{\langle (phase) \rangle_{phase\ quench}}\end{aligned}$$

usually the reweighting does not work in practice...

- violent phase fluctuation
  → both numerator and denominator becomes almost zero.   0/0 = ??
- vacua of full and phase-quenched model can disagree

'overlapping problem'



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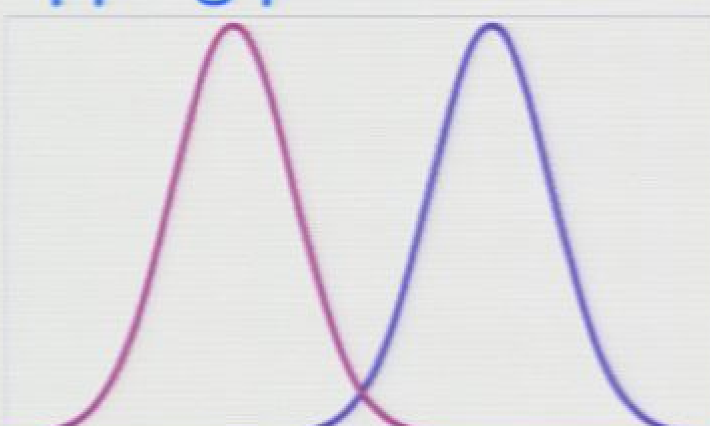

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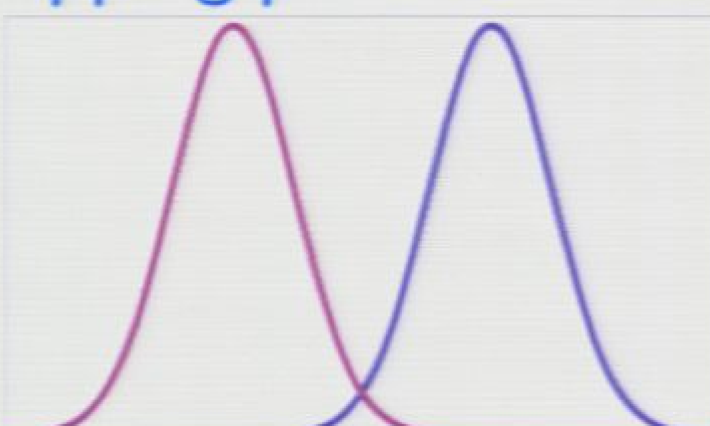

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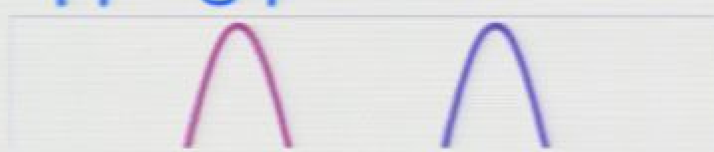


Miracles happen in SYM!

Tomorrow : application to SYM

Keywords

- Markov Chain
- detailed balance condition
- importance sampling
- Metropolis algorithm
- ‘Dynamical fermion’ vs ‘Quench’
- Sign problem

Tomorrow :
application to SYM

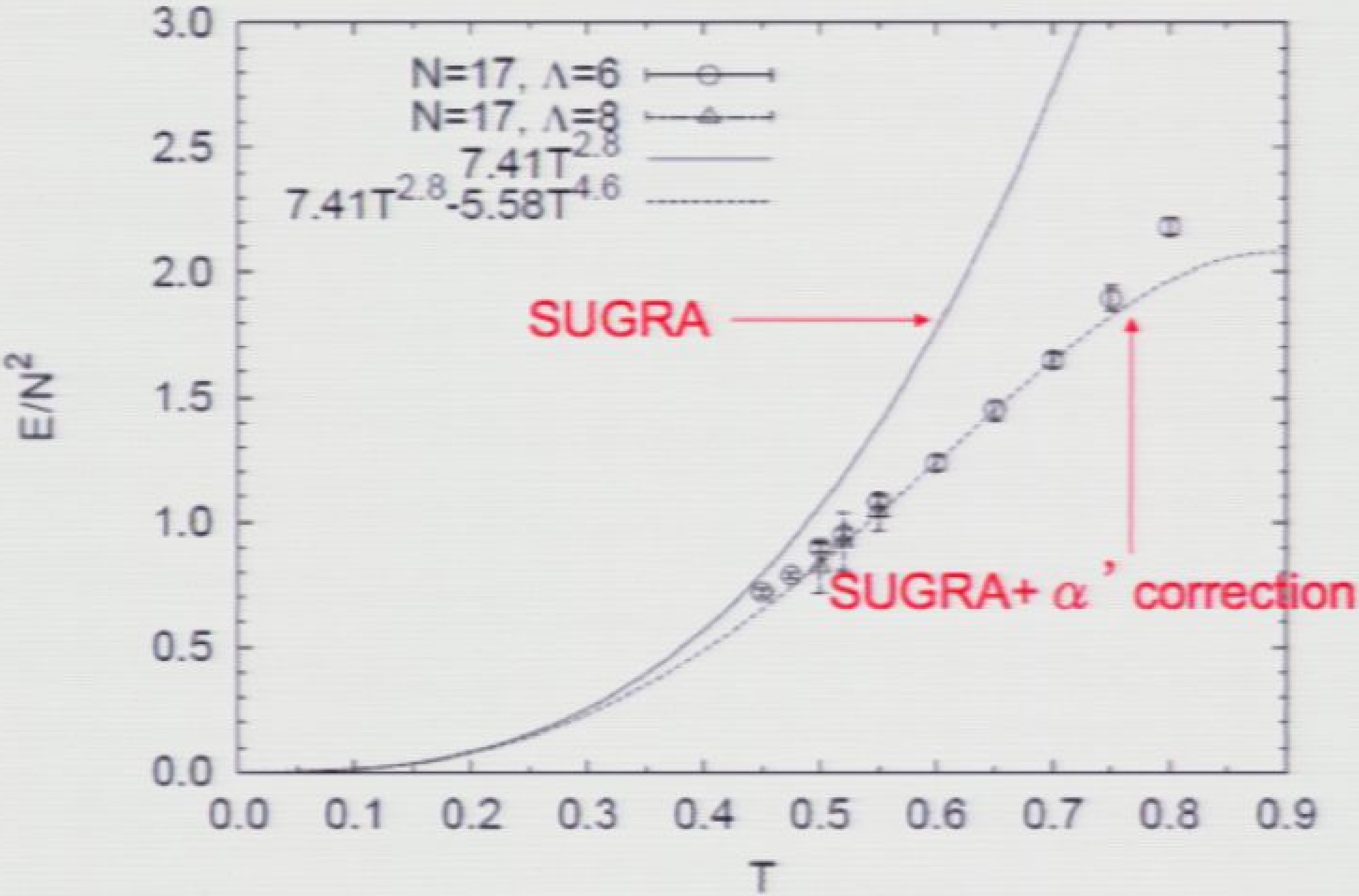
The principle of Monte-Carlo

- Consider field theory on Euclidean spacetime with the action $S[\phi]$.
- Generate field configurations with probability $e^{-S[\phi]}$. Then,

$$\langle \mathcal{O} \rangle = \frac{\int [d\phi] \mathcal{O}[\phi] e^{-S[\phi]}}{\int [d\phi] e^{-S[\phi]}} \simeq \frac{1}{n} \sum_{i=1}^{n} \mathcal{O}[\phi_i]$$

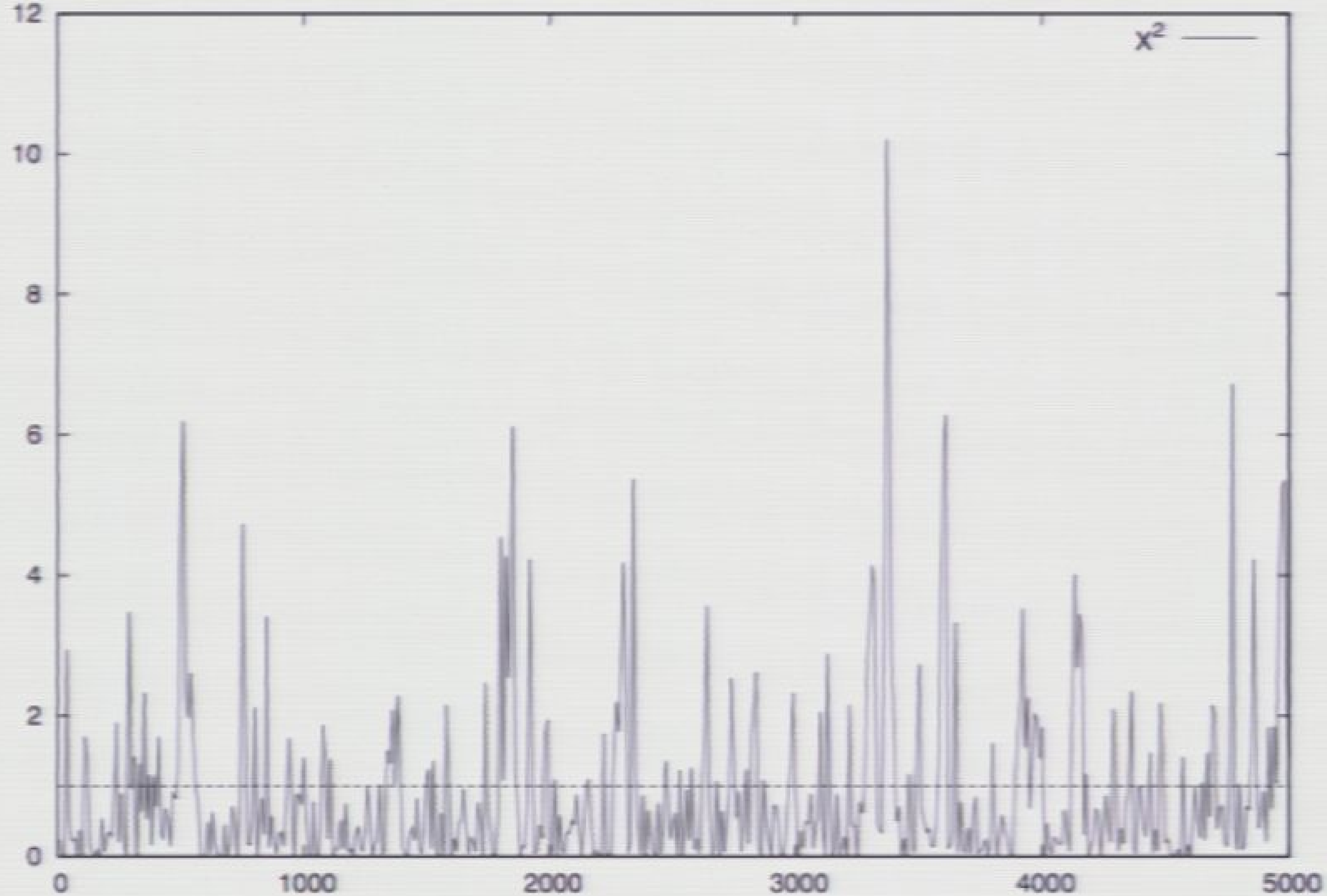
- Such a set of configurations can be generated as long as $e^{-S[\phi]} > 0$ (not 'probability' otherwise...)

simulation result (1d)



Anagnostopoulos-M.H.-Nishimura-Takeuchi, 2007

M.H.-Hyakutake-Nishimura -Takeuchi, 2008



history of x^2

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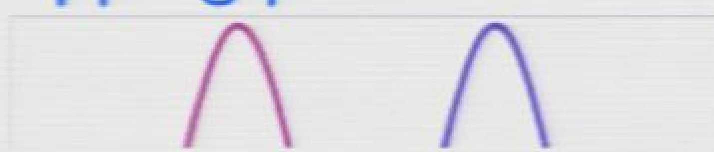

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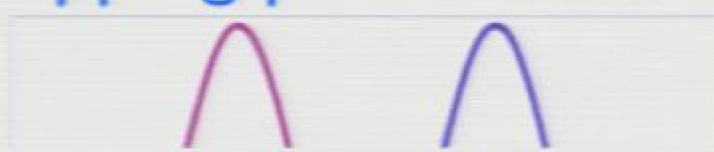

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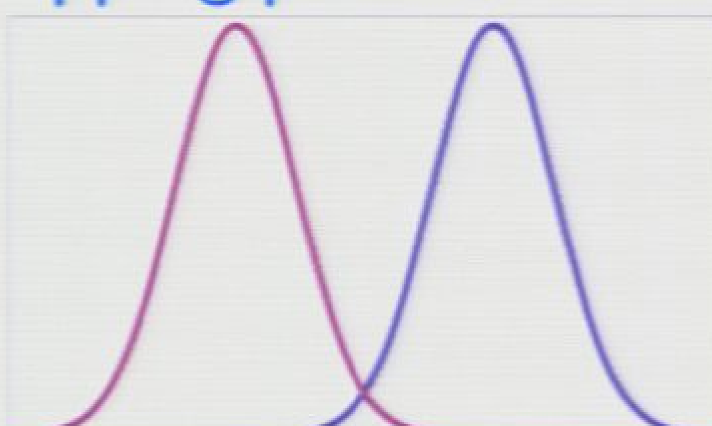

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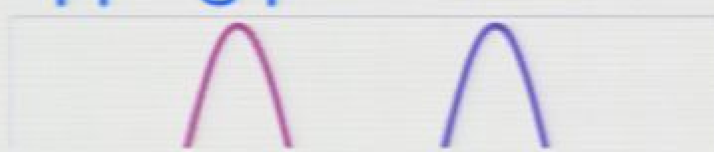

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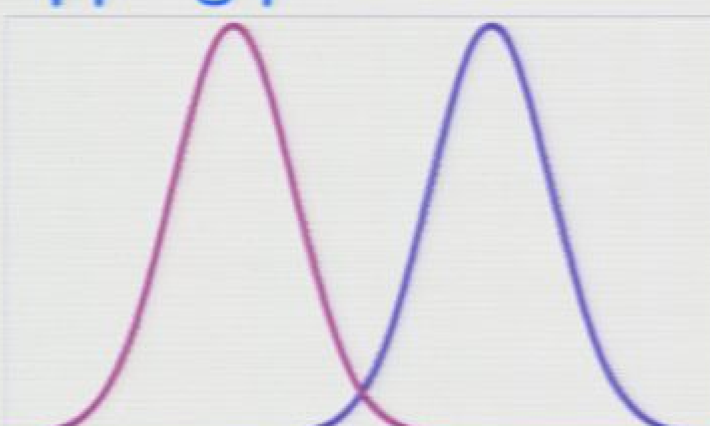

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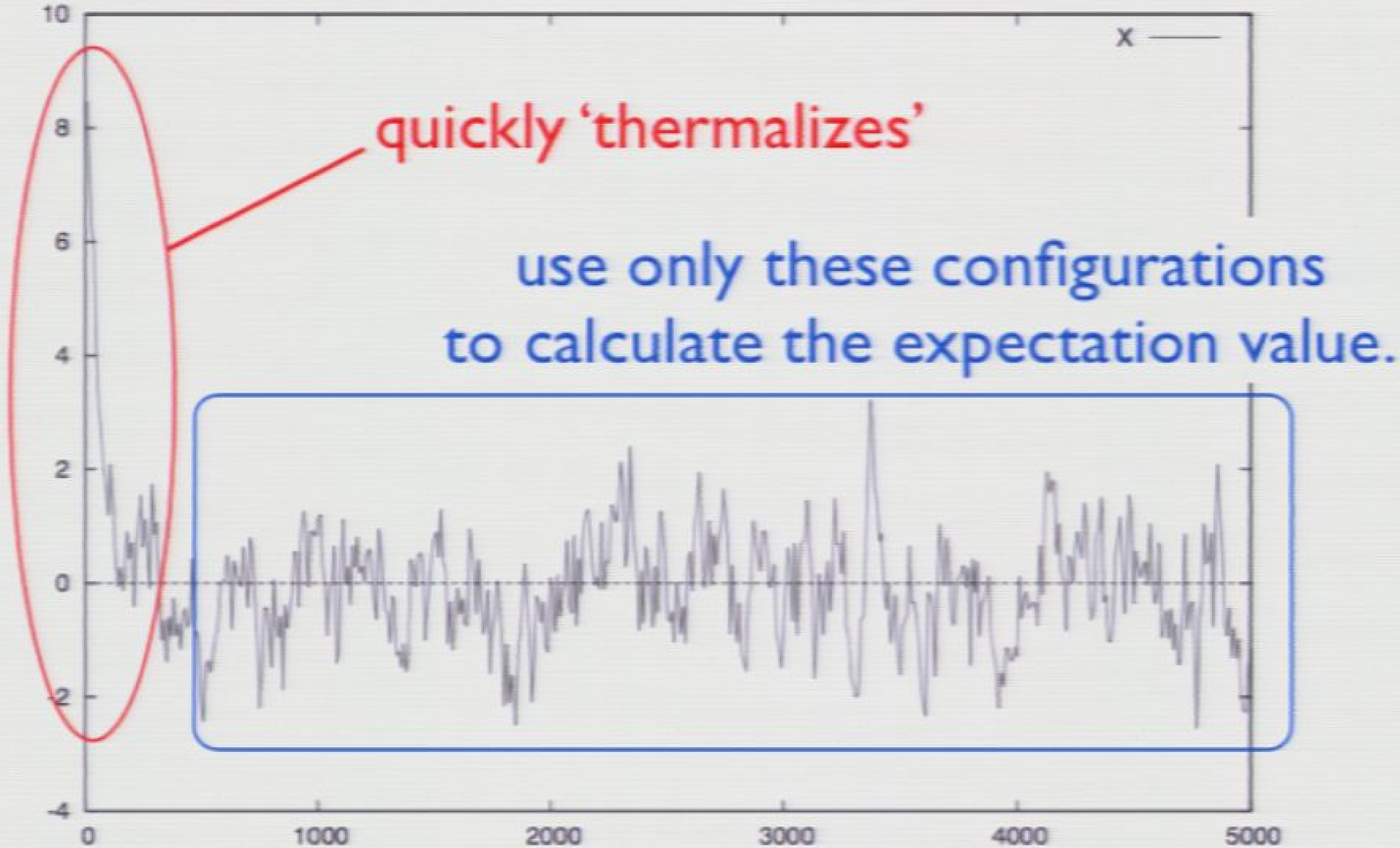

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Metropolis (cont'd)

- Ergodicity is satisfied.
- Detailed balance is also OK:

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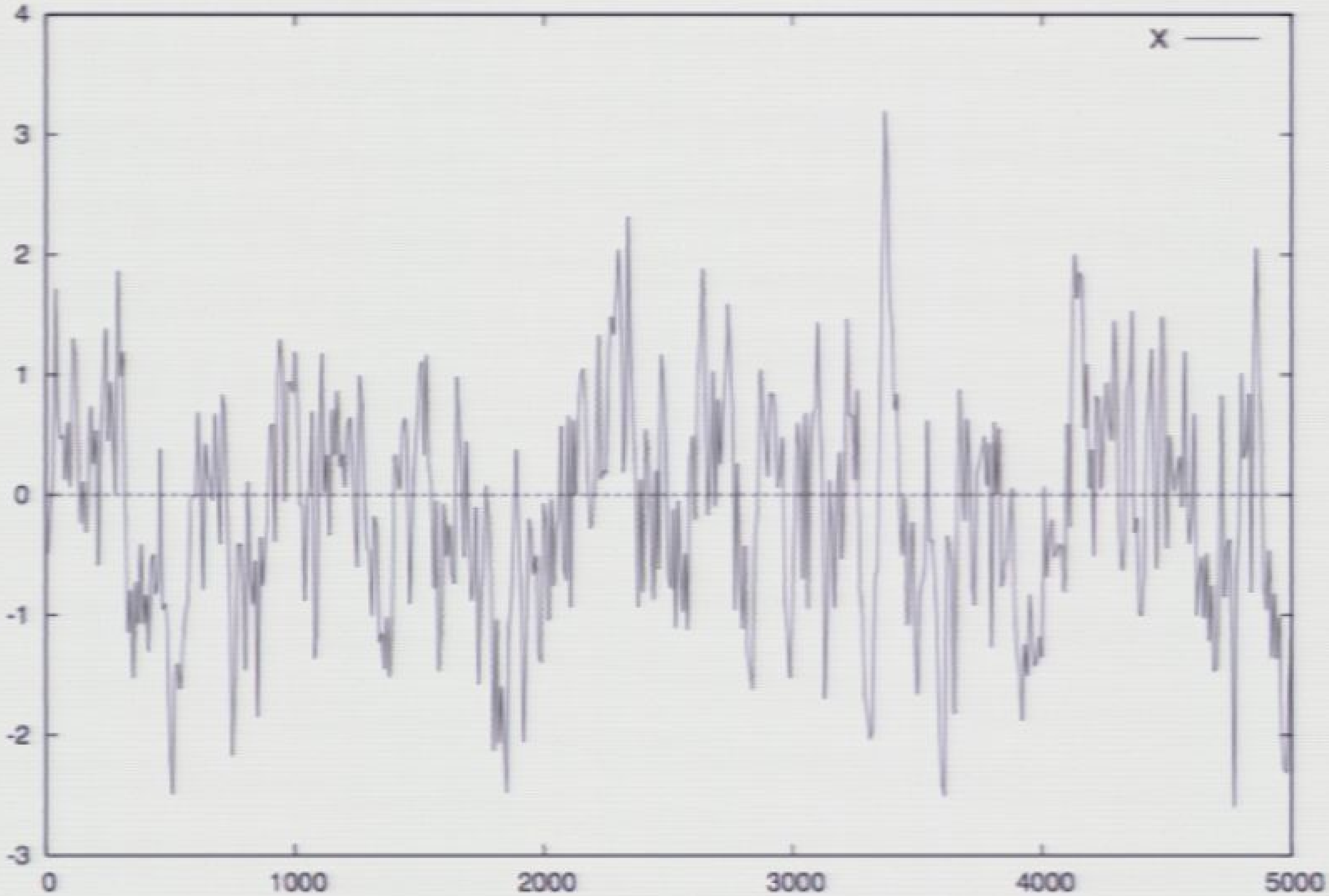

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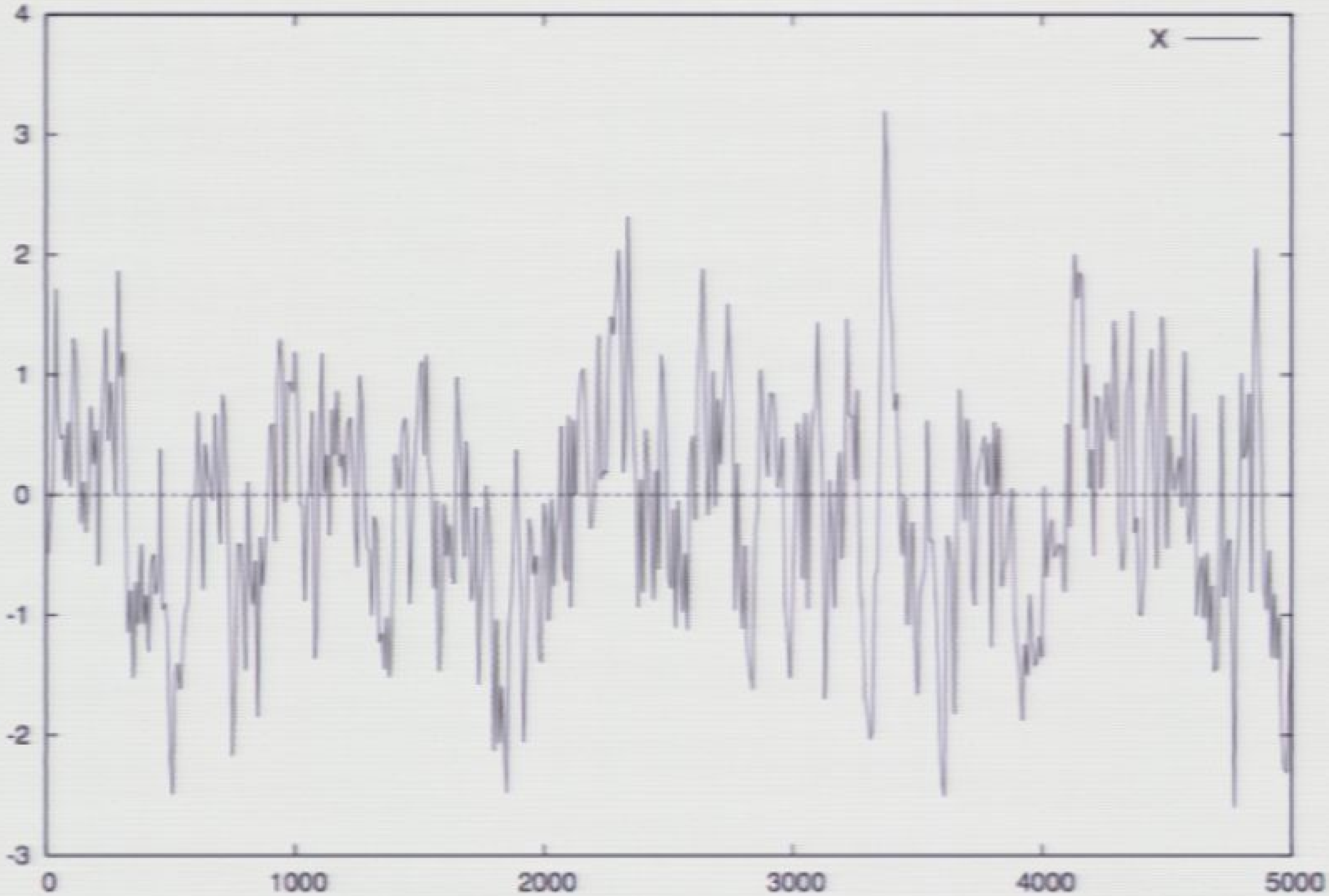

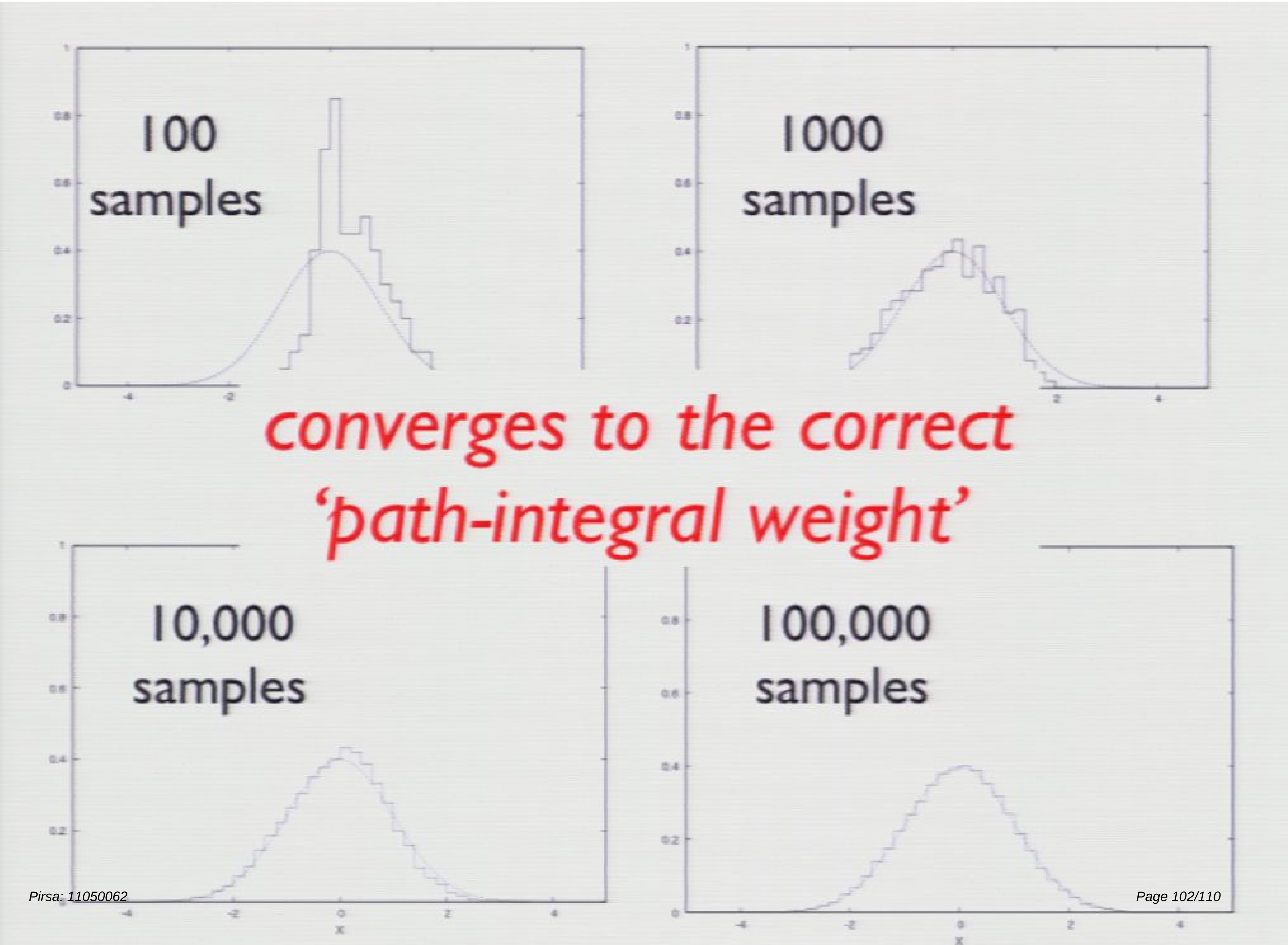
100
samples
1000
samples
converges to the correct
'path-integral weight'
10,000
samples
100,000
samples
x

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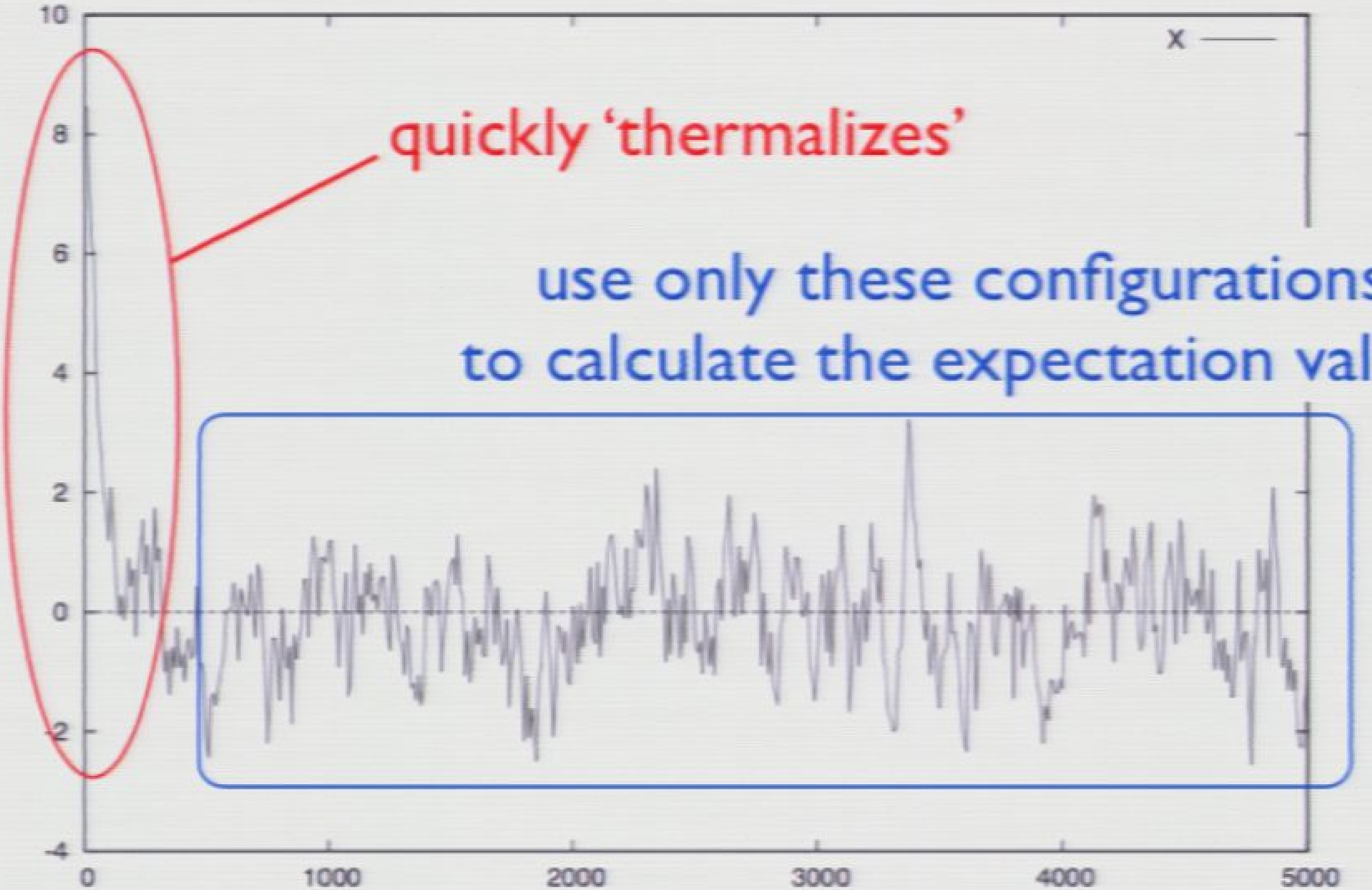

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