Title: Uncertainty, nonlocality & complementarity

Date: May 12, 2011 11:40 AM

URL: http://pirsa.org/11050054

Abstract:

Nonlocality,
uncertainty
& complementarity

Nonlocality,

uncertainty

& complementarity

The nonlocality of quantum theory is determined by the uncertainty principle

J. Oppenheim, S. Wehner

Science, 33, 1072 (2010)

Both uncertainty and nonlocality are about storage and retrieval of information (Alice stores, Bob retrieves)

In quantum nonlocality, control over the information stored requires Alice to cram too much info in too small a system and Bob's retrieval is thus restricted by uncertainty and complementarity.

$\boxed{\text{Uncertainty}}$

Random Access Code

$$\Delta x \, \Delta p \geq \hbar/2$$

$\boxed{\text{Nonlocality}}$ (Bell inequality)

$$|a_0(b_0+b_1) + a_1(b_0-b_1)| \leq 2$$

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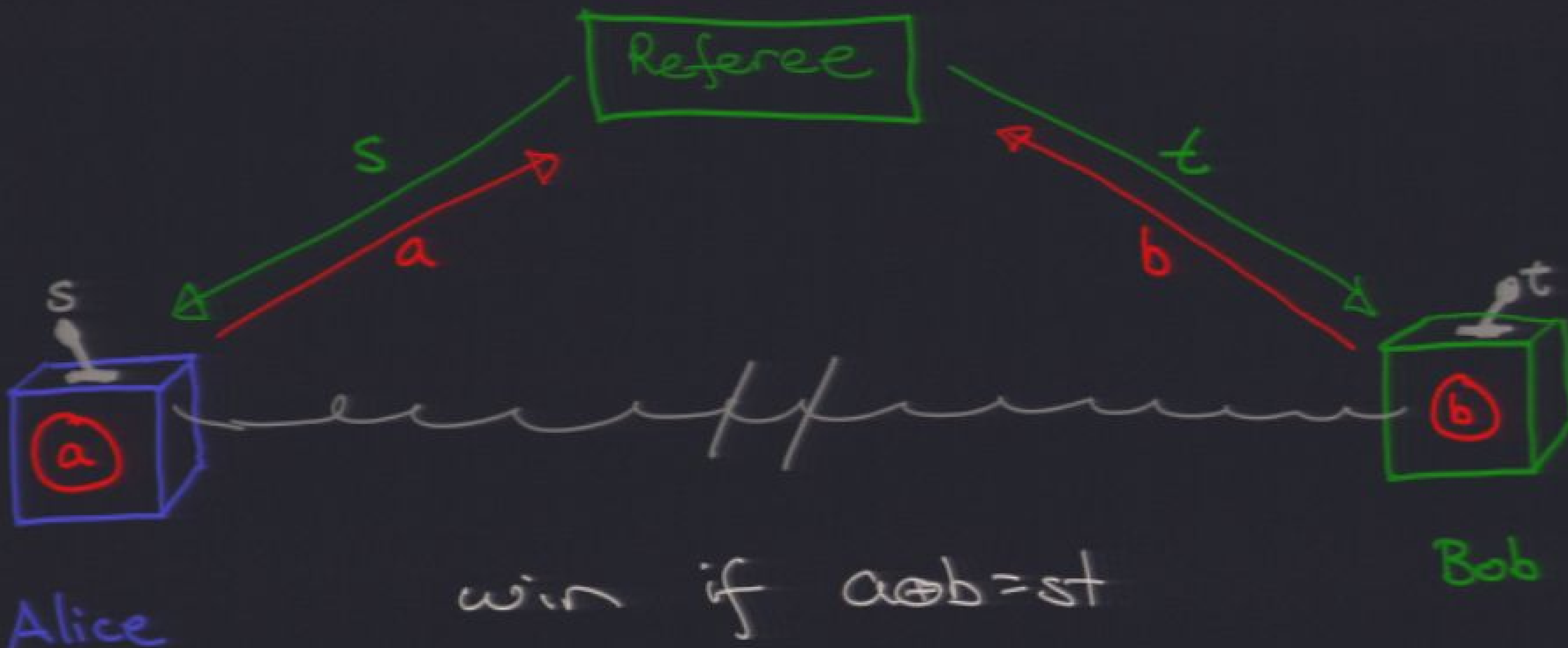
Nonlocality as a game

CHSH game



win if $a \oplus b = st$

Classical World: $P_{win} = 3/4$

deterministic strategy (a_s, b_t):

$a_0 = 0,\ a_1 = 0$

$b_0 = 0,\ b_1 = 1$

$$a_s \oplus b_t = s \cdot t$$

$$\left.\begin{array}{l} a_0 \oplus b_0 = 0 \\ a_0 \oplus b_1 = 0 \\ a_1 \oplus b_0 = 0 \\ a_1 \oplus b_1 = \underline{1} \end{array}\right\}$$

add mod 2

$0 = \underline{1}$

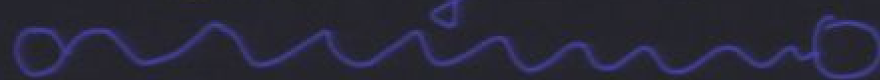
$$|a_0(b_0+b_1)+a_1(b_0-b_1)| \leq 2$$

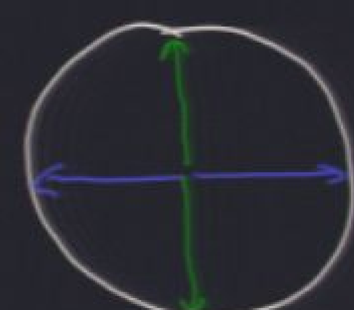
Classical World: $P_{win} = 3/4$

Quantum World: $P_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$

$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$ $\sigma_0^A, \sigma_1^A, \sigma_0^B, \sigma_1^B$

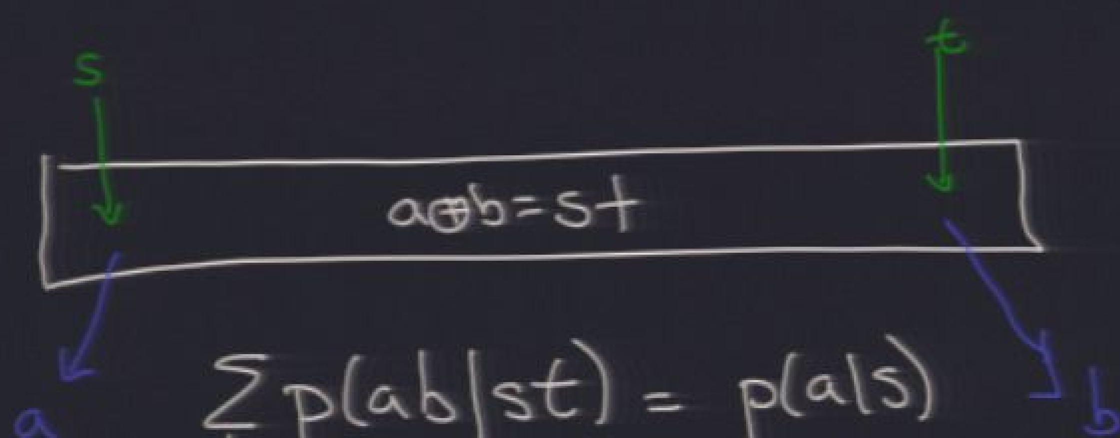
entanglement





Post quantum World: $P_{win} = 1$

PR-box (no-signalling theory)



$$\sum_b p(ab|st) = p(a|s)$$

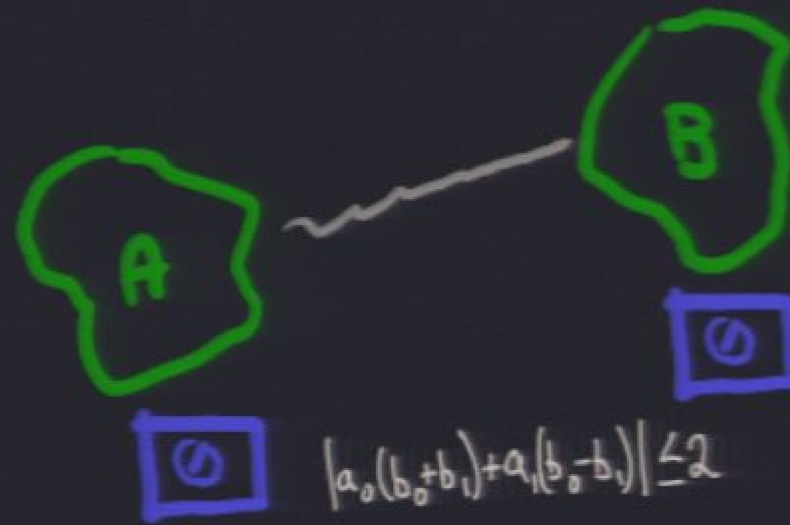
$$\sum_a p(ab|st) = p(b|t)$$

Quantum theory could be even spookier!

What restricts P_{win} ?

$\Delta x \Delta p \geq \hbar/2$

Uncertainty



$|a_0(b_0+b_1)+a_1(b_0-b_1)| \leq 2$

Non-locality

① All theories: non-locality determined by uncertainty + steering

② Quantum Mechanics: non-locality determined by uncertainty
(CHSH, XOR, retrieval, mod 3)

③ There are theories which are as non-local and as uncertain as Q.M. but are less complementary

Post quantum World: $P_{win} = 1$

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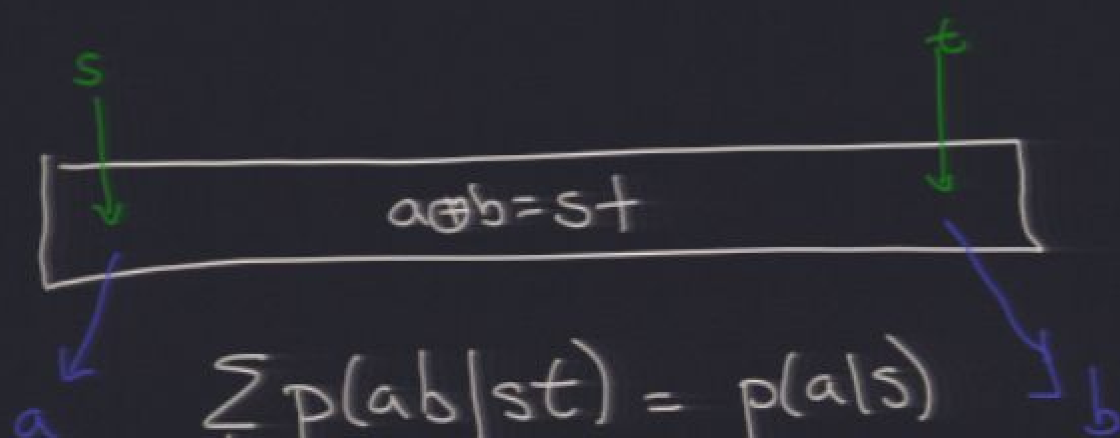

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$$\sum_a p(ab|st) = p(b|t)$$

Quantum theory could be even spookier!

What restricts P_{win} ?

Q.M. exhibits correlations which cannot be explained by any local hidden variable theory.

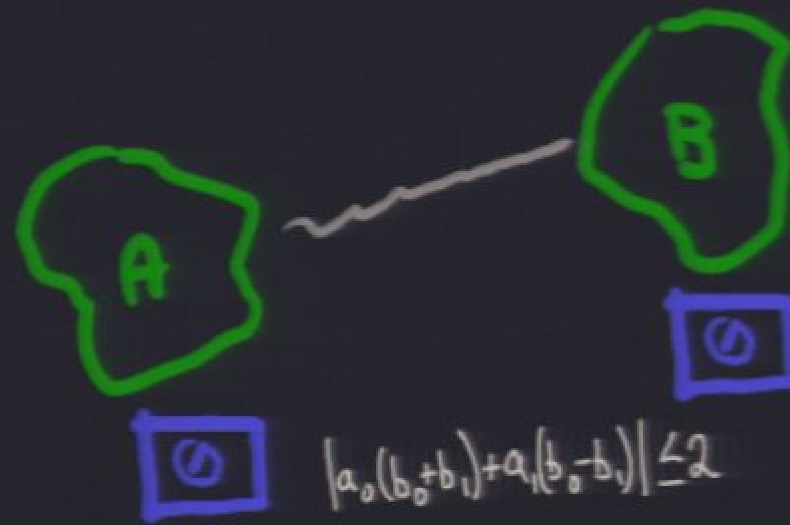
What restricts this nonlocality?

No-Signalling (NS) is not enough
(Popescu-Rohrlich, Tsirelson, 93)

Information causality, communication complexity, nonlocal computation, don't appear to be enough
(Pawlaski et al, van Dam, Brassard et al, Linden et al)

$\Delta x \Delta p \geq \hbar/2$

Uncertainty



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Random Access Code

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Random Access Code

Ambainis, Nayak, Ta-Shma, Vazirani (99), Nayak (99)

$$\bar{x} = x^{(0)}\, x^{(1)} \cdots \boxed{x^{(t)}} \cdots x^{(n)}$$

uniform

$\bar{x}$

t

Alice encodes $\bar{x}$

Bob must retrieve $x^{(t)}$

$$P^{RAC} = \sum_t \frac{1}{|T|} P\left(x_B^{(t)} = x_A^{(t)}\right)$$

Uncertainty

$$\Delta x \, \Delta p \geq \hbar/2$$

Random Access Code

Nonlocality

(Bell Inequality)

$$|a_0(b_0+b_1) + a_1(b_0-b_1)| \leq 2$$

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uncertainty

Robertson: $\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$

$$\Delta x \Delta p \geq \hbar/2$$

Entropic: $\sum_t H(t, \sigma)\, p(t) \geq c$ (Bialynicki-Birula, Mycielski)

$$H(t, \sigma) := -\sum_b P_\sigma(b|t) \log P_\sigma(b|t)$$

eg:

$$H_\infty(t) := -\log \max_b p(b|t)$$

Uncertainty

$$\Delta x \Delta p \geq \hbar/2$$

$$\boxed{\begin{array}{l} p(x = x_0) = 1 \\ p(p = p_0) = 1 \end{array}}$$

forbidden

Spin: $\sigma_x \in \{0,1\}$, $\sigma_z \in \{0,1\}$

if $p(\sigma_x = 1) = 1$ then $p(\sigma_z = 0) = \frac{1}{2}$

$$\frac{1}{2} p(\sigma_x) + \frac{1}{2} p(\sigma_z) \leq \xi = \frac{1}{2} + \frac{1}{2\sqrt{2}} < 1$$

Fine-grained uncertainty relations

$\frac{1}{2}[p(\sigma_x=1) + p(\sigma_z=1)] \leq \xi^{11}$ $\quad x=11 \quad \sigma_{11}$

$\frac{1}{2}[p(\sigma_x=1) + p(\sigma_z=0)] \leq \xi^{10}$ $\quad x=10 \quad \sigma_{10}$

$\frac{1}{2}[p(\sigma_x=0) + p(\sigma_z=1)] \leq \xi^{01}$ $\quad x=01 \quad \sigma_{01}$

$\frac{1}{2}[p(\sigma_x=0) + p(\sigma_z=0)] \leq \xi^{00}$ $\quad x=00 \quad \sigma_{00}$

Most certain states

$$\boxed{\xi^{\bar{x}} = \frac{1}{2} + \frac{1}{2\sqrt{2}} < \underline{1}}$$

Uncertainty relation as a Random Access Code

$$\frac{1}{2}\left[p(\sigma_x = 1) + p(\sigma_z = 1)\right] \le \xi^{11} \qquad x = 11$$

$$\frac{1}{2}\left[p(\sigma_x = 1) + p(\sigma_z = 0)\right] \le \xi^{10} \qquad x = 10$$

$$\frac{1}{2}\left[p(\sigma_x = 0) + p(\sigma_z = 1)\right] \le \xi^{01} \qquad x = 01$$

$$\frac{1}{2}\left[p(\sigma_x = 0) + p(\sigma_z = 0)\right] \le \xi^{00} \qquad x = 00$$

p^{RAC} (with most certain state)

Fine-grained Uncertainty Relations

$t \in \mathcal{T}$

$$P^{cert}_{\mathcal{T}}(\sigma_x) = \sum p(t)\, p_{\sigma}(x^{(t)}|t) \le \zeta_{\bar{x}}$$

outcomes $x^{(1)}, x^{(2)} \cdots x^{(|\mathcal{T}|)}$

$$\mathcal{U} = \left\{ \sum p(t)\, p_{\sigma_x}(x^{(t)}|t) \le \zeta_{\bar{x}} \;\middle|\; \forall \bar{x} \in B^{\times n} \right\}$$

$$\zeta_{\bar{x}} = \max_{\sigma_x} P^{cert}_{\mathcal{T}}(\sigma_x)$$

$\boxed{\text{Most certain State } \sigma_x}$

$$\sum H_{\infty}(t|\sigma) \ge -\log \max \zeta_{\bar{x}}$$

$\boxed{\text{Uncertainty}}$

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Random Access Code

$\boxed{\text{Nonlocality}}$ (Bell inequality)

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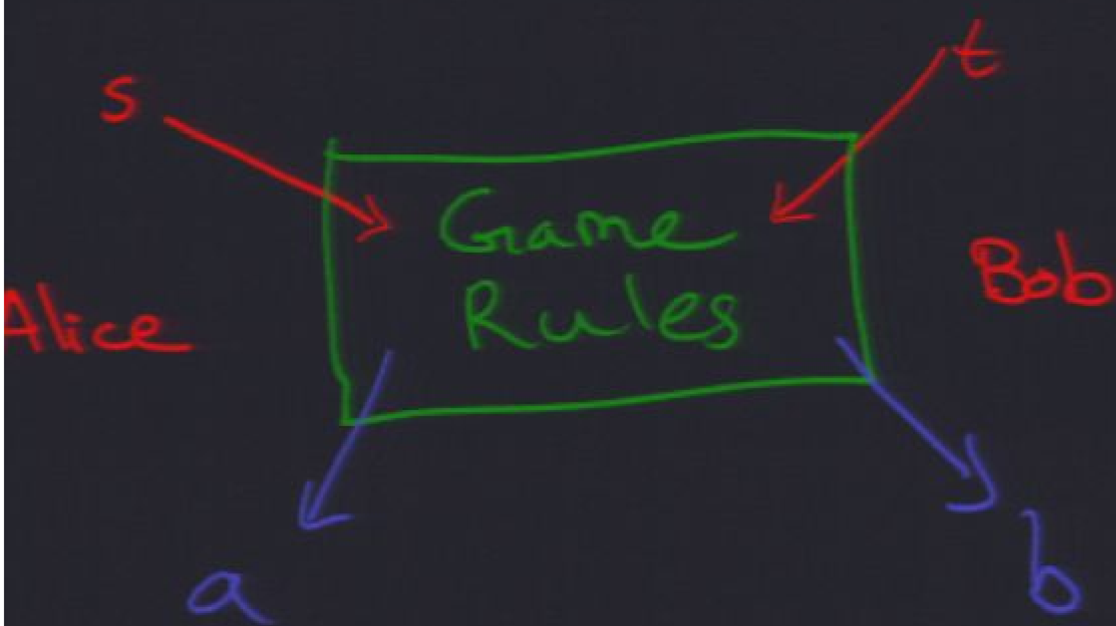
Non-Locality

$$p(b|t) \longrightarrow p(ab|st)$$



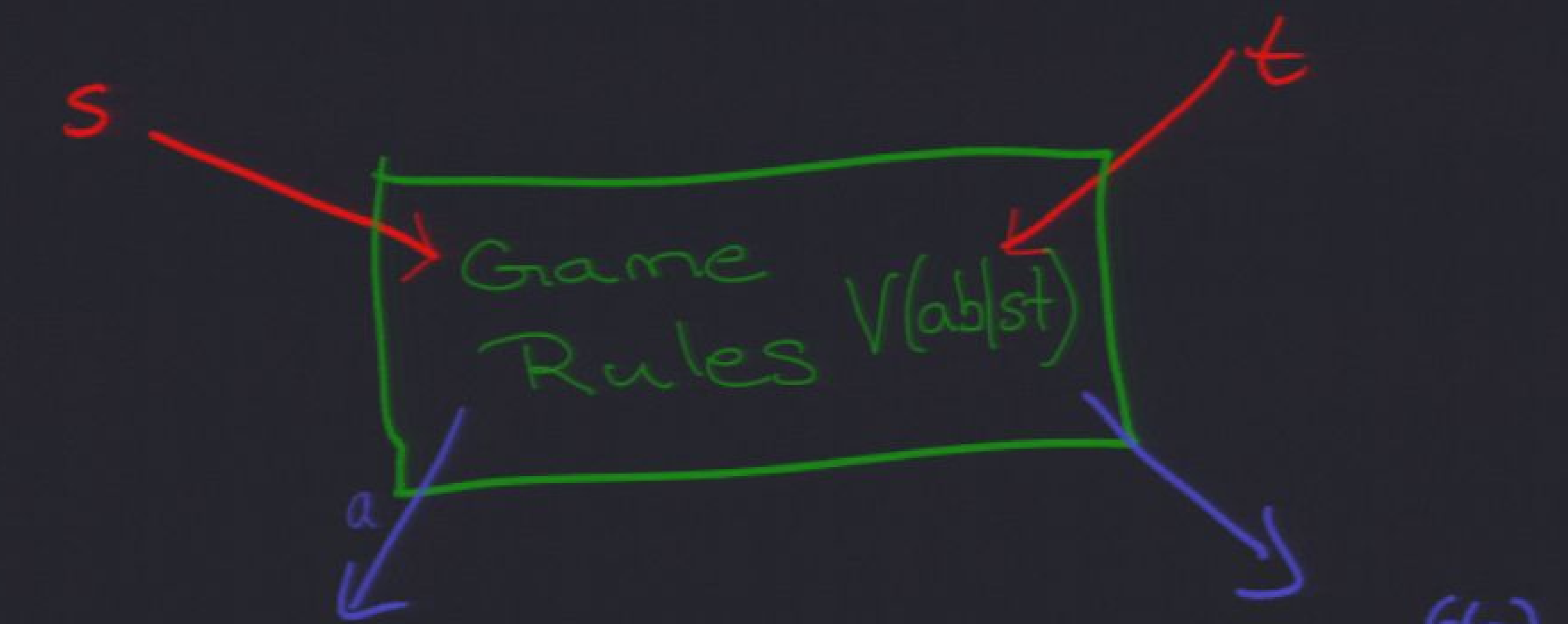
Win!
$s \cdot t = a \oplus b$

Classically $P_{win} \leq 3/4$

Quantumly $P_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$

Post-quantumly $P_{win} = 1$

Unique Games



$$X_{sa} = X^{(1)}_{sa} X^{(2)}_{sa} \ldots X^{(n)}_{sa}$$

$$X^{(t)}_{sa}$$

winning answer for question t

like a Random Access Code

$$\text{CHSH} = |a_0(b_0+b_1) + a_1(b_0-b_1)| \leq 2$$

win: $\underline{1}$ if $a \oplus b = st$

lose: 0 otherwise

$\vec{X}_{s,a}$

00

s=0 11

s=1

01

$\boxed{10}$

Bob needs to answer $\underline{1}$ if t=0 and 0 if t=1

$$P^{win}_{QM} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Steering to most certain states?

$P^{win} = P^{RAC}$ (optimized Υ, steering)

$\xi = P^{RAC}$ (with most certain state)

$$p(\sigma_x = 0) + p(\sigma_z = 1) \leq \xi^{01}$$

σ_B^{01} most certain state

Can Alice steer to it?

$$CHSH = |a_0(b_0+b_1) + a_1(b_0-b_1)| \leq 2$$

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$\vec{x}_{s_j a}$

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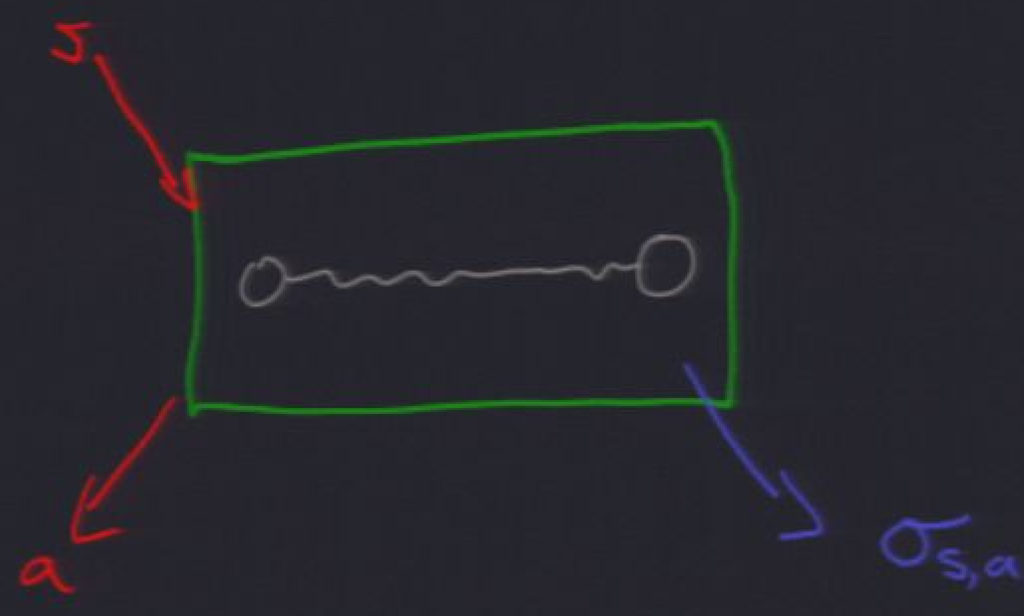
Steering

$$\mathcal{E}_s = \{p(a,s), \sigma_{s,a}\}_a$$

N. Sig $\quad \sum p(a,s)\,\sigma_{s,a} = \sigma_B$



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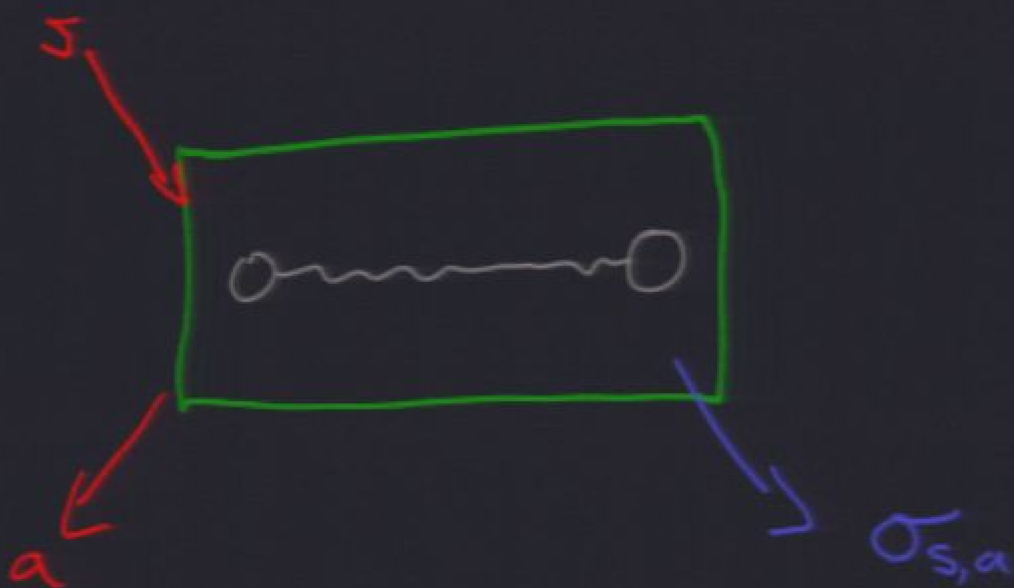
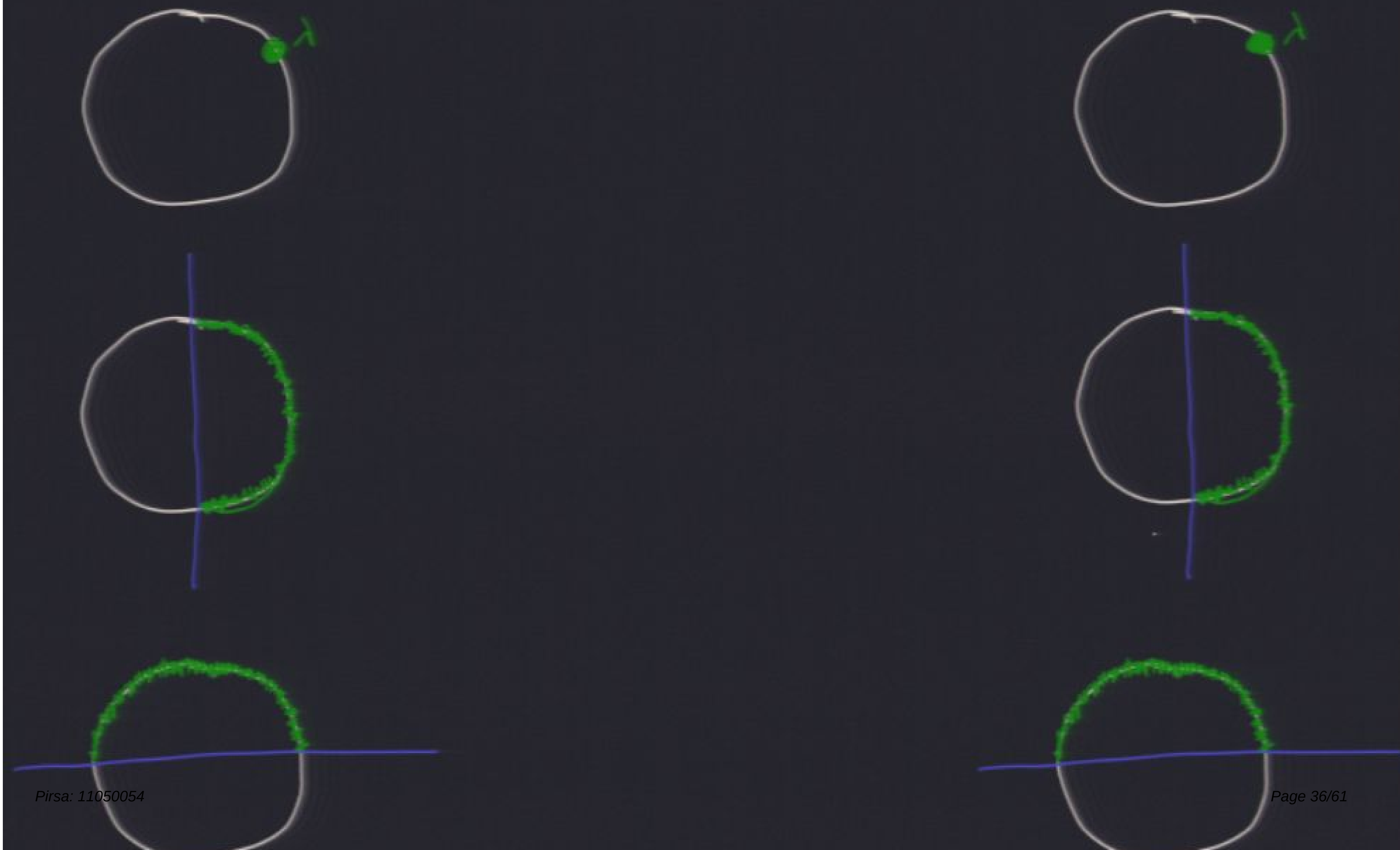


Steering in classical mechanics



Uncertainty : $\xi^x = P^{RAC}$

Nonlocality : Bell $\longrightarrow$ $P^{win} = P^{RAC}$

$\longrightarrow \quad P^{win} = \xi^x$

For CHSH $\quad P^{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \quad (\sigma_x, \sigma_z)$

$$\xi^x(\sigma_x, \sigma_z) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

NL and UN for gen theories

$$P^{game}(S, \Upsilon, \sigma_{AB}) = \sum p(st) \sum_{ab} V(ab|st)\, P_{\sigma_{AB}}(ab|st)$$

$$= \sum p(s) \sum p(a|s)\, p\left(x_{as}^{(t)}, \sigma_{sa}\right) p(t)$$

$$= \sum p(s) \sum p(a|s)\, P_{\Upsilon}^{cert}(\sigma_{sa})$$

$$P^{game}_{max} = \max_{\{\Sigma^s\}_s, \Upsilon} \sum p(s) \sum p(a|s)\, P_{\Upsilon}^{cert}(\sigma_{sa})$$

$$= \sum p(s)\, p(a|s) \xi_x$$ (for XOR, mod 3, CHSH...) $\because$ can steer to max. certain state!

for retrieval, eg CHSH $\longleftrightarrow \sum p(t)\, H(t)$

Eg. CHSH

	steering	uncertainty ξ^x	p^{win}
QM	perfect	$\frac{1}{2}+\frac{1}{2\sqrt{2}}$	$\frac{1}{2}+\frac{1}{2\sqrt{2}}$
CM probabilistic	perfect	$3/4$	$3/4$
PR	perfect	$\underline{1}$	$\underline{1}$
CM deterministic	none	$\underline{1}$	$\frac{1}{2}+\frac{1}{4}=3/4$

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NL and UN for gen theories

$$P^{game}(S, r, \sigma_{AB}) = \sum p(st) \sum_{ab} V(ab|st)\, P_{\sigma_{AB}}(ab|st)$$

$$= \sum p(s) \sum p(a|s)\, p\left(x^{(t)}_{as}, \sigma_{sa}\right) p(t)$$

$$= \sum p(s) \sum p(a|s)\, P^{cert}_{r}(\sigma_{sa})$$

$$P^{game}_{max} = \max_{\{\{\Sigma^S\}_{s}, r\}} \sum p(s) \sum p(a|s)\, P^{cert}_{r}(\sigma_{sa})$$

$$= \sum p(s)\, p(a|s) \xi_x$$

(for XOR, mod 3, CHSH...)
∴ can steer to max. certain state!

for retrieval, eg CHSH $\longleftrightarrow$ $\sum p(t)\, H(\ldots)$

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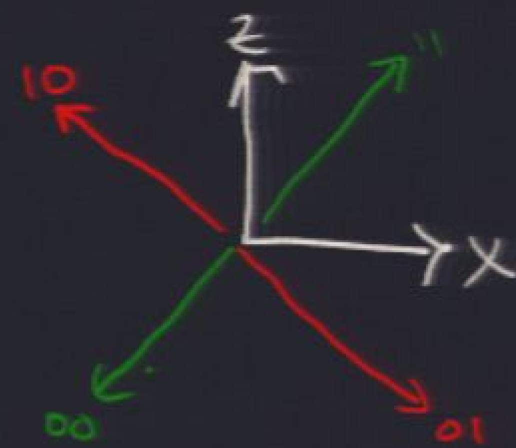
CM deterministic	none	$\underline{1}$	$\frac{1}{2}+\frac{1}{4}=3/4$

Ex/ CHSH

QM/ steering
uncertainty

$s \cdot t = a \oplus b$

$s=0$	0	0
	1	1
$s=1$	1	0
	0	1



$$\xi^{\bar{x}}_{s,a} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

$$P^{game}_{max} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

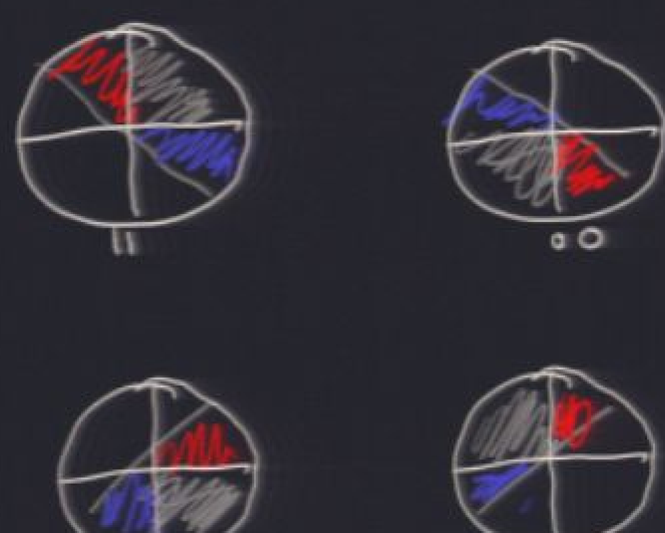
Classical mechanics

determanistic $\longrightarrow$ no steering
certainty

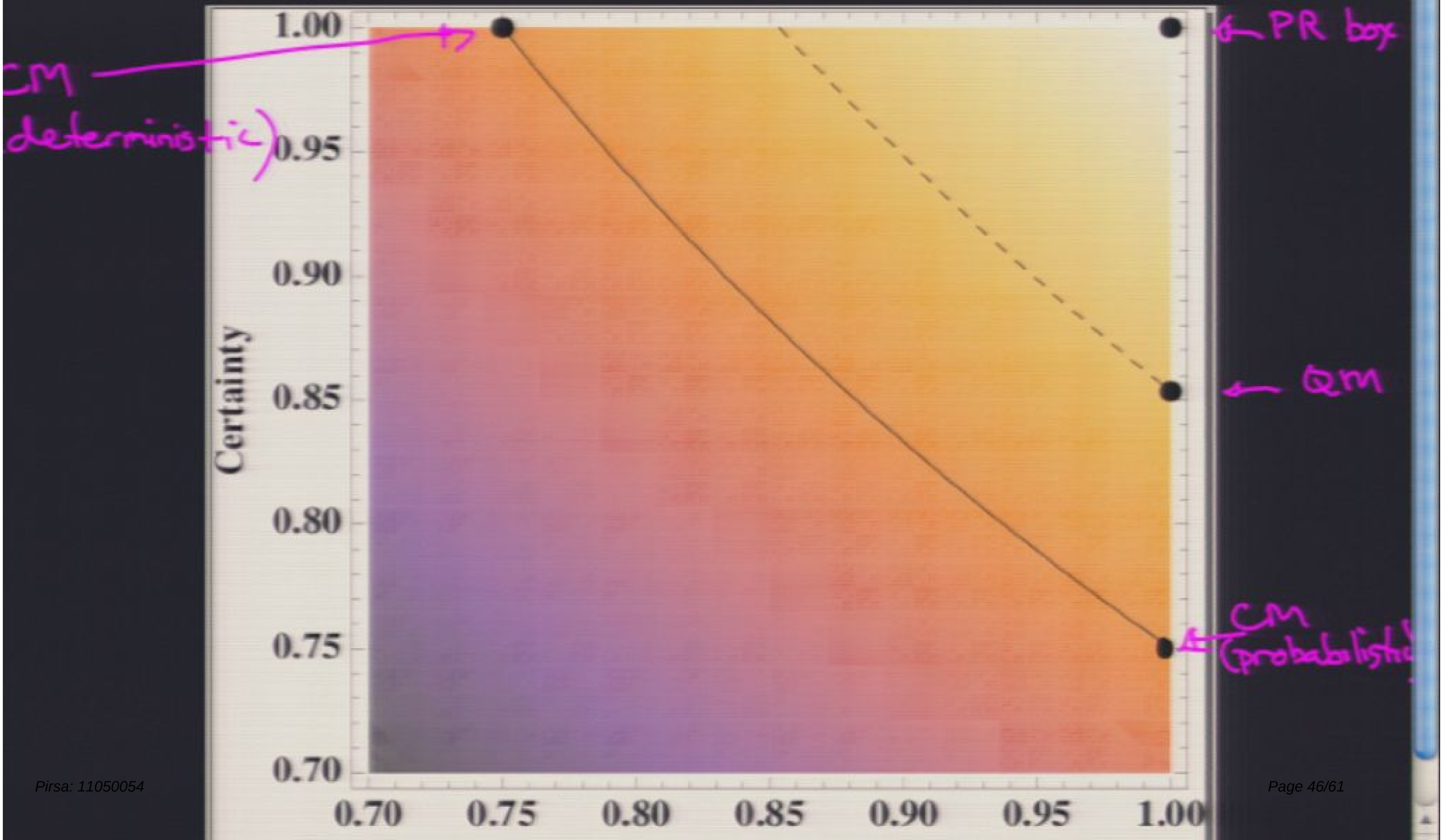
$$P^{game}_{max} = 3/4$$

probabilistic
(ie. LHV)

limited steering
some uncertainty



Theories: certeinty + steering = nonlocality



Complementarity

There are theories which are as uncertain and non-local as quantum theory, but have less complementarity.

$\vec{x}_{s,a}$

$s=0$ 00   11

$s=1$ 01 $\boxed{10}$

$$P^{win}_{QM} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

What if Bob wants to retrieve both bits?

Complementarity

"complementarity is when a red spider meets a green spider and their legs fall off." –Bob Coecke, PI, May 10, 2011

Information complementarity

	red	green
$s=0$	0	0
	1	1
$s=1$	$\boxed{0}$	1
	1	0

"other bit"

No signalling

PR boxes	$p(\text{"first"}) = 1$	$p(\text{other}) = 1/2$
... M...	$p(\text{"first"}) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$	$p(\text{other}) = 1/2$

Complementarity

There are theories which are as uncertain and non-local as quantum theory, but have less complementarity.

$$\vec{X}_{s,a}$$

	00
$s=0$	11
$s=1$	01
	$\boxed{10}$

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Information complementarity

$s=0$	0 1	0 1
$s=1$	$\boxed{0}$ 1	1 0

"other bit"

No signalling

PR boxes	$p(\text{"first"}) = 1$	$p(\text{other}) = 1/2$
[illegible]	$p(\text{"first"}) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$	$p(\text{other}) = 1/2$

Complementarity

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Information complementarity

$S=0$ 0 0   1 1

$S=1$ $\boxed{0}$ 1   1 0

"other bit"

No signalling

PR boxes	$p(\text{"first"}) = 1$	$p(\text{other}) = 1/2$
c) M...	$p(\text{"first"}) = \frac{1}{2} + \frac{1}{2\sqrt{2}}$	$p(\text{other}) = 1/2$

No signalling

$p(\text{parity}) = 1/2$

$p(\text{right}, \text{right}) + p(\text{wrong}, \text{wrong}) = 1/2$

If Bob tries to retrieve one bit then the other, we have:

$p(\text{right})\ p(\text{right} | \text{right}) + p(\text{wrong})\, p(\text{right} | \text{wrong}) = 1/2$

we can construct a N.S. theory which has same amount of nonlocality as QM but less complementarity

ie. $p(\text{right}) = \xi^*$ "first"

$p(\text{right}) > 1/2$ other

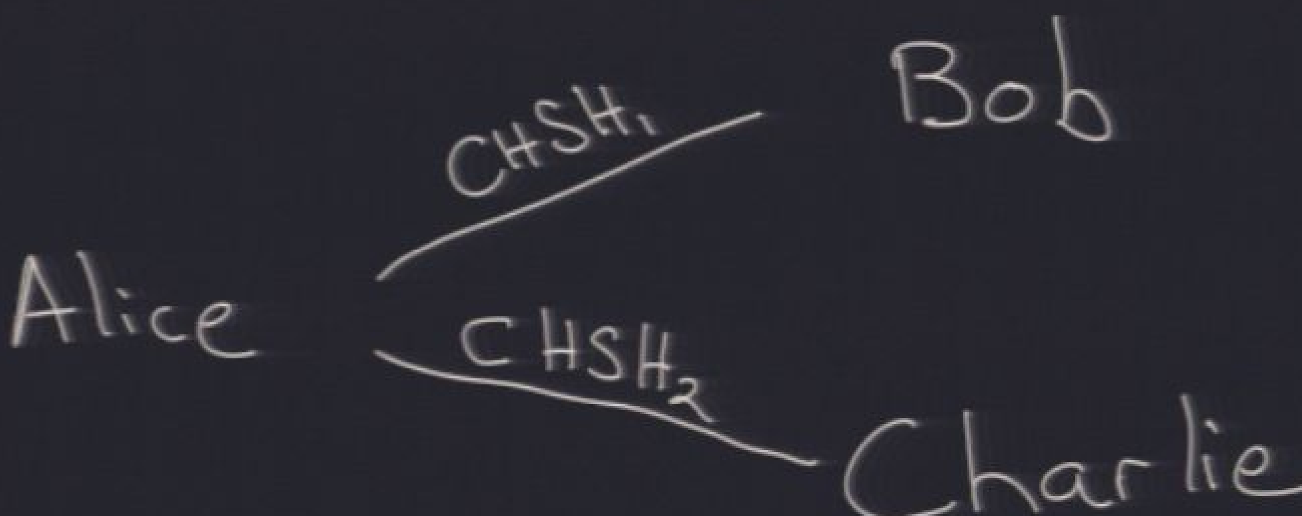
$$p(\text{right}|\text{wrong}) = 1$$

$$p(\text{right}|\text{right}) = \tfrac{1}{2}\xi^*$$

$$\underbrace{p(\text{right})\, p(\text{right}|\text{right})}_{1/2} + p(\text{wrong})\, \cancel{p(\text{right}|\text{wrong})} = 1/2$$

$\to 0$

Monogamy of nonlocality is a special case of information complementarity



(Toner, Verstraete)

Information Complementarity

$$\hat{H}(Y|\sigma) := \inf_M \bar{H}(Y|M(\sigma))$$

M a measurement on state σ

$\bar{H}$ any entropy (H_{min}, H etc)

$$\hat{\Delta}_I := \hat{H}(Y_1 Y_2 \cdots Y_n|\sigma) - \sum_i \hat{H}(Y_i|\sigma)$$

$$\hat{\Delta}_{max} := \max_C \hat{H}(Y_1 \cdots Y_n|\sigma) - \hat{H}(Y_C|\sigma C)$$

Complementarity and Noncontextuality

$\hat{\Delta}_I$ positive only for contextual measurements

related to positivity of
$I(X:Y) := H(X) + H(Y) - H(XY)$

$\hat{\Delta}_{max}$ can be arbitrarily large for $n=2$ (Vidick, Wehner)

Summary

Q.M. cannot be more nonlocal with measurements which respect the original uncertainty principle

For general theories, steering + uncertainty determines nonlocality.

If steering only restricted by N.S. the local state structure determines nonlocality

The complementarity of QM does not determined by N.S. + uncertainty

Summary

Q.M. cannot be more nonlocal with measurements which respect the original uncertainty principle

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If steering only restricted by N.S. the local state structure determines nonlocality

The complementarity of QM does not determined by N.S + uncertainty

Executive Summary

It might be useful to break down nonlocality into its constituent parts

- storage of alot of info in low capacity systems (steering)
- restrictions on retrieval (uncertainty & complementarity)

storage vs retrieval trade-off

Open Questions

In QM, does steering factor out for all nonlocal games? which?

What restricts complementarity?

What properties fully characterize quantum theory?

multiple measurements/parties

role of dimension

Thank you
for your
attention