Title: Non-contextual correlations in probabilistic models

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Abstract: Non-contextuality is presented as an abstraction and at the same time generalisation of locality. Rather than in correlations, the underlying physical model leaves its signature in collections of expectation values, which are contrained by inequalities much like Bell's or Tsirelson's inequalities. These non-contextual inequalities reveal a deep connection to classic topics in graph theory, such as independence numbers, Lovasz numbers and other graph parameters. By considering the special case of bi-local experiments, we arrive at a semidefinite relaxation (and indeed a whole hierarchy of such relaxations) for the problem of determining the maximum quantum violation of a given Bell inequality.

Non-contextual correlations in probabilistic models

Andreas Winter (Univ. Bristol & CQT Singapore)

[w/ A. Cabello, S. Severini — arXiv: 1010.2163]

1. Gleason theorem and Kochen–Specker
2. Compatibility structures
3. Example: pentagon
4. Models and inequalities
5. Commuting observables; Bell inequalities
6. ~~Hierarchy~~ omitted
7. Conclusion

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Born rule $\langle X \rangle = \mathrm{Tr}\,\rho X$

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((Classical) Hidden Variables, but of a weird kind:

contextual — the same outcome P (e.g. a projector) is represented by multiple events $E \subset \Omega$ in the prob. space depending on the context in which it occurs, i.e. the different POVMs of which P is an element.

3

Gleason thm. In dim ≥ 3, only functions $v(P) \in [0;1]$ on projectors P s.t. $\begin{cases} \text{whenever } \sum_{i=1}^{n} P_i = \mathbb{1}, \\ \text{then } \sum_{i=1}^{n} v(P_i) = 1, \end{cases}$

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[Interpretation: There can be no non-contextual classical hidden variable model for the propositions of quantum mechanics!]

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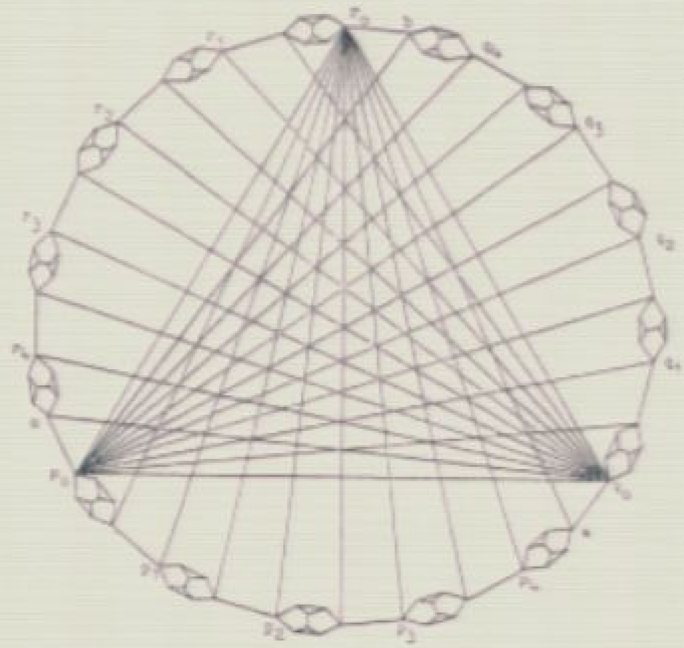

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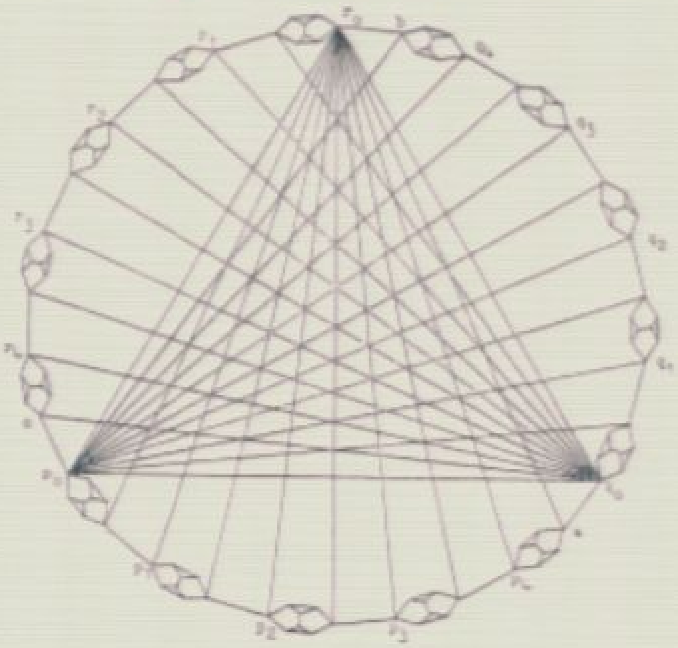

Other arrangements:

$\left.\begin{array}{l}\text{Peres (33)}\\ \text{Conway–Kochen (31)}\end{array}\right\}\ d=3$

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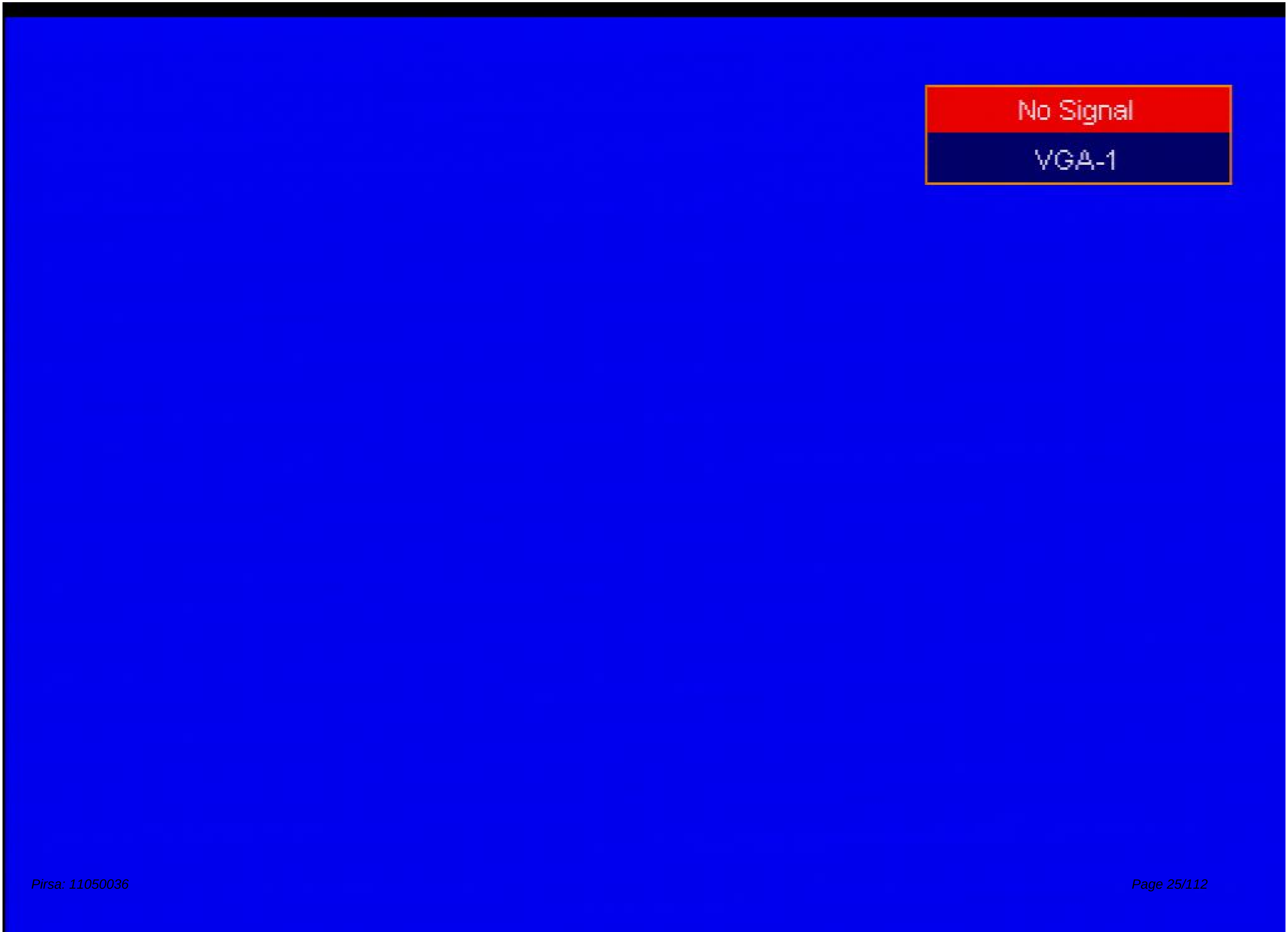
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This structure is then realised / interpreted in various probabilistic models.

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Classical / Non-contextual model:

...given by a measurable space Ω and measurable subsets $E_i \subset \Omega$ $(i \in V)$ s.t. for all $C \in \Gamma$ and all $i \neq j \in C$ $E_i \cap E_j = \emptyset$.

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Aka fractional packing of the hypergraph Γ.

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... given by a generalised probabilistic theory [with observable space $\mathcal{O}$ and positive cone $\mathcal{O}_+ \subset \mathcal{O}$]

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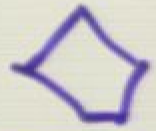
... if $tP \leq u$ implies $t \leq 1$ ...

3. Example: the pentagon 

$V = \{1, 2, 3, 4, 5\}$

$\Gamma = \{12, 23, 34, 45, 15\}$

A classical model: $\Omega = V = \{1, 2, 3, 4, 5\}$

$E_i = \{\text{neighbours of } i\}$

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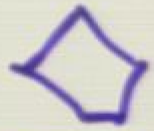
3. Example: the pentagon 

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(A state would be any distribution on Ω ...)

but note that always

$$\sum_i \langle P_i \rangle \leq 2.$$

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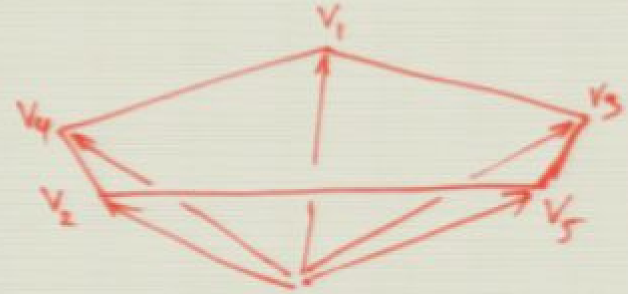
A quantum model : [Lovász 1979 ; Wright 1978 ; Klyachko et al. 2008]

$\mathcal{H} = \mathbb{C}^3$ and $P_i = |v_i\rangle\langle v_i|$



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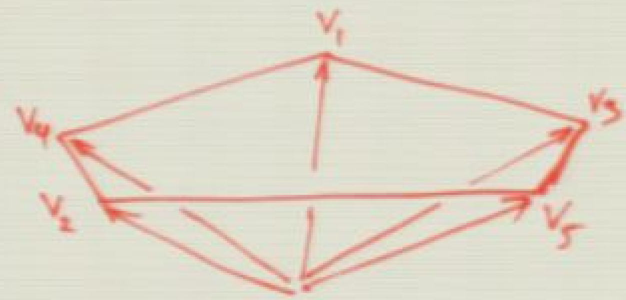

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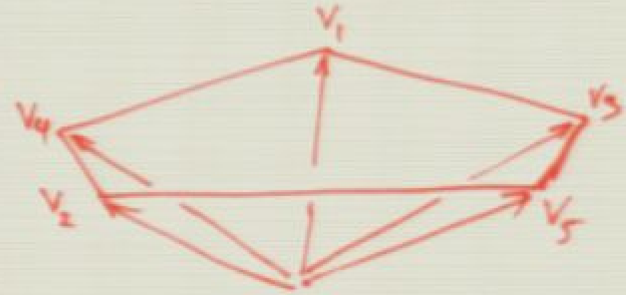

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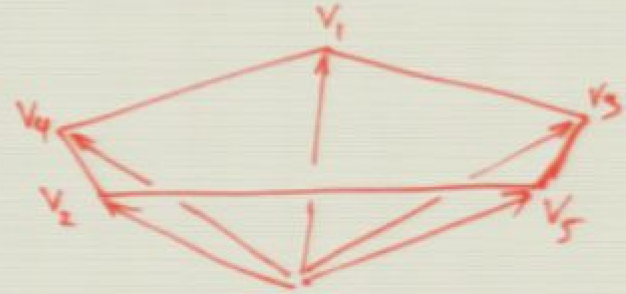

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Simply assign consistent weights $w_i \geq 0$ to $i = 1,2,3,4,5$:

E.g. $w_i = \frac{1}{2} \quad \forall i \in V$

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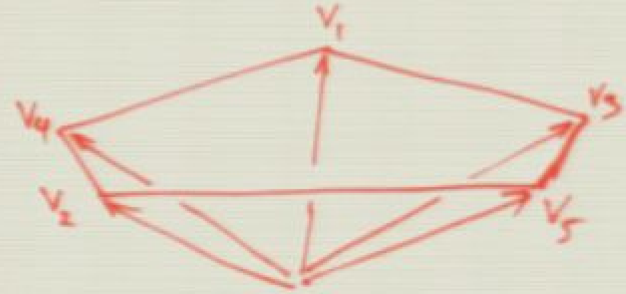

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$\beta_c(\Gamma)$, $\beta_{QM}(\Gamma)$, $\beta_{GPT}(\Gamma)$: maxima over respective sets.

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[Need only consider det. models, i.e. $w_i \in \{0,1\}$. These are the consistent valuations/colourings of Kochen-Specker: If $i \sim j$, at most one of w_i, w_j can be 1.
$S = \{i : w_i = 1\} \subset V$ independent ...]

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[Need only consider det. models, i.e.
$\omega_i \in \{0,1\}$. These are the consistent
valuations/colourings of Kochen-Specker:
If $i \sim j$, at most one of ω_i, ω_j can be 1.
$S = \{i : \omega_i = 1\} \subset V$ independent ...

More generally:

$\mathcal{E}_c(\Gamma) = \text{conv}\{1_S : S \subset V \text{ independent}\}$.

Computations on $\mathcal{E}_c$ — in particular $\beta_c(\Gamma)$ — are NP-hard.

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With adjacency graph G on V [edge $i \sim j$ iff $i,j \in C \in \Gamma$]:

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In contrast, $\mathcal{E}_{GPT}$ is easy (polynomial time). E.g.

$$\beta_{GPT}(\Gamma) = \alpha^*(\Gamma) \quad \text{"fractional packing number"}$$

$$= \max \sum_{i \in V} w_i \quad \text{s.t.} \quad w_i \geq 0$$

$$\forall C \in \Gamma \ \sum_{i \in C} w_i \leq 1$$

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Finally, quantum value(s):

$$\beta_{QM}(\Gamma) = \vartheta(G) \quad \text{"Lovász number"}$$

$$= \max \left\| \sum_{i \in V} |v_i\rangle\langle v_i| \right\| \quad \text{s.t. the } |v_i\rangle$$

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$\beta = \sum_i \langle P_i \rangle = \mathrm{Tr}\, \rho \sum_i P_i$ is maximised on a pure state $\rho = |\psi\rangle\langle\psi|$. Then can restrict to 1-dim. projectors $P_i = |v_i\rangle\langle v_i|$ & compatibility enforces orthogonality.

So, $\beta = \sum_i |\langle\psi|v_i\rangle|^2 = \langle\psi| \sum_i |v_i\rangle\langle v_i| \,|\psi\rangle$

and maximum is largest eigenvalue of $\sum_i |v_i\rangle\langle v_i|$.

Lovász (1979): $\vartheta(G)$ is a semidefinite programme.

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More generally, maximizing a linear function $\sum_i \lambda_i w_i$ over $\vec{w} \in \mathcal{E}_{QM}(\Gamma)$ [w.l.o.g. $\lambda_i \geq 0$!]

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More generally, maximising a linear function $\sum_i \lambda_i \omega_i$ over $\vec{\omega} \in \mathcal{E}_{QM}(\Gamma)$ [w.l.o.g. $\lambda_i \geq 0$!]

$$\max_{\text{QM-model}} \sum_i \lambda_i \langle P_i \rangle = \max \operatorname{Tr} \Lambda T \quad \text{s.t.} \quad T \geq 0,\ \operatorname{Tr} T = 1$$

and $i \sim j \Rightarrow T_{ij} = 0$,

where $\Lambda_{ij} = \sqrt{\lambda_i \lambda_j}$.

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Cor. Since these are semidefinite programmes, computations on $\mathcal{E}_{QM}$ are polynomial-time.

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5. Commuting observables & Non-locality

Also non-locality experiments give rise to compatibility structures: any observable of Alice commutes with any of Bob.

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Alice

settings $x \in \mathcal{X}$

outcomes $a \in \mathcal{A}$

projectors $P_{a|x}$

Bob

settings $y \in \mathcal{Y}$

outcomes $b \in \mathcal{B}$

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Alice	Bob
settings $x \in \mathcal{X}$	settings $y \in \mathcal{Y}$
outcomes $a \in \mathcal{A}$	outcomes $b \in \mathcal{B}$
projectors $P_{a\|x}$	projectors $P_{b\|y}$

atomic events $ab|xy$ with associated projectors $P_{ab|xy} = P_{a|x} \cdot P_{b|y} = P_{b|y} \cdot P_{a|x}$

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Compatibility structure:

$V = \{abxy\} = \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y}$

G has an edge $abxy \sim a'b'x'y'$ iff $(x = x'$ and $a \neq a')$
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Hypergraph Γ of all cliques of G

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Hypergraph Γ of all <u>cliques</u> of G

<u>Note</u>: G (and Γ) captures coexistence and mutual exclusiveness of elementary events $abxy$. But <u>not</u> that for given xy, <u>one</u> of $abxy$ $(a \in \mathcal{A}, b \in \mathcal{B})$ has to occur with probability 1 ...

(20

So, add the constraint that $\forall xy \sum_{ab} w_{ab|xy} = 1$:

$$\mathcal{E}_{GPT}(\Gamma)$$
$$\cup$$
$$\mathcal{E}_{QM}(\Gamma)$$
$$\cup$$
$$\mathcal{E}_{C}(\Gamma)$$

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\begin{array}{ccc}
\Sigma_{GPT}(\Gamma) & \supset & \Sigma^1_{GPT}(\Gamma) \\
\cup & & \cup \\
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Proposition: • $\Sigma^1_C(\Gamma)$ are exactly the local (hidden variable) corr.

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Furthermore, maximum values of expressions like

$$\sum_{abxy} \lambda_{abxy} \Pr\{ab|xy\}$$

[w.l.o.g. with $\lambda_{abxy} \geq 0$] over local / no-signalling correlations, i.e. over $\mathcal{E}'_C$ / $\mathcal{E}'_{GPT}$ are the same as maximised over $\mathcal{E}_C$ / $\mathcal{E}_{GPT}$.

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I.e. local (and no-signalling) values of a Bell-type correlator can be interpreted 1-1 as non-contextual (and Gleason) values of a canonically associated compatibility structure.

Maximum quantum violations can be upper bounded by the (efficiently computable!) maximum over $\mathcal{E}'_{QM}$.

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Example 1: CHSH $\left(\mathcal{A} = \mathcal{B} = \mathcal{X} = \mathcal{Y} = \{0,1\} \right)$

Expression to maximise is

$$\sum_{\substack{abxy \\ \text{s.t.} \\ a\oplus b = xy}} w_{ab|xy} \quad . \quad \text{I.e.} \quad \lambda_{abxy} = \begin{cases} 1 \text{ if } a\oplus b = x\cdot y, \\ 0 \text{ otherwise} \end{cases} .$$

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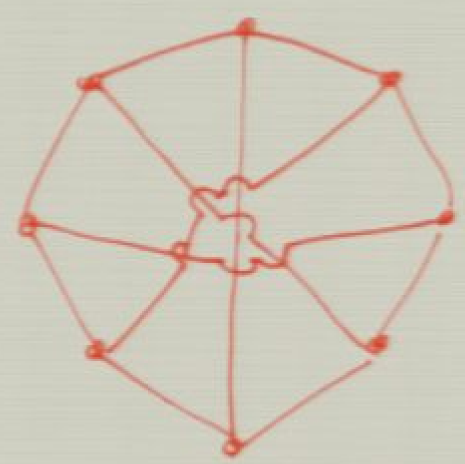
Turns out that local, quantum and no-signalling values of this function are just

$\alpha(G) = 3$,

$\vartheta(G) = 2+\sqrt{2}$,

$\alpha^*(\Gamma) = 4$, resp.

of the following 8-vertex graph $G = C_8(1,4)$:



Example 2 : I_{3322} [Collins/Gisin, J.Phys A 2004] (23

$(\mathcal{A} = \mathcal{B} = \{0,1\};\ \mathcal{X} = \mathcal{Y} = \{0,1,2\})$

Maximise $\sum_{abxy} \lambda_{abxy}\, w_{ab|xy}$ where:

$\vec{\lambda} =$

xa \ yb	00	01	10	11	20	21
00	1	0	1	0	1	0
01	0	0	1	1	1	1
10	1	1	1	0	0	1
11	0	1	1	1	1	1
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Local value (max over Σ_c') is 6, the independence number of a 20-vertex graph.

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Local value (max over Σ_C^1) is 6, the independence number of a 20-vertex graph.

Maximum over Σ_{QM}^1 is ≈ 6.25147

[Currently best upper bound on quantum value is 6.25087556 using the Navascués-Acín-Pironio hierarchy of SDP relaxations of Σ_{QM}^{1} (Pal/Vértesi, arXiv:1006.3032); lower bound 6.250[illegible]38]

7. Conclusion / Questions

- Noncontextual inequalities give a new look on Gleason- and Kochen-Specker theorems.

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- Noncontextual inequalities give a new look on Gleason- and Kochen-Specker theorems.

- Unlike Bell inequalities, it is very easy to find examples of very large violations:

  $\exists$ graphs on n vertices with $\alpha(L^{(n)}) = O(\log n)$,
  but $\vartheta(L^{(n)}) \approx \sqrt{n}$;

  and $\alpha(C^{(n)}) = 3$, but $\vartheta(C^{(n)}) \approx \sqrt[4]{n}$.

- Open: Can any violation of a non-contextual
(quantum)

inequality be converted into a (comparably large?) violation of a Bell inequality?

- Open: Can any (quantum) violation of a non-contextual inequality be converted into a (comparably large?) violation of a Bell inequality?

- How different are $\mathcal{E}'_{QM}$ and $\mathcal{E}^{1}_{EQM}$ in the non-local setting?
How good is the maximum over $\mathcal{E}'_{QM}$ of a Bell inequality as upper bound on the maximum quantum violation?

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No Signal

VGA-1

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Increasing
entangle...111.pdf

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