Title: Symmetry, Self-Duality and the Jordan Structure of Quantum Theory

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Abstract: This talk reviews recent and on-going work, much of it joint with Howard Barnum, on the origins of the Jordan-algebraic structure of finite-dimensional quantum theory. I begin by describing a simple recipe for constructing highly symmetrical probabilistic models, and discuss the ordered linear spaces generated by such models. I then consider the situation of a probabilistic theory consisting of a symmetric monoidal *-category of finite-dimensional such models: in this context, the state and effect cones are self-dual. Subject to a further ``steering" axiom, they are also homogenous, and hence, by the Koecher-Vinberg Theorem, representable as the cones of formally real Jordan algebras. Finally, if the theory contains a single system with the structure of a qubit, then (by a result of H. Hanche-Olsen), each model in the category is the self-adjoint part of a C*-algebra.

Symmetry, Self-Duality, and the Structure of Quantum Theory

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(includes work in progress with Howard Barnum)

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Perimeter Institute, May 2011

Two Questions

(A) Plausibly, any physical system is associated with an ordered vector space **E** (in fact, an order-unit space), representing its observables. In QM, **E** is the Hermitian part of a C^* algebra. *Why?*

(B) Plausibly, any physical *theory* is associated with a category $\mathscr{C}$ of ordered vector spaces (morphisms representing processes). In QM, $\mathscr{C}$ is a dagger-symmetric monoidal category. *Why?*

Requiring that systems be highly symmetrical (in a sense to be discussed) casts light on both questions.

Outline

1) Background on ordered linear spaces
2) Operational probabilistic models
3) Fully symmetric models
4) Categories of FS models

1. Background on ordered linear spaces

An **ordered vector space** is a real vector space **E** fitted with a *positive cone* $\mathbf{E}_+ \subseteq A$:

- $\alpha, \beta \in \mathbf{E}_+$ and $s, t \in \mathbb{R}_+ \Rightarrow s\alpha + t\beta \in \mathbf{E}_+$;
- $\mathbf{E}_+ \cap -\mathbf{E}_+ = \{0\}$;
- $\mathbf{E}_+$ spans A;
- $\mathbf{E}_+$ is closed (topologically).

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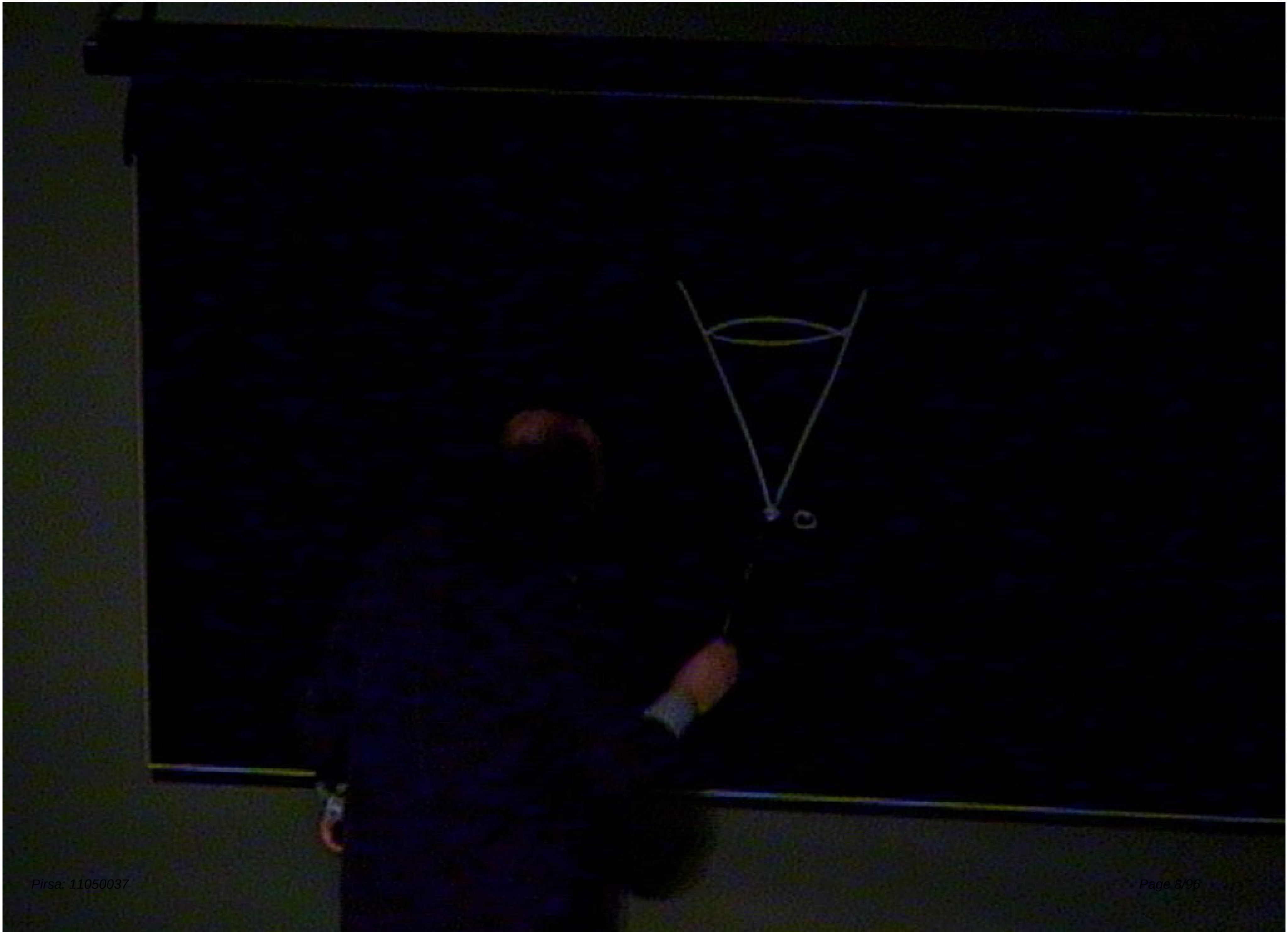

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Standard examples:

In QM, **E** is the Hermitian part of a complex matrix algebra, with its usual operator-theoretic order.

In classical probability theory, **E** is the space $\mathbb{R}^E$ of random variables on a (say, discrete) outcome-set E, ordered pointwise on E.

If **E**, **F** are ordered vector spaces, a linear mapping $f : \mathbf{E} \to$ is *positive* iff $f(\mathbf{E}_+) \subseteq_+$. The set of positive mappings is a cone, $\mathscr{L}_+(\mathbf{E},)$, the span of which is then an ordered linear space.

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Self-duality and homogeneity

Given an inner product on $\mathbf{E}$, the *internal dual* of $\mathbf{E}_+$ is the cone

$$\mathbf{E}^+ := \{b \in A | \langle b, a \rangle \geq 0 \; \forall \in \mathbf{E}_+\}.$$

One calls $\mathbf{E}$

- *self-dual* iff $\exists$ an inner product on $\mathbf{E}$ with $\mathbf{E}_+ = \mathbf{E}^+$.
- *homogeneous* iff $\forall a, b$ in the *interior* of $\mathbf{E}_+$, $\exists$ an isomorphism $\mathbf{E}_+ \to \mathbf{E}_+$ taking a to b.

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Two Theorems

Theorem (M. Koecher, E. B. Vinberg)

Any finite-dimensional homogeneous self-dual cone is isomorphic to the positive cone of a formally real Jordan algebra

Theorem (H. Hanche-Olsen)

If $\mathbf{E}$ *is a formally real Jordan algebra, and* $\mathbf{E} \otimes \mathbf{M}_2(\mathbb{C})$ *admits a Jordan structure such that*

$$(a \otimes x) \bullet (b \otimes y) = (a \bullet b) \otimes (x \bullet y)$$

then A is a complex matrix algebra

Can we motivate homogeneity and self-duality? I'll focus on self-duality.

2. Probabilistic Operational Models

We start with the following basic operational ideas:

A *test space* is a pair $(X, \mathfrak{A})$ where X is a non-empty set of "outcomes", and $\mathfrak{A}$ is a covering of X by non-empty sets, called *tests*, understood as the outcome-sets of various "measurements".

Two outcomes $x, y \in X$ are *distinguishable* iff there exists a test $E \in \mathfrak{A}$ with $x, y \in E$. (Optimistic) notation: $x \perp y$

A *state* or *probability weight* on $\mathfrak{A}$ is a mapping $\alpha : X \to [0, 1]$ with

$$\sum_{x \in E} \alpha(x) = 1$$

for all $E \in \mathfrak{A}$.

Definition: A *probabilistic model* (hereafter: model) is a structure $(A, \mathfrak{A}, \Omega)$ where $(X, \mathfrak{A})$ is a test space and Ω is a compact convex set of states on $(X, \mathfrak{A})$.

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Square: Let $\mathfrak{A} = \{\{x, x'\}, \{y, y'\}\}$. Geometrically, the space Ω of all states on $\mathfrak{A}$ is a square. Any closed convex subset thereof can serve as Ω. Call this model a *square bit* ("squit").

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Let $(X, \mathfrak{A}, \Omega)$ be a probabilistic model. Every outcome $x \in X$ defines a functional $x : \Omega \to \mathbb{R}$ by evaluation (that is, $x(\alpha) = \alpha(x)$ for all $\alpha \in \Omega$). It is harmless to identify x with this functional.

Definition: The *effect space*, $\mathbf{E} := \mathbf{E}(X, \mathfrak{A}, \Omega)$, associated with a model $(X, \mathfrak{A}, \Omega)$ is the span of X in $\mathbb{R}^\Omega$, ordered by the cone

$$\mathbf{E}_+ := \{\sum_i t_i x_i | x_i \in X, t_i \geq 0\}.$$

The *unit effect* in $\mathbf{E}$ is the vector $u \in \mathbf{E}_+$ given by $u(\alpha) \equiv 1$ for all $\alpha\Omega$. Equivalently,

$$u := \sum_{x \in E} x$$

where $E \in \mathfrak{A}$.

Standing Assumption: $\mathbf{E}$ is finite-dimensional.

Models linearized

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Composite systems I

If $\mathfrak{A}$ and $\mathfrak{B}$ are test spaces, let $\mathfrak{A} \times \mathfrak{B} = \{E \times F | E \in \mathfrak{A}, F \in \mathfrak{B}\}$.

A state $\omega \in \Omega(\mathfrak{A} \times \mathfrak{B})$ is **non-signaling** iff the marginal states

$$\omega_1(x) := \sum_{y \in F} \omega(x,y) \quad \text{and} \quad \omega_2(y) := \sum_{x \in E} \omega(x,y)$$

are independent of $E \in \mathfrak{A}$, $F \in \mathfrak{B}$.

Example: a *product state* $(\alpha \otimes \beta)(x,y) = \alpha(x)\beta(y)$. In general, there will exist many *entangled* non-signaling states not arising as mixtures of product states.

Composites II

Definition: A *product* of two models $A = (X, \mathfrak{A}, \Omega)$ and $B = (Y, \mathfrak{B}, \Gamma)$ is a model $(Z, \mathfrak{C}, \Theta)$ plus an injective mapping $X \times Y \to Z$, say $x, y \mapsto xy$, such that, for $E \in \mathfrak{A}$ and $F \in \mathfrak{B}$, $E \times F \mapsto EF \in \mathfrak{C}$, and

(1) states on $\mathfrak{C}$ restrict to non-signaling states on $\mathfrak{A} \times \mathfrak{B}$;

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(3) states on $\mathfrak{C}$ are uniquely determined by their restrictions to $\mathfrak{A} \times \mathfrak{B}$ ("local tomography"").

Lemma: *Let* $\mathbf{E}_A$ *and* $\mathbf{E}_B$ *be the effect spaces associated with models A and B. If AB is a non-singaling composite of A and B, then* $\mathbf{E}_{AB} = \mathbf{E}_A \otimes \mathbf{E}_B$.

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3. Models with symmetry

Let G be a compact group acting on X so as to preserve tests (that is, if $E \in \mathfrak{A}$, then $gE \in \mathfrak{A}$ for every $g \in G$. Call the triple $(X, \mathfrak{A}, G)$ a *G-test space*.

Definition: A *fully symmetric model* is a structure $(X, \mathfrak{A}, \Omega, G)$ where $(X, \mathfrak{A})$ is a G-test space, Ω is a separating set of states invariant under G, and

- every test has cardinality n (the *rank* of the model)
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Examples: Classial, quantum, square — all are FS.

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Digression: a recipe

One can *construct* such models as follows:

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Let G be a compact group acting on X so as to preserve tests (that is, if $E \in \mathfrak{A}$, then $gE \in \mathfrak{A}$ for every $g \in G$. Call the triple $(X, \mathfrak{A}, G)$ a *G-test space.*

Definition: A *fully symmetric model* is a structure $(X, \mathfrak{A}, \Omega, G)$ where $(X, \mathfrak{A})$ is a G-test space, Ω is a separating set of states invariant under G, and

- every test has cardinality n (the *rank* of the model)
- for all tests $E, F \in \mathfrak{A}$, every bijection $f : E \to F$ has the form $f(x) = gx$ for some $g \in G$
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Two parameters associated with any positive, invariant, normalized form B on $\mathbf{E}$:

- $r^2 \equiv B(x,x) \; \forall x \in X$
- $c \equiv B(x,y) \; \forall x \perp y \in X$

Lemma: *With notation as above,*

(a) $B(x,u) \equiv 1/n$

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Definition: A (positive, invariant, normalized) symmetric bilinear form B on $\mathbf{E}$ is *orthogonalizing* iff $c = 0$ — in other words, iff $B(x, y) = 0$ for all $x \perp y$ in X.

Lemma: *Let B be orthogonalizing. Then $\delta_x := nB(x, \cdot)$ is a state on $(X, \mathfrak{A})$ with $\delta_x(x) = 1$.*

Proof: The positivity of B makes $\delta_x(y) \geq 0$ for all $y \in X$. By (a) of the previous Lemma, $\delta_x(u) = 1$, so δ_x is a state. If B is orthogonalizing, then $\delta_x(y) = 0$ fr all $y \perp x$, so that $\delta_x(x) = 1$. □

Definition: Call $(X, \mathfrak{A})$ *sharp* iff, for every $x \in X$, there exists a unique state δ_x on $(X, \mathfrak{A})$ with $\delta_x(x) = 1$.

Lemma: *A fully-symmetric, sharp model is orthogonal iff it is SD.*

Proof: $(\Rightarrow)$ If the model is orthogonal, $n\langle x|$ takes value 1 at x. Hence, $n\langle x| = \delta_x$. By fully symmetry, every pure state has this form. Thus, $\mathbf{E}^+ \subseteq \mathbf{E}_+$.

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Irreducible models

Lemma: *Let B and B' be positive, invariant, symmetric, normalized bilinar forms on* **E**. *Then, for all $a \in \mathbf{E}$, $B(a,u) = 0$ iff $B(a',u) = 0$.*

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Example: QM; classical PT — but *not* the "square bit".

By the real form of Schur's Lemma, any symmetric bilinear form on **E** has the form

$$B_{\lambda}(a,b) := \lambda\langle a,b\rangle + (1-\lambda)\langle a,u\rangle\langle b,u\rangle$$

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$*$-monoidality

A little informally: our category $\mathscr{C}$ is $*$-*monoidal* iff it is equipped with

(a) a symmetric, associative, bi-functorial *monoidal product* $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, with $A \otimes B$ a non-signaling composite for all $A, B \in \mathscr{C}$;
(b) an involutive (=contravariant) functor $* : \mathscr{C} \to \mathscr{C}$ with $A^* = A$ for $A \in \mathscr{C}$, and $(f \otimes g)^* = f^* \otimes g^*$ for morphisms f, g.

Strong Assumptions:

(a) $\mathscr{C}$ is $*$-monoidal,
(b) I'll also requie that (a) for every $g \in G_A$, $g^* = g^{-1}$;
(c) the mapping $A \mapsto \mathbf{E}_A$ extends to a $*$-monoidal functor from $\mathscr{C}$ to a $*$-monoidal category $\mathscr{E}$ of ordered spaces and positive linear mappings.

For each $A \in \mathscr{C}$, we now have a canonical positive, G_A-invariant, postitive, symmetric bilinear form given by

$$\langle a, b \rangle := a \circ b^*$$

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Remark: One can generate such a category from (essentially) any functor from sets to groups, extending the functor $E \mapsto S(E)$. (Details: AW 2011)

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Lemma: *With notation as above, this canonical form is orthogonalizing on every system $A \in \mathscr{C}$ of rank ≥ 2.*

Proof: Let r and c be the parameters associated with the canonical form on $\mathbf{E}_A$, and c_2, the c-parameter for $\mathbf{E}_{A \otimes A}$. Let $x \perp y$ in X_A. Then $x \otimes x \perp x \otimes y \perp y \otimes x$. Hence, by monoidality,

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Hence, $r^2 c = c^2$, whence, either $c = 0$ or $c = r^2$. In the latter case, $r^2 = 1/n^2$, and $\dim(\mathbf{E}) = 1$, whence, the rank of $\mathfrak{A}$ is 1. □

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Corollary: Let $A \in \mathscr{C}$ with $\mathbf{E}_A$ irreducible.

(a) $\langle , \rangle$ is an inner product;
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To cope with reducible systems, one can require that the image (suitably defined) of any model in $\mathscr{C}$ under a surjective map of models, again belong to $\mathscr{C}$. This *image-closure* condition is operationally plausible.

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Homogeneity

One can motivate the assumption that $\mathbf{E}_+$ is homogeneous in various ways — e.g. that every state of any system in $\mathscr{C}$ be the marginal of a "steering" bipartite state) are enough to secure homogeneity. Subject to such a steering axiom, the foregoing tells us that every system in $\mathscr{C}$ is Jordan-algebraic. (More details: BGW 09)

If, in addition, $\mathscr{C}$ contains a system having the structure of a qubit, then Hanche-Olsen's result tells us that every system in $\mathscr{C}$ is not only Jordan-algebraic, but C^*-algebraic. (More details: Howard Barnum's talk later today!)

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